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A Truthmaker Semantics for Wansing's C

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Introduction

The aim of this thesis is to provide a truthmaker semantics for the propositional connexive logic C introduced by Heinrich Wansing [28].

The basic idea of thruthmaker semantics is that the truth of propositions are necessitated by states that are wholly relevant to the truth of said propositions. Note that this "state" can be a state of affairs, a state of facts, or anything else, as long as there is a mereology which is defined in chapter 2. An overview of truthmaker semantics can be found in [12].

There are many results achieved for truthmaker semantics in the past few years; a few are highlighted here:

- There is a truthmaker semantics for first degree entailment which is essentially due to Bas van Fraassen [15]. Kit Fine in [10] established a modernized version of this result.
- Fine also defined a truthmaker semantics for full intuitionistic logic [9].
- Mark Jago did the same for relevant logic [16].

Truthmaker semantics is a new shared semantic underpinning for these logics and constructing a truthmaker semantics for Wansing's C provides a way to compare it to the rest of these logics. C is a promising four-valued semantics for connexive logic. Logic is fundamental for artificial intelligence and database reasoning. As argued by Anderson and Belnap, four-valued logic is needed to be able to make inferences in inconsistent or incomplete databases [2]. The reason to use connexive logic is that the negation for the implication in connexive logic is semantically highly intuitive for English speakers [20]. C is preferred over other semantics for connexive logic as it is FDE based [30].

This is a technical thesis and the focus is mainly on the semantic result: how does one model the semantics for the logic C? Naturally, the use of truthmaker semantics brings forth many philosophically interesting questions, as well as proof-theoretical challenges. These aspects will only be briefly addressed throughout this paper.

In order to get a truthmaker semantics for C, we first combine parts of the previously established results for first degree entailment and full intuistionistic logic. We therefore review these logics, as well as C, and briefly mention the relation between them. This is done in the first chapter. In the second chapter, we introduce truthmaker semantics, along with the underlying mathematical foundation and some philosophical context. In the third chapter, we go through the truthmaker semantics for first degree entailment. In the next chapter, we do the same for positive intuitionistic logic. The main result and the conclusion of the thesis are found in chapter 5.

Throughout the thesis, we work with the logical connectives neg (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication). We use Greek letters

$\phi, \psi, \chi, \ldots$ for well formed formulas and small letters $p, q, r, \ldots$ to talk about atoms or propositional letters. When describing relations, we use the infix notation wRv to indicate that $(w, v) \in R$.

1 Connexive Logic

1.1 Introduction

Connexive logic is a branch of non-classical logic. An overview of connexive logic can be found in [23]. Modern connexive logic is founded by Angell [3] and McCall [19]. It is based on the intuitive idea that if some formula ϕ implies some (maybe other) formula ψ, that it would be wrong to have that ϕ also implies the negation of ψ. In general, a system of logic is called a *(weakly) connexive logic* if and only if the following four laws are provable:

Aristotle's Laws: $\neg(\phi \to \neg\phi),\ \neg(\neg\phi \to \phi)$.

Boethius' Laws: $(\phi \to \psi) \to \neg(\phi \to \neg\psi),\ (\phi \to \neg\psi) \to \neg(\phi \to \psi)$.

These laws are a direct realisation of the idea of connexive logic and the idea indeed dates back to Aristotle's Prior Analytics 57b14 and De Syllogismo Hypothetico 843D. Note that Aristotle's theses follow from Boethius' [30].

In paraconsistent logics, we do not necessarily have that a formula is not true whenever its negation is true. This is why Kapsner [18] proposed a notion of strong connexivity. A logic is called *strongly* connexive if and only if in the logic:

Unsat-Aristotle: $(\phi \to \neg\phi)$ is unsatisfiable.

Unsat-Boethius: $\phi \to \psi$ and $\phi \to \neg\psi$ are not jointly satisfiable.

Moreover, for the system to be interesting from a connexivity point of view, it should not have $(\phi \to \psi) \to (\psi \to \phi)$ as a valid schema [19]. In this thesis we mainly concern ourselves with the notion of weak connexivity. There are multiple systems qualified as (weakly) connexive logics that can be found in [23]. One logic that satisfies the aforementioned requirements is known as *Wansing's C* [28]. C can be obtained by adding the De Morgan's laws and a special clause for negation for the implication based on Boethius' laws to *Positive Intuitionistic Logic* (I^+). I^+ in its turn can be viewed as the positive (negation-free) part of *First Degree Entailment* with an extra clause for the implication ($\to$). This chapter consists of a brief outline of these three logics.

1.2 First Degree Entailment

First Degree Entailment (FDE) can be seen as the core of relevance and entailment logics. It is developed by Anderson and Belnap in the late 1950s. For more details see [1] and for an overview one can look at [22]. Some known semantics are the $*$ semantics as introduced by Routley [27], relational semantics as given by Dunn [7] and the Four-valued semantics characterized through truth tables due to Smiley [[5] p.16]. In this paper we will make use of the impossible world semantics [24], which will be defined later on, as it embeds most naturally in this project.

Let us first fix the language:

Definition 1.1 (Language FDE)**.** *We will work with a propositional language $\mathcal{L}_{\neg,\wedge,\vee}$ which only has the connectives $\neg, \wedge, \vee$. The propositional letters are $\mathcal{P}$.*

We define a model:

Definition 1.2 (*FDE* model)**.** *An* FDE *model is a structure $\mathcal{M} = \langle W, \nu^+, \nu^- \rangle$ where:*

1. *$W \neq \emptyset$ is a non-empty set of entities ('worlds').*
2. *$\nu^+ : \mathcal{P} \to \wp(W)$ is an interpretation function sending each propositional letter to the set of worlds where the corresponding atom is true.*
3. *$\nu^- : \mathcal{P} \to \wp(W)$ is an interpretation function sending each atom to the set of worlds where that atom is false.*

We use $w, v, u, \ldots$ (possibly indexed) as meta-variables ranging over worlds. By $w \vDash \phi$ we mean that formula ϕ is true in world w in some model $\mathcal{M}$. Strictly speaking, truth as we use it here is always relative to a model. We omit the $\mathcal{M}$ for readability however. By $w \dashv \phi$ we mean that ϕ is false in w. We also write $w \nvDash \phi$ and $w \not\dashv \phi$ to indicate that a formula is not true and not false respectively at world w. In classical logics $w \nvDash \phi$ implies $w \dashv \phi$, but in the logics we use, this is not the case. It is precisely the difference between falsity and absence of truth. Lastly, we use $w \vDash \phi, \psi$ (idem for $\dashv$) to indicate both $w \vDash \phi$ and $w \vDash \psi$.

We define truth and falsity:

Definition 1.3 (Truth clauses FDE)**.** *Let $\mathcal{M} = \langle W, \nu^+, \nu^- \rangle$ be an FDE model. We then define by the following double recursion for all $w \in W$:*

$$
\begin{aligned}
w \vDash p &\Leftrightarrow w \in \nu^+(p),\\
w \dashv p &\Leftrightarrow w \in \nu^-(p),\\[4pt]
w \vDash \neg\phi &\Leftrightarrow w \dashv \phi\\
w \dashv \neg\phi &\Leftrightarrow w \vDash \phi,\\[4pt]
w \vDash \phi \wedge \psi &\Leftrightarrow w \vDash \phi \text{ and } w \vDash \psi,\\
w \dashv \phi \wedge \psi &\Leftrightarrow w \dashv \phi \text{ or } w \dashv \psi,\\[4pt]
w \vDash \phi \vee \psi &\Leftrightarrow w \vDash \phi \text{ or } w \vDash \psi,\\
w \dashv \phi \vee \psi &\Leftrightarrow w \dashv \phi \text{ and } w \dashv \psi.
\end{aligned}
$$

Note that by using two separate valuation functions for truth and falsity, we can distinguish four different degrees of truth for each formula in a world:

TRUE Just $w \vDash \phi$. ϕ is only true in world w.

BOTH Both $w \vDash \phi$ and $w \dashv \phi$. ϕ is both true and false in w, making the logic paraconsistent.

NEITHER Neither $w \vDash \phi$ nor $w \dashv \phi$. ϕ is true nor false in w, making the logic paracomplete.

FALSE Just $w \dashv \phi$. ϕ is only false in world w.

Priest [26] has a more in-depth analysis of the four truth-values.

Consequence in *FDE* is defined as truth-preservation across worlds in all models:

Definition 1.4 (FDE consequence)**.** *$\phi \vDash_{FDE} \psi$ iff for all FDE models $\mathcal{M} = \langle W, \nu^+, \nu^- \rangle$, for all $w \in W$, if $w \vDash \phi$, then $w \vDash \psi$.*

There are a lot of things that are interesting to note about FDE, of which I will highlight some.

Firstly, in classical propositional logic, $(p \wedge \neg p)$ is false in every model. Hence, whenever $(p \wedge \neg p)$ is true in some model, anything is true in that model. In FDE however, we do not have this explosion. For example: $(p \wedge \neg p) \nvDash_{FDE} q$.

Proof. Consider an FDE model $\mathcal{M} = \langle \{w\}, \nu^+, \nu^- \rangle$ where $\nu^+(p) = \nu^-(p) = \{w\}$ and $\nu^+(q) = \nu^-(q) = \emptyset$. As $w \in \nu^-(p)$ we get using definition 1.3 $w \dashv p$ and hence $w \vDash \neg p$. As $w \vDash p$ and $w \vDash \neg p$, the antecedent is obviously true. (It is false as well since $w \nvDash p$, but that is not important for now.) However, as $w \notin \nu^+(q)$, we get that $w \nvDash q$. Therefore by definition 1.4 we get $(p \wedge \neg p) \nvDash_{FDE} q$. □

Furthermore, FDE is purely inferential: it has no theorems or anti-theorems in the sense that there are no formulas that are made true in all worlds in all models and no formulas that are made false in all worlds in all formulas. This can be shown by induction over the complexity of formulas in a world in a model where every proposition is neither true nor false.

There are several sound and complete proof systems for *FDE*: there is a Hilbert-style system, a Gentzen-style system (both due to Josep Maria Font [14]), a natural deduction calculus in the style of Gentzen and Prawitz due to Graham Priest [25], p. 309, and a tableau system, also due to Priest [26].

1.3 Positive Intuitionistic Logic

Intuisionistic logic is built on the idea that the meaning of a sentence is not given by the conditions under which the sentence is true, but rather by the conditions under which the sentence is proved, where a proof is a (mental) construction of some kind [26]. This idea seems to be captured best in a possible world semantics; for this purpose we will describe a Kripke semantics. For a more detailed exposition of full intuitionistic logic, see [21].

Positive Intuisionistic Logic (I^+) consists of only the positive part of full intuitionistic logic. It does not deal with negation, but, as we also want it to capture the idea of entailment, has an implication.

Definition 1.5 (Language I^+)**.** *We will work with a propositional language $\mathcal{L}_{\wedge,\vee,\rightarrow}$ which only has the connectives $\wedge, \vee, \rightarrow$. The propositional letters are $\mathcal{P}$.*

We define a model:

Definition 1.6 (I^+ model)**.** *An I^+ model is a structure $\mathcal{M} = \langle W, R, \nu \rangle$ such that:*

1. *$W \neq \emptyset$ is a set of entities ('worlds'),*
2. *R is a partial order on W, and*
3. *$\nu : \mathcal{P} \to \wp(W)$ is an interpretation function subject to the following monotonicity constraint:*

$$\forall w, v \in W (\textit{if } w \in \nu(p) \textit{ and } wRv, \textit{ then } u \in \nu(p)).$$

We again use $w, v, u, \ldots$ (possibly indexed) as meta-variables ranging over worlds.

The worlds in an intuitionistic model can be viewed as stages of construction of truth. When you look at any world w and you consider any world v such that wRv, you see that truth is preserved: whenever something is true in w, it must also be true in v; this is due to the monotonicity constraint.

We define truth:

Definition 1.7 (Truth clauses I^+)**.** *Given an I^+ model $\mathcal{M} = \langle W, R, \nu \rangle$, we define for all $w, v \in W$:*

$$\begin{array}{ll} w \vDash p & \Leftrightarrow w \in \nu(p), \\ w \vDash \phi \wedge \psi & \Leftrightarrow w \vDash \phi \textit{ and } w \vDash \psi, \\ w \vDash \phi \vee \psi & \Leftrightarrow w \vDash \phi \textit{ or } w \vDash \psi, \\ w \vDash \phi \to \psi & \Leftrightarrow \forall v (\textit{if } wRv \textit{ and } v \vDash \phi, \textit{ then } v \vDash \psi). \end{array}$$

Note that the first three clauses are the exact same clauses as the truth-clauses in *FDE*. The last clause gives us a condition for an implication. It comes down to the fact that a formula of the form $(\phi \to \psi)$ is true in a world w if and only if in all worlds reached by w (in all later stages of the construction) where ϕ is true, ψ also is true. It can be easily shown that the monotonicity constraint extends to all formulas.

Lemma 1.1. *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a positive intuitionistic model and let $w, v \in W$. Furthermore, let wRv. We then have $v \vDash \phi$ whenever $w \vDash \phi$.*

Proof. The proof is done by induction on the complexity of formulas. $v \vDash p$ whenever $w \vDash p$ follows directly from the definition of the monotonicity constraint. Our induction hypothesis is that $v \vDash \phi, \psi$ whenever $w \vDash \phi, \psi$. We go on to show that $v \vDash \phi \wedge \psi$ under the assumption that $w \vDash \phi \wedge \psi$. The assumption gives us by definition 1.7 that $w \vDash \phi$ as well as $w \vDash \psi$. By our induction hypothesis, we get $v \vDash \phi, \psi$, which using the same definition leads to $v \vDash \phi \wedge \psi$.

To show $v \vDash \phi \vee \psi$ whenever $w \vDash \phi \vee \psi$, assume the latter again. By definition 1.7, we get that either $w \vDash \phi$ or $w \vDash \psi$. The induction hypothesis gives us that $v \vDash \phi$ or $v \vDash \psi$ respectively. Both lead to $v \vDash \phi \vee \psi$ by definition 1.7.

To establish $v \vDash \phi \to \psi$ whenever $w \vDash \phi \to \psi$, again assume the latter. Consider any $u \in W$ such that vRu and $u \vDash \phi$. As R is a partial ordering, it is transitive and we have both wRv and vRu. Hence we get wRu. By definition of $\phi \to \psi$, we get that $u \vDash \psi$ as $u \vDash \phi$ and $w \vDash \phi \to \psi$. Given that u was arbitrary, we have that for all u such that uRv and $u \vDash \phi$ we also have $u \vDash \psi$. This means by definition 1.7 that $v \vDash \phi \to \psi$. $\square$

Validity is again defined as truth-preservation at all worlds in all models:

Definition 1.8 (I^+ consequence)**.** *$\phi \vDash_{I^+} \psi$ iff for all I^+ models $\mathcal{M} = \langle W, R, \nu^+ \rangle$, for all $w \in W$, if $w \vDash \phi$, then $w \vDash \psi$.*

Proof systems for full intuitionistic logic can also be used for positive intuitionistic logic when leaving out the rules for negation. Most notable are a contraction-free sequent calculus by Dyckhoff [8] and a tableaux system by Priest [26].

1.4 Wansing's C

C was first introduced in 2005 by Heinrich Wansing and the semantics is described as the "first known intuitively plausible interpretation of a system of connexive logic" [28]. It can be viewed as FDE, to which the intuitionistic arrow is added, with a special clause for its negation.

Definition 1.9 (Language C)**.** *We will work with a propositional language $\mathcal{L}_{\neg,\wedge,\vee,\to}$ which has the connectives $\neg, \wedge, \vee$ and $\to$. The propositional letters are $\mathcal{P}$.*

Definition 1.10 (C model)**.** *A model in* C *is a structure $\mathcal{M}_C = \langle W, R, \nu^+, \nu^- \rangle$, where:*

1. *$W \neq \emptyset$ is non-empty a set of entities ('worlds'),*
2. *R is a partial order on W,*
3. *$\nu^+ : \mathcal{P} \to \wp(W)$ is an interpretation function sending each atom to the set of worlds in which the atom is true, and*
4. *$\nu^- : \mathcal{P} \to \wp(W)$ is an interpretation function from each atom to sets of worlds in which the atom is false.*
5. *Both interpretation functions are, just like in I^+, subject to a monotonicity constraint:*

$$\forall w, v \in W (\text{if } w \in \nu^\circ(p) \text{ and } wRv, \text{ then } v \in \nu^\circ(p)) \qquad \text{for } \circ \in \{+, -\}.$$

Again we use $w, v, u, \ldots$ (possibly indexed) as meta-variables ranging over worlds.

The monotonicity constraint for falsehood is just as meaningful as the one for truth: once something is established to be false, it remains false. We state without proof that the constraint here extends to all formulas as well.

Truth and falsehood in a model in a world are again defined by a double recursion:

Definition 1.11 (Truth clauses C)**.** *Let $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$ be a C model. We define for all $w \in W$:*

$$\begin{array}{ll} w \vDash p & \Leftrightarrow w \in \nu^+(p), \\ w \Dashv p & \Leftrightarrow w \in \nu^-(p), \\ w \vDash \neg\phi & \Leftrightarrow w \Dashv \phi, \\ w \Dashv \neg\phi & \Leftrightarrow w \vDash \phi, \\ w \vDash \phi \wedge \psi & \Leftrightarrow w \vDash \phi \text{ and } w \vDash \psi, \\ w \Dashv \phi \wedge \psi & \Leftrightarrow w \Dashv \phi \text{ or } w \Dashv \psi, \\ w \vDash \phi \vee \psi & \Leftrightarrow w \vDash \phi \text{ or } w \vDash \psi, \\ w \Dashv \phi \vee \psi & \Leftrightarrow w \Dashv \phi \text{ and } w \Dashv \psi, \\ w \vDash \phi \to \psi & \Leftrightarrow \forall v (\text{if } wRv \text{ and } v \vDash \phi, \text{ then } v \vDash \psi), \\ w \Dashv \phi \to \psi & \Leftrightarrow \forall v (\text{if } wRv \text{ and } v \vDash \phi, \text{ then } v \Dashv \psi). \end{array}$$

All clauses but one should look familiar from FDE or I^+. The vital clause here is the one for the negation of the implication; it is derived directly from Boethius' laws.

Validity is defined the usual way.

Definition 1.12 (C consequence)**.** *$\phi \vDash_C \psi$ iff for all C models $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$, for all $w \in W$, if $w \vDash \phi$, then $w \vDash \psi$.*

We first prove that **Unsat-Aristotle** fails by showing $\vDash_C (p \wedge \neg p) \to \neg(p \wedge \neg p)$. This makes the logic not strongly connexive.

Proof. The proof uses only definition 1.11. Consider any world w in any C model where $w \vDash p \wedge \neg p$. This means that $w \vDash \neg p$, which leads to $w \Dashv p$. By definition 1.11 this means $w \vDash \neg(p \wedge \neg p)$. □

Unsat-Boethius fails as well as naturally $\vDash_C (p \wedge \neg p) \to (p \wedge \neg p)$ holds.

To see that the logic is weakly connexive, one needs to show that both of Aristotle's as well as Boethius' laws hold. We will show the latter. Aristotle's laws are left to the reader.

Proof. This proof also uses 1.11 a lot. We first prove $\vDash_C (\phi \to \psi) \to \neg(\phi \to \neg\psi)$. Let w be any world in any C model and consider any world v such that wRv and $v \vDash (\phi \to \psi)$. We get that for any u such that uRv and $u \vDash \phi$, $u \vDash \psi$. The latter gets us that $u \Dashv \neg\psi$, which means that any u such that vRu and $u \vDash \phi$ gets us that $u \Dashv \neg\psi$, which, as u was arbitrary, means that $v \Dashv \phi \to \neg\psi$. That just leads to $v \vDash \neg(\phi \to \neg\psi)$, which means, as v also was arbitrary, that any v such that wRv and $v \vDash \phi \to \psi$, will mean that $v \vDash \neg(\phi \to \neg\psi)$, which is just the definition of $w \vDash (\phi \to \psi) \to \neg(\phi \to \neg\psi)$. As w was arbitrary, we get $\vDash_C (\phi \to \psi) \to \neg(\phi \to \neg\psi)$, as desired.

Now for $\vDash_C (\phi \to \neg\psi) \to \neg(\phi \to \psi)$, consider any world w in any C model and let v be any world such that wRv and $v \vDash (\phi \to \neg\psi)$. Now consider any world u such that vRu and $u \vDash \phi$. We get $u \vDash \neg\phi$ which means $u \Dashv \phi$. Using similar reasoning as above, we then get $v \vDash \neg(\phi \to \psi)$ and thus $\vDash_C (\phi \to \neg\psi) \to \neg(\phi \to \psi)$. □

Known sound and complete proof systems for C exist in the form of a natural deduction system [29] and a cut-free sequent calculus by Kamide and Wansing [17].

With the logics defined as we are going to use them, we will continue in the next chapters by giving an outline of truthmaker semantics and providing truthmaker semantics for FDE, I^+ and C.

2 Truthmaker semantics

Truthmaker semantics (*TMS*) was first introduced by Kit Fine in [9]. Instead of working with (im)possible worlds, truthmaker semantics work with states. A state makes a propsition true whenever it is wholly relevant for its truth and necessitates that truth. Note that a state can represent anything (a fact, a state of affairs) as long as it has the necessary mereology. It might be intuitively more appealing to work with truthmakers than with impossible worlds, as combining contradictory facts happens in the real world, whereas worlds that contain contradictions should only be able to exist in one's mind.

In this chapter we will first introduce the basis needed for truthmaker semantics. We will see an example of a non-residuated state space and then the concept of exact truthmaking is introduced.

2.1 State spaces

Definition 2.1 (State Space)**.** *In the following, under a* state space *we understand a structure* $\mathcal{S} = \langle S, \sqsubseteq \rangle$*, where:*

1. $S \neq \emptyset$ *is a non-empty set of entities called* states.
2. $\sqsubseteq \, \subseteq S^2$ *is a binary partial ordering known as the parthood relation or simply* parthood.
3. *For each set* $X \subseteq S$ *there exists a unique least-upper-bound* $\bigsqcup X \in S$*, called the* fusion *of* X*.*

We use $s, t, u, \ldots$ (possibly indexed) as variables ranging over the states and $X, Y, Z, \ldots$ as variables ranging over sets of states. For a finite collection $\{s_1, \ldots, s_n\} \subseteq S$ of states, we write $s_1 \sqcup \ldots \sqcup s_n$ instead of $\bigsqcup\{s_1, \ldots, s_n\}$. For an indexed family of states $\{s_i : i \in I\}$, we also write $\bigsqcup_{i \in I} s_i$ instead of $\bigsqcup\{s_i : i \in I\}$.

It is routine to establish the following properties of fusions:

$$\bigsqcup\{s\} = s. \qquad \text{(Idempotence)}$$

$$\bigsqcup\{\bigsqcup X_i : i \in I\} = \bigsqcup \bigcup_{i \in I} X_i. \qquad \text{(Associativity)}$$

Similarly, it is good practice to establish the following characterization of $\sqsubseteq$ in terms of $\sqcup$:

$$s \sqsubseteq t \text{ iff } s \sqcup t = t. \qquad (\sqsubseteq\text{-Def})$$

These proofs are left to the interested reader.

Furthermore, we define the null state as the least upper bound of nothing and the full state as the least upper bound of every state in S:

Definition 2.2 (Null state)**.** *In a state space $\mathcal{S}$, we define the null state as:*

$$\square := \bigsqcup \emptyset.$$

Definition 2.3 (Full state)**.** *Let $\mathcal{S} = \langle S, \sqsubseteq \rangle$ be a state space. We define the full state as:*

$$\blacksquare := \bigsqcup S.$$

This leads to the following lemma:

Lemma 2.1. *Let $\mathcal{S} = \langle S, \sqsubseteq \rangle$ be a state space. We then have for all states $s \in S$:*

$$\square \sqsubseteq s,$$
$$s \sqsubseteq \blacksquare.$$

Proof. This follows immediately from associativity and the definition of parthood. $\square$

In a state space, we can define the dual operation of fusion, the *greatest common part* of $X \subseteq S$, as the unique least upper bound that is part of every state in X:

Definition 2.4 ($\bigsqcap$)**.** *Let $\mathcal{S} = \langle S, \sqsubseteq \rangle$ be a state space and let $X \subseteq S$ We define:*

$$\bigsqcap X := \bigsqcup \{t : \forall s \in X(t \sqsubseteq s)\}.$$

The notational conventions for $\bigsqcup$ carry over to $\bigsqcap$ in the obvious way.

We need one more definition to be able to construct a truthmaker semantics for positive intuitionistic logic. This definition is due to Fine[9].

Definition 2.5 (Conditional states)**.** *Let $\mathcal{S} = \langle S, \sqsubseteq \rangle$ be a state space and let $s, t \in S$. We define:*

$$s \to t := \bigsqcap \{u : s \sqcup u \sqsupseteq t\}.$$

Intuitively, the presence of the conditional state $s \to t$ can indicates the presence of t whenever s is added [9]. Note that although $s \to t$ is well-defined for all $s, t \in S$, the state may fail to satisfy the residuation condition:

$$s \sqcup (s \to t) \sqsupseteq t. \qquad \text{(Residuation condition)}$$

A counterexample is easily found: consider a state space $\langle S, \sqsubseteq \rangle$ with $S = \{\square, s, t, u, \blacksquare\}$. For an illustration of the state space, see figure 1. The figure is generally considered easier to understand than $\sqsubseteq = \{(\square, \square), (\square, s), (\square, t), (\square, u), (\square, \blacksquare), (s, s), (s, \blacksquare), (t, t), (t, \blacksquare), (u, u), (u, \blacksquare), (\blacksquare, \blacksquare)\}$. Moreover, let $s \sqcup t = \blacksquare$ and let $s \sqcup u = \blacksquare$. Then what is needed to get from s to $\blacksquare$ is either t, u, or

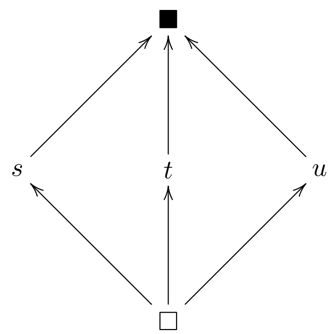

Figure 1: There are a few things to note here. First of all, the arrow shows the parthood relation. However, the reflexive and transitive arrows are not shown here. To find the result of the fusion of any number of states, you need to find the lowest state that all states you want to fuse point to.

$\blacksquare$. The greatest common part of all these states is exactly nothing or in other words: $s \rightarrow \blacksquare = \square$. But as the null state is part of s (because it is part of every state), the fusion of s and the null state is just s, which clearly does not contain $\blacksquare$ as a part.

Now that we know this, we can intuitively define a residuated state space:

Definition 2.6 (Residuated state space)**.** *Let $\mathcal{S} = \langle S, \sqsubseteq \rangle$ be a state space. $\mathcal{S}$ is called* residuated *iff the Residuation condition holds for all states $s, t \in S$.*

Lastly, we can define a model:

Definition 2.7 (Truthmaker model)**.** *A truthmaker model is a structure $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$, where:*

1. *$\langle S, \sqsubseteq \rangle$ is a state space,*
2. *$\nu^+ : \mathcal{P} \rightarrow \wp(S)$ is a so called 'truthmaker assignment',*
3. *$\nu^- : \mathcal{P} \rightarrow \wp(S)$ is a 'falsemaker assignment', and*
4. *both assignments are subject to the following condition:*

$$\textit{if } s, t \in \nu^\circ(p), \textit{ then } s \sqcup t \in \nu^\circ(p) \textit{ for } \circ = +, -. \qquad \text{(Closure)}$$

The interpretation functions ν^+ and ν^- simply send the propositions to the states that are wholly relevant for their truth and falsity respectively. When multiple states are wholly relevant to the truth of some atom, then the fusion of these states is too.

2.2 Inclusive semantics

The inclusive semantics is an exact truthmaker semantics. It is the most characteristic semantics for TMS and it is of course due to Kit Fine[9]. For a more detailed exposition, see [13] and for more background, see [11]. The semantics captures the idea that a state makes a formula true only if that state is wholly relevant for the truth of the formula [11].

Let us first fix the language:

Definition 2.8. *In the following, we'll work with a propositional language $\mathcal{L}_{\neg,\wedge,\vee}$ which only has the connectives $\neg, \wedge, \vee$. The propositional letters are $\mathcal{P}$.*

We use $p, q, r \ldots$ as variables ranging over the set of propositional letters and we use $\phi, \psi, \chi, \ldots$ to range over formulas in $\mathcal{L}_{\neg,\wedge,\vee}$.

Let us now define truthmaking:

Definition 2.9 (Truth clauses)**.** *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. We define truth and falsity for all $s \in S$ by means of the following double recursion:*

$$
\begin{array}{lcl}
s \Vdash p & \Leftrightarrow & s \in \nu^+(p), \\
s \dashv\mid p & \Leftrightarrow & s \in \nu^-(p), \\
s \Vdash \neg\phi & \Leftrightarrow & s \dashv\mid \phi, \\
s \dashv\mid \neg\phi & \Leftrightarrow & s \Vdash \phi, \\
s \Vdash \phi \wedge \psi & \Leftrightarrow & \exists t, u(s = t \sqcup u, t \Vdash \phi, \text{ and } u \Vdash \psi), \\
s \dashv\mid \phi \wedge \psi & \Leftrightarrow & s \dashv\mid \phi, s \dashv\mid \psi, \text{ or } s \dashv\mid \phi \vee \psi, \\
s \Vdash \phi \vee \psi & \Leftrightarrow & s \Vdash \phi \text{ or } s \Vdash \psi, \text{ or } s \Vdash \phi \wedge \psi, \\
s \dashv\mid \phi \vee \psi & \Leftrightarrow & \exists t, u(s = t \sqcup u, t \dashv\mid \phi, \text{ and } u \dashv\mid \psi).
\end{array}
$$

The idea of wholly relevance is preserved in all clauses and is most clear in the clause for conjunction. A state is a truthmaker for a conjunction only if it is composed of the states that are wholly relevant for the truth of both conjuncts, making the fusion wholly relevant for the truth of the conjunction whilst also necessitating it. This concept is exactly where this semantics differs from FDE. We now introduce some useful notation:

Definition 2.10 (The set of all truthmakers of a formula)**.** *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model, let $s \in S$ be a state and let ϕ be a formula. We define:*

$$[\phi]^+ = \{s : s \Vdash \phi\},$$

$$[\phi]^- = \{s : s \dashv\mid \phi\}.$$

It turns out that the closure condition generalizes to all formulas $\phi \in \mathcal{L}_{\neg,\wedge,\vee}$:

Lemma 2.2. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model and let $s, t \in S$ be states. Then if $s, t \in [\phi]^\circ$, then $s \sqcup t \in [\phi]^\circ$ for $\circ = +, -$.*

Proof. The proof is a straightforward induction on the complexity of formulas.

The base case is a direct consequence of the closure condition for atoms.

The induction hypothesis in the next step is that $s \sqcup t \in [\psi]^-$ whenever $s, t \in [\psi]^-$. To get $s \sqcup t \in [\phi]^+$ whenever $s, t \in [\phi]^+$ for $\phi = \neg\psi$, assume $s, t \in [\neg\phi]^+$. This means $s, t \in [\psi]^-$, which by the induction hypothesis leads to $s \sqcup t \in [\psi]^-$. We then get by definition 2.9 that $s \sqcup t \in [\neg\psi]^+$. The case for $s \sqcup t \in [\neg\psi]^-$ whenever $s, t \in [\neg\psi]^-$ is proved in a similar way using $s \sqcup t \in [\psi]^+$ whenever $s, t \in [\psi]^+$ as the induction hypothesis.

We are now going to prove that the condition holds for $\phi = \psi \wedge \chi$.
For the positive case, suppose states $s, t \in [\psi \wedge \chi]^+$. The induction hypothesis is that $s, t \in [\phi]^+$ implies that $s \sqcup t \in [\phi]^+$ for $\phi = \psi, \chi$. Now we know that $s = s_1 \sqcup s_2$ and $t = t_1 \sqcup t_2$ where $s_1, t_1 \in [\psi]^+$ and $s_2, t_2 \in [\chi]^+$. By induction hypothesis, we know that $u_1 = s_1 \sqcup t_1 \in [\phi]^+$ and $u_2 = s_2 \sqcup t_2 \in [\chi]^+$. Using associativity, we infer that $u = s \sqcup t = s_1 \sqcup s_2 \sqcup t_1 \sqcup t_2 = s_1 \sqcup t_1 \sqcup s_2 \sqcup t_2 = u_1 \sqcup u_2$. As $u_1 \in [\psi]^+$ and $u_2 \in [\chi]^+$, we now know that $u = u_1 \sqcup u_2 \in [\phi \wedge \chi]^+$.

Now for the negative case, suppose $s, t \in [\psi \wedge \chi]^-$. The induction hypothesis here is that $s, t \in [\phi]^-$ implies that $s \sqcup t \in [\phi]^-$ for $\phi = \psi, \chi$. Now there are nine different cases; I will show three, as all other cases are very similar to these ones:

1. $s, t \dashv\!| \, \psi$ and thus, $s, t \in [\psi]^-$. Then by induction hypothesis, $s \sqcup t \in [\psi]^-$ and hence $s \sqcup t \in [\psi \wedge \chi]^-$.

2. $s \dashv\!| \, \psi$, $t \dashv\!| \, \chi$. Then $u = s \sqcup t$ and as $s \dashv\!| \, \psi$ and $t \dashv\!| \, \chi$, follows that $u \dashv\!| \, \psi \vee \chi$. Hence $u \dashv\!| \, \psi \wedge \chi$, so $u \in [\psi \wedge \chi]^-$.

3. $s \dashv\!| \, \psi$, $t \dashv\!| \, \psi \vee \chi$. Now follows that $t = t_1 \sqcup t_2$ where $t_1 \dashv\!| \, \psi$ and $t_2 \dashv\!| \, \chi$. As $s \in [\psi]^-$ and $t_1 \in [\psi]^-$, follows by induction hypothesis that $v = s \sqcup t_1 \in [\psi]^-$. We get that $u = s \sqcup t = s \sqcup t_1 \sqcup t_2 = v \sqcup t_2$. As $v \dashv\!| \, \psi$ and $t_2 \dashv\!| \, \chi$, $u \dashv\!| \, \phi \vee \psi$. As we did before we get that $u \in [\psi \wedge \chi]^-$.

The case for $\psi \vee \chi$ is DeMorgan dual to $\psi \wedge \chi$ and is left to the interested reader. □

The logic of exact entailment (*ExE)* is based on the inclusive semantics. *ExE* is a purely inferential logic in the sense that the logic has no theorems or anti-theorems: there are no formulas that are made true by all states in all models and no formulas that are never made true by any state in any model. A sound and complete proof system has not been published yet.[1]

Next, now the basics of truthmaker semantics are clear, we are going to define truthmaker semantics for the logics FDE, I^+ and C.

[1] A sound and complete proof system for *ExE* is handed to me in personal communication by Johannes Korbmacher. For reasons of space it is omitted here.

3 Truthmaker Semantics and First Degree Entailment

In this section we are going to present a truthmaker semantics for First degree entailment. Firstly, we will define the notion of inexact truthmaking and inexact consequence. Afterwards, we are going to show how we can construct an FDE model from a truthmaker model and vice versa, in such a way that truth and falsity are both preserved. Lastly, we will prove van Fraassen's characterization theorem [15]. This result is in a different way than here also proved in [10] with the notion of Truthmaker semantics as it is known now.

A truthmaker semantics for First Degree Entailment is essentially due to Bas van Fraassen [15]. Although *TMS* did not exist as such, van Fraassen proposed an idea which is best described as *inexact truthmaking*. A state s makes a formula ϕ inexactly true if s has some state t as a part which makes ϕ exactly true:

Definition 3.1 (Inexact truthmaking)**.** *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model, let $s, t \in S$ be states and let $\phi \in \mathcal{L}_{\neg,\wedge,\vee}$ be a formula. We define:*

$$s \Vvdash \phi \text{ iff } \exists t(t \sqsubseteq s \text{ and } t \Vdash \phi). \qquad (\Vvdash\text{-Def})$$

$$s \dashVvert \phi \text{ iff } \exists t(t \sqsubseteq s \text{ and } t \dashv\vert \phi). \qquad (\dashVvert\text{-Def})$$

These definitions lead directly to the following lemma, which will be very useful for showing that the inexact semantics is indeed a semantics for FDE:

Lemma 3.1. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. Then for all $s \in S$:*

$$\begin{array}{rlcl} i. & s \Vvdash \neg\phi & \Leftrightarrow & s \dashVvert \phi, \\ ii. & s \dashVvert \neg\phi & \Leftrightarrow & s \Vvdash \phi, \\ iii. & s \Vvdash \phi \wedge \psi & \Leftrightarrow & s \Vvdash \phi \text{ and } s \Vvdash \psi, \\ iv. & s \dashVvert \phi \wedge \psi & \Leftrightarrow & s \dashVvert \phi \text{ or } s \dashVvert \psi, \\ v. & s \Vvdash \phi \vee \psi & \Leftrightarrow & s \Vvdash \phi \text{ or } s \Vvdash \psi, \\ vi. & s \dashVvert \phi \vee \psi & \Leftrightarrow & s \dashVvert \phi \text{ and } s \dashVvert \psi. \end{array}$$

Proof. The proof is straightforward and uses definitions 2.9 and 3.1.

i. $\Rightarrow$ Suppose $s \Vvdash \neg\phi$. Then by definition of $\Vvdash$, there exists a $t \sqsubseteq s$ such that $t \Vdash \neg\phi$. That means using definition 2.9 that $t \dashv\vert \phi$ and as t is a part of s, $s \dashVvert \phi$.

$\Leftarrow$ For the other direction we suppose that $s \dashVvert \phi$. We get that there is a $t \sqsubseteq s$ such that $t \dashv\vert \phi$, so we know that $t \Vdash \neg\phi$. As $t \sqsubseteq s$, we have $s \Vvdash \neg\phi$.

ii. $\Rightarrow$ Suppose $s \dashv\!\!|\!|\, \neg\phi$. We then know that $\exists t$ such that $t \sqsubseteq s$ and $t \dashv\!|\, \neg\phi$. This just means $t \Vdash \neg\phi$ from which we derive that $t \dashv\!|\, \phi$, which leads to $s \Vvdash \phi$.

$\Leftarrow$ For the other direction assume $s \Vvdash \phi$. We get that $\exists t$ such that $t \sqsubseteq s$ and $t \Vdash \phi$. This means that $t \dashv\!|\, \neg\phi$, which gives us that $s \Vvdash \neg\phi$.

iii. $\Rightarrow$ Suppose $s \Vvdash \phi \wedge \psi$. We get that $\exists t$ such that $t \sqsubseteq s$ and $t \Vdash \phi \wedge \psi$. That gives us that $t = t_1 \sqcup t_2$ where $t_1 \Vdash \phi$ and $t_2 \Vdash \psi$. As parthood is transitive, we get that $t_1 \sqsubseteq s$ and hence $s \Vvdash \phi$ and $t_2 \sqsubseteq s$, so $s \Vvdash \psi$.

$\Leftarrow$ For the other direction, assume $s \Vvdash \phi$ and $s \Vvdash \psi$. The former means that there is a $s_1 \sqsubseteq s$ such that $s \Vdash \phi$. The latter means that there is a $s_2 \sqsubseteq s$ such that $s \Vdash \psi$. Due to (Completeness), we now get that $s_1 \sqcup s_2 \sqsubseteq s$, and as $s_1 \sqcup s_2 \Vdash \phi \wedge \psi$, we get that $s \Vvdash \phi \wedge \psi$.

iv. $\Rightarrow$ Suppose $s \dashv\!\!|\!|\, \phi \wedge \psi$. We get that there is a $s_1 \sqsubseteq s$ such that $s_1 \dashv\!|\, \phi \wedge \psi$. We now have three cases:

(a) $s_1 \dashv\!|\, \phi$. As $s_1 \sqsubseteq s$, we get that $s \dashv\!\!|\!|\, \phi$.

(b) $s_1 \dashv\!|\, \psi$. As $s_1 \sqsubseteq s$, we get that $s \dashv\!\!|\!|\, \psi$.

(c) $s_1 \dashv\!|\, \phi \wedge \psi$. This means that there is a t, u such that $s_1 = t \sqcup u$ and $t \dashv\!|\, \phi$. Due to transitivity and $t \sqsubseteq s_1$ and $s_1 \sqsubseteq s$, we get that $t \sqsubseteq s$, which means that $s \dashv\!\!|\!|\, \phi$.

In all cases, $s \dashv\!\!|\!|\, \phi$ or $s \dashv\!\!|\!|\, \psi$, as desired.

$\Leftarrow$ Assume $s \dashv\!\!|\!|\, \phi$ or $s \dashv\!\!|\!|\, \psi$. Without loss of generality, we only work out the former here. $s \Vvdash \phi$ means by definition that there is a $t \sqsubseteq s$ such that $t \dashv\!|\, \phi$. Hence, $t \dashv\!|\, \phi$ or $t \dashv\!|\, \psi$ and thus, according to definition 2.9, we get that $t \dashv\!|\, \phi \wedge \psi$. As $t \sqsubseteq s$, we get $s \dashv\!\!|\!|\, \phi \wedge \psi$.

v. and *vi.* are the de Morgan duals of *iii* and *iv* and are left for the interested reader.

□

With inexact truthmaking we lose the wholly relevance of the state for the truth of the propositions. However, the states still necessitate the truth. The truth-clauses are already very similar to the ones of FDE.

Consequence is defined in a natural way:

Definition 3.2 (Inexact Consequence)**.** *$\phi \Vvdash \psi$ iff for all models $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$, for all $s \in S$, if $s \Vvdash \phi$, then $s \Vvdash \psi$.*

We are now going to show that every truthmaker model has an equivalent first degree entailment model.

Lemma 3.2. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. Now let $\mathcal{M}' = \langle W, \nu^{+'}, \nu^{-'} \rangle$ be an FDE model such that:*

1. $W = S$,
2. $s \in \nu^{+'}(p)$ *iff there exists a* $t \in S$ *such that* $t \sqsubseteq s$ *and* $t \in \nu^{+}(p)$,
3. $s \in \nu^{-'}(p)$ *iff there exists a* $t \in S$ *such that* $t \sqsubseteq s$ *and* $t \in \nu^{-}(p)$.

We then have that for all $s \in S$, $s \Vvdash \phi$ *in* $\mathcal{M}$ *iff* $s \vDash_{FDE} \phi$ *in* $\mathcal{M}'$.

Proof. The proof is done by induction on the complexity of formulas. We will make use of the definition 3.1, as well as lemma 3.1 and the truth and falsity clauses for FDE as defined in definition 1.3 on page 4. For the base cases, we are going to show that $s \Vvdash p$ iff $s \vDash p$ for $p \in \mathcal{P}$. The negative case is left to the reader as it is very similar to the positive one.

- $\Rightarrow$ Suppose $s \Vvdash p$. We get that there is a state $t \sqsubseteq s$ such that $t \Vdash p$. This means that $t \in \nu^{+}(p)$ and, as $t \sqsubseteq s$, we get $s \in \nu^{+'}(p)$, which means $s \vDash p$.
- $\Leftarrow$ Suppose $s \vDash p$. We get that $s \in \nu^{+'}(p)$ and hence that there is a $t \sqsubseteq s$ such that $t \in \nu^{+}(p)$. This means that $t \Vdash p$ and hence, as $t \sqsubseteq s$, we get that $s \Vvdash p$.

Our induction hypotheses are that $s \Vvdash \phi, \psi$ iff $s \vDash \phi, \psi$ (IH1) and $s \dashv\!\Vert\, \phi, \psi$ iff $s \Dashv \phi, \psi$ (IH2). To show $s \Vvdash \neg\phi$ iff $s \Vdash \neg\phi$, we use the latter hypothesis. The negative case is omitted again, as it is done in a similar fashion as the positive one.

- $\Rightarrow$ Suppose $s \Vvdash \neg\phi$. By lemma 3.1 this means that $s \dashv\!\Vert\, \phi$, which using (IH2) leads to $s \Dashv \phi$, which is just equivalent to $s \vDash \neg\phi$ by definition 1.3.
- $\Leftarrow$ Suppose $s \vDash \neg\phi$. That means that $s \Dashv \phi$. Using the same induction hypothesis again, we get $s \dashv\!\Vert\, \phi$, which in its turn, using lemma 3.1 leads to $s \Vvdash \neg\phi$.

To show $s \Vvdash \phi \wedge \psi$ iff $s \vDash \phi \wedge \psi$, we use (IH1):

- $\Rightarrow$ Assume $s \Vvdash \phi \wedge \psi$. Through lemma 3.1 we get $s \Vvdash \phi$ and $s \Vvdash \psi$. This gets us that $s \vDash \phi$ and $s \vDash \psi$ respectively by use (IH1). But that just means $s \vDash \phi \wedge \psi$ by definiton 1.3.
- $\Leftarrow$ Assume $s \vDash \phi \wedge \psi$. We get that $s \vDash \phi$ and $s \vDash \psi$ by definition 1.3. We derive $s \Vvdash \phi$ and $s \Vvdash \psi$ using our induction hypothesis, leading to $s \Vvdash \phi \wedge \psi$ by lemma 3.1.

For $s \dashv\!\Vert\, \phi \wedge \psi$ iff $s \Dashv \phi \wedge \psi$, we make use of (IH2) again.

- $\Rightarrow$ Let $s \dashv\!\Vert\, \phi \wedge \psi$. We now have $s \dashv\!\Vert\, \phi$ or $s \dashv\!\Vert\, \psi$ according to lemma 3.1. In the first case, by (IH2) we get $s \Dashv \phi$. In the second case through similar reasoning we get $s \Dashv \psi$. In both cases, by definition 1.3 we then get $s \Dashv \phi \wedge \psi$.
- $\Leftarrow$ Assume $s \Dashv \phi \wedge \psi$. By definition 1.3, we get $s \Dashv \phi$ or $s \Dashv \psi$. In the former case, we get $s \dashv\!\Vert\, \phi$ by (IH2); in the latter we get $\dashv\!\Vert\, \psi$. Both lead to $s \dashv\!\Vert\, \phi \wedge \psi$ using lemma 3.1.

The cases for the disjunction are DeMorgan duals of the conjunction and will be left to the interested reader. □

Now we are going to prove that for every FDE model, there is also a corresponding TMS model in which there is an equivalent state for each world in the FDE model.

Lemma 3.3. *Let $\mathcal{M} = \langle W, \nu^+, \nu^- \rangle$ be an FDE model. Now let $\mathcal{M}' = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ be a TMS model such that:*

1. *$S = \wp(W)$.*
2. *$\sqsubseteq = \subseteq$. (It is easy to see that $\wp(W)$ is closed under fusion, which in our case is just union.)*
3. *$\nu^{+'} : \mathcal{P} \to S$ is defined by $\{w\} \in \nu^{+'}(p)$ iff $w \in \nu^+(p)$.*
4. *$\nu^{-'} : \mathcal{P} \to S$ is defined by $\{w\} \in \nu^{-'}(p)$ iff $w \in \nu^-(p)$.*

We then have that for all $w \in W$, $w \vDash \phi$ iff $\{w\} \Vdash \phi$.

Proof. This is shown by induction on the complexity of formulas. The base cases are a direct consequence of the definitions of $\nu^{+'}$ and $\nu^{-'}$, lemma 3.1 and definition 1.3 . Now our induction hypotheses will be that $w \Dashv \phi$ iff $\{w\} \dashv\!\!|| \phi$ (IH1) and $w \vDash \phi, \psi$ iff $\{w\} \Vdash \phi, \psi$ (IH2). We are now going to show that $w \vDash \neg\phi$ iff $\{w\} \Vdash \neg\phi$.

- ⇒ Suppose $w \vDash \neg\phi$. This means that $w \Dashv \phi$ according to definition 1.3 and by (IH1) then follows that $\{w\} \dashv\!\!|| \phi$. Using lemma 3.1 we get that $\{w\} \vDash \neg\phi$.
- ⇐ Suppose $\{w\} \Vdash \neg\phi$. We get that $\{w\} \dashv\!\!|| \phi$ according to lemma 3.1. This gets us that $w \Dashv \phi$ by (IH1) and using definition 1.3, we get that $w \Dashv \neg\phi$, as desired.

For $w \Dashv \neg\phi$ iff $\{w\} \dashv\!\!|| \neg\phi$:

- ⇒ Assume $w \Dashv \neg\phi$. We get $w \vDash \phi$ by definition 1.3 and by (IH2) we derive $\{w\} \Vdash \phi$, which is equivalent to $\{w\} \dashv\!\!|| \neg\phi$ according to lemma 3.1.
- ⇐ Assume $\{w\} \dashv\!\!|| \neg\phi$. We get $\{w\} \Vdash \phi$ by lemma 3.1 and by (IH2) we get $w \vDash \phi$. By definition 1.3 this gets us $w \Dashv \neg\phi$.

For $w \vDash \phi \wedge \psi$ iff $\{w\} \Vdash \phi \wedge \psi$:

- ⇒ Assume $w \vDash \phi \wedge \psi$. By definition 1.3 we get that $w \vDash \phi$ and that $w \vDash \psi$. Using (IH2), this leads to $\{w\} \Vdash \phi$ and $\{w\} \Vdash \psi$ respectively. Lemma 3.1 then gets us that $\{w\} \Vdash \phi \wedge \psi$, as desired.
- ⇐ Suppose $\{w\} \Vdash \phi \wedge \psi$. We get that $\{w\} \Vdash \phi$ and $\{w\} \Vdash \psi$ according to lemma 3.1. This gets us using (IH2) that $w \vDash \phi$ and $w \vDash \psi$. By definition 1.3, we get $w \vDash \phi \wedge \psi$.

For $w \Dashv \phi \wedge \psi$ iff $\{w\} \dashv\!\!|| \phi \wedge \psi$:

⇒ Assume $w \Dashv \phi \wedge \psi$. By definition 1.3, we get a case distinction: either $w \Dashv \phi$ or $w \Dashv \psi$. By (IH1) we get that $\{w\} \mathrel{\dashv\!\!\|} \phi$ in the former case and $\{w\} \mathrel{\dashv\!\!\|} \psi$ in the latter. Both get us that $\{w\} \mathrel{\dashv\!\!\|} \phi \wedge \psi$ using lemma 3.1.

⇐ Suppose $\{w\} \mathrel{\dashv\!\!\|} \phi \wedge \psi$. By lemma 3.1, we get that either $\{w\} \mathrel{\dashv\!\!\|} \phi$ or $\{w\} \mathrel{\dashv\!\!\|} \psi$. By (IH1) we get $w \dashv\!| \phi$ or $w \dashv\!| \psi$ respectively. By definition 1.3 we get $w \dashv\!| \phi \wedge \psi$ in both cases.

The cases for the disjunction are again DeMorgan duals of the conjunction and are left to the interested reader. □

Now this is all out of the way, we can prove the main theorem of this section.

Theorem 3.4. *$\phi \vDash_{FDE} \psi$ iff $\phi \Vvdash \psi$.*

Proof. Both directions proceed via contraposition and the subscript for FDE consequence is omitted again:

⇒ Assume $\phi \not\Vvdash \psi$. This means by definition 3.2 that there is a truthmaker model $\mathcal{M} = \langle S, \sqsubseteq, \nu^{+}, \nu^{-} \rangle$ such that $s \Vvdash \phi$ but $s \not\Vvdash \psi$ for some $s \in S$. Now we construct an FDE model $\mathcal{M}' = \langle W, \nu^{+'}, \nu^{-'} \rangle$ out of $\mathcal{M}$ as done in lemma 3.2. We get $s \vDash \phi$ but $s \nvDash \psi$ for $s \in W$, which by definition 1.4 gets us $\phi \nvDash_{FDE} \psi$, as desired.

⇐ Assume $\phi \nvDash_{FDE} \psi$. We get by definition 1.4 that there is an FDE model $\mathcal{M} = \langle W, \nu^{+}, \nu^{-} \rangle$ such that $w \vDash \phi$ but $w \nvDash \psi$ for some $w \in W$. We now construct a truthmaker model $\mathcal{M}' = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ as done in lemma 3.3. We get that $\{w\} \Vvdash \phi$ but $\{w\} \not\Vvdash \psi$, which by definition 3.2 means that $\phi \not\Vvdash \psi$, as desired.

□

Hence we have proven that inexact truthmaking gives us a truthmaker semantics for FDE. We will build on this result to get the semantics for I^{+} in the next section.

4 Truthmaker Semantics and Positive Intuisionistic Logic

A truthmaker semantics for full intuitionistic logic is given by Fine in [9]. Our result is obtained in an approach that is similar to but nonetheless slightly different from the one Fine used. Firstly we define the truthmaker semantics for positive intuitionistic logic. Then we show how to turn every positive truthmaker model into a positive intuitionistic model and afterwards we establish that we can make a positive truthmaker model for every I^+ one such that we preserve truth. We do this all in order to show that the truthmaker semantics that we introduce here is indeed a semantics for positive intuitionistic logic, which is proved at the very end of this section.

The language we use here is the same as the one we used before for I^+ (definition 1.5 on page 5). We remind the reader that the language, the models, the truth clauses and consequence can be found in section 1.3.

We now define the truthmaker equivalent for I^+ models:

Definition 4.1 (Positive Truthmaker Model)**.** *A positive truthmaker model is a structure* $\mathcal{M} = \langle S, \sqsubseteq, \nu \rangle$ *such that:*

1. $\langle S, \sqsubseteq \rangle$ *is a residuated state space,*
2. $\nu : \mathcal{P} \to \wp(W)$ *is an interpretation function subject to the following constraint:*
$$\textit{if } s, t \in \nu(p), \textit{ then } s \sqcup t \in \nu(p). \qquad \text{(Closure)}$$

Furthermore, we will make use of conditional states (definition 2.5) and the set of all truthmakers of a formula (definition 2.10).

The following property can now easily be established:

Lemma 4.1. *Let* s, t *be states. If* $s \sqsupseteq t$*, then* $s \to t = \Box$*.*

Proof. Suppose $s \sqsupseteq t$. Note that $s \sqcup \Box \sqsupseteq s \sqsupseteq t$. Hence, $\Box \in \{u : s \sqcup u \sqsupseteq t\}$. By defenition of $\bigsqcap$, we then get that $\bigsqcap\{u : s \sqcup u \sqsupseteq t\} \sqsubseteq \Box$ and as the null state has only itself as a part, we get that $\bigsqcap\{u : s \sqcup u \sqsupseteq t\} = \Box$. $\Box$

Let us define truth in a positive truthmaker model. We use the same definition of the arrow as Fine [9]. This way a state s verifies a claim $\phi \to \psi$ just in case it can tell us how to pass from any verifier of the antecedent to some verifier of the consequent. Because the state is wholly composed of states that are relevant to the truth of the conditional, this definition is plausibly exact.

Definition 4.2 (Truthmaker clauses)**.** *Let* $\mathcal{M} = \langle S, \sqsubseteq, \nu \rangle$ *be a positive truthmaker model. We then define by recursion for all* $s \in S$*.*

$$
\begin{array}{ll}
s \Vdash p & \Leftrightarrow s \in \nu(p), \\
s \Vdash \phi \wedge \psi & \Leftrightarrow \textit{there exist } s_1, s_2 \in S, \textit{ s.t. } s = s_1 \sqcup s_2 \textit{ such that } s_1 \Vdash \phi \textit{ and } s_2 \Vdash \psi, \\
s \Vdash \phi \vee \psi & \Leftrightarrow s \Vdash \phi \textit{ or } s \Vdash \psi \textit{ or } s \Vdash \phi \wedge \psi, \\
s \Vdash \phi \rightarrow \psi & \Leftrightarrow \textit{There is a function } f : [\phi]^+ \rightarrow [\psi]^+, \textit{ such that } s = \bigsqcup_{t \in [\phi]^+} t \rightarrow f(t).
\end{array}
$$

The truthmaker of the implication is hence an insurance that whenever there is a truthmaker for the antecedent present, we also have a corresponding truthmaker for the conclusion.

For the next part, recall the definitions of inexact truthmaking 3.1 and inexact consequence 3.2 on pages 15 and 16.

We will now show that formulas in the positive truthmaker model behave in the same way as in an I^+ model, which will help us to show that we are indeed defining a truthmaker semantics for positive intuisionistic logic.

Lemma 4.2. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu \rangle$ be a positive truthmaker model. Then for all $s \in S$:*

i. $s \Vvdash \phi \wedge \psi$ *iff* $s \Vvdash \phi$ *and* $s \Vvdash \psi$,

ii. $s \Vvdash \phi \vee \psi$ *iff* $s \Vvdash \phi$ *or* $s \Vvdash \psi$,

iii. $s \Vvdash \phi \rightarrow \psi$ *iff* $\forall t \in S$(*if* $s \sqsubseteq t$ *and* $t \Vvdash \phi$, *then* $t \Vvdash \psi$).

Proof. For cases *i.* and *ii.*, see lemma 3.1 on page 15. We now show case *iii.*

$\Rightarrow$ Suppose that $s \Vvdash \phi \rightarrow \psi$, $s \sqsubseteq t$ and $t \Vvdash \phi$. Because $s \Vvdash \phi \rightarrow \psi$, we know by definition 3.1 that there exists a state s' such that $s' \sqsubseteq s$ and $s' \Vdash \phi \rightarrow \psi$. By the same definition we know that there must be some state $t' \sqsubseteq t$ such that $t' \Vdash \phi$. As $s' \Vdash \phi \rightarrow \psi$, by definition 4.2 it must have the residual $t' \rightarrow u$ as a part, for some u such that $u \Vdash \psi$. Because the state space is residuated, we know that the residuation condition holds and thus that $(t' \rightarrow u) \sqcup t' \sqsupseteq u$. Hence as $u \Vdash \psi$, we get that $((t' \rightarrow u) \sqcup t') \Vvdash \psi$ (definition 3.1). As $t' \rightarrow u \sqsubseteq s' \sqsubseteq s \sqsubseteq t$ and $t' \sqsubseteq t$, we have, using associativity, that $(t' \rightarrow u \sqcup t') \sqsubseteq t$ and so that $u \sqsubseteq t$. Now because $u \Vdash \psi$, we obtain $t \Vvdash \psi$ by definition 3.1 again, as desired.

$\Leftarrow$ For the other direction, we assume that for all $t \sqsupseteq s$, if $t \Vvdash \phi$, then $t \Vvdash \psi$. We are going to show that there exists some $s' \sqsubseteq s$ such that $s' \Vdash \phi \rightarrow \psi$. Now we know by definition 4.2 that $s' \Vdash \phi \rightarrow \psi$ iff $s' = \bigsqcup\{u \rightarrow f(u) : u \Vdash \phi\}$ where $f : [\phi]^+ \rightarrow [\psi]^+$ is some function from truthmakers of ϕ to truthmakers of ψ. Now take any $u \Vdash \phi$ and consider $u \sqcup s$. Because $u \sqcup s \sqsupseteq s$ by definition of $\sqsubseteq$ and $u \sqcup s \Vvdash \phi$ by definition 3.1, we get $s \sqcup u \Vvdash \psi$ by our assumption. Hence by definition 3.1, there exists a $v \sqsubseteq s \sqcup u$, such that $v \Vdash \psi$. As this goes for every $u \Vdash \phi$, we can define our function f as $f(u) = v$, where $v \sqsubseteq s \sqcup u$ and $v \Vdash \psi$.
We are now going to show that $\bigsqcup_{u \in [\phi]^+}\{u \rightarrow f(u)\} \sqsubseteq s$. Take an arbitrary $u \in [\phi]^+$. We make a case distinction: either $u \sqsupseteq s$ or not:

(1) If $u \sqsupseteq s$, then $u \sqcup s = u$. We then get that $f(u) \sqsubseteq u$ and hence, by lemma 4.1 $u \to f(u) = \Box$. As $\Box \sqsubseteq s$, we get that $u \to f(u) \sqsubseteq s$.

(2) If not $u \sqsupseteq s$, then note that $u \to f(u)$ is defined as $\bigsqcap\{x : x \sqcup u \sqsupseteq f(u)\}$. Now recall that $f(u) = v \sqsubseteq s \sqcup u$. Hence we get that $s \in \{x : x \sqcup u \sqsupseteq f(u)\}$. Now by definition of $\bigsqcap$, we get that $\bigsqcap\{x : x \sqcup u \sqsupseteq f(u)\} \sqsubseteq s$, so we get that $u \to f(u) \sqsubseteq s$.

As u was arbitrary, we now know that for all $u \in [\phi]^+$, $u \to f(u) \sqsubseteq s$ and hence, as s is closed under fusion, $\bigsqcup_{u \in [\phi]^+}\{u \to f(u)\} \sqsubseteq s$. By definition 4.2, $\bigsqcup_{u \in [\phi]^+}\{u \to f(u)\} \Vdash \phi \to \psi$, so we get that $s \Vvdash \phi \to \psi$ by the definition of inexact truthmaking.

$\square$

We can now straightforwardly construct an I^+ model out of a positive truthmaker model in such a way that truth is preserved.

Lemma 4.3. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu\rangle$ positive truthmaker model. Then $\mathcal{M}$ is an I^+ model and viewed as such, we get for all $s \in S$: $s \Vvdash \phi$ iff $s \vDash \phi$.*

Proof. Let $\mathcal{M} = \langle S, \sqsubseteq, \nu\rangle$ be a truthmaker model and let $\mathcal{M}' = \langle W_S, R, \nu'\rangle$ be its associated I^+ model, where $W_S = S$. $R = \{(s,t) : s \sqsubseteq t\}$ and $\nu' : \mathcal{P} \to W_S$ is such that $s \in \nu'(p)$ if and only if $s \in \nu(p)$. The proof proceeds by induction over the complexity of formulas.
For the base case, we need to show that $s \Vvdash p$ iff $s \vDash p$ for $p \in \mathcal{P}$:

$\Rightarrow$ To show that $s \vDash p$ whenever $s \Vvdash p$, assume there is some state s such that $s \Vvdash p$. We then get that there exists some $s' \sqsubseteq s$ such that $s' \Vdash p$. So $s' \in \nu(p)$ and hence $s' \in \nu'(p)$. By definition of R, we get that $s'Rs$ and then due to the monotonicity constraint we get that $s \in \nu'(p)$, and hence $s \vDash p$.

$\Leftarrow$ For the other direction, suppose $s \vDash p$. We then get that $s \in \nu'(p)$ and therefore $s \in \nu(p)$. So $s \Vdash p$ and because $s \sqsubseteq s$, we get that $s \Vvdash p$.

Our induction hypothesis is that $s \Vvdash \phi, \psi$ iff $s \vDash \phi, \psi$.

$w \Vvdash \phi \wedge \psi$ iff $s \vDash \phi \wedge \psi$ is proved in the exact same way as in lemma 3.2.

For the $\phi \vee \psi$ case:

$\Rightarrow$ Assume $s \Vvdash \phi \vee \psi$. Due to our lemma 4.2, we can distinguish two cases: $s \Vvdash \phi$ and $s \Vvdash \psi$. In the former case we get due to our induction hypothesis that $s \vDash \phi$ and in the latter we get that $s \vDash \psi$. Both directly lead to $s \vDash \phi \vee \psi$, as desired.

$\Leftarrow$ Assume $s \vDash \phi \vee \psi$. We distinguish two cases: $s \vDash \phi$ and $s \vDash \psi$. In the former, due to our induction hypothesis, we get $s \Vvdash \phi$. The latter results in $s \Vvdash \psi$. By lemma 4.2, we get that in both cases, $s \Vvdash \phi \vee \psi$.

For the $\phi \to \psi$ case:

- $\Rightarrow$ Assume $s \Vdash \phi \to \psi$. Suppose for contradiction that not $s \vDash \phi \to \psi$. We then get that there is some $t \in W_s$ such that sRt and $t \vDash \phi$, but not $t \vDash \psi$. Since sRt, we get that $s \sqsubseteq t$ and hence, because $s \Vdash s \to t$ we get by lemma 4.2 that if $t \Vdash \phi$, that then also $t \Vdash \psi$. By our induction hypothesis, we get that $t \Vdash \phi$, as $t \vDash \phi$, and hence $t \Vdash \psi$. But by means of the induction hypothesis, this means that $t \vDash \psi$, which is a contradiction. Therefore we must have $s \vDash \phi \to \psi$.
- $\Leftarrow$ Assume $s \vDash \phi \to \psi$. Assume for contradiction that $s \Vdash \phi \to \psi$ is not the case. Then, using lemma 4.2, there must be some $t \sqsupseteq s$ such that $t \Vdash \phi$, but not $t \Vdash \psi$. Consider t. We know that sRt as $s \sqsubseteq t$. We also know that $t \vDash \phi$ due to our induction hypothesis. Because of these things and the fact that $s \vDash \phi \to \psi$, we get that $t \vDash \psi$. According to our induction hypothesis, we then get that $t \Vdash \psi$, which is a contradiction and hence $s \Vdash \phi \to \psi$ must hold.

□

To construct a truthmaker model out of a positive intuitionistic model is a bit more challenging. Firstly, we need to make sure that our obtained state space is closed under arbitrary fusions. In order to do this, we make use of finite tree models.

Definition 4.3 (Finite tree model). *A finite tree model $\mathcal{M} = \langle W, R, \nu \rangle$ is an I^+ model such that:*

1. *W is finite.*
2. *there is a unique distinguishable world $r \in W$ such that $\forall w \in W(\neg wR'r)$, where R' is the reflexive transitive reduct of R. This world r is called the root.*
3. *$\forall w \in W$, there is a unique path $(rR'...R'w)$ from root r to w, where R' is the reflexive transitive reduct of R.*

In short, the reflexive transitive reduct R' of the partial ordering R is the set of edges of a tree with W as the nodes. We will see that for every I^+ countermodel, there exists an equivalent finite tree one.

Lemma 4.4 (Finite model property). *If there exists an I^+ model $\mathcal{M} = \langle W, R, \nu \rangle$ such that for some world $w \vDash \phi$ and $w \nvDash \psi$, then there exists a model $\mathcal{M}' = \langle W', R', \nu' \rangle$ such that $\mathcal{M}'$ is a finite tree model with root $r \in W'$ and $r \vDash \phi$ and $r \nvDash \psi$.*

Proof. It is a well known fact that I^+ is complete with respect to finite rooted tree frames. For an example of the proof, see [6] (Theorem 5.12). From this follows directly that if $\mathcal{M}, \phi \vDash \psi$ for some I^+ model $\mathcal{M}$, then there exists a finite tree model $\mathcal{M}' = \langle W, R, \nu \rangle$ such that $\mathcal{M}', \phi \vDash \psi$. This means that for some world $w \in W$, we have that $w \vDash \phi$, but not that $w \vDash \psi$. Now note that truth of a formula in a world w is independent with respect to worlds that stand in relation to w and that trees are closed under subtrees. This means that we can take the subtree with root w and that subtree would still be a valid countermodel. □

The just established property gives us just the tools we need to show that for every I^+ countermodel, there is an equivalent finite tree one. We make use of such countermodels to show by contraposition that anything we can infer using positive truthmaker models, we can also get in a positive intuitionistic logic. However, we do not have arbitrary fusions just yet. To get those, we introduce the concept of downward closed sets:

Definition 4.4 (Downward closed set)**.** *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a finite tree model. A set $X \subseteq W$ is downward closed iff for all $w, v \in W$, if $w \in X$ and vRw, then $v \in X$. We denote the set of all downward closed sets of W by $\downarrow W$.*

Note that empty set is downward closed as well. A special kind of a downward closed set is a principal downward closed set.

Definition 4.5 (Principal downward closed set)**.** *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a truthmaker model and let $X \subseteq \downarrow W$ be a downward closed set. We have that X is principal iff $X = \emptyset$, or X is of the form $\{v : vRw\}$ for $w, v \in W$.*

For nonempty principal downward closed sets of the form $\{v : vRw\}$ we write $\overline{w}$.

Principal downward closed sets have the following useful property:

Lemma 4.5. *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a finite tree model and let $\overline{w}$ be any principal downward closed set. We get that every nonempty downward closed $\overline{v} \subseteq \overline{w}$ is also principal.*

Proof. We use proof by contraposition. Assume $\overline{v}$ is non-principal. This means that there is a $w, v \in \overline{v}$ such that neither wRv nor vRw. Because $\overline{v} \subseteq \overline{w}$, we have that $w, v \in \overline{w}$ as well and hence, $\overline{w}$ is also non-principal. □

Now we have our state space:

Lemma 4.6. *Let $\mathcal{M}$ be a finite tree model. Then $\langle \downarrow W, \subseteq \rangle$ is a residuated state space.*

Proof. Firstly, note that $\subseteq$ is a partial ordering on $\downarrow W$. We need to show two things:

1. $\downarrow W$ is closed under fusions. In our case, fusion is just union. So we are going to show that the union of two downward closed sets is also a downward closed set.

Proof. Consider any $w \in A \cup B$ for nonempty downward closed sets A, B. (If A and B are both empty, then their union is also empty and therefore vacuously downward closed. If either A or B is empty, their union is just the nonempty set and is therefore by assumption downward closed.) By definition of $\cup$, we know that either $w \in A$ or $w \in B$. Without loss of generality, assume $w \in A$. Now consider any $v \in W$ such that vRw. If vRw, then by our assumption that A is downward closed, $v \in A$ and hence $v \in A \cup B$. As $v \in W$ and $w \in A \cup B$ was arbitrary, we have show that for all $w \in A \cup B$, if vRw for some $v \in W$, we have $v \in A \cup B$ and hence, $A \cup B$ is a downward closed set. ☐

2. The state space is residuated. For the residuation property, we need to prove: $s \cup \bigcap\{u : s \cup u \supseteq t\} \supseteq t$.

   Proof. Take an arbitrary $x \in t$. We need to show that $x \in s \cup \bigcap\{u : s \cup u \supseteq t\}$. We distinguish two cases:

   (a) $x \in s$. Then $x \in s \cup X$ for any X, in particular for $X = \bigcap\{u : s \cup u \supseteq t\}$.

   (b) $x \notin s$. It is sufficient to show that $x \in \bigcap\{u : s \cup u \supseteq t\}$. Take any u such that $s \cup u \supseteq t$. As $x \in t$, we need to have $x \in s \cup u$. As $x \notin s$, we get $x \in u$. As u was arbitrary such that $s \cup u \supseteq t$, we get $x \in u$ for all u such that $s \cup u \supseteq t$. Therefore, $x \in \bigcap\{u : s \cup u \supseteq t\}$.

   Since in both cases $x \in s \cup \bigcap\{u : s \cup u \subseteq t\}$, and since $x \in t$ was arbitrary, we get that for all x, if $x \in t$, then $x \in s \cup \bigcap\{u : s \cup u \subseteq t\}$. Therefore $s \cup \bigcap\{u : s \cup u \supseteq t\} \supseteq t$. ☐

Hence we have shown that $\langle \downarrow W, \subseteq \rangle$ is a residuated state space. ☐

Finally, we can construct our positive truthmaker model out of a finite tree model:

Definition 4.6 (Associated positive truthmaker model)**.** *For $\mathcal{M} = \langle W, R, \nu \rangle$ a finite tree model, we define $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu' \rangle$ as the associated positive truthmaker model by:*

1. *$S = \downarrow W \setminus \emptyset$,*
2. *$\sqsubseteq = \subseteq$, and*
3. *$\nu' : \mathcal{P} \to S$ is defined by $X \in \nu'(p)$ iff either X is not a principal downward closed set or for some $w \in X$, $w \in \nu(p)$, for $X \in S$ and $p \in \mathcal{P}$.*

The empty set is excluded from the state space as it as well does not correspond to any world in the I^+ model (and we already have the root as the null-state). The definition of ν' leads to all formulas being true in all non-principal sets. We need it like this so we can directly translate the intuitionistic implication to the truthmaker implication. When we fuse two distinct branches, we get a non-principal set which does not correspond to any world in the I^+ model. Making that non-principal set a truthmaker of everything is then convenient, as it can already be a truthmaker of anything.

Lemma 4.7. *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a finite tree model and consider the associated positive truthmaker model $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu' \rangle$ as constructed in definition 4.6. We have $X \Vdash \phi$ for all non-principal sets $X \in S$.*

Proof. The proof is done by a straightforward induction over the complexity of formulas. The base case follows directly from the definition of ν' in 4.6 combined with definitions 4.2 and 3.1.

The induction hypothesis is that $X \Vdash \phi, \psi$ for all non-principal sets X. $X \Vdash \phi \wedge \psi$ and $X \Vdash \phi \vee \psi$ follow directly from the induction hypothesis and lemma 4.2. In order to show that $X \Vdash \phi \to \psi$, we consider any Y such that $X \sqsubseteq Y$. Note that because X is non-principal, there is some $w, v \in X$ such that neither wRv nor vRw. As $X \sqsubseteq Y$, we get that $X \subseteq Y$ by definition 4.6 and hence $w, v \in Y$, so Y is also non-principal. Hence by induction hypothesis we get that $Y \Vdash \psi$. As Y was arbitrary such that $Y \supseteq X$, we get that for all $Y \Vdash \psi$, from which follows that $X \Vdash \phi \to \psi$ by definition 4.2.

□

Before we prove that truth is preserved under the construction of the truthmaker model, we need to show one more thing.

Lemma 4.8. *Let $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu' \rangle$ be a truthmaker model based on a finite tree model $\mathcal{M} = \langle W, R, \nu \rangle$ as in definition 4.6. We then have that $\overline{w} \sqsubseteq \overline{v}$ iff wRv.*

Proof. We show both directions directly.

- $\Rightarrow$ Suppose $\overline{w} \sqsubseteq \overline{v}$. This just means that $\overline{w} \subseteq \overline{v}$. Recall that $\overline{w} = \{u : uRv\}$. Because R is reflexive, $w \in \overline{w}$. Because of that and $\overline{w} \subseteq \overline{v}$, we get that $w \in \overline{v} = \{u : uRv\}$, so we get wRv.
- $\Leftarrow$ Suppose wRv. Take any $u \in \overline{w}$. Because R is transitive and uRw by definition of $\overline{w}$, we get that uRv and thus $u \in \overline{v}$ by definition of $\overline{v}$. Since u was arbitrary such that $u \in \overline{w}$, we get that this goes for all $u \in \overline{w}$. Hence we have shown that $\overline{w} \subseteq \overline{v}$ and so $\overline{w} \sqsubseteq \overline{v}$.

□

Now we show that all formulas that are true in some arbitrary finite tree model, are also true in the associated positive truthmaker model.

Lemma 4.9. *Let $\mathcal{M} = \langle W, R, \nu \rangle$ be a finite tree model and consider the associated positive truthmaker model $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu' \rangle$ as constructed in definition 4.6. We then get $w \vDash \phi$ iff $\overline{w} \Vdash \phi$.*

Proof. The proof is done by induction over the complexity of formulas.
For the base case we show that $w \vDash p$ iff $\overline{w} \Vdash p$ for $p \in \mathcal{P}$.

- $\Rightarrow$ Suppose $w \vDash p$ and consider $\overline{w}$. Because R is reflexive, $w \in \overline{w}$ and so by definition of ν', we get that $\overline{w} \Vdash p$.

- $\Leftarrow$ Suppose $\overline{w} \Vdash p$. By the definition of inexact truthmaking (3.1), this means that there is some $s \in S$ such that $s \sqsubseteq \overline{w}$ and $s \Vdash p$. Because $s \sqsubseteq \overline{w}$, just means that $s \subseteq \overline{w}$, we get that s is a principal downward closed set by lemma 4.5, and hence it is of the form $\overline{v}$ for some $v \in W$. By definition 4.2, we then get that $\overline{v} \in \nu'(p)$. By definition of ν', we then have that there is some $t \in \overline{v}$, such that $t \in \nu(p)$. As $t \in \overline{v}$ and $\overline{v} \subseteq \overline{w}$, we have $t \in \overline{w}$ and hence tRw. The monotonicity constraint then gives us that $w \in \nu(p)$ and hence $w \vDash p$.

The induction hypothesis is that $w \vDash \phi, \psi$ iff $\overline{w} \Vdash \phi, \psi$. Now for the $\phi \wedge \psi$ case:

- $\Rightarrow$ Suppose $w \vDash \phi \wedge \psi$. By definition of conjunction in I^+ models (definition 1.7), we get that $w \vDash \phi$ and $w \vDash \psi$. By our induction hypothesis, we derive that $\overline{w} \Vdash \phi$ and $\overline{w} \Vdash \psi$. Now by lemma 4.2, we get that $\overline{w} \Vdash \phi \wedge \psi$.

- $\Leftarrow$ Suppose $\overline{w} \Vdash \phi \wedge \psi$. By lemma 4.2, we get that $\overline{w} \Vdash \phi$ and $\overline{w} \Vdash \psi$. Hence, by induction hypothesis, we get $w \vDash \phi$ and $w \vDash \psi$. Then, by definition of conjunction, we get $w \vDash \phi \wedge \psi$.

For $\phi \vee \psi$:

- $\Rightarrow$ Suppose $w \vDash \phi \vee \psi$. By definition of disjunction, we get that either $w \vDash \phi$ or $w \vDash \psi$. By the induction hypothesis, we get that $\overline{w} \Vdash \phi$ in the former case, and $\overline{w} \Vdash \psi$ in the latter. In both cases however, we get by lemma 4.2 that $\overline{w} \Vdash \phi \vee \psi$.

- $\Leftarrow$ Suppose $\overline{w} \Vdash \phi \vee \psi$. By lemma 4.2, we get $\overline{w} \Vdash \phi$ or $\overline{w} \Vdash \psi$. By our induction hypothesis, we get $w \vDash \phi$ in the former case and $w \vDash \psi$ in the latter. Hence, by definition of disjunction, we get $w \Vdash \phi \vee \psi$ in both cases.

For $\phi \to \psi$:

- $\Rightarrow$ Assume $w \vDash \phi \to \psi$ and consider any $s \in S$ such that $\overline{w} \sqsubseteq s$ and $s \Vdash \phi$. We either have that s is principal or that it is not.

  1. If s is principal, it is of the form $\overline{v}$ for some $v \in W$. Consider any $\overline{v}$ such that $\overline{w} \sqsubseteq \overline{v}$. As $\overline{w} \sqsubseteq \overline{v}$, we get that wRv by lemma 4.8. Our induction hypothesis then gives us that $v \vDash \phi$. Since wRv, $v \vDash \phi$ and $w \vDash \phi \to \psi$, we get $v \vDash \psi$. Therefore, using the induction hypothesis again, we get $\overline{v} \Vdash \psi$.

  2. If s is non-principal, then by lemma 4.7 we get that $s \Vdash \psi$.

  Hence for all $s \in S$ such that $\overline{w} \sqsubseteq s$, we get that if $s \Vdash \phi$, then $s \Vdash \psi$, which according to lemma 4.2 leads to $\overline{w} \Vdash \phi \to \psi$.

- $\Leftarrow$ Assume $\overline{w} \Vdash \phi \to \psi$. Consider any v such that wRv and $v \vDash \phi$. Due to our induction hypothesis, we get that $\overline{v} \Vdash \phi$. Using lemma 4.8, we get that $\overline{w} \sqsubseteq \overline{v}$. Since also $\overline{w} \Vdash \phi \to \psi$ as well as $\overline{v} \Vdash \phi$, we infer that $\overline{v} \Vdash \psi$ (using lemma 4.2). Using the induction hypothesis again, we get $v \vDash \psi$. As v was arbitrary such that wRv and $v \vDash \phi$, we get that for all v such that wRv, if $v \vDash \phi$ then $v \vDash \psi$, which is the definition of $w \vDash \phi \to \psi$.

□

From here, it is not hard to show that we have indeed a truthmaker semantics for positive intuitionistic logic.

Theorem 4.10. *$\phi \vDash_{I^+} \psi$ iff $\phi \Vdash \psi$.*

Proof. We are going to prove both directions by contraposition:

- $\Rightarrow$ Assume $\phi \nVdash \psi$. This means that there is a positive truthmaker model $\mathcal{M} = \langle S, \sqsubseteq, \nu \rangle$ such that there is a state $s \in S$ so that $s \Vdash \phi$ and $s \nVdash \psi$. Now consider the I^+ model $\mathcal{M}' = \langle W_S, R, \nu' \rangle$ constructed as in lemma 4.3. We then get that $s \vDash \phi$ and $s \nvDash \psi$, which means $\phi \nvDash_{I^+} \psi$, as desired.
- $\Leftarrow$ Assume $\phi \nvDash_{I^+} \psi$. This means that there is an I^+ model $\mathcal{M} = \langle W, R, \nu \rangle$ such that there is a world $w \in W$ so that $w \vDash \phi$ and $w \nvDash \psi$. We take the corresponding finite tree model $\mathcal{M}' = \langle W', R', \nu' \rangle$, which we know exists due to lemma 4.4. We then get that $r \in W'$ is so that $r \vDash \phi$ and $r \nvDash \psi$. We then translate $\mathcal{M}'$ into a truthmaker model $S(\mathcal{M}') = \langle S, \sqsubseteq, \nu'' \rangle$ as done in lemma 4.9. We get that $\overline{r} \Vdash \phi$ and $\overline{r} \nVdash \psi$, which leads to $\phi \nVdash \psi$ as desired.

□

Hence we have proven the main theorem of this section. Now we have a truthmaker semantics for both FDE and I^+, we continue with defining one for C.

5 Truthmaker semantics and Wansing's C

The goal of this section is to construct a truthmaker semantics for Wansing's C. For this purpose, we will use the results from the previous sections to give the semantics itself and prove that it works.

The language, models, truth clauses and consequence for C are defined in section 1.4 starting on page 7.

For the truthmaker semantics, we use all truth and falsity clauses of the truthmaker semantics for FDE. The truth clause for the arrow will be from I^+. Furthermore, the falsity clause for the arrow will be a direct translation of the one used in the known semantics for C as stated above. We use the standard truthmaker model as defined in definition 2.7 on page 12. We will define the exact truthmaker semantics first. The only new clause is, as expected, the negation for the implication. It is just as in C defined as an implication from the antecedent to the negation of the conclusion.

Definition 5.1 (Truth clauses)**.** *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. We define truth and falsity for all $s \in S$ by means of the following double recursion:*

$$
\begin{array}{lcl}
s \Vdash p & \Leftrightarrow & s \in \nu^+(p), \\
s \dashv\!\vert\, p & \Leftrightarrow & s \in \nu^-(p), \\[4pt]
s \Vdash \neg\phi & \Leftrightarrow & s \dashv\!\vert\, \phi, \\
s \dashv\!\vert\, \neg\phi & \Leftrightarrow & s \Vdash \phi, \\[4pt]
s \Vdash \phi \wedge \psi & \Leftrightarrow & \exists t, u(s = t \sqcup u, t \Vdash \phi, \text{ and } u \Vdash \psi), \\
s \dashv\!\vert\, \phi \wedge \psi & \Leftrightarrow & s \dashv\!\vert\, \phi, s \dashv\!\vert\, \psi, \text{ or } s \dashv\!\vert\, \phi \vee \psi, \\[4pt]
s \Vdash \phi \vee \psi & \Leftrightarrow & s \Vdash \phi \text{ or } s \Vdash \psi, \text{ or } s \Vdash \phi \wedge \psi, \\
s \dashv\!\vert\, \phi \vee \psi & \Leftrightarrow & \exists t, u(s = t \sqcup u, t \dashv\!\vert\, \phi, \text{ and } u \dashv\!\vert\, \psi), \\[4pt]
s \Vdash \phi \to \psi & \Leftrightarrow & \text{There is a function } f : [\phi]^+ \to [\psi]^+, \text{ such that } s = \bigsqcup_{t \in [\phi]^+} t \to f(t), \\
s \dashv\!\vert\, \phi \to \psi & \Leftrightarrow & \text{There is a function } f : [\phi]^+ \to [\psi]^-, \text{ such that } s = \bigsqcup_{t \in [\phi]^+} t \to f(t).
\end{array}
$$

A state s is a falsemaker for $\phi \to \psi$ if it is the fusion of residual states form all truthmakers of ϕ to falsemakers of ψ.

As done in the previous sections, we will again make use of inexact truthmaking (definition 3.1) on 15.

We get the following lemma, in which clause *viii.* is the only new one. Its proof however is similar to the proof of *vii* which is found in the proof of lemma 4.2.

Lemma 5.1. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. Then for all*

$s \in S$:

$$
\begin{array}{rlcl}
i. & s \Vdash \neg\phi & \Leftrightarrow & s \dashV \phi, \\
ii. & s \dashV \neg\phi & \Leftrightarrow & s \Vdash \phi, \\
iii. & s \Vdash \phi \wedge \psi & \Leftrightarrow & s \Vdash \phi \text{ and } s \Vdash \psi, \\
iv. & s \dashV \phi \wedge \psi & \Leftrightarrow & s \dashV \phi \text{ or } s \dashV \psi, \\
v. & s \Vdash \phi \vee \psi & \Leftrightarrow & s \Vdash \phi \text{ or } s \Vdash \psi, \\
vi. & s \dashV \phi \vee \psi & \Leftrightarrow & s \dashV \phi \text{ and } s \dashV \psi, \\
vii. & s \Vdash \phi \rightarrow \psi & \Leftrightarrow & \forall t \in S(\text{ if } s \sqsubseteq t \text{ and } t \Vdash \phi, \text{ then } t \Vdash \psi), \\
viii. & s \dashV \phi \rightarrow \psi & \Leftrightarrow & \forall t \in S(\text{ if } s \sqsubseteq t \text{ and } t \Vdash \phi, \text{ then } t \dashV \psi).
\end{array}
$$

Proof. For cases $i. - vi.$, see lemma 3.1 on page 15. For case $vii.$, see lemma 4.2 on page 21. We show case $viii$ here:

- $\Rightarrow$ Suppose that $s \dashV \phi \rightarrow \psi$. Consider any t such that $s \sqsubseteq t$ and $t \Vdash \phi$. As $s \dashV \phi \rightarrow \psi$, we get by definition 3.1 that there is some state $s' \sqsubseteq s$ such that $s' \dashv \phi \rightarrow \psi$. We also know by the same definition that there exists a $t' \sqsubseteq t$ such that $t' \Vdash \phi$. As $s' \dashv \phi \rightarrow \psi$ and $t' \Vdash \phi$, we know that the residual state $t' \rightarrow u$ is a part of s' by definition 5.1, for some $u \in S$ such that $u \dashv \psi$. Because the state space is residuated, we know that $(t' \rightarrow u) \sqcup t' \sqsupseteq u$ and hence we get, as $u \dashv \psi$, by definition 3.1 that $((t' \rightarrow u) \sqcup t') \dashV \psi$. As $(t' \rightarrow u) \sqsubseteq s' \sqsubseteq s \sqsubseteq t$ and $t' \sqsubseteq t$, we get using associativity that $((t' \rightarrow u) \sqcup t' \sqsubseteq t$, and so $u \sqsubseteq t$. Hence, as $u \dashV \psi$, we get $t \dashV \psi$ by definition 3.1, as desired.
- $\Leftarrow$ For the other direction, we assume that for all $t \sqsupseteq s$, if $t \Vdash \phi$, then $t \dashV \psi$. We are going to show that there exists some $s' \sqsubseteq s$ such that $s' \dashv \phi \rightarrow \psi$. Now by definition 5.1 that $s' \dashv \phi \rightarrow \psi$ iff $s' = \bigsqcup\{u \rightarrow f(u) : u \Vdash \phi\}$, where $f : [\phi]^+ \rightarrow [\psi]^-$ is a function from truthmakers of ϕ to falsemakers of ψ. Now take any u such that $u \Vdash \phi$ and consider $u \sqcup s$. Because $u \sqcup s \sqsupseteq s$ by defintion of $\sqsubseteq$, and $u \sqcup s \Vdash \phi$ by defintion 3.1, we get that $u \sqcup s \dashV \psi$ by our assumption. Hence, by definition 3.1, we get that there exists a $v \in S$ such that $v \sqsubseteq s \sqcup u$ and $v \dashv \psi$. As such a v exists for every u such that $u \vDash \phi$, we can define our function f as $f(u) = v$, where $v \sqsubseteq u \sqcup s$ and $v \dashv \psi$. As shown in the proof of lemma 4.2, $\bigsqcup_{u \in [\phi]^+}\{u \rightarrow f(u)\} \sqsubseteq s$. This state is by definition 5.1 a falsemaker for $\phi \rightarrow \psi$ and hence by definition 3.1, we get that $s \dashV \phi \rightarrow \psi$.

□

We can now construct a C model out of any truthmaker model under preservation of truth and falsity.

Lemma 5.2. *Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model. Then $\mathcal{M}$ is a C model and viewed as such, we get for all $s \in S$: $s \Vdash \phi$ iff $s \vDash \phi$ and $s \dashV \phi$ iff $s \Dashv \phi$.*

Proof. Let $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$ be a truthmaker model and let $\mathcal{M}' = \langle W_S, R, \nu^{+'}, \nu^{-'} \rangle$ be the associated C model, where $W_S = S$, $R = \{(s,t) : s \sqsubseteq t\}$ and $\nu^{\circ'} : \mathcal{P} \to W$ is such that $s \in \nu^{\circ'}$ iff $s \in \nu^{\circ}$ for $\circ = +, -$. The proof is done by induction over the complexity of formulas. For $s \Vvdash p$ iff $s \vDash p$, $s \Vvdash \phi \wedge \psi$ iff $s \vDash \phi \wedge \psi$, $s \Vvdash \phi \vee \psi$ iff $s \vDash \phi \vee \psi$ and $s \Vvdash \phi \to \psi$ iff $s \vDash \phi \to \psi$, see the proof for lemma 4.3 on page 22. The base case $s \dashv\!|\!| p$ iff $s \Dashv p$ is shown here:

- $\Rightarrow$ Suppose $s \dashv\!|\!| p$. We get by definition 3.1 that there is some $s' \in S$ such that $s' \sqsubseteq s$ and $s' \dashv\!| p$. By lemma 5.1, we get that $s' \in \nu^-(p)$. By definition of $\nu^{-'}$, we then have that $s' \in \nu^{-'}(p)$. The monotonicity constraint then gives us that $s \in \nu^{-'}$ whenever $s'Rs$, which we have because $s' \sqsubseteq s$. Therfore we have $s \Dashv p$ as desired.
- $\Leftarrow$ Suppose $s \Dashv p$. We get that $s \in \nu^{-'}(p)$ and hence by definition of $\nu^{-'}$, we get that $s \in \nu^-(p)$ and so $s \dashv\!| p$. As $s \sqsubseteq s$, we have that $s \dashv\!|\!| p$ as well.

To show $s \Vvdash \neg\phi$ iff $s \vDash \neg\phi$, we have $s \dashv\!|\!| \phi$ iff $s \Dashv \phi$ as our induction hypothesis.

- $\Rightarrow$ To show $s \vDash \neg\phi$ whenever $s \Vvdash \neg\phi$, assume $s \Vvdash \neg\phi$. By definition 5.1, we get that $s \dashv\!|\!| \phi$. Our induction hypothesis then gives us $s \Dashv \phi$, which leads to $s \vDash \neg\phi$ by definition 1.10.
- $\Leftarrow$ Assume $s \vDash \neg\phi$. By definition 1.10, we get $s \Dashv \phi$, which by our induction hypothesis leads to $s \dashv\!|\!| \phi$. Definition 5.1 then gives us that $s \Vvdash \neg\phi$, as desired.

With $s \Vvdash \phi$ iff $s \vDash \phi$ as induction hypothesis, the result for $s \dashv\!|\!| \neg\phi$ iff $s \Dashv \neg\phi$ is obtained similar to the result above and is hence left to the reader.

The induction hypothesis $s \dashv\!|\!| \phi, \psi$ iff $s \Dashv \phi, \psi$ will be used for the false conjunction.

- $\Rightarrow$ To show $s \dashv\!| \phi \wedge \psi$ whenever $s \dashv\!|\!| \phi \wedge \psi$, assume the latter. By definition 5.1 we get a case distinction: either $s \dashv\!|\!| \phi$ or $s \dashv\!|\!| \psi$. Using the induction hypothesis, the former gets us $s \Dashv \phi$ and the latter gets us $s \Dashv \psi$. Both lead to $s \Dashv \phi \wedge \psi$ according to definition 1.11.
- $\Leftarrow$ For the other direction, suppose $s \Dashv \phi \wedge \psi$. 1.11 gets us that either $s \dashv\!| \phi$ or $s \Dashv \psi$. Using the induction hypothesis again, the former gives us $s \dashv\!|\!| \phi$ and the latter gives us $s \dashv\!|\!| \psi$. Both lead to $s \dashv\!|\!| \phi \wedge \psi$ using definition 5.1.

The induction hypothesis for the next step is $s \dashv\!|\!| \phi, \psi$ iff $s \Dashv \phi, \psi$. We will show $s \dashv\!|\!| \phi \vee \psi$ iff $s \Dashv \phi \vee \psi$.

- $\Rightarrow$ Suppose $s \dashv\!|\!| \phi \vee \psi$. Definition 5.1 gives us that $s \dashv\!|\!| \phi$ and $s \dashv\!|\!| \psi$. The induction hypothesis then gets us that $s \Dashv \phi$ and $s \Dashv \psi$, which in its turn lead to $s \Dashv \phi \vee \psi$ by definition 1.11.
- $\Leftarrow$ Suppose $s \Dashv \phi \vee \psi$. Definition 1.11 then gives us that $s \Dashv \phi$ and $s \Dashv \psi$, which lead to $s \dashv\!|\!| \phi$ and $s \dashv\!|\!| \psi$ respectively by our induction hypothesis. Using definition 5.1 we then get $s \dashv\!|\!| \phi \vee \psi$, as desired.

To show $s \dashv\!\!\!\Vdash \phi \to \psi$ iff $s \dashv \phi \to \psi$, we use both $s \Vdash\!\!\vdash \phi$ iff $s \vDash \phi$ (IH1) as well as $s \dashv\!\!\!\Vdash \psi$ iff $s \dashv \psi$ (IH2) as our induction hypotheses.

⇒ We use contraposition. Assume $s \nvDash \phi \to \psi$. By definition 1.11, we get that there is some $t \in W$ such that sRt, $t \vDash \phi$, but $t \nvDash \psi$. As sRt, we get $s \sqsubseteq t$ by definition of R. Furthermore, by (IH1), we know that $t \Vdash\!\!\vdash \phi$ and by (IH2) we know that $t \dashv\!\!\!\Vdash \psi$. Hence by definition 5.1, we do not have that $s \dashv\!\!\!\Vdash \phi \to \psi$, as desired.

⇐ We will also prove the contrapositive here. Assume $s \dashv\!\!\!\Vdash \phi \to \psi$ is not the case. By definition 5.1, we get that there is a t such that $s \sqsubseteq t$, $t \Vdash\!\!\vdash \phi$, but $t \dashv\!\!\!\Vdash \psi$. By using (IH1) and (IH2), we get $t \vDash \phi$ and $t \dashv \psi$. Furthermore, by definition of R, we get sRt. Hence, according to definition 1.11 we get that $s \dashv \phi \to \psi$, as desired.

□

Now we want to construct truthmaker models out of models for connexive logic. The aim is to do so in similarly to the way we did for positive connexive logic. This means that we need to be able to transform every C countermodel into an equivalent (finite) tree model (see definition 4.3 on page 23). Although there is no proof that such a transformation exists for C, we have strong reasons to believe that it can be done. As shown by Wansing [28], every C formula can be transformed into an equivalent formula in negation normal form.

Definition 5.2 (Negation normal form)**.** *A formula ϕ is in negation normal form (NNF) iff all negations are in front of an atom. In other words, if $\neg\psi$ is a subformula of ϕ, then ψ is of the form p.*

In a formula in NNF, we can substitute each negated atom $\neg p$ with a new atom p'. This way, we basically have a positive intuitionistic model, which we know we can transform into a finite tree model.

Furthermore, a C model can be regarded as two "positive" intuitionistic models, one for truth and the other for falsity, which interact with each other only through negation. This view also contributes to the plausibility that such a transformation exists.

The proof however highly likely is done in a similar way to the proof for intuitionistic logic and is outside the scope of this thesis. Hence, we state the property as a conjecture.

Conjecture 5.3. *For every C countermodel $\mathcal{M}$ for $\phi \vDash \psi$, there exists a finite tree one which is also a countermodel for $\phi \vDash \psi$.*

When we have our tree model, we construct a truthmaker model out of it as we have done in lemma 4.6. The established conventions for ν^+ carry over to ν^- in the obvious way.

Definition 5.3 (Associated truthmaker model)**.** *Let $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$ be a finite tree model. We define $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ as the associated truthmaker model by:*

1. *$S = \downarrow W \setminus \emptyset$,*
2. *$\sqsubseteq = \subseteq$*
3. *$\nu^{\circ'} : \to S$ is defined by $X \in \nu^{\circ'}(p)$ iff either X is not a principal downward closed set, or for some $w \in X$, $w \in \nu^{\circ'}(p)$, for $X \in S$, $p \in \mathcal{P}$ and $\circ = -, +$.*

Note that the only new thing here is the addition of $\nu^{-'}$. The conventions and properties of $\nu^{+'}$ carry over to $\nu^{-'}$ without any surprises though. Non-principal sets are now also falsemakers as well as truthmakers of everything.

Lemma 5.4. *Let $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$ be a finite tree model and consider its associated truthmaker model $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ as constructed in definition 5.3. Then, for all non-principal $X \in S$, we have both $X \Vdash \phi$ and $X \dashv\!\!| \phi$ for all ϕ.*

Proof. The proof is again done by induction. The base cases for $\phi = p$ follow directly from the definitions of $\nu^{+'}$ and $\nu^{-'}$.

Our induction hypotheses are that $X \Vdash \phi, \psi$ (IH1) and that $X \dashv\!\!| \phi, \psi$ (IH2) for all non-principal sets X. $X \Vdash \neg\phi$, $X \dashv\!\!| \phi \wedge \psi$ and $X \dashv\!\!| \phi \vee \psi$ follow directly from (IH2) using lemma 5.1. $X \dashv\!\!| \neg\phi$, $X \Vdash \phi \wedge \psi$ and $X \Vdash \phi \vee \psi$ are a direct consquence of (IH1) with the same lemma 5.1. $X \Vdash \phi \to \psi$ is established in the exact same way as in lemma 4.7. In order to show $X \dashv\!\!| \phi \to \psi$, we consider any Y such that $X \sqsubseteq Y$. Note that because X is non-principal, there is some $w, v \in X$ such that neither wRv nor vRw. As $\sqsubseteq$ is just $\subseteq$ by definition 5.3, we get that those $w, v \in X$ are also elements of Y. Therefore Y is non-principal as well. By (IH2), we then get that $Y \dashv\!\!| \psi$ for all Y such that $X \sqsubseteq Y$. Hence by definition 5.1, we get that $X \dashv\!\!| \phi \to \psi$. $\square$

As we have only added an interpretation function for falsemaking, the structure of the associated truthmaker model is the same as the structure of a positive associated truthmaker model. Hence, $\langle \downarrow W, \sqsubseteq \rangle$ is still a residuated state space.

Now we can show that the construction of an associated truthmaker model out of a finite tree one, preserves both truth and falsity. Recall that for nonempty principal downward closed sets of the form $\{v : vRw\}$, we write $\overline{w}$.

Lemma 5.5. *Let $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$ be a finite tree model and consider the associated truthmaker model $S(\mathcal{M}) = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ as constructed in definition 5.3. We then have that $w \vDash \phi$ iff $\overline{w} \Vdash \phi$ as well as $w \mathrel{=\!\!|} \phi$ iff $\overline{w} \dashv\!\!| \phi$.*

Proof. The proof proceeds by induction. For the proofs of $w \vDash \phi$ iff $\overline{w} \Vdash \phi$, where ϕ is of the form p, $\psi \vee \chi$, $\psi \wedge \chi$, $\psi \to \chi$, we refer to lemma 4.9. For the remaining case where ϕ is of the form $\neg\psi$, we have $w \mathrel{=\!\!|} \psi$ iff $\overline{w} \dashv\!\!| \psi$ as the induction hypothesis. The base case for this hypothesis is proved directly afterwards.

- ⇒ Assume $w \vDash \neg\psi$. By definition 1.11, we get that $w \Dashv \psi$, which by our induction hypothesis leads to $\overline{w} \dashv\!\vert\!\vert\, \psi$. However, that just means that $\overline{w} \Vvdash \neg\psi$ by definition 5.1, as desired.
- ⇐ Assume $\overline{w} \Vvdash \neg\psi$. By definition 5.1, we get that $\overline{w} \dashv\!\vert\!\vert\, \psi$. This leads to $w \Dashv \psi$ by the induction hypothesis. Definition 1.11 then gives us $w \Dashv \neg\psi$, as desired.

We now show the preservation of falsehood. For the base case, we prove that $w \Dashv p$ iff $\overline{w} \dashv\!\vert\!\vert\, p$ for $p \in \mathcal{P}$.

- ⇒ Suppose $w \Dashv p$ and consider $\overline{w}$. Because R is reflexive, $w \in \overline{w}$ and hence by definition of $\nu^{-'}$, we get that $\overline{w} \dashv\!\vert\!\vert\, p$.
- ⇐ Assume $\overline{w} \dashv\!\vert\!\vert\, p$. By definition 3.1, there is some state $s \in S$ such that $s \sqsubseteq \overline{w}$ and $s \dashv\!\vert\, p$. As $s \sqsubseteq \overline{w}$ just means $s \subseteq \overline{w}$, we know by lemma 4.5 that s is a principal downward closed set and is hence of the form $\overline{v}$ for some $v \in W$. By definition 5.1, we then get that because $s \dashv\!\vert\, p$, $s \in \nu^{-'}(p)$. By definition of $\nu^{-'}$, we then have that there is some $t \in \overline{v}$ such that $t \in \nu^{-}(p)$. As $\overline{v} \subseteq \overline{w}$, we have that as $t \in \overline{v}$, $t \in \overline{w}$. Therefore, by definition of $\overline{w}$, we have tRw. Now because $t \in \nu^{-}(p)$, the monotonicity constraint then gives us $w \in \nu^{-}(p)$, which using definition 1.11 leads to $w \Dashv p$, as desired.

To show $w \Dashv \neg\phi$ iff $\overline{w} \dashv\!\vert\!\vert\, \neg\phi$, we use $w \vDash \phi$ iff $\overline{w} \Vvdash \phi$ as our induction hypothesis.

- ⇒ Assume $w \Dashv \neg\phi$. Definition 1.11 then gives us that $w \vDash \phi$. By our induction hypothesis, we then know that $\overline{w} \Vvdash \phi$, which by definition 5.1 just means that $\overline{w} \dashv\!\vert\!\vert\, \neg\phi$, as desired.
- ⇐ Suppose $\overline{w} \dashv\!\vert\!\vert\, \neg\phi$. Definition 5.1 then gives us that $\overline{w} \Vvdash \phi$. Using the induction hypothesis gives then gives us $w \vDash \phi$, which according to definition 1.11 leads to $w \Dashv \neg\phi$.

For the remaining cases, the induction hypothesis is $w \Dashv \phi, \psi$ iff $\overline{w} \dashv\!\vert\!\vert\, \phi, \psi$. First we show $w \Dashv \phi \wedge \psi$ iff $\overline{w} \dashv\!\vert\!\vert\, \phi \wedge \psi$.

- ⇒ Assume $w \vDash \phi \wedge \psi$. Using definition 1.11, we get that either $w \Dashv \phi$ or $w \Dashv \psi$. Using the induction hypothesis, the former leads to $\overline{w} \dashv\!\vert\!\vert\, \phi$ and the latter to $\overline{w} \dashv\!\vert\!\vert\, \psi$. Using definition 5.1, both cases lead to $\overline{w} \dashv\!\vert\!\vert\, \phi \wedge \psi$.
- ⇐ Suppose $\overline{w} \dashv\!\vert\!\vert\, \phi \wedge \psi$. Definition 5.1 gives us that either $\overline{w} \dashv\!\vert\!\vert\, \phi$ or $\overline{w} \dashv\!\vert\!\vert\, \psi$. Using the induction hypothesis, we get $w \Dashv \phi$ in the former case and $w \Dashv \psi$ in the latter. Definition 1.11 gives us $w \Dashv \phi \wedge \psi$ in both cases.

To show $w \Dashv \phi \vee \psi$ iff $\overline{w} \dashv\!\vert\!\vert\, \phi \vee \psi$:

- ⇒ Suppose $w \vDash \phi \vee \psi$. We get $w \vDash \phi$ as well as $w \vDash \psi$. Using the induction hypothesis, this means that $\overline{w} \dashv\!\vert\!\vert\, \phi$ and $\overline{w} \dashv\!\vert\!\vert\, \psi$. This gives us $\overline{w} \dashv\!\vert\!\vert\, \phi \vee \psi$, as desired.

⇐ Suppose $\overline{w} \dashv\!\!\Vert\, \phi \vee \psi$. We get $\overline{w} \dashv\!\!\Vert\, \phi$ and $\overline{w} \dashv\!\!\Vert\, \psi$ by definition 5.1. The induction hypothesis then gives us $w \dashv \phi$ as well as $w \dashv \psi$, which just means that $w \dashv \phi \vee \psi$.

To show $w \dashv \phi \to \psi$ iff $\overline{w} \dashv\!\!\Vert\, \phi \to \psi$, we use $w \vDash \phi$ iff $\overline{w} \Vdash\!\!\vdash \phi$ (IH1) as well as $w \dashv \psi$ iff $\overline{w} \dashv\!\!\Vert\, \psi$ (IH2) as our induction hypotheses:

⇒ Suppose $w \dashv \phi \to \psi$ and consider any $s \in S$ such that $s \sqsupseteq \overline{w}$ and $s \Vdash\!\!\vdash \phi$. s is either principle or it is not.

   (a) If it is principal, then it is of the form $\overline{v}$ for some $v \in W$. As $\overline{w} \sqsubseteq \overline{v}$, we get wRv by lemma 4.8. Furthermore, by (IH1) we know that $v \vDash \phi$. Since $w \dashv \phi \to \psi$ by assumption, we know that $v \dashv \psi$. (IH2) then gives us that $\overline{v} \dashv\!\!\Vert\, \psi$.

   (b) If s is non-principal, we have by lemma 5.5 that $s \dashv\!\!\Vert\, \psi$.

   Hence, in both cases $s \dashv\!\!\Vert\, \psi$. Because s was arbitrary such that $\overline{w} \sqsubseteq s$ and $s \Vdash\!\!\vdash \phi$, we get that $\overline{w} \dashv\!\!\Vert\, \phi \to \psi$, as desired.

⇐ Assume $\overline{w} \dashv\!\!\Vert\, \phi \to \psi$ and consider any v such that wRv and $v \vDash \phi$. (IH1) gives us that $\overline{v} \Vdash\!\!\vdash \phi$. Lemma 4.8 gives us that $\overline{w} \subseteq \overline{v}$ and hence we know that $v \dashv\!\!\Vert\, \psi$ by definition 5.1. (IH2) then gets us that $v \dashv \psi$. As v was arbitrary such that $v \vDash \phi$ and wRv, we know that $w \vDash \phi \to \psi$.

□

We now show that we have a truthmaker semantics for C.

Theorem 5.6. *$\phi \vDash \psi$ iff $\phi \Vdash\!\!\vdash \psi$.*

Proof. We are going to prove both directions by contraposition.

⇒ Suppose $\phi \not\Vdash\!\!\vdash \psi$. This means that there is a truthmaker model $\mathcal{M} = \langle S, \sqsubseteq, \nu^+, \nu^- \rangle$, such that there is a state $s \in S$ for which $s \Vdash\!\!\vdash \phi$ as well as $s \not\Vdash\!\!\vdash \psi$. Now consider the C model $\mathcal{M} = \langle W, R, \nu^{+'}, \nu^{-'} \rangle$ as constructed in lemma 5.2. We then have that $s' \vDash \phi$, but $s' \nvDash \psi$, which means $\phi \nvDash \psi$ by definition 1.12.

⇐ Suppose $\phi \nvDash \psi$. Hence there is a C model $\mathcal{M} = \langle W, R, \nu^+, \nu^- \rangle$ such that there is a world w in which $w \vDash \phi$ but $w \nvDash \psi$. Consider the associated finite tree countermodel $\mathcal{M}'$ (conjecture 5.3) and construct the associated truthmaker model $S(\mathcal{M}') = \langle S, \sqsubseteq, \nu^{+'}, \nu^{-'} \rangle$ out of it as done in lemma 5.3. We have that for some $s \in S$, that $s \Vdash\!\!\vdash \phi$ but $s \not\Vdash\!\!\vdash \psi$. Hence, by definition 3.2, we get $\phi \not\Vdash\!\!\vdash \psi$, as desired.

□

5.1 Conclusion

We have now provided a truthmaker semantics for Wansing's C. However, the result is contingent on the conjecture that there is a tree countermodel for every C countermodel. A logical next step would be to prove this conjecture.

By providing a truthmaker semantics for C, we now have a new shared semantic underpinning for (full) intuitionistic logic, FDE and C. In this framework, these logics and their applications can be analyzed and compared more. The more logics get their truthmaker semantics, the merrier of couse. Moreover, the truthmaker semantics for C provides a way to understand the connexive arrow and its negation in terms of the conditional state, which is also done for full intuitionistic logic by Kit Fine.

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