Utrecht University

An Introduction to Geometric Quantization

Author:
Artim BASSANT
6183530

Supervisor:
Dr. Thomas GRIMM
Dr. Gil CAVALCANTI

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"What I am going to tell you about is what we teach our physics students in the third or fourth year of graduate school... It is my task to convince you not to turn away because you don't understand it. You see my physics students don't understand it... That is because I don't understand it. Nobody does."

Richard Feynman

UNIVERSITY UTRECHT

Abstract

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In this thesis, we will be attempting to derive the quantum operators from their classical analogue. It begins by clearly defining what an observable should be in a classical system with the help of symplectic geometry. This turns out to be a continuous function on the space of the system. The observable will, therefore, generate a vector field and a flow.

Now we search for the mapping procedure of classical observables to quantum operators. Dirac defined some properties for quantum operator and this will lead us to the mapping procedure of prequantization. The mapping procedure closely resembles a connection working on a section of a Hermitian line bundle. Therefore we will need to prove the existence of such a structure and this will lead us to the Weil's integration condition. We have constructed prequantization on a symplectic manifold with sections that represent the wavefunctions of a quantum system. But we find that prequantization fails a lot of the cases we introduce it to.

The problem is that the sections are dependant on all coordinates and this is just not the case in quantum mechanics. Thus a restriction is needed and therefore polarization was introduced. This brings us to polarized sections where the operators work on. The polarizations induce their own problems. They need to be preserved when applying the operator. Or else the sections become "depolarized" and are projected in a new space of different polarized sections. Observables that do preserve the polarization are correctly mapped to their corresponding operators. An example of an observable that doesn't preserve the polarization would be the Hamiltonian of a free particle.

The BKS construction gives us some hope in recovering the Hamiltonian of the free particle.

We conclude that there are many weaknesses, nonetheless a very insightful and mostly mathematical rigorous procedure to find correct operators.

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Chapter 1

Introduction

When someone has done a course on quantum mechanics they derive the quantum observables by using symmetries of the space to predict the correct operators. But we are constrained by the symmetries of space. So it may be interesting to provide a way of finding observables where we are (almost) free to choose our observables from the classical systems and give a more mathematical approach to quantum mechanics.

Quantization is the attempt at producing these operators from their classical observables. It may be counterintuitive since quantum mechanics is a refinement of classical systems and this will have its consequences.

This idea started in 1927 when Hermann Weyl proposed the first quantization called the Weyl quantization. The attempt was to produce a mapping to simple operators for phase space, but it did give some nonphysical answers. H.J. Groenewold added more insight into this idea in 1946. He showed why the quantization procedure is so restricted. Modern quantization is developed by Bertram Kostant and Jean-Marie Souriau in the 1970s. There are many more who contributed to Geometric quantization besides these big names.

Geometric quantization will not only be a mathematical backbone to some quantum mechanics. It also fulfils the desire to make some sense of how the operators are produced. We use advanced mathematics, but sometimes it is too sophisticated for a bachelor thesis. Then we will use physical consequences to argue our way to geometric quantization. Firstly it is important to identify observables of the classical system and a way to do this is by using symplectic geometry. When we establish the mathematical objects in classical systems we can begin with the prequantization. The prequantization is the very first attempt at mapping the observables to their corresponding operators. The mapping is developed by finding a mapping that satisfies all of Dirac's conditions.

A quick analysis of the mapping shows that it is dependant on the symplectic potential and therefore the mapping is not unique and not global. This can be corrected by introducing a gauge transformation and a Hermitian line bundle such that the mapping gives us an operator that works on sections rather then functions on the manifold. When we have fixed the superficial problems we find that prequantization fails simple examples. It is clear that prequantization is doomed to fail and the correction we then apply will result in geometric quantization.

The correction will be to introduce polarizations. This will not be the end of the procedure because when we make this correction new problems arise such as depolarization and inner products that do not converge. Depolarization will result in a restriction of observables and the BKS construction. The inner product problem will be fixed by using half-forms.

Along the way, we will also show that the Fourier transform between position space and momentum space is just a result of mapping polarizations onto each other.

I will heavily rely on the book from N.M.J. Woodhouse called *Geometric Quantization* **[1]**. It is a very inspiring book and shows how much mathematics is involved in physics. The book treats this subject in way more detail then I do, so if the reader is interested in this subject I would highly recommend to read it.

Chapter 2

Symplectic geometry

2.1 A brief introduction to symplectic geometry

I will assume that the reader does have some knowledge of differential geometry. But a brief introduction to symplectic geometry may ease the reader a bit. (Some key definitions of differential geometry will be added in appendix A)

Definition 1 *A* ***symplectic vector space*** *is a finite dimensional vector space V with a closed non-degenerate 2-form $\omega \in \Omega^2(V)$ defined on it. The tangent space at a point in V is equal to the vector space. The symplectic 2-form acts on the elements of the vector space as,*

- $\omega(X,Y) = -\omega(Y,X)$ *for every* $X,Y \in V$,
- $X \lrcorner \omega = 0$ *only if* $X = 0$,
- ω *is bilinear.*

A symplectic vector space is often denoted as the pair (V,ω) and has even dimensions.

An example of a symplectic vector space would be the cotangent space. Let $V = T^*\mathbb{R}^n$ and $\{p_a, q^a\}$ for $a = 1 \dots n$ is a coordinate system where $\{q^a\}$ are the coordinates of $\mathbb{R}^n$ and $\{p_a\}$ are the coordinates of $T^*_x\mathbb{R}$ for $x \in \mathbb{R}$. The ***canonical symplectic 2-form*** is $\omega = dp_a \wedge dq^a$. This space is very useful. It holds all the variables that we need for an object moving through flat euclidean space.

Definition 2 *A* ***symplectic manifold*** *is a smooth manifold M and has a closed non-degenerate 2-form $\omega \in \Omega^2(M)$ defined on it. In other words,*

$$d\omega = 0, \tag{2.1}$$

and the map,

$$T_m M \to T^*_m M : X \to X \lrcorner \omega \tag{2.2}$$

is a linear isomorphism at each point in M.

A symplectic manifold is often denoted as the pair (M,ω).

Proposition 1 (Symplectic frame) *Let (V, ω) be a 2n-dimensional symplectic vector space. Then V has a basis $\{p_a, q^a\}$ for all $a = 1, \ldots, n$ such that,*

$$\omega(q^a, q^b) = 0, \quad (2.3) \qquad \omega(p_a, p_b) = 0, \quad (2.4)$$

$$2\omega(q^a, p_b) = \delta^a_b. \quad (2.5)$$

Definition 3 *Let (V, ω) be a symplectic vector space with a symplectic 2-form. For any $X \in V$ then $Y \in V$ is **symplectic orthogonal** to X if $\omega(X, Y) = 0$.*

Definition 4 *Let (V, ω) be a symplectic vector space with a symplectic 2-form. A subspace $F^{\perp}$ of V is called the **symplectic complement** of $F \subset V$ if $F^{\perp} = \{X \in V | \omega(X, Y) = 0 \quad \forall Y \in F\}$.*

Definition 5 *Let (V, ω) be a symplectic vector space with a symplectic 2-form. A subspace $L \subset V$ is a **Lagrangian subspace** if $L = L^{\perp}$.*

Proposition 2 *Let (V, ω) be a symplectic vector space and $L \subset V$ be a subspace of V. L is a Lagrangian subspace if and only if $\omega|_L = 0$ and $\dim L = 1/2 \dim V$.*

We can generalise the Definition of the Lagrangian subspace to a symplectic manifold.

Definition 6 *Let (M, ω) be a symplectic manifold and $L \subset M$ a submanifold. Then if $T_m L \subset T_m M$ has corresponding properties to Definition 5. Then we can identify L as a Lagrangian submanifold on M.*

Notice that a Lagrangian subspace has the property that $\omega(X, Y) = 0$ for all $X, Y \in L$ and has exactly half the dimensions of the symplectic vector space.

Definition 7 *Let (V, ω) be a symplectic vector space with a symplectic 2-form. A **canonical transformation** of a symplectic vector space (V, ω) is a linear map $\rho : V \to V$ such that,*

$$\omega(\rho X, \rho Y) = \omega(X, Y). \quad (2.6)$$

For every $X, Y \in V$.

Definition 8 *Given two symplectic manifolds (M, ω_M), (N, ω_N) a diffeomorphism $\rho : M \to N$ is a **symplectomorphism** if $\rho^*(\omega_N) = \omega_M$.*

These results can also be applied when a vector space is complex finite-dimensional. This will make it easier to introduce complex manifolds in section 5.2.3. In the next chapter, we will show how symplectic manifolds describe a classic system. A very important theorem will be needed.

Theorem 1 (Darboux's theorem) *Let (M, ω) be a 2n-dimensional symplectic manifold and let $m \in M$. Then there exists a neighbourhood U of m and a coordinate system $\{p_a, q^a\}$ with $a, b = 1, 2 \ldots n$ such that $\omega = dp_a \wedge dq^a$ in U.*

The proof will be added in the appendix. The take away message of this theorem is that any symplectic manifold holds some local similarities with cotangent spaces just like in the example. A classical system can be well described when considering its cotangent space because it can hold position space and momentum space. That's why we often choose the symplectic frame to be $\{p_a, q^a\}$. Those are the degrees of freedom of space and their conjugate momentum.

Definition 9 *Let M be a smooth manifold. A* ***(real) distribution*** *is a subbundle of the tangent bundle. Let D be a distribution of M, then the fibre D_m varies smoothly over $m \in M$.*

Definition 10 *Let M be a smooth manifold. An* ***integrable distribution*** *is a distribution that is involutive. Let D be an integrable distribution of M. Then $[X, Y] \in D_m$ for all $X, Y \in D_m$ for every point $m \in M$.*

Definition 11 *Let M be an n-dimensional smooth manifold. A* ***decomposition*** *of M are disjoint, connected, nonempty, immersed k-dimensional submanifolds Λ_α of M and they are called* ***leaves****.*

$$M = \cup_\alpha \Lambda_\alpha \tag{2.7}$$

Definition 12 *Let M be a n-dimensional smooth manifold and there is a coordinate system $\{q^a\}$ for $a = 1, \ldots, n$. A set of leaves are called a* ***foliation*** *if there is for every neighbourhood of a point in M a smooth chart (U, φ) on M which $\varphi(U)$ is a cube in $\mathbb{R}^n$. Such that for each leaf Λ_α intersects U in either the empty set or a countable union of k-dimensional surfaces of constant $q^{k+1}, q^{k+2}, \ldots q^n$.*

The next theorem tells us that when we have an integrable distribution, then we have a foliation.

Theorem 2 (Global Frobenius theorem) *Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

An example would be: Consider a smooth manifold C and a 2-form σ. Then we can define,

$$K_m = \{X | X \lrcorner \sigma = 0\} \subset T_m C \tag{2.8}$$

If σ is defined such that K_m is dimensionally constant as m varies over C. Then K is is a distribution on C and called the ***characteristic distribution*** of σ.

Let $X, Y \in V_K(C)$ and for all $W \in V(C)$,

$$d\sigma(X, Y, W) = -\sigma([X, Y], W)$$[1]

Thus whenever σ is closed then the distribution has for every $X, Y \in K_c$ that $[X, Y] \lrcorner \sigma \in K_c$ for every point $c \in C$. We conclude that the characteristic distribution of σ is

[1] This follows from the identity $d\alpha(X_i, X_k, X_l) = \frac{1}{3!}\sum sgn(\sigma) X_{\sigma(i)} \alpha(X_{\sigma(k)}, X_{\sigma(l)}) - \frac{3}{2}\frac{1}{3!}\sum sgn(\sigma)\alpha([X_{\sigma(i)}, X_{\sigma(k)}], X_{\sigma(l)})$

integrable. Thus by the Global Frobenius theorem also a foliation. Then we call it the ***characteristic foliation***.

Definition 13 *Let C be a smooth manifold and K is a foliation on C. A foliation is called* ***reducible*** *if the space C/K of leaves is a Hausdorff manifold.*

Definition 14 *Let the pair (C, σ) be a smooth manifold and a 2-form that is closed and of constant rank. (C, σ) is a* ***presymplectic manifold*** *whenever the characteristic foliation is reducible.*

Definition 15 *Let C be a smooth manifold and K is a distribution on C. The set of vector fields that are symplectic orthogonal to every element of distribution K is denoted as $V_K(C)$.*

Chapter 3

Classical systems

3.1 Lagrangian and Hamiltonian mechanics

In this section, we will reveal what an observable should be in a classical system. Such that we can develop the mapping from classical observables to quantum operators. We will also prove that there always exists a local symplectic potential. This will bring us to the local generating function which can describe a Lagrangian submanifold. After that, a quick introduction on how to calculate a symplectic 2-form for a system with a Lagrangian.

In the last section, we showed the basics of symplectic geometry. We have chosen to use this mathematics because it is a very natural way of describing how classical systems behave. A classical observable is some parameter of the classical system that can be measured and it should generate a set of canonical transformations. For example, the Hamiltonian of a classical system generates time evolution. Given a symplectic manifold (M, ω) that describes a given classical system. Then these classical observables are smooth functions on the phase space $f \in C^\infty(M)$.

When searching for the canonical transformation of a classical observable then we have to introduce a vector field X_f generated by f.

Definition 16 *Let (M, ω) be a symplectic manifold with a symplectic 2-form. Given a classical observable $f \in C^\infty(M)$. Then the* ***Hamiltonian vector field*** *is defined by,*

$$X_f \lrcorner\, \omega - df = 0. \tag{3.1}$$

An argument for this definition can be made when the observable is the Hamiltonian $h \in C^\infty(M)$. Let us consider a region in M small enough such that Darboux's theorem can be applied with the coordinate system $\{p_a, q^a\}$ with $a, b = 1, 2 \dots n$, therefore the symplectic 2-form has the form $\omega = dp_a \wedge dq^a$. Let the Hamiltonian vector field be of the form $X_h = g\frac{\partial}{\partial p_a} + f\frac{\partial}{\partial q^a}$ with $g, f \in C^\infty(M)$. Then the equation (3.1) gives the equality,

$$-gdq^a + fdp_a - dh = -gdq^a + fdp_a - \frac{\partial h}{\partial p_a}dp_a - \frac{\partial h}{\partial q^a}dq^a = 0. \tag{3.2}$$

This equation implies that $X_h = -\frac{\partial h}{\partial q^a}\frac{\partial}{\partial p_a} + \frac{\partial h}{\partial p_a}\frac{\partial}{\partial q^a}$. When calculating the flow we find,

$$\frac{\partial h}{\partial q^a} = -\dot{p}_a, \tag{3.3}$$

$$\frac{\partial h}{\partial p_a} = \dot{q}^a. \tag{3.4}$$

The dot represents the time derivative. These equations are exactly Hamilton's equations. The space of Hamiltonian vector fields is denoted by $V_H(M)$.

Consider $\gamma(t)$ an integral curve of X_h, where h is the Hamiltonian. For an observable f to be conserved in time, it has to satisfy $\frac{\mathrm{d}}{\mathrm{dt}}(f \circ \gamma) = 0$. This can also be seen as the Poisson brackets $\{f, h\} := \omega(X_f, X_h) = df(X_h) = 0$.

Lemma 1 (Local Exactness of Closed forms) *Let M be a smooth manifold with or without a boundary. Each point of M has a neighbourhood on which every closed form is exact.*

This is corollary 17.15 from source [3].

I will give a quick proof of how a closed 1-form $\phi \in \Omega^1(M)$ is exact in a neighbourhood of a point $m \in M$.

Lemma 2 *Let there be a closed covector field $\phi \in \Omega^1(M)$ on a smooth manifold M. Then every point of M has a neighbourhood on which ϕ is exact.*

Proof Let $m \in M$ be arbitrairy and $\phi = \phi_i du^i$ for $i = 1, \ldots, n$ is closed. Then choose a ball $U \subset M$ that is in the neighbourhood of m and containing m. A ball is convex, thus also simply connected. Choose a $c \in U$. Apply a translation to U such that $c = 0$. A translation is a diffeomorphism, therefore closed forms stay closed forms and exact ones stay exact. Since U is simply connected we can define a path for any point $u \in U$ such that $\gamma_u : [0,1] \to U$ with the expression $\gamma_u(t) = tu$. The image of the path for every u is fully contained in U. Now we can use this line segment γ_u to define an integral. Define a function $f : U \to \mathbb{R}$ by,

$$f(u) = \int_{\gamma_u} \phi. \tag{3.5}$$

The integral exists because the line segment is smooth and ϕ is bounded.

We need to show that f is a potential of ϕ. This implies that $\frac{\partial f}{\partial u^j} = \phi_j$ for $j = 1, \ldots, n$. Let's compute f and use the summation convention,

$$f(u) = \int_0^1 \phi_{\gamma_u(t)}(\gamma_u'(t))dt = \int_0^1 \phi_i(tu)u^i dt. \tag{3.6}$$

Now let's compute the partial derivatives.

$$\frac{\partial f}{\partial u^j} = \int_0^1 t\frac{\partial \phi}{\partial u^j}u^i + \phi_j(tu)dt \tag{3.7}$$

Note that the integral and the partial derivative can be changed places because the integral is smooth over all of its variables. Remark that $\frac{\mathrm{d}}{\mathrm{dt}}(t\phi_j(tu)) = t\frac{\partial \phi}{\partial u^j}u^i + \phi_j(tu)$.

$$\frac{\partial f}{\partial u^j} = \int_0^1 \frac{\mathrm{d}}{\mathrm{dt}}(t\phi_j(tu))dt = \left[t\phi_j(tu)\right]_0^1 = \phi_j(u) \tag{3.8}$$

Hereby we have shown that there exists a locally exact form of a 1-form on a smooth manifold. ■

If we consider (M, ω) a symplectic manifold then lemma 1 tells us that every symplectic 2-form is exact in a neighbourhood of every point in M.

Definition 17 *Let (M, ω) be a 2n-dimensional symplectic manifold and a symplectic 2-form. Then a* ***symplectic potential*** *is a 1-form $\theta \in \Omega^1(M)$ such that $d\theta = \omega$ in a neighbourhood of a point in M.*

When Darboux's theorem is applicable then the symplectic potential $\theta \in \Omega^1(M)$ can be of the form $\theta = p_a dq^a$ in the coordinate system $\{p_a, q^a\}$. Note that we can add du with $u \in \Omega^0(M)$ to the symplectic potential without changing the symplectic 2-form ω. This will be a concern when dealing with prequantization.

With this new insight we can again look at a Lagrangian submanifold $L \subset M$ where $\omega|_L = 0$ and $\dim L = (\dim M)/2$. The 2-form vanishes on the Lagrangian submanifold with the consequence that $0 = \omega|_L = d\theta|_L$ the symplectic potential is closed on the Lagrangian submanifold. Thus lemma 1 implies that there exists an $S \in C^\infty(M)$ such that $\theta|_L = dS|_L$ on a neighbourhood for every point in L.

Definition 18 *Given (M, ω) a symplectic manifold with a symplectic potential $\theta \in \Omega^1(M)$ and let L be a Lagrangian submanifold on M. Then a* ***generating function*** *is a function $S \in C^\infty(M)$ such that $dS|_L = \theta|_L$ in a neighbourhood of every point in L.*

Let L be the Lagrangian submanifold of a symplectic manifold M. When the region on L is small enough, then we have that the symplectic potential is $\theta = p_a dq^a$ and $\theta|_L = \frac{\partial S}{\partial q^a} dq^a|_L$. There exists such a generating function S for a neighbourhood of every point in L, thus we can define the Lagrangian submanifold also as $L = \{p_a = \frac{\partial S}{\partial q^a}\}$. Notice that this is the formula for the Hamilton-Jacobi theory. Once S is identified with canonical transformation produced by the Hamiltonian, one can identify S as the action integral.

3.1.1 Application and limits of the symplectic 2-form

One might ask after reading this how it can be applied to any classical system. Well, when dealing with a space of motion we can make it into an even dimensionally space by also considering the tangent spaces or the space of velocities. We establish the coordinate system $\{q^1, \ldots, q^n, v^1, \ldots, v^n\}$ and let $v^a = \dot{q}^a$. In any classical system we have an action integral that governs the motion of the objects.

$$I = \int_{t_1}^{t_2} L(q, v) dt \tag{3.9}$$

Where $L \in C^\infty(\mathbb{R}^{2n})$ is the Lagrangian defined as the kinetic energy minus the potential energy. Any object in the system tries to minimise its action. We can find the differential equation that minimises the action via the Euler-Lagrange equation.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial q^a} = 0 \tag{3.10}$$

This implies that there are trajectories that objects follow. These trajectories usually fill the space. Thus every point has one trajectory with a derivative in tangent space. These vectors combined create a vector field X. This vector field has to satisfy,

$$X \lrcorner \, \omega_L + dh = 0. \tag{3.11}$$

Where we define the closed 2-form as,

$$\omega_L = \frac{\partial^2 L}{\partial q^a \partial v^b} dq^a \wedge dq^b + \frac{\partial^2 L}{\partial v^a \partial v^b} dv^a \wedge dq^b. \tag{3.12}$$

And define the Hamiltonian as the Legendre transformation of the Lagrangian.

$$h = v^a \frac{\partial L}{\partial v^a} - L \tag{3.13}$$

ω defines a symplectic structure in the space of the classical system C. But this definitely doesn't have to be the case!

Definition 19 *Let C be a smooth vector space and ω_L a closed 2-form on C. C is a classical system where there is a Lagrangian $L \in C^\infty(C)$ defined. The Lagrangian is **regular** whenever ω_L is everywhere non-degenerate or equivalently,*

$$det\left[\frac{\partial^2 L}{\partial v^a \partial v^b}\right] \neq 0. \tag{3.14}$$

When L is regular then (3.11) is equivalent to (3.10).

When L is irregular (or does not satisfy Definition 19). Then equation (3.11) doesn't have to be satisfied and ω_L is degenerate! This can be solved in some cases by a special choice of vector field. But when the space resembles a presymplectic space (C, ω_L) you could try to reduce it to a symplectic space. Where ω_L projects on some symplectic structure ω' on $M' = C/K$. We call (M', ω') the *reduction* of (C, ω_L) or the *reduced phase space*. I consider this one of the first weaknesses of using this method. When it is not possible to reduce the space to a symplectic space, then it can't be quantized.

Chapter 4

Prequantization

4.1 Dirac's quantization conditions

Quantization has been developed to show that quantum mechanical observables can (in some cases) be derived from there classical counterparts. Let us take a step back and look at what there has to be done. We already established that any observable in classical mechanics is a smooth function on a symplectic manifold. Somehow we have to find a way to link the classical observables to their quantum operators such that we have a map of $Q : C^\infty(M) \longrightarrow \mathcal{O}$. Where $\mathcal{O}$ is the set of all operators that can act on wavefunctions in the Hilbert space $\mathcal{H}$ of the quantum system.

Dirac suggested some conditions for the map that are crucial ([1]). $Q : C^\infty(M) \longrightarrow \mathcal{O} : f \longrightarrow \hat{f}$ and let $\hat{f}$ denote the quantum operator associated with the classical observable f.

- ($\mathcal{Q}1$) If f is constant, then $\hat{f}$ corresponds to multiplication with f.
- ($\mathcal{Q}2$) The map $f \longrightarrow \hat{f}$ is linear ($\widehat{(\lambda f + \nu g)} = \lambda \hat{f} + \nu \hat{g}$ with $f, g \in C^\infty(M)$ and $\lambda, \nu \in \mathbb{C}$).
- ($\mathcal{Q}3$) If $\{f, g\} = h$, then $[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f} = -i\hbar\hat{h}$

$\mathcal{Q}1$ and $\mathcal{Q}2$ are quite simple conditions and they are very natural when working with quantum mechanics. But $\mathcal{Q}3$ implies that Poisson brackets are the classical analogue of the commutators. There are just three conditions and one could probably imagine more of them.

We will see that these conditions alone will lead to reducible representations of quantum operators. Thus we will inevitably have to impose more conditions. Quoting A. Carosso [2]: "We note by the deep Groenewold-van Hove (GvH) theorem, no such "Dirac" map is sufficient for the construction of irreducible representation; extra conditions must be imposed, and this will be the central failure of our first attempt at quantization." Woodhouse [1] explained it as if $\mathcal{H}$ is way too large and that restrictions are needed. Before we impose extra conditions or restrictions to $\mathcal{H}$ we will make do with the standard conditions given by Dirac. This is called prequantization and keep in mind that it will fail at some point.

4.2 Mapping observables

Let f be a classic observable on the symplectic manifold (M, ω) and $\hat{f}$ is the observable in quantum systems. Observables in quantum systems are operators that act on the Hilbert space of that quantum system. We call elements of the Hilbert space that are square integrable wavefunctions ψ. We will be guessing and tweaking our way to the mapping procedure.

A good guess would be $\hat{f}\psi = -i\hbar X_f \lrcorner d\psi$ with $\psi \in \mathcal{H}$. The map would satisfy $\mathcal{Q}2$ and $\mathcal{Q}3$. But when f would be constant then it's vector field is 0 and doesn't produce the multiplication. Thus another guess would be $\hat{f}\psi = (-i\hbar X_f \lrcorner d + f)\psi$. It fixes $\mathcal{Q}1$, but now it fails $\mathcal{Q}3$.

When checking $\mathcal{Q}3$ we find, $[\hat{f}_1, \hat{f}_2]\psi = -i\hbar(-i\hbar X_{f_3} \lrcorner d + f_3)\psi - i\hbar f_3\psi \neq -i\hbar \hat{f}_3\psi$. We have to get rid of the term $-i\hbar f_3$. We can fix this by adding another term,

$$\hat{f}\psi = -i\hbar\left[X_f \lrcorner d\psi - \frac{i}{\hbar}(X_f \lrcorner \theta)\psi\right] + f\psi. \tag{4.1}$$

Where θ is the symplectic potential. This is called the **Kostant-Souriau prequantum operator**. This mapping procedure satisfies all of Dirac's conditions. But notice that the quantization is now dependant on the symplectic potential. Some ambiguities arise here.

When we introduced the symplectic potential, we said that it is not unique and there could always be some du with $u \in C^\infty(M)$ added without affecting the symplectic 2-form. This will be a problem because now the operator can be dependant on du. Therefore $\hat{f}$ is not unique and then there is no reason to choose which "variant" is physically correct. But there is still some hope. If we allow gauge invariance we could make it work. Gauge invariance was introduced to the English language by Hermann Weyl and came from the german word Eichinvarianz. the goal for this invariance is that we can change a parameter of the equation and if we accordingly change the rest, we would still have the same physics.

Let $\theta' = \theta + du$ and f is a classical observable. Then we can compute $\hat{f}'$ by θ' and $\hat{f}$ by θ. Then there follows $\hat{f}' = \hat{f} - (X_f \lrcorner du)$. When we make a phase change $e^{iu/\hbar}$ to the wavefunction then,

$$\hat{f}'(e^{iu/\hbar}\psi) = e^{iu/\hbar}(\hat{f}'(\psi) + (X_f \lrcorner du)\psi) = e^{iu/\hbar}\hat{f}(\psi). \tag{4.2}$$

We can see that the gauge transformation is paired with some kind of phase change. If we combine the gauge change $\theta' \longrightarrow \theta$ with the phase change $\psi' \longrightarrow e^{iu/\hbar}\psi$. Then $\hat{f}$ becomes unique, but at the cost that the phase of a wavefunction is ambiguous.

Another ambiguity is that the symplectic potential doesn't have to be globally defined.

There is some way to clear things up by recognising that $X_f \lrcorner d - \frac{i}{\hbar}(X_f \lrcorner \theta)$ closely resembles a directional connection $X_f \lrcorner \nabla = \nabla_{X_f}$ on a Hermitian line bundle.

This implies that $\hat{f}$ must not act on just wavefunctions but rather on sections of Hermitian line bundles over the manifold.

The structure we need is a Hermitian line bundle with a connection that mimics the operator mapping we just found. We give a quick introduction to Hermitian line bundles in appendix B. The sections on the Hermitian line bundle will serve as wavefunctions and the connection of the Hermitian line bundle is chosen such that it represents the term between the square brackets in the Kostant-Souriau prequantum operator.

Definition 20 (Prequantization) *(M, ω) is a symplectic manifold and when there exists a projection $\pi : B \longrightarrow M$ that is a Hermitian line bundle with a connection $\nabla = d - \frac{i}{\hbar}\theta$ and Hermitian metric $(\cdot, \cdot)$. Then the manifold can be quantized. Let $f \in C^\infty(M)$ be a classical observable with the Hamiltonian vector field X_f. $s \in \Gamma(B)$[1] is the section of B. Then the **quantization map** is,*

$$Q : f \longrightarrow \hat{f} : Q(f)(s) = \hat{f}(s) = -i\hbar\nabla_{X_f} s + fs. \tag{4.3}$$

We call this Definition the prequantization, because this is the mapping procedure defined from only Dirac's conditions. We can construct any section by multiplying a wavefunction and a unit section $\mathfrak{u}$. Such that $s := \psi\mathfrak{u} : M \longrightarrow B$ with $(\mathfrak{u}, \mathfrak{u}) = 1$. This is a natural way of thinking about these sections because ψ is a wavefunction. Therefore we consider the sections to be the elements of the Hilbert space. The Hilbert space is an infinite-dimensional vector space with an inner product. Let's define this for sections as,

$$\langle s, s' \rangle := \int_M (s, s')\varepsilon. \tag{4.4}$$

With ε as the Liouville measure of the symplectic manifold and $(\cdot, \cdot)$ the Hermitian metric.

$$\varepsilon = \left(\frac{1}{2\pi\hbar}\right)^n \omega \wedge \cdots \wedge \omega = \left(\frac{1}{2\pi\hbar}\right)^n \bigwedge_{i=1}^{n} \omega \tag{4.5}$$

The factor $(\frac{1}{2\pi\hbar})^n$ can be explained because ω has physical units that have to be cancelled out such that the wavefunction does not have any relation with physical units. This choice does resemble the typical quantum mechanics inner product when $s = \psi\mathfrak{u}$ and $s' = \psi'\mathfrak{u}$.

$$\langle s, s' \rangle = \langle \psi, \psi' \rangle = \int_M \bar{\psi}\psi'\varepsilon \tag{4.6}$$

And because of this only square-integrable wavefunction (and thus also sections) are the only viable wavefunctions.

[1] $\Gamma(B)$ is the space of sections $s : M \longrightarrow B$.

4.2.1 Weil's integration condition

In our definition of prequantization we said that when there exists such a Hermitian line bundle then we can use prequantization. This alludes to some condition that has to be satisfied. Because there doesn't have to exist a Hermitian line bundle with the desired connection.

The condition we will be introducing is the Weil's integration condition.

Theorem 3 (The existence of an Hermitian line bundle) *Let (M, ω) be a symplectic manifold. The class of $(2\pi\hbar)^{-1}\omega \in \check{H}^2(M, \mathbb{R})$ lies in the image of $\check{H}^2(M, \mathbb{Z})$ if and only if there exists an Hermitian line bundle and a connection ∇ with curvature $\hbar^{-1}\omega$.*

Proof

Let's begin with the assumption that the class of $(2\pi\hbar)^{-1}\omega$ is an element of $\check{H}^2(M, \mathbb{Z})$. There exists a good cover $\mathcal{U} = \{U_i | i \in I\}$ in M. On every U_i there exists a symplectic potential $\theta_i \in \Omega^1(U_i)$. The set $\{\theta_i | i \in I\}$ is an element of $\check{C}^0(M, \Omega^1(M))$.

When $U_{ij} = U_i \cap U_j \neq \emptyset$ (this convention will be used throughout the proof) then on this intersection we can define $\theta_i - \theta_j$. This expression is exact because $d\theta_i - d\theta_j = \omega - \omega = 0$. Thus there exists another primitive $df_{ij} = \theta_i - \theta_j$. The set $\{f_{ij} | i, j \in I\}$ is an element of $\check{C}^1(M, \Omega^0(M))$.

Now consider when three patches overlap $U_{ijk} \neq \emptyset$, then define $c_{ijk} = (2\pi\hbar)^{-1}(f_{ij} + f_{jk} + f_{ki})$ restricted to the intersections of their patches.

When we take the exterior derivative we find $(2\pi\hbar)^{-1}d(f_{ij} + f_{jk} + f_{ki}) = (2\pi\hbar)^{-1}(\theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i) = 0$.

Thus we find that $c_{ijk} = (2\pi\hbar)^{-1}(f_{ij} + f_{jk} + f_{ki})$ has to be constant on U_{ijk}. The set $\{c_{ijk} | i, j, k \in I\}$ is then an element of $\check{C}^2(M, \mathbb{R})$ and an element of the class of $(2\pi\hbar)^{-1}\omega$ and because of the assumption also an element of $\check{C}^2(M, \mathbb{Z})$. Then $(2\pi\hbar)^{-1}(f_{ij} + f_{jk} + f_{ki})$ has to be an integer!

Now we can focus on proving the existence of a Hermitian line bundle.

Construct a transition function $g_{ij} = \exp[if_{ij}/\hbar]$ on U_{ij}. Also let $(2\pi\hbar)^{-1}(f_{ij} + f_{jk} + f_{ki}) = n$. The transition function is skew symmetric $g_{ii} = e^0 = 1$ and is cocyclic $g_{ij}g_{jk}g_{ki} = \exp[i/\hbar(f_{ij} + f_{jk} + f_{ki})] = \exp[2\pi\hbar i n/\hbar] = \exp[2\pi i n] = 1$ because n is an integer.

Proposition 2.1 (*Notes on the masterclass differential geometry from Gil Cavalcanti* **[4]**) shows us that there exists a line bundle $B \longrightarrow M$ whenever a transition function is constructed and satisfies the skew-symmetric and cocyclic condition.

Because the transition functions are all complex and of unit length, it suffices to use the norm of $\mathbb{C}$ on each patch U_i to define a Hermitian structure on B. Such that B becomes a Hermitian line bundle.

The rule (Appendix B.3) on how the connection changes depending on the transition function is,

$$d \log g_{ij} = \frac{i}{\hbar} df_{ij} = i(\theta_i/\hbar - \theta_j/\hbar). \tag{4.7}$$

Then $\theta_i/\hbar$ is the connection which leads us to a curvature $\omega/\hbar$. ■

Conversely, suppose we have a Hermitian line bundle $B \longrightarrow M$ with a connection with curvature $\omega/\hbar$. We have a local trivialisation on B relative to some open cover

$\{U_i\}$. Let those transition functions be $\{g_{ij}\}$ on $U_i \cap U_j \neq \emptyset$. Then define on $U_{ijk} \neq \emptyset$,

$$z_{ijk} = \frac{1}{2\pi i}(\log g_{ij} + \log g_{jk} + \log g_{ki}). \tag{4.8}$$

The transition functions satisfy the cocycle condition, thus $z_{ijk} \in \mathbb{Z}$. And when we look at the exterior derivative of z then,

$$dz_{ijk} = \frac{1}{2\pi i}\frac{i}{\hbar}(d\log g_{ij} + d\log g_{jk} + d\log g_{ki}) = \frac{1}{2\pi\hbar}(\theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i) = 0. \tag{4.9}$$

We used the rule how transition function change the connection ($d\log g_{ij} = i(\theta_i/\hbar - \theta_j/\hbar)$). It is logical we would find 0 because z_{ijk} is constant, but $2\pi\hbar z$ does this by representing a cocycle of the class $(2\pi\hbar)^{-1}\omega$ in $\check{H}^2(M, \mathbb{R})$. And this shows that it is needed that $(2\pi\hbar)^{-1}\omega$ lies in $\check{H}^2(M, \mathbb{Z})$ such that these z_{ijk} can exist. ∎

When M is simply connected, then we can refer to a simpler condition that follows from the theorem above.

Theorem 4 (Weil's integration condition) *For a 2n-dimensional symplectic manifold (M, ω) with M simply connected. If,*

$$\int_\Sigma \omega = 2\pi\hbar n. \tag{4.10}$$

For an orientable closed 2-surface $\Sigma \subset M$ and $n \in \mathbb{Z}$. Then there exists a Hermitian line bundle with the appropriate connection ($\nabla = d - \frac{i}{\hbar}\theta$) and Hermitian structure that allows for quantization.

4.2.2 not simply connected manifolds

Let us consider when (M, ω) is a symplectic manifold and M is not simply connected. Then we cannot use Weil's integration condition. Not simply connected manifolds can still satisfy theorem 3. In these cases, we will see that the connection and Hermitian line bundles are not unique up to equivalence and an example that we are going to look at is $M = T^*S^1 \cong \mathbb{R} \times S^1$ (from source [6]).

M is clearly a smooth manifold. Let $\{p\}$ be the basis in $\mathbb{R}$ and $\{\phi\}$ be the basis in S^1 such that we can define a symplectic 2-form $\omega = dp \wedge d\phi$.

This 2-form is exact and has a globally defined symplectic potential $\theta = pd\phi$. When we consider a line bundle $B \longrightarrow M$ with connection $\nabla = d - \frac{i}{\hbar}\theta$. We can add $\lambda d\phi$ with λ a real constant to the symplectic potential without changing the symplectic 2-form.

$$\theta_\lambda = \theta - \hbar\lambda d\phi, \tag{4.11}$$

$$\nabla^{(\lambda)} = d - \frac{i}{\hbar}\theta + i\lambda d\phi. \tag{4.12}$$

Which implies that there exists a family of line bundles B_λ. Let's apply the quantization mapping of definition 20 to p with the new connection $\nabla^{(\lambda)} = d - \frac{i}{\hbar}\theta + i\lambda d\phi$.

$$\hat{p}^{(\lambda)} = -i\hbar\frac{\partial}{\partial\phi} + \hbar\lambda \tag{4.13}$$

These operators have spectrums which are sets of eigenvalues. We will now look for the set of eigenvalues of $\hat{p}^{(\lambda)}$. We know that a function on M should be periodic in the ϕ basis, thus an eigenfunction would be of the form $\psi \propto e^{in\phi}$. This function gives us the eigenvalues $\{\hbar(n+\lambda)|n \in \mathbb{Z}\}$. Where λ only takes values between $[0,1)$ or else there will be multiple equivalent eigenvalues. This set shows us that the set B_λ gives for all $\lambda \in [0,1)$ inequivalent $\hat{p}^{(\lambda)}$.

There are some physical consequences from this such as the Aharanov-Bohm effect.

4.3 Quantization for flows of complete Hamiltonian vector fields

We already established that working with a Hermitian line bundle $\pi : B \longrightarrow M$ on the symplectic manifold (M, ω) is the right choice. We call B the ***prequantum bundle***. So for this chapter we will assume that the symplectic manifold (M, ω) has a class of $(2\pi\hbar)^{-1}\omega \in \check{H}^2(M, \mathbb{R})$ that lies in the image of $\check{H}^2(M, \mathbb{Z})$. Such that theorem 3 reassures us that a Hermitian line bundle with the desired connection exists.

Let z be an element on a fibre of B. Then lift the Hamiltonian vector field generated by $f \in C^\infty(M)$ to,

$$V_f = X_f + i\hbar^{-1}L_f z\partial_z \quad V_f \in TB. \tag{4.14}$$

Where X_f is the Hamiltonian vector field of f and $L_f = X_f \lrcorner \theta - f$. Assume that X_f is complete, then V_f is also complete. Notice that it doesn't change the vector field in M. Thus $\pi_* V_f = X_f$.

We will argue that this vector field in the tangent space of the prequantum bundle is well chosen because it preserves some important structures. A very important one is that it is gauge invariant. This will result in uniqueness for every observable. V_f has the flow ξ_t that preserves the fibres of B and projects on the canonical flow.

For a generic vector field in the tangent space of the prequantum bundle that projects on the Hamiltonian vector field would have the explicit form $V = X_f + \dot{z}\partial_z$

We can solve for $\dot{z}(t) = \frac{i}{\hbar}L_f(t)z(t)$. This will result in $z(t) = z(0)\exp\left[\frac{i}{\hbar}\int_0^t L_f(t)d\tau\right]$, where the integral is over a path in X_f. B can typically be trivialised by $M \times \mathbb{C}$. Then we could write the flow as a coordinate in M and one in $\mathbb{C}$,

$$\xi_t(m, z) = \left(\rho_t m, z_0 \exp\left[\frac{i}{\hbar}\int_0^t L_f(t)d\tau\right]\right). \tag{4.15}$$

Thus when the action of ξ_t is applied to sections $s \in \Gamma(B)$ we get,

$$\xi_t[s(m)] = s(m)\exp\left[\frac{i}{\hbar}\int_0^t L_f(t)d\tau\right]. \tag{4.16}$$

The flow of the quantization will then be defined as,

$$\xi_t[\hat{\rho}_t s(m)] := s(\rho_t m). \tag{4.17}$$

We can now use 4.16 to find a more explicit form of 4.17,

$$\hat{\rho}_t s(m) = s(\rho_t m)\exp\left[-\frac{i}{\hbar}\int_0^t L_f(t)d\tau\right]. \tag{4.18}$$

The integral is over the path produced by $\rho_t(m)$. Notice that when the observable is the Hamiltonian of a system then L_h is the proper Lagrangian. We just found that the flow of a quantized Hamiltonian produces the action integral. The flow of the Hamiltonian should produce time evolution, but remember that the standard time evolution in quantum mechanics is given by $\exp[-it\hat{h}/\hbar]$. This is one of the first discrepancies we will see.

4.4 Failure of the prequantization

4.4.1 Prequantization on replicating canonical quantization

After all this mathematical machinery and arguments we have not checked if it is consistent with the known theory.

A useful baseline is canonical quantization. It produces experimentally correct answers and thus is definitely true (as far as we know). Canonical quantization predicts that momentum (p_a) and position (q^a) are the quantum operators,

$$\hat{p}_a = -i\hbar\frac{\partial}{\partial q^a}, \qquad (4.19) \qquad\qquad \hat{q}^a = q^a. \qquad (4.20)$$

Now we established what we should find. We can look at what prequantization predicts by using definition 20. First we define our space to be $M = T^*Q$ with Q as the configuration space (Throughout this thesis, $Q = \mathbb{R}^n$). Where $\{q^a\}$ are coordinates of Q and $\{p_a\}$ coordinates of the cotangent space of Q. Together they make a symplectic frame $\{p_a, q^a\}$. Define the symplectic 2-form to be $\omega = dp_a \wedge dq^a$ and the symplectic potential is $\theta = p_a dq^a$. The existence of a prequantum bundle with the correct connection is trivially shown by the Weil's integration condition. The connection will be defined as $\nabla := d - i\hbar^{-1}\theta$. First we have to find the vector field generated by p_a and q^a. Let such a vector field have the form $X_f = x_f\partial/\partial q^a + y_f\partial/\partial p_a$. Then we can compute the vector field for p_a,

$$X_{p_a} \lrcorner\, \omega - dp_a = x_p dp_a - y_p dq^a - dp_a = 0. \qquad (4.21)$$

There follows that $x_p = 1$ and $y_p = 0$. Thus the vector field is $X_{p_a} = \partial/\partial q^a$.

$$\hat{p}_a = -i\hbar\left[\frac{\partial}{\partial q^a} - \frac{i}{\hbar}\left(\frac{\partial}{\partial q^a} \lrcorner\, p_a dq^a\right)\right] + p_a = -i\hbar\frac{\partial}{\partial q^a} - p_a + p_a = -i\hbar\frac{\partial}{\partial q^a}. \qquad (4.22)$$

This is exactly what we would expect from the quantization of p_a.

For q^a we will do the same computation,

$$X_{q^a} \lrcorner\, \omega - dq^a = x_q dp_a - y_q dq^a - dq^a = 0. \qquad (4.23)$$

There follows that $x_q = 0$ and $y_q = -1$. Thus the vector field is $X_{q^a} = -\partial/\partial p_a$.

$$\hat{q}^a = -i\hbar\left[-\frac{\partial}{\partial p_a} - \frac{i}{\hbar}\left(-\frac{\partial}{\partial p_a} \lrcorner\, p_a dq^a\right)\right] + q^a = i\hbar\frac{\partial}{\partial p_a} + q^a \neq q^a \qquad (4.24)$$

This is a discrepancy with the canonical quantization. We have an extra dependence on p_a what causes this problem. The solution is to restrict the sections such that in this case it isn't dependant on p_a and q^a at the same time. This will be further treated in the next chapter.

4.4.2 prequantization on replicating a Hamiltonian operator

When we consider the same space as before and we define the Hamiltonian to be $h = \frac{1}{2m}p^2$. The standard physics textbook on quantum mechanics will tell you that this will give the operator,

$$\hat{h} = -\frac{\hbar^2}{2m}\nabla^2. \tag{4.25}$$

Let's again use definition 20 to compute the operator produced by prequantization. The Hamiltonian vector field will be,

$$X_h \lrcorner\, \omega - dh = x_h dp_a - y_h dq^a - \frac{1}{m}p_a dp_a = 0. \tag{4.26}$$

This implies that $x_h = \frac{1}{m}p_a$ and $y_h = 0$. Now we can compute the operator,

$$\hat{h} = -i\hbar\left[\frac{1}{m}p_a\frac{\partial}{\partial q^a} - \frac{i}{\hbar}\left(\frac{1}{m}p^2\right)\right] + \frac{1}{2m}p^2 = \frac{-i\hbar}{m}p_a\frac{\partial}{\partial q^a} - \frac{1}{2m}p^2. \tag{4.27}$$

Again we have found that prequantisation fails at this point. It is maybe not directly obvious, but what went wrong is that prequantisation struggles with operators of degrees higher than one. In the next chapter we will use BKS construction to deal with Hamiltonian's that are dependant on a momentum term of degree 2.

Chapter 5

Geometric Quantization

5.1 Introduction

After the semi-success of prequantization. It still fails to predict one of the simplest operators. But in section 4.4 we concluded what went wrong and in this chapter, we will make an effort to correct these problems and make sure it reproduces what first failed. The first problem was that the sections are dependant on all the coordinates and this is not the case when dealing with quantum systems. So we have a natural way to restrict these section by introducing polarizations. When we introduce polarizations we immediately want to introduce complex structures, because some manifolds behave exceptionally well when they are complex with a positive complex structure.

The polarizations will bring their own problem and we will attempt to fix it. The problem is that when the vector field of a classical observable doesn't preserve the polarization then the section who are restricted may not have the same restrictions after applying the operator of that observable and this is nonphysical. It will result in a restriction of possible observables and the BKS construction. Another problem is that when defining the inner product of sections on a manifold with a polarizations with real directions then there is the possibility that the inner product will never converge for non-vanishing polarized sections. This is discussed in section 5.5. We solve this by modifying the prequantum bundle.

5.2 Polarization

5.2.1 Real polarizations

Definition 21 *Consider a symplectic manifold (M, ω), then a **real polarization** $P \subset TM$ is an integrable distribution where the fibres are Lagrangian subspaces of $T_m M$ for all $m \in M$. Thus a polarization is a distribution that satisfies,*

- *Fibre wise Lagrangian,*
- *Integrable.*

An example for when $M = T^*\mathbb{R}^n$ is the cotangent space and we have the coordinate system $\{p_a, q^a\}$ for $a = 1 \dots n$. M is a symplectic manifold when we define the symplectic 2-form to be $\omega = dp_a \wedge dq^a$. A possible distribution is the subbundle $D \subset TM$ that is spanned by $\{\partial/\partial p_a\}$ for $a = 1 \dots n$. This subbundle is involutive and therefore also integrable. By the global Frobenius theorem this distribution implies a foliation on M. This is the **vertical foliation** and it is the union of surfaces of constant q.

For every two elements of the distribution $X, Y \in D_m$ at a point $m \in M$. We can show that $\omega(X, Y) = 0$ and $\dim D_m = 1/2 \dim T_m M$ for all $m \in M$, thus Lagrangian. The vertical foliation is, therefore, a polarization.

When a function f is constant on every leaf of the polarization we can show that it's vector field has to be symplectic orthogonal to the leaves. When f is constant on a leaf Λ, then $df|_\Lambda = 0$. Thus the formula for the vector field becomes $0 = df|_\Lambda = X_f \lrcorner \omega|_\Lambda$. This implies that X_f is symplectic orthogonal to the polarization.

5.2.2 Complex Manifolds

Life does sometimes have to be a bit more "complex" then only real dimensions. Introducing complex manifolds will give us insight into the Kähler manifold. The Kähler manifold is particularly interesting for geometric quantization because it is extraordinarily well-behaved.

Definition 22 *A n-(complex)dimensional **complex manifold** M is a manifold with a complete complex atlas,*

$$A = \{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi) | \alpha \in I\}. \tag{5.1}$$

Where I is the index set, $M = \bigcup_\alpha \mathcal{U}_\alpha$, $\mathcal{V}_\alpha$ are open subsets of $\mathbb{C}^n$, and the maps $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha$. These maps have the property that $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is biholomorphic. Which is that $\psi_{\alpha\beta}$ is a bijection where both the map and the inverse of the map are holomorphic.

Definition 23 *Let M be a smooth manifold. A **complex structure** on M is a linear transformation on the tangent space.*

$$m \longrightarrow J_m : T_m M \longrightarrow T_m M \tag{5.2}$$

And $J_m^2 = -Id$.

When multiplying complex scalars $(x + iy) \in \mathbb{C}$ with $X \in TM$ then we compute it as,

$$(x + iy)X = xX + yJX. \tag{5.3}$$

Vectors spaces can be complexified. Let C be a vector space with a complex structure J. Then the complexified vector space can be constructed from $X + JY \in C_\mathbb{C}$ for all $X, Y \in C$.

Definition 24 *Let (M, ω) be a symplectic manifold and a symplectic 2-form with a complex structure J. The complex structure is **compatible** with the symplectic 2-form when we can define a positive nondegenerate symmetric bilinear form on M,*

$$m \longrightarrow g_m : T_m M \times T_m M \longrightarrow \mathbb{R} : g_m(X, Y) = \omega_m(X, JY). \tag{5.4}$$

For every point m in M.

Definition 25 *Let (M, ω) be a symplectic manifold with a complex structure J. J is positive if the nondegenerate symmetric bilinear form g is positive definite.*

Suppose that M is a complex manifold and a complex structure J. Let U be a neighbourhood of a point m in M with real coordinates $\{p_a, q^a\}$ for $a = 1 \dots n$ and complex coordinates $z^a = p_a + iq^a$.

$$T_m M = \mathbb{R} - \text{span}\left\{ \frac{\partial}{\partial q^a}\bigg|_m, \frac{\partial}{\partial p_a}\bigg|_m \right\} \tag{5.5}$$

$$T_m M \otimes \mathbb{C} = \mathbb{C} - \text{span}\left\{ \frac{\partial}{\partial q^a}\bigg|_m, \frac{\partial}{\partial p_a}\bigg|_m \right\} \tag{5.6}$$

$$= \mathbb{C} - \text{span}\left\{ \frac{1}{2}\left(\frac{\partial}{\partial p_a} - i\frac{\partial}{\partial q^a}\right)\bigg|_m \right\} \oplus \mathbb{C} - \text{span}\left\{ \frac{1}{2}\left(\frac{\partial}{\partial p_a} + i\frac{\partial}{\partial q^a}\right)\bigg|_m \right\} \tag{5.7}$$

The first term of (5.7) denotes the vectors with eigenvalues i of J and are called the (1,0)-vectors and the eigenspace is denoted by $T^{(1,0)}$.

$$J\frac{1}{2}\left(\frac{\partial}{\partial p_a} - i\frac{\partial}{\partial q^a}\right) = \frac{i}{2}\left(\frac{\partial}{\partial p_a} - i\frac{\partial}{\partial q^a}\right) \tag{5.8}$$

The second term of (5.7) denotes the vectors with eigenvalues $-i$ of J and are called the (0,1)-vectors and the eigenspace is denoted by $T^{(0,1)}$.

$$J\frac{1}{2}\left(\frac{\partial}{\partial p_a} + i\frac{\partial}{\partial q^a}\right) = \frac{-i}{2}\left(\frac{\partial}{\partial p_a} + i\frac{\partial}{\partial q^a}\right) \tag{5.9}$$

Definition 26 *M is a complex manifold and a complex structure J. Let U be a neighbourhood of a point m in M with real coordinates $\{p_a, q^a\}$ for $a = 1 \dots n$. Define,*

$$\frac{\partial}{\partial z^a} = \frac{1}{2}\left(\frac{\partial}{\partial p_a} - i\frac{\partial}{\partial q^a}\right), \tag{5.10}$$

$$\frac{\partial}{\partial \bar{z}^a} = \frac{1}{2}\left(\frac{\partial}{\partial p_a} + i\frac{\partial}{\partial q^a}\right). \tag{5.11}$$

We can define something very similar for the dual of the tangent space.

Definition 27 *M is a complex manifold and a complex structure J. Let U be a neighbourhood of a point m in M with real coordinates $\{p_a, q^a\}$ for $a = 1 \dots n$. Define,*

$$dz^a = dp_a + idq^a, \tag{5.12}$$

$$d\bar{z}^a = dp_a - idq^a. \tag{5.13}$$

There are different possible forms on U. For 1 forms,

$$\Omega^{(1,0)}(U;\mathbb{C}) = \{\sum_a b_a dz^a | b_a \in C^\infty(U,\mathbb{C})\},$$

$$\Omega^{(0,1)}(U;\mathbb{C}) = \{\sum_a b_a d\bar{z}^a | b_a \in C^\infty(U,\mathbb{C})\}.$$

For 2 forms,

$$\Omega^{(2,0)(U;\mathbb{C})} = \{\sum_{a<k} b_{a,k} dz^a \wedge dz^k | b_{a,k} \in C^\infty(U,\mathbb{C})\},$$

$$\Omega^{(1,1)}(U;\mathbb{C}) = \{\sum_{a,k} b_{a,k} dz^a \wedge d\bar{z}^k | b_{a,k} \in C^\infty(U,\mathbb{C})\}.$$

$$\Omega^{(0,2)}(U;\mathbb{C}) = \{\sum_{a<k} b_{a,k} d\bar{z}^a \wedge d\bar{z}^k | b_{a,k} \in C^\infty(U,\mathbb{C})\}.$$

Then the 2-forms on U are $\Omega^2(U;\mathbb{C}) = \bigoplus_{k+l=2}\Omega^{(k,l)}(U;\mathbb{C})$. $\Omega^2(U;\mathbb{C})$ can act like a vector bundle on one of it's decompositions. Let $\pi^{(k,l)} : \Omega^2(U;\mathbb{C}) \longrightarrow \Omega^{(k,l)}(U;\mathbb{C})$ for $k+l=2$.

We can easily generalise this for all n-forms $\Omega^n(U;\mathbb{C}) = \bigoplus_{k+l=n}\Omega^{(k,l)}(U;\mathbb{C})$ and $\pi^{(k,l)} : \Omega^n(U;\mathbb{C}) \longrightarrow \Omega^{(k,l)}(U;\mathbb{C})$ for $k+l=n$.

Dolbeault operators

When we apply the exterior derivative to $\Omega^{(k,l)}$ then we know $d\Omega^{(k,l)} \subset \Omega^{n+1}$ if $k+l=n$.

Definition 28 *M is an almost complex manifold and a complex structure J. Let U be a neighbourhood of a point m in M. The Dolbeault operators that act on $\Omega^{(k,l)}(U;\mathbb{C})$ are defined as,*

$$\partial = \pi^{(k+1,l)} \circ d : \Omega^{(k,l)}(U;\mathbb{C}) \longrightarrow \Omega^{(k+1,l)}(U;\mathbb{C}), \tag{5.14}$$

$$\bar{\partial} = \pi^{(k,l+1)} \circ d : \Omega^{(k,l)}(U;\mathbb{C}) \longrightarrow \Omega^{(k,l+1)}(U;\mathbb{C}). \tag{5.15}$$

These operators are in local coordinates for $\alpha = \sum_{(k,l)} b_{k,l}dz^k \wedge d\bar{z}^l \in \Omega^{(k,l)}(U;\mathbb{C})$.

$$\bar{\partial}\alpha = \sum_{|M|,|N|}\sum_{a}\frac{\partial b_{M,N}}{\partial z^a}dz^a \wedge dz^M \wedge d\bar{z}^N \tag{5.16}$$

$$\partial\alpha = \sum_{|M|,|N|}\sum_{a}\frac{\partial b_{M,N}}{\partial \bar{z}^a}d\bar{z}^a \wedge dz^M \wedge d\bar{z}^N \tag{5.17}$$

Where $M = (m_1, \dots, m_k)$ such that $m_1 < \cdots < m_k$, $dz^M = dz^{m_1} \wedge \cdots \wedge dz^{m_k}$ and $N = (n_1, \dots, n_l)$ such that $n_1 < \cdots < n_l$, $d\bar{z}^N = d\bar{z}^{n_1} \wedge \cdots \wedge d\bar{z}^{n_l}$.

Proposition 3 *When M is a complex manifold, then the Dolbeault operators have the properties that,*

$$d = \partial + \bar{\partial}, \tag{5.18}$$

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{5.19}$$

5.2.3 Complex polarizations

A good argument to include complex structures is that real polarizations produce nowhere vanishing vector fields on a two-dimensional surface. When we consider M to be S^2, then a real polarization should give a non-vanishing vector field and this is a contradiction with the hairy ball theorem. Thus we cannot find real polarizations for S^2, but we can find complex polarizations.

Definition 29 *Let M be an n-dimensional smooth manifold on which a complex structure is defined. We can complexify the tangent space of M. This is $T_{\mathbb{C}}M$. A* ***complex distribution*** *D is a subbundle of the complexified tangent bundle $T_{\mathbb{C}}M$. The fibre D_m varies smoothly over $m \in M$.*

The definition is very similar to the definition of the real distribution.

Now we have all the ingrediënts to define a complex polarization,

Definition 30 *Let (M, ω) be a symplectic manifold, then a* ***complex polarization*** *P is an integrable lagrangian distribution of $T_{\mathbb{C}}M$.*

Definition 31 *Let (M, ω) be a symplectic manifold with a polarization P. A local symplectic potential θ is* ***adapted*** *to polarization P whenever $X \lrcorner \theta = 0$ for all $X \in V_P(M)$.*

Definition 32 *Let (M, ω) be a symplectic manifold with a polarization P. P is* ***admissible*** *if there exists for every neighbourhood of a point in P an adapted symplectic potential.*

5.2.4 Kähler Forms

Definition 33 *A* ***Kähler manifold*** *is a symplectic manifold (M, ω) equipped with a positive compatible complex structure J. The symplectic 2-form is in this case a* ***Kähler form****.*

A Kähler manifold is a complex manifold and therefore are the Dolbeault operators as in Proposition 3.

Let M be a Kähler manifold with Kähler form ω and $\dim_{\mathbb{C}} M = n$. A Kähler form ω should be a form that has the following properties.

- The Kähler form is a 2-form.

$\omega \in \Omega^2(M;\mathbb{C}) = \Omega^{(2,0)} \oplus \Omega^{(1,1)} \oplus \Omega^{(0,2)}$. Therefore on a local complex chart $(U, z^1, \ldots, z^n)$ it has the form $\omega = \sum_{l<k} a_{l,k} dz^l \wedge dz^k + \sum_{l,k} b_{l,k} dz^l \wedge d\bar{z}^k + \sum_{l<k} c_{l,k} d\bar{z}^l \wedge d\bar{z}^k$ with $a_{l,k}, b_{l,k}, c_{l,k} \in C^\infty(M)$.

- The Kähler form is compatible with the complex structure.

ω is compatible with the complex structure and this implies that J is a symplectomorphism. This is only true when $a_{l,k} = 0 = c_{l,k}$ for all l, k.

- The Kähler form is closed.

M has the property that $0 = d\omega = \partial\omega + \bar{\partial}\omega$. $\partial\omega$ is a (2,1)-form and $\bar{\partial}\omega$ is a (1,2)-form. They can't annihilate each other, thus we conclude that $\partial\omega = 0$ and $\bar{\partial}\omega = 0$.

- The Kähler form is real-valued.

If ω is real valued then we know that $\omega = \bar{\omega}$. Let $b_{l,k} = ir_{l,k}$,

$$\bar{\omega} = -i \sum_{l,k} \bar{r}_{l,k} d\bar{z}^l \wedge dz^k = i \sum_{l,k} \bar{r}_{l,k} dz^l \wedge d\bar{z}^k = i \sum_{l,k} r_{l,k} dz^l \wedge d\bar{z}^k = \omega. \tag{5.20}$$

Thus $\bar{r}_{l,k} = r_{l,k}$ and therefore has to be real for all l, k.

- The Kähler form is non-degenerate.

If ω is non-degenerate then $\det_{\mathbb{C}}(r_{l,k})_{l,k \in I} \neq 0$ with I the index set.

- $\omega_m(X, JX) > 0 \ \ \forall X \in T_m M$ for every point m.

$\omega_m(X, JX) > 0$ implies that the matrix $(r_{l,k})_{l,k\in I}$ is positive definite.

We conclude that the Kähler form ω on U should have the form,

$$\omega = i \sum_{l,k} \bar{r}_{l,k} dz^l \wedge d\bar{z}^k. \tag{5.21}$$

Where $(r_{l,k})_{l,k\in I}$ is positive definite for every point in U.

Proposition 4 *Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$. A* ***Kähler potential/scalar*** *is a real valued function $\mathcal{K} \in C^\infty(M)$ such that,*

$$\det\left(\frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^a}\right) > 0. \tag{5.22}$$

For all $a \in [1, n] \cap \mathbb{N}$ and on each local chart $(U, z^1, \dots, z^n)$. Then,

$$\omega = i\partial\bar{\partial}\mathcal{K}. \tag{5.23}$$

Is a Kähler form.

Theorem 5 *Let M be a complex manifold with a closed real-valued (1,1)-form ω defined on it. Then there exists for every neighbourhood of a point in M a Kähler potential such that,*

$$\omega = i\partial\bar{\partial}\mathcal{K}. \tag{5.24}$$

Let (M, ω) be a Kähler manifold. Then by theorem 5, the Kähler form has a local Kähler potential $\omega = i\partial\bar{\partial}\mathcal{K}$.

Let's take a look at the case that $M = \mathbb{C}^n$. We have a global coordinate system $\{z^a, \bar{z}^a\}$ where we can define a global Kähler potential.

$$\omega = i\partial\bar{\partial}\mathcal{K} = i\frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^a} dz^a \wedge d\bar{z}^a \quad \text{with} \quad \mathcal{K} = \frac{1}{2} z^a \bar{z}^a \tag{5.25}$$

Notice that M has two polarizations. A ***holomorphic polarization*** spanned by $\partial/\partial z^a$ by all $a = 1, \dots n$ and a ***antiholomorphic polarization*** spanned by $\partial/\partial \bar{z}^a$ by all $a = 1, \dots n$. We can find two global adapted symplectic potentials. $\theta = -i\partial\mathcal{K}$ is adapted to the antiholomorphic polarization and $\theta = i\bar{\partial}\mathcal{K}$ is adapted to the holomorphic polarization.

M satisfies Weil's integration condition. Thus there exists a prequantum bundle B with a connection. The connection 1-form is in our case $i\hbar^{-1}\theta$ and this has to be only imaginary. Thus we need to choose a symplectic potential that assures the connection 1-form to be only imaginary. Our preference for symplectic potential is one that is adapted to our chosen polarization.[1] In this case, it will be $\theta = -i\partial\mathcal{K}$ and the antiholomorphic polarization. The symplectic potential θ could be imaginary and therefore would the connection 1-form have a real part and this is a problem. We can solve this by only using the real part of the symplectic potential.

$$\theta_0 = \frac{1}{2}(\theta + \bar{\theta}) \tag{5.26}$$

[1] This is for calculation purposes and in examples we will see that also the adapted polarization will give us the familiar operators.

This is still a symplectic potential $d\theta_0 = \frac{1}{2}(d\theta + d\bar{\theta}) = \frac{1}{2}(\omega + \bar{\omega}) = \omega$. We used that the Kähler form is real-valued thus $\omega = \bar{\omega}$. The new symplectic potential is explicitly,

$$\theta_0 = \frac{1}{2}(-i\partial\mathcal{K} + i\bar{\partial}\mathcal{K}) = -i\partial\mathcal{K} + \frac{1}{2}(i\partial\mathcal{K} + i\bar{\partial}\mathcal{K}) = \theta + \frac{i}{2}d\mathcal{K}. \tag{5.27}$$

Therefore the real symplectic potential has the convenient form of the desired symplectic potential plus the derivative of the Kähler potential. We can now use the same trick as in section 4.2 where we used a gauge transformation such that the transformation between symplectic potentials is paired with a transformation of sections.

Thus we are interested in sections s in the θ gauge, but only sections s' in the θ_0 gauge will work. Because then we have a purely imaginary connection 1-form. We can relate these sections by $s' = s\exp[-\mathcal{K}/2\hbar]$. Let $\nabla = d - i\hbar^{-1}\theta_0$ which is the proper connection and define $\nabla^* = d - i\hbar^{-1}\theta$ as a dummy connection.

$$\nabla s' = (d - i\hbar^{-1}\theta_0)s' = (d - i\hbar^{-1}\theta_0)se^{-\mathcal{K}/2\hbar} = e^{-\mathcal{K}/2\hbar}(d - i\hbar^{-1}\theta)s = e^{-\mathcal{K}/2\hbar}\nabla^* s \tag{5.28}$$

Thus in the rest of this thesis, we will use the dummy connection for calculation with in mind that we actually should calculate $\exp[\mathcal{K}/2\hbar]\nabla(\exp[-\mathcal{K}/2\hbar]s)$.

A consequence of this gauge is that we have a modified inner product. Define the sections of the θ gauge as $s_1 = \phi_1\mathfrak{u}$ and $s_2 = \phi_2\mathfrak{u}$ where $\mathfrak{u}$ is the unit section and $\phi \in C^\infty(M)$. Because we have to use θ_0 for the connection then we should also use the θ_0 sections for our Hermitian metric. Let s'_1 and s'_2 be in the θ_0 gauge, which are equal to $s_1\exp[-\mathcal{K}/2\hbar]$ and $s_2\exp[-\mathcal{K}/2\hbar]$.

Then the natural inner product takes the form,

$$\langle s'_1, s'_2\rangle = \int_M (s'_1, s'_2)\varepsilon = \int_M (\phi_1\mathfrak{u}, \phi_2\mathfrak{u})e^{-\mathcal{K}/\hbar}\varepsilon = \int_M \bar{\phi}_1\phi_2 e^{-\mathcal{K}/\hbar}\varepsilon =: \langle s_1, s_2\rangle. \tag{5.29}$$

This makes the Kähler manifold easy to handle because the inner product naturally makes the integral converge (as long as the functions ϕ and ϕ' behave).

A quick note. The Kähler potential is not unique. We can solve this as in a similar way as the trick in section 4.2. I won't include this, because it is very analogues to the trick in section 4.2 and does not add to the understanding of geometric quantization.

5.3 Polarized sections

Now we have the mathematical tools to force our sections to be polarized. One could see this as a restriction on sections such that the sections depend on half the coordinates of the symplectic manifold. Like in quantum mechanics we have to deal with positions $\psi(q)$ and momentum $\psi(p)$, but never a wavefunction of both $\cancel{\psi(q,p)}$.

Definition 34 *Let (M,ω) be a symplectic manifold that satisfies Weil's integration condition and has a polarization P. M has a Hermitian line bundle $B \longrightarrow M$ with a connection ∇. A* ***polarized section*** *is a smooth section $s : M \longrightarrow B$ with the property that,*

$$\nabla_X s = 0\text{[2]}. \tag{5.30}$$

For all $X \in V_P(M)$.

The set of sections that satisfy this condition is the restricted Hilbert space $\mathcal{H}_P$. Our second concern is that now we have all the sections we need, but when applying an operator then there is no guarantee that the new sections are still polarized!

$$\nabla_X s = 0 \quad \not\Longrightarrow \quad \nabla_X \hat{f} s = 0 \tag{5.31}$$

Given a symplectic manifold (M,ω) that satisfies Weil's integration condition and has a polarization P. Let $f \in C^\infty(M)$ be a classical observable then $\hat{f} = Q(f)$ is an operator on sections of the prequantum bundles where the mapping $Q : f \longrightarrow \hat{f}$ is defined in Definition 20. Then let's compute what conditions f should have to keep the polarized sections polarized.

$$\nabla_X \hat{f} s = -i\hbar \nabla_X \nabla_{X_f} s + f \nabla_X s = \hat{f}(\nabla_X s) - i\hbar \nabla_{[X,X_f]} s = -i\hbar \nabla_{[X,X_f]} s \tag{5.32}$$

Thus whenever the commutator $[X, X_f] \in P$ for every $X \in V_P(M)$, then $\hat{f}$ keeps polarized sections polarized with the right polarization.

In the cotangent space, we can find some indication what form an observable should have such that it preserves the polarization. Let $M = T^*\mathbb{R} \cong \mathbb{R}^{2n}$ with coordinates $\{p_a, q^a\}$ for $a \in [1,n] \cap \mathbb{N}$ and the symplectic 2-form $\omega = dp_a \wedge dq^a$. We use the vertical foliation as polarization P. Let $f \in C^\infty(T^*Q)$ be an observable which generates the vector field $X_f = -\frac{\partial f}{\partial q^a}\frac{\partial}{\partial p_a} + \frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a}$. Then straight forward calculation of the commutator of the vector field X_f and the elements of P.

$$[X_f, \partial/\partial p_k] = \frac{\partial^2 f}{\partial p_k \partial q^a}\frac{\partial}{\partial p_a} - \frac{\partial^2 f}{\partial p_k \partial p_a}\frac{\partial}{\partial q^a} \in P \quad \forall k \in [1,n] \cap \mathbb{N} \tag{5.33}$$

The elements of the polarization are spanned by $\partial/\partial p_k$, so therefore the $\partial/\partial q^a$ term has to vanish. We have that $\frac{\partial^2 f}{\partial p_k \partial p_a} = 0$ and this is only true when $f(q,p) = f_0(q) + f^k(q) p_k$. Where $f^k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $f_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$. We can see that when an observable is dependant on squared momentum, then it does not preserve the polarization. An example of such an observable would be the Hamiltonian of the free particle.

[2] Remember that $\nabla_X s = X \lrcorner \nabla s$

If the manifold is Kähler then the vector field X_f that is generated by an observable f should be a ***Killing vector field***. We will keep our computations as general as possible, so therefore we will keep computing the commutator.

When (M,ω) is a symplectic manifold with a polarization P. The set of classical observables that preserve the given polarization is $C_P^\infty(M) \subset C^\infty(M)$. The preservation of polarization is a heavy condition and it limits our possible observables. The Hamiltonian from section 4.4.2 does not satisfy the preservation of any polarization because it's not linear in p. Luckily there is a solution to this. the BKS-construction gives us some hope of recovering the right quantization.

Definition 35 *For a symplectic manifold (M,ω) that satisfies the prequantization conditions (Definition 20) and has a polarization $P \subset TM$ on it. Then the set of square-integrable, polarized sections of B denoted by $\mathcal{H}_P$ are the **wavefunctions**.*

There may be a problem that there may not exist non vanishing square-integrable polarized sections on a symplectic manifold. This is mostly a concern for real polarizations because there already exists some natural measure on the Kähler manifold that converges. But for now, we assume that non-vanishing square-integrable polarized sections exist.

Let us work out some examples of geometric quantization.

5.4 Holomorphic quantization

A very nice application of geometric quantization is to apply it to a Kähler manifold. This example will also argue that the Hermitian structure is a good measure of probability.

5.4.1 Kähler manifold

Let (M,ω) be a symplectic n-(complex)dimensional flat Kähler manifold with canonical coordinates $\{p_a, q^a\}$ for $a = 1,\dots n$. Also does (M,ω) satisfy Weil's integration condition. We can construct holomorphic coordinates $z^a = p_a + iq^a$ and $\bar{z}^a = p_a - iq^a$ such that $\{z^a, \bar{z}^a\}$ becomes the coordinate system. Let the Kähler potential be $\mathcal{K} = \frac{1}{2}\bar{z}^a z^a$ such that,

$$\omega = dp_a \wedge dq^a = \frac{i}{2} dz^a \wedge d\bar{z}^a = -i\partial\bar{\partial}\mathcal{K}. \tag{5.34}$$

The symplectic potential is $\theta = \frac{i}{2}\bar{z}^a dz^a = -i\partial\mathcal{K}$. There exists an Hermitian line bundle $B \longrightarrow M$ with a Hermitian metric $(\cdot,\cdot)$ and a connection $\nabla = d - i\hbar^{-1}\theta_0$ as defined in section 5.2.4. We are going to use the dummy connection $\nabla^* = d - i\hbar^{-1}\theta$ for calculations such that we get the desired sections. Like in the remark of definition 20, any two non-vanishing polarized sections $s, s' \in \Gamma(B)$ can be related as $s' = \phi s$ with ϕ a holomorphic function. Thus let $\mathfrak{u}$ be the unit section ($(\mathfrak{u},\mathfrak{u}) = 1$ and because it is constant, it is polarized), then any section can be written as $s = \psi\mathfrak{u}$ with ψ a holomorphic function.

If we choose P to be the antiholomorphic polarization then θ is adapted to P. When $X \in V_P(M)$ then $\nabla^*_X = X \lrcorner (d - i\hbar^{-1}\theta) = X \lrcorner d$. Then for any polarized section $\nabla^*_{\partial/\partial\bar{z}^a} s = \partial/\partial\bar{z}^a \lrcorner d\psi\mathfrak{u} = 0$ and this is equivalent to $\frac{\partial}{\partial\bar{z}^a}\psi = 0$. This implies we successfully restraint $\psi(z,\bar{z})$ to $\psi(z)$ because it has to be constant in the $\bar{z}$ coordinate and is thus independent of it.

Let $s = \psi u$ and $s' = \psi' u$ be sections of the prequantum bundle in the θ gauge. ψ and ψ' are both holomorphic functions. We defined in section 5.2.4 that the inner product between the sections in θ gauge is,

$$\langle s, s'\rangle = \int_M \bar{\psi}\psi' \exp\left[-\frac{z^a\bar{z}^a}{2\hbar}\right]\varepsilon. \tag{5.35}$$

Where $\varepsilon = \omega^n/(2\pi\hbar)^n$ like in equation 4.4. From this, we can easily construct a space of all wavefunctions of this system. $\mathcal{H}_P$ is the space of polarized sections and the sections should have a finite solution for 5.35 which is also the inner product.

Let's now apply the mapping procedure on this space. Some useful observables are z^a, $\bar{z}^a$ and $z^a\bar{z}^a$.

We want to first find the vector fields of those observable, let X_f have the form $X_f = x_f\partial/\partial z^a + y_f\partial/\partial\bar{z}^a$. Use equation 3.1 to find for X_{z^a},

$$dz^a = \left(x_{z^a}\frac{\partial}{\partial z^a} + y_{z^a}\frac{\partial}{\partial\bar{z}^a}\right) \lrcorner \frac{i}{2}dz^a \wedge d\bar{z}^a = \frac{i}{2}[-x_{z^a}d\bar{z}^a + y_{z^a}dz^a]. \tag{5.36}$$

We conclude that the vector field is $X_{z^a} = -2i\partial/\partial\bar{z}^a$. The other vector fields are,

$$X_{\bar{z}^a} = 2i\frac{\partial}{\partial z^a}, \tag{5.37}$$

$$X_{z^a\bar{z}^a} = 2i\left(z^a\frac{\partial}{\partial z^a} - \bar{z}^a\frac{\partial}{\partial\bar{z}^a}\right). \tag{5.38}$$

We have to check if these preserve the antiholomorphic polarization. Remember that a vector field preserves the polarization when $[X, X_f] \in P$ for every $X \in V_P(M)$.

$$[X, X_{z^a}] = -2i\frac{\partial}{\partial\bar{z}^a}\frac{\partial}{\partial\bar{z}^a} + 2i\frac{\partial}{\partial\bar{z}^a}\frac{\partial}{\partial\bar{z}^a} = 0. \tag{5.39}$$

Partial derivatives commute, thus X_{z^a} preserves the polarization. The same argument can be applied to $X_{\bar{z}^a}$. For $X_{z^a\bar{z}^a}$ we find,

$$[X, X_{z^a\bar{z}^a}] = 2i\left(z^a\frac{\partial}{\partial\bar{z}^a}\frac{\partial}{\partial z^a} - \frac{\partial}{\partial\bar{z}^a} - \bar{z}^a\frac{\partial}{\partial\bar{z}^a}\frac{\partial}{\partial\bar{z}^a}\right) - 2i\left(z^a\frac{\partial}{\partial z^a}\frac{\partial}{\partial\bar{z}^a} - \bar{z}^a\frac{\partial}{\partial\bar{z}^a}\frac{\partial}{\partial\bar{z}^a}\right) = -2i\frac{\partial}{\partial\bar{z}^a}. \tag{5.40}$$

$-2i\frac{\partial}{\partial\bar{z}^a}$ is an element of the polarization, thus $X_{z^a\bar{z}^a}$ also preserves the polarization.

We have shown that the observables preserve polarization such that their operator variant can keep polarized sections polarized. Thus let us apply the quantization mapping on these observables and use the dummy connection $\nabla^* = d - i\theta/\hbar$ for calculation. Remember that the actual connection is defined as in 5.2.4. Let us also remember that we have shown that sections are not dependant on the $\bar{z}^a$ coordinate.

$$\hat{z}^a s = -i\hbar\left[-2i\frac{\partial}{\partial\bar{z}^a} - \frac{i}{\hbar}(0)\right]s + z^a s = -2\hbar\frac{\partial}{\partial\bar{z}^a}s + z^a s = z^a s \tag{5.41}$$

For section in $\mathcal{H}_P$. The same computation will show us $\hat{\bar{z}}^a s = 2\hbar\frac{\partial}{\partial z^a}s$. These two operators act like raising and lowering operators just as in the quantum harmonic oscillator.

For the last observable we show that,

$$\widehat{z^a\bar{z}^a}s = -i\hbar\left[2i\left(z^a\frac{\partial}{\partial z^a} - \bar{z}^a\frac{\partial}{\partial \bar{z}^a}\right) - \frac{i}{\hbar}z^a\bar{z}^a\right]s + z^a\bar{z}^a s = 2\hbar\left(z^a\frac{\partial}{\partial z^a} - \bar{z}^a\frac{\partial}{\partial \bar{z}^a}\right)s - z^a\bar{z}^a s + z^a\bar{z}^a s = 2\hbar z^a\frac{\partial}{\partial z^a}s. \tag{5.42}$$

Now we have a few operators and we want to test them. A very nice example of this will be the harmonic oscillator. Like at the beginning of this section we will express it first in $\{p_a, q^a\}$ coordinates, and then in $\{z^a, \bar{z}^a\}$ coordinates. The harmonic oscillator has the Hamiltonian,

$$h = (p^2 + q^2)/2 = z\bar{z}/2. \tag{5.43}$$

The Hamiltonian of a harmonic oscillator in quantum mechanics has the form $\hat{h} = \hbar z^a\frac{\partial}{\partial z^a} + \hbar\frac{1}{2}$, where z^a acts as a raising operator and $2\hbar\frac{\partial}{\partial z^a}$ as a lowering operator.

We have already shown that $z\bar{z}$ preserves the polarization, thus h also preserves it. We find for h the operator,

$$\hat{h} = \hbar z^a\frac{\partial}{\partial z^a}. \tag{5.44}$$

This is very close to the operator form we are used too. We just missed the term $\hbar\frac{1}{2}$. This motivates another correction for this theory called the metaplectic correction. We will not include this correction in this thesis so we have to be satisfied with our solution. Nonetheless, it is a very good solution.

5.5 Quantization for manifolds with real directions

I will first motivate why we need a different approach for polarizations with real directions. Let Q be the configuration space of n-dimensions and $M = T^*Q \cong \mathbb{R}^{2n}$.

On it we can choose the 2-form $\omega = dp_a \wedge dq^a$ such that (M, ω) is a symplectic vector space. $\mathbb{R}^{2n}$ satisfies the Weil's integration condition. There exists a prequantum bundle with the appropriate connection and Hermitian metric. Then a natural polarization is the vertical foliation. Let's call the polarization P and the vectors in P are spanned by $\{\partial/\partial p_a\}$ with $a \in [1, n] \cap \mathbb{N}$. Notice that this polarization has an adapted symplectic potential $\theta = p_a dq^a$. This polarization decomposes TM in surfaces of constant q. This will mean that for a polarized section of the form $s = \psi\mathfrak{u}$,

$$0 = \nabla_X s = X \lrcorner (d\psi)\mathfrak{u} - \frac{i}{\hbar} X \lrcorner \theta s = \frac{\partial \psi}{\partial p_a}\mathfrak{u}. \tag{5.45}$$

For every $\partial/\partial p_a \in P$. We conclude that ψ is constant on the leafs of P (constant in the p_a coordinates). Let $s = \psi\mathfrak{u}$ and $s' = \psi'\mathfrak{u}$ be polarized sections of the prequantum bundle with $\psi, \psi \in C^\infty(M)$ and $\mathfrak{u}$ is the unit section. When we compute the standard inner product integral, we see that it doesn't converge when M is non-compact[3].

$$\langle s, s' \rangle = \int_M (\psi\mathfrak{u}, \psi'\mathfrak{u})\varepsilon = \int_M \bar{\psi}\psi'\varepsilon \longrightarrow \infty \tag{5.46}$$

Because on every leaf there is a constant function that will never converge for such an integral. For the Kähler manifold we could construct some natural inner product, but that won't work for this. The idea will be to work with the quotient space M/P with M a symplectic vector space and P the polarization such that we don't have to integrate over constant functions. For this, to work we need that $V = M/P$ is an orientable Hausdorff manifold.

Let us again work with the general 2n-dimensional symplectic vector space (M, ω) that satisfies Weil's integration condition and a polarization P that has real directions. Let $V = M/P$ be an orientable Hausdorff manifold. V is n-dimensional because the dimensions of leaves of P are half the dimensions of M. Our goal will be to add some kind of rooted volume form to the wavefunction such that when we are calculating the inner product that it becomes a finite integral over V. The determinant bundle $\det(V) = \bigwedge^n T^*_{\mathbb{C}}V$ has complex n-forms α. We can use the pullback of the projection $\mathrm{pr} : M \longrightarrow V$ to define these forms on M.

Let $\beta = \mathrm{pr}^*\alpha \in \Omega^n_{\mathbb{C}}(M)$.

Then β satisfies the properties,

$$X \lrcorner \beta = 0, \tag{5.47}$$

$$X \lrcorner d\beta = 0. \tag{5.48}$$

For every $X \in V_P(M)$.

Define the set $K_P = \mathrm{pr}^*(\det(V))$.

We would like the "square root" of the sections of K_P. This can be done by trivialising K_P and square rooting the positive transition functions. This will lead to the set δ_P where the square rooted sections map to. Such a section is denoted by $\nu \in \Gamma(\delta_P)$. If $\nu, \nu' \in \Gamma(\delta_P)$ then $\nu\nu'$ is a section of K_P.

[3] A simple example would be the tangent space of the configuration space.

We define the covariant derivative ∇_X on the sections $\beta \in K_P$ as $\nabla_X\beta = X \lrcorner \beta$ for every $X \in V_P(M)$. The Lie derivative on the sections of K_P should map K_P to K_P by the typical computation of the Lie-derivative. When $Z \in P$ then the lie derivative is automatic $\mathcal{L}_Z = \nabla_Z$.

The covariant derivative and Lie-derivative on $\nu \in \Gamma(\delta_P)$ is defined as,

$$\nabla_X\nu^2 = 2\nu\nabla_X\nu, \qquad \mathcal{L}_X\nu^2 = 2\nu\mathcal{L}_X\nu.$$

Now we will modify the prequantum bundle B to $B_P = B \otimes \delta_P \longrightarrow M$. This will lead to new sections of the form $\tilde{s} = s\nu$, where $\tilde{s} \in \Gamma(B_P)$, $s \in \Gamma(B)$ and $\nu \in \Gamma(\delta_P)$.

Definition 36 *Let (M,ω) be a 2n-dimensional symplectic vector space that satisfies Weil's integration condition and a polarization P that has real directions. Sections from the prequantum bundle $B_P = B \otimes \delta_P$ are called **P-wavefunctions**.*

Definition 37 ***Polarizaed P-wavefunctions** are P-wavefunctions that satisfy,*

$$\nabla_X\tilde{s} = (\nabla_X s)\nu + s(\nabla_X\nu) = 0 \quad \textit{For every} \quad X \in V_P(M). \tag{5.49}$$

When $\tilde{s} = s\nu$ and $\tilde{s}' = s'\nu'$ are P-wavefunctions. Then let the Hermitian metric[4] be,

$$(\tilde{s},\tilde{s}') := (s,s')\nu\nu'. \tag{5.50}$$

Notice that $\nabla_X(\tilde{s},\tilde{s}') = (\nabla_X\tilde{s},\tilde{s}') + (\tilde{s},\nabla_X\tilde{s}') = 0\ \forall X \in V_P(M)$. Therefore the connection is compatible with the new Hermitian metric.

Recall that we said that $\nu\nu'$ is a section of K_P and conclude that the Hermitian metric defines an n-form on V. Exactly as we wanted. Thus now we want to define the inner product as,

$$\langle\tilde{s},\tilde{s}'\rangle := \int_V (s,s')\nu\nu'. \tag{5.51}$$

We have constructed a new Hilbert space. Therefore we need an updated mapping such that the new quantum observable can properly act on the P-wavefunctions and satisfy all of Dirac's conditions.

Definition 38 *Let (M,ω) be a symplectic manifold that satisfies Weil's integration condition and has a polarization P. M/P is an orientable Hausdorff manifold. Then there exists a modified prequantum bundle $B_P = B \otimes \delta_P$ with sections $\tilde{s} : B_P \longrightarrow M$. Let $f \in C^\infty_P(M)$ be a polarization preserving classical observable and let $\tilde{s} = s\nu$ be a polarized P-wavefunction with $s \in \Gamma(B)$ and $\nu \in \Gamma(\delta_P)$. Then the **half form mapping** $\mathfrak{Q} : C^\infty_P(M) \longrightarrow \mathcal{O}$ is,*

$$\mathfrak{Q} : f \longrightarrow \tilde{f} : \mathfrak{Q}(f)\tilde{s} = \tilde{f}\tilde{s} = -i\hbar(\nabla_{X_f}s + fs)\nu - i\hbar s\mathcal{L}_{X_f}\nu. \tag{5.52}$$

We use from now on the convention to write our quantum observables that act on P-wavefunctions with a tilde. The half-form mapping is equivalent to,

$$\mathfrak{Q} : f \longrightarrow \tilde{f} : \tilde{f}(\tilde{s}) = \hat{f}(s)\nu - i\hbar s\mathcal{L}_{X_f}\nu \tag{5.53}$$

If the hamiltonian vector field X_f is complete for a classical observable f. We can also define the flow that is generated by $\tilde{f}$.

[4]This might be a bit confusing, because B_P is not a Hermitian line bundle. That's why we define the new "Hermitian metric" to only act on the B part of the P-wavefunctions.

$$\tilde{\rho}\tilde{s} = \hat{\rho}_t(s)\rho_t^*(\nu) \tag{5.54}$$

Let us get back to the example in the beginning and derive the results of canonical quantization. $Q = \mathbb{R}^n$ is the configuration space and $M = T^*Q \cong \mathbb{R}^{2n}$ with a symplectic 2-form $\omega = dp_a \wedge dq^a$ (the symplectic potential will be $\theta = p_a dq^a$). The polarization we will use is the vertical foliation P. Now consider the prequantum bundle $B_P \longrightarrow M$ with polarized sections $\tilde{s} = s\nu$. Let the section on B be $s = \psi u$ where $(u, u) = 1$ and $\psi \in C^\infty(M)$. Define the connection on B as $\nabla = d - i\theta/\hbar$. We already saw that equation 5.45 implies that $\psi(q^a, p_a) = \psi(q^a)$. The sections of δ_P will be defined as the square root of the standard volume form on $V = M/P = Q$, thus $\nu^2 = d^n q$.

The inner product becomes,

$$\langle \tilde{s}, \tilde{s} \rangle = \int_V \bar{\psi}\psi d^n q. \tag{5.55}$$

Let us compute the operators for the observables q^a and p_a. We can use the same vector fields that we found in section 4.4. Thus $X_{q^a} = -\partial/\partial p_a$ and $X_{p_a} = \partial/\partial q^a$. They have to satisfy $[X, X_f] \in V_P(M)$. They do because partial derivatives commute.

We can compute the operators,

$$\tilde{q}^a \tilde{s} = -i\hbar(\nabla_{X_{q^a}} s + q^a s)\nu - i\hbar s \mathcal{L}_{X_{q^a}} \nu = q^a \tilde{s}. \tag{5.56}$$

We used that $X_{q^a} \in V_P(M)$, thus $X_{q^a} \lrcorner \theta = 0$ and $\mathcal{L}_{X_{q^a}} \nu = 0$.

For $\tilde{p}_a$ follows,

$$\tilde{p}_a \tilde{s} = -i\hbar(\nabla_{X_{p_a}} s + p_a s)\nu - i\hbar s \mathcal{L}_{X_{p_a}} \nu = -i\hbar \frac{\partial}{\partial q^a}. \tag{5.57}$$

We produced both canonical quantization operators correctly! [5]

5.6 Pairing

Consider a symplectic manifold (M, ω) that satisfies Weil's integration condition and we are dealing with two polarizations. Then we can construct two Hilbert spaces, $\mathcal{H}_P$ and $\mathcal{H}_{P'}$. There is a way to link these two by a ***pairing map*** $\langle\!\langle \cdot, \cdot \rangle\!\rangle$: $\mathcal{H}_P \times \mathcal{H}_{P'} \longrightarrow \mathbb{C}$. From the pairing map we will deduct a map $\Pi : \mathcal{H}_{P'} \longrightarrow \mathcal{H}_P$ such that,

$$\langle \tilde{s}, \Pi \tilde{s}' \rangle := \langle\!\langle \tilde{s}, \tilde{s}' \rangle\!\rangle. \tag{5.58}$$

With $\langle \cdot, \cdot \rangle$ as the inner product defined on $\mathcal{H}_P$. also $\tilde{s}' \in \mathcal{H}_{P'}$ and $\tilde{s}, \Pi\tilde{s}' \in \mathcal{H}_P$. This will prove to be very useful in the BKS construction when observables don't preserve the polarization.

In this section, we will consider only the case of two real polarizations P and P'. We assume that these are transverse such that $TM = P \times P'$ with M a real symplectic manifold with a symplectic 2-form ω. This implies that we can split the space into

[5] You might ask why we needed the $\mathcal{L}_{X_f}\nu$ term in the operator map. It is not relevant for the canonical quantization but when X_f is linear in one of its coefficients, then $\mathcal{L}_{X_f}\nu \neq 0$.

two configuration spaces $M = Q \times Q'$. For example, in the typical phase space then $Q = \mathbb{R}^n$ would be position space and $Q' = \mathbb{R}^n$ momentum space. Let $\{q^a\}$ for $a = 1 \dots n$ be the coordinate system of Q and $\{q'^a\}$ for $a = 1 \dots n$ be the coordinate system of Q'.

Let the local symplectic potential θ be adapted to P. Let $S \in \Omega^0(M)$ be a local generating function (18) on the leafs Λ of P such that $0 = \theta|_\Lambda = dS$. Λ' is a leaf of P' and we know that ω vanishes on P' thus $d\theta|_{\Lambda'} = 0$. This implies that there exists a potential of the local symplectic potential. Modify S to also satisfy $\theta|_{\Lambda'} = dS$, such that it varies smoothly over Λ'. We introduced the coordinates $\{q^a\}$ and $\{q'^a\}$ where the polarzation P has leaves of constant q and P' has leaves of constant q'. Then we can deduce that,

$$\theta = \frac{\partial S}{\partial q^a} dq^a, \qquad \omega = \frac{\partial^2 S}{\partial q^a \partial q'^b} dq'^b \wedge dq^a. \tag{5.59}$$

As result S satisfies a non-degeneracy condition.

$$\det\left(\frac{\partial^2 S}{\partial q^a \partial q'^b}\right) \neq 0. \tag{5.60}$$

From the last section, we know that we also have to work with the modified prequantum bundle B_P. Define a pairing between two bundles K_P and $K_{P'}$ by,

$$(\beta, \beta')\varepsilon := \beta \wedge \beta'. \tag{5.61}$$

Where (β, β') should represent an element of $C^\infty(M)$. For the rooted sections ν and ν' we define $(\nu, \nu') := \sqrt{(\nu^2, \nu'^2)} \in C^\infty(M)$.

The pairing map is of the form for polarized P-wavefunctions $\tilde{s} = s\nu \in \Gamma(B_P)$ and $\tilde{s}' = s'\nu' \in \Gamma(B_{P'})$,

$$\langle\!\langle \tilde{s}, \tilde{s}' \rangle\!\rangle = \int_M (s, s')(\nu, \nu')\varepsilon. \tag{5.62}$$

Before we substitute our definition of pairing between two bundles, we want to look at some of the restrictions that are induced on $s = \psi \mathrm{u} \in B$ and $s' = \psi' \mathrm{u} \in B'$ by polarization.

We will assume that both ψ and ψ' are dependent on q^a and q'^a at first.

These sections should be polarized with their respective polarizations and the connection will be $\nabla = d - i\theta/\hbar = d - i\hbar^{-1}(\partial S/\partial q^a)dq^a$. Thus for s' and $X' = \partial/\partial q'^a \in P$,

$$0 = \nabla_{X'} s' = \frac{\partial}{\partial q'^a} \lrcorner\, d\psi' = \frac{\partial \psi'}{\partial q'^a}. \tag{5.63}$$

This is a clear implication that $\psi'(q, q') = \psi'(q)$. For s and $X = \partial/\partial q^a \in P$,

$$0 = \nabla_X s = \frac{\partial \psi}{\partial q^a} - \frac{i}{\hbar}\frac{\partial S}{\partial q^a}\psi. \tag{5.64}$$

From this, it isn't immediately clear what restriction there is. But we want that $\psi(q, q')$ is restricted to $\psi(q')$. Because the symplectic potential doesn't vanish on this polarization we have to deal with an extra term. Let $\psi(q, q') = \phi(q') \exp[R(q, q')]$ then the differential equation becomes.

$$\frac{\partial R}{\partial q^a} = \frac{i}{\hbar}\frac{\partial S}{\partial q^a} \tag{5.65}$$

This implies that $\psi(q,q') = \phi(q')\exp[\frac{i}{\hbar}S(q,q')]$.

Now for the pairing between the bundles K_P and $K_{P'}$. First remark that when $D = \det\left(\frac{\partial^2 S}{\partial q^a \partial q'^b}\right)$, then ε has the form,

$$\varepsilon = \frac{1}{(2\pi\hbar)^n}\omega^n = \frac{1}{(2\pi\hbar)^n}D^n d^n q' \wedge d^n q. \tag{5.66}$$

If we let $\nu = \sqrt{dq^a}$ and $\nu = \sqrt{dq'^a}$ then,

$$d^n q' \wedge d^n q = (\nu^2, \nu'^2)\varepsilon = (\nu^2, \nu'^2)\frac{1}{(2\pi\hbar)^n}D^n d^n q' \wedge d^n q. \tag{5.67}$$

Concude that $(\nu^2, \nu'^2) = (2\pi\hbar)^n / D^n$, thus $(\nu, \nu')\varepsilon = \frac{1}{(2\pi\hbar)^{n/2}}\sqrt{D}^n d^n q' \wedge d^n q$. Now we have everything to finish the pairing,

$$\langle\!\langle \tilde{s}', \tilde{s} \rangle\!\rangle = \frac{1}{(2\pi\hbar)^{n/2}}\int_M \bar{\psi}'(q)\phi(q')e^{\frac{i}{\hbar}S}\sqrt{D}^n d^n q' \wedge d^n q. \tag{5.68}$$

When we consider the phase space $M = T^*Q \cong \mathbb{R}^{2n}$ such that q is the position (the space Q) and $q' = p$ is the momentum (the space $T_x Q = Q'$ for $x \in Q$). The polarizations are described by $S = p \cdot q$. Then the pairing becomes,

$$\langle\!\langle \tilde{s}', \tilde{s} \rangle\!\rangle = \frac{1}{(2\pi\hbar)^{n/2}}\int_Q \bar{\psi}'(q)\left[\int_{Q'} \phi(p)e^{\frac{i}{\hbar}p\cdot q}d^n p\right]d^n q. \tag{5.69}$$

Thus we conclude that in this case,

$$(\Pi\tilde{s})(q) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{Q'} \phi(p)e^{\frac{i}{\hbar}p\cdot q}d^n p. \tag{5.70}$$

This is exactly the traditional way of computing wavefunctions from momentum space to position space by the Fourier transform.

5.7 BKS construction

Now we have all the tools to reproduce the operator form of the simplest Hamiltonian, $h = p^2/(2m)$. What you immediately see is that the Hamiltonian is quadratic in momenta and after a quick calculation we see that its vector field will never preserve the polarization. We are going to fix it by introducing the idea to apply the flow of the Hamiltonian to a wavefunction for a small time interval $\delta t = t' - t$. Then sending the new wavefunction in $\mathcal{H}_{P'}$ back to the old Hilbert space $\mathcal{H}_P$ by the pairing map. This will relate to the time evolution of the wavefunction.

This construction is introduced by Blatnerr, Kostant, and Sternberg (BKS).

$$\left\langle \frac{d\tilde{s}_t}{dt}, \tilde{s}' \right\rangle = -\frac{d}{dt'} \langle\!\langle \tilde{\rho}_{\delta t} \tilde{s}_t, \tilde{s}' \rangle\!\rangle |_{t'=0} \tag{5.71}$$

For every time independent $\tilde{s}' \in \mathcal{H}_P$. Where the time dependant wavefunction is $\tilde{s}_t \in \mathcal{H}_P$. On the left, we have the inner product on $\mathcal{H}_P$. On the right we have pairing between the polarization P and P', but for the pairing to be well defined we have to assume that they are transverse. $\tilde{\rho}$ is the flow generated by $\tilde{h}$.

So let us immediately apply this to the symplectic space $M = T^*Q$ with Q the typical configuration space. The symplectic frame is $\{p_a, q^a\}$ and therefore $\omega = dp_a \wedge dq^a$. We will assume that the configuration space is a flat n-dimensional euclidean space $Q = \mathbb{R}^n$, thus $M = TQ \cong \mathbb{R}^{2n}$. Euclidean space trivially satisfies the Weil's integration condition, thus there exists a prequantum bundle with the desired connection. The polarization that we will use is the vertical foliation. We know that M/P is an orientable Hausdorff manifold. Thus we have a prequantum bundle $B_P = B \otimes \delta_P$. The polarization implies that the polarized sections $\tilde{s} = \psi(q,p)\mathrm{u}\nu$ and $\tilde{s}' = \phi(q,p)\mathrm{u}\nu'$ are only dependant on the coordinate system $\{q^a\}$. $\phi, \psi \in C^\infty(M)$, u is the unitairy section of the prequantum bundle B and $\nu \in \Gamma(\delta_P)$. Choose $\nu = \nu' = \sqrt{d^n q}$ and let $\mu = \nu^2 = d^n q$. The Hamiltonian we will be working with is $h = p^2/(2m)$. This is the Hamiltonian of the free particle and consequently produces flows that are straight lines.

$$\rho_t(q,p) = (q^a(t), p_a(t)) = \left(q^a + \frac{p_a}{m} t, p_a\right) \tag{5.72}$$

So let's compute the pairing before the time derivative and let $t = 0$,

$$\langle\!\langle \tilde{\rho}_t \tilde{s}, \tilde{s}' \rangle\!\rangle = \int_M (\overline{\rho_t^* \psi})(q,p)\phi(q) \exp\left[i\hbar^{-1} \int_0^t (L \circ \gamma)(t) dt' \right] \sqrt{(\rho_t^* \mu, \mu)} \varepsilon. \tag{5.73}$$

Where L is the Lagrangian and in this case $L = h$ and γ is a path created by the flow of the Hamiltonian. We said in section 18 that the generating function S is the action integral when identified by the canonical transformations of the Hamiltonian. That's why we have the action integral in the exponent.

Let us calculate everything separately. First $(\rho_t^* \psi)(q,p) = \psi(\rho_t(q,p)) = \psi(q(t), p(t)) = \psi\big(q^a + \frac{p_a}{m} t, p_a\big)$. Now the integral,

$$\int_0^t (L \circ \gamma)(t') dt' = \int_0^t \frac{p^2}{2m} dt' = \frac{p^2}{2m} t.. \tag{5.74}$$

The last part is $\sqrt{(\rho_t^* \mu, \mu)} \varepsilon$. First compute $\rho_t^* \mu = \rho_t^* d^n q = d^n(q + \frac{t}{m} p)$. Then,

$$(\rho_t^* \mu, \mu)\varepsilon := \rho_t^* \mu \wedge \mu = d^n(q + \frac{t}{m} p) \wedge d^n q = \left(\frac{t}{m}\right)^n d^n p \wedge d^n q. \tag{5.75}$$

ε has the usual form $\varepsilon = (2\pi\hbar)^{-n} d^n p \wedge d^n q$, thus $\sqrt{(\rho_t^* \mu, \mu)} = (2\pi\hbar t/m)^{n/2}$.

When substituting our results we get,

$$\langle\!\langle \tilde{\rho}_t \tilde{s}, \tilde{s}' \rangle\!\rangle = \left(\frac{t}{2m\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} \bar{\psi}(q(t), p)\phi(q) \exp\left[it\hbar^{-1} \frac{p^2}{2m} \right] d^n p \wedge d^n q. \tag{5.76}$$

First focus on the momentum integral. The only momentum dependant functions are $\bar{\psi}(q(t),p)$ and $\exp\left[itp^2/(2m\hbar)\right]$.

$$\int_{\mathbb{R}^n} \bar{\psi}(q(t),p)\exp\left[it\hbar^{-1}\frac{p^2}{2m}\right]d^n p \tag{5.77}$$

Let us first expand $\psi(q(t),p)$ about q.

$$\psi(q(t),p) = \psi(q,p) + \frac{t}{m}p_a\frac{\partial}{\partial q^a}\psi(q,p) + \frac{t^2}{2m^2}p_a p_b\frac{\partial^2}{\partial q^a \partial q^b}\psi(q,p) + \mathcal{O}(t^3) \tag{5.78}$$

The integral vanishes for linear terms of p_a, because of symmetry reasoning. Thus higher orders don't vanish when $a = b$. Let us only consider the first 2 non-vanishing expansion terms. Then the integral becomes,

$$\bar{\psi}(q,p)\int_{\mathbb{R}^n}\exp\left[it\frac{p^2}{2m\hbar}\right]d^n p + \frac{t^2}{2m^2}\nabla^2\bar{\psi}(q,p)\int_{\mathbb{R}^n}p^2\exp\left[it\frac{p^2}{2m\hbar}\right]d^n p. \tag{5.79}$$

Where $\nabla^2 = \frac{\partial^2}{\partial q^a \partial q^a}$. We can evaluate the latter integral further. We will use the result of source [2].

$$\int_{\mathbb{R}^n}p^2\exp\left[it\frac{p^2}{2m\hbar}\right]d^n p = \frac{im\hbar}{t}\left(\frac{2\pi i m\hbar}{t}\right)^{n/2} \tag{5.80}$$

Now substituting this back into the pairing integral.

$$\langle\!\langle \tilde{\rho}_t\tilde{s},\tilde{s}'\rangle\!\rangle = \langle\!\langle \tilde{\rho}_0\tilde{s},\tilde{s}'\rangle\!\rangle + i^{n/2}\frac{i\hbar t}{2m}\int_{\mathbb{R}^n}\nabla^2\bar{\psi}(q,p)\phi(q)d^n q \tag{5.81}$$

Let us now apply the time derivative on $\langle\!\langle \tilde{\rho}_t\tilde{s},\tilde{s}'\rangle\!\rangle$ and substitute this result in equation 5.71 and use that $i^{n/2} = e^{i\pi n/4}$.

$$\int_{\mathbb{R}^n}\frac{\partial\bar{\psi}}{\partial t}\phi d^n q = \left\langle\frac{d\tilde{s}}{dt},\tilde{s}'\right\rangle = -\frac{d}{dt'}\langle\!\langle \tilde{\rho}_t\tilde{s},\tilde{s}'\rangle\!\rangle|_{t'=0} = -e^{i\pi n/4}\frac{i\hbar}{2m}\int_{\mathbb{R}^n}\nabla^2\bar{\psi}\phi d^n q \tag{5.82}$$

This is true for every $\tilde{s}' \in \mathcal{H}$, thus also for $\tilde{s}' = u\nu$. We conclude that,

$$\frac{\partial\bar{\psi}}{\partial t} = -e^{i\pi n/4}\frac{i\hbar}{2m}\nabla^2\bar{\psi} \tag{5.83}$$

This would produce an extra constant. For now, we can let the constant be zero. When we take the complex conjugate and rearrange constants, then we get the known differential equation.

$$i\hbar\frac{\partial\psi}{\partial t} = -e^{-i\pi n/4}\frac{\hbar^2}{2m}\nabla^2\psi \tag{5.84}$$

We successfully predicted the operator form of the Hamiltonian of the free particle correctly except for an extra phase change. This phase change will disappear when applying the metaplectic correction. Then it will be absorbed in the new definition of pairing.

Chapter 6

Conclusion

The whole quantization process went as followed. Starting with a classical system of only position space Q. We can make this into a phase space by considering its tangent space $M = T^*Q$. Assume that M is real. Now we have to define a symplectic 2-form and search for a polarization P. When this is successful, we have to check if there exists a Hermitian line bundle $B \longrightarrow M$ with the right connection and check if M/P is an orientable Hausdorff manifold. We define our sections of B_P to be polarized and decide which classical observables preserve the polarization $f \in C_P^\infty(M)$. Then we can find the operator form of that observable by the half form map. The operator $\mathfrak{Q}(f) = \tilde{f}$ acts on polarized P-wavefunctions.

After all this, we predicted canonical quantization correctly and the operator form of the free particle up to a phase with the help of the BKS-construction. It's clear that this theory is built upon correcting the last one and to finalize geometric quantization there still is a metaplectic correction.

In the case of the Kähler manifold, the geometric quantization procedure was straight forward and uses the quantization map. Although there is still a metaplectic correction to be made for the correct prediction of the Hamiltonian of the harmonic oscillator.

Along the way, we found many weaknesses. These weaknesses lie mostly in the amount of construction that went into geometric quantization. Beginning with a classical system. It is not always possible to define a non-degenerate 2-form. Then for prequantization is that Weil's integration condition doesn't have to be satisfied. Therefore M does not guarantee that the desired prequantum bundle exist to define our sections on. After that, we found that we needed restricted sections. Here we dealt with polarizations, but some symplectic manifolds don't have an appropriate polarization or don't have any. When we have a polarization we can define polarized sections although we are not sure if there exists square-integrable non-vanishing polarized sections on the given symplectic manifold. Finally, the observables that can undergo geometric quantization are massively restricted, because they have to preserve the polarization or else we may have to try the BKS construction. But when the leading momentum term in a Hamiltonian is cubic then the BKS construction won't solve it. Luckily we are mostly interested in the trivial cases where everything works.

We successfully predicted the harmonic oscillator in complex space up to a constant, canonical quantization, and the Hamiltonian for a free particle. All are very impressive feats of this theory. It doesn't add any new concepts in the world of physics or mathematics, but rather provides a link between the classical systems described by symplectic geometry and the quantum systems described by quantizations. It gives us some insight on how quantum mechanics works and hopefully understand nature a little bit better.

Appendix A

Recap of some differential geometry keypoints

A.1 n-dimensional smooth Manifolds

Manifolds are the generalisation of n-dimensional space with some topological aspects.

Definition 39 *An **n-dimensional manifold** M is a space that satisfies,*

- *Hausdorff space,*
- *Second countable,*
- *Locally Euclidean of dimension n.*

Some simple examples would be any open interval on the real line or a plane in 3-dimensional euclidean space.

Definition 40 *Let M be a manifold. A **chart** is a pair (φ, U) where U is an open subset in M and $\phi : U \longrightarrow U' \subset \mathbb{R}^n$ is a homeomorphism. U' is open in $\mathbb{R}^n$.*

Definition 41 *Let M be a manifold. An **atlas** $\mathcal{A}$ is a collection of charts $(\varphi_\alpha, U_\alpha)$ and U_α for all α covers M. A **smooth atlas** is an atlas with every two charts smoothly compatible.*

$$\varphi_\alpha \circ \varphi_\beta^{-1} \text{ is smooth whenever } (\varphi_\alpha, U_\alpha) \text{ and } (\varphi_\beta, U_\beta) \text{ are charts of atlas } \mathcal{A} \tag{A.1}$$

Definition 42 *M is a **smooth manifold**, whenever M is a manifold with a maximal smooth atlas.*

Definition 43 *Let M be a manifold. A **curve** is a continuous map $\gamma(t) : I \longrightarrow M$ with $t \in I \subset \mathbb{R}$.*

Let M be a smooth manifold. Consider a curve $\gamma(t) : I \longrightarrow M$ with $I = [-1, 1]$. When $\gamma(0) = m \in M$, the curve induces a vector $v = \frac{\mathrm{d}}{\mathrm{dt}}\gamma(t)|_{t=0}$ at m and the space of these vector is called the **tangent space** in m denoted as $T_m M$. Let $\{x^a\}$ be the coordinate system, then any vector $v \in T_m M$ can be written as the sum over $a = 1, \dots n$. (we will use the summation convention)

$$v = v^a \frac{\partial}{\partial x^a} \tag{A.2}$$

We can unify all the tangent spaces at different $m \in M$ as $\bigsqcup_{m \in M} T_m M = TM$, called the **tangent bundle**.

Definition 44 *Let M be a smooth manifold. A **vector field** is a smooth map $X : M \longrightarrow TM : m \longrightarrow X_p$. For every point $m \in M$, $X_m \in TM$ that varies smoothly over M.*

Definition 45 *Let M be a smooth manifold with a vector field X. A **flow** is a map $\rho_t(m) : I \times M \longrightarrow M$ with $I \subset \mathbb{R}$. The flow should have the property that $X(\rho_t(m)) = \frac{d}{dt}\rho_t(m)$ for every $t \in I$. When the set of flows of a vector field covers the whole manifold, then the vector field is called **complete**.*

We want to define another object on the smooth manifold, a **covector** at $m \in M$. It should behave as $\phi : T_m M \longrightarrow \mathbb{R}$ and the space of covectors is called the **dual space** of $T_m M$ or the **cotangent space** $T_m^* M$. Like in the tangent space case, we can define a **cotangent bundle** $T^* M = \bigsqcup_{m \in M} T_m^* M$ and from here we can define in the same way a **covector field**.

An example for such an object would be the differential (gradient) of a smooth function on the manifold, $df(X)(m) \in \mathbb{R}$ for $f \in C^\infty(M)$ and $X \in V(M)$. This implies that the differential is some sort of operator. We can write a covector as $\phi = \phi_a dx^a$.

This operator will be referred to as the **exterior derivative**. We use the **wedge product**[1] $\wedge$ to combine these covectors and it is antisymmetric, $dx \wedge dy = -dy \wedge dx$. Now we have established that this operator can be applied multiple times on covectors and this brings us the idea of **n-forms**. e.g. dx is a one form, $dx \wedge dy$ is a two form ...

A set of n-forms is denoted by $\Omega^n(M)$. Some properties that the exterior derivative should have is,

- $d \circ d = 0$,
- d is linear over $\mathbb{R}$,
- d commutes with pullbacks,
- If $\omega \in \Omega^n(M)$ and $\eta \in \Omega^k(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^n \omega \wedge d\eta$.

Definition 46 *Let M be a smooth manifold. If $\omega \in \Omega^n(M)$ and $d\omega = 0$, we call it **closed**. If there exists a $\theta \in \Omega^{n-1}(M)$ and $\omega = d\theta$, then ω is **exact** and θ is the **potential**.*

Definition 47 *Let M be a smooth manifold with X a vector field and ω an n-form. An **interior derivative** is the operation $X \lrcorner\, \omega = i_X \omega = \omega(X, \dots)$.*

Now we introduce the **Lie-derivative** $\mathcal{L}_X$ for $X \in V(M)$. This derivative works on functions $f \in C^\infty(M)$, vector fields $Y \in V(M)$ and forms $\omega \in \Omega^n(M)$.

- $\mathcal{L}_X f := df(X)$,
- $\mathcal{L}_X Y := [X, Y]$,
- $\mathcal{L}_X \omega := d(i_X \omega) + i_X d\omega$.

The last one is often referred to as Cartan's magic formula.

[1]For a more detailed explanation I would recommend to read **Introduction to Smooth Manifold** by J. M. Lee.

Integration

An important application of covector fields is to make coordinate independent integrals. For example an integral over $\theta \in \Omega^n(M)$ through a path γ. Let v be the tangent vectors of γ then,

$$\int_\gamma \theta := \int_0^t v \lrcorner \theta dt = \int_0^t \theta_a v^a dt. \tag{A.3}$$

Appendix B

Vector bundles and connections

Definition 48 *A **vector bundle of rank k** over a manifold M is a manifold E together with a map $\pi : E \longrightarrow M$. It should satisfy,*

- *$\forall m \in M$ then $E_m := \pi^{-1}(m)$ has the structure of k-dimensional vector space.*
- *Let U be a neighbourhood around $m \in M$ then there should be a diffeomorphism $\Phi : E_U \longrightarrow M \times \mathbb{R}^k$ with $E_U := \pi^{-1}(U)$.*

Φ is called the **transition function** and the operation is called **trivialisation**. In this thesis we will be mostly focusing on **line bundles**, where $k = 1$.

A **section** is a smooth map $s : M \longrightarrow E$ and has to pick for every $m \in M$ an element of $s(m) \in E_m$. The space of sections of E is denoted as $s \in \Gamma(E)$. We consider these almost as smooth functions on the manifold, but some mathematical objects like the exterior derivative don't make sense. For this we introduce,

Definition 49 *Let $E \longrightarrow M$ be a vector bundle ad then define the **connection** as a linear differential operator $\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$. ∇ has the properties,*

- *For constants a, b and $s, s' \in \Gamma(E)$, we have that $\nabla(as + bs') = a\nabla s + b\nabla s'$,*
- *Let $f \in C^\infty(M)$ and $s \in \Gamma(E)$ then $\nabla(fs) = df \otimes s + f\nabla s$ (Leibniz rule).*

We will often use it as $\nabla_X = X \lrcorner \nabla$. Our goal of the connection was to mimic the exterior derivative, but it fails when we compute $\nabla^2 = F_\nabla$. ∇^2 does not have to equal zero. We call the squared ∇ the curvature and it is important for the Weil's integration condition because the curvature induces an integral class.

Vector bundle can be complex. **Complex line bundles** trivialise to $M \times \mathbb{C}$.

We can define a metric on vector bundles and we are particularly interested in the metric on a complex line bundle because this will play an important role in defining an inner product on the Hilbert space. The metric on a complex vector bundle is called the **Hermitian structure** or the **Hermitian metric**.

Definition 50 *A **Hermitian metric** on a complex line bundle $E \longrightarrow M$ is a map $g : E \times E \longrightarrow \mathbb{R}$ such that or every $m \in M$ and $X \in E_m \backslash \{0\}$,*

- *$g(X, X) > 0$,*
- *$g(X, Y) = \overline{g(Y, X)}$.*

g will often be denoted as $(\cdot,\cdot)$. A connection is compatible with a Hermitian metric if,

$$\nabla_X(s,s') = (\nabla_X s, s') + (s, \nabla_X s'). \tag{B.1}$$

We call $\mathfrak{u} \in \Gamma(E)$ a unitary section if $(\mathfrak{u},\mathfrak{u}) = 1$. The **connection 1-form** is determined by its action on the unit section.

$$\nabla_X \mathfrak{u} := -i\Theta(X)\mathfrak{u} \tag{B.2}$$

The connection 1-form in the text will always be the symplectic potential divided by $\hbar$.

Consider a Hermitian line bundle $E \longrightarrow M$ and we have two intersecting neighbourhoods $U_i \cap U_j \neq \varnothing$. We define a transition function $g_{ij}(E_{U_i \cap U_j}) = U_i \cap U_j \times \mathbb{C}$. Then we can find the connection 1 form through,

$$d\log(g_{ij}) = i(\Theta_i - \Theta_j). \tag{B.3}$$

Appendix C

Darboux's theorem

Recall the theorem.

Theorem 6 (Darboux's theorem) *Let (M, ω) be a 2n-dimensional symplectic manifold and let $m \in M$. Then there exists a neighbourhood U of m and a coordinate system $\{p_a, q^a\}$ with $a, b = 1, 2 \ldots n$ on U such that $\omega = dp_a \wedge dq^a$ in U.*

We will first introduce **time-dependant vector fields** and **time-dependent differential forms**.

Let M be a smooth manifold with coordinates $\{x^a\}$ and let $m \in M$. A time-dependant vector field is just a vector field with time-dependent components, $X(m,t) = X^a(m,t)\frac{\partial}{\partial x^a}$.

We should also consider the time derivative of such a time-dependant vector field. $\partial_t X = (\frac{\partial}{\partial t} X^a(m,t))\frac{\partial}{\partial x^a}$.

The associated vector field of a time-dependant vector field is $\tilde{X} \in V(M \times \mathbb{R})$.

$$\tilde{X} = X^a(m,t)\frac{\partial}{\partial x^a} + \frac{\partial}{\partial t} \tag{C.1}$$

The time-dependant k-forms are very similar defined as the vector fields. The components of the k-form are time-dependant and when taking the time derivative, it is the time derivative of every component.

The consequence for the Lie-derivative is,

$$\mathcal{L}_X \alpha = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha) + \partial_t \alpha \tag{C.2}$$

Where X is a time dependant vector field and α a time dependent k-form.

First, we will sketch the proof of a lemma that we will use.

Lemma 3 *Let ω and ω' be symplectic structures on a smooth 2n-dimensional manifold M and $m \in M$. If $\omega(m) = \omega'(m)$, then there are neighbourhoods U and V of m and a diffeomorphism $\rho : U \longrightarrow V$ such that $\rho(m) = m$ and $\rho^*(\omega') = \omega$.*

This is not a waterproof proof, but rather an idea on how to prove it. We know from the assumptions that $\omega(m) = \omega'(m)$. Therefore there exists a neighbourhood W around m such that $d(\omega - \omega') = 0$. Thus we can find a local potential α such that $d\alpha = \omega - \omega'$ and let $\alpha(m) = 0$.

Then define a time dependant 2-form Ω,

$$\Omega = \omega + t(\omega' - \omega). \tag{C.3}$$

Note that $\Omega(m,t) = \omega(m)$ for all $t \in \mathbb{R}$. Thus Ω is non-degenerate in a neighbourhood inside W, just like the symplectic structure ω. Then there is a well-defined time-dependent vector field X such that,

$$X \lrcorner \Omega + \alpha = 0. \tag{C.4}$$

For the Lie-derivative we find $\mathcal{L}_X\Omega = 0$.

From this follows that if $\rho_{tt'}$ is the flow of X, then $\rho^*_{tt'}\Omega(t') = \Omega(t)$. Thus $\rho^*_{01}\omega' = \omega$. $\rho(m) = m$, because $X(m) = 0$. Let $U \subset W$, then if U is small enough, $\rho_{01}(U) = V \subset W$.

Hereby we conclude the lemma to be proven.

Proof of Darboux's theorem

Let (M,ω) be a symplectic manifold with a symplectic 2-form. Then because of proposition 1 in chapter 2 we have a neighbourhood of m where we can choose a symplectic frame $\{y_a, x^a\}$ as coordinate system and let $\omega' = dy_a \wedge dx^a$. Let ρ be as in lemma 3. Define $p_a := y_a \circ \rho$ and $q^a := x^a \circ \rho$. Then because of the earlier lemma. ω has the form $\omega = \rho^*\omega' = dp_a \wedge dq^a$. ∎

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