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Eigenvalues, eigenvectors, and random-matrix theory

Master Thesis

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Abstract

We study methods for calculating eigenvector statistics of random matrix ensembles, and apply one of these methods to calculate eigenvector components of Toeplitz ± Hankel matrices. Random matrix theory is a broad field with applications in heavy nuclei scattering, disordered metals and quantum billiards. We study eigenvalue distribution functions of random matrix ensembles, such as the n-point correlation function and level spacings. For critical systems, with eigenvalue statistics between Poisson and Wigner-Dyson, the eigenvectors can have multifractal properties. We explore methods for calculating eigenvector component expectation values. We apply one of these methods, referred to as the eigenvector-eigenvalue identity, to show that the absolute values of eigenvector components of certain Toeplitz and Toeplitz±Hankel matrices are equal in the limit of large system sizes.

Contents

1 Introduction

Random Matrix theory is the study of matrices with random entries and where were first introduced by Wishart in 1928 [1]. The first application in physics was found in the 1950s when Wigner used this theory to describe Hamiltonians of heavy nuclei [2], [3], replacing the complex interactions with a random matrix satisfying appropriate symmetries. For these complicated systems it is inconceivable to write down all interactions within the nucleus. It is assumed that these interactions are sufficiently chaotic, such that they are modeled well by a random Hamiltonian. Then, predictions, in the form of expectation values and variances, can be made without knowing the precise form of the interactions.

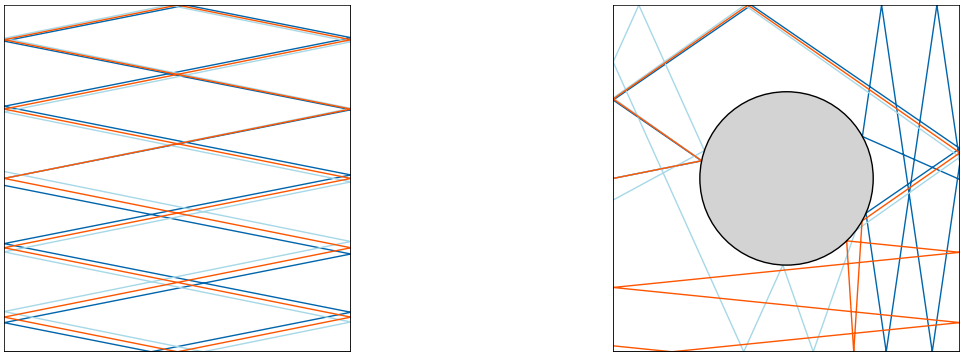
The set of possible matrices and associated probability density function (PDF) is called the *ensemble*. We will first look at three important examples, together called the Gaussian ensembles, and see how they come up in quantum mechanical systems with certain symmetries.

Using Random Matrix theory machinery one can calculate, among others, the level density (also called eigenvalue density, spectral density or 1-point correlation function), and the n-point level correlation function. The latter determines the probability of finding eigenvalues at n given points, while keeping the other $(N - n)$ undetermined, where N is the matrix size. In particular, the N-point correlation function is just the eigenvalue PDF.

For quantum systems, whose classical counterpart is chaotic, it is conjectured that the level spacings satisfy random matrix statistics [4]. This is the Bohigas-Giannoni-Schmit (BGS) conjecture. An important example are quantum billiards, where the Hamiltonian is proportional to the Laplacian with either Dirichlet (wave function vanishes at the boundary) or Neumann boundary conditions (the derivative of the wave function vanishes at the boundary). The classical counterpart is a system where a particle moves in this domain and bounces elastically off the boundary. For the quantum Sinai Billiard, the classical counterpart is chaotic, which is depicted in Figure 1b: trajectories that deviate slightly from each other initially drift apart quickly for chaotic systems. For the square billiard [Figure 1a], on the other hand, the trajectories remain close to each other. This is an example of an integrable system. Integrable systems with more than one degree of freedom, are conjectured (Berry-Tabor) to satisfy Poisson energy level statistics, where the level spacing distribution $p(s) = e^{-s}$ [5]. See Figure 2 for a comparison of these two types of level spacings.

Random matrix theory is not limited to the study of purely chaotic or purely integrable systems. It can also be used to describe systems with *intermediate statistics* where eigenvalue statistics, as expressed by level spacings and the n-point correlation functions, lie between the Poisson and Wigner-Dyson results. For some of these systems, summed moments of eigenvector components, called inverse participation ratios, satisfy a particular scaling relation with respect to the system size. This has the interpretation of eigenvector multifractality. The phenomenon occurs for example in the Rosenzweig-Porter model, as well as in the translational invariant, that is, Toeplitz variant of the Rosenzweig-Porter model [7]. An example of a physical system with wave function multifractality is the Anderson transition at the critical energy [8].

Toeplitz±Hankel matrices, which can describe Hamiltonians of translational invariant 1 dimensional systems with boundary effects, are closely related to random matrix theory. They appear in formulas for calculating orthogonal polynomials and in the probability of having a gap of a certain length in the spectrum [9, Chapter 18]. Furthermore, integral



(a) (b)

Figure 1: Two examples of billiards, (a) the Square Billiard and (b) the Sinai Billiard. In both cases we draw the trajectories (the first few bounces) of three particles originating from the same point with slightly different angles. The quantum analog is conjectured to have Wigner-Dyson statistics in the chaotic case (b), and Poisson statistics in the integrable case (a).

representations of Toeplitz and Hankel matrices are related to, for example, Chern-Simons theory on S^3 [10], the Gross-Witten-Wadia model [11], [12] and the Brezin-Gross-Witten model [11], [13].

The structure of the thesis is as follows. We start by introducing two important examples of random matrix ensembles, the Gaussian and Circular ensembles. We show the Dyson's threefold way, which states that there are three possible classes of Gaussian and Circular ensembles, Orthogonal, Unitary and Symplectic. The classification of a system in one of these three classes is done using its behaviour under time reversal. We continue by introducing the Coulomb gas analogy and the interpretation of Brownian motion of eigenvalues. In the next part the method of orthogonal polynomials is discussed. We highlight the relation with Toeplitz and Hankel matrices, and its integral representation. Then, we discuss inverse participation ratios and the interpretation of multifractality. We discuss the eigenvalue-eigenvector identity [14], and show how this method can be used to calculate eigenvector components. Finally, we introduce Toeplitz±Hankel matrices and calculate determinants of Toeplitz±Hankel matrices via the integral expression using the Cauchy identity and Schur orthogonality. This method allows us to show that the ratio of determinants used in the eigenvalue-eigenvector identity does not depend on the classical group we integrate over. From this we obtain our main result, namely, that the absolute values of the components of eigenvectors of Toeplitz and Toeplitz±Hankel matrix are equal in the limit of a large system size. This allows us to conclude, for example, that the effects of the boundary, described by the Hankel part, for a 1 dimensional hopping model are irrelevant in the large system size.

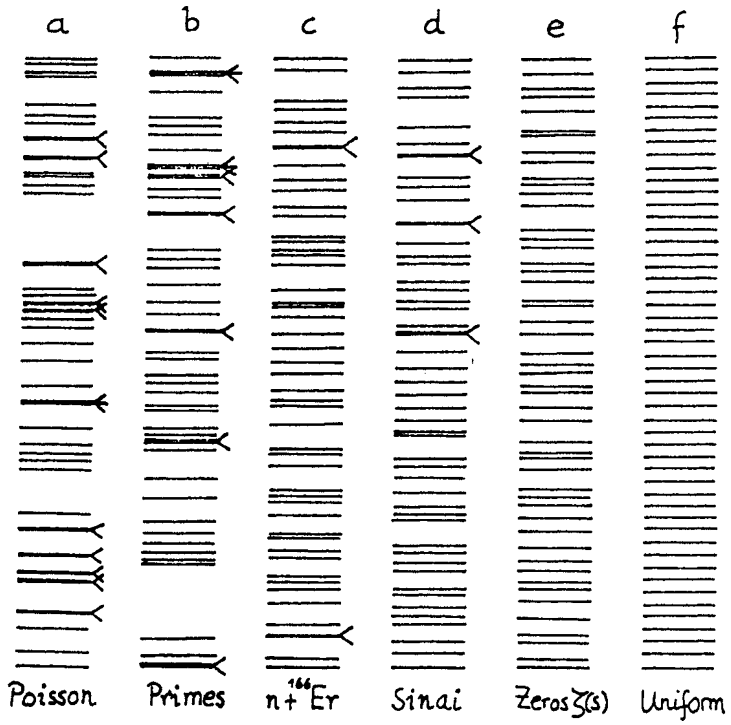


Figure 2: Example of level spacings: (a) Distribution with no correlation (Poisson). (b) Prime numbers. (c) Slow neutron resonance of Erbium ^{166}Er, a heavy nucleus. (d) Energy levels of a particle in Sinai's billiard table (Figure 1). (e) Zeroes of the Riemann zeta function on the Re$z = 1/2$ line. (f) Uniform distribution.
The ^{166}Er, Sinai and Zeros $\zeta(s)$ distributions resemble each other and are more uniformly distributed than the Poisson distribution. Figure taken from Bohigas, Giannoni and Schmit [6].

2 Random matrix ensembles

In this section, we will introduce two important examples of Random Matrix Ensembles, the Gaussian and the Circular ensembles. These ensembles have found a number of applications, from describing the energy statistics of heavy nuclei to the statistics of quantum chaotic billiards. Matrices in the Gaussian ensembles are Hermitian, and hence well suited to describe Hamiltonians of time invariant quantum mechanical systems. The Gaussian matrices will be introduced in Section 2.1, where we show that behaviour under time reversal symmetry can put additional constraint on the Hamiltonians. Three cases are distinguished; this is Dyson's threefold way [15]. Another method of deriving this threefold way is the subject of Section 2.2. This method relates better to the 10-fold way of Atland and Zirnbauer [16].

In Section 2.3 we introduce the Circular ensembles. Finally, in Section 2.4, we discuss the probability density of the Gaussian ensembles and its properties. For example, we show that it is the unique rotational invariant probability density function with independent matrix entries with zero mean and unit variance, up to the constraints due to the Hermitian property.

2.1 Symmetries and the Gaussian ensembles

Quantum mechanical systems are described by Hamiltonians, which are often Hermitian matrices. There may be conserved quantum numbers, such as total spin or parity, present. If one chooses a basis using the eigenvectors of these conserved numbers, the $M \times M$ Hamiltonian will be in block diagonal form, with say $N \times N$ dimensional matrices on the diagonal, for which the precise form is unknown, other than possibly which symmetries it has [9]. The latter are the objects studied with random matrix theory, and will just be called Hamiltonians from now on.

To obtain the Gaussian ensembles, we first make the assumption that the entries are independently distributed,

$$P_G(H) = \prod_{i \leq j} f_{ij}(H_{ij}),$$

where $P_G(H)$ is the PDF of H, and f_{ij} is the PDF of the individual entries. Secondly, some rotational invariance is assumed. It will turn out that these rotational symmetries, together with the condition that the entries are statistically independent, imply that the components are Gaussian distributed. We follow the reasoning by Mehta [9, Chapter 2] to give a classification based on whether the Hamiltonian has time-reversal symmetry or spin-rotation symmetry.

The time-reversal operator T is an antiunitary operator and can hence be expressed as $T = KC$, where K is unitary and C is the complex conjugation operation. The time reversal ψ^R of a state ψ is then given by $\psi^R = T\psi = K\psi^*$. From the requirement

$$\langle \varphi, A\psi \rangle = \langle \psi^R, A^R \varphi^R \rangle = \langle T\psi, A^R T\varphi \rangle = \langle K^\dagger (A^R)^\dagger KC\psi, C\varphi \rangle = \langle \varphi, K^T (A^R)^T K^* \psi \rangle,$$

it follows that the time-reversal symmetry can be expressed as

$$KA^T K^{-1} = K(K^T (A^R)^T K^*)^T K^\dagger = KK^\dagger A^R KK^\dagger = A^R. \tag{2.1}$$

Consider a unitary transformation $\psi \to U\psi$, then since

$$\langle \varphi, T\psi \rangle = \langle U\varphi, (UTU^{-1})U\psi \rangle,$$

for all ψ, φ, it follows that T transforms as $T \to UTU^\dagger$, and hence $KC \to UKCU^\dagger = UKU^TC$. Therefore K transforms as

$$K \to UKU^T. \tag{2.2}$$

Reversing time twice should do nothing, i.e. $T^2 = \alpha \cdot 1$, $|\alpha| = 1$. This implies for K that

$$\alpha K^T = T^2 K^T = KCKCK^T = KK^*CCK^T = KK^*K^T = K(KK^\dagger)^* = K,$$

so $K = \alpha(\alpha K^T)^T = \alpha^2 K$. This results in $T^2 = KK^* = \pm 1$. This condition corresponds, respectively, to integer $(+1)$ or half-integer (-1) total angular momentum [9, Chapter 2], which are described, respectively, by the Gaussian Orthogonal Ensemble (GOE) and Gaussian Symplectic Ensemble(GSE).

2.1.1 Gaussian Orthogonal Ensemble (GOE)

Suppose $T^2 = 1$. In this case, we have $K = UU^T$, for some unitary U [9, Chapter 2]. Indeed, let $K = \exp(iA)$, then A is symmetric, as $\exp(iA) = K = K^T = \exp\big(iA^T\big)$. Choose $U = \exp(iA/2)$, then this satisfies our constraints. Perform a unitary transformation $\psi \to U\psi$, then K transforms as $K \to UKU^T$ [Eq. (2.2)], so this fixes $K = 1$. Our assumption is that H is invariant under time reversal, so $H = H^R$. Eq. (2.1) with $K = 1$ then implies that $H^R = H^T$, i.e. H is real symmetric, hence we require that the PDF is invariant under transformations of the orthogonal group.

Definition 2.1.1 (Gaussian Orthogonal Ensemble)**.** The Gaussian orthogonal ensemble E_{1G} is the set T_{1G} of real symmetric matrices H together with a PDF $P_G(H)$ satisfying two constraints:

a) *(Invariance)* With the volume element $dH = \prod_{i \le j} dH_{ij}$, the probability $P_G(H)dH$ is invariant under transformations of the orthogonal group (the automorphisms of T_{1G}).

$$P_G(H')dH' = P_G(H)dH, \qquad H' = O^THO = O^{-1}HO, \quad O \in O(N).$$

b) *(Independence)* The entries are statistically independent,

$$P_G(H) = \prod_{i \le j} f_{ij}(H_{ij}). \tag{2.3}$$

An example H of the GOE can be drawn as follows. First construct an $N \times N$ matrix $\tilde{H}$ with real entries drawn from independent Gaussian distributions, with PDF

$$P_G(\tilde{H}) = \prod_{i,j=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\tilde{H}_{ij}^2}{2\sigma^2}\right).$$

Here, σ is the standard deviation of the Gaussian distribution where the entries are drawn from.

Now, symmetrize and define $H = (\tilde{H} + \tilde{H}^T)/2$. The result is a real symmetric matrix. Note that the diagonal entries stay the same and the off-diagonal entries are a sum of two normal distributed variables. The following is a standard result in probability theory.

Lemma 2.1.2. *Let X and Y be two independent random variables drawn from a normal distribution with mean respectively μ_X and μ_Y and variance σ_X^2 resp. σ_Y^2. Then the sum $Z = X + Y$ is normal distributed, with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.*

Dividing by 2 results in a total variance of $2\left(\frac{\sigma}{2}\right)^2 = \frac{\sigma^2}{2}$ for the off diagonal entries, hence the PDF for the upper right $N(N-1)/2$ entries of H is given by [17, Chapter 1]

$$P_G(H) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{H_{ii}^2}{2\sigma^2}\right) \prod_{i<j} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{H_{ij}^2}{\sigma^2}\right). \tag{2.4}$$

This shows that the f_{ij} in Eq. (2.3) will actually depend on i and j.

Note that $\operatorname{tr}(H^2) = \sum_{i,j} H_{ij}H_{ji} = \sum_{i,j} H_{ij}^2$, so we can also write Eq. (2.4) as

$$P_G(H) = \exp\left(-\frac{1}{2\sigma^2} \operatorname{tr} H^2 + c\right),$$

for some normalization constant c.

2.1.2 Gaussian Symplectic Ensemble (GSE)

We again follow Mehta [9, Chapter 2] for this section. Suppose $T^2 = -1$. This means that $KK^* = -1$, hence $K^T = -K$. All eigenvalues of H are doubly degenerate, this is Kramer's degeneracy. To see this let ψ be an eigenvector of H with eigenvalue E. Then as H is time reversal invariant, $T\psi$ is also an eigenvector with eigenvalue E. We will show that these two eigenvectors are orthogonal. Indeed,

$$\langle \psi | T\psi \rangle = \overline{\langle T\psi | T^2\psi \rangle} = -\overline{\langle T\psi | \psi \rangle} = -\langle \psi | T\psi \rangle = 0.$$

The doubly degeneracy implies that the Hilbert space is even dimensional. Let us relabel and denote the size of K with $2N$ for this case. Then we can replace each 2×2 block by a quaternion number, using the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a+d) \cdot 1 - \frac{i}{2}(a-d)e_1 + \frac{1}{2}(b-c)e_2 - \frac{i}{2}(b+c)e_3, \tag{2.5}$$

where $e_1 = i\sigma_1$, $e_2 = i\sigma_2$ and $e_3 = i\sigma_3$, σ_1, σ_2 and σ_3 are the Pauli matrices. The matrices e_1, e_2 and e_3 satisfy the quaternion relations $e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$. Quaternions extend the complex numbers with two additional units, usually denoted by j and k satisfying $i^2 = j^2 = k^2 = ijk = -1$. We can therefore identify $j = \sigma_2$ and $k = \sigma_3$, and σ_1 as the first quaternion unit.

A general quaternion number can then be written as

$$q = q^{(0)} + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3,$$

with $q^{(i)}$ complex coefficients. A quaternion q is called 'real' if the $q^{(i)}$ are real. Define the 'quaternion conjugate' $\overline{q}$ by

$$\overline{q} := q^{(0)} - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3$$

and 'complex conjugate' q^*

$$q^* := (q^{(0)})^* + (q^{(1)})^*e_1 + (q^{(2)})^*e_2 + (q^{(3)})^*e_3.$$

Combining both results in the Hermitian conjugate $q^\dagger := \overline{q}^*$. Note that q is real if $q^* = q$. Now consider a $2N \times 2N$-dimensional matrix A, and write it using quaternions as a $N \times N$ matrix Q with quaternionic elements. Using Eq. (2.5), we can express the transpose Q^T, Hermitian conjugate $Q^\dagger$ and time reversal Q^R [Eq. (2.1) with $K = e_2 \cdot 1$] in terms of the quaternion and complex conjugate,

$$\begin{aligned}(Q^T)_{ij} &= -e_2\overline{Q_{ji}}e_2,\\ (Q^\dagger)_{ij} &= (Q_{ij})^\dagger,\\ (Q^R)_{ij} &= e_2(Q^T)_{ij}e_2^{-1} = -e_2e_2\overline{Q_{ji}}e_2e_2^{-1} = \overline{Q_{ji}}.\end{aligned} \tag{2.6}$$

If $Q^R = Q$, then Q is called *self-dual*.

Unlike before, we cannot do a transformation so that $K = 1$, as this does not solve $KK^* = -1$, but we can do such a transformation to get $K = Z$, where Z is a $N \times N$ matrix with e_2 on the diagonal. Once such a unitary transformation is chosen, further transformations $\psi \to B\psi$ can be made with unitary B, provided it satisfies $Z = BZB^T$. The matrices B form the N-dimensional symplectic group $\mathrm{Sp}(N)$ and satisfy $B^R = B^\dagger = B^{-1}$ [9].

The Hermitian matrices H that we are considering are also assumed to have time-invariance, i.e. are self-dual $H^R = H$. Eq. (2.6) together with the Hermitian property implies that H is a real symmetric quaternionic matrix.

Definition 2.1.3 (Gaussian Symplectic Ensemble)**.** The Gaussian symplectic ensemble E_{4G} is the set T_{4G} of self-dual Hermitian matrices together with a PDF $P_G(H)$ satisfying two constraints:

a) *(Invariance)* Let the volume element be $dH = \prod_{i\le j} dH_{ij}^{(0)} \prod_{k=1}^{3} \prod_{i<j} dH_{ij}^{(k)}$. The probability $P_G(H)dH$ is invariant under transformations of the symplectic group (the automorphisms of T_{4G}).

$$P_G(H')dH' = P_G(H)dH, \quad H' = B^\dagger HB = B^{-1}HB, \quad B \in Sp(N).$$

b) *(Independence)* The entries are statistically independent,

$$P_G(H) = \prod_{i\le j} f_{ij}^{(0)}\left(H_{ij}^{(0)}\right) \prod_{k=1}^{3}\prod_{i<j} f_{ij}^{(k)}\left(H_{ij}^{(k)}\right).$$

2.1.3 Gaussian Unitary Ensemble (GUE)

In this case the Hamiltonian has no time reversal invariance. This can, for example, be achieved by applying a strong external magnetic field to the system [9, Chapter 2]. The only assumption is that the Hamiltonians are Hermitian.

Definition 2.1.4 (Gaussian Unitary Ensemble)**.** The Gaussian unitary ensemble E_{2G} is the set T_{2G} of Hermitian matrices H together with a PDF $P_G(H)$ satisfying two constraints:

a) *(Invariance)* Let the volume element be $dH = \prod_{i\le j} dH_{ij}^{(0)} \prod_{i<j} dH_{ij}^{(1)}$, with $H_{ij}^{(0)}$ and $H_{ij}^{(1)}$ the real and imaginary parts of H_{ij}. The probability $P_G(H)dH$ is invariant under transformations of the unitary group (the automorphisms of T_{2G}).

$$P_G(H')dH' = P_G(H)dH, \quad H' = U^\dagger HU = U^{-1}HU, \quad U \in U(N).$$

b) *(Independence)* The entries are statistically independent,

$$P_G(H) = \prod_{i \leq j} f_{ij}^{(0)}\left(H_{ij}^{(0)}\right) \prod_{i<j} f_{ij}^{(1)}\left(H_{ij}^{(1)}\right).$$

2.2 Dyson's three-fold way

In this section, we will show an alternative method of deriving the threefold way, which arises from first principles when discussing the symmetries of density matrices. This method emphasizes which assumptions are made to arrive at the threefold way, and while producing no new results compared to the previous section, can lead to better insight in why exactly there are three classes of Gaussian ensembles, and also three classes of Circular ensembles, but for example four classes of Bogoliubov-de Gennes ensembles [18]. Furthermore, this method can be extended to arrive at the 10-fold classification originally due to Atland and Zirnbauer [16]. Kitaev [19] used this result to classify the topology of free fermion Hamiltonians based on the symmetry class and the dimension of the system. This is the periodic table for topological insulators and superconductors. This result is, however, beyond the scope of this thesis, but we refer an interested reader to Freed and Moore [20]. The current section is also based on this paper.

A quantum state is described by a wave vector $|\psi\rangle$ in a complex separable Hilbert space $\mathcal{H}$. Usually, this wave vector is required to be normalized to one, $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$. We will not require this, but instead consider $z|\psi\rangle$ for all $z \in \mathbb{C}^*$ to be equivalent quantum states, belonging to the same equivalence class $[|\psi\rangle]$ in the projective space $\mathbb{P}\mathcal{H}$. The transition probability p from states $[|\psi\rangle], [|\psi'\rangle] \in \mathbb{P}\mathcal{H}$ is given by

$$\begin{array}{rccccc} p: & \mathbb{P}\mathcal{H} & \times & \mathbb{P}\mathcal{H} & \to & [0,1], \\ & ([|\psi\rangle] & , & [|\psi'\rangle]) & \mapsto & \frac{|\langle\psi|\psi'\rangle|^2}{\|\psi\|\|\psi'\|}. \end{array} \tag{2.7}$$

Alternatively, there is a $1-1$ correspondence between elements $[|\psi\rangle]$ in the projective Hilbert space and rank 1 projection operators $\mathcal{P}_\psi$, also known as density matrices, given by

$$\mathcal{P}_\psi = |\psi\rangle \frac{1}{\langle\psi|\psi\rangle} \langle\psi|.$$

The operator $\mathcal{P}_\psi$ is a rank 1 projection, or equivalently a pure state, because $\mathcal{P}_\psi^2 = \mathcal{P}_\psi$. The transition probability Eq. (2.7) can now be written as

$$\begin{array}{rccccc} p: & \mathbb{P}\mathcal{H} & \times & \mathbb{P}\mathcal{H} & \to & [0,1], \\ & (\mathcal{P}_\psi & , & \mathcal{P}_{\psi'}) & \mapsto & \mathrm{tr}_{\mathcal{H}}(\mathcal{P}_\psi \mathcal{P}_{\psi'}). \end{array} \tag{2.8}$$

Every potential symmetry of a quantum system should at least preserve transition probabilities. This is the minimal requirement for a transformation of the Hilbert space to be called a symmetry. We will call such potential symmetry a *projective quantum symmetry* or *quantum automorphism*, and it is defined as an invertible map $\mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ (an automorphism), which preserves p. Denote the set of quantum automorphism with $\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$.

A norm preserving linear map $u : \mathcal{H} \to \mathcal{H}$ can be extended to a map $\bar{f}_u : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ by defining $\bar{f}_u([|\psi\rangle]) = [u|\psi\rangle]$. Equivalently, acting on density matrices, one can define $\bar{f}_u$ as $\bar{f}_u : \mathcal{P}_\psi \mapsto u\mathcal{P}_\psi u^\dagger$. Both unitary and antiunitary maps $\mathcal{H} \to \mathcal{H}$ preserve norms, and Eq. (2.7) (or Eq. (2.8))[1]. A linear map $S : \mathcal{H} \to \mathcal{H}$ is antiunitary if $\langle S\psi, S\varphi\rangle = \overline{\langle\psi, \varphi\rangle}$

[1]Both unitary and antiunitary operators u satisfy $u^\dagger u = uu^\dagger = 1$

for all $\psi, \varphi \in \mathcal{H}$. Let $\mathrm{Aut}_{qtm}(\mathcal{H})$ be the set of unitary and antiunitary operators $\mathcal{H} \to \mathcal{H}$. Wigner's theorem states that these are all the quantum automorphisms.

Theorem 2.2.1 (Wigner)**.** *Every quantum automorphism is induced by a unitary or antiunitary operator $u : \mathcal{H} \to \mathcal{H}$. The surjective map $\pi : \mathrm{Aut}_{qtm}(\mathcal{H}) \twoheadrightarrow \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ is given by $u \mapsto \bar{f}_u$. Furthermore, the kernel of π are the complex numbers of norm 1, $U(1)$. Hence, we have the following short exact sequence,*

$$1 \longrightarrow U(1) \hookrightarrow \mathrm{Aut}_{qtm}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \longrightarrow 1. \tag{2.9}$$

Let us assign a sign to maps S in $\mathrm{Aut}_{qtm}(\mathcal{H})$ based on whether S is unitary $\varphi_{\mathcal{H}}(S) = 1$ or antiunitary $\varphi_{\mathcal{H}}(S) = -1$. Then $\varphi_{\mathcal{H}}$ is a map $\mathrm{Aut}_{qtm}(\mathcal{H}) \to \{\pm 1\} = \mathbb{Z}_2$. Similarly, we can define $\varphi_{\mathbb{P}\mathcal{H}} : \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \to \mathbb{Z}_2$. We will denote both these maps with φ. For $\lambda \in U(1)$ and $S \in \mathrm{Aut}_{qtm}(\mathcal{H})$, by the properties of linear and antilinear maps, we have

$$S\lambda = \begin{cases} \lambda S & \text{if } \varphi_{\mathcal{H}}(S) = 1, \\ \bar{\lambda} S & \text{if } \varphi_{\mathcal{H}}(S) = -1. \end{cases} \tag{2.10}$$

We will give the short exact sequence Eq. (2.9) together with the map φ a name.

Definition 2.2.2 (φ-twisted $U(1)$ central extension)**.** Let G be a topological group and $\varphi : G \to \mathbb{Z}_2$ a continuous homomorphism Then a *φ-twisted $U(1)$ central extension of G*, denoted by G^τ, is the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1, \\ & & & & \downarrow{\scriptstyle \varphi^\tau} & \swarrow{\scriptstyle \varphi} & & & \\ & & & & \mathbb{Z}_2 & & & & \end{array}$$

such that for all $S \in G^\tau$ and $\lambda \in U(1)$,

$$S\lambda = \begin{cases} \lambda S & \text{if } \varphi(S) = 1, \\ \bar{\lambda} S & \text{if } \varphi(S) = -1, \end{cases}$$

and where $\varphi^\tau : G^\tau \to \mathbb{Z}_2$ is a continuous homomorphism.

For $\varphi \equiv 1$ this is called $U(1)$ central extension.

Example 2.2.3. Let $G = M_2 = \{1, \overline{T}\}$. Choose a certain element $T \in G^\tau$ in the pre-image of $\overline{T}$ of the map $\pi : G^\tau \to M_2$. Let $z = T^2$, then because $\pi(z) = \pi(T)^2 = \overline{T}^2 = 1$, it must hold that $z \in U(1)$, where we see $U(1)$ as a subset of G^τ. We distinguish two cases:

a) $\varphi(T) = +1$: For any $g \in G^\tau$, we must have $\varphi^\tau(g) = \varphi \circ \pi(g) = 1$. In particular for $g = T$ this implies that $zT = Tz$. This allows us to define the map $j : M_2 \to G^\tau$, $1 \mapsto 1, \overline{T} \mapsto \sqrt{\overline{z}}T$, which is an homomorphism, because

$$j(T^2) = j(1) = 1 = \overline{z}z = \sqrt{\overline{z}}z\sqrt{\overline{z}} = \sqrt{\overline{z}}T\sqrt{\overline{z}}T = j(T)j(T).$$

Therefore, the sequence is split, $G^\tau \simeq U(1) \times M_2$, and we can always choose T such that $T^2 = 1$.

b) $\varphi(T) = -1$: Let $z = T^2$, then $\pi(z) = \pi(T)^2 = 1$, so $\varphi^\tau(z) = 1$. As $\varphi^\tau(T) = -1$, we have for $z \in U(1)$, $Tz = \overline{z}T$. Then

$$T^3 = T^2T = zT,$$
$$T^3 = TT^2 = Tz = \overline{z}T,$$

so we must have $z = \overline{z} = \pm 1$. Therefore, G^τ is the group

$$G^\tau = M_2^\pm := \{z, zT \mid Tz = \overline{z}T, T^2 = \pm 1, z \in U(1)\}.$$

Note that $\varphi(z) = 1$ and $\varphi(T) = -1$.

△

In an actual physical system, these potential symmetries $\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ need not all be actual symmetries. The set of actual symmetries forms a topological group G and we have a continuous homomorphism

$$\rho : G \to \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}).$$

By composition, there exists also a map $\varphi = \varphi_{\mathcal{H}} \circ \rho : G \to \mathbb{Z}_2$, such that $\rho(g)$ is linear if $\varphi(g) = 1$ and antilinear if $\varphi(g) = -1$. With

$$G^\tau = \{(S, g) \in \mathrm{Aut}_{qtm}(\mathcal{H}) \times G : \pi(S) = \rho(g)\}$$

and projections $\rho^\tau : G^\tau \to \mathrm{Aut}_{qtm}(\mathcal{H})$ and $\tilde{\pi} : G^\tau \to G$, we can write down the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G^\tau & \xrightarrow{\tilde{\pi}} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho^\tau} & & \downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & \mathrm{Aut}_{qtm}(\mathcal{H}) & \xrightarrow{\pi} & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) & \longrightarrow & 1 \end{array}$$

called the *pullback extension*. The map ρ^τ is an example of a (φ, τ)-twisted representation of G.

Definition 2.2.4 ((G^τ, φ)-representation)**.** Let G be a topological group and G^τ a φ-twisted central extension (Definition 2.2.2). A *(G^τ, φ)-representation* (V, ρ^τ) is a complex vector space V, together with a homomorphism $\rho^\tau : G^\tau \to \mathrm{End}(V_\mathbb{R})$ to the group of endomorphisms of the underlying real vector space $V_\mathbb{R}$ of V, such that

$$\rho^\tau(g) = \begin{cases} \text{complex linear} & \text{if } \varphi(g) = +1, \\ \text{complex antilinear} & \text{if } \varphi(g) = -1. \end{cases}$$

Our goal now is to classify the (G^τ, φ)-representations. Much of the $\mathbb{Z}_2$-graded representation theory is the same as the regular representation theory. For starters, one can define *intertwiners*.

Definition 2.2.5 (Intertwiner)**.** Let (V_1, ρ_1^τ) and (V_2, ρ_2^τ) be two (G^τ, φ)-representations. An *intertwiner* $\mathcal{O}$ is a complex linear map $\mathcal{O} : V_1 \to V_2$, such that, for all $g \in G^\tau$, the following diagram commutes

$$\begin{array}{ccc} (V_1, \rho_1^\tau) & \xrightarrow{\mathcal{O}} & (V_2, \rho_2^\tau) \\ \downarrow{\scriptstyle \rho_1^\tau(g)} & & \downarrow{\scriptstyle \rho_2^\tau(g)} \\ (V_1, \rho_1^\tau) & \xrightarrow{\mathcal{O}} & (V_2, \rho_2^\tau) \end{array}$$

The set of intertwiners is denoted by $\mathrm{Hom}_{\mathbb{C}}^{G^\tau}(V_1, V_2)$. Furthermore, we denote $\mathrm{End}_{\mathbb{C}}^{G^\tau}(V) = \mathrm{Hom}_{\mathbb{C}}^{G^\tau}(V, V)$.

The (V_1, ρ_1^τ) and (V_2, ρ_2^τ) are called *equivalent* if there exists such an $\mathcal{O}$ which is also an isomorphism.

Furthermore, there is a version of Schur's lemma.

Theorem 2.2.6 (Schur's lemma)**.**

a) *An intertwiner between two irreducible (G^τ, φ)-representations is either zero or an isomorphism.*

b) *If (V, ρ^τ) is an irreducible (G^τ, φ)-representation, $\mathrm{End}_{\mathbb{C}}^{G^\tau}(V)$ is a real division algebra.*

Proof. For the proof we refer to [20, Theorem C1]. □

Recall that a division algebra D over a field κ is a nonzero algebra D over κ, such that for every nonzero $a \in D$ there exists an inverse a^{-1}. Fortunately, there are only three division algebras.

Theorem 2.2.7 (Frobenius)**.** *There exists precisely three real associative division algebras, namely $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.*

Proof. For the proof we refer to [21, p.26]. □

Each of these three division algebra corresponds to a class in the Dyson's threefold way. Let β be the number of real variables needed to specify an element of the division algebra D, that is $\beta = 1$ for $D = \mathbb{R}$, $\beta = 2$ for $D = \mathbb{C}$ and $\beta = 4$ for $D = \mathbb{H}$. The number β is called the *Dyson index*.

If G^τ is compact, every (G^τ, φ)-representation can be decomposed into irreducible representations, for which we can apply Schur's lemma.

Theorem 2.2.8 (Weyl)**.** *Suppose G^τ is compact, then any (G^τ, φ)-representation is* completely reducible*. Let $\{(V_\lambda, \rho_\lambda^\tau)\}$ be a complete list of inequivalent irreducible (G^τ, φ)-representations. Then we can decompose V as*

$$V \simeq \bigoplus_\lambda S_\lambda \otimes V_\lambda \simeq \bigoplus_\lambda \underbrace{V_\lambda \otimes \cdots \otimes V_\lambda}_{s_\lambda \ \ times}, \tag{2.11}$$

where $\rho_\lambda^\tau(g)$ is the identity on S_λ. The space S_λ is called the degeneracy space, $s_\lambda = \dim S_\lambda$ the degeneracy.

Proof. For the proof we refer to [22, p.60]. □

Using respectively the first and second part of Schur's lemma,

$$\mathrm{Hom}_{\mathbb{C}}^{G^\tau}(V_\lambda, V_\lambda') = \delta_{\lambda,\lambda'} \mathrm{End}_{\mathbb{C}}^{G^\tau}(V_\lambda) = \delta_{\lambda,\lambda'} D_\lambda,$$

where D_λ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Decompose $\mathrm{Hom}_{\mathbb{C}}(V, V)$ using Eq. (2.11)

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(V, V) &\simeq V^* \otimes_{\mathbb{C}} V \\
&\simeq \bigoplus_{\lambda,\lambda'} (S_\lambda^* \otimes_{\mathbb{R}} S_{\lambda'}) \otimes_{\mathbb{R}} (V_\lambda^* \otimes_{\mathbb{C}} V_{\lambda'}) \\
&\simeq \bigoplus_{\lambda,\lambda'} \mathrm{Hom}_{\mathbb{R}}(S_\lambda, S_{\lambda'}) \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\lambda'}).
\end{aligned}$$

Now take the G^τ-invariant part to get

$$\mathrm{End}_{\mathbb{C}}^{G^\tau}(V) = \bigoplus_\lambda \mathrm{End}_{\mathbb{R}}^{G^\tau}(S_\lambda) \otimes_{\mathbb{R}} D_\lambda = \bigoplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda),$$

where $\mathrm{Mat}_N(D)$ are $N \times N$-matrices with entries in the field D.

Suppose now that the quantum system is time translation invariant. Then there is a one parameter subgroup $\mathbb{R} \to G^\tau$, which via ρ^τ gives a one parameter subgroup $\mathbb{R} \to \mathrm{Aut}_{qtm}(\mathcal{H})$. Time translation by 0 is the identity, a unitary symmetry, so by continuity, the one-parameter subgroup $t \mapsto U(t) = e^{-itH/\hbar}$ consists of unitary transformations. The self-adjoint operator H is called the Hamiltonian.

Corollary 2.2.9 (Schrödinger equation)**.** *The time dependent wave function $|\psi(t)\rangle$ is given by $U(t)|\psi\rangle$. Derive with respect to t to get*

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Because the system is time translation invariant, any symmetry $g \in G^\tau$ can either reverse or preserve the arrow of time. Define $\tau : G^\tau \to \mathbb{Z}_2$ by[2]

$$\tau(g) = \begin{cases} +1 & \text{if } g \text{ preserves the arrow of time,} \\ -1 & \text{if } g \text{ reverses the arrow of time.} \end{cases}$$

The time translations commute or anticommute with $g \in G^\tau$ according to $\tau(g)$, so for all $g \in G^\tau$,

$$\rho^\tau(g)U(t)\rho^\tau(g^{-1}) = U(\tau(g)t).$$

By deriving at $t = 0$ we get $\rho^\tau(g)iH\rho^\tau(g^{-1}) = i\tau(g)H$ Now $\rho^\tau(g)i = i\varphi(g)\rho^\tau(g)$, hence

$$\rho(g)H\rho(g)^{-1} = \chi(g)H, \tag{2.12}$$

where $\chi(g) = \tau(g)\varphi(g)$. We say that a Hamiltonian H is *compatible with the dynamics* if it satisfies Eq. (2.12).

If for some $g \in G^\tau$, $\chi(g) = -1$, then $\rho^\tau(g)$ inverts the spectrum of H. This is not possible if H is bounded from below but not from above, so in most systems we must require $\chi(g) = 1$ for all $g \in G^\tau$, so that also $\varphi = \tau$. In this case, the following diagram commutes for all $g \in G^\tau$

$$\begin{array}{ccc} (\mathcal{H}, \rho^\tau) & \xrightarrow{H} & (\mathcal{H}, \rho^\tau) \\ \downarrow{\scriptstyle \rho^\tau(g)} & & \downarrow{\scriptstyle \rho^\tau(g)} \\ (\mathcal{H}, \rho^\tau) & \xrightarrow{H} & (\mathcal{H}, \rho^\tau) \end{array},$$

that is, H is an intertwiner,

$$H \in \mathrm{End}_{\mathbb{C}}^{G^\tau}(\mathcal{H}) = \bigoplus_\lambda \mathrm{Mat}_{s_\lambda}(D_\lambda).$$

The Hamiltonian is also self-adjoint, so the space $\mathcal{E}$ of Hamiltonians compatible with the dynamics is given by

$$\mathcal{E} = \bigoplus_\lambda \mathrm{Herm}_{s_\lambda}(D_\lambda), \tag{2.13}$$

[2]This τ is not related to the τ superscript of G^τ and ρ^τ.

where

$$\mathrm{Herm}_N(D) = \begin{cases} N \times N \text{ symmetric matrix} & \text{if } D = \mathbb{R}, \\ N \times N \text{ Hermitian matrix} & \text{if } D = \mathbb{C}, \\ N \times N \text{ quaternion Hermitian matrix} & \text{if } D = \mathbb{H}. \end{cases} \tag{2.14}$$

In other words, the Hamiltonian H can be written in block diagonal form, with each block either a symmetric, Hermitian or quaternion Hermitian matrix. This is Dyson's threefold way.

2.2.1 Real and quaternionic structures

We have shown that there are three symmetry classes. We will now show to which class (real, complex or quaternionic) a Hamiltonian belongs based on the existence of and behaviour under time reversal symmetry. This will be the topic of this subsection. First we need to discuss real and quaternionic structures on complex vector spaces.

Definition 2.2.10 (Real structure on complex vector space)**.** Let V be a complex vector space, then a *real structure* is a complex antilinear map $K : V \to V$ such that $K^2 = 1$.

We denote with $V_+ = \{v \in V \mid K(v) = v\}$ the set of *real vectors* and with $V_- = \{v \in V \mid K(v) = -v\}$ the set of *imaginary vectors*.

The set of all real structures on V is denoted by $\mathbb{R}\mathrm{Str}(V)$.

The map K can be thought of as a complex conjugation.

Example 2.2.11. Let $V = \mathbb{C}$ with underlying real space $V_\mathbb{R} = \mathbb{R}^2$, then for any $\varphi \in [0, \pi)$, the map

$$K : x + iy \mapsto e^{i\varphi}(x - iy),$$

defines a real structure on V. In this case V_+ is a line with angle φ to the x-axis, hence $\varphi \in [\pi, 2\pi)$ does not result in any new real structures. We have $\mathbb{R}\mathrm{Str}(\mathbb{C}) = U(1)/O(1)$. △

The same holds for all $n > 0$,

$$\mathbb{R}\mathrm{Str}(\mathbb{C}^n) = U(n)/O(n).$$

Similar to how a complex vector space is a real vector space with a module for the complex numbers, that is, a method of multiplying vectors with a complex number, we can define a quaternionic vector space as follows.

Definition 2.2.12 (Quaternionic vector space)**.** A *quaternionic vector space* is a real vector space V together with three linear maps $I, J, K : V \to V$ satisfying the quaternion relations.

Multiplying an element $v \in V$ with a quaternion number is then defined as follows

$$(x_0 + \mathfrak{i}x_1 + \mathfrak{j}x_2 + \mathfrak{k}x_3)v := x_0 v + x_1 Iv + x_2 Jv + x_3 Kv,$$

where $\mathfrak{i}, \mathfrak{j}$ and $\mathfrak{k}$ are the generators of the quaternion algebra, satisfying the quaternion relations

$$\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = \mathfrak{ijk} = -1.$$

Definition 2.2.13 (Quaternionic structure on complex vector space)**.** Let V be a complex vector space, then a *quaternionic structure* is a complex antilinear map $K : V \to V$ such that $K^2 = -1$.

This makes $V_{\mathbb{R}}$ into a quaternionic vector space by defining I multiplication with $\sqrt{-1}$ and $J = KI$.

The space of all quaternionic structures on a complex vector space is denoted by $\mathbb{H}\mathrm{Str}(V)$.

The space of quaternionic structures is the symmetric space

$$\mathbb{H}\mathrm{Str}(\mathbb{C}^{2n}) = U(2n)/USp(2n).$$

Here $USp(2n) = Sp(2n) \cap U(2n)$ is the compact symplectic group.

Let us now classify Hamiltonians based on the time reversal symmetries. By considering a single block of the Hamiltonian if needed, we may assume that the entire Hamiltonian has the same time reversal behaviour. Suppose first that the Hamiltonian has time reversal symmetry, and let $\overline{T}$ be the time reversal operator. Then, $\overline{T}$ must satisfy $\tau(\overline{T}) = \varphi(\overline{T}) = -1$ and the group $M_2 = \{1, \overline{T}\}$ is a subset of G. Restrict the homomorphism $\rho : G \to \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ to a homomorphism $\rho|_{M_2} : M_2 \to \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$. From Example 2.2.3 it follows that there are precisely two φ-twisted $U(1)$ central extensions of M_2, namely $M_2^\tau = M_2^\pm$. This results in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & M_2^\pm & \xrightarrow{\tilde{\pi}} & M_2 & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho^\tau|_{M_2^\pm}} & & \downarrow{\scriptstyle \rho|_{M_2}} & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & \mathrm{Aut}_{qtm}(\mathcal{H}) & \xrightarrow{\pi} & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) & \longrightarrow & 1 \end{array}$$

Let $K = \rho^\tau|_{M_2^\pm}(T)$, where T is a chosen element in the pre-image $\tilde{\pi}^{-1}(\overline{T})$, as in Example 2.2.3. Then K is anti-linear ($\varphi(K) = -1$) and $K^2 = \rho^\tau(T^2) = \rho^\tau(\pm 1) = \pm 1$. Therefore, there are two possibilities. For M_2^+, K is a real structure on $\mathcal{H}$, $D_\lambda = \mathbb{R}$ and since the Hamiltonian H is assumed to be compatible with the dynamics, H commutes with K, so that H has real entries. For M_2^- on the other hand, K is a quaternionic structure on $\mathcal{H}$, $D_\lambda = \mathbb{H}$ and H has quaternionic matrix entries.

Finally, if the Hamiltonian has no time reversal symmetry, this results in the simple commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & U(1) & \longrightarrow & 1 & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho^\tau|_{U(1)}} & & \downarrow & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & \mathrm{Aut}_{qtm}(\mathcal{H}) & \xrightarrow{\pi} & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) & \longrightarrow & 1 \end{array}$$

No additional structure is placed on $\mathcal{H}$, hence this is the unitary ($\beta = 2$, complex) case.

Remark 2.2.14. In the literature, one finds associated to each $D = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, a symmetric space. This is the space where the time evolution operator $U(t)$ lives, and is the unitary part of $\mathrm{Aut}_{qtm}(\mathcal{H})$. In the case of $D = \mathbb{C}$, this is $U(N)$, and for $D = \mathbb{R}$ and $D = \mathbb{H}$ the above discussion tells us that the spaces are respectively $U(N)/O(N)$ and $U(2N)/USp(2N)$. The result is summarized in Table 1.

	T^2	D	β	**Cartan label**	**Symmetric space**	G^*/K
Unitary		$\mathbb{C}$	2	A	$U(N)$	$SL(N,\mathbb{C})/SU(N)$
Orthogonal	$+1$	$\mathbb{R}$	1	AI	$U(N)/O(N)$	$SL(N,\mathbb{R})/SO(N)$
Symplectic	-1	$\mathbb{H}$	4	AII	$U(2N)/USp(2N)$	$SU^*(2N)/USp(2N)$

Table 1: A summary of the three symmetry classes of the threefold way. The first column is the name used in the ensemble names. The second column describes the behaviour under time-reversal symmetry. If it is empty, there is no time-reversal symmetry. A ± 1 indicates that the time reversal operator squares to ± 1. The third column is the field of matrix entries. The fourth column is the Dyson index. Finally, there is the Cartan label of the symmetric space, the symmetric space of compact type, where the time evolution operator lives, and the symmetric space of noncompact type.

To summarize, in this section we have looked at the Hamiltonians of time translation invariant systems. There are three possible behaviours under time reversal. The system could not be invariant under time reversal at all, it could be invariant because all states are time reversal invariant, or it could be invariant because states come in pairs (Kramer's pairs) which are mapped to each other under time reversal. For the latter two this resulted in restrictions on the Hamiltonians. We were able to deduce that the Hamiltonians must belong to the Hermitian matrices, symmetric matrices or quaternion Hermitian matrices, respectively.

Random matrix theory takes this threefold classification Eq. (2.14), but acknowledges that one does not know the precise form of the Hamiltonian $H \in \mathcal{E}$, where $\mathcal{E}$ is given by Eq. (2.13), and instead places a probability measure on $\mathcal{E}$. It is always assumed that the Hamiltonian consists of a single block, that is, $\mathcal{E} = \mathrm{Herm}_N(D)$. Results of random matrix theory are probability density functions and expectation values of such ensembles. If one chooses the measure on $\mathcal{E}$ to maximize entropy, or alternatively assumes independent matrix entries, one obtains the important Gaussian ensembles. This will be the subject of Section 2.4, but first we consider the circular ensembles.

2.3 The Circular ensembles

Consider an irregular cavity, where plane waves are entering and exiting through a lead which only permits N distinct wave states. Denote the amplitudes of the N incoming states with $\vec{I}$ and the outgoing with $\vec{O}$. Assume there is no absorption or dissipation. Then, the scattering matrix S is such that

$$\vec{O} = S\vec{I}.$$

All such S are unitary, since flux conservation requires $|\vec{I}|^2 = |\vec{O}|^2$, hence $\langle S\vec{I}, S\vec{I}\rangle = \langle \vec{I}, \vec{I}\rangle$. If there is no time-invariance, this is the only constraint. Otherwise, either S is symmetric, $S = S^T$, for $\beta = 1$ or S is self-dual, $S = S^R$ for $\beta = 4$ (see Beenakker [23] or Forrester [24]). We will denote the set of symmetric, unitary or self-dual matrices by

$$\mathrm{Sym}_N(D) = \begin{cases} S \in U(N) \text{ such that } S^T = S & \text{if } D = \mathbb{R}, \\ S \in U(N) & \text{if } D = \mathbb{C}, \\ S \in U(N) \text{ such that } S^R = S & \text{if } D = \mathbb{H}. \end{cases}$$

Definition 2.3.1 (Circular ensembles)**.** The *circular ensembles* are the spaces $\mathrm{Sym}_N(D)$ with *uniformly* distributed measure $\mathrm{d}\mu(S)$, where $D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. With uniformly distributed we mean satisfying $\mathrm{d}\mu(S) = \mathrm{d}\mu(USV)$, for all $U, V \in U(N)$ for which $S' = USV$ is again an element of $\mathrm{Sym}_N(D)$. This implies $V = U^T$ for $D = \mathbb{R}$ and $V = U^R$ for $D = \mathbb{H}$, so we might also say that the measure $\mathrm{d}\mu(S)$ is invariant under the similarity transformations

$$S \to USU^{-1} \quad (\beta = 1, 4) \qquad S \to VSU \quad (\beta = 2). \tag{2.15}$$

The circular ensembles are called the *Circular Orthogonal Ensemble (COE)* for $D = \mathbb{R}$, *Circular Unitary Ensemble (CUE)* for $D = \mathbb{C}$ and *Circular Symplectic Ensemble (CSE)* for $D = \mathbb{H}$.

The transformation property Eq. (2.15) of the measure can be used to transform the space $\mathrm{Sym}_N(D)$ to eigenvalue space, by diagonalizing $S \in \mathrm{Sym}_N(D)$. This is the content of the following two lemmas.

Lemma 2.3.2. *Let S be a unitary and symmetric matrix. Then, there exists an orthogonal matrix R such that $S = R^{-1}ER$, where E is a diagonal with complex entries $e^{i\theta_j}$.*

Proof. Write $S = X + iY$, with $X = (S + S^*)/2$ and $Y = (S - S^*)/2i$ real symmetric matrices. Here, M^* denotes the complex conjugation of matrix entries $(M^*)_{ij} = \overline{M_{ij}}$, M^T is the matrix transpose and $M^\dagger = (M^*)^T$ for some matrix M. Then

$$1 = S^\dagger S = X^2 + Y^2 + i(XY - YX),$$

so that X and Y commute. This implies that X and Y can be simultaneously diagonalized by some real orthogonal matrix R. With $E := R(X + iY)R^T$ the result follows. □

Lemma 2.3.3. *Let S be a unitary and self-dual ($S = S^R$) quaternion matrix. Then, there exists a symplectic matrix B such that $S = B^{-1}EB$, where E is a diagonal with complex entries $e^{i\theta_j}$.*

Proof. The proof is the same as for Lemma 2.3.2, with the transpose replaced by the dual. □

In the following proposition we calculate the Jacobian of the transformation from $\mathrm{Sym}_N(D)$ to eigenvalue space.

Proposition 2.3.4. *The probability density function $P_C(\theta_1, \ldots, \theta_N)$ of obtaining the eigenvalues $e^{i\theta_j}$, $1 \le j \le N$, for the Circular ensembles, $\beta = 1, 2, 4$ is given by*

$$P_C(\theta_1, \ldots, \theta_N) = C_{N\beta} \prod_{1 \le i < j \le N} |e^{i\theta_j} - e^{i\theta_i}|^\beta, \tag{2.16}$$

where $C_{N\beta}$ is a normalization constant.

Proof. In this proof we follow Mehta [9, Lemma 10.4.4]. We only consider the $\beta = 1$ case. The cases $\beta = 2$ and $\beta = 4$ are similar.

Circular Orthogonal Ensemble Let $\beta = 1$, then $S = R^{-1}ER$.

Since $d(e^{i\theta}) = e^{i\theta} i\mathrm{d}\theta$, $dE = iEd\theta$. Also, because of $RR^T = 1$, $-R(dR)^T = (dR)R^T =: dA$, which is real anti-symmetric. Furthermore, differentiating $S = R^{-1}ER$ results in

$$RdSR^{-1} = -dAE + iEd\theta + EdA. \tag{2.17}$$

Let F be a diagonal matrix with $e^{i\theta_j/2}$ on the diagonal, then $F^2 = E$. With $U := FR$, S is written as $S = U^T U$ and dM defined by $dS = iU^T dM\ U$ is Hermitian. We need to calculate dM, as $d\mu(S) = \frac{1}{C}(dM)$. Eq. (2.17) becomes

$$\begin{aligned} iFdMF &= -dAF^2 + iF^2 d\theta + F^2 dA, \\ dM &= d\theta + i(F^{-1}dAF - FdAF^{-1}). \end{aligned}$$

The il component of $F^{-1}dAF$ is given by

$$(F^{-1}dAF)_{il} = \sum_{j,k} F^{-1}_{ij} dA_{jk} F^{kl} \delta_{ij}\delta_{kl} = F^{-1}_{ii} dA_{il} F^{ll} = e^{i(\theta_l - \theta_i)/2} dA_{il}.$$

Therefore, the components of dM are

$$\begin{aligned} dM_{ii} &= d\theta_i, \\ dM_{ij} &= 2\sin\left(\frac{\theta_i - \theta_j}{2}\right) dA_{ij}. \end{aligned} \tag{2.18}$$

This results in the measure

$$d\mu(S) = \prod_j d\theta_j \prod_{i<j} \left|2\sin\left(\frac{\theta_i - \theta_j}{2}\right)\right| dA_{ij} = \prod_j d\theta_j \prod_{i<j} |e^{i\theta_j} - e^{i\theta_i}| dA_{ij}.$$

Integrate over dA_{ij} to get the PDF for Circular Orthogonal Ensemble. □

2.3.1 Weyl integration

For the case $\beta = 2$, $\mathrm{Sym}_N(\mathbb{C}) = U(N)$ is a Lie Group, and the invariant measure is the Haar measure of the unitary group. This allows us to make use of Weyl Integration. The Weyl integral formula reduces an integral over a Lie group G with integrand a class function, to an integral over the maximal torus T.

Theorem 2.3.5 (Weyl Integral Formula)**.** *Let G be a compact connected Lie Group with maximal torus T. Let $f : G \to \mathbb{C}$ be a continuous function, then*

$$\int_G f(g)\mathrm{d}g = \frac{1}{|W|} \int_T J(t)|_{\mathfrak{g}/\mathfrak{t}} \int_{G/T} f(gtg^{-1})\,\mathrm{d}(gT)\,\mathrm{d}t,$$

where $|W|$ is the order of the Weyl group and $J(t) = \det\big(\mathrm{Ad}(t^{-1})|_{\mathfrak{g}/\mathfrak{t}} - I\big)$. Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{t}$ is the Lie algebra of T and Ad *is the adjoint action of G. Furthermore, $J(t) = |\Delta(t)|^2$, where $\Delta(t)$ is the Weyl denominator*

$$\Delta(e^H) = \prod_{\alpha \in R^+} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right),$$

with R^+ the set of positive roots.

Group	Haar measure on maximal torus
$U(r)$	$\frac{1}{(2\pi)^r r!}\int_0^{2\pi}\prod_{1\le j<k\le r}\lvert e^{i\theta_j}-e^{i\theta_k}\rvert^2 \mathrm{d}\theta_1\dots\mathrm{d}\theta_r$
$SU(r+1)$	$\frac{1}{(2\pi)^r (r+1)!}\int_0^{2\pi}\prod_{1\le j<k\le r+1}\lvert e^{i\theta_j}-e^{i\theta_k}\rvert^2 \mathrm{d}\theta_1\dots\mathrm{d}\theta_r,\ \theta_{r+1}=-\sum_{j=1}^r\theta_j$
$SO(2r+1)$	$\frac{2^{r^2}}{(2\pi)^r r!}\int_0^{2\pi}\prod_{1\le j<k\le r}(\cos\theta_j-\cos\theta_k)^2\prod_{n=1}^r\sin^2\left(\frac{1}{2}\theta_n\right)\mathrm{d}\theta_n$
$USp(2r)$	$\frac{2^{r^2}}{(2\pi)^r r!}\int_0^{2\pi}\prod_{1\le j<k\le r}(\cos\theta_j-\cos\theta_k)^2\prod_{n=1}^r\sin^2\theta_n\mathrm{d}\theta_n$
$SO(2r)$	$\frac{2^{(r-1)^2}}{(2\pi)^r r!}\int_0^{2\pi}\prod_{1\le j<k\le r}(\cos\theta_j-\cos\theta_k)^2\mathrm{d}\theta_1\dots\mathrm{d}\theta_r$

Table 2: Haar measure on the maximal torus after Weyl integration for the classical groups. Table 38 in Hanany and Kalveks [26].

Proof. For the proof we refer to [25, Theorem 11.30 and Proposition 12.24] □

For the Circular unitary ensemble, the density $f = 1$ is trivially invariant under conjugation ($f(gtg^{-1}) = f(t)$), that is, f is a *class function*.

Corollary 2.3.6. *Let $G, T, \Delta, |W|$ and f be as in Theorem 2.3.5. Suppose that f is also a class function, then*

$$\int_G f(g)\mathrm{d}g = \frac{1}{|W|}\int_T |\Delta(t)|^2 f(t)\mathrm{d}t.$$

For later use we have summarized the results for the classical groups in Table 2. It is useful to change variables $x_j = e^{i\theta_j}$, which replaces $\int_0^{2\pi}\mathrm{d}\theta_j$ by a contour integral $-\oint i/x_j \mathrm{d}x_j$. This replacement is summarized in Table 3. The resulting integral can be calculated by taking residues.

For the unitary groups $U(N)$, $\Delta(t)$ is the Vandermonde determinant, as we shall see now.

Second proof of Proposition 2.3.4 for $\beta = 2$. The maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{su}(n+1)$ is the set of diagonal matrices in $\mathfrak{su}(n+1)$,

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_{n+1} \end{pmatrix} \middle| \quad \theta_i \in \mathbb{R},\ \sum_i \theta_i = 0 \right\}.$$

Let E_{ij} be the matrix with a 1 on the i-th row and j-th column, and let H be diagonal, $H = \mathrm{diag}(\lambda_1, \dots, \lambda_{n+1}) \in \mathfrak{t}_{\mathbb{C}}$, then

$$[H, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}.$$

This shows that the roots are the linear functionals α_{ij}, $i \neq j$, which assign to each $H \in \mathfrak{t}_{\mathbb{C}}$ the quantity $\lambda_i - \lambda_j$. We can choose the ones where $j > i$ to form a positive set of roots R^+. Apply Corollary 2.3.6 with $f(t) = 1$ to get the result for the Circular Unitary Ensemble, and with $f(\mathrm{diag}(\lambda_1, \dots, \lambda_{n+1})) = \exp\left(\frac{1}{2}\sum_i \lambda_i^2\right)$ for the Gaussian Unitary Ensemble. □

Group	Haar measure on maximal torus
$U(r)$	$\frac{1}{(2\pi i)^r r!}\oint \prod_{1\le j<k\le r} \lvert x_j - x_k\rvert^2 \prod_{n=1}^{r} \frac{\mathrm{d}x_n}{x_n}$
$SU(r+1)$	$\frac{1}{(2\pi i)^r (r+1)!}\oint \prod_{1\le j<k\le r+1} \lvert x_j - x_k\rvert^2 \prod_{n=1}^{r} \frac{\mathrm{d}x_n}{x_n} \quad x_{r+1} = \prod_{j=1}^{r} x_j^{-1}$
$SO(2r+1)$	$\frac{1}{2^r(2\pi i)^r r!}\oint \prod_{1\le j<k\le r} \frac{(x_j - x_k)^2(1-x_jx_k)^2}{x_j^2x_k^2} \prod_{n=1}^{r} \frac{(1-x_n)(x_n-1)}{x_n}\frac{\mathrm{d}x_n}{x_n}$
$USp(2r)$	$\frac{1}{2^r(2\pi i)^r r!}\oint \prod_{1\le j<k\le r} \frac{(x_j - x_k)^2(1-x_jx_k)^2}{x_j^2x_k^2} \prod_{n=1}^{r} \frac{(1-x_n^2)(x_n^2-1)}{x_n^2}\frac{\mathrm{d}x_n}{x_n}$
$SO(2r)$	$\frac{1}{2^{r-1}(2\pi i)^r r!}\oint \prod_{1\le j<k\le r} \frac{(x_j - x_k)^2(1-x_jx_k)^2}{x_j^2x_k^2} \prod_{n=1}^{r} \frac{\mathrm{d}x_n}{x_n}$

Table 3: Haar measure on the maximal torus after Weyl integration for the classical groups written as contour integrals. Table 39 in Hanany and Kalveks [26]. The contour integrals are over the region where $|x_i| = 1$. Note that this implies that $x_i^* = x_i^{-1}$.

2.3.2 Associated symmetric spaces

The method of Section 2.3.1 only works for the unitary case, because only in this case the integration manifold is a Lie Group. The orthogonal and symplectic integration manifold are a slight generalization to this, and are examples of *symmetric spaces*, and are of the form G/K where G is a Lie group and K is a maximal compact subalgebra. We will discuss symmetric spaces in greater detail in Section 3.1. In this section, we identify the integration manifold of the three circular ensembles (see [18, Section 8.3.1]).

COE Any unitary symmetric matrix S can be written as $S = U^TU$, that is, we have a surjective map $\pi : U(N) \to \mathrm{Sym}_N(\mathbb{R}), U \mapsto U^TU$, with kernel $\ker\pi = \pi^{-1}(\{1\}) = \{R \in U(N) \mid R^TR = 1\} = O(N)$. We conclude that $\mathrm{Sym}_N(\mathbb{R}) \simeq U(N)/O(N)$.

CUE By definition $\mathrm{Sym}_N(\mathbb{C}) = U(N)$.

CSE Any self-dual unitary quaternion matrix S an be written as $S = U^RU$, with U a $2N \times 2N$ unitary matrix, that is, we have a surjective map $\pi : U(2N) \to \mathrm{Sym}_N(\mathbb{H}), U \mapsto U^RU$, with kernel $\ker\pi = \{B \in U(2N) \mid B^RB = 1\} = USp(2N)$. We conclude that $\mathrm{Sym}_N(\mathbb{H}) \simeq U(2N)/USp(2N)$.

We can now write the partition function as

$$Z = \int_{G/K} \mathrm{d}\mu(S),$$

for appropriate G and K.

2.4 The Gaussian ensembles

Random matrix theory is about replacing an unknown Hamiltonian by an ensemble of Hamiltonians with a certain probability measure. In Section 2.2, we showed that it is natural to consider the set $\mathrm{Herm}_N(D)$, with $D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. In this section we provide reasons for choosing the Gaussian probability measure on $\mathrm{Herm}_N(D)$. With the new notation we can define the Gaussian ensembles as follows.

Definition 2.4.1 (Gaussian ensembles)**.** The *Gaussian ensembles* is set of real symmetric ($D = \mathbb{R}, \beta = 1$), Hermitian ($D = \mathbb{C}, \beta = 2$) or self-dual Hermitian ($D = \mathbb{H}, \beta = 4$) $N \times N$ matrices $\mathrm{Herm}_N(D)$ together with probability density function $P_G(H) = \exp\left(-\frac{\beta}{2} \operatorname{tr} H^2\right)$.

The ensembles are named Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE), respectively.

The parameter β is the Dyson index.

There are a number of reasons to choose this particular probability density function. For one, it is the unique probability density function on $\mathrm{Herm}_N(D)$ such that the measure is invariant under automorphisms of $\mathrm{Herm}_N(D)$ and such that the matrix entries are independently distributed. Let us first elaborate on that.

Invariance The probability density function $P_G(H)$ and measure $\mathrm{d}H$ are both separately invariant under conjugation by elements U,

$$H \to UHU^{-1},$$

where U is an element of the orthogonal group $O(N)$ for $D = \mathbb{R}$, the unitary group $U(N)$ for $D = \mathbb{C}$ or the symplectic group $Sp(N)$ for $D = \mathbb{H}$. This follows directly from the properties of the trace. Furthermore, the combination $P_G(H)\mathrm{d}H$ is invariant under the transformation

$$H \to UHU^{-1} + H'$$

where $H' \in \mathrm{Herm}_N(D)$. This is because the factor H' will be absorbed in the normalization.

The measure $\mathrm{d}H$ can be written as

$$\mathrm{d}H = \begin{cases} \prod_{i \le j} \mathrm{d}H_{ij} & \text{for } D = \mathbb{R}, \\ \prod_{i \le j} \mathrm{d}H_{ij}^{(0)} \prod_{i<j} \mathrm{d}H_{ij}^{(1)} & \text{for } D = \mathbb{C}, \\ \prod_{i \le j} \mathrm{d}H_{ij}^{(0)} \prod_{k=1}^{3} \prod_{i<j} \mathrm{d}H_{ij}^{(k)} & \text{for } D = \mathbb{H}, \end{cases} \tag{2.19}$$

where $H_{ij}^{(0)}$ is the real component of H_{ij}, and $H_{ij}^{(k)}$ for $k \ge 1$ are the imaginary or quaternion imaginary parts of H_{ij}. These are all the independent components due to constraints by the Hermitian property.

Independence The entries $H_{ij}^{(k)}$ are statistically independent, that is, there exists functions $f_{ij}^{(0)}$ for $1 \le i \le j \le N$ and functions $f_{ij}^{(k)}$ for $1 \le i < j \le N$ and $1 \le k < \beta$ such that

$$P_G(H) = \prod_{i \le j} f_{ij}^{(0)}\left(H_{ij}^{(0)}\right) \prod_{k=1}^{\beta-1} \prod_{i<j} f_{ij}^{(k)}\left(H_{ij}^{(1)}\right). \tag{2.20}$$

In fact, as mentioned earlier, these two properties do not leave much freedom to choose $P_G(H)$. This is a theorem by Porter and Rosenzweig [27].

Theorem 2.4.2. *Suppose for all $i \leq j$ and for all α, $f_{ij}^{(\alpha)}(H_{ij}^{(\alpha)})$ is a smooth, strictly positive function. Suppose also that $N \geq 2$. Then, the PDF $P_G(H)$ in the definitions of the Gaussian ensembles is of the form*

$$P_G(H) = \exp\bigl(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\bigr), \tag{2.21}$$

for some $a, b, c \in \mathbb{R}$ and $a \geq 0$.

Proof. By diagonalizing H, $P_G(H)$ can be written as a function of the eigenvalues λ_k, $k = 1, \ldots, N$. The function is symmetric in the eigenvalues, because of the remaining freedom of ordering the eigenvalues. Any symmetric function can be written in terms of the first N elementary symmetric functions p_j,

$$p_j = \sum_{k=1}^{N} \lambda_k^j = \operatorname{tr} H^j.$$

We get a result of Weyl [28];

Claim 1. Any conjugation invariant function $f(H)$ can be expressed in terms of the first N powers of the traces of H.

Therefore, there exists some smooth function φ in N variables such that

$$P_G(H) = \exp\bigl(\varphi(\operatorname{tr}(H), \operatorname{tr}\bigl(H^2\bigr), \ldots, \operatorname{tr}\bigl(H^N\bigr))\bigr).$$

Furthermore, there exists smooth $g_{kj}^{(\alpha)}$ such that $f_{kj}^{(\alpha)}(H_{kj}^{(\alpha)}) = \exp\Bigl(g_{kj}^{(\alpha)}(H_{kj}^{(\alpha)})\Bigr)$. The independence then implies

$$P_G(H) = \exp\left(\sum_{\alpha, k \leq j} g_{kj}^{(\alpha)}(H_{kj}^{(\alpha)})\right).$$

Therefore, we look for solutions of

$$\varphi(\operatorname{tr}(H), \ldots, \operatorname{tr}\bigl(H^N\bigr)) = \sum_{\alpha, k \leq j} g_{kj}^{(\alpha)}(H_{kj}^{(\alpha)}), \tag{2.22}$$

which hold for all H.

As H is orthogonal, unitary or symmetric,

$$\operatorname{tr}\bigl(H^2\bigr) = \sum_{i,j} H_{ij} H_{ji} = \sum_{i,j} |H_{ij}|^2,$$

so both $\operatorname{tr}(H^2)$ and $\operatorname{tr}(H)$ are sums over terms containing the same variable H_{ij}. This is not the case for traces of higher powers of H, as is already clear for the third power;

$$\operatorname{tr}\bigl(H^3\bigr) = \sum_{i,j,k} H_{ij} H_{jk} H_{ki}.$$

Therefore, $\operatorname{tr}(H^3)$ cannot be present in φ.

Claim 2. φ does not depend on $\operatorname{tr}(H^3)$. Write $\partial_i\varphi$ for $\partial\varphi\left(\operatorname{tr}(H),\ldots,\operatorname{tr}\left(H^N\right)\right)/\partial\operatorname{tr}(H^i)$. Derive Eq. (2.22) with respect to H_{ij}, then H_{jk} and finally H_{ki}. If the variables H_{ij}, H_{jk}, H_{ki} are all distinct variables, after taking the derivative, the right-hand side will be zero. Therefore,

$$\begin{aligned} 0 &= \left.\frac{\partial^3\varphi\left(\operatorname{tr}(H),\ldots,\operatorname{tr}\left(H^N\right)\right)}{\partial H_{ij}\partial H_{jk}\partial H_{ki}}\right|_{H=0} \\ &= \partial_3\varphi\frac{\partial^3\operatorname{tr}(H^3)}{\partial H_{ij}\partial H_{jk}\partial H_{ki}}, \end{aligned} \tag{2.23}$$

where $\partial_k\varphi := \partial\varphi/\partial\operatorname{tr}\left(H^k\right)$. The fact that this is the only term remaining can be seen as follows. There can only be terms proportional to $\partial_3\varphi$, $\partial_1\varphi\partial_2\varphi$ and $\partial_1^3\varphi$. Other derivatives are still proportional to at least one H_{mn}, which will be set to zero. Note that i,j,k must be all distinct for H_{ij}, H_{jk} and H_{ki} to be distinct variables. In particular, all variables H_{ij} are off diagonal, so that there are no terms proportional to $\partial_1\varphi$. Hence, indeed Eq. (2.23) holds.

There exists choices of i,j,k such that $\partial^3\operatorname{tr}(H^3)/\partial H_{ij}H_{jk}H_{ki}$ is nonzero. Therefore, from Eq. (2.23) we can now conclude that $\partial_3\varphi|_H = 0$ for all H, so φ does not depend on $\operatorname{tr}(H^3)$.

Similarly, one shows that φ does not depend on $\operatorname{tr}\left(H^k\right)$ for all $k = 3,\ldots,N$. The same reasoning can also be applied to show that $\partial_1^k\partial_2^l\varphi = 0$ for $k+l \geq 2$, $k,l \in \mathbb{Z}_{\geq 0}$. Upon Taylor expansion, φ can hence be written as $\varphi\left(\operatorname{tr}(H),\operatorname{tr}(H^2)\right) = a\operatorname{tr}(H^2) + b\operatorname{tr}(H) + c$, where $b = \partial_1\varphi$, $a = \partial_2\varphi$ and $c = \varphi(0,0)$.

The condition on a is to ensure that $P_G(H)$ can be normalized to 1. PDF's are real valued functions, so a, b and c must be real. □

An alternative proof can be found in Mehta [9, Theorem 2.6.3] or Porter and Rosenzweig [27].

Out of these options, the Gaussian PDF $P_G(H)$ is the one with zero mean and unit variance. Therefore, the functions $f_{ij}^{(k)}(z)$ in Eq. (2.20) are as follows

$$f_{ij}^{(k)}(z) = \begin{cases} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right) & \text{if } i = j, \\ \frac{1}{\sqrt{\pi}}\exp(-z^2) & \text{otherwise,} \end{cases}$$

and reflect the fact that diagonal elements occur once, and non-diagonal elements occur twice. Here $1 \leq i \leq j \leq N$ and $0 \leq k \leq \beta$.

The probability density function $P_G(H)$ is also the unique PDF that maximizes entropy, or alternatively, minimizes information, subject to the constraint of zero mean and unit variance [29]. We will outline this reasoning in the remaining of this section. With the amount of information associated to a probability density $P(H)$ we mean

$$I[P(H)] := \int \mathrm{d}H P(H)\log P(H). \tag{2.24}$$

Note that this does depend on the measure $\mathrm{d}H$, which can be constructed as follows. Let x_μ be independent variables for H. For example, for the Hermitian matrices, $x =$

$(H_{11}^{(0)}, \dots, H_{NN}^{(0)}, H_{12}^{(0)}, \dots)$, and the other matrix entries are fixed by the Hermitian property. Let $g_{\mu\nu}$ be such that

$$\mathrm{d}s^2 = \operatorname{tr} \delta H \delta H^\dagger = g_{\mu\nu} \delta x_\mu \delta x_\nu.$$

Then $\mathrm{d}H$ is the induced measure by this metric $g_{\mu\nu}$,

$$\mathrm{d}H = (\det g_{\mu\nu})^{\frac{1}{2}} \prod_\mu \mathrm{d}x_\mu.$$

For the Hermitian matrices we have

$$\mathrm{d}s^2 = \sum_i \left(\delta H_{ii}^{(0)}\right)^2 + 2 \sum_{i<j} \sum_{k=0}^{\beta} \left(\delta H_{ij}^{(k)}\right)^2,$$

so that we obtain Eq. (2.19)

$$\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{ii}^{(0)} \prod_{i<j} \prod_{k=0}^{\beta-1} \mathrm{d}H_{ij}^{(k)}.$$

The goal is now to find the probability distribution $P(H)$ satisfying the given constraints, and minimizing the amount of information $I[P(H)]$. The constraints are expectation values of some property f_i,

$$\langle f_i \rangle = \int f_i(H) P(H) \mathrm{d}H. \tag{2.25}$$

Minimizing $I[P(H)]$ subject to these constraints results in the equation

$$\int \delta P(H) \left(\log P(H) - \sum_i \lambda_i f_i(H) \right) \mathrm{d}H = 0,$$

so that

$$P(H) = \exp\left(\sum_i \lambda_i f_i(H) \right). \tag{2.26}$$

Here the Lagrange multipliers λ_i can be found by filling Eq. (2.26) into Eq. (2.25).

Example 2.4.3 (Circular ensembles)**.** The PDF should be normalized, so the only constraint is $\langle f \rangle = 1$, with $f(H) = 1$. This results in $P(H) = 1$. △

Example 2.4.4 (Gaussian ensembles)**.** The PDF should have total norm 1, zero mean and unit variance,

$$\begin{aligned} \langle 1 \rangle &= 1, \\ \langle \operatorname{tr} H \rangle &= 0, \\ \langle \operatorname{tr} H^2 \rangle &= 1. \end{aligned}$$

This results in the PDF $P_G(H) = \exp(\lambda_1 + \lambda_2 \operatorname{tr} H + \lambda_3 \operatorname{tr} H^2)$, where $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \beta/2$. △

2.4.1 Eigenvalue distribution

In this section, the Jacobian of the transformation from $\mathrm{Herm}_N(D)$ to eigenvalue space is explicitly calculated. This is a diagonalization process, which is just a change of variables, so we can calculate the eigenvalue distribution by calculating the right Jacobian. This results in the following important theorem.

Theorem 2.4.5. *The eigenvalues of an $N \times N$ Gaussian matrix are distributed according to the PDF*

$$P_G(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\frac{1}{2}\beta \sum_{\nu=1}^{N} x_\nu^2\right) \prod_{\alpha<\gamma} |x_\gamma - x_\alpha|^\beta, \tag{2.27}$$

with

$$Z_{N,\beta} = (2\pi)^{N/2} \beta^{-N/2-\beta N(N-1)/4} \prod_{j=1}^{N} \frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}.$$

Here β is the Dyson index*, which corresponds to how many real values are needed to specify an entry of the matrix. For GOE $\beta = 1$. For GUE there is a real and imaginary part, so $\beta = 2$. A quaternion number can be written using 4 real numbers, hence $\beta = 4$ for the Gaussian Symplectic Ensemble.*

Ignoring the product term, this distribution is again Gaussian. The product term, however, has a *repulsive* effect; the probability of finding eigenvalues close to one another goes to zero. This causes the eigenvalues to be distributed approximately uniformly. We will perform a more detailed study in Section 4.6. Therefore, the spectral density is not Gaussian, but semicircular in the large N limit. This is called *Wigner's semicircle law* and will be discussed later in Section 3.2.2. Let us first give a proof of Eq. (2.27).

Partial proof of Theorem 2.4.5. This proof follows Mehta [9, Chapter 3.1].

The calculation of $Z_{N,\beta}$ is postponed until Section 4.2, where we will calculate $Z_{N,\beta}$ for $\beta = 2$ in Eq. (4.25). For the calculation of $Z_{N,\beta}$ in the case $\beta = 1$ and $\beta = 4$, we will refer to Mehta [9, Chapter 3.1].

We will prove Eq. (2.27) for the three cases $\beta = 1, 2, 4$.

Case $\beta = 1$ (GOE). Do a change of variables and express H_{ij} in terms of N eigenvalues λ_ν and $[N(N-1)/2] - N$ other variables p_μ, which are used to specify the orthogonal U in the diagonalization

$$H = U\Lambda U^T. \tag{2.28}$$

Here, Λ is the matrix with the eigenvalues λ_ν on the diagonal. As U does not depend on λ_ν and Λ does not depend on p_μ, the following derivatives hold,

$$\begin{aligned} U^T \frac{\partial H}{\partial \lambda_\nu} U &= U^T U \frac{\partial \Lambda}{\partial \lambda_\nu} U^T U = \frac{\partial \Lambda}{\partial \lambda_\nu}, \\ U^T \frac{\partial H}{\partial p_\mu} U &= U^T \frac{\partial U}{\partial p_\mu} \Lambda + \Lambda \frac{\partial U^T}{\partial p_\mu} U. \end{aligned} \tag{2.29}$$

Furthermore, differentiate $1 = U^T U$ with respect to p_μ,

$$\frac{\partial U^T}{\partial p_\mu} U + U^T \frac{\partial U}{\partial p_\mu} = 0,$$

so that with $S^{(\mu)} := U^T \frac{\partial U}{\partial p_\mu}$, the second relation in Eq. (2.29) becomes

$$U^T \frac{\partial H}{\partial p_\mu} U = S^{(\mu)} \Lambda - \Lambda S^{(\mu)}.$$

In coordinates, Eq. (2.29) is

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \lambda_\nu} U_{j\alpha} U_{k\gamma} = \frac{\partial \Lambda_{\alpha\gamma}}{\partial \lambda_\nu} = \delta_{\alpha\gamma}\delta_{\alpha\nu}, \tag{2.30a}$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial p_\mu} U_{j\alpha} U_{k\gamma} = S^{(\mu)}_{\alpha\gamma}(\lambda_\gamma - \lambda_\alpha). \tag{2.30b}$$

Apart from the double sum, which can be solved by picking some arbitrary ordering, this system of equations can be written in matrix form. The summation is done as follows: First sum over the diagonal elements, then sum over the non-diagonal elements in the upper right triangle, which we need to count twice. For example, for fixed α, γ and ν, Eq. (2.30a) can be written as

$$\begin{pmatrix} \frac{\partial H_{11}}{\partial \lambda_\nu} & \dots & \frac{\partial H_{NN}}{\partial \lambda_\nu} & \frac{\partial H_{12}}{\partial \lambda_\nu} & \dots & \frac{\partial H_{1N}}{\partial \lambda_\nu} & \frac{\partial H_{23}}{\partial \lambda_\nu} & \dots & \frac{\partial H_{(N-1)N}}{\partial \lambda_\nu} \end{pmatrix} \begin{pmatrix} U_{1\alpha}U_{1\gamma} \\ \vdots \\ U_{N\alpha}U_{N\gamma} \\ 2U_{1\alpha}U_{2\gamma} \\ \vdots \\ 2U_{(N-1)\alpha}U_{N\gamma} \end{pmatrix} = \delta_{\alpha\gamma}\delta_{\alpha\nu}$$

If all values of ν, α and γ and Eq. (2.30b) are also included, the entire system of equations Eq. (2.30) can then be written compactly as

$$\begin{pmatrix} \frac{\partial H_{jj}}{\partial \lambda_\nu} & \frac{\partial H_{jk}}{\partial \lambda_\nu} \\ \frac{\partial H_{jj}}{\partial p_\mu} & \frac{\partial H_{jk}}{\partial p_\mu} \end{pmatrix} \begin{pmatrix} U_{j\alpha}U_{j\gamma} \\ 2U_{j\alpha}U_{k\gamma} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\gamma}\delta_{\alpha\nu} \\ S^{(\mu)}_{\alpha\gamma}(\lambda_\gamma - \lambda_\alpha) \end{pmatrix}, \tag{2.31}$$

where each of the entries are entire matrices themselves. The left most matrix actually is the Jacobi matrix, of size $[N + N(N-1)/2] \times [N + N(N-1)/2]$, which can be expressed in terms of determinants and absolute values:

$$J(\lambda, p) \begin{vmatrix} U_{j\alpha}U_{j\gamma} \\ 2U_{j\alpha}U_{k\gamma} \end{vmatrix} = \prod_{\alpha<\gamma} |\lambda_\gamma - \lambda_\alpha| \begin{vmatrix} \delta_{\alpha\gamma}\delta_{\alpha\nu} \\ S^{(\mu)} \end{vmatrix}.$$

In the last step, we could take $(\lambda_\gamma - \lambda_\alpha)$ outside the determinant. Indeed, for $\gamma \neq \alpha$, $\delta_{\alpha\gamma}$ is zero, so $(\lambda_\gamma - \lambda_\alpha)$ is a common factor of this entire column, and it is a property of the determinant that we can take this factor out. The remaining determinants only depend on p_μ, hence

$$J(\lambda, p) = \prod_{\alpha<\gamma} |\lambda_\gamma - \lambda_\alpha| f(p), \tag{2.32}$$

for some f. Finally, integrate over p_μ and rewrite the traces of Eq. (2.21) in terms of sums over eigenvalues λ_ν. This leads to

$$P_G(\lambda_1, \dots, \lambda_N) = \exp\left(-\sum_{\nu=1}^{N} (a\lambda_\nu^2 - b\lambda_\nu - c) \right) \prod_{\alpha<\gamma} |\lambda_\gamma - \lambda_\alpha|.$$

Here, a, b are still arbitrary constants and c is fixed by the normalization condition. We may perform yet another change of variables $x_\nu = \theta_\nu + b/2a$, and then scale x_ν such that for $N = 1$ we get a Gaussian distribution with standard deviation 1. This proves the theorem for $\beta = 1$.

Case $\beta = 2$ (GUE). The case for $\beta = 2$ is analogous, but the real and imaginary parts need to be separated, so instead of Eq. (2.30), we have the following system of equations

$$\begin{aligned}
\sum_{j,k} \frac{\partial H^{(0)}_{jk}}{\partial \lambda_\nu} U^*_{j\alpha} U_{k\gamma} &= \delta_{\alpha\gamma} \sigma^{(0)}_{\alpha\nu}, \\
\sum_{j,k} \frac{\partial H^{(1)}_{jk}}{\partial \lambda_\nu} U^*_{j\alpha} U_{k\gamma} &= \delta_{\alpha\gamma} \sigma^{(1)}_{\alpha\nu}, \\
\sum_{j,k} \frac{\partial H^{(0)}_{jk}}{\partial p_\mu} U^*_{j\alpha} U_{k\gamma} &= S^{(0\mu)}_{\alpha\gamma} (\lambda_\gamma - \lambda_\alpha), \\
\sum_{j,k} \frac{\partial H^{(1)}_{jk}}{\partial p_\mu} U^*_{j\alpha} U_{l\gamma} &= S^{(1\mu)}_{\alpha\gamma} (\lambda_\gamma - \lambda_\alpha),
\end{aligned}$$

for some $\sigma^{(0)}_{\alpha\nu}, \sigma^{(1)}_{\alpha\nu}, S^{(0\mu)}_{\alpha\gamma}$ and $S^{(1\mu)}_{\alpha\gamma}$. The same arguments hold as in the $\beta = 1$ case, but in this case two columns proportional to $(\lambda_\gamma - \lambda_\alpha)$ appear in the right-hand side of Eq. (2.31), instead of only one for the $\beta = 1$ case.

Case $\beta = 4$ (GSE) Similar arguments as in the $\beta = 2$ case apply also here, but with $\beta = 4$. For a more detailed analysis, we refer to Mehta [9, Chapter 3.2]. □

We finally remark that the same calculation can also be done for the Circular ensembles, but here we prefer to use $\lambda_j = e^{i\theta_j}$. This will result in the Jacobian

$$J_\beta(\{\theta_i\}) \propto \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^\beta.$$

The partition function for both Gaussian and Circular ensembles can now be written as

$$Z_{N,\beta} = \int_{\gamma^N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} e^{-\beta V(\lambda_i)} \mathrm{d}\lambda_i, \tag{2.33}$$

where γ is the contour in the complex plane, which is the real line $\mathbb{R}$ for the Gaussian ensembles and the unit circle S^1 for the circular ensembles. The potential $V(\lambda)$ is $\lambda^2/2$ for the Gaussian ensembles and 0 for the Circular ensemble.

3 Transition to chaos

In the previous sections symmetric spaces appeared a number of times. In fact, these spaces are closely related to random matrix theory. We will give an introduction to symmetric spaces in Section 3.1, introduce the Calogero-Sutherland models and it's connection to symmetric spaces. This in turn can be used to discuss Dyson Brownian motion and the Coulomb gas analogy. The language of symmetric spaces and zonal spherical functions, allows us to see Dyson Brownian motion as the transition from diagonal Hamiltonians to the Gaussian ensembles. Therefore, Dyson Brownian motion becomes a method to probe the energy statistics in the transition from integrable (diagonal) to chaotic (Gaussian) Hamiltonians. These intermediate statistics are important for the concept of multifractality.

3.1 Cartan symmetric spaces

In this section we follow the review by Caselle and Magnea [18] closely.

Generally, a symmetric space is of the form G/K, a *coset space*. Here K is a subgroup of G and elements of G/K are sets of the form gK for $g \in G$. There is an action of G on G/K given by $g_1(gK) = (g_1 g)K$.

Suppose G acts on a space V. Then we define the *isotropy subgroup* G_{v_0} as the elements $g \in G$ that leave v_0 fixed, $G_{v_0} = \{g \in G \mid gv_0 = v_0\}$. The *orbit* of G at v_0, denoted by Gv_0, is the set that can be reached from v_0, $Gv_0 = \{gv \mid g \in G\}$. In particular, the orbit of G_{v_0} at v_0 is $G_{v_0}v_0 = \{v_0\}$. If the orbit of a point v_0 is the entire space V, that is, $Gv_0 = V$, we say that G *acts transitively* on V. In this case we have an isomorphism $G/G_{v_0} \simeq V$.

Example 3.1.1. Let $G = SO(3)$, $V = S^2$ be the unit sphere inside $\mathbb{R}^3$, then G acts transitively on V. Let v_0 be the north pole, then the subgroup $K = SO(2)$ leaves v_0 fixed. Therefore, we have the isomorphism $SO(3)/SO(2) \simeq S^2$. △

3.1.1 Lie algebras and root spaces

In this subsection we give a quick recap of Lie algebras and root spaces.

Definition 3.1.2 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$ of characteristic 0, together with a multiplication denoted by the bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying for $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{K}$,

a) $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$,

b) $[X, Y] = -[Y, X]$,

c) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

A Lie algebra $\mathfrak{g}$ generates a Lie group G using the exponential map, $\exp(t^i X_i) \in G$ for $t^i \in \mathbb{K}$, $\{X_i\}$ a basis of the Lie algebra $\mathfrak{g}$.

An *ideal* $\mathfrak{i}$ is a subalgebra which is closed under taking brackets $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. A Lie algebra is called *simple* if it has no ideals other than $\{0\}$ and $\mathfrak{g}$ and *semisimple* if it is a direct sum of simple Lie algebras.

Definition 3.1.3 (Lie algebra representation)**.** A *Lie algebra representation* (ρ, V) of $\mathfrak{g}$ is a vector space V together with a *Lie algebra homomorphism* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, that is, a homomorphism satisfying $\rho([X,Y]) = [\rho(X), \rho(Y)]$. Here $\mathfrak{gl}$ is the space $\mathrm{End}(V)$ of linear maps $V \to V$ with bracket $[\cdot,\cdot]$ the regular commutator bracket $[X,Y] = XY - YX$.

Partial application of $[\cdot,\cdot]$ defines the *adjoint map* $\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}, Y \mapsto [X,Y]$. This is an example of a Lie algebra representation. Suppose $\{X_i\}$ is a basis for $\mathfrak{g}$, then $[X_i, X_j] = C_{ij}^k X_k$ can again be written in terms of this basis. The coefficients C_{ij}^k are called the *structure constants*.

An abelian subalgebra $\mathfrak{h}$ satisfies $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$. A maximal abelian subalgebra is called a *Cartan subalgebra*. If the Lie algebra is semisimple and of rank r, there is a basis $\{H_1, \ldots, H_r\}$ of $\mathfrak{h}$. Furthermore, there exists *root vectors* E_α, labeled by $\alpha = (\alpha_1, \ldots, \alpha_r)$, which are the shared eigenvectors of the H_i in the adjoint representation,

$$[H_i, E_\alpha] = \alpha_i E_\alpha = \alpha(H_i) E_\alpha,$$

where $\alpha : \mathfrak{h} \to \mathbb{K}$ is the functional defined on the basis by $\alpha(H_i) = \alpha_i$ and extended linearly. Such functional is called a *root*. For a general representation (ρ, V) we call such functionals *weights*; a *weight* μ is a linear functional $\mathfrak{h} \to \mathbb{K}$ such that $\rho(H)v = \mu(H)v$, for $H \in \mathfrak{h}, v \in V$. There exists a common eigenbasis of V for $\mathfrak{h}$, and we denote this basis by $|\mu\rangle$, hence

$$\rho(H_i)|\mu\rangle = \mu(H_i)|\mu\rangle = \mu_i|\mu\rangle,$$

where $\mu_i = \mu(H_i)$.

For any root α and weight μ the following holds,

$$\frac{2\alpha \cdot \mu}{\alpha \cdot \alpha} = -(p - q),$$

where p and q are positive integers such that $E_\alpha|\mu + p\alpha\rangle = 0$ and $E_{-\alpha}|\mu - q\alpha\rangle = 0$. This severely limits the possibilities for the adjoint representation, where μ is also a root and allows a complete classification of simple root systems. There are the classical root systems A_n, B_n, C_n and D_n related to the classical Lie groups $SU(n+1), SO(2n+1), Sp(2n)$ and $SO(2n)$. There are also 5 exceptional root systems E_6, E_7, E_8, F_4 and G_2, but we will not consider these in this review.

Let $\mathcal{V}$ be the space spanned by the roots. With the canonical basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$, the root systems are as follows [18, Section 2.7]

A_{n-1} $\mathcal{V}$ is the hyperplane with normal vector $(1, 1, \ldots, 1)$. The root lattice contains $\{e_i - e_j, i \neq j\}$.

B_n $\mathcal{V} = \mathbb{R}^n$ and the roots are $\{\pm e_i, \pm e_i \pm e_j, i \neq j\}$.

C_n $\mathcal{V} = \mathbb{R}^n$ and the roots are $\{\pm 2e_i, \pm e_i \pm e_j, i \neq j\}$.

D_n $\mathcal{V} = \mathbb{R}^n$ and the roots are $\{\pm e_i \pm e_j, i \neq j\}$.

BC_n $\mathcal{V} = \mathbb{R}^n$ and the roots are $\{\pm e_i, \pm 2e_i, \pm e_i \pm e_j, i \neq j\}$.

Here we included BC_n, which belongs to a super-algebra, for later use, however BC_n is not a root system because it contains scalar multiples of e_i other than $-e_i$, namely $2e_i$. It will be used when discussing restricted root systems.

3.1.2 Symmetric spaces

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a lie algebra automorphism (a homomorphism to itself), such that $\sigma^2 = 1$. Such an automorphism is called *involutive*. In the following let $\mathfrak{g}$ always be a compact simple Lie algebra, and σ an involutive automorphism. Define the positive and negative eigenspaces[3]

$$\mathfrak{k} := \{X \in \mathfrak{g} \mid \sigma(X) = X\},$$
$$\mathfrak{p} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\},$$

so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We have the commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \tag{3.1}$$

therefore, $\mathfrak{k}$ is a subalgebra, called a *symmetric subalgebra*, but $\mathfrak{p}$ is not.

By multiplying $\mathfrak{p}$ by i, called the *Weyl unitary trick*, we get a noncompact algebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$. Now $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}^*$. The corresponding *symmetric spaces* are the coset spaces G/K and G^*/K. The group K is called a *symmetric subgroup*.

The involutive automorphism σ can be extended to an automorphism $G \to G$, by defining $\sigma(e^X) = e^{\sigma(X)}$. Then $\sigma(k) = k$ for $k \in K$. Suppose that there are no other elements in G that satisfy $\sigma(g) = g$. It can be shown that $gp\sigma(g^{-1}) \in P$ for $p \in P$. Then this defines an action of G on P, which is transitive. Furthermore, there exists a fixed point $x_0 \in P$ such that $G_{x_0} = K$ so that $G/K \simeq P$ and the tangent space of G/K at the origin is spanned by $\mathfrak{p}$.

Example 3.1.4. Consider the algebra $\mathfrak{so}(3)$ with generators

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

These satisfy the commutation relations $[L_i, L_j] = \varepsilon_{ijk} L_k$, where ε_{ijk} is the Levi-Chivita symbol. The element $\exp(t^i L_i) \in SO(3)$ is the rotation around the axis $\vec{t}$ with angle $\|t\|$. Therefore, the generator L_3 spans the subalgebra $\mathfrak{k} = \mathfrak{so}(2)$ which keeps the north pole fixed

$$\exp\big(t^3 L_3\big) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We see that $SO(3)/SO(2) \simeq S^2$. Furthermore, the coordinates t_1 and t_2 can also be used to move the north pole to any point on the unit sphere, that is, we have the map $\pi : SO(3) \to S^2, g \mapsto g x_0$. Here x_0 is the north pole, so the kernel of π is $SO(2)$, and we establish the isomorphism $SO(3)/SO(2) \simeq S^2$.

The generators L_1 and L_2 do not commute, the subalgebra $\mathfrak{h}$ spanned by L_2 is a maximal abelian subalgebra in $\mathfrak{p}$. To summarize,

$$\mathfrak{k} =< L_3 > \quad \mathfrak{p} =< L_1, L_2 > \quad \mathfrak{h} =< L_2 > \ ,$$

where $< X_1, \ldots, X_N >$ denotes the set generated by linear combinations of $X_1, \ldots, X_N$. The raising and lowering operators E_α need to satisfy $[H_i, E_\alpha] = \alpha_i E_i$, for $i = 1, \ldots, r$,

[3] $\mathfrak{k}$ is the fraktur variant of k.

where r is the rank of $\mathfrak{h}$. In this case $r = 1$ and $H_1 = L_2$, and $E_{\pm 1} = \frac{1}{\sqrt{2}}(-L_1 \pm L_3)$. Indeed

$$[L_2, E_{\pm 1}] = \frac{1}{\sqrt{2}}\left([L_1, L_2] \pm [L_2, L_3]\right) = \frac{1}{\sqrt{2}}\left(L_3 \pm L_1\right) = \pm E_{\pm 1}.$$

Any element $p \in P$ can be written as $p = khk^{-1}x_0$, where $h = e^H$, $H \in \mathfrak{h}$ and $k \in K$. The group K leaves x_0 fixed, hence $p = khx_0$. Usually t^3 is denoted by φ and is called the azimuthal angle, and t^2 is denoted by θ and is called the polar angle.[4] The coordinates $(t^3, t^2) = (\varphi, \theta)$ are called spherical coordinates. △

The statements in the previous example are in fact general; Let $\mathfrak{h} \subset \mathfrak{p}$ be a maximal abelian subalgebra in $\mathfrak{p}$. Then every element $p \in P \simeq G/K$ can be written as $p = khk^{-1}x_0 = khx_0$, where $k \in K$ and $h = e^H$, where $H \in \mathfrak{h}$. The point x_0 is the fixed point of K. Both k and h are not unique, k is defined up to an element in M, h is defined up to an element in M'/M, where M and M' are respectively the centralizer and normalizer of K,

$$M := \{k \in K \mid kHk^{-1} = H, H \in \mathfrak{h}\},$$
$$M' := \{k \in K \mid kHk^{-1} = H', H, H' \in \mathfrak{h}\}.$$

The coordinates $(k(x), h(x))$ are called *spherical coordinates*, $k(x)$ is called the angular coordinate and $h(x)$ is called the spherical radial coordinate.

Alternatively, there is also the *Iwasawa decomposition*. Decompose $\mathfrak{g}$ into the spaces $\mathfrak{k}, \mathfrak{h}$ and $\mathfrak{n}$, where

$$\mathfrak{n} = \bigoplus_{\alpha \in R+} \mathfrak{g}_+^{(\alpha)},$$

is the sum of root spaces of positive roots, with $\mathfrak{g}_+^{(\alpha)}$ the space generated by E_α. Then the Iwasawa decomposition is $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{k}$. Let N be the Lie group generated by $\mathfrak{n}$, then the Iwasawa decomposition of G is $G = NHK$.[5]

To any compact symmetric space there is also an associated noncompact symmetric space G^*/K.

Example 3.1.5. Let $\mathfrak{g} = \mathfrak{so}(3)$, $\mathfrak{k} = \mathfrak{so}(2)$, with generators L_1, L_2 and L_3 as in Example 3.1.4. We have

$$\exp\left(it^1 L_1\right)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cosh(t^1) & \sinh(t^1)\\ 0 & \sinh(t^1) & \cosh(t^1)\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\ \sinh(t^1)\\ \cosh(t^1)\end{pmatrix},$$
$$\exp\left(it^2 L_2\right)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\cosh(t^2) & 0 & \sinh(t^2)\\ 0 & 1 & 0\\ \sinh(t^2) & 0 & \cosh(t^2)\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sinh(t^1)\\ 0\\ \cosh(t^1)\end{pmatrix},$$

so this is a two sheeted hyperboloid H^2 as the points satisfy

$$x^2 + y^2 - z^2 = -1.$$

△

Note that the sphere $S^2 \simeq SO(3)/SO(2)$ has constant positive curvature and the hyperboloid $H^2 \simeq SO(3)^*/SO(2)$ has constant negative curvature.

[4] This is the ISO convention commonly used in physics. In mathematical texts the meaning of φ and θ is often switched.

[5] In the literature one usually finds the notation $G = KAN$

3.1.3 Curvature

It is possible to define a metric on a Lie algebra $\mathfrak{g}$. This is used to define a metric on the symmetric space G/K, which is in turn needed to calculate the Jacobian. The metric is defined using the Killing form. In terms of the basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$ this is

$$g_{ij} = K(X_i, X_j) = \operatorname{tr}(\operatorname{ad} X_i, \operatorname{ad} X_j) = C^r_{is} C^s_{jr}.$$

Example 3.1.6. The metric on $\mathfrak{so}(3)$ is

$$g_{ij} = C^r_{is} C^s_{jr} = \varepsilon_{isr}\varepsilon_{jrs} = -2\delta_{ij}.$$

△

Note that the metric is negative definite. In fact, a theorem by Weyl states that a simple Lie group is compact if and only if the Killing form is negative definite. We will typically add a minus sign to make the metric positive definite. The metric can be extended to the symmetric space G/K. At the identity the metric $g_{ij}(I)$ is defined using the restriction of g_{ij} to the tangent space $\mathfrak{p}$. For other points $M \in G/K$ we can apply a group transformation to get back to the identity, and define

$$g_{rs}(M) = g_{ij}(I) \frac{\partial x^i(I)}{\partial x^r(M)} \frac{\partial x^j(I)}{\partial x^s(M)}.$$

Example 3.1.7. The metric of $SO(3)/SO(2)$ at (φ, θ) is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{3.3}$$

The metric on $SO(3)^*/SO(2)$ at (φ, θ) is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}. \tag{3.4}$$

△

It can be shown, for example in Chapter IV of Helgason [30], that the Riemann curvature tensor in the basis $\{X_i\}$ is given by

$$R^n_{ijk} = C^n_{im} C^m_{jk}.$$

Using the Jacobi identity, this results in

$$\begin{aligned} R_{ijkl} = g_{im} R^m_{jkl} &= C^r_{is} C^m_{jn} C^s_{mr} C^n_{kl} = -C^r_{is} C^m_{jn} (C^s_{kr} C^n_{lm} - C^s_{lr} C^n_{km}) \\ &= -(C^r_{is} C^s_{kr} C^n_{lm} C^m_{jn} - C^r_{is} C^s_{lr} C^n_{km} C^m_{jn}) = -(g_{ik} g_{lj} - g_{il} g_{kj}). \end{aligned}$$

This implies that the symmetric space G/K has constant sectional curvature equal to -1, but can also be chosen $+1$ by multiplying the metric with -1. For the noncompact symmetric space G^*/K, due to the factor i of the Weyl unitary trick, the sectional curvature changes sign when going from G/K to G^*/K. Associated to the same subgroup K there is therefore both a positively and negatively curved symmetric space.

There is also a zero-curvature symmetric space G^0/K, which can be seen as the limit of either G/K or G^*/K where the radius of curvature goes to infinity. We refer to Helgason Chapter V [30] for a more complete discussion of zero-curvature symmetric spaces. The group G^0 is defined as the semi-direct product $G^0 = K \ltimes \mathfrak{p}$, where $g_0 = (k, p) \in G^0$ acts on G^0/K as

$$g_0(x) = \mathrm{Ad}(k)x + p, \quad k \in K, x, p \in G_0/K.$$

In contrast to G/K and G^*/K, G^0/K can be identified with a subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$, instead of the Lie group G.

Example 3.1.8. Any unitary matrix U can be written as $U = \exp(iH)$, where iH is an anti-Hermitian matrix. An arbitrary $N \times N$ matrix M can be written in terms of an Hermitian matrix H' and an anti-Hermitian matrix iH,

$$M = H' + iH \quad \text{where} \quad H' = \tfrac{1}{2}(M + M^\dagger) \quad \text{and} \quad iH = \tfrac{1}{2}(M - M^\dagger).$$

In other words, we have a decomposition of the $N \times N$ matrices $\mathfrak{gl}_N$, into subspaces of anti-Hermitian matrices $\mathfrak{k}$ and Hermitian matrices $\mathfrak{p}$,

$$\mathfrak{gl}_N = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the positive and negative eigenspaces of the involution

$$\begin{array}{rccc} \sigma: & \mathfrak{gl}_N & \to & \mathfrak{gl}_N \\ & M & \mapsto & -M^\dagger \end{array}$$

Now let $G = GL(N, \mathbb{C})$, $K = U(N)$ and $\mathfrak{p}$ be the space of Hermitian matrices. Then we may identify G^0/K with $\mathfrak{p}$, where $G^0 = K \ltimes \mathfrak{p}$ and we have the action of $g_0 = (U, H') \in G^0$ on G^0/K defined by

$$g_0(H) = UHU^{-1} + H' \quad \text{for} \quad H \in G^0/K.$$

This is also precisely the symmetry group of measure $P_G(H)\mathrm{d}H$ of the Gaussian Unitary Ensemble. Therefore, the Gaussian Unitary Ensemble corresponds to the zero-curvature symmetric space associated to $G = GL(N, \mathbb{C})$ and $K = U(N)$. Similar statements can be made for the Gaussian Orthogonal Ensemble and Gaussian Symplectic Ensemble, for which we refer to Chapter 8 of Caselle and Magnea [18]. △

3.1.4 Restricted root systems

Suppose $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Recall that *roots* of $\mathfrak{g}$ relative to $\mathfrak{h}$ are nonzero linear functionals $\alpha \in \mathfrak{h}^*$ such that there exists an $E_\alpha \in \mathfrak{g}$, called a root vector or raising/lowering operator, satisfying

$$[H, E_\alpha] = \alpha(H)E_\alpha \quad \text{for all } H \in \mathfrak{h}. \tag{3.5}$$

A *restricted root system* is a root system of $\mathfrak{g}$ relative to a, not necessarily maximal, abelian subalgebra $\mathfrak{a}$. In the case of symmetric spaces we consider the subset $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ of some Cartan subalgebra $\mathfrak{h}$, which lies in $\mathfrak{p}$. Because the subalgebra $\mathfrak{a}$ need not be maximal, there might be multiple independent root vectors $E_\alpha^1, \ldots, E_\alpha^{m_\alpha}$ satisfying Eq. (3.5) for the same root α. In this case we say that α has multiplicity m_α.

3.2 The Laplacian

When solving the Schrödinger equation for the hydrogen atom, one uses separation of variables and write the solution $\psi(r,\theta,\varphi)$ as $\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, where $R(r)$ is the radial part and $Y(\theta,\varphi)$ the angular part. For the angular part $Y(\theta,\varphi)$, the Schrödinger equation reduces to the Laplace equation in spherical coordinates

$$\Delta_{S^2} Y(\theta,\varphi) = \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y(\theta,\varphi),$$

for which we can find eigenfunctions Y_l^m with eigenvalues $l(l+1)$, $l \in \mathbb{Z}_{\geq 0}$, $m = -l, -l+1, \ldots, l$. These eigenfunctions are called spherical harmonics or zonal spherical functions. There are a couple of things to note about the spherical harmonics. Firstly, they form a basis of the square integrable functions on S^2, $L^2(S^2)$. This is the Peter-Weyl theorem. Consequently, they can be used to do an analog of Fourier transforms on S^2. Secondly, the spherical harmonics are given in terms of the Legendre polynomials $P_l^m(z)$ (See for example [31, p. 391])

$$Y_l^m(\theta,\varphi) \propto P_l^m(\cos\theta)e^{im\varphi}.$$

Finally, the numbers l and m are quantum numbers, related to $L^2 = L_1^2 + L_2^2 + L_3^2$ and L_3 respectively, where L_1, L_2 and L_3 are the angular momentum operators. The operator L^2 is called the *Casimir operator*.

Similar statements as these statements about the eigenfunctions of the Laplace operator on the symmetric space $S^2 \simeq SO(3)/SO(2)$ can be made for all symmetric spaces. This is the subject of this subsection.

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r with basis $\{X_i\}_{i=1}^n$, where n is the dimension of $\mathfrak{g}$. A *Casimir operator* C is a symmetric homogeneous polynomial satisfying

$$[C, X_i] = 0 \quad \text{for all } X_i \in \mathfrak{g}.$$

The Casimir operators are elements of the polynomial algebra in X_i, $\mathbb{K}[X_1, \ldots, X_n]$. The bracket is defined as if $[Z, X] = ZX - XZ$; For C of the form $C = X_{i_1} \ldots X_{i_k}$ and $Y \in \mathfrak{g}$,

$$[C, Y] := X_{i_1} \ldots X_{i_{k-1}}[X_{i_k}, Y] + \cdots + [X_{i_1}, Y]X_{i_2} \ldots X_{i_k}. \tag{3.6}$$

Here we used the identity $[AB, Y] = A[B, Y] + [A, Y]B$ for the regular commutator bracket $[A, Y] = AY - YA$. The definition Eq. (3.6) is extended linearly for general Casimir operators C.

Let g^{ij} be the inverse metric on $\mathfrak{g}$. The simplest quadratic Casimir operator associated with $\mathfrak{g}$ is

$$C = g^{ij}X_iX_j. \tag{3.7}$$

In general, a linear independent basis of Casimir operators consists of completely homogeneous polynomials in X_i, $i = 1, \ldots, n$. We refer the reader to Hall [25, Theorem 9.10] for a proof of this statement. Therefore, as will be explained in Section 4.3, a basis of Casimir operators can be found as follows. Consider the characteristic polynomial

$$\det\left(\sum_{i=1}^n t^i \operatorname{ad}(X_i) - \lambda I_n\right) = \sum_{k=0}^n (-\lambda)^{n-k}\varphi_k(t^i)$$

Here $\varphi_k(t^i)$ are symmetric homogeneous polynomials in t^i. In general there will be r independent coefficients $\varphi_k(t^i)$. Choose the r independent ones and make the substitution $t^i \to X_i$. We remark that the degree 2 term results in Eq. (3.7).

Example 3.2.1. Consider the rank 1 Lie algebra $\mathfrak{so}(3)$. Due to the commutation relations $[L_i, L_j] = \varepsilon_{ijk} L_k$, we see that the matrices of the adjoint representation are given by Eq. (3.2). Let $t = (t^1, t^2, t^3)$ and $L = (L_1, L_2, L_3)$, then the characteristic polynomial is

$$\det(t \cdot L - \lambda I_3) = \begin{vmatrix} -\lambda & t^3 & t^2 \\ -t^3 & -\lambda & t^1 \\ -t^2 & -t^1 & -\lambda \end{vmatrix} = (-\lambda)^3 + t^2(-\lambda)$$

Therefore, $\varphi_1(t) = t^2$ can be used to obtain the Casimir operator

$$C = L_1^2 + L_2^2 + L_3^2.$$

Here we have taken the freedom to rescale with a constant. This Casimir operator is also denoted with L^2. △

Elements X of the Lie algebra are left invariant vector fields, so they define a tangent vector X_g at any point $g \in G$, but are uniquely defined by the value at the identity, $X = X_e \in T_e G$. Tangent vectors are differential operators in the following way. Let f be a smooth function on G, $g \in G$ then X_g is the derivation along a curve through g with tangent vector X_g,

$$X_g(f) = \frac{d}{dt} f(\exp(tX)g)|_{t=0}.$$

Here we chose a natural curve $\gamma(t) = \exp(tX)g$ through g.

This can be extended to the symmetric space G/K in the following way. Let $\pi : G \to G/K, g \mapsto gK$ be the projection map. Then the *pullback* map π^* is defined as

$$\pi^* : C(G/H) \to C(G), f \mapsto f \circ \pi,$$

where $C(G)$ are the continuous functions on G. The map π^* is injective; if $f, g \in G$ are equal at all points $g \in G$, then in particular they are equal at representatives in G/K. The image of π^* are the K-invariant functions on G, $C(G)^K$, on which we can with elements $X \in \mathfrak{g}$. For elements $X \in \mathfrak{k}$, due to the invariant property, $X(f) = 0$.

For example, consider $S^2 \simeq SO(3)/SO(2)$ and the tangent vector $L_1 = x^2\partial_3 - x^3\partial_2$. Let x be local coordinates from the embedding of S^2 in $\mathbb{R}^3$. In these coordinates, the curve $\gamma(t) = \exp(tL_1)x$ becomes

$$\gamma(t) = \exp(tL_1)x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \cos t - x^3 \sin t \\ x^2 \sin t + x^3 \cos t \end{pmatrix},$$

so that, using the chain rule,

$$L_1 f = \left.\frac{\mathrm{d}(x^2 \cos t - x^3 \sin t)}{\mathrm{d}t}\right|_{t=0} \partial_2 f + \left.\frac{\mathrm{d}(x_2 \sin t + x_3 \cos t)}{\mathrm{d}t}\right|_{t=0} \partial_3 f = (x^2\partial_3 - x^3\partial_2) f.$$

Expressed in local coordinates, Casimir operators are called *Laplace operators* and usually denoted by Δ. For the special case Eq. (3.7) it is called the *Laplace-Beltrami operator* Δ_B,

$$\Delta_B f = g^{ij} \nabla_i \nabla_j f = g^{ij}(\partial_i \partial_j - \Gamma_{ij}^k \partial_k) f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x^j} f, \tag{3.8}$$

where $g = \det g_{ij}$ and $\nabla_i x^j = \partial_i x^j + \Gamma^j_{ki} x^k$.

All independent Laplace operators on a symmetric space G/K can be found as follows. First one finds all Casimir operators on $\mathfrak{g}$, and then one notes that for $X \in \mathfrak{k}$, $X(f) = 0$, so to obtain independent operators we might as well set $X_\gamma = 0$ for all $X_\gamma \in \mathfrak{k}$. This results in r independent Laplace operators, where r is the rank of the symmetric space, that is, the number of generators in the maximally commuting subalgebra $\mathfrak{h} \subset \mathfrak{p}$.

Example 3.2.2. With the metric Eq. (3.3), the Laplace-Beltrami operator becomes

$$\Delta_B = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{3.9}$$

The same expression is found when rewriting $L_1^2 + L_2^2 + L_3^2$, with $L_1 = x^2\partial_3 - x^3\partial_2$, etc., in spherical coordinates. This is expected, because $SO(3)/SO(2)$ is of rank 1, so L^2 is the only Casimir operator. The coordinate θ is the radial coordinate of the spherical coordinates $(k, h) = (\varphi, \theta)$. The radial part of Δ'_B is therefore the first term in Eq. (3.9), and ∂^2_φ is the transversal part. △

In general, the radial part Δ'_B of the Laplace-Beltrami operator Δ_B is given by

$$\Delta'_B = \frac{1}{J^{(j)}}\sum_{\alpha=1}^{r'}\frac{\partial}{\partial q^\alpha}J^{(j)}\frac{\partial}{\partial q^\alpha}, \tag{3.10}$$

where $J^{(j)} = \sqrt{|g|}$ and $q = \log h(x)$ are the radial coordinates. This is because for the radial coordinates x_i, the metric g_{ij} is the Kronecker delta δ_{ij}, as the corresponding tangent vectors X_i commute with every other tangent vector in $\mathfrak{p}$, by definition of $\mathfrak{h}$. Let us elaborate on that. Let $H_i \in \mathfrak{h}$, and let $\{H_1, \ldots, H_r, E_{\alpha_1}, E_{-\alpha_1}, \ldots, E_{\alpha_m}, E_{-\alpha_m}\}$ be a basis of $\mathfrak{g}$, where $E_{\pm\alpha_1}, \ldots, E_{\pm\alpha_m}$ are root vectors associated with $\mathfrak{h}$. That is, the following relations hold

$$[H_i, H_j] = 0 \quad \text{and} \quad [H_i, E_{\pm\alpha_k}] = \pm\alpha_i(H_k)E_{\pm\alpha_k} \quad \text{for all } 1 \le i, j \le r, 1 \le k \le m.$$

Then in the adjoint representation, H_i has the form

$$H_i = \operatorname{diag}(\underbrace{0, \ldots, 0}_{r \text{ times}}, \alpha_1(H_i), -\alpha_1(H_i), \ldots, \alpha_m(H_i), -\alpha_m(H_i)),$$

so that the group element e^{tH_i} has the form

$$e^{tH_i} = \operatorname{diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, e^{t\alpha_1(H_i)}, \ldots, e^{-t\alpha_m(H_i)}),$$

and a general element $e^{\vec{t}\cdot\vec{H}} \in H$, where $\vec{t} = (t^1, \ldots, t^r)$ and $\vec{H} = (H_1, \ldots, H_r)$ has the form

$$e^{\vec{t}\cdot\vec{H}} = \operatorname{diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, e^{\vec{t}\cdot\alpha_1}, \ldots, e^{\vec{t}\cdot\alpha_m}),$$

where $\vec{t}\cdot\alpha_j := \sum_i t^i\alpha_j(H_i)$. The same reasoning will apply for restricted root systems, but now the roots α_j are in the restricted root lattice. It can now be seen that the radial part of the metric is the identity, hence Eq. (3.9) reduces to Eq. (3.10).

The Jacobian takes the form [18, Equation (6.30)]

$$J^{(0)}(q) = \prod_{\alpha\in R^+} (\vec{q}\cdot\alpha)^{m_\alpha},$$
$$J^{(-)}(q) = \prod_{\alpha\in R^+} (\sinh(\vec{q}\cdot\alpha))^{m_\alpha},$$
$$J^{(+)}(q) = \prod_{\alpha\in R^+} (\sin(\vec{q}\cdot\alpha))^{m_\alpha},$$

for symmetric spaces of zero (0), negative (−) and positive (+) curvature.

Example 3.2.3. The radial parts of the Laplace-Beltrami operator on S^2 and H^2 are given by

$$\Delta'_B = \frac{1}{\sinh\theta}\partial_\theta \sinh\theta\ \partial_\theta \quad \text{for } H^2,$$
$$\Delta'_B = \frac{1}{\sin\theta}\partial_\theta \sin\theta\ \partial_\theta \quad \text{for } S^2.$$

△

3.2.1 Calogero-Sutherland models

Let us start with the observation that the Vandermonde determinant $\Delta_r(q) = \prod_{i<j}(q^i - q^j)$ satisfies

$$\sum_{i=1}^{r} \frac{\partial^2\Delta_r}{\partial(q^i)^2} = 0.$$

In other words, it is the ground state of the r particle Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{r} p_i^2, \tag{3.11}$$

where $p_i = -i\frac{\partial}{\partial q^i}$. This is the simplest case of the Calogero-Sutherland Hamiltonian,

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{r} p_i^2 + \sum_{\alpha\in R^+} g_\alpha^2 v(q\cdot\alpha). \tag{3.12}$$

Here Eq. (3.11) corresponds to the case where $g_\alpha = 0$ for all $\alpha \in R^+$. The potential v is usually taken one of the following

$$v_I(z) = z^{-2},$$
$$v_{II}(z) = \sinh^{-2}(z),$$
$$v_{III}(z) = \sin^{-2}(z).$$

For these potentials the Calogero-Sutherland Hamiltonian is related to the Laplace-Beltrami operator on a symmetric space with zero curvature for $v = v_I$, negative curvature for $v = v_{II}$ and positive curvature for $v = v_{III}$. Let $\rho \in \mathfrak{h}^*$ be the functional

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+} m_\alpha\alpha,$$

with norm squared

$$\rho^2 = \rho \cdot \rho = \sum_{i=1}^{r} \rho(H_i)\rho(H_i).$$

Furthermore, let $\xi(q)$ be the square root of the Jacobian, $\xi^2(q) = J(q)$. Then the Calogero-Sutherland Hamiltonian is given in terms of the radial part of the Laplace-Beltrami operator by [18, Equation (7.5)]

$$\mathcal{H} = \xi(q)\tfrac{1}{2}(\Delta'_B \pm (-1)\rho^2)\xi^{-1}(q) \quad (- \text{ for II}, + \text{ for III}, \rho = 0 \text{ for I})$$

if and only if the coupling constants g_α are given by

$$g_\alpha = \frac{m_\alpha(m_\alpha + 2m_{2\alpha} - 2)|\alpha|^2}{8}. \tag{3.13}$$

Now indeed $\xi(q)$ trivially is an eigenfunction of the Calogero-Sutherland Hamiltonian with eigenvalue $\pm\rho^2/2$. Furthermore, the Jacobian $J(q)$ is the probability density function of such ground state $\xi(q)$ of the Calogero-Sutherland Hamiltonian.

Given a eigenfunction φ_λ of Δ'_B, with eigenvalue $-\lambda^2 \pm \rho^2$, the function

$$\psi_\lambda(q) := \xi(q)\varphi_\lambda(q) \tag{3.14}$$

is an eigenfunction of the Calogero-Sutherland Hamiltonian with eigenvalue $-\lambda^2/2$. The eigenvalues of Δ'_B will be discussed in Section 3.3.1.

Now suppose for all roots α, $m_\alpha = 2$, and suppose furthermore that 2α is not a root, that is, $m_{2\alpha} = 0$. In this case the coupling constants Eq. (3.13) are all zero, and the Calogero-Sutherland Hamiltonian Eq. (3.12) is a sum of single particle Hamiltonians $\mathcal{H}_0$,

$$\mathcal{H} = \sum_{i=1}^{r} \frac{1}{2}\frac{\partial^2}{\partial(q^i)^2} =: \sum_{i=1}^{r} \mathcal{H}_0. \tag{3.15}$$

3.2.2 Coulomb gas analogy and Fokker-Planck

The Calogero-Sutherland Hamiltonian is a quantum analogy. There is also a classical analogy called the Coulomb gas analogy. Instead writing down a differential equation for the wave amplitude ξ, this should be a differential equation for the probability density function, the square of the wave function. Inspired by Eq. (3.15) let F be the following differential operator

$$F = J\Delta'_B J^{-1}$$

Then of course the Jacobian J is an eigenfunction of F with eigenvalue 0. Similarly as in, Eq. (3.14) for any eigenfunction $\varphi_\lambda(q)$ of the radial part of Laplace-Beltrami operator Δ'_B we get an eigenfunction of F, namely $J(q)\varphi_\lambda(q)$.

We will now give F the interpretation of the Focker-Planck operator, so that the obtain Dyson Brownian motion in the Coulomb gas analogy. Consider the partition function Eq. (2.33)

$$Z_{N,\beta} = \int_{\gamma^N} e^{-\beta W(\{x_i\})} \prod_{i=1}^{N} \mathrm{d}x_i = \int_{\gamma^N} P(\{x_i\}) \prod_i \mathrm{d}x_i,$$

where

$$W(\{x_i\}) = \sum_{i=1}^{N} V(x_i) - \sum_{i<j} \log|x_i - x_j|. \tag{3.16}$$

Then with Eq. (3.10) this results in

$$\begin{aligned} FP &= \sum_{j=1}^{N} \frac{\partial}{\partial x_j} J \frac{\partial}{\partial x_j} J^{-1} P \\ &= \sum_{j=1}^{N} \left(\frac{\partial^2 P}{\partial x_j^2} - \beta \frac{\partial}{\partial x_j} [E(x_j) P] \right), \end{aligned}$$

where $E(x_j) = -\partial W(\{x_j\})/\partial x_j$, which for the Gaussian ensembles is

$$E(x_j) = -x_j + \sum_{i \neq j} \frac{1}{x_j - x_i}.$$

When introducing a fictitious time dependence of P, such that

$$\lim_{t\to\infty} P(\{x_j\}, t) = P(\{x_j\}) = J(\{x_j\}),$$

the equation

$$\frac{\partial P}{\partial t} = FP = \sum_{j=1}^{N} \left(\frac{\partial^2 P}{\partial x_j^2} - \frac{\partial}{\partial x} [\beta E(x_j) P] \right)$$

describes Brownian motion of N particles in 1D with position x_j with drift $\beta E(\lambda_j)$ and diffusion constant 1. This equation is called the Fokker-Planck equation. The analogy with particles in 1D can also already be seen from the partition function Eq. (3.16), which consists of a Coulomb interaction, confined to 2 dimensions, $\sum_{i\neq j} \log|x_i - x_j|$ and a background potential V. Therefore, the eigenvalues of random matrix ensembles can be seen as infinity long charged rods confined to move in 1 dimension. In this analogy, level repulsion is the repulsion of like charges.

For the Gaussian ensembles, this background potential prevents particles from drifting off to infinity. Let us rescale the positions with $\sqrt{N}$: $x \to \sqrt{N}x$, then the new partition function becomes

$$Z_{N,\beta} = \int e^{-N^2 \beta S[x]} \prod_{i=1}^{N} \mathrm{d}x_i,$$

where $S[x]$, which can be seen as the action, is given by

$$S[x] = \frac{1}{N} \sum_{i=1}^{N} V(x_i) - \frac{1}{N^2} \sum_{i<j} \log|x_i - x_j|. \tag{3.17}$$

The parameter β plays the role of inverse temperature. In the limit of $N \to \infty$, only stationary paths of the action S contribute. This allowed Wigner to calculate the asymptotic behaviour of the spectral density $\rho(x)$, the Wigner's semicircle law. We need a lemma to relate the resolvent to the spectral density first.

Lemma 3.2.4. *Let $\delta(x)$ be the Dirac delta. Then*

$$\frac{1}{2\pi i} \lim_{\varepsilon\to 0} \left[\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right] = \delta(x).$$

Proof. This identity is an identity of distributions, and hence only makes sense under the integral sign. Let us integrate the left-hand side over some interval $[a, b]$. Suppose f is a holomorphic function and consider the integral

$$\begin{aligned}\frac{1}{2\pi i}\int_a^b \left[\frac{1}{y-i\varepsilon}-\frac{1}{y+i\varepsilon}\right] f(y)\mathrm{d}y &= \frac{1}{2\pi i}\left[\int_a^b \frac{f(y)}{y-i\varepsilon}\mathrm{d}y + \int_b^a \frac{f(y)}{y+i\varepsilon}\mathrm{d}y\right] \\ &= \frac{1}{2\pi i}\oint_C \frac{f(y)}{y}\mathrm{d}y + \mathcal{O}(\varepsilon),\end{aligned}$$

where C is the rectangle with vertices $(a, \pm\varepsilon), (b, \pm\varepsilon)$. There is an order ε contribution due to the fact that the two sides of C of length 2ε were missing and due to the fact that we shifted the evaluation of f from y to $y + i\varepsilon$. The residue theorem gives

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_a^b \left[\frac{1}{y-i\varepsilon}-\frac{1}{y+i\varepsilon}\right] f(y)\mathrm{d}y = \begin{cases} f(0) & \text{if } 0\in[a,b], \\ 0 & \text{otherwise.}\end{cases}$$

□

From the lemma it also follows that

$$\lim_{\varepsilon\to 0}\operatorname{Im}\frac{1}{x-i\varepsilon} = \pi\delta(x).$$

This implies

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x-i\varepsilon-x_i} = \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i) = \varrho(x). \tag{3.18}$$

Proposition 3.2.5 (Wigner's semicircle law)**.** *The spectral density $\rho(x)$ of a Gaussian ensemble approaches a semi-elliptical shape,*

$$\lim_{N\to\infty}\sqrt{N}\rho\left(\sqrt{N}x\right) = \frac{1}{2\pi}\sqrt{4-x^2}. \tag{3.19}$$

Proof. In this proof we follow [32, Chapter 3.1] and [33, Chapter 1.2]. Note that the action Eq. (3.17) scales with N as $\mathcal{O}(1)$. This allows us to do a semi-classical / saddle point approximation and assume that, as $N \to \infty$, the only contribution to the partition function is from the saddle points. These saddle points satisfy the equations of motion. The equations of motion, obtained by varying the action with respect to x_i, are

$$V'(x_i) = \frac{2}{N}\sum_{i\neq j}\frac{1}{x_i-x_j}, \qquad \text{for } i = 1,\dots,N.$$

Define the *resolvent*, a.k.a. *Green's function*

$$G(x) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{x-x_i}.$$

This is the *Stieltjes transform* of the spectral density $\varrho(x)/N$. The spectral density can be extracted from this using Eq. (3.18),

$$\frac{1}{N}\varrho(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im} G(x-i\varepsilon).$$

Let us compute

$$G(x)^2 + \frac{1}{N}G'(x) = \frac{1}{N^2}\left(\sum_{i,j=1}^{N}\frac{1}{x-x_i}\frac{1}{x-x_j} - \sum_{i=1}^{N}\frac{1}{(x-x_i)^2}\right)$$
$$= \frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{x-x_i}\frac{1}{x-x_j}$$

With

$$\frac{1}{x-x_i}\frac{1}{x-x_j} = \left[\frac{1}{x-x_i} - \frac{1}{x-x_j}\right]\frac{1}{x_i-x_j},$$

we obtain, together with the equations of motions,

$$G(x)^2 + \frac{1}{N}G'(x) = \frac{2}{N^2}\sum_{i=1}^{N}\frac{1}{x-x_i}\sum_{j\neq i}\sum_{1}x_i - x_j = \frac{1}{N}\sum_{i=1}^{N}\frac{V'(x_i)}{x-x_i}.$$

Using the rational function

$$P(x) = \frac{1}{N}\sum_{i=1}^{N}\frac{V'(x)-V'(x_i)}{x-x_i},$$

the equation for $G(x)$ can be written as

$$G(x)^2 + \frac{1}{N}G'(x) = V'(x)G(x) - P(x). \tag{3.20}$$

This equation is called the *Riccati equation*. Let us take the $N\to\infty$ limit. The term $G'(x)/N$ drops out. Let us write $\overline{G}(x) = \lim_{N\to\infty}G(x)$ and similarly define $\overline{P}$, then the differential equation reduces to an algebraic equation,

$$\overline{G}^2 - V'\overline{G} + \overline{P} = 0,$$

with solution

$$\overline{W} = \frac{1}{2}\left(V' - \sqrt{(V')^2 - 4\overline{P}}\right),$$

so the (rescaled) spectral density is given by

$$\varrho(x) = \frac{1}{2\pi}\sqrt{4\overline{P} - (V')^2}.$$

In the Gaussian case $\overline{P} = 1$ and $V' = x$. □

Remark 3.2.6. From the Riccati equation, which is a differential equation for the Green's function, a differential equation for the characteristic polynomial can be derived. Let $\tilde{\varphi}_N(x)$ be the characteristic polynomial of the matrix with eigenvalues x_i,

$$\tilde{\varphi}_N(x) = \prod_{i=1}^{N}(x-x_i), \tag{3.21}$$

then

$$\frac{1}{N}\frac{\partial}{\partial x}\log\tilde{\varphi}_N(x) = \frac{1}{N}\frac{\tilde{\varphi}'_N(x)}{\tilde{\varphi}_N(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{x-x_i} = G(x).$$

Here we remark that this equation holds for eigenvalues x_i such that the action Eq. (3.17) is minimized, and a saddle point approximation of the integral can be done. In other words, we have, up to prefactors,

$$\tilde{\varphi}_N(x) \approx \langle\det(x-H)\rangle$$

This is called *Heine's formula*.

The Riccati equation for $G(x)$ Eq. (3.20) becomes a differential equation for $\tilde{\varphi}_N(x)$,

$$\frac{1}{N^2}\frac{\tilde{\varphi}'_N(x)^2}{\tilde{\varphi}_N(x)^2} + \frac{1}{N^2}\frac{\tilde{\varphi}''_N(x)\tilde{\varphi}_N(x) - \tilde{\varphi}'_N(x)^2}{\tilde{\varphi}_N(x)^2} = \frac{1}{N}V'(x)\frac{\tilde{\varphi}'_N(x)}{\tilde{\varphi}_N(x)} - P(x),$$
$$\frac{1}{N^2}\tilde{\varphi}''_N(x) - \frac{1}{N}V'(x)\tilde{\varphi}'_N(x) - P(x)\tilde{\varphi}_N(x) = 0.$$

For the Gaussian unitary ensemble, upon rescaling, this is the Hermite differential equations, and the solution is

$$\tilde{\varphi}_N(x) = N^{-N/2}H_N(x\sqrt{N}).$$

3.3 Two-matrix model

The Brownian motion of the Coulomb gas is a powerful analogy to understand the eigenvalue statistics of a Random Matrix Ensemble. In the previous sections we have seen how the Wigner-Dyson statistics corresponds to the equilibrium, $t \to \infty$ limit, of the Coulomb gas. We will now introduce the two matrix model, which will allow us to extend the analogy to finite t, and explore a transition from Poisson statistics to Wigner-Dyson statistics.

3.3.1 Zonal spherical functions

This subsection is based on Chapter 6 of Caselle and Magnea [18] and Ben Saïd and Ørsted [34]. The Fokker-Plank equation for Dyson Brownian motion could be written as a conjugation of the radial part of the Laplace-Beltrami operator Δ'_B. In order to explore the Brownian at finite times t, we will now consider the eigenfunctions of the radial part of the Laplace-Beltrami operator.

Let $\mathcal{D}(G/K)$ be the G-invariant differential operators on G/K, and consider the eigenvalue equation for $D \in \mathcal{D}(G/K)$

$$D\varphi_\lambda(x) = \gamma_D(\lambda)\varphi_\lambda(x) \quad \text{for all } x \in G/K,$$

where $\lambda \in \mathfrak{h}^*$ is a label and $\gamma_D(\lambda)$ is the eigenvalue. Because the Casimir operators form a commutative algebra, the Laplace operators have common eigenfunctions, bi-invariant under the subgroup K, $\varphi_\lambda(kxk') = \varphi_\lambda(x)$ for $x \in G/K, k, k' \in K$. Such eigenfunctions, satisfying also $\varphi_\lambda(e) = 1$ are called zonal spherical functions.

Example 3.3.1. The eigenfunctions of the Laplace operator $L^2 = L_1^2 + L_2^2 + L_3^2$ on $SO(3)/SO(2)$ are the Legendre polynomials $P_l(\cos\theta)$, with eigenvalue $-l(l+1)$,

$$L^2 P_l(\cos\theta) = -l(l+1)P_l(\cos\theta).$$

Note that these functions do not depend on the angular coordinate φ, and hence are invariant under the rotations $K = SO(2)$ which keep the north pole fixed. △

Due to the invariance property $\varphi_\lambda(kxk') = \varphi_\lambda(x)$, the zonal spherical functions only depend on the radial coordinate h. To define a radial coordinate h for $x \in G$, write x in the Iwasawa decomposition, $x = nhk$ for $h \in H, k \in K$ and $n \in$, and define $H(x) \in \mathfrak{h}$ to be such that $h = \exp(H(x))$. There is an integral representation of the φ_λ due to Harish-Chandra [35].

Theorem 3.3.2. *All spherical functions on G are given by*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)H(kg)}\mathrm{d}k,$$

where λ runs through $\mathfrak{h}^*$. The vector ρ is the half sum of positive roots,

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+} m_\alpha \alpha.$$

Furthermore, $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = \omega\mu$ for some $\omega \in W$ in the Weyl group.

The eigenvalues of the zonal spherical functions for the radial part of the Laplace-Beltrami operator are given by

$$\Delta'_\lambda \varphi_\lambda = (-\lambda^2 \pm \rho^2)\varphi_\lambda,$$

where $\pm$ is a $+$ for positively curved spaces and an $-$ for negatively curved spaces. Using this formula, the eigenvalues of the Calogero-Sutherland Hamiltonian Section 3.2.1 can now also be given, which justifies the assumption about the eigenvalues of Δ'_B made in Eq. (3.14).

Example 3.3.3. For the positively curved symmetric space $SO(3)/SO(2)$, there is one short root of length 1, so $\rho^2 = 1/4$, hence the eigenvalues are $-\lambda^2 + 1/4 = -l(l+1)$ and we conclude that $\lambda = l + 1/2$. △

The Dyson Brownian motion was used by Dyson for the Gaussian ensembles, however, the current discussion did not yet consider the zero-curvature case. We will therefore now introduce a radius of curvature and consider the limit to infinity.

Definition 3.3.4 (Deformation of $\mathfrak{g}$)**.** Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Lie Algebra and Cartan decomposition. Then define $\mathfrak{g}_\varepsilon$ as the set $\mathfrak{g}$ together with the Lie bracket $[\cdot,\cdot]_\varepsilon$ given by

$$\begin{aligned}
[X, X']_\varepsilon &:= [X, X'] \quad \text{for} \quad X, X' \in \mathfrak{k} \\
[Y, Y']_\varepsilon &:= \varepsilon^2 [Y, Y'] \quad \text{for} \quad Y, Y' \in \mathfrak{p} \\
[X, Y]_\varepsilon &:= [X, Y] \quad \text{for} \quad X \in \mathfrak{k}, Y \in \mathfrak{p}
\end{aligned}$$

This deformation is therefore a deformation of the structure constants. As the metric is defined using structure constants for vectors in $\mathfrak{p}$, one can see that indeed sectional curvature is rescaled by a factor of ε^2. The following lemma, due to Ben Saïd and Ørsted [34], states how the roots and root spaces are rescaled.

Lemma 3.3.5. *Let $\mathfrak{g}_\varepsilon$ as in Definition 3.3.4, $\varepsilon > 0$. Let $R(\mathfrak{g}, \mathfrak{h})$ be the set of roots in $\mathfrak{g}$ for the Cartan subalgebra $\mathfrak{h}$. Finally, denote with $\mathfrak{g}^{(\alpha)}$ the root space of the root α, that is, the space generated by the root vectors $\{E_{\pm\alpha}\}$. Then,*

a) *If $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, then $\varepsilon\alpha \in R(\mathfrak{g}_\varepsilon, \mathfrak{h})$.*

b) *Given a root $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, the root space $\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)}$ is given by*

$$\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)} = \{\varepsilon X + Y \mid X + Y \in \mathfrak{g}^{(\alpha)} \text{ where } X \in \mathfrak{k}, Y \in \mathfrak{p}\}$$

Proof. Let σ be the Cartan involution, which splits $\mathfrak{g}$ into positive and negative eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$. Then any element Z in the root space $\mathfrak{g}^{(\alpha)}$ can be decomposed using σ

$$Z = \frac{1}{2}(Z + \sigma(Z)) + \frac{1}{2}(Z - \sigma(Z)) =: X + Y \in \mathfrak{k} \oplus \mathfrak{p}.$$

By definition of roots, $[H, Z] = \alpha(H)Z$ for $\alpha \in R(\mathfrak{g}, \mathfrak{h})$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, and because $H \in \mathfrak{p}$ we can conclude from $[H, X + Y] = \alpha(H)(X + Y)$ that

$$[H, X] = \alpha(H)Y, \quad [H, Y] = \alpha(H)X.$$

We need to show that $[H, \varepsilon X + Y]_\varepsilon = \varepsilon\alpha(H)(\varepsilon X + Y)$. Indeed,

$$[H, \varepsilon X + Y]_\varepsilon = [H, \varepsilon X]_\varepsilon + [H, Y]_\varepsilon = [H, \varepsilon X] + \varepsilon^2[H, Y] = \varepsilon\alpha(H)(Y + \varepsilon X).$$

□

Let G_ε be the Lie group with Lie algebra $\mathfrak{g}_\varepsilon$, and $\Delta'_{B,\varepsilon}$ be the Laplace-Beltrami operator on G_ε/K. It's limit is given by the following lemma.

Lemma 3.3.6. *Let $\Delta'_{B,0}$ be defined as*

$$\Delta'_{B,0} := \frac{1}{J^{(0)}} \sum_{\alpha=1}^{r'} \frac{\partial}{\partial q^\alpha} J^{(0)} \frac{\partial}{\partial q^\alpha},$$

where $q = \log h(x)$ are the radial coordinates and $J^{(0)}(q) = \prod_{\alpha \in R^+} (\vec{q} \cdot \alpha)^{m_\alpha}$. Then the following limit holds

$$\lim_{\varepsilon \to 0} \varepsilon^2 \Delta'_{B,\varepsilon} = \Delta'_{B,0}.$$

Proof. For the proof we refer to Ben Saïd and Ørsted [34]. □

The main theorem of this section is an integral representation of the zonal spherical functions for the zero-curvature symmetric spaces, that is, for the limit $\varepsilon \to 0$.

Theorem 3.3.7. *Let $\varepsilon > 0$, $g_\varepsilon = k \exp(\varepsilon X) \in G$ for some fixed $k \in K$ and $X \in \mathfrak{p}$. Define*

$$\psi_\lambda(X) := \lim_{\varepsilon \to 0} \varphi_{\lambda/\varepsilon}(g_\varepsilon).$$

Then this limit $\psi_\lambda(X)$ exists and has integral representation the Harish-Chandra-Itzykson-Zuber integral

$$\psi_\lambda(X) = \int_K e^{iK(\mathrm{Ad}(k)X, A_\lambda)} \mathrm{d}k,$$

where $K(\cdot, \cdot)$ is the Killing form, and A_λ is the algebra element determined by $K(A_\lambda, H) = \lambda(H)$.

Furthermore, $\psi_\lambda(X)$ are the eigenfunctions of $\Delta'_{B,0}$ with eigenvalue $-\lambda^2$,

$$\Delta'_{B,0}\psi_\lambda(X) = -\lambda^2\psi_\lambda(X).$$

Proof. This proof is based on the proof given by Ben Saïd and Ørsted [34]. Since $\varphi_\lambda(x)$ is K invariant, we may assume $g_\varepsilon = \exp(\varepsilon X)$. Because of the Iwasawa decomposition, $H(gk^{-1}) = H(g)$ for $k \in K$. Therefore, $H(k\exp(\varepsilon X)) = H(\exp(\varepsilon \mathrm{Ad}(k)X))$, where $\mathrm{Ad}(k)X = kXk^{-1}$. By definition of A_λ, $iK(\mathrm{Ad}(k)X, A_\lambda) = i\lambda(\mathbb{P}(\mathrm{Ad}(k)X))$, where $\mathbb{P} : \mathfrak{p} \to \mathfrak{h}$ is the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{h}$. It remains to show that

$$\lim_{\varepsilon\to 0} \frac{\lambda}{\varepsilon} H(\exp(\varepsilon \mathrm{Ad}(k)X)) = \lambda \mathbb{P}(\mathrm{Ad}(k)X),$$

or equivalently

$$\frac{d}{d\varepsilon} H(\exp(\varepsilon \mathrm{Ad}(k)X))|_{\varepsilon=0} = \mathbb{P}(\mathrm{Ad}(k)X). \tag{3.22}$$

Using $\mathbb{P}$ we can decompose $\mathrm{Ad}(k)X = \mathbb{P}(\mathrm{Ad}(k)X) + Y \in \mathfrak{h} \oplus \mathfrak{h}^\perp$. The Iwasawa decomposition decomposes the part Y further into $Y = Y_k + Y_n \in \mathfrak{k} \oplus \mathfrak{n}$. Using the Baker-Campbell-Hausdorff formula this results in

$$\exp(\varepsilon Y_n)\exp(\varepsilon \mathbb{P}(\mathrm{Ad}(k)X))\exp(\varepsilon Y_k) = \exp\big(\varepsilon \mathrm{Ad}(k)X + \mathcal{O}(\varepsilon^2)\big). \tag{3.23}$$

The right hand side is in NHK form, so we have

$$H(\exp(\varepsilon Y_n)\exp(\varepsilon \mathbb{P}(\mathrm{Ad}(k)X))\exp(\varepsilon Y_k)) = \varepsilon \mathbb{P}(\mathrm{Ad}(k)X). \tag{3.24}$$

Apply $H(\cdot)$ to both sides of Eq. (3.23), use Eq. (3.24) and then derive with respect to ε at $\varepsilon = 0$ to obtain Eq. (3.22). This proves the first part. The second part follows from the fact that $\varphi_\lambda(g_\varepsilon)$ is an eigenfunction of $\Delta'_{B,\varepsilon}$ with eigenvalue $-\lambda^2+\rho^2$ and Lemma 3.3.6. □

Example 3.3.8. Consider the Gaussian Unitary ensemble. Here $K = U(N)$, $\mathfrak{p}$ are the Hermitian matrices and the Cartan subalgebra $\mathfrak{h}$ is generated by matrices H_i, which have a 1 on the ii-th position and 0 elsewhere. The element A_λ is the matrix with λ_i on the diagonal and the Killing form reduces to the matrix trace. This results in the formula

$$\psi_\lambda(X) = \int_{U(N)} e^{i\,\mathrm{tr}\left(UXU^\dagger A_\lambda\right)} \mathrm{d}U = I(iA_\lambda, X; 1),$$

where $I(\Lambda, X)$ is the Itzykson-Zuber integral,

$$I(\Lambda, X; \beta) = \int_{U(N)} e^{\beta\,\mathrm{tr}\left(\Lambda UXU^\dagger\right)} \mathrm{d}U. \tag{3.25}$$

△

3.3.2 The matrix model

The Itzykson-Zuber integral is fundamental to the study of random matrices of the form $H = H_0 + \sqrt{2t}H_1$, where H_0 is a fixed Hermitian matrix, and H_1 a member of the Gaussian Unitary Ensemble. Usually H_0 is chosen to be diagonal to simplify calculations. Let $y_1, \ldots, y_N$ be the eigenvalues of H_0, and write A_y for the matrix with $y_1, \ldots, y_N$ on the diagonal, then we have $H_0 = A_y$. The matrix H_0 describes the initial conditions of the system. In the $t \to \infty$ limit, the contribution from H_0 can be ignored and we recover the Gaussian Unitary Ensemble. Because the Gaussian ensembles are associated to quantum systems whose classical counterpart is chaotic, we will call this transition the *transition to chaos*.

The matrix model $H_0 + \sqrt{2t}H_1$ was in fact already introduced in the original paper by Itzykson and Zuber [36]. Furthermore, in this paper Brownian motion was used to give a closed expression for the integral Eq. (3.25), known as the Itzykson-Zuber-Harish-Chandra integral formula.

The probability density function of the combined H is given by

$$P(H) \propto \exp\bigl(-\operatorname{tr} H_1^2\bigr) = \exp\biggl(-\frac{1}{2t}\operatorname{tr}(H-H_0)^2\biggr).$$

Let us write explicitly the dependence on H_0 and t and make sure the total probability is one. Define $P_t(H|H_0)$ as

$$P_t(H|H_0) = \frac{1}{(2\pi t)^{N(N-1)/2}} \exp\biggl(-\frac{1}{2t}\operatorname{tr}(H-H_0)^2\biggr).$$

The notation $P_t(H|H_0)$ is introduced to make the dependence on t and H_0 explicit. The matrix H can be written as UA_xU^{-1}, where $x_1, \ldots, x_N$ are the eigenvalues of H. Change variables from H to $x_1, \ldots, x_N$ and U, and do the unitary integral,

$$P_t(x|y) = \frac{1}{N!}\frac{1}{(2\pi)^N}\Delta_N(x)^2 \int_{U(N)} P_t(UA_xU^\dagger|A_y)\mathrm{d}U.$$

This can be rewritten in terms of the Itzykson-Zuber integral as,

$$P_t(x|y) = \frac{1}{N!}\frac{t^{-N(N-1)/2}}{(2\pi)^{N^2/2}}\Delta_N(x)^2 \exp\biggl(-\frac{1}{2t}\sum_{i=1}^N (x_i^2+y_i^2)\biggr) I(A_y, A_x; t^{-1})$$

This is the heat kernel of the Fokker-Plank equation and can be exactly solved [37]. The two point correlation function is shown in Figure 3. This shows that for finite t, this model is an example of a model with intermediate statistics.

With the same calculation we have

$$|U_{i,j}|^2 P(x;t) = \frac{1}{N!}\frac{t^{-N(N-1)/2}}{(2\pi)^{N^2/2}}\Delta_N(x)^2 e^{-\frac{1}{2t}\sum_{i=1}^N (x_i^2+y_i^2)} \int_{U(N)} |U_{i,j}|^2 P_t(UA_xU^\dagger|A_y)\mathrm{d}U,$$

which is, using Morozov's formula [38],

$$|U_{i,j}|^2 P(x;t) = \frac{1}{N!}\frac{t^{-N(N-1)/2}}{(2\pi)^{N^2/2}}\Delta_N(x)^2 e^{-\frac{1}{2t}\sum_{i=1}^N (x_i^2+y_i^2)} \operatorname{Res}_{x\to x_i, y\to y_i} \frac{\det\Bigl(E + \frac{1}{x_i-x}E\frac{t}{y_i-y}\Bigr)}{\det(E)},$$

where $E = \bigl[e^{x_i y_j/t}\bigr]_{i,j=1}^N$. The advantage of this expression over an expectation value using the eigenvalue-eigenvector identity to be discussed in Section 5.4 is that in this case the unitary integral has already been done. More general versions of Morozov's formula for higher moments also exist [32, p.180].

Remark 3.3.9. The method of calculating probability density functions by writing down a diffusion equation can be extended for other probability density functions as well, notably for eigenfunction statistics as well, for example the probability density of obtaining specific values of eigenfunction components, pair functions and inverse participation ratios. This is done in a paper by Shukla [39].

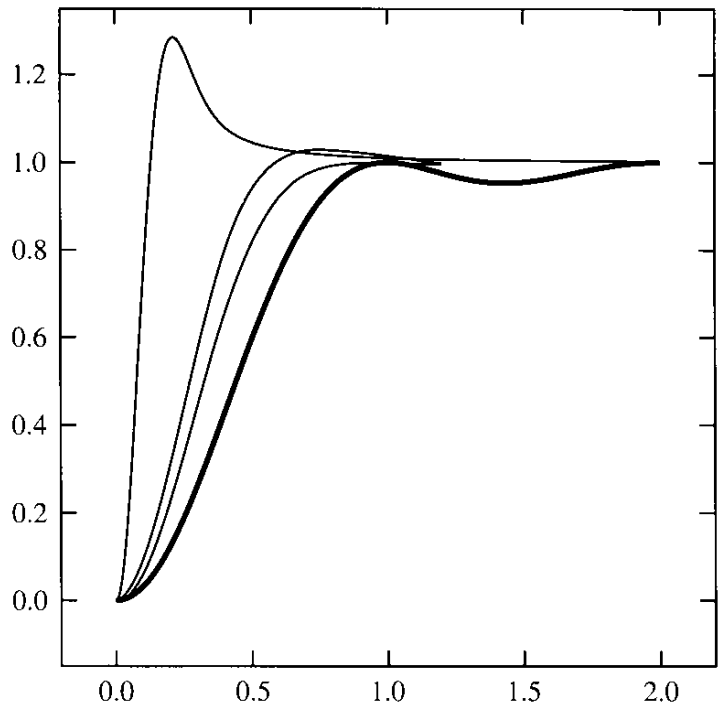


Figure 3: Two level correlation function for the Gaussian ensembles (bold, and the two graphs close to it) and the two matrix model $H_0 + tV$ (left most graph). Figure is from Guhr [37, Fig. 1].

4 Integrable systems

4.1 Andréief's identity

The following identity due to Andréief [40] is the basis of the orthogonal polynomial method of random matrix theory. It turns a complex integral into a determinant of simpler integrals, which can be solved more easily with computer algebra and is also of theoretical significance.

Lemma 4.1.1 (Andreiéf)**.** *Let $g_1, \dots, g_N$, $h_1, \dots, h_N$ be functions on a measure space (γ, σ), then*

$$\frac{1}{N!}\int_{\gamma^N} \det[\overline{g_j}(x_k)]_{j,k=1}^N \det[h_j(x_k)]_{j,k=1}^N \prod_{k=1}^N \mathrm{d}\sigma(x_k) = \det\left(\int_\gamma \overline{g_j}(x)h_k(x)\mathrm{d}\sigma(x)\right)_{j,k=1}^N,$$

whenever both sides are well defined.

In the rest of this section, we assume that the measure $\sigma(x)$ is of the form $\sigma(x) = w(x)\mathrm{d}x$, where $w(x)$ is a positive function which goes to zero as $|x| \to \infty$ sufficiently fast, that is to say that the integral $\int_\gamma x^k w(x)\mathrm{d}x$ is finite for all $k \geq 0$. The function $w(x)$ is called the *weight* function.

For the method of orthogonal polynomials, the integrand will be the Vandermonde determinant.

Definition 4.1.2 (Vandermonde determinant)**.** Let Δ_N be the product of differences,

$$\Delta_N(x) := \prod_{1\leq j<k\leq N} (x_k - x_j).$$

This can be expressed using a determinant, namely the Vandermonde determinant,

$$\Delta_N(x) = \det\left[x_j^{i-1}\right] = \det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{pmatrix}. \tag{4.1}$$

Let now $\varphi_k(x)$ be any monic polynomial of degree k,

$$\varphi_k(x) = x^k + \text{lower order terms.}$$

Then, by adding rows in the Vandermonde matrix, the Vandermonde determinant can also be written as

$$\Delta_N(x) = \det[\varphi_{i-1}(x_j)]_{i,j=1}^N$$

and the Andréief's identity results in

$$\int_{\gamma^N} |\Delta_N(x)|^2 \prod_{k=1}^N w(x_k)\mathrm{d}x_k = N!\det\left(\int_\gamma \overline{\varphi_{j-1}}(x)\varphi_{k-1}(x)w(x)\mathrm{d}x\right)_{j,k=1}^N.$$

It is therefore convenient to choose φ_k such that

$$\langle \varphi_j, \varphi_k\rangle = \int_\gamma \overline{\varphi_j}(x)\varphi_k(x)w(x)\mathrm{d}x = h_j\delta_{jk}, \tag{4.2}$$

resulting in the name *orthogonal polynomial*. The constant h_j is needed because the orthogonal polynomials $\varphi_k(x)$ need to be monic. The partition function can now be easily evaluated to be

$$Z_{N,2} = N!\det(h_{j-1}\delta_{jk})_{j,k=1}^{N} = N!\prod_{k=0}^{N-1} h_k. \tag{4.3}$$

Let now $K_N(x,y)$, called the kernel, be defined by

$$K_N(x,y) = \sum_{k=0}^{N-1} \frac{1}{h_k}\sqrt{w(x)}\overline{\varphi_k}(x)\sqrt{w(y)}\varphi_k(y) = \sum_{k=0}^{N-1} \overline{\psi_k}(x)\psi_k(y), \tag{4.4}$$

where $\psi_k(x) = \sqrt{w(x)}\varphi_k(x)/\sqrt{h_k}$. Then due to the determinant properties $|\det A|^2 = \det A^\dagger A = \det\left(\sum_{i=1}^{N} \overline{A_{ki}}A_{kj}\right)$, and $\lambda\det(\vec{v}_1,\dots,\vec{v}_l,\dots,\vec{v}_N) = \det(\vec{v}_1,\dots,\lambda\vec{v}_l,\dots,\vec{v}_N)$ it can be seen that

$$|\Delta_N(x)|^2\prod_{k=1}^{N} w(x_k) = \det[K(x_i,x_j)]_{i,j=1}^{N}\prod_{k=0}^{N-1} h_k. \tag{4.5}$$

The kernel has some important properties, which will be used in Dyson's Theorem 4.2.7.

Lemma 4.1.3 (Properties of $K_N(x,y)$)**.** *Let $K_N(x,y)$ be as in Eq. (4.4), then*

a) $K_N(x,y) = \overline{K_N(y,x)}$,

b) $\int_\gamma K_N(x,x)\mathrm{d}x = N$ *and*

c) $\int_\gamma K_N(x,y)K_N(y,z)\mathrm{d}y = K_N(x,z)$.

Proof. Properties a) and b) follow directly from the definition of K and the orthogonality property of $\varphi_k(x)$,Eq. (4.2). For c) we have

$$\begin{aligned}\int_\gamma K_N(x,y)K_N(y,z)\mathrm{d}y &= \sum_{k,l=0}^{N-1}\frac{1}{h_k h_l}\int_\gamma \sqrt{w(x)}w(y)\sqrt{w(z)}\overline{\varphi}_k(x)\varphi_k(y)\overline{\varphi}_l(y)\varphi_l(z)\mathrm{d}y\\ &= \sum_{k,l=0}^{N-1}\frac{1}{h_k h_l}\sqrt{w(x)}\sqrt{w(y)}\overline{\varphi}_k(x)\varphi_l(z)h_k\delta_{kl}\\ &= \sum_{k=0}^{N-1}\frac{1}{h_k}\sqrt{w(x)}\sqrt{w(z)}\overline{\varphi}_k(x)\varphi_k(z) = K_N(x,z).\end{aligned}$$

□

4.2 Orthogonal polynomials

In this section, we will show one method for obtaining orthogonal polynomials. Furthermore, some properties of the orthogonal polynomials are discussed. We follow Chapter 5 of Eynard, Kimura and Ribault [32] unless stated otherwise.

The method for obtaining the φ_i uses the Gram-Schmidt orthogonalization procedure. Write

$$G_n = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}, \quad \text{where } a_n := \int_\gamma x^n w(x) \mathrm{d}x$$

for the Gram-Schmidt determinant. Note that when $\gamma = S^1$ is the unit circle in the complex plane, a_{-n} is the n-th Fourier coefficient of w. The orthogonal polynomials φ_n are given by

$$\varphi_n(x) = \frac{1}{G_{n-1}} \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}. \tag{4.6}$$

With this definition φ_n is monic. The φ_n are also indeed orthogonal, since the inner product

$$\langle \varphi_n(x), x^k \rangle = \frac{1}{G_{n-1}} \det \begin{pmatrix} a_0 & \cdots & a_n \\ \vdots & \cdots & \vdots \\ a_{n-1} & \cdots & a_{2n-1} \\ a_k & \cdots & a_{n+k} \end{pmatrix}$$

is zero for $k < n$ because the same row appears twice. For $m < n$, $\varphi_m(x)$ is a linear combination of x^k with $k < n$, so $\varphi_m(x)$ is orthogonal to $\varphi_n(x)$ for $m < n$. By interchanging labels this orthogonality holds for all $k \neq n$.

Note that G_n is the determinant of a matrix with integrals as entries. Let us apply Andréief's identity to G_n,

$$\begin{aligned} G_{n-1} &= \det\left(\int_\gamma x^{j-1} x^{k-1} w(x) \mathrm{d}x \right)_{j,k=1}^n \\ &= \frac{1}{n!} \int_{\gamma^n} \det\left[x_k^{j-1}\right]_{j,k=1}^n \det\left[x_k^{j-1}\right]_{j,k=1}^n \prod_{k=1}^N w(x_k) \mathrm{d}x_k \\ &= \frac{1}{n!} \int_{\gamma^k} |\Delta_n(x)|^2 \prod_{k=1}^n w(x_k) \mathrm{d}x_k. \end{aligned}$$

Similarly, the integral expression for the determinant in Eq. (4.6) is

$$\det \begin{pmatrix} a_0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-1} \\ 1 & \cdots & z^n \end{pmatrix} = \frac{1}{n!} \int_{\gamma^n} |\Delta_n(x)|^2 \prod_{k=1}^n (z - x_k) w(x_k) \mathrm{d}x_k. \tag{4.7}$$

This can be shown as follows. Define $g_j(x) = x^{j-1}$ for $1 \leq j \leq n$ and $g_{n+1}(x) = w(x)^{-1}\delta(x - z)$. Let $h_k(x) = x^{k-1}$, then

$$\det\left(\int_\gamma g_j(x) h_k(x) w(x) \mathrm{d}x \right)_{j,k=1}^{n+1} = \det \begin{pmatrix} a_0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-1} \\ 1 & \cdots & z^n \end{pmatrix}.$$

Furthermore,

$$\det[g_j(x_k)]_{j,k=1}^{n+1} = \det \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \\ w(x_1)^{-1}\delta(x_1 - z) & \cdots & w(x_n)^{-1}\delta(x_n - z) \end{pmatrix}.$$

Expand this determinant around the last row, rewrite $\det[h_j(x_k)] = \Delta(x)$ with the observation that $\Delta_n(x)\prod_{k=1}^n(z-x_k) = \Delta_{n+1}(x_1,\ldots,x_n,z)$, and do some relabeling of integration variables and Eq. (4.7) follows.

Corollary 4.2.1 (Heine's formula)**.** *The orthogonal polynomials are the expectation values of the characteristic polynomial,*

$$\varphi_k(x) = \langle \det(x-H)\rangle_{k\times k-\text{matrix}} = \frac{\int_{\gamma^k} |\Delta(x)|^2 \prod_{k=1}^n (x-x_k) w(x_k) \mathrm{d}x_k}{\int_{\gamma^k} |\Delta(x)|^2 \prod_{k=1}^n w(x_k)\mathrm{d}x_k}. \tag{4.8}$$

Remark 4.2.2. The matrix G_{N-1} is an example of a Hankel determinant, the determinant of a Hankel matrix. A Hankel matrix is a matrix with the same entries on each diagonal going from left below to right above, that is, the matrix entries only depend on the sum of the coordinates. A Toeplitz matrix is a matrix with the same entries on each regular diagonal. Every Toeplitz matrix is the reflection of a Hankel matrix:

$$\begin{pmatrix} d_{-N+1} & \cdots & d_0 \\ \vdots & \ddots & \vdots \\ d_0 & \cdots & d_{N-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = \begin{pmatrix} d_0 & \cdots & d_{-N+1} \\ \vdots & \ddots & \vdots \\ d_{N-1} & \cdots & d_0 \end{pmatrix}.$$

Therefore, the determinants of the Toeplitz and Hankel matrices are the same, so Eq. (4.6) can also be written as a sum of ratios of Toeplitz determinants, instead of as a sum of ratios of Hankel determinants. Let $f(z) = \sum_i d_i z^i$ be a function with Fourier coefficients d_i. Define the Toeplitz determinants $D_N^{\lambda,\mu}(f)$ and $D_N(f)$ generated by f as

$$\begin{aligned} D_N^{\lambda,\mu}(f) &:= \det\left[d_{j+\lambda_j^r-k-\mu_k^r}\right]_{j,k=1}^N \\ D_N(f) &:= D_N^{0,0}(f) = \det[d_{j-k}]_{j,k=1}^N, \end{aligned}$$

where $\lambda = (\lambda_1,\ldots,\lambda_N)$ and $\mu = (\mu_1,\ldots,\mu_N)$ are partitions of length N, and $\lambda^r = (\lambda_N,\ldots,\lambda_1)$ is the reverse partition. The partitions $\lambda = (1)^r$ and $\mu = (1)^s$, where

$$(1)^r = (\underbrace{1,\ldots,1}_{r \text{ times}}, \underbrace{0,\ldots,0}_{N-r \text{ times}}),$$

can be used to write the r,s-th minor of the $N\times N$ Toeplitz matrix generated by f as $D_{N-1}^{(1)^{N-r},(1)^{N-s}}(f)$.

Write $a(z) = \sum_i a_i z^i$, for the generating function of the coefficients a_i. Then, Eq. (4.6) can be written as

$$\varphi_n(x) = \det \begin{pmatrix} a_n & a_{n-1} & \cdots & a_0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & \cdots & a_{n-1} \\ x^n & x^{n-1} & \cdots & 1 \end{pmatrix} \Big/ \begin{pmatrix} a_{n-1} & \cdots & a_0 \\ \vdots & \ddots & \vdots \\ a_{2n-1} & \cdots & a_{n-1} \end{pmatrix} = \sum_{j=0}^n x^j \frac{D_n^{0,(1)^j}(z^{-n}a)}{D_n(z^{-n+1}a)}. \tag{4.9}$$

This is still a monic polynomial due to the identity

$$D_N^{\lambda,(1)^N}(a) = D_N^{\lambda,0}(za),$$

which holds because $s_{(1^N)}(x_1,\dots,x_N) = e_N(x_1,\dots,x_N) = x_1\dots x_N$, and will be shown in Section 6.

We remark that Eq. (4.9) is nothing more than Heine's formula Eq. (4.8) together with the integral representation of $D_N^{\lambda,\mu}(f)$ Lemma 6.1.4 and the Cauchy identity for elementary symmetric polynomials Eq. (4.16). One needs to write the characteristic polynomial in therms of elementary symmetric polynomials. This is worked out in a different context in Example 6.0.3.

Remark 4.2.3. Let $\mathbf{t} = (t_1, t_2, \dots)$ be an infinite vector of coefficients, and suppose $w(x) = \exp(-V(x))$ where $V(x) = \sum_{k>0} t_k x^k/k$. Let us introduce the following notation,

$$\tau_N(\mathbf{t}) = \int_{H_N(\gamma)} \exp(-\operatorname{tr} V(H))\mathrm{d}H,$$

where $H_N(\gamma)$ are the Hermitian matrices when $\gamma = \mathbb{R}$ and $H_N(\gamma) = U(N)$ for $\gamma = S^1$. By writing the determinant as $\exp(\operatorname{tr}\log(x-H))$ and then Taylor expanding the log we have

$$\int_{H_N(\gamma)} \det(x-H)\exp(-V(H))\mathrm{d}H = \int_{H_N(\gamma)} x^N \exp\left(-\operatorname{tr}\sum_{k>0}\left(t_k - \frac{1}{x^k}\right)\frac{H^k}{k}\right)\mathrm{d}H.$$

With Sato's notation for infinite vectors

$$[x] := \left(-\frac{1}{x}, -\frac{1}{x^2}, \dots\right),$$

Heine's formula Eq. (4.8) becomes

$$\varphi_N(x) = x^N \frac{\tau_N(\mathbf{t}+[x])}{\tau_N(\mathbf{t})}.$$

This is the *Sato formula* for writing the *Baker-Akhiezer function* in terms of tau functions, see [32, Section 5.3.3].

Note that $x\varphi_k(x)$ is a monic polynomial of degree $k+1$, so it can be decomposed into the basis of orthogonal polynomials, that is, there exists $\hat{Q}_{k,j}$ for $0 \le j \le k+1$ such that

$$x\varphi_k(x) = \sum_{j=0}^{k+1} \hat{Q}_{k,j}\varphi_j(x), \tag{4.10}$$

with $\hat{Q}_{k,k+1} = 1$. Of these $\hat{Q}_{k,k+1}$, there are only three nonzero.

Lemma 4.2.4. *The φ_k satisfy a three-term recursion relation*

$$\varphi_{k+1}(x) = \left(x - \hat{Q}_{k,k}\right)\varphi_k(x) - \hat{Q}_{k,k-1}\varphi_{k-1}(x),$$

where

$$\hat{Q}_{k,k} = \frac{1}{h_k}\langle\varphi_k(x), x\varphi_k(x)\rangle, \quad \hat{Q}_{k,k-1} = \frac{h_k}{h_{k-1}}.$$

In particular, if the weight $w(x)$ is an even function, $B_k = 0$.

Proof. The φ_i are linearly independent and of degree i. The polynomial $\varphi_{k+1}(x) - x\varphi_k(x)$ is of degree $\leq k$, and can be written as

$$x\varphi_k(x) = \varphi_{k+1}(x) + \sum_{j=0}^{k} \hat{Q}_{k,j}\varphi_j(x).$$

Using the orthogonality relation $\langle \varphi_j(x), \varphi_i(x)\rangle = h_i\delta_{ij}$, we can now extract $\hat{Q}_{k,j}$ for $1 \leq j \leq k$ as

$$\langle x\varphi_k(x), \varphi_j(x)\rangle = \langle \varphi_{k+1}(x) + \sum_{i=0}^{k} \hat{Q}_{k,i}\varphi_i(x), \varphi_j(x)\rangle = \langle \hat{Q}_{k,j}\varphi_j(x), p_j(x)\rangle = \hat{Q}_{k,j}h_j.$$

Since

$$\langle x\varphi_k(x), x^j\rangle = \langle \varphi_k(x), x^{j+1}\rangle = 0, \quad \text{for } k = 0, 1, \ldots, n-2,$$

it follows that also $0 = \langle x\varphi_k(x), \varphi_j(x)\rangle = \hat{Q}_{k,j}h_j$ for $0 \leq k \leq n-2$ and therefore,

$$x\varphi_k(x) = \varphi_{k+1}(x) + \hat{Q}_{k,k}\varphi_k + \hat{Q}_{k,k-1}\varphi_{k-1}. \tag{4.11}$$

Use this recursion relation to calculate $\hat{Q}_{k,k-1}$,

$$\begin{aligned}\hat{Q}_{k,k-1}h_{k-1} &= \langle x\varphi_k(x), \varphi_{k-1}(x)\rangle = \langle \varphi_k(x), x\varphi_{k-1}(x)\rangle \\ &= \langle \varphi_k(x), \varphi_k(x) + \hat{Q}_{k,k-2}\varphi_{k-2} + \hat{Q}_{k,k-1}\varphi_{k-1}\rangle = \langle \varphi_k(x), \varphi_k(x)\rangle = h_n.\end{aligned}$$

The result follows by rearranging terms in Eq. (4.11). □

Corollary 4.2.5. *The wave functions $\psi_k(x) = \sqrt{w(x)}\varphi_k(x)/\sqrt{h_k}$ satisfy the three term recursion*

$$x\psi_k(x) = Q_{k,k+1}\psi_{k+1}(x) + Q_{k,k}\psi_k(x) + Q_{k,k-1}\psi_{k-1}(x),$$

where

$$Q_{k,j} := \sqrt{\frac{h_j}{h_k}}\hat{Q}_{k,j}.$$

Note that $Q_{k,k-1} = \sqrt{h_k/h_{k-1}} = Q_{k-1,k}$ so that for all k, j we have $Q_{k,j} = Q_{j,k}$.

The recursion formula allows us to express the kernel using only the top two 'wave-functions' ψ_{N-1} and ψ_N. This is the *Christoffel-Darboux formula* and is useful to describe the large N limit of expectation values, if the asymptotic behavior of the orthogonal polynomials is known.

Theorem 4.2.6 (Christoffel-Darboux formula)**.** *The orthogonal polynomials $\psi_n(x)$ satisfy*

$$K_N(x, y) = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(y) = \sqrt{\frac{h_N}{h_{N-1}}}\frac{\psi_{N-1}(x)\psi_N(y) - \psi_{N-1}(y)\psi_N(x)}{y - x}.$$

Proof. Corollary 4.2.5 gives the following two equations

$$\begin{aligned}x\psi_k(x)\psi_k(y) &= Q_{k,k+1}\psi_{k+1}(x)\psi_k(y) + Q_{k,k}\psi_k(x)\psi_k(y) + Q_{k,k-1}\psi_{k-1}(x)\psi_k(y), \\ y\psi_k(y)\psi_k(x) &= Q_{k,k+1}\psi_{k+1}(y)\psi_k(x) + Q_{k,k}\psi_k(y)\psi_k(x) + Q_{k,k-1}\psi_{k-1}(y)\psi_k(x),\end{aligned}$$

which subtracted results in

$$\begin{aligned}(x-y)\psi_k(x)\psi_k(y) = &Q_{k,k+1}[\psi_{k+1}(x)\psi_k(y) - \psi_{k+1}(y)\psi_k(x)] \\ &- Q_{k,k-1}[\psi_k(x)\psi_{k-1}(y) - \psi_k(y)\psi_{k-1}(x)].\end{aligned}$$

Since $Q_{k,k-1} = Q_{k-1,k}$ this results in the telescoping sum

$$\begin{aligned}(x-y)\sum_{k=0}^{N-1}\psi_k(x)\psi_k(y) &= \sum_{k=0}^{N-1} Q_{k,k+1}\left[\psi_{k+1}(x)\psi_k(y) - \psi_{k+1}(y)\psi_k(x)\right] \\ &\quad - \sum_{k=0}^{N-1} Q_{k-1,k}\left[\psi_k(x)\psi_{k-1}(y) - \psi_k(y)\psi_{k-1}(x)\right] \\ &= Q_{k,k+1}\left[\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)\right] \\ &\quad - Q_{-1,0}\left[\psi_0(x)\psi_{-1}(y) - \psi_0(y)\psi_{-1}(x)\right],\end{aligned}$$

where the last term is zero because either $\psi_{-1}(y) = 0$ or $\psi_{-1}(x) = 0$. Divide by $(x-y)$ to obtain the result. □

If we interpret the set $\{\psi_k(x)\}_{k\geq 1}$ as basis vectors of the space of polynomials in x, Corollary 4.2.5 allows us to write the multiplication by x operator Q as the tridiagonal matrix

$$Q = \begin{pmatrix} S_k & \gamma_1 & 0 & \dots \\ \gamma_1 & S_1 & \gamma_2 & \dots \\ 0 & \gamma_2 & S_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{where} \quad \gamma_k = Q_{k,k-1} \quad \text{and} \quad S_k = Q_{k,k}.$$

The characteristic polynomial of the $k \times k$ upper left submatrix $\Pi_k Q \Pi_k$, where Π_k is the matrix

$$\Pi_k = \begin{pmatrix} \mathrm{Id}_{k\times k} & 0_{\infty\times k} \\ 0_{k\times\infty} & 0_{\infty\times\infty} \end{pmatrix},$$

can be calculated by expansion in the $k+1$-th row. Let $p_k(x) = \det(x - \Pi_k Q \Pi_k)$, then

$$p_{k+1}(x) = (x - Q_{k,k})p_k(x) - Q_{k,k-1}^2 p_{k-1}(x),$$

which is the same recursion relation as the one for $\varphi_n(x)$ of Lemma 4.2.4. Furthermore, $p_0(x) = 1$ and $p_{-1}(x) = 0$, so that the initial conditions of $p_k(x)$ and $\varphi_k(x)$ are the same. This results in the third formula for $\varphi_k(x)$:

$$\varphi_k(x) = \det_{k\times k \text{ submatrix}}(x - Q).$$

Suppose $w(x) = e^{-V(x)}$ for some polynomial potential V of degree d. Then, the derivative ψ_k' of ψ_k can, like Eq. (4.10), be written in terms of ψ_j with degree $\leq k + d$

$$\psi_k'(x) = \sum_{j=0}^{k+d} P_{k,j}\psi_j(x). \tag{4.12}$$

The matrix P is antisymmetric. Indeed,

$$\int_\gamma \psi_i(x)\psi_k'(x)\mathrm{d}x = \sum_{j=0}^{k+d} P_{k,j}\int_\gamma \psi_i(x)\psi_j(x)\mathrm{d}x = P_{k,i},$$

then partial integration implies

$$P_{k,i} = \int_\gamma \psi_i(x)\psi_k'(x)\mathrm{d}x = -\int_\gamma \psi_i'(x)\psi_k(x)\mathrm{d}x = -P_{i,k}.$$

Since $\psi_k(x) = \sqrt{w(x)}\varphi_k(x)/\sqrt{h_k}$, Eq. (4.12) implies

$$\begin{aligned}\varphi_k'(x) &= \frac{\partial}{\partial x}e^{\frac{1}{2}V(x)}\sqrt{h_k}\psi_k(x) = \sqrt{h_k}e^{\frac{1}{2}V(x)}\left[\frac{1}{2}V'\psi_k(x) + \psi_k'(x)\right] \\ &= \sqrt{h_k}e^{\frac{1}{2}V(x)}\sum_{j=0}^{k+d}\left(\frac{1}{2}V'(Q) + P\right)_{kj}\psi_j(x) \\ &= \sum_{j=0}^{k+d}\left(\frac{1}{2}V'(Q) + P\right)_{kj}\varphi_j(x).\end{aligned}$$

Here, we note that the matrix Q is multiplication by x. Now, $\varphi_k'(x)$ has degree $k-1$, so the matrix $V'(Q)/2 + P$ must be strictly lower triangular. Denote with M_+, M_0 and M_- the upper triangular, diagonal and lower triangular parts of M, then we have

$$0 = \left(\frac{1}{2}V'(Q) + P\right)_+.$$

Then, by antisymmetry, $P- = V'(Q)_-/2$, so that

$$P = -\frac{1}{2}\left(V'(Q)_+ - V'(Q)_-\right).$$

4.2.1 Dyson's theorem

Dyson's theorem is a recursion formula to be able to calculate integrals of the form

$$\int\left[\prod_{k=1}^{N}w(x_k)\right]\left[\prod_{1\le j<k\le N}|x_k - x_j|^\beta\right]\mathrm{d}x_{m+1}\ldots\mathrm{d}x_N, \tag{4.13}$$

for $\beta = 2$.

The general procedure for calculating Eq. (4.13) is as follows: Given a weight function $w(x)$, we construct a kernel $K_\beta(x,y)$ such that

a) $\overline{K}_\beta(x,y) = K_\beta(y,x)$,

b) $\int K_\beta(x,y)K_\beta(y,z)\mathrm{d}y = K_\beta(x,z) + \lambda K(x,z) - K(x,z)\lambda$,

c) $\det[K_\beta(x_i,x_j)]_N \propto |\Delta_N(x)|^\beta\prod_{k=1}^N w(x_k)$.

Then, the following theorem gives a recursion formula for calculating the integral.

Theorem 4.2.7 (Dyson, 1970)**.** *Let $K(x,y) : \mathbb{K}\times\mathbb{K}\to\mathbb{K}$ be a function, with $\mathbb{K}$ the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$, such that*

$$\overline{K}(x,y) = K(y,x),$$

with $\overline{K}$ being K if $\mathbb{K} = \mathbb{R}$, the complex conjugate of K if $\mathbb{K} = \mathbb{C}$ or the dual of K if $\mathbb{K} = \mathbb{H}$. Suppose furthermore that

$$\int K(x,y)K(y,z)\mathrm{d}y = K(x,z) + \lambda K(x,z) - K(x,z)\lambda,$$

where $\lambda \in \mathbb{H}$, so λ vanishes if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ due to commutativity. Denote with $[K(x_i,x_j)]_N$ the $N \times N$ matrix with (i,j) entry $K(x_i,x_j)$. Then

$$\int \det[K(x_i,x_j)]_{i,j=1}^{N} \mathrm{d}x_N = (c - N + 1)\det[K(x_i,x_j)]_{i,j=1}^{N-1},$$

where $c = \int K(x,x)\mathrm{d}x$.

Proof. The Leibniz formula for the determinant gives the following sum over all pairings

$$\det[K(x_i,x_j)]_{i,j=1}^{N} = \sum_P \sigma(P)K(x_1,x_{P_1})K(x_2,x_{P_2})\dots K(x_N,x_{P_N}).$$

Either $P_N = N$ and the integration results in the scalar c, or $P_N \neq N$. In the latter case, which occurs $N-1$ times, the integration reduces it to a product of K's of length $N-1$ for the permutation $\left(\begin{smallmatrix} 1 & \cdots & k & \cdots & N-1 \\ P_1 & \cdots & P_N & \cdots & P_{N-1} \end{smallmatrix}\right)$, where k is such that $P_k = N$ ($k \neq N$). This permutation has opposite sign. The terms containing λ sum to zero. Adding these two contributions proves the theorem. □

Using Lemma 4.1.3 it now follows that

$$\int_\gamma \det[K(x_i,x_j)]_{i,j=1}^{l} \mathrm{d}x_k = (N - l + 1)\det\big[K_(x_i,x_j)\big]_{i,j=1}^{l-1},$$

so that

$$Z_{N,2} = \int \left[\prod_{k=1}^{N} w(x_k)\right] |\Delta_N(x)|^2 \mathrm{d}x_1 \dots \mathrm{d}x_N = \prod_{l=1}^{N}(N-l+1) \prod_{k=0}^{N-1} c_k = N! \prod_{k=0}^{N-1} c_k. \quad (4.14)$$

This is the same result as Eq. (4.3). Similarly, the n-point correlation function can be calculated using this recursion in $N-n$ steps.

$$\begin{aligned} R_n(x_1,\dots,x_n) &= \frac{1}{Z_{N,2}} \frac{N!}{(N-n)!} \int \left[\prod_{k=1}^{N} w(x_k)\right] |\Delta_N(x)|^2 \mathrm{d}x_{n+1} \dots \mathrm{d}x_N \\ &= \frac{1}{(N-n)!} \int \det[K(x_i,x_j)]_{i,j=1}^{N} \mathrm{d}x_{n+1} \dots \mathrm{d}x_N \\ &= \det[K(x_i,x_j)]_{i,j=1}^{n}. \end{aligned} \quad (4.15)$$

In particular, the 1-point correlation function is just $K(x,x)$.

4.3 Newton's relations

In the following, let A be a diagonalizable $N \times N$ matrix with eigenvalues $x_1,\dots,x_N$. By Cayley-Hamilton, the matrix A is a solution of its own characteristic polynomial $P(\lambda)$,

$$P(\lambda) = \det(\lambda - A) = \lambda^N \det\big(1 + (-\lambda^{-1})A\big) = \lambda^N \sum_{k=0}^{N} e_k(x) z^k = \sum_{n=0}^{N} (-1)^{N-n} e_{N-n}(x) \lambda^n,$$

where $z = -1/\lambda$ and $k = N - n$. The coefficients $e_k(x)$ are defined by this equation, and do depend on the eigenvalues x. This dependence is denoted by $e_k(x)$. We can relate these coefficients $e_k(x)$ to the traces of A as follows. The Taylor expansion of the logarithm around 1 is

$$\log(1-x) = -\sum_{k>0} \frac{1}{k} x^k,$$

from which it follows that

$$\det(\lambda - A) = \prod_{i=1}^{N} (\lambda - \lambda_i) = \lambda^N \prod_{i=1}^{N} \left(1 - \frac{\lambda_i}{\lambda}\right) = \lambda^N \exp\left(-\sum_{k>0} \frac{p_k(x)}{k} \lambda^{-k}\right),$$

where $p_k(x) = \sum_{i=1}^{N} x_i^k = \operatorname{tr}\left(A^k\right)$. With $z = -1/\lambda$ we get

$$E(x;z) := \prod_{j=1}^{N} (1 + x_j z) = \sum_{k=0}^{N} e_k(x) z^k = \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{p_k(x)}{k} z^k\right). \tag{4.16}$$

Here, $E(x;z)$ is called the generating function for $e_k(x)$. This equation is $\det(\lambda - A) = \exp \operatorname{tr} \log(\lambda - A)$ rewritten in a slightly different form. One can now derive Eq. (4.16) with respect to z and set $z = 0$ to obtain for example

$$\begin{aligned} e_0(x) &= 1, \\ e_1(x) &= p_1(x), \\ 2e_2(x) &= p_1(x)^2 - p_2(x), \\ &\dots \end{aligned}$$

These equations can then also be used to write $p_k(x)$ in terms of $e_k(x)$. In general k this can be written as (see for example (5.15.1) in Haake [41]),

$$e_k(x) = (-1)^k \sum_{\substack{m_1 + 2m_2 + \dots + km_k = k \\ m_1 \geq 0, \dots, m_k \geq 0}} \prod_{l=1}^{k} \frac{(-p_k(x))^{m_l}}{m_l! l^{m_l}}, \tag{4.17a}$$

$$p_k(x) = (-1)^k k \sum_{\substack{m_1 + 2m_2 + \dots + km_k = k \\ m_1 \geq 0, \dots, m_k \geq 0}} (-1)^k (m_1 + \dots + m_k - 1)! \prod_{l=1}^{k} \frac{(-e_l(x))^{m_l}}{m_l!}. \tag{4.17b}$$

The same relations hold for general variables x, not necessarily eigenvalues of some matrix A. Furthermore, from the identity $\prod_{j=1}^{N}(1 + x_j z) = \sum_{k=0}^{N} e_k(x) z^k$ it can be seen that

$$e_k(x) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} x_{j_1} \cdots x_{j_k}.$$

The polynomials $e_k(x)$ are called the *elementary symmetric polynomials*. The $p_k(x)$ are the *power sum symmetric polynomials*. Finally, we have the *complete homogeneous symmetric polynomials*

$$h_m(x_1, \dots, x_N) := \sum_{\substack{j_1 + j_2 + \dots + j_N = m \\ j_i \geq 0}} x_1^{j_1} \cdots x_N^{j_N}.$$

For these polynomials, we have a similar equation as Eq. (4.16),

$$H(x;z) = \prod_{j=1}^{\infty} \frac{1}{1-x_j z} = \sum_{k=0}^{\infty} h_k(x) z^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(x)}{k} z^k \right). \tag{4.18}$$

A good reference for the theory of symmetric polynomials is Macdonald [42].

4.4 Correlation functions

Random Matrix Theory is a statistical theory, so we must calculate expectation values or ensemble averages $\langle \cdots \rangle$,

$$\langle f \rangle := \int f(x_1, \ldots, x_N) P_N(x_1, \ldots, x_N) \mathrm{d}x_1 \ldots \mathrm{d}x_N.$$

We assume that $P_N(x_1, \ldots, x_N)$ is such that $\langle 1 \rangle = 1$, and that $P_N(x_1, \ldots, x_N)$ is a symmetric function, that is, invariant under change of ordering of the $x_1, \ldots, x_N$ (the remaining Weyl symmetry). This section goes as follows. We start by giving definitions of some important expectation values, and present a relation between them. We follow Mehta [9] in this section, but the reader may also look in the appendix of Nishigaki [43], where they use these expressions to give analytical results for level spacing distributions of orthogonal, unitary and symplectic ensembles with an extra potential which introduces multifractality. Then, we continue by presenting the technique of orthogonal polynomials, which will allow us to calculate the n-point correlation functions in the specific case of the Gaussian Unitary Ensemble. In the third subsection, we will combine the knowledge of the first two subsections to calculate other distributions for the GUE. Finally, we will consider the circular and the multifractal ensemble.

4.4.1 The distributions and their relations

The goal of this section is to present some *eigenvalue* distributions. One of the most important ones is the n-point correlation.

Definition 4.4.1 (n-point correlation function)**.** The *n-point correlation function* is the probability density function of finding a level around the points $x_1, \ldots, x_n$ with the other levels unspecified and given by [44]

$$R_n(x_1, \ldots, x_n) := \frac{N!}{(N-n)!} \int P_N(x_1, \ldots, x_N) \mathrm{d}x_{n+1} \ldots \mathrm{d}x_N.$$

In particular $R_1(x)$ is the level density.

Because we have assumed that $P_N(x_1, \ldots, x_N)$ is symmetric under change of ordering of the x_i, it can be seen that $R_n(x_1, \ldots, x_n) = \langle \varrho(x_1) \ldots \varrho(x_n) \rangle$. Here $\varrho(x) = \sum_{i=1}^{N} \delta(x - x_i)$. Furthermore, the large N limit of $R_1(x) = \rho(x) = \langle \varrho(x) \rangle$ for the Gaussian ensembles is given by the Wigner semicircle law. The density $\varrho(x)$ is normalized to N, $\int \varrho(x) \mathrm{d}x = N$.

For uncorrelated systems, one would find $R_2(x_1, x_2) = R_1(x_1) R_1(x_2)$. The clustering of the levels breaks this relation. Then *n-level cluster function* attempts to measure this.

Definition 4.4.2 (n-level cluster function)**.** The *n-level cluster function* describes the grouping of n levels in subgroups, which we will call clusters. It is defined as [9, Section 6.1.1],

$$T_n(x_1, \ldots, x_n) := \sum_G (-1)^{n-|G|}(|G|-1)! \prod_{j=1}^{|G|} R_{|G_j|}(x_k, \text{ with } k \text{ in } G_j). \tag{4.19}$$

Here, $\sum_G$ is the sum over partitions G of the levels $(1, 2, \ldots, n)$ into $|G|$ sets $(G_1, \ldots, G_{|G|})$. An example to make the notation clear; let $G_j = \{1, 3, 4\}$, then $R_{|G_j|}(x_k, \text{ with } k \text{ in } G_j) = R_3(x_1, x_3, x_4)$.

The formula Eq. (4.19) is a Newton's identity (Eq. (4.17b)) like formula. In particular, $T_1(x_1) = R_1(x_1)$ and

$$T_2(x_1, x_2) = -R_2(x_1, x_2) + R_1(x_1)R_1(x_2). \tag{4.20}$$

With the observations made before,

$$R^K(x_1, x_2) := \frac{R_2(x_1, x_2)}{\rho(x_1)\rho(x_2)} = 1 - \frac{T_2(x_1, x_2)}{\rho(x_1)\rho(x_2)}.$$

This quantity is what Kravtsov calls the *two-level correlation function* [45, Section X]. Consider some average of $R^K(x_1, x_2)$,

$$\begin{aligned}\int \mathrm{d}x_2 \rho(x_1)\rho(x_2)R^K(x_1, x_2) &= \int \mathrm{d}x_2 \langle \varrho(x_1)\varrho(x_2)\rangle = \langle \varrho(x_1) \int \mathrm{d}x_2 \varrho(x_2)\rangle \\ &= N\langle \varrho(x_1)\rangle = NR_1(x_1),\end{aligned}$$

due to the normalization of $\varrho(x_1)$. Therefore, we obtain the *normalization sum rule*,

$$\alpha := \int \mathrm{d}x_2 T_2(x_1, x_2) = -\int \mathrm{d}x_2 \rho(x_1)\rho(x_2)(R^K(x_1, x_2) - 1) = 0.$$

Due to non-commuting limit, α may not be 0 in general in the limit $N \to \infty$. This is called the *deficiency of the normalization sum rule*.

Kravtsov also defines the following.

Definition 4.4.3 (Level number variance)**.** The *level number variance* is the variance in the level numbers n,

$$\Sigma(\bar{n}, x) = \langle n_x^2\rangle - \bar{n}_x^2,$$

where n_x is the fluctuating number of levels in an interval δ_x of length s around some base point x, $\delta_x = [x - s/2, x + s/2]$, and $\bar{n}_x$ is the average number of levels of an interval of length s. The length s is determined by $\bar{n}_x$ by this requirement, so actually we would write $\delta_x = \delta_x(\bar{n})$.

We can write n_x as[6]

$$n_x = \int_{\delta_x} \varrho(x_1)\mathrm{d}x_1,$$

[6]Kravtsov [45] writes ρ here instead of ϱ, but I believe that is a mistake.

so that $\bar{n}_x$ is given by

$$\bar{n}_x = \langle n_x \rangle = \int_{\delta_x} \langle \varrho(x_1) \rangle \mathrm{d}x_1 = \int R_1(x_1) \chi_{\delta_x}(x_1) \mathrm{d}x_1.$$

Here, we have introduced the characteristic function of an interval,

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

It can now be seen that [45, Section X],

$$\Sigma(\bar{n}, x) = \int_{\delta_x} \int_{\delta_x} \rho(x_1)\rho(x_2)[R^K(x_1, x_2) - 1]\mathrm{d}x_1\mathrm{d}x_2 = \int_{\delta_x} \int_{\delta_x} T_2(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2. \quad (4.21)$$

In the following, we will drop the explicit base point dependence in our notation. This is because there usually is a natural base point to pick, such as 0, or we expect that in the large N limit, these quantities do not depend on the base point anymore. We will therefore write $\Sigma(\bar{n})$ for $\Sigma(\bar{n}, x)$ and n for n_x.

An important quantity to characterize the degree of level repulsion is how much this variance changes if we pack more levels together.

Definition 4.4.4 (Level compressibility)**.** The *level compressibility* is the derivative of the level number variance with respect to $\bar{n}$.

$$\chi(\bar{n}) := \frac{d\Sigma(\bar{n})}{d\bar{n}}.$$

In the small $\bar{n}$ limit, we have that $\chi \sim (1 - d_2/d)/2$, where d_2 is the second multifractal dimension and d is the dimension [46].

Typically, one wants to take expectation values of the correlation of cluster functions, that is, integrate over $x_1, \ldots, x_n$ over some interval I. In general, one is interested in the quantities

$$r_{0,I} := 1, \quad (4.22)$$

$$r_{n,I} := \int R_n(x_1, \ldots, x_n) \prod_{i=1}^{n} (\chi_I(x_i)\mathrm{d}x_i), \quad 1 \leq n \leq N, \quad (4.23)$$

$$t_{n,I} := \int T_n(x_1, \ldots, x_n) \prod_{i=1}^{n} (\chi_I(x_i)\mathrm{d}x_i), \quad 1 \leq n \leq N \quad (4.24)$$

where

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{else} \end{cases}$$

is the characteristic function for the interval I. For $n > N$, we set both $r_{n,I} = t_{n,I} = 0$. Let us remark that χ_I is just a projection, as $\chi_I^2 = \chi_I$.

Using the new notation, we see that $\bar{n} = r_{1,\delta_x}$ and furthermore, looking at the equation for $\Sigma(\bar{n})$ in Eq. (4.21),

$$\Sigma(\bar{n}) = t_{2,\delta_x(\bar{n})}.$$

It is useful to define a generating function for the quantities t_n and r_n,

$$R(z,I) := \sum_{n=0}^{\infty} (-1)^n \frac{r_{n,I}}{n!} z^n,$$
$$T(z,I) := -\sum_{n=1}^{\infty} \frac{t_{n,I}}{n!} z^n.$$

As for $n > N$, $r_{n,I} = t_{n,I} = 0$, this is in fact a finite sum. These generating functions are related.

Lemma 4.4.5.

$$R(z,I) = \exp(T(z,I)).$$

Proof. For convenience, drop the I labels everywhere. Fill in Eq. (4.19) for $T_n(x_1,\ldots,x_n)$ in the equation for t_n, Eq. (4.24),

$$\begin{aligned} t_n &= \sum_G (-1)^{n-m}(m-1)! \int \prod_{j=1}^{m} R_{|G_j|}(x_k, \text{ with } k \text{ in } G_j) \prod_{i=1}^{n} (\chi_I(x_i)\mathrm{d}x_i) \\ &= \sum_G (-1)^{n-m}(m-1)! \prod_{j=1}^{m} r_{|G_j|}. \end{aligned}$$

Observe that this formula does not depend anymore on the precise division over the sets G_i, but only on the sizes $\ell_i := |G_i|$, with multiplicity $n!/|G_1|!\ldots|G_m|!$ for the number of ways of subdividing the elements into sets of lengths $|G_i|$. We add a factor of $1/m!$ to include the freedom of changing the ordering of the sets G_i. This results in

$$t_n = \sum_{\ell} (-1)^{n-m} \frac{(m-1)!n!}{m!} \prod_{j=1}^{m} \frac{r_{\ell_j}}{\ell_j!},$$

where the sum is over all partitions ℓ of n such that $n = \ell_1 + \cdots + \ell_m$.

$$\begin{aligned} T(z) &= -\sum_{n=1}^{\infty} \frac{t_n}{n!} z^n \\ &= -\sum_{n=1}^{\infty} \sum_{\ell(n)} \frac{(-1)^{n-m}}{m} \prod_{j=1}^{m} \frac{r_{\ell_j}}{\ell_j!} z^{\ell_j} \\ &= -\sum_{m,\ell_1,\ldots,\ell_m=1}^{\infty} \frac{(-1)^{n-m}}{m} \prod_{j=1}^{m} \frac{r_{\ell_j}}{\ell_j} z^{\ell_j}. \end{aligned}$$

In the last step, we combined the two summations. We now have

$$T(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\sum_{n=1}^{\infty} \frac{r_n}{n!} (-z)^n \right)^m = \log R(z).$$

For more info, see [9, A.7.11]. □

The level spacings are what characterize Wigner-Dyson statistics. A key result, the Wigner surmise, gives a clear distinction between such statistics and Poisson statistics. A step towards obtaining this result is the general question of what the probability is of finding m levels in an interval I and the other levels outside.

Definition 4.4.6 (Spacing function)**.** The *spacing function* is the probability function of finding the first n levels inside an interval I and the other $N-n$ outside I [9, Section 5.9],

$$E(n,I) := \binom{N}{n} \int_I \mathrm{d}x_1 \dots \mathrm{d}x_n \int_{I^c} \mathrm{d}x_{n+1} \dots \mathrm{d}x_N P_N(x_1, \dots, x_N),$$

where I^c is the complement of I.

It can also be written as

$$E(n,I) = \binom{N}{n} \int \mathrm{d}x_1 \dots \mathrm{d}x_N \prod_{i=1}^{n} \chi_I(x_i) \prod_{j=n+1}^{N} (1-\chi_I(x_j)) P_N(x_1, \dots, x_N).$$

This can be expressed in terms of the generating functions $R(z,I)$ (or $T(z,I)$).

Lemma 4.4.7.

$$E(n;I) = \frac{(-1)^n}{n!} \left(\frac{d}{dz}\right)^n R(z,I)|_{z=1}.$$

In particular $E(0;I) = R(1,I)$.

Proof. Let us expand the product $\prod_{j=n+1}^{N}(1-\chi_I(x_j))$ and relabel

$$\begin{aligned}
E(n;I) &= \frac{1}{n!} \sum_{k=n} \frac{N!}{(N-n)!} \binom{N-n}{k-n} (-1)^{k-n} \int \mathrm{d}x_1 \dots \mathrm{d}x_N P_N(x_1, \dots, x_N) \prod_{i=1}^{k} \chi_I(x_i) \\
&= \frac{1}{n!} \sum_{k=n} \frac{(-1)^{k-n}}{(k-n)!} \frac{N!}{(N-k)!} \int \mathrm{d}x_{k+1} \dots \mathrm{d}x_N R_k(x_1, \dots, x_k) \prod_{i=1}^{k} \chi_I(x_i) \mathrm{d}x_i \\
&= \frac{(-1)^n}{n!} \sum_{k=n}^{N} \frac{(-1)^k}{(k-n)!} r_{k,I} \\
&= \frac{(-1)^n}{n!} \left(\frac{d}{dz}\right)^n \sum_{k=n} \frac{(-1)^k}{k!} r_{k,I} z^k|_{z=1} \\
&= \frac{(-1)^n}{n!} \left(\frac{d}{dz}\right)^n R(z,I)|_{z=1}.
\end{aligned}$$

These calculations were also given by Mehta [9, A.7.17]. □

For these spacings, we are generally not interested in the particular starting point of the interval, but only care about the length of the interval I. We will just pick the interval I to be centered around 0 and write $E(n;s) = E(n;[-s/2, s/2])$.

Definition 4.4.8 (Level spacing distribution)**.** The *level spacing distribution* $p(n;s)$ is the probability density function that the distance between an arbitrary level E_i and the $n+1$-th successive level E_{i+n+1} is s. In other words, $p(n;s)\mathrm{d}s$ is the probability that the distance between E_i and E_{i+n+1} lies between s and $\mathrm{d}s$.

Let us now introduce $\tilde{F}(n;s)$, which we define to be the probability that if we choose a level at random (with uniform probability $1/N$), the interval starting at this level with a length of s contains exactly n levels, not counting the level we selected. Then, $p(0;s)\mathrm{d}s$ is the probability that the interval of length s contains no levels, minus the probability

that $s + \mathrm{d}s$ still contains no levels, as in this case we must conclude that $\mathrm{d}s$ also contains no levels. In other words, $p(0;s)\mathrm{d}s = F(0;s) - F(0;s+ds)$, or

$$p(0;s) = -\frac{d}{ds}\tilde{F}(0;s).$$

Similarly, we get $\tilde{F}(0;s)$ as a derivative of $E(0;s)$ by fixing the starting point,

$$\tilde{F}(n;s) = -\frac{d}{ds}E(0;s).$$

More generally,

$$p(n;s) = \frac{d^2}{ds^2}\sum_{j=0}^{n}(n-j+1)E(j;s).$$

It is clear that $R(z,I)$ is an important quantity to calculate. If we know how $R(z,I)$ looks like, we are able to calculate all other distributions defined in this section. Orthogonal polynomials allows us to do that for the Gaussian Unitary Ensemble.

4.4.2 The Gaussian Unitary case

In the Gaussian Unitary case, we have now found (Eq. (4.15))

$$R_n(x_1,\ldots,x_n) = \det[K(x_i,x_j)]_n.$$

This implies that

$$T_n(x_1,\ldots,x_n) = K(x_1,x_2)K(x_2,x_3)\ldots K(x_N,x_1) + \ldots,$$

where the dots indicate similar terms for other orderings of x_i. For example, we have

$$\begin{aligned}
T_1(x_1) &= K(x_1,x_1),\\
T_2(x_1,x_2) &= K(x_1,x_2)K(x_2,x_1),\\
T_3(x_1,x_2,x_3) &= K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1),\\
&\ldots
\end{aligned}$$

Hence, if we think of $K(x_1,x_2)$ as an arrow from x_1 to x_2, $T_n(x_1,\ldots,x_n)$ expresses all possible loops one can draw through the points x_i, $i=1,\ldots,n$, where the pen does not leave the paper and each point is visited once. Then,

$$\begin{aligned}
t_{n,I} &= (n-1)!\int K(x_1,x_2)\chi_I(x_2)K(x_2,x_3)\chi_I(x_3)\ldots K(x_N,x_1)\chi_I(x_1)\mathrm{d}x_1\ldots\mathrm{d}x_N\\
&= (n-1)!\,\mathrm{tr}((K\Pi_I)^n).
\end{aligned}$$

Here, we regard K as an integral operator, where Π_I is the projection operator on the interval I,

$$\Pi_I := \int_I \mathrm{d}x|x\rangle\langle x|,$$

so that $\langle x|\Pi_I|y\rangle = \delta(x-y)\chi_I(x)$. The sum $T(z,I)$ is then a Taylor expansion for a logarithm and

$$T(z,I) = \mathrm{tr}\log(1 - zK\Pi_I),$$

so that

$$R(z, I) = \exp(\operatorname{tr} \log(1 - zK\Pi_I)) = \det(1 - zK\Pi_I).$$

This is Equation A7a from Nishigaki [43], which is known as the Fredholm Determinant.

If one would know the eigenvalues λ_i of $K\Pi_I$, this determinant could be evaluated as $\prod_i (1 - z\lambda_i)$. Eigenvalues are numbers, such that the following holds,

$$\lambda_i \psi_i(x) = \int_I K(x, y)\psi_i(y)\mathrm{d}y.$$

We refer to Mehta [9, Chapter 18] for a calculation of the asymptotics of $R(0, I)$ using Toeplitz matrices and the Szegö limit theorem.

4.4.3 Circular ensemble

For the Circular Unitary Ensemble, we can also use the method of orthogonal polynomials, and pick the polynomials $\varphi_j(x) = x^j$. Then, Eq. (4.2) is satisfied with $h_i = 2\pi$:

$$\int \varphi_k(x)\overline{\varphi}_j(x)w(x)\mathrm{d}x = \int_0^{2\pi} e^{i(k-j)\theta}\mathrm{d}\theta = 2\pi\delta_{kj}$$

Now, use the results from Section 4.4 with

$$K_2(e^{i\theta_i}, e^{i\theta_j}) = \frac{1}{2\pi}\sum_{k=0}^{N-1} e^{ik(\theta_j - \theta_i)} =: S_N(\theta_j - \theta_i).$$

Therefore, the n-level correlation function is

$$R_n(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \det[S_N(\theta_j - \theta_i)]_n.$$

We will also write $R_n(\theta_1, \ldots, \theta_n)$ to mean $R_n(e^{i\theta_1}, \ldots, e^{i\theta_n})$. For example, the level density (1-level correlation) and 2-level correlation function are

$$\begin{aligned} R_1(\theta) &= S_N(0) = \frac{N}{2\pi}, \\ R_2(\theta, \varphi) &= (S_N(0))^2 - (S_N(\theta - \varphi))^2. \end{aligned}$$

The limit as $N \to \infty$ is denoted by Y_n

$$Y_n(y_1, \ldots, y_n) = \lim_{N\to\infty} \alpha^n T_n(x_1, \ldots, x_n),$$

where y_i are energy levels in units of mean level spacing α, $y_i = x_i/\alpha$.

Using Eq. (4.20), it follows that the two-level cluster function for the Circular Unitary Ensemble is

$$T_2(\theta - \varphi) = (S_N(\varphi - \theta))^2 = \left\{\frac{\sin[N(\theta - \varphi)/2]}{2\pi \sin[(\theta - \varphi)/2]}\right\}^2.$$

With mean level spacing $\alpha = 2\pi/N$, we get

$$Y_2(\xi, \eta) = \lim_{N\to\infty} \alpha^2 T_2(\theta, \varphi) = \left(\frac{\sin(\pi|\xi - \eta|)}{\pi|\xi - \eta|}\right)^2.$$

4.5 The Gaussian Unitary Ensemble

In this section, the orthogonal polynomial technique is applied to the Gaussian Unitary Ensemble. For the Gaussian Unitary Ensemble, the orthogonal polynomials are the Hermite polynomials

$$\varphi_j(x) = \overline{\varphi}_j(x) = 2^{-j} H_j(x),$$

where the 2^{-j} prefactor is to ensure that $\varphi_i(x)$ are monic. The Hermite polynomials satisfy $\int H_j(x) H_k(x) \exp(-x^2) \mathrm{d}x = \sqrt{\pi} 2^j j! \delta_{jk}$, so when taking the prefactor into account, $c_j = \sqrt{\pi} 2^{-j} j!$. In particular, this allows us to finally calculate the normalization constant $Z_{N,\beta}$ in Theorem 2.4.5, at least for $\beta = 2$, using Eq. (4.14),

$$\begin{aligned} Z_{N,\beta} &= N! \prod_{j=0}^{N-1} \sqrt{\pi} 2^j j! = \pi^{N/2} 2^{\sum_{j=0}^{N-1} j} N! \prod_{j=1}^{N} (j-1)! \\ &= (2\pi)^{N/2} 2^{-N/2} 2^{N(N-1)/2} \prod_{j=1}^{N} j\Gamma(j) = (2\pi)^{N/2} \beta^{-N/2+\beta N(N-1)/4} \prod_{j=1}^{N} \Gamma(1+j). \end{aligned} \quad (4.25)$$

Example 4.5.1 (Hermite polynomials)**.** The Hermite polynomials are supposed to be the orthogonal polynomials for the weight $w(x) = \exp(-x^2)$, with $h_i = \sqrt{\pi} 2^{-j} j!$ and $q_i = 2^i$, hence $h_n / h_{n-1} = n/2$. The weight $w(x)$ is an even function, so $B_n = 0$. The recursion formula is

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x),$$

which is indeed the recursion formula for the Hermite polynomials. △

Example 4.5.2 (More about Hermite polynomials)**.** Hermite polynomials are also eigenvectors of the differential equation

$$0 = \left(e^{-x^2} \varphi'\right)' + 2n e^{-x^2} \varphi = e^{-x^2} \left(\varphi'' - 2x\varphi' + 2n\varphi\right),$$

which is the Sturm-Liouville eigenvalue problem,

$$L\varphi := -\frac{1}{w(x)} \left[\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + q(x)\varphi\right] = \lambda\varphi,$$

for $w(x) = p(x) = e^{-x^2}$, $\lambda = 2n$ and $q(x) = 0$. Solutions of such a differential equation with different eigenvalues are always orthogonal. This follows from the self-adjointness of L, which can be seen by partial integrating twice,

$$\begin{aligned} \langle L\varphi, \psi\rangle &= -\int \frac{1}{w(x)} (p(x)\varphi'(x))' \psi(x) w(x) \mathrm{d}x - \int \frac{1}{w(x)} q(x)\varphi(x)\psi(x)w(x)\mathrm{d}x \\ &= \int p(x)\varphi'(x)\psi'(x)\mathrm{d}x - \int q(x)\varphi(x)\psi(x)\mathrm{d}x \\ &= \int \varphi(x)(p(x)\psi'(x))'\mathrm{d}x - \int q(x)\varphi(x)\psi(x)\mathrm{d}x = \langle \varphi, L\psi\rangle. \end{aligned}$$

The kernel of this differential operator is the function $G(x,y)$, which satisfies $LG(x,y) = \delta(x-y)$ and can be written in terms of the eigenbasis as

$$G(x,y) = \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{2n h_n},$$

where $h_n = \sqrt{\pi} 2^{-n} n!$ ensures the $H_n(x)$ are properly normalized. △

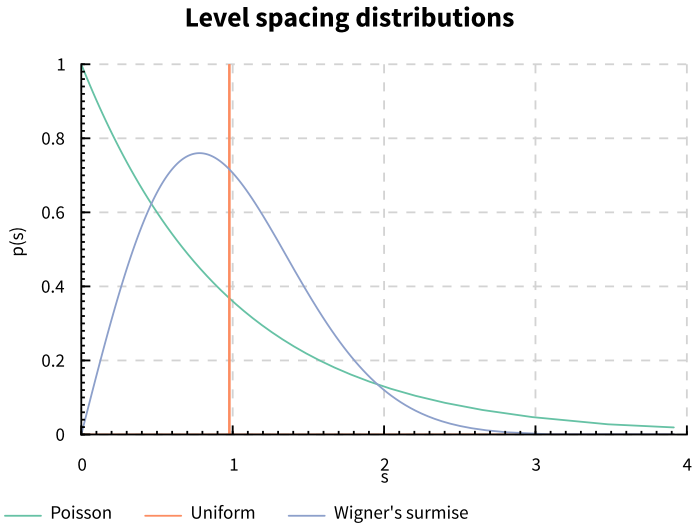


Figure 4: Level spacing distributions for (blue) uncorrelated levels (Poisson), (red) uniformly distributed levels and (purple) eigenvalue distribution of the 2×2 Gaussian Orthogonal Ensemble. Unlike the Poisson distribution, which is peaked at 0, the probability density of the GOE eigenvalue distribution is zero at $s = 0$.

4.6 Level spacing distribution

Due to the term $|x_k - x_j|$, the probability for a random matrix to have two eigenvalues close to one another goes to zero. This is called *level repulsion* and can be seen in Figure 4.

Let $E_1, E_2, \ldots$ be the ordered (energy) levels of a matrix in the ensemble, with spacings $S_i = E_{i+1} - E_i$ and mean spacing $D := \langle S_i \rangle$. Let $s_i := S_i/D$ be the relative spacing. The probability density function $p(s)$ for the s_i is called the *level spacing distribution*. We will consider the three examples depicted in Figure 4.

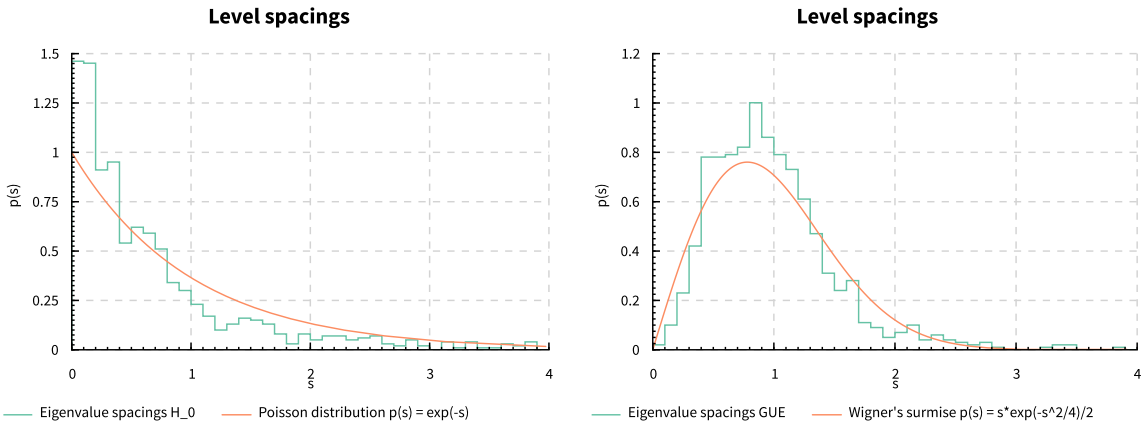
Uniform distribution The spacing is always equal to the mean spacing, that is, $p(s) = \delta(s-1)$, where δ is the Dirac delta.

Uncorrelated energy levels Suppose the energy levels E_i are not correlated. Then, the probability that E_i lies in the interval $(E, E + dE)$ is $\rho \mathrm{d}E$, with $\rho = \frac{1}{D}$ such that the mean spacing is D. Divide the interval $(E, E + S)$ into m sub-intervals [9, Chapter 1.4], then the probability that each of those m sub-intervals of length S/m does not contain a level is approximately $1 - \rho S/m$. In the limit $m \to \infty$,

$$\lim_{m\to\infty} \left(1 - \rho\frac{S}{m}\right)^m = e^{-\rho S},$$

is the probability that the interval $(E, E + S)$ does not contain a level. The probability that $(E, E + S)$ does not contain a level, and $(E + S, E + S + dS)$ does, is hence given by $e^{-\rho S}\rho \mathrm{d}S$. With $s = S/D$, this results in $p(s)\mathrm{d}s = e^{-s}\mathrm{d}s$, the Poisson distribution.

GOE with $N = 2$. The case $\beta = 1$ and $N = 2$ is simple enough for us to calculate, while showing the general principle of level repulsion. This result is called the *Wigner's*



(a) (b)

Figure 5: Example of level spacings of (a) a diagonal matrix with Gaussian distributed entries and (b) a matrix drawn from the GUE.

surmise. Using Eq. (2.27), the PDF for S is given by

$$\begin{aligned} p(S) &= \int P(x_1,x_2)\delta(S-|x_2-x_1|)\mathrm{d}x_1\mathrm{d}x_2 \\ &= \left(2\pi\frac{\Gamma(2)}{\Gamma(3/2)}\right)^{-1}\int \exp\left(-\frac{1}{2}(x_1^2+x_2^2)\right)|x_2-x_1|\delta(S-|x_2-x_1|)\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{2}{4\sqrt{\pi}}\int_{x_2\geq x_1} S\exp\left(-\frac{S^2}{2}-x_1x_2\right)\delta(S-(x_2-x_1))\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{S}{2\sqrt{\pi}}\int \exp\left(-\frac{S^2}{2}-x_1^2-x_1S\right)\mathrm{d}x_1 = \frac{S}{2\sqrt{\pi}}\sqrt{\pi}\exp\left(\frac{S^2}{4}-\frac{S^2}{2}\right) \\ &= \frac{S}{2}e^{-\frac{S^2}{4}}. \end{aligned}$$

We have filled in $x_2 = S + x_1$ in this calculation. The current mean value of S is

$$D = \langle S\rangle = \int_0^\infty S\cdot\frac{S}{2}e^{-\frac{S^2}{4}}\mathrm{d}S = \int_0^\infty e^{-\frac{S^2}{4}}\mathrm{d}S = \frac{1}{2}\sqrt{4\pi} = \sqrt{\pi},$$

hence $S = \sqrt{\pi}s$ and $\mathrm{d}S = \sqrt{\pi}\mathrm{d}s$. In the second step, we used integration by parts twice. The *Wigner's surmise* is then

$$p(s) = \frac{\pi s}{2}e^{-\frac{\pi s^2}{4}}. \tag{4.26}$$

4.6.1 Level number variance

The *level number variance*, Eq. (4.21), characterizes the difference between Poisson and the more evenly spread random matrix statistics, such as the Riemann Zeta zeroes and level statistics of the Erbium atom and Sinai Billiard in Figure 2. In the latter, one finds that the levels lie more closely to the uniform average, so $\Sigma(\bar{n}) \ll 1$.

Consider Eq. (4.21) and take the limit $N \to \infty$. Then, we may assume that the level density is homogeneous $\rho(x) = \rho_0$ and hence $R^N(y,z) = R_\infty(z-y)$. Note that δx contains on average $\rho_0\delta x$ levels, so $\delta x = \bar{n}/\rho_0$. Let $s := (z-y)\rho_0/2$ and integrate over y

at fixed s;

$$\Sigma(\bar{n}) = \int_{\delta x} \rho_0 \mathrm{d}y \int_{\delta x} \rho_0 [R_\infty(z-y) - 1] \mathrm{d}z = \int_{-\bar{n}}^{\bar{n}} (\bar{n} - |s|)[R_\infty(s) - 1] \mathrm{d}s.$$

The term $\bar{n} - |s|$ is the length of the interval over which y is integrated. The level compressibility is given by

$$\chi(\bar{n}) = \int_{-\bar{n}}^{\bar{n}} [R_\infty(s) - 1] \mathrm{d}s.$$

There is no contribution due to the factors of $\bar{n}$ in the integration bounds because for $s = \pm\bar{n}$, $(\bar{n} - |s|)(R_\infty(s) - 1) = 0$. If the normalization sum rule holds, then $\lim_{\bar{n}\to\infty} \chi(\bar{n}) = 0$ and conversely if $\lim_{\bar{n}\to\infty} \chi(\bar{n}) = 0$, the limits $N \to \infty$ and $\bar{n} \to \infty$ commute and the normalization sum rule holds. This might not always be the case. This is called an *anomaly* and it is just a deficiency of the normalization sum rule,

$$\int_{-\infty}^{\infty} [R_\infty(s) - 1] \mathrm{d}s = \alpha, \quad \alpha \neq 0. \tag{4.27}$$

In the absence of the anomaly, the level number variance for the Gaussian ensembles is [45, Section X]

$$\Sigma(\bar{n}) = \frac{2}{\pi^2 \beta} \log \bar{n} + \mathcal{O}(1), \tag{4.28}$$

which is a smaller variance than in the Poisson case,

$$\Sigma(\bar{n}) = \bar{n}. \tag{4.29}$$

5 Multifractality

Random Matrix Theory has historically been a study of the energy level statistics of random Hamiltonians. The model H_0+tV from Section 3.3, which describes the transition from Poisson statistics to Wigner-Dyson (Gaussian) statistics, raises the question what happens with the eigenvectors at this transition. Because H_0 is assumed to be diagonal, supposing all diagonal entries (the eigenvalues) are distinct, then the eigenvectors are the basis vectors of $\mathbb{C}^N$. These eigenvectors are localized. Furthermore, eigenvectors of the Gaussian Unitary Ensemble, due to rotational invariance, are, on average, delocalized. This transition from Poisson to Wigner-Dyson statistics is a localization to delocalization transition. How can we describe the intermediate states?

An important example of a localized to delocalized transition is the *Anderson transition*, which occurs in systems of dimension 1 or 2. This is the transition from a metallic phase to an insulating phase due to either an increase in disorder or a decrease of the energy. In the metallic phase the eigenstates are delocalized, and the eigenvalues satisfy Wigner-Dyson statistics. We say that this is the 'chaotic' phase, not to be confused with 'disorder'. On the other hand, in the insulating phase, the eigenstates are exponentially localized, and the eigenvalues are uncorrelated, hence have Poisson statistics. We say that this is the 'integrable' phase. For sufficiently high disorder, therefore, metallic states become localized. This is *Anderson localization*, the absence of diffusion of wave states because of a sufficiently disordered medium [47].

The phase transition occurs at the critical energy E_c, called the *mobility edge*. At this critical energy, the generalized diffusion propagator $\mathcal{D}(q_1, q_2; \omega)$ changes from Goldstone form

$$\mathcal{D}(k, \omega) = \frac{2\pi\rho(E)}{Dk^2 - i\omega},$$

with D the diffusion constant, $\sigma = e^2\rho D$, in the metallic phase, to a massive form in the insulating phase [8],

$$\mathcal{D}(q_1, q_2; \omega) \approx \frac{2\pi\rho(E)}{-i\omega} \exp(-|q_1 - q_2|/\xi).$$

The generalized diffusion propagator is defined as

$$\mathcal{D}(q_1, q_2; \omega) = \langle G^R(q_1, q_2; E + \omega/2) G^A(q_2, q_1; E - \omega/2) \rangle, \tag{5.1}$$

where $\langle \cdot \rangle$ is the disorder average, that is the ensemble average.

Furthermore, observables such as the localization length ξ in the insulating phase ($E < E_c$), or the conductivity σ in the metallic phase ($E > E_c$), satisfy a certain scaling relation for energies close to the critical energy,

$$\xi \propto (E_c - E)^{-\nu}, \quad \sigma \propto (E - E_c)^{\tau}.$$

The critical exponents ν and τ satisfy $\tau = \nu(d-2)$ [8, Section II.A], first derived by Wegner [48]. A scaling relation also holds for moments of eigenvector components. This leads to the concept of multifractality of wave functions. For this we introduce *generalized dimensions*.

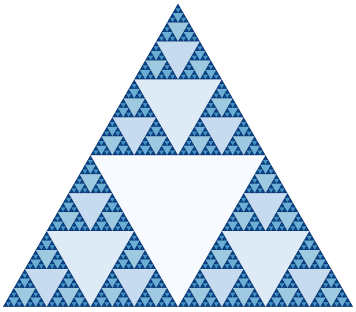


Figure 6: The Sierpinski triangle, obtained recursively by cutting out the center triangle, and repeating this process for the remaining three triangles. It has Hausdorff dimension $\log 3/\log 2$.

5.1 Generalized dimensions

There are many measures of fractal dimensions of a fractal embedded in a d-dimensional space. In this section, we will consider some of these measures. These divide the space into d-dimensional cubes of size b^d first. Assume these cubes are labeled by some index j. Let $B(b)$ be the number of boxes which contain points of the fractal. For a n-dimensional plane embedded in this d-dimensional space, the number of boxes needed is b^{-n}. In general, the *box-counting dimension* is roughly speaking the exponent D, defined by

$$D = -\lim_{b\to 0} \frac{\log B(b)}{\log b}. \tag{5.2}$$

In the small b limit we therefore have $B(b) \sim b^{-D}$. Let p_j be the fraction of points present in the j-th cube. This can be calculated as follows. Approximate the fractal with a finite set of points M and let M_j be the number of points of the fractal in the j-th cube. Then $p_j = \lim_{M\to\infty} M_j/M$. We note that

$$\lim_{q\to 0} p_j^q = \begin{cases} 1 & \text{if the } j\text{-th cube contains points of the fractal,} \\ 0 & \text{if the } j\text{-th cube does not contain points of the fractal,} \end{cases}$$

so that the box counting dimension can also be written as

$$D = -\lim_{q\to 0}\lim_{b\to 0} \frac{1}{1-q} \frac{\log\left(\sum_j p_i^q\right)}{\log b}.$$

For most fractals, the box-counting dimension is equal to the Hausdorff dimension, defined as follows for a self similar object. Let V and L be two parameters describing the object, where V describes the number of lattice sites and L the scale. Then, the Hausdorff dimension is $d_H = \log V/\log L$. For example, consider the Sierpinski triangle show in Figure 6. When multiplying the system size L with 2, the total volume increases by a factor of 3, since the original triangle appears three times in the rescaled version. The Hausdorff dimension is $\log 3/\log 2$.

Other than the box-counting dimension and the Hausdorff dimension, there are two other generalized dimensions, namely the information dimension and correlation dimension. The *information dimension* σ is the critical exponent in the scaling of the Shannon entropy,

$$\sigma = -\lim_{b\to 0} \frac{-\sum_j p_j \log p_j}{\log b}.$$

Finally, the *correlation dimension* ν is defined by

$$\nu = \lim_{b\to 0} \lim_{N\to\infty} \frac{\log C(b)}{\log b}, \tag{5.3}$$

where $C(b)$ counts the number of pairs of points on the fractal with distance less than b,

$$C(b) = \frac{1}{M^2} \sum_{k\neq i} \theta(b - |x_k - x_i|),$$

where the sum is over all points of the fractal, θ is the Heavyside function and x_k is the position of point k. Up to corrections of $\mathcal{O}(1)$, we can assume that $C(b)$ counts the number of pairs of points that belong to the same box,

$$C(b) = \sum_j p_j^2.$$

Hentschell and Procaccia [49] showed that there are in fact an infinite number of generalized dimensions D_q that characterize fractals. Furthermore, they showed that the three dimensions D, σ and ν are special cases of the D_q, namely

$$\begin{aligned} D &= \lim_{q\to 0} D_q, \\ \sigma &= \lim_{q\to 1} D_q, \\ \nu &= D_2. \end{aligned}$$

The generalized dimensions are defined via a generalization of the Shannon entropy, the Rényi entropy [50], which is defined as

$$S_R(q) = \frac{1}{1-q} \log\left(\sum_j p_j^q\right), \tag{5.4}$$

where $0 < q < \infty$ and p_j is the probability of state j occurring. L'Hôpital's rule shows that it can be extended to $q = 1$, where it is the *Shannon entropy*

$$\lim_{q\to 1} S_R(q) = -\sum_j p_j \log p_j.$$

When p_j is the probability that a microstate labeled by j occurs in a thermodynamic system, this is also called the *Gibbs entropy*, assuming $k_B = 1$. The limiting cases $\lim_{q\to 0} S_R(q)$ and $\lim_{q\to\infty} S_R(q)$ are called the *max-entropy* and *min-entropy*, for the reason shown later in Proposition 5.1.1. For quantum systems, a natural choice is $p_j = |\psi_j|^{2q}$, where ψ_j is the j-th component of some distinguished state ψ. Another example of p_j is the number of times M_i a time series visits the i-th box, as a fraction of the length of the time series M and defining $p_j = \lim_{N\to\infty} M_i/M$.

The generalized dimensions are now defined by [49]

$$D_q = -\lim_{b\to 0} \frac{S_R(q)}{\log b}.$$

It can indeed be seen that $D = \lim_{q\to 0} D_q$ and that $\sigma = \lim_{q\to 1} D_q$. Alternatively, since $b \propto L^{-1}$, one could keep the box size fixed b and let the system size L go to ∞,

$$D_q = \lim_{L\to\infty} \frac{S_R(q)}{\log L}. \tag{5.5}$$

Denote with P_q the sum of probability moments

$$P_q = \sum_j p_j^q,$$

then the generalized dimensions discussed in Section 5.1 are expressed as $D_q = \lim_{b\to 0}[\log(P_q)/((q-1)\log b)]$.

We will now look at some properties of these generalized dimensions. Firstly, there are bounds on the values D_q can take.

Proposition 5.1.1. *The fractal dimensions are decreasing,*

$$D_q \geq D_{q'}, \quad \textit{for } q < q'.$$

Proof. Let $q < q'$, $\|\cdot\|_q$ be the q-norm, then we have the following bound on a vector $p \in \mathbb{C}^n$,

$$\|p\|_{q'}^{q'} \leq N^{1-q'/q}\|p\|_q^q.$$

If $p \propto (1, 1, \ldots, 1)$ this inequality becomes an equality, and otherwise the inequality is strict. The norm inequality can be shown using Hölder's inequality

$$\|fg\|_1 \leq \|f\|_{r/(r-1)}\|g\|_r,$$

with $f = 1$, $r = q/q'$, and g with components $g_j = p_j^{q'}$, $1 \leq j \leq N$. Furthermore, if $q' > q$,

$$\frac{1}{q'-1} < \frac{1}{q-1}.$$

These two inequalities can be combined to give the inequality

$$\frac{1}{q'-1}\log\|p\|_{q'}^{q'} \leq \frac{1}{q-1}\log\|p\|_q^q,$$

from which the result follows. □

Corollary 5.1.2. *The generalized dimensions D_q are bounded between D_0 and D_∞.*

Remark 5.1.3. If one looks at the proof of Proposition 5.1.1, there are two important cases where the D_q are not all different, namely the homogeneous case $p = (N^{-1}, N^{-1}, \ldots, N^{-1})$ and the fully localized case $p = (0, \ldots, 0, 1, 0, \ldots, 0)$, as in the latter case $\|p\|_q = 1$ for all q, hence $\log\|p\|_q^q = 0$ for all q. In the former, $D_q = d$ for all q, while in the latter $D_q = 0$ for all q.

In other cases, there may be an infinite set of generalized dimensions. This is called *multifractality*.

Let us give an interpretation of D_q for integer $q > 2$. As in Eq. (5.3), the integer generalized dimensions can also be written as a scaling law on some correlation function [49]. Define the higher order correlation functions $C_n(l)$ by

$$C_n(l) = \lim_{M\to\infty} \frac{1}{M^n}[\text{Number of } n\text{-tuples of points } (i_1, \ldots, i_n) \text{ whose pairwise distances } |x_{i_\alpha} - x_{i_\beta}| \text{ are less than } l \text{ for all } \alpha, \beta. \ ]$$

A more geometric interpretation of D_q is found using the *singularity spectrum*, following Halsey et al. [51]. For this, we look at the scaling relation of each box individually. Let indices α_i be the scaling exponent of the i-th box,

$$p_i^q \sim b^{\alpha_i q}.$$

Halsey et al. [51] suggest that the number of times α_i takes a value between α and $\alpha + \mathrm{d}\alpha$ is of the form

$$b^{-f(\alpha)}\mathrm{d}\alpha, \tag{5.6}$$

so that P_q becomes

$$P_q = \int b^{q\alpha - f(\alpha)}\mathrm{d}\alpha.$$

In the small b limit, one can do a saddle point approximation and obtain

$$P_q \propto b^{q\tilde{\alpha} - f(\tilde{\alpha})}, \tag{5.7}$$

where $\tilde{\alpha}$ is a minimum of $q\alpha - f(\alpha)$, hence $\tilde{\alpha}$ is a solution of $q = f'(\tilde{\alpha})$. From Eq. (5.4), it follows that $(q-1)D_q = q\tilde{\alpha} - f(\tilde{\alpha})$. This is a Legendre transformation.

The box counting dimension Eq. (5.2) of the set of boxes i which have equal α_i, Eq. (5.6), is given by

$$D = -\lim_{b\to 0} \frac{\log\left(b^{-f(\alpha)}\ \mathrm{d}\alpha\right)}{\log b} = f(\alpha),$$

which gives $f(\alpha)$ the interpretation of the box counting dimension of the set i, where p_i scales as b^α. Since the box counting dimension usually agrees with the Hausdorff dimension, also called just the fractal dimension, in some papers, for example [8, Section C2], this interpretation is found with 'box counting dimension' replaced by 'fractal dimension' or 'Hausdorff dimension'.

5.1.1 Inverse participation ratios

Let us now turn to the quantum random matrix case. We specialize to $p_j = |\psi_j|^2$ for some eigenstate ψ_j with energy E_j, and add an ensemble average in the definitions of fractal dimensions. To make the dependence on E_j explicit, we write $P_{n,i}$ instead of P_n. The Rényi entropy becomes

$$S_R(n) = \frac{1}{1-n}\log\left(\sum_j |\psi_{i,j}|^{2n}\right).$$

Let us also take the continuous limit and replace the sum over j by an integral. Then, we can write

$$S_R(n) = \frac{1}{1-n}\log(P_{n,i}),$$

where

$$P_{n,i} = \int \mathrm{d}^d q |\psi_i(q)|^{2n} = \|\psi_i\|_{L^{2n}}^{2n},$$

where $\|\cdot\|_{L^p}$ is the p-norm on the Lebesgue space L^p. The quantity $P_{n,i}$ is called the *Inverse Participation Ratio* (IPR). Usually, interesting phenomena will only occur if disorder is added. One would then take a disorder average $\langle\cdot\rangle$ of these quantities, and look at eigenstates with a certain (critical) energy,

$$F_n(E) := \langle \sum_i P_{n,i} \delta(E_i - E) \rangle = \int \mathrm{d}^d q \sum_i \langle |\psi_i(q)|^{2n} \delta(E_i - E) \rangle.$$

Under the assumption that the ensemble average, also called the disorder average, kills the q dependence of $\langle |\psi_i(q)|^{2n} \rangle$, we can do the integral over q and obtain

$$F_n(E_c) = L^d \sum_i \langle |\psi_i|^{2n} \delta(E_i - E_c) \rangle, \tag{5.8}$$

where we write $|\psi_i|$ for $|\psi_i(q)|$. The generalized dimensions will now describe the scaling of the inverse participation ratios, and by definition we have

$$F_n(E) \propto L^{-\tau_n},$$

where $\tau_n = D_n(n-1)$ and L is the system size.

One of the most prominent examples of where multifractality of the eigenfunctions occurs, is the Anderson transition, where near the mobility edge E_c, the fractal dimensions D_n are nontrivial [52]. These critical exponents depend only on the basic symmetries of the Hamiltonian H and on the dimension of the space d. We say that $P_{n,i}$ shows an anomalous scaling with the system size L. For a metallic state, we have $D_n = d$ and for the insulating state $D_n = 0$. The difference Δ_n with the scaling of extended states is called the *anomalous dimensions*, and defined by the relation

$$\tau_n = d(n-1) + \Delta_n.$$

Consider now the eigenvector probability density $\mathcal{P}(|\psi|^2)$, that is, let $\mathcal{P}(|\psi|^2)$ be such that

$$\langle |\psi|^{2n} \rangle = \int_0^\infty s^n \mathcal{P}(s) \mathrm{d}s,$$

where one should think of s being $|\psi|^2$. Define α by $s = L^{-\alpha}$, that is, $\alpha = -\log s / \log L$, then $\mathrm{d}s/\mathrm{d}\alpha = -L^{-\alpha} \log L$. With a change of variables from s to α we have

$$\langle P_n \rangle = L^d \int \log(L) L^{-(n+1)\alpha} \mathcal{P}(L^{-\alpha}) \mathrm{d}\alpha.$$

The $\log(L)$ is unimportant for the asymptotics, and we drop this factor. The analog of Eq. (5.6) is the following. The singularity spectrum f is such that [8, Section C2]

$$\mathcal{P}(s) \sim \frac{1}{s} L^{-d + f(-\log s / \log L)},$$

then

$$\langle P_n \rangle = \int L^{-n\alpha + f(\alpha)} \mathrm{d}\alpha. \tag{5.9}$$

As in Eq. (5.7) $f(\alpha)$ is related to $\tau_n = D_n(q-1)$ by a Legendre transform, and has the interpretation of being the box counting dimension of the set of points q, where $|\psi_i(q)|^2$ scales as $|\psi_i(q)|^2 \sim N^{-\alpha}$.

Example 5.1.4 (Completely delocalized states)**.** The ℓ^p-norm of completely delocalized states ψ, with absolute values of components $|\psi_i| = N^{-1/2}$, such that $\|\psi\|_{\ell^2} = 1$, is given by

$$\|\psi\|_{\ell^p}^p = \sum_{j=1}^{N} |\psi_j|^p = NN^{-p/2} = N^{1-p/2}.$$

△

Example 5.1.5 (Gaussian Orthogonal Ensemble)**.** The Gaussian Orthogonal Ensemble describes a metallic state, where $D_n = d$. We have the following estimate for the p-norms [53, Theorem 2.2].

Theorem 5.1.6 (ℓ^p-norm)**.** *The eigenvectors of the GOE are random vectors uniformly distributed in the unit sphere S^{N-1}. For any such random vector v and $p \geq 1$, there is a $c_p > 0$, such that*

$$\|v\|_{\ell^p}^p = N^{1-p/2}c_p + \textit{lower orders in } N,$$

for almost all $v \in S^{N-1}$.

Take $n = 2p$, $E_c = E_i$ for some i and $v = \psi_i$. Eq. (5.8) holds with $D_n = 1$.

An important ingredient in this proof is the fact that the entries are independently distributed from a PDF, which has a Gaussian tail and mean 0. △

Example 5.1.7 (Gaussian Unitary Ensemble)**.** The probability density $P_{GUE}(s)$ of one component $|\psi_{i,j}|^2$ to take value s is given by [41, (5.10.13)]

$$P_{GUE}(s) = \begin{cases} (N-1)(1-s)^{N-2} & \text{for } 0 \leq s \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that after normalization $\eta = sN$ this results in the Porter-Thomas distribution

$$P_{GUE}(\eta) = e^{-\eta}.$$

The eigenfunction moments are now, for real $n > 0$,

$$\begin{aligned} \sum_{i=1}^{N} \langle |\psi_{i,j}|^{2n} \rangle &= N(N-1) \int_0^1 s^n (1-s)^{N-2} \mathrm{d}s = (N-1)B(n+1, N-1) \\ &= N(N-1)\frac{\Gamma(n+1)\Gamma(N-1)}{\Gamma(n+N)} = \frac{\Gamma(n+1)\Gamma(N+1)}{\Gamma(n+N)}. \end{aligned}$$

Here, $B(\cdot,\cdot)$ is the beta function, and $\Gamma(x)$ is the gamma function. Then, the finite N approximations of the fractal dimensions are [54, (35)]

$$\begin{aligned} D_n(N) &= \frac{1}{1 - n\log N} \log\left(\frac{\Gamma(n+1)\Gamma(N+1)}{\Gamma(n+N)}\right) \\ &\approx 1 - \frac{\log(\Gamma(n+1))}{(n-1)\log N}. \end{aligned}$$

Indeed, as $N \to \infty$, $D_n \to 1$, with logarithmic corrections. △

Example 5.1.8 (Diagonal matrices)**.** Diagonal matrices have eigenvectors e_i, $1 \leq i \leq N$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^N$ or $\mathbb{C}^N$. The eigenvectors are completely localized, which is seen by writing e_i as $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$. We have, independent of N,

$$\|e_i\|_{\ell^p}^p = 1, \quad \text{for all } p > 0.$$

Let $\langle \cdot \rangle$ be an ensemble average over a certain set of diagonal matrices, such that the subset of diagonal matrices where there is an energy with multiplicity greater than 1 is of measure 0. We can conclude that

$$F_n(E) \propto N^0 \quad \text{for all } n > 0.$$

Therefore, $D_n = 0$ for all $n > 0$. △

Example 5.1.9 (Random band matrices)**.** The Random band matrices are a transition between the diagonal matrices and the Gaussian ensembles. The Random band matrices of band width L can be constructed as follows. First sample a matrix H from a Gaussian ensemble. Then set the (i, j)-entry in H to zero for all i, j sufficiently off-diagonal $|i-j| \leq L$. For $L = 1$ we get diagonal matrices, the case $L = N$ results in the Gaussian ensemble. The special case $L = 1$ are the tri-diagonal matrices.

A variant of this are the Power Law Banded Random matrices, where standard deviation depends on the value of $|i - j|$. △

5.1.2 Rényi entropy

In this subsection, we put forward a relation presented by Chen et al. [55] between the multifractal spectrum and the Rényi entanglement entropies. The relation should provide more insight in the nature of integrable to chaotic transition, namely that the system becomes more entangled. The Rényi entanglement entropies differ slightly from the Rényi entropies used to define the multifractal dimensions. This subsection is based on Chen et al. [55].

In the following, let $\rho = |\psi\rangle\langle\psi|$ be the density matrix. Here, ψ is some quantum state, for example one may choose $\psi = \psi_i$, the i-th eigenvector. As this is a pure state, $\rho^2 = \rho$ and $\operatorname{tr}(\rho) = 1$, and hence both the von-Neumann entropy and the Rényi entropy are zero,

$$S = -\operatorname{tr}(\rho \log \rho) = 0,$$
$$S_q = \frac{1}{1-q} \log(\operatorname{tr} \rho^q) = 0.$$

The *entanglement entropy*, on the other hand, will not be zero. Partition the system into two subregions A and B, and only observe part A. For example, we may choose A to be the single site j, and B to be all other sites. The entanglement entropy is the entropy of the reduced density matrix $\rho_A = \operatorname{tr}_B \rho$, where the trace is over all possible configurations of region B,

$$\hat{S}_q = \frac{1}{1-q} \log \operatorname{tr} \rho_A^q.$$

The hat indicates that this is an entanglement entropy, to avoid confusion with the general Rényi entropy definition Eq. (5.4), where one still has to choose the probabilities p_j. It can be shown that [55],

$$\operatorname{Tr} \rho_A^q = p_A^q + p_B^q,$$

where p_A and p_B are the probabilities that ψ lies in region A and B, respectively. In the case A is a single site j, we have $p_A = |\psi_j|^2$ and $p_B = 1 - |\psi_j|^2$. The site j is not more special than any other site, so we look at the average over all sites j, denoted by $[\cdot]_j$,

$$\begin{aligned}[\operatorname{tr}\rho_A^n]_j &= \frac{1}{N}\sum_j \left((1-|\psi_j|^2)^q + |\psi_j|^{2q}\right) \\ &= \frac{1}{N}\left[\sum_{m=0}^{N}\binom{q}{m}(-1)^m \sum_j |\psi_j|^{2m} + \sum_j |\psi_j|^{2n}\right]\end{aligned}$$

Now, $\sum_j |\psi_j|^{2q}$ is the inverse participation ratio P_q, which is assumed to scale as $L^{-\tau_n}$, where $N = (a/L)^d$. Here a is the lattice constant, L the system size and d the system dimension. For $|q| < 1$, we now have the approximation

$$[\operatorname{tr}\rho_A^q]_j \approx 1 - q(a/L)^d + (a/L)^{\tau_q + d},$$

which results in the following approximation for the site averaged Rényi entanglement entropy [55],

$$(1-n)S_q \sim -q\Big(\frac{a}{L}\Big)^d + \Big(\frac{a}{L}\Big)^{\tau_q+d}.$$

This relates the multifractal dimensions to the scaling of the site averaged Rényi entanglement entropy.

5.2 Level compressibility

Wave function multifractality is a characteristic of the intermediate state between localized (non-diffusive) and extended (diffusive) wave functions. It occurs amongst others in the Anderson transition if the wave function has a certain energy, called the *mobility edge*. We expect that the level compressibility at criticality lies in between the Poisson result ($\chi = 1$, fully compressible) and the Wigner-Dyson result ($\chi = 0$, incompressible). This is derived for disordered metals at the mobility edge using the diffusion propagator, and this derivation is originally due to Chalker, Lerner and Smith [56], which we also follow in this section.

We consider the following Brownian motion path through an ensemble of disordered metals, with a fictitious time parameter τ,

$$\begin{aligned}H(\tau) &= H_0 + \int_0^\tau \mathrm{d}\tau' V(\tau', \vec{r}), \\ H_0 &= -\frac{\hbar^2}{2m}\nabla^2 + U(\vec{r}),\end{aligned}$$

where $U(\vec{r})$ and $V(\tau, \vec{r})$ are two Gaussian distributed potentials, with zero mean and standard deviation

$$\langle U(\vec{r})U(\vec{r}')\rangle = \frac{\hbar}{2\pi\rho t_{el}}\delta(\vec{r}-\vec{r}') \tag{5.10a}$$

$$\langle V(\tau,\vec{r})V(\tau',\vec{r}')\rangle = v^2 L^d \delta(\tau-\tau')\delta(\vec{r}-\vec{r}'). \tag{5.10b}$$

Here, t_{el} is the time scale for electron scattering on impurities, and ρ is the average spectral density, $\rho = \langle \varrho(E)\rangle$. Using perturbation theory, we have

$$\delta E_n = V_{nn} + \sum_{m\neq n}\frac{V_{mn}V_{nm}}{E_n - E_m},$$

where

$$V_{nm} = \int_{\tau}^{\tau+\delta\tau} \mathrm{d}\tau' \langle n|V(\tau', \vec{r})|m\rangle,$$
$$\langle \vec{r}|n\rangle = \psi_n(\tau', \vec{r}),$$

with $\psi(\tau', \vec{r})$ the eigenfunctions of $H(\tau')$. The average over all V, using Eq. (5.10b), results in [56]

$$\langle V_{nm}V_{mn}\rangle = \langle V_{nn}V_{mn}\rangle = v^2 \int \mathrm{d}\tau c_{nm}(\tau),$$

where the correlation functions $c_{nm}(\tau)$ are given by

$$c_{nm}(\tau) = L^d \int \mathrm{d}^d r |\psi_n(\tau, r)|^2 |\psi_m(\tau, r)|^2.$$

From this, it follows that the energy levels satisfy the following Langevin equation,

$$\frac{\mathrm{d}E_n(\tau)}{\mathrm{d}\tau} v^2 \sum_{l\neq 0} \frac{c_{n,n+l}(\tau)}{E_n(\tau) - E_{n+l}(\tau)} + \xi_n(\tau),$$

for some random variable $\xi_n(\tau)$ with zero mean and variance $\langle \xi_n(\tau)\xi_m(\tau')\rangle = v^2\delta(\tau - \tau')c_{nm}(\tau)$. This Langevin equation differs from the Langevin equation from Dyson Brownian motion in that the variances of V_{nm} depend on the eigenvectors of $H(\tau)$. If this would not be the case, it could also be solved exactly using the Fokker-Planck equation. Chalker, Lerner and Smith [56] showed that the spectral form factor,

$$K(t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} R_\infty(E) e^{-iEt/\hbar} \mathrm{d}E,$$

can be expressed in terms of the return probability $p(t)$ as

$$K(t) = \frac{|t|p(t)}{(2\pi)^2 + 4\pi \int_0^t p(t')\mathrm{d}t'}, \tag{5.11}$$

if some approximations are made. This relation holds for times shorter than the Heisenberg time $t_H = \hbar/\Delta$, where Δ is the mean level spacing. Note that the level compressibility is given by $\chi = 2\pi K(0)$.

We will now follow [46] and show a relation between D_2 and the deficiency of the normalization sum rule α (Eq. (4.27)), but much of the derivation was also already present in Chalker, Lerner and Smith [56]. The return probability $p(t)$ can be expressed in terms of the diffusion propagator $P(\vec{q}, \omega)$,

$$p(t) = \frac{2}{\beta} \int \frac{d\omega}{2\pi} \sum_{\vec{k}} P(\vec{k}, \omega) e^{-i\omega t/\Delta}.$$

For metals and sufficiently small $\omega \ll D/L^2$, a delta shaped wave packet at the origin will spread out homogeneously throughout the system, hence the sum over all momenta reduces to only the $\vec{k}$ contribution (there are no nonzero modes). The diffusion propagator is given by $P(\vec{k}, \omega) = (Dk^2 - i\omega)^{-1}$, so $p(t) = 2/\beta$. We generalize the diffusion coefficient to $D(k, \omega)$ and require that it is independent of k at $kL_\omega \ll 1$, and independent of ω at

$kL_\omega \gg 1$, where $L_\omega := (-i\rho\omega)^{-1/d}$. This results in $D(k,\omega) \propto k^{d-2}$, where $k = |\vec{k}|$, for large k. Let us now take multifractality into account. It should also be visible in the large k limit, so a natural assumption is

$$k^2 D(k,\omega) = C k^{d-\eta} L_\omega^{-\eta}.$$

With this assumption, for large k, the return probability is given by

$$p(t) \sim \frac{(\Lambda L)^\eta}{t^{1-\frac{\eta}{d}}}.$$

Let us fill this in into Eq. (5.11), with $Q = (\Lambda L)^\eta$, and impose the thermodynamic limit, in which Q is divergent,

$$\begin{aligned} K(t) &= |t| Q t^{\frac{\eta}{d}-1} \left((2\pi)^2 + 4\pi \int_0^t Q t'^{\frac{\eta}{d}-1} \mathrm{d}t' \right)^{-1} \\ &= |t| Q t^{\frac{\eta}{d}-1} \left((2\pi)^2 + 4\pi \frac{d}{\eta} Q t^{\frac{\eta}{d}} \right)^{-1} \\ &\xrightarrow[Q\to\infty]{} \frac{|t| t^{\frac{\eta}{d}-1}}{4\pi \frac{d}{\eta} t^{\frac{\eta}{d}}} = \frac{|t|}{t} \frac{1}{4\pi \frac{d}{\eta}}, \end{aligned}$$

hence

$$\chi = 2\pi K(0) = \frac{\eta}{2d}.$$

This is the relation between the violation of the normalization sum rule and multifractality. Furthermore, we see that for critical disordered metals, the level compressibility lies between values of Gaussian ensembles, $\chi = 0$ (Eq. (4.28)) and the Poisson result $\chi = 1$ (Eq. (4.29)).

5.3 A multifractal ensemble

Let us now give an example of a deformed ensemble satisfying multifractal statistics. To this end, consider the space of Hermitian matrices H with PDF [57]

$$P(H) \propto \exp(-\beta \operatorname{tr} V(H)), \qquad V(x) = \sum_{n=0}^{\infty} \log[1 + 2q^{n+1}(1+2x^2) + q^{2(n+1)}],$$

with $0 < q < 1$ a control parameter. For large $|x|$, the potential converges to a $\log^2$,

$$V(x) \to A \log^2 |x|, \quad A = \frac{2}{\log(q^{-1})}.$$

It turns out that the kernel $K(s-s')$ for $\log q^{-1} \ll 1$ is [58]

$$K(s-s') = \pi\kappa \frac{\sin(\pi(s-s'))}{\sinh(\pi^2\kappa(s-s'))},$$

for some κ, so that the two level correlation function (which is the determinant of a 2×2 matrix of K's) for $N \to \infty$ is

$$R_\infty(s-s') = 1 - \pi^2\kappa^2 \frac{\sin^2(\pi(s-s'))}{\sinh^2(\pi^2\kappa(s-s'))}.$$

The deficiency of the normalization sum rule α,

$$\alpha = \int_{-\infty}^{\infty} \pi^2\kappa^2 \frac{\sin^2(\pi s)}{\sinh^2(\pi^2\kappa s)} \mathrm{d}s$$

lies between Wigner-Dyson and Poisson statistics $0 < \alpha < 1$. Indeed, it is clear that $\alpha > 0$ and furthermore

$$\alpha < \int_{-\infty}^{\infty} \pi^2\kappa^2 \frac{\sin^2(\pi s)}{(\pi^2\kappa s)^2} \mathrm{d}s = \int_{-\infty}^{\infty} \frac{\sin^2(\pi s)}{\pi^2 s^2} \mathrm{d}s = 1.$$

The precise form of $V(x)$ allows for an exact solution in terms of q^{-1}-Hermite polynomials. For this calculation we refer the reader to [58] and [59]. We will only write the weight function in terms of the third Jacobi theta function here. For this, denote with $(a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1})$ the q-Pochhammer symbol or q-shifted factorial. Furthermore, write $(a_1, a_2, \ldots; q)_n = (a_1, q)_n (a_2, q)_n \ldots$ for the product of q-shifted factorials. With u defined in terms of x by $x = \sinh(u)$, the weight function $w(x;q) = \exp(-V(x))$ can be written as [58, Eq. (2.2)]

$$\begin{aligned} w(x;q) &= \exp\left(-\sum_{n=0}^{\infty} \log\bigl(1 + 2q^{n+1}(1+2x^2) + q^{2(n+1)}\bigr) \right) \\ &= \left[1 + 2q^k \cosh(2u) + q^{2k}\right]^{-1} \\ &= \left[\prod_{k=1}^{\infty} (1 + q^k e^{-2u})(1 + q^k e^{2u}) \right]^{-1} \\ &= (-qe^{-2u}, -qe^{2u}; q)_{\infty}^{-1}. \end{aligned}$$

Jacobi's third theta function is defined as (see for example [31, p. 463])

$$\theta_3(z;q) := \sum_{n\in\mathbb{Z}} q^{n^2/2} z^n. \tag{5.12}$$

It satisfies a triple product expansion (see for example [31, p. 469-472]),

$$\begin{aligned} \theta_3(z,q) &= (q, -q^{1/2}z, -q^{1/2}z^{-1})_{infty} = (q;q)_{\infty}(q; -q^{1/2}z)_{\infty}(q; -q^{1/2}z^{-1})_{\infty} \\ &= (q;q)_{\infty} \prod_{k=1}^{\infty} (1 + q^{k-1/2}z)(1 + q^{k-1/2}z^{-1}). \end{aligned} \tag{5.13}$$

Choose now $z = q^{1/2}e^{2u}$ to obtain

$$\frac{1}{w(x;q)} = \frac{\theta_3(q^{1/2}e^{2u}; q)}{(q;q)_{\infty}},$$

which establishes the relation with Jacobi's third theta function.

5.4 Methods for calculating eigenvectors

Random matrix theory has been successful for describing eigenvalue statistics. The main ingredient for this success is the conjugation invariance of the probability measures, which

reduces the problem significantly. Furthermore, the resulting Jacobian contained a Vandermonde determinant. This has allowed us to derive the Coulomb gas analogy in Section 2.4, resulting in the Wigner semicircle law, and is the basis of the orthogonal polynomial method via Dyson's theorem, as will be discussed in Section 4.1. Furthermore, Guhr [37] also used the Coulomb gas analogy to obtain exact results for the transition from Poisson to Gauss, see Section 3.3.

It is therefore no surprise that the calculation of eigenvector statistics, primarily the inverse participation ratios, has proved to be more difficult. The nature of this problem kills the unitary invariance; the eigenvectors are the column vectors of the unitary matrix used to diagonalize the Hermitian matrix. In this section, we will explore methods which try to use only the eigenvalues of the random matrices, but still provide useful results about eigenvector statistics.

In the following, assume H is a Hermitian matrix with eigenvalues $\lambda_1(H), \dots, \lambda_N(H)$. As the set of degenerate matrices H is usually of measure zero, we will assume that all eigenvalues are distinct. Furthermore, it is assumed that the eigenvalues are ordered $\lambda_1(H) < \cdots < \lambda_N(H)$. We will also sometimes drop the explicit H, and write λ_i for $\lambda_i(H)$. Let v_i be the eigenvector corresponding to $\lambda_i(H)$ for $i = 1, \dots, N$, with components $v_{i,j}$, $j = 1, \dots, N$. Assume the eigenvectors have norm 1, $\|v_i\|^2 = \sum_j |v_{i,j}|^2 = 1$. Finally, let M_{ij} be the submatrix of H obtained by deleting the i-th row and j-th column, with eigenvalues $\lambda_1(M_{ij}), \dots, \lambda_{N-1}(M_{ij})$ also ordered non-decreasingly.

The *resolvent* of H is the matrix $G(\lambda; H) = (\lambda I - H)^{-1}$, which can be written in terms of the eigenbasis as

$$G(\lambda; H) = \sum_{j=1}^{N} \frac{v_j v_j^\dagger}{\lambda - \lambda_j(H)}.$$

The key observation is now that the projection on the i-th eigenspace can be extracted by taking the residue

$$v_i v_i^\dagger = \frac{1}{2\pi i} \oint_{C_{\lambda_i}} (\lambda I - H)^{-1} \mathrm{d}\lambda, \tag{5.14}$$

where C_{λ_i} is a contour in the complex plane around the eigenvalue λ_i and λ_i only. In the following, we will just drop the C and write $\oint_{\lambda_i}$ for $\oint_{C_{\lambda_i}}$. Let now e_j be the j-th basis vector of $\mathbb{C}^N$, and define $B_j = e_j e_j^\dagger$, then the j-th component of v_i can be calculated using Eq. (5.14) as

$$|v_{i,j}|^2 = \mathrm{tr}\Big(v_i v_i^\dagger B_j\Big). \tag{5.15}$$

Using Cramer's rule, the resolvent can be written as the ratio of two determinants. A ratio of two determinants gives rise to a nonlinear sigma model, by writing both determinants as a Gaussian integral with either commuting (bosonic) or anti-commuting (fermionic) variables. Also, with Cramer's rule, Eq. (5.15) results in an old elementary identity, popularized again in 2019 by Denton et al. [14], and called the *eigenvalue-eigenvector identity*. In my opinion, this name is misleading because the identity also involves the determinant of the ij-th submatrix. This is important because this minor introduces the preferred basis. Therefore, we will call this identity the *projected Cramer's rule*.

Theorem 5.4.1 (Cramer's rule)**.** *The inverse of a matrix A, if it exists, is given by*

$$A^{-1} = \frac{1}{\det A} \mathrm{adj}(A),$$

where $\mathrm{adj}(A)$ *is the adjugate matrix, which has entries* $(\mathrm{adj}(A))_{ij} = (-1)^{i+j} \det M_{ji}$*, where* M_{ji} *is the* ji*-th submatrix of* H*.*

Corollary 5.4.2 (Projected Cramer's rule)**.** *Let* $B_j = e_j e_j^\dagger$*,* $i, j = 1, \ldots, N$*, then*

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{N-1}(\lambda_i(H) - \lambda_k(M_{jj}))}{\prod_{k=1}^{N}(\lambda_i(H) - \lambda_k(H))}. \tag{5.16}$$

Proof. Fill in Eq. (5.14) into Eq. (5.15), and replace $(\lambda - H)^{-1}$ using Cramer's rule, with $A = \lambda - H$,

$$|v_{i,j}|^2 = \frac{1}{2\pi i} \oint_{\lambda_i(H)} \frac{\mathrm{tr}(\mathrm{adj}(\lambda - H)B_j)}{\det(\lambda - H)} \mathrm{d}\lambda = \mathrm{Res}_{\lambda \to \lambda_i(H)} \frac{\det(\lambda - M_{jj})}{\det(\lambda - H)},$$

from which Eq. (5.16) can be deduced by writing the determinant as the product over the eigenvalues. □

Remark 5.4.3. Use Cramer's rule again, with $A = H - \lambda$ to get

$$(H - \lambda)\mathrm{adj}(H - \lambda) = \det(H - \lambda) I.$$

For eigenvalues $\lambda = \lambda_i(H)$, $\det(H - \lambda_i) = 0$. Then, each column vector of $\mathrm{adj}(H - \lambda_i(H))$ satisfies the eigenvector equation, so each nonzero column of $\mathrm{adj}(H - \lambda_i(H))$ is an eigenvector of H. There exists at least one such nonzero column because $\mathrm{adj}(H - \lambda_i(H))$ has rank equal to the degeneracy of $\lambda_i(H)$. This fact will be used in Section 6.2. Here, we assumed that H is not proportional to the identity matrix, that is $H \neq cI$ for all $c \in \mathbb{C}$. In this case any vector is an eigenvector. We ignore this case, because for the ensembles we consider, this set is of measure zero.

Without taking the residue, Eq. (5.16) is

$$\sum_{i=1}^{N} \frac{|v_{i,j}|^2}{\lambda - \lambda_i(H)} = \frac{\det(\lambda - M_{jj})}{\det(\lambda - H)}. \tag{5.17}$$

Furthermore, by doing a row expansion around the j-th row of $\det(\lambda - H)$, we see that

$$\frac{\partial}{\partial J} \det(\lambda - H - JB_j) = \det(\lambda - M_{jj}).$$

This is the starting point of the nonlinear sigma model, where the partition function

$$\mathcal{Z}_j(x, J) := \frac{\det i(\lambda - H - JB_j)}{\det i(\lambda - H)}, \tag{5.18}$$

is calculated by writing the determinants as Gaussian integrals over bosonic and fermionic fields. For example, for the denominator the Gaussian integral is

$$\sqrt{\frac{(2\pi)^N}{\det A}} = \int \exp\left(-\frac{1}{2} x^T A x\right) \mathrm{d}^N x,$$

where x can be thought of as a bosonic field. In Appendix A, this technique is used to derive the inverse participation ratios for the Gaussian Unitary Ensemble, but the method can be extended to allow for perturbative corrections on the GUE result.

Another application of Eq. (5.14) is *Morozov's formula*. Suppose here that H is already diagonalized, namely let $X = \operatorname{diag}(\lambda_1(H), \dots, \lambda_N(H))$ and let U be the unitary matrix of eigenvectors, such that $H = UXU^\dagger$. As e_j for $j = 1, \dots, N$ are eigenvectors of X, the projection on the eigenspace Eq. (5.14) becomes

$$e_i e_i^\dagger = \frac{1}{2\pi i} \oint_{\lambda_i(H)} (\lambda I - X)^{-1} \mathrm{d}\lambda.$$

Suppose also that Y is another matrix with eigenvalues $\lambda_1(Y), \dots, \lambda_N(Y)$ all distinct. Then, we have

$$|U_{i,j}|^2 = \frac{1}{(2\pi i)^2} \oint_{\lambda_i(X)} \mathrm{d}x \oint_{\lambda_j(Y)} \mathrm{d}y \operatorname{tr}\left(U^\dagger \frac{1}{x - X} U \frac{1}{y - Y} \right).$$

For the unitary group with measure $\mathrm{d}U e^{XU^\dagger YU}$, the expectation value of the integrand $\operatorname{tr}\big(U(x - X)^{-1}U^\dagger(y - Y)^{-1}\big)$ evaluates to [38],

$$\int_{U(N)} \mathrm{d}U \operatorname{tr}\left(U^\dagger \frac{1}{x - X} U \frac{1}{y - Y} \right) e^{\operatorname{tr}\left(UXU^\dagger Y\right)} = \frac{\det E - \det\left(E - \frac{1}{x-X} E \frac{1}{y-Y}\right)}{\Delta(X)\Delta(Y)}, \tag{5.19}$$

where $\Delta(X)$ is the Vandermonde determinant and E is the matrix with entries $E_{ij} = e^{x_i y_j}$. Eq. (5.19) is called *Morozov's formula*. This formula can be applied to calculate expectation values of eigenvector components in the Poisson to GUE transition using the model described in Section 3.3.

More generally, let $\pi, \rho \in S_n$ be two permutations of length n. Choose two sets of indices, $\{i_1, \dots, i_n\} \subset \{1, \dots, N\}$ and $\{j_1, \dots, j_n\} \subset \{1, \dots, N\}$. Then [32, Eq. (6.1.21)],

$$\begin{aligned} &U_{i_1, j_{\pi(1)}} \dots U_{i_n, j_{\pi(n)}} U^\dagger_{j_{\rho(1)}, i_1} \dots U^\dagger_{j_{\rho(n)}, i_n} \\ &\qquad = \frac{1}{(2\pi i)^{2n}} \oint_{X_{i_1}} \mathrm{d}x_1 \cdots \oint_{X_{i_n}} \mathrm{d}x_n \oint_{Y_{i_1}} \mathrm{d}y_1 \cdots \oint_{Y_{i_n}} \mathrm{d}y_n H(\vec{x}, \vec{y}; UXU^\dagger, Y)_{\pi,\rho}, \end{aligned}$$

where $X_i := C_{\lambda_i(X)}$ is a contour around $\lambda_i(X)$ and

$$H(\vec{x}, \vec{y}; X, Y)_{\pi,\rho} = \prod_{c = \text{cycle of } \pi \circ \rho^{-1}} \left(\delta_{\text{length}(c),1} + \operatorname{tr} \prod_{i \in c} \frac{1}{x_{\rho(i)} - X} \frac{1}{y_i - Y} \right).$$

A generalization of Eq. (5.19) also exists, for which we refer the reader to Eynard et al. [32, Section 6.1.1].

Finally, the form of Eq. (5.17) suggests that if the matrix H is of such a form that one has both an expression of the determinant of the jj-th minor of H as well as the determinant of H itself, by taking residues, eigenvector components can be calculated. One such case are Toeplitz±Hankel matrices. Toeplitz matrices are matrices where the matrix entries only depend on the diagonal it is on, that is, the matrix is of the form

$$H = [d_{j-k}]_{j,k=1}^N,$$

for some d_i, $i = -N, \dots, N$. Toeplitz matrices, therefore, describe 1-dimensional hopping models, with nearest, next-to-nearest,... neighbor hopping, with strength $d_{\pm 1}, d_{\pm 2}, \dots$. In a Hankel matrix, the entries only depend on the sum of coordinates $[d_{i+j}]_{i,j=1}^N$. This

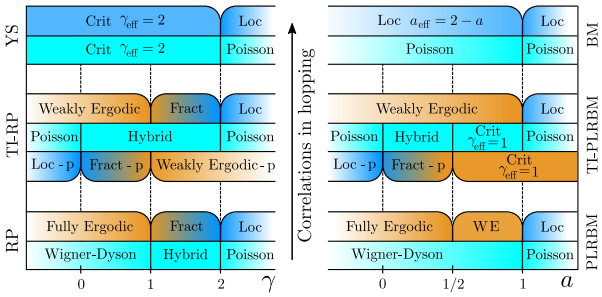


Figure 7: Phase diagrams of three variants of the RP (left) and PLRBM (right) models. TI stands for translational invariant. The eigenvalue statistics is drawn in cyan, the eigenvector statistics is the top row and the bottom row, in the case of a TI model, are the eigenvector statistics in momentum space. The TI models require a correlation of $j_{nm} = j_{kl}$ for $n - m = k - l$, therefore upper models are more correlated. This figure is taken from Nosov, Khaymovich and Kravtsov [7, Figure 3].

describes a boundary repulsion or attraction in the 1-dimensional hopping model. In Section 6, determinants of Toeplitz ± Hankel matrices will be calculated using techniques also used in the context of orthogonal polynomials (Section 4).

The study of Toeplitz Hamiltonians, while allowing for diagonal perturbations as in Section 3.3.2, is further motivated by a recent paper by Nosov, Khaymovich and Kravtsov [7], where the singularity spectrum of such a model is studied. In this paper, the free particle Hamiltonian H with coefficients $H_{nm} = \varepsilon_m \delta_{nm} + j_{nm} = H_0 + V$ is considered, where ε_m are random on-site energies with zero mean and variance $\langle \varepsilon_m^2 \rangle = \Delta^2$. The random parameters $j_{nm} = j_{mn}^*$ are the hopping parameters. These may be taken deterministically, in which case we say that the variance $\langle |j_{mn}|^2 \rangle$ is zero. It is assumed that the hopping parameters are of Toeplitz form, that is, $j_{nm} = j_{n-m}$. Nosov, Khaymovich and Kravtsov consider two kinds of variances $\langle |j_{n-m}|^2 \rangle$, the Rosenzweig-Porter (RP) model, where $\langle |j_{n-m}|^2 \rangle = \Delta^2 N^{-\gamma}$, for some γ, and the Power Law Random Banded (PLRBM) model, where $\langle |j_{n-m}|^2 \rangle = (1 - \delta_{nm})/|n - m|^{2a}$, for some a. In both cases the mean of j_{n-m} is zero. Their results are as follows [7]. For both the RP and PLRBM models, the translational and regular variants have the same critical values of γ and a. The eigenstates still transition from localized to ergodic, albeit to only weakly ergodic in the translational invariant case. With fully ergodic is meant that the eigenfunctions have Porter-Thomas statistics. Weakly ergodic states are states where the eigenfunction support set scales with N, but also the set where the eigenfunction is zero. The statistics of the eigenvalues, on the other hand, is not Wigner-Dyson in the ergodic case, but Poisson. These statements are summarized in Figure 7.

6 Toeplitz and Hankel matrices

Toeplitz and Hankel matrices are closely related to random matrix theory. A Toeplitz matrix H is a matrix where the entries are equal on the diagonals, that is, it is of the form

$$H = \begin{pmatrix} d_0 & d_{-1} & \cdots & d_{-N} \\ d_1 & d_0 & \cdots & d_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_N & d_{N-1} & \cdots & d_0 \end{pmatrix} = [d_{j-k}]_{j,k=1}^{N} =: T_N(f),$$

where $f : S^1 \to \mathbb{C}$ is the generating function of $T_N(f)$, defined by

$$f(z) = \sum_m d_m z^m.$$

In other words, d_m are the Fourier coefficients of f. We could also pick any sufficiently smooth f and define a Toeplitz matrix $T_N(f)$ generated by f, using these Fourier coefficients;

$$T_N(f) := [d_{j-k}]_{j,k=1}^{N}, \quad d_m := \frac{1}{2\pi} \int_{S^1} \overline{z} f(z) dz.$$

Examples of Toeplitz matrices are nearest-neighbor hopping Hamiltonians.

Toeplitz matrices are, in a way, related to the unitary group, as will be explored in Section 6.1. Much of the theory extends to the classical Lie groups $SO(2N+1), Sp(2N)$ and $SO(2N)$, where one obtains Toeplitz±Hankel matrices. Note that the Cartan labels corresponding to these groups are B, C and D respectively.

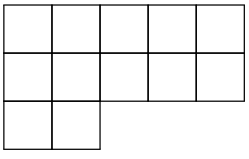
Hankel matrices are matrices where the entries depend only on the position on the other diagonal,

$$\begin{pmatrix} d_{-N} & d_{-N+1} & \cdots & d_0 \\ d_{-N+1} & d_{-N+2} & \cdots & d_1 \\ \vdots & \vdots & \ddots & \vdots \\ d_0 & d_1 & \cdots & d_N \end{pmatrix}$$

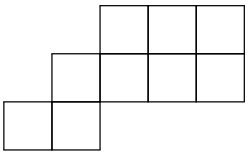
6.0.1 Symmetric polynomials

Symmetric and in particular Schur polynomials play an important role in the study of Toeplitz matrices. These are reviewed in the current subsection. A good reference for symmetric polynomials is Macdonald [42].

A *partition* λ is a sequence of non-negative, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, in descending order $\lambda_1 \geq \lambda_2 \geq \ldots$. Here, $\ell(\lambda)$ is the length of the partition. Partitions can be drawn as a diagram of boxes, called a Young diagram, such that the i-th row contains λ_i boxes. For example, the partition $(5, 5, 2)$ is drawn as



We say, for two partitions λ and μ, that λ contains μ, $\lambda \subset \mu$, if λ contains μ as a diagram, that is $\lambda_i \geq \mu_i$ for all $i \geq 1$. For such λ and μ, we define the *skew diagram* λ/μ as the set difference of diagrams of λ and μ. For example, with $\lambda = (5, 5, 2)$ and $\mu = (2, 1)$ we have



A column of width w and length l will be denoted with (w^l). For example, the diagram of (1^3) is

The transpose λ' of a partition λ is the mirror image in the diagonal. For example, the transpose of $(1^3) = (3)$ is

We say that a partition λ is *even* if all coefficients are even, that is λ_i is even for all $i \geq 1$. Finally, a partition λ of length $\ell(\lambda)$ can be reversed to obtain a vector of length $\ell(\lambda)$ in ascending order, $\lambda^r = (\lambda_{\ell(\lambda)}, \lambda_{\ell(\lambda)-1}, \ldots, \lambda_1)$.

Our only use of partitions is to label Schur polynomials.

Definition 6.0.1 (Schur polynomials)**.** Schur functions for a partition λ is defined as the symmetric function in variables $x = (x_1, \ldots, x_N)$

$$s_\lambda(x) := \frac{\det\left(x_j{}^{\lambda_i - i + N}\right)_{i,j}}{\det\left(x_j^{-i+N}\right)_{i,j}} \quad \text{for } \ell(\lambda) \leq N,$$

where $\ell(\lambda)$ is the length of the partition λ, and zero otherwise.

The first and second Jacobi-Trudi formulas express the Schur polynomials in terms of the elementary symmetric and completely homogeneous symmetric polynomials,

$$s_\lambda(x) = \det(h_{j-k+\lambda_k}(x))_{j,k=1}^{\ell(\lambda)} \tag{6.1a}$$

$$s_\lambda(x) = \det\left(e_{j-k+\lambda'_k}(x)\right)_{j,k=1}^{\ell(\lambda')}. \tag{6.1b}$$

In particular, for $\lambda = (n)$ a partition consisting of a single row of length n, this results in $s_{(n)}(x) = h_n(x)$ and $s_{(1^n)} = e_n(x)$. Therefore, Schur polynomials can be seen as a generalization of both the elementary symmetric and the completely homogeneous polynomials. There is also a generalization of Eq. (4.16) and Eq. (4.18), which are called the Cauchy-identities (Eq. (6.16)).

A Young tableau is a diagram with boxes filled with positive integers. It is called semistandard if the integers are nondecreasing on the rows and strictly increasing on the columns. The following is an example of a skew semistandard tableau,

		1	1	2
	2	3	3	
1	3			

(6.2)

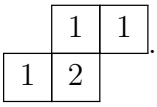
The weight of a tableau is the partition $\mu = (\mu_1, \mu_2, \dots)$, where μ_i is the number of times the symbol i occurs, $i \geq 1$.

Let λ, μ, ν be three partitions. Define the coefficients $c^{\lambda}_{\mu\nu}$ by

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda. \tag{6.3}$$

The $c^{\lambda}_{\mu\nu}$ are called Littlewood-Richardson coefficients. The Littlewood-Richardson rule computes these coefficients, and goes as follows. Let T be a tableau, then the derived word $w(T)$ of T is the word obtained by reading, for each line, the contents of the boxes from right to left, as in Arabic, and concatenating the results. For example, the word of Eq. (6.2) is 21133231. We call a word $w(T) = a_1 a_2 \dots a_N$, where $a_i \in \{1, 2, \dots, n\}$, a *lattice permutation* if for each prefix $a_1 a_2 \dots a_r$, $1 \leq r \leq N$, the symbol i occurs not less than $i+1$, for $1 \leq i \leq n-1$. For example, the word 21133231 is not a lattice permutation because, the coefficients $c^{\lambda}_{\mu\nu}$ are the number of skew semistandard tableaux T of shape λ/μ and weight ν such that the word of T, $w(T)$ is a lattice permutation.

Example 6.0.2. Let $\lambda = (3,2)$, $\mu = (1)$, $\nu = (3,1)$. As the right most box in the first row must be a 1, the entire first row must be 1, but this means that the right most box of the second row must be a 2. The remaining box must be filled with a 1, to make sure that the tableau contains the symbol 1 three times. This results in $c^{\lambda}_{\mu\nu} = 1$, since the only valid Littlewood-Richardson tableau is



△

In terms of the Littlewood-Richardson coefficients, one can define *skew Schur functions* as

$$s_{\lambda/\mu} = \sum_\nu c^{\lambda}_{\mu\nu} s_\nu.$$

The skew Schur functions satisfy a similar equation to the Jacobi-Trudi formulas Eq. (6.1),

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^{\ell(\lambda)},$$
$$s_{\lambda'/\mu'} = \det\left(e_{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^{\ell(\lambda)}.$$

6.0.2 Generators

We will concern ourselves with $f(z)$ of the form

$$f(z) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i z} \prod_{i=1}^{\infty} \frac{1}{1 - y_i z} = H(x; z) H(y; z^{-1}), \tag{6.4a}$$

or

$$f(z) = \prod_{i=1}^{\infty} (1 + x_i z) \prod_{i=1}^{\infty} (1 + y_i z^{-1}) = E(x; z) E(y; z^{-1}), \tag{6.4b}$$

or

$$f(z) = \exp\left(\sum_{m>0} \frac{1}{m} \left(p_m z^m + p^*_m z^{-m} \right) \right), \tag{6.4c}$$

where $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$, $p = \{p_i\}_{i=1}^{\infty}$ and $p^* = \{p_i^*\}_{i=1}^{\infty}$ are arbitrary independent parameters. This is because for these we can perform explicit calculations using the Cauchy identity or dual Cauchy identity.

In principle, a choice of f in the form Eq. (6.4) is without loss of generality, because both homogeneous and elementary symmetric polynomials $h_k(x)$ and $e_k(x)$ are algebraically independent. Therefore, it is always possible to find x such that the Fourier coefficients d_k of f satisfy $d_k = h_k(x)$ or $d_k = e_k(x)$. Let us now give some examples of $f(z)$ which are already written in the form Eq. (6.4).

Example 6.0.3 (Vieta's formulas)**.** Let f be a Laurent polynomial $f(z) = \sum_{k=p-w}^{p} d_k z^k$, with $p \in \mathbb{Z}_{\geq 0}$ and $w \in \mathbb{Z}_{>0}$. The band width of the Toeplitz matrix is $w+1$. Also, $P(z) := z^{w-p} f(z)$ is a polynomial of degree w, say with zero's $\alpha_1, \ldots, \alpha_w$, and can be written

$$P(z) = d_p \prod_{j=1}^{w} (z - \alpha_j) = d_p z^w \prod_{j=1}^{w} (1 - \alpha_j z^{-1}) = d_p z^w E(-\alpha; z^{-1}) = d_p z^w \sum_{k=0}^{\infty} e_k(-\alpha) z^{-k}.$$

Note that $e_k(\alpha_1, \ldots, \alpha_w)$ is the sum over all distinct products of k distinct variables. Therefore, $e_k(\alpha) = 0$ for $k > w$ and $e_k(-\alpha) = (-1)^k e_k(\alpha)$. This results in

$$P(z) = d_p \sum_{j=0}^{w} (-1)^{w-j} e_{w-j}(\alpha) z^j. \tag{6.5}$$

Plugging Eq. (6.5) back in the formula for f results in

$$f(z) = d_p \sum_{k=p-w}^{p} (-1)^{p-k} e_{p-k}(\alpha) z^k = d_p z^p E(\alpha, -z^{-1}).$$

In particular, we have $e_{p-k}(\alpha) = (-1)^{p-k} d_k / d_p$. These relations are called *Vieta's formulas*. This can be used to write the Toeplitz matrix $T_N(f)$ generated by f

$$T_N(f) = d_p [e_{p+k-j}(-\alpha)]_{j,k=1}^{N}.$$

This can be written in terms of Schur functions using the Jacobi-Trudy formula's Eq. (6.1b) as $\det T_N(f) = d_p^N s_{(N^p)}(-\alpha)$. △

Example 6.0.4 (Tridiagonal Toeplitz matrix)**.** [60, Section 4.1] Let $f(z) = E(y; z^{-1}) E(x; z)$, where x and y are single variables, that is

$$f(z) = E(y; z^{-1}) E(x; z) = (1 + yz^{-1})(1 + xz) = yz^{-1} + (1 + xy) + xz.$$

Then, $T_N(f)$ is the tridiagonal matrix

$$T_N(f) = \begin{pmatrix} 1+xy & y & \ldots \\ x & 1+xy & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}.$$

For the special case $x = y = -1$, this generates the finite element method approximation of the Laplacian. △

Example 6.0.5 (Jacobi's third theta function)**.** [10, Section 3.1] Let $|q| < 1$, then Jacobi's third theta can be written as (Eqs. (5.12) and (5.13))

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{n^2/2} z^n &= (q;q)_\infty \prod_{k=1}^{\infty} (1+q^{k-1/2}z)(1+q^{k-1/2}z^{-1}) \\ &= (q;q)_\infty E(q^{1/2}, q^{3/2}, \ldots; z) E(q^{1/2}, q^{3/2}, \ldots; z^{-1}).\end{aligned}$$

Then, $f(z)$ defined as

$$f(z) := \Theta(z) := E(q^{1/2}, q^{3/2}, \ldots; z) E(q^{1/2}, q^{3/2}, \ldots; z^{-1}),$$

generates a Toeplitz matrix with coefficients $d_k = (q;q)_\infty^{-1} q^{k^2/2}$. The Jacobi's third theta function $\Theta(U)$ is related to Chern-Simons theory on S^3 as follows. The integrals

$$\langle W_\mu \rangle_{G(N)} = \frac{1}{Z_{G(N)}} \int \chi^\mu_{G(N)}(U)\Theta(U)\mathrm{d}U$$

and

$$\langle W_{\lambda\mu} \rangle_{G(N)} = \frac{1}{Z_{G(N)}} \int_{G(N)} \chi^\lambda_{G(N)}(U^{-1})\chi^\mu_{G(N)}(U)\Theta(U)\mathrm{d}U,$$

are proportional to the expectation values of the Wilson loop of the unknot and Hopf links of Chern-Simons theory on S^3 [10, Section 3.1]. Here $Z_{G(N)}$ is the partition function,

$$Z_{G(N)} = (q;q)_\infty^N \int_{G(N)} \Theta(U)\mathrm{d}U.$$

△

The same calculations can be applied to the other three Theta functions.

Example 6.0.6 (Gross-Witten-Wadia model)**.** [11], [12] The Gross-Witten-Wadia model is characterized by [10, Section 3.2]

$$f_{GWW}(z) = \exp\bigl(-\beta(z+z^{-1})\bigr).$$

This fits the general form $f(z) = \exp\bigl(\sum_{m>0}(p_m z + p^*_m z^{-1})/m\bigr)$ with $p_1 = p^*_1 = -\beta$ and $p_i = p^*_i = 0$ for $i > 1$. △

Example 6.0.7 (Brezin-Gross-Wittem model)**.** [11], [13] The general f defined as

$$f(z) = \exp\left(\sum_{m>0} \frac{1}{m}\left(p_m z^m + p^*_m z^{-m}\right)\right)$$

appears in the context of the Brezin-Gross-Witten model, where the parameters p and p^* are called *coupling constants*. △

Let us denote for some diagonalizable matrix U, $f(U) = \prod_{i=1}^N f(z_i)$, where z_i are the eigenvalues of U. Furthermore, write $D_N(f) = \det(d_{j-k})_{j,k=1}^N$, and more generally $D_N^{\lambda,\mu}(f) = \det\bigl(d_{j-\lambda_j-k+\mu_k}\bigr)_{j,k=1}^N$, so that $D_N(f) = D_N^{\emptyset,\emptyset}(f)$. Also note that the determinant of the j,k-th minor of $T_N(f)$ is given by $D_{N-1}^{(1^j),(1^k)}(f)$.

In order to calculate integrals over the unitary, orthogonal or symplectic group, it will be important to write the integrands in terms of Schur functions, because then the integrals will be easy due to Schur orthogonality. We need the Cauchy identities to do so.

Lemma 6.0.8 (Cauchy identities)**.** *Let $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$, then we have the Cauchy identity*

$$\sum_{\nu} s_\nu(x)s_\nu(y) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}$$

and dual Cauchy identity

$$\sum_{\nu} s_\nu(x)s_{\nu'}(y) = \prod_{i,j=1}^{\infty} (1 + x_i y_i),$$

where $\sum_\nu$ is the sum over all partitions ν of any length. Here, ν' is the transpose partition of ν.

Proof. For the proof we refer to Macdonald [42, p.63]. □

Note that the equalities in Eq. (4.18) and Eq. (4.16) are special cases of the Cauchy identities.

The Cauchy identity allows us to rewrite

$$\prod_{i=1}^{N} H(x; z_i) = \prod_{j=1}^{\infty}\prod_{i=1}^{N} \frac{1}{1 - x_j z_i} = \sum_{\ell(\nu)\leq N} s_\nu(x)s_\nu(z),$$

where partitions with length $> N$ do not contribute, because $z_i = 0$ for $i > N$, hence $s_\nu(z) = 0$ for $\ell(\nu) > N$. In particular, for z_i eigenvalues of some matrix U,

$$H(x; U) = \sum_{\ell(\nu)\leq N} s_\nu(x)s_\nu(U),$$

where we write $s_\nu(U)$ for $s_\nu(z_1, \ldots, z_N)$. A similar expression holds for $E(x; U)$. As a result, when f is expressed as Eq. (6.4a) we get

$$\begin{aligned}
\int_{U(N)} f(U)\mathrm{d}U &= \int H(x; U)H(y; U^{-1})\mathrm{d}U \\
&= \int \left(\sum_{\nu} s_\nu(x)s_\nu(U)\right)\left(\sum_{\mu} s_\mu(y)\overline{s_\mu(U)}\right)\mathrm{d}U \\
&= \sum_{\nu,\mu} s_\nu(x)s_\mu(y)\delta_{\mu\nu} \\
&= \sum_{\ell(\nu)\leq N} s_\nu(x)s_\nu(y) \qquad (6.6)
\end{aligned}$$

Here we used Schur orthogonality and the fact that the Schur polynomials are the characters of the unitary group Proposition 6.1.2. Again a similar expression holds for Eq. (6.4b), the difference being that the sum is over partitions, for which the transpose has length $\leq N$, $\ell(\nu') \leq N$. Similar expressions can also be found for f of the form $f(U) = H(x; U)E(y; U^{-1})$, but we do not consider this possibility here.

If f is of the form Eq. (6.4c), we cannot directly use the Cauchy identity. If one can find variables $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$ such that

$$p_m = \sum_{i=1}^{\infty} x_i^m = p_m(x) \qquad (6.7a)$$

and

$$p_m^* = \sum_{i=1}^{\infty} y_i^m = p_m(y), \tag{6.7b}$$

we have that $f(z) = H(x;z)H(y;z^{-1})$ and we can use Eq. (6.6) to get, in the limit of $N \to \infty$,

$$\begin{aligned}\lim_{N\to\infty} \int_{U(N)} f(U)\mathrm{d}U &= \lim_{N\to\infty} \sum_{\ell(\nu)\le N} s_\nu(x)s_\mu(y) = \prod_{i,j=1}^{\infty} \frac{1}{1-x_iy_j} \\ &= \exp\left(\sum_{m>0} \frac{1}{m} p_m(y)p_m(x)\right) = \exp\left(\sum_{m>0} \frac{1}{m} p_m^* p_m\right). \end{aligned} \tag{6.8}$$

This method extends to slightly more general f, such as

$$f(z) = \exp\left(\sum_{m>0} \frac{(-1)^{m+1}}{m} \left(p_m z^m + p_m^* z^{-m}\right)\right).$$

In this case, the result will be the same and is called the strong Szegö limit. It will be discussed in Theorem 6.1.7 also for the unitary and symplectic groups.

If variables p and x are related by Eq. (6.7), it will be useful to introduce some notation for the Schur polynomial $s_\lambda(x)$ depending on the variables x. The function $s_\lambda(\mathbf{p})$, with a bold typeset argument $\mathbf{p}$, is defined by

$$s_\lambda(\mathbf{p}) := s_\lambda(x),$$

where the variables x are related to p by Eq. (6.7). That is, $s_\lambda(\mathbf{p})$ is a Schur polynomial in the variables x, defined implicitly by p.

To summarize, we have stated which kind of functions f we will consider, stated examples of functions f written in this form and have shown a method for calculating integrals over the unitary, symplectic or orthogonal group for such f. In the next section, we will see that this is in fact relevant, because it is possible to calculate determinants of Toeplitz ± Hankel matrices as integrals over the unitary, symplectic or orthogonal group.

6.1 Determinants as integrals

For the unitary group, we can apply Lemma 4.1.1 directly.

Corollary 6.1.1. *Let $f(z)$ be a function on the measure space $(S^1, \mathrm{d}\theta/2\pi)$, that is, a function on the unit circle, then*

$$\begin{aligned}\int_{U(N)} f(U)\mathrm{d}U &= \frac{1}{N!} \int_{[0,2\pi]^N} \det V(e^{-i\theta}) \det V(e^{i\theta}) \prod_{k=1}^{N} f(e^{i\theta_k}) \frac{\mathrm{d}\theta_k}{2\pi} \\ &= \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-ij\theta} e^{ik\theta} f(e^{i\theta})\mathrm{d}\theta\right)_{j,k=1}^{N} = \det(d_{j-k})_{j,k=1}^{N} = D_N(f), \end{aligned}$$

where

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(e^{i\theta})\mathrm{d}\theta$$

is the k-th Fourier coefficient of f and $V(x) = [x_j^{N-j}]_{j,k=1}^{N}$ is the Vandermonde matrix, so that $\det V(x) = \Delta(x)$ is the Vandermonde determinant.

More generally, we write $M^{\lambda}_{U(N)}(x)$ for the generalized Vandermonde matrix

$$M^{\lambda}_{U(N)}(x) = [x_j^{N-k+\lambda_k}]^N_{j,k}, \tag{6.9}$$

so that the Schur polynomial can be written as

$$s_\lambda(x) = \frac{\det M^{\lambda}_{U(N)}(x)}{\det M_{U(N)}(x)}.$$

This formula is also the Weyl Character formula.

Proposition 6.1.2 (Characters of the unitary group)**.** *The Schur functions are the irreducible characters of the unitary group.*

Proof. This proof is based on [61, A.5]. The character of semisimple Lie algebra's is given in terms of the Weyl character formula as

$$\chi^\lambda(x) = \frac{A_{\lambda+\rho}}{A_\rho}, \qquad \text{where } A_\mu = \sum_{w\in W} \operatorname{sgn}(w) e^{w(\mu)},$$

where W is the Weyl group of the Lie algebra, and ρ the sum over all fundamental weights. Using the Weyl denominator formula, A_ρ can also be written as

$$A_\rho = \prod_{\alpha\in R^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right), \tag{6.10}$$

where R^+ is the set of positive roots.

In the case of $\mathfrak{gl}_n$, the Cartan subalgebra consists of the diagonal matrices $\varepsilon_i = E_{ii}$. Note that we do not consider $\mathfrak{sl}_n$ here due to the technical difficulty that ε_i is not an element of $\mathfrak{sl}_n$. Decompose $\lambda = \sum_{i=1}^N \lambda_i \varepsilon_i$. The set of positive roots is given by $\varepsilon_i - \varepsilon_j$ for $i < j$, and hence

$$\rho = \frac{1}{2}\sum_{i<j}(\varepsilon_i - \varepsilon_j) = \sum_{i=1}^N \left(\frac{N+1}{2} - i\right)\varepsilon_i.$$

The second step follows simply by reordering the sum. The Weyl group is S_n, as we could still reorder the basis. Write $x_i = e^{\varepsilon_i}$, then $A_{\lambda+\rho}$ happens to be a determinant

$$A_{\lambda+\rho} = \sum_{w\in S_n} \operatorname{sgn}(w) x_{w(1)}^{\lambda_1+\frac{N+1}{2}-1} \dots x_{w(N)}^{\lambda_N+\frac{N+1}{2}-N} = \det\left(x_j^{\lambda_i+\frac{N+1}{2}-i}\right)_{i,j}.$$

Remember that A_ρ in Eq. (6.10) is just the Vandermonde determinant, therefore the character can be written as

$$\chi^\lambda_{U(N)}(U) = \frac{\det\left(x_j^{\lambda_i+\frac{N+1}{2}-i}\right)_{i,j}}{\det\left(x_j^{\frac{N+1}{2}-i}\right)_{i,j}} = \frac{\det\left(x^{\lambda_i+N-i}\right)_{i,j}}{\det\left(x_j^{N-i}\right)_{i,j}} = s_\lambda(x),$$

where $x_1, \dots, x_N$ are the eigenvalues of U. Therefore, the character of the unitary group is equal to the Schur polynomial of the eigenvalues. □

Corollary 6.1.1 can now be generalized to the groups $Sp(2N), SO(2N)$ and $SO(2N+1)$, with $\det M_{G(N)}(x)$ be given by [62, Lemma 2]

$$\det M_{U(N)}(x) = \det\left(x_j^{N-k}\right)_{j,k=1}^{N} = \prod_{1\le j<k\le N}(x_j - x_k), \tag{6.11a}$$

$$\det M_{SO(2N+1)} = \det\left(x^{N-k+\frac{1}{2}} - x_j^{-N+k-\frac{1}{2}}\right)_{j,k=1}^{N} = \prod_{j<k}(x_j - x_k)(1 - x_j x_k)\prod_{j=1}^{N}(x_j - 1)x_j^{-N+\frac{1}{2}}, \tag{6.11b}$$

$$\det M_{Sp(2N)}(x) = \det\left(x_j^{N-k+1} - x_j^{-N+k-1}\right)_{j,k=1}^{N} = \prod_{j<k}(x_j - x_k)(1 - x_j x_k)\prod_{j=1}^{N}(x_j^2 - 1)x_j^{-N}, \tag{6.11c}$$

$$\det M_{SO(2N)}(x) = \det\left(x_j^{N-k} + x_j^{-N+k}\right)_{j,k=1}^{N} = 2\prod_{j<k}(x_j - x_k)(1 - x_j x_k)\prod_{j=1}^{N}x_j^{-N+1}. \tag{6.11d}$$

Define $M^{\lambda}_{G(N)}(x)$ similarly to Eq. (6.9), by replacing $k \to k - \lambda_j$, for example,

$$M^{\lambda}_{SO(2N+1)}(x) := [x^{N-k+\lambda_j+\frac{1}{2}} - x^{N+k-\lambda_j-\frac{1}{2}}].$$

For $SO(2N)$, we must assume here that $\lambda_N = 0$. The general case would require some minor modifications.

Corollary 6.1.3. *Let $f(z)$ be a measurable function on the unit circle, with Fourier coefficients d_k, then [10, Eq. (4)-(7)]*

$$\begin{aligned}
\int_{U(N)} f(U)\mathrm{d}U &= \det(d_{j-k})_{j,k=1}^{N},\\
\int_{SO(2N+1)} f(U)\mathrm{d}U &= \frac{1}{2^N}\det(d_{j-k} + d_{k-j} - d_{1-j-k} - d_{j+k-1})_{j,k=1}^{N},\\
\int_{Sp(2N)} f(U)\mathrm{d}U &= \frac{1}{2^N}\det(d_{j-k} + d_{k-j} - d_{-j-k} - d_{j+k})_{j,k=1}^{N},\\
\int_{SO(2N)} f(U)\mathrm{d}U &= \frac{1}{2^{N-1}}\det(d_{j-k} + d_{k-j} + d_{2-j-k} + d_{j+k-2})_{j,k=1}^{N}.
\end{aligned}$$

We will denote this integral with $D_{G(N)}(f)$. Therefore, $D_N(f) = D_{U(N)}(f)$.

Proof. For the case $G(N) = SO(2N)$ the following holds,

$$\begin{aligned}
&\int_{SO(2N)} f(U)\mathrm{d}U\\
&= \frac{2}{2^N}\det\left(\frac{1}{2\pi}\int_0^{2\pi}(e^{-i\theta(N-j)} - z^{i\theta(N-j)})(e^{i\theta(N-k)} - z^{-i\theta(N-k)})f(e^{i\theta})\mathrm{d}\theta\right)_{j,k=1}^{N}\\
&= \frac{2}{2^N}\det\left(\frac{1}{2\pi}\int_0^{2\pi}(e^{-i\theta(k-j)} - z^{-i\theta(j+k-2N)} - e^{-i\theta(2N-j-k)} + z^{-i\theta(j-k)})f(e^{i\theta})\mathrm{d}\theta\right)_{j,k=1}^{N}\\
&= \frac{2}{2^N}\det(d_{k-j} - d_{j+k-2N} - d_{2N-j-k} + d_{j-k})_{j,k=1}^{N}\\
&= \frac{2}{2^N}\det(d_{j-k} - d_{2-j-k} - d_{j+k-2} + d_{k-j})_{j,k=1}^{N}.
\end{aligned}$$

In the last step, we replaced $k \to N+1-k$, $j \to N+1-j$. Similarly, for $Sp(2N)$,

$$\begin{aligned}\int_{Sp(2N)} f(U)\mathrm{d}U &= \frac{1}{2^N}\det(d_{j-k} - d_{j+k-2N-2} - d_{2N+2-j-k} + d_{j-k})_{j,k=1}^N \\ &= \frac{1}{2^N}\det(d_{j-k} - d_{-j-k} - d_{j+k} + d_{k-j})_{j,k=1}^N,\end{aligned}$$

and finally for $SO(2N+1)$,

$$\begin{aligned}\int_{SO(2N+1)} f(U)\mathrm{d}U &= \frac{1}{2^N}\det(d_{j-k} - d_{j+k-2N-1} - d_{2N+1-j-k} + d_{j-k})_{j,k=1}^N \\ &= \frac{1}{2^N}\det(d_{j-k} - d_{1-j-k} - d_{j+k-1} + d_{k-j})_{j,k=1}^N.\end{aligned}$$

In the special case that $d_k = d_{-k}$, one recovers Equation (4) − (7) of García-García and Tierz [10]. □

Proposition 6.1.2 is generalized to the expression

$$\chi^\lambda_{G(N)}(U) = \frac{\det M^\lambda_{G(N)}(x)}{\det M_{G(N)}(x)} =: \mathfrak{s}^\lambda_{G(N)}(x), \tag{6.12}$$

with $x_1, \ldots, x_N$ the eigenvalues of U. Therefore,

$$\begin{aligned}&\int_{G(N)} \chi^\lambda_{G(N)}(U^{-1})\chi^\mu_{G(N)}(U)f(U)\mathrm{d}U \\ &\qquad = C_{G(N)}\frac{1}{N!}\int_{[0,2\pi]^N} \det M^\lambda_{G(N)}(e^{-i\theta})\det M^\mu_{G(N)}(e^{i\theta})\prod_{k=1}^N f(e^{i\theta_k})\frac{\mathrm{d}\theta_k}{2\pi},\end{aligned}$$

where constants $C_{G(N)}$ are

$$C_{U(N)} = 1, \quad C_{Sp(N)} = C_{SO(2N+1)} = \frac{1}{2^N}, \quad C_{SO(2N)} = \frac{1}{2^{N-1}}.$$

This expression can again be calculated using Andreiéf's identity Lemma 4.1.1. We will just state the results

Lemma 6.1.4. *Let f be as in Corollary 6.1.1, then*

$$\begin{aligned}D^{\lambda,\mu}_{U(N)}(f) &= \det\Big(d_{j+\lambda^r_j-k-\mu^r_k}\Big)_{j,k=1}^N, \\ D^{\lambda,\mu}_{SO(2N+1)}(f) &= \frac{1}{2^N}\det\Big(d_{j+\lambda^r_j-k-\mu^r_k} + d_{k+\mu^r_k-j-\lambda^r_j} - d_{1-j-\lambda^r_j-k-\mu^r_k} - d_{j+\lambda^r_j+k+\mu^r_k-1}\Big)_{j,k=1}^N, \\ D^{\lambda,\mu}_{Sp(2N)}(f) &= \frac{1}{2^N}\det\Big(d_{j+\lambda^r_j-k-\mu^r_k} + d_{k+\mu^r_k-j-\lambda^r_j} - d_{-j-\lambda^r_j-k-\mu^r_k} - d_{j+\lambda^r_j+k+\mu^r_k}\Big)_{j,k=1}^N, \\ D^{\lambda,\mu}_{SO(2N)}(f) &= \frac{1}{2^{N-1}}\det\Big(d_{j+\lambda^r_j-k-\mu^r_k} + d_{k+\mu^r_k-j-\lambda^r_j} + d_{2-j-\lambda^r_j-k-\mu^r_k} + d_{j+\lambda^r_j+k+\mu^r_k-2}\Big)_{j,k=1}^N.\end{aligned}$$

Here,

$$D^{\lambda,\mu}_{G(N)} := \int_{G(N)} \overline{\chi^\lambda_{G(N)}(U)}\chi^\mu_{G(N)}(U)f(U)\mathrm{d}U \tag{6.13}$$

and

$$\lambda^r := (\lambda_{N-j+1})_j = (\lambda_N, \ldots, \lambda_1).$$

In the special case of $d_k = d_{-k}$, this result reduces to Theorem 1 of García-García and Tierz [10].

Finally, let us discuss some facts about the Schur polynomials. First some notation.

Definition 6.1.5. The Schur polynomials for the groups $SO(2N+1)$, $Sp(2N)$ and $SO(2N)$ are denoted as follows;

$$\begin{aligned} s_\lambda(x) &= \mathfrak{s}_{U(N)}(x), \\ o_\lambda^{odd}(x) &= \mathfrak{s}_{SO(2N+1)}(x), \\ sp_\lambda(x) &= \mathfrak{s}_{Sp(2N)}(x), \\ o_\lambda^{even}(x) &= \mathfrak{s}_{SO(2N)}(x). \end{aligned}$$

Baker [63] showed that these characters are related to the regular Schur polynomials by the action of an operator. The reader can also consult Equation (13) of Van de Leur and Orlov [61]. Namely, write $\tilde{\partial} = (\partial_{p_1}, 2\partial_{p_2}, 3\partial_{p_3}, \dots)$ and define

$$\Omega_\mp(\mathbf{p}) = \sum_{m>0} \left(-\frac{1}{2m} p_m^2 \mp \frac{1}{2m} p_{2m} \right),$$

so that $\Omega_\mp(\tilde{\partial})$ is given by

$$\Omega_\mp(\tilde{\partial}) = \sum_{m>0} \left(-\frac{m}{2} (\partial_m)^2 \mp \partial_{2m} \right).$$

Then, the Schur polynomials are related by

$$o_\lambda^{odd}(\mathbf{p}) = e^{\Omega_-(\tilde{\partial})} s_\lambda(\mathbf{p}), \quad sp_\lambda(\mathbf{p}) = e^{\Omega_+(\tilde{\partial})} s_\lambda(\mathbf{p}).$$

In particular, if we act with $\Omega_\mp(\tilde{\partial}^*)$ on the Cauchy identity, we get new Cauchy identities

$$\begin{aligned} \tau_-(\mathbf{p}|\mathbf{p}^*) &= \sum_\lambda o_\lambda^{odd}(\mathbf{p}^*) s_\lambda(\mathbf{p}), \\ \tau_+(\mathbf{p}|\mathbf{p}^*) &= \sum_\lambda sp_\lambda(\mathbf{p}^*) s_\lambda(\mathbf{p}), \end{aligned}$$

where $\tau_\mp(\mathbf{p}|\mathbf{p}^*)$ are given by

$$\tau_\mp(\mathbf{p}|\mathbf{p}^*) = \exp\left(-\frac{1}{2} \sum_{m>0} \frac{1}{m} p_m^2 \mp \sum_{m>0} \frac{1}{2m} p_{2m} + \sum_{m>0} \frac{1}{m} p_m p_m^* \right) = e^{\Omega_\mp(\mathbf{p})} \tau_0(\mathbf{p}|\mathbf{p}^*) \quad (6.14)$$

with τ_0 a Toda lattice tau function

$$\tau_0(\mathbf{p}|\mathbf{p}^*) := \exp\left(\sum_{m>0} \frac{1}{m} p_m^* p_m \right). \quad (6.15)$$

Note that $\tau_\mp(\mathbf{p}|\mathbf{p}) = e^{-\Omega_\pm(\mathfrak{p})}$. By reintroducing x and y variables and identifying some Taylor series, these Cauchy identities can also be written as

$$\sum_\nu s_\nu(y) s_\nu(x) = \prod_{i,j=1}^\infty \frac{1}{1 - y_i x_j}, \quad (6.16a)$$

$$\sum_\nu o_\nu^{odd}(y) s_\nu(x) = \prod_{i \le j} (1 - x_i x_j) \prod_{i,j=1}^\infty \frac{1}{1 - y_i x_j} \frac{1}{1 - y_i^{-1} x_j} \prod_{j=1}^\infty \frac{1}{1 - x_j}, \quad (6.16b)$$

$$\sum_\nu sp_\nu(y)s_\nu(x) = \prod_{i<j}(1-x_ix_j)\prod_{i,j=1}^{\infty}\frac{1}{1-y_ix_j}\frac{1}{1-y_i^{-1}x_j}, \tag{6.16c}$$

$$\sum_\nu o_\nu^{even}(y)s_\nu(x) = \prod_{i\le j}(1-x_ix_j)\prod_{i,j=1}^{\infty}\frac{1}{1-y_ix_j}\frac{1}{1-y_i^{-1}x_j}. \tag{6.16d}$$

These Cauchy identities can be found in Van de Leur and Orlov [61, Equation (14), (15), (31), (32) and (33)] or García-García and Tierz [10, Equation (62)-(65)].

The Cauchy identity Eq. (6.16a), written in terms of the variables $\mathbf{p}$ and $\mathbf{p}^*$ is

$$\sum_\nu s_\nu(y)s_\nu(x) = \prod_{j=1}^{\infty}H(y;x_j) = \prod_{j=1}^{\infty}\exp\left(\sum_{k>0}\frac{p_k(y)}{k}x_j^k\right) = \exp\left(\sum_{k>0}\frac{p_k(y)p_k(x)}{k}\right).$$

With the notation $p_k(x) = p_k$ and $p_k(y) = p_k^*$ this becomes

$$\sum_\nu s_\nu(\mathbf{p})s_\nu(\mathbf{p}^*) = \exp\left(\sum_{k>0}\frac{1}{k}p_k^*p_k\right).$$

Let now U be a unitary matrix with eigenvalues $x_1,\dots,x_N$, and $Z\in G(N)$ with the following eigenvalues

$$\begin{array}{ll} y_1,y_1^{-1},\dots,y_N,y_N^{-1} & \text{if } Z\in Sp(2N) \text{ or } Z\in SO(2N)\\ y_1,y_1^{-1},\dots,y_N,y_N^{-1},1 & \text{if } Z\in SO(2N+1)\\ y_1,\dots,y_N & \text{if } Z\in U(N).\end{array}$$

Then, using Eq. (6.16), we have

$$\sum_\nu s_\nu(Z)s_\nu(x) = H(x;Z) = \tau_0(\mathbf{p}|Z) \tag{6.17a}$$

$$\sum_\nu o_\nu^{odd}(Z)s_\nu(x) = H(x;Z)H(x;Z^{-1})H(x;1)\prod_{i\le j}(1-x_ix_j) \tag{6.17b}$$

$$\sum_\nu sp_\nu(Z)s_\nu(x) = H(x;Z)H(x;Z^{-1})\prod_{i<j}(1-x_ix_j) = \tau_+(\mathbf{p}|Z) \tag{6.17c}$$

$$\sum_\nu o_\nu^{even}(Z)s_\nu(x) = H(x;Z)H(x;Z^{-1})\prod_{i\le j}(1-x_ix_j) = \tau_-(\mathbf{p}|Z). \tag{6.17d}$$

This is because $H(x;Z)$ is defined as the product over the independent eigenvalues of Z,

$$H(x;Z) = \prod_{i=1}^{N}H(x;y_i).$$

Here, we used that

$$\prod_{i<j}(1-x_ix_j) = \left[\prod_{i\neq j}(1-x_ix_j)\right]^{\frac{1}{2}} = \left[\prod_{j=1}^{\infty}\frac{1}{1-x_jx_j}\right]^{\frac{1}{2}}\exp\left(-\frac{1}{2}\sum_{k>0}\frac{1}{k}p_k^2(x)\right),$$

and that, with $x^2 = (x_1^2,x_2^2,\dots)$,

$$\prod_{j=1}^{\infty}\frac{1}{1-x_jx_j} = H(x^2;1) = \exp\left(\sum_{k>0}\frac{1}{k}p_k(x^2)\right) = \exp\left(\sum_{k>0}\frac{1}{k}p_{2k}(x)\right).$$

For completeness, we also state the dual Cauchy identities, in the same form as Eq. (6.17),

$$\sum_{\nu} s_{\nu'}(Z)s_{\nu}(x) = E(x;Z), \tag{6.18a}$$

$$\begin{aligned}\sum_{\nu} o^{odd}_{\nu'}(Z)s_{\nu}(x) &= E(x;Z)E(x;Z^{-1})E(x;1)\prod_{i<j}(1-x_ix_j)\\ &= E(x;Z)E(x;Z^{-1})E(x;1)e^{\Omega_+(\mathbf{p})},\end{aligned} \tag{6.18b}$$

$$\sum_{\nu} sp_{\nu'}(Z)s_{\nu}(x) = E(x;Z)E(x;Z^{-1})\prod_{i\le j}(1-x_ix_j) = E(x;Z)E(x;Z^{-1})e^{\Omega_-(\mathbf{p})}, \tag{6.18c}$$

$$\sum_{\nu} o^{even}_{\nu'}(Z)s_{\nu}(x) = E(x;Z)E(x;Z^{-1})\prod_{i<j}(1-x_ix_j) = E(x;Z)E(x;Z^{-1})e^{\Omega_+(\mathbf{p})}. \tag{6.18d}$$

The reader can find these dual Cauchy identities written out completely in García-García and Tierz [10, Eq. (66)-(69)].

For the groups $G(N) = SO(2N+1), Sp(2N)$ and $SO(2N)$, both the Cauchy identities Eq. (6.17) and dual Cauchy identities Eq. (6.18) share a common term, namely $H(x;Z)H(x;Z^{-1})$ and $E(x;Z)E(x;Z^{-1})$, respectively. To ease calculations, we will introduce the following notation for the group dependent term,

$$\Xi_{G(N)}(x) = \begin{cases} H(x;1)\prod_{i\le j}(1-x_ix_j) & G(N) = SO(2N+1),\\ \prod_{i<j}(1-x_ix_j) & G(N) = Sp(2N),\\ \prod_{i\le j}(1-x_ix_j) & G(N) = SO(2N).\end{cases} \tag{6.19}$$

For the variables p, this becomes

$$\Xi_{G(N)}(\mathbf{p}) = \begin{cases} H(x;1)e^{\Omega_-(\mathbf{p})} & G(N) = SO(2N+1),\\ e^{\Omega_+(\mathbf{p})} & G(N) = Sp(2N),\\ e^{\Omega_-(\mathbf{p})} & G(N) = SO(2N).\end{cases} \tag{6.20}$$

Similarly, one can define such a factor for the dual case. We will frequently drop the x or $\mathbf{p}$ if it is clear from the context, and just write $\Xi_{G(N)}$. Note that although N is present in the notation $\Xi_{G(N)}$, the value of $\Xi_{G(N)}$ does not depend on N, not even on it's parity. We only have to include it in the notation to be able to distinguish between the different groups. In calculations, however, we will often use the following variation, which does depend on N,

$$\Xi_{G(N),N}(x) := \Xi_{G(N)}(x_1,\dots,x_N,0,\dots). \tag{6.21}$$

Note that $\Xi_{G(N),N}(x) \to \Xi_{G(N)}(x)$ as $N\to\infty$.

6.1.1 Integrals evaluated

In this section, $D_{G(N)}(f)$ and $D^{\lambda,\mu}_{G(N)}(f)$ for the four groups $G(N) = U(N), SO(2N+1), Sp(2N), SO(2N)$ shall be calculated. For $G(N) = U(N)$, this is the following proposition.

Proposition 6.1.6. *Let $f(z) = H(x;z)H(y;z^{-1})$ for some $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$. Then*

$$D^{\lambda,\mu}_N(f) = \int_{U(N)} \chi^{\lambda}_{U(N)}(U^{-1})\chi^{\mu}_{U(N)}(U)H(x;U)H(y;U^{-1})\mathrm{d}U = \sum_{\ell(\nu)\le N} s_{\nu/\lambda}(y)s_{\nu/\mu}(x).$$

Proof. The result for $\lambda = \mu = \emptyset$ is precisely Eq. (6.6). In general,

$$\begin{aligned}
D^{\lambda,\mu}_{U(N)}(f) &= \int \chi^{\lambda}_{U(N)}(U^{-1})\chi^{\mu}_{U(N)}(U)H(y;U^{-1})H(x;U)\mathrm{d}U \\
&= \sum_{\ell(\rho),\ell(\sigma)\leq N} s_\rho(y)s_\sigma(x)\int_{U(N)} \chi^{\lambda}_{U(N)}(U^{-1})\chi^{\mu}_{U(N)}(U)s_\rho(U^{-1})s_\sigma(U)\mathrm{d}U \\
&= \sum_{\ell(\rho),\ell(\sigma)\leq N} s_\rho(y)s_\sigma(x)\int_{U(N)} s_\lambda(U^{-1})s_\rho(U^{-1})s_\mu(U)s_\sigma(U)\mathrm{d}U \\
&= \sum_{\ell(\rho),\ell(\sigma)\leq N} s_\rho(y)s_\sigma(x)\sum_{\nu,\tau} c^{\nu}_{\lambda\rho}c^{\tau}_{\mu\sigma}\int_{U(N)} s_\nu(U^{-1})s_\tau(U)\mathrm{d}U \\
&= \sum_{\ell(\rho),\ell(\sigma)\leq N}\sum_{\nu} c^{\nu}_{\lambda\rho}c^{\nu}_{\mu\sigma}s_\rho(y)s_\sigma(x).
\end{aligned}$$

Here, we used the Littlewood-Richardson rule Eq. (6.3). This concludes the proof. □

Theorem 6.1.7 (Szegö, Johansson)**.** *Let* $f(z) = \exp\big(\sum_{m>0}\frac{1}{m}(p_m z^m + p_m z^{-m})\big)$ *with* $\sum_m |p_m| < \infty$ *and* $\sum_m m|p_m| < \infty$*, then*

$$\begin{aligned}
\lim_{N\to\infty}\int_{U(N)} f(U)\mathrm{d}U &= \exp\left(\sum_{m>0}\frac{1}{m}p_m^2\right) = \tau(\mathbf{p}|\mathbf{p}) \\
\lim_{N\to\infty}\int_{O(2N+1)} f(U)\mathrm{d}U &= \exp\left(\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 - \sum_{m>0}\frac{1}{2m-1}p_{2m-1}\right) \\
\lim_{N\to\infty}\int_{Sp(2N)} f(U)\mathrm{d}U &= \exp\left(\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 - \sum_{m>0}\frac{1}{2m}p_{2m}\right) = \tau_-(\mathbf{p}|\mathbf{p}) \\
\lim_{N\to\infty}\int_{O(2N)} f(U)\mathrm{d}U &= \exp\left(\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 + \sum_{m>0}\frac{1}{2m}p_{2m}\right) = \tau_+(\mathbf{p}|\mathbf{p}).
\end{aligned}$$

Proof. The unitary case is shown in Eq. (6.8) together with Eq. (6.15). It remains to prove the statement for the groups $SO(2N+1), Sp(2N)$ and $SO(2N)$. Using the Cauchy-identities Eq. (6.17), we have

$$\begin{aligned}
\int_{G(N)} f(U)\mathrm{d}U &= \int_{G(N)} H(x;U)H(x;U^{-1})\mathrm{d}U \\
&= \Xi^{-1}_{G(N),N}(x)\int_{G(N)}\sum_{\ell(\nu)\leq N}\chi^{\nu}_{G(N)}(U)s_\nu(x)\mathrm{d}U \\
&= \Xi^{-1}_{G(N),N}(x)s_\emptyset(x) = \Xi^{-1}_{G(N),N}(x),
\end{aligned} \tag{6.22}$$

where $\Xi^{-1}_{G(N),N}$ is defined by Eq. (6.21). In the limit $N\to\infty$, this becomes

$$\lim_{N\to\infty}\int_{G(N)} f(U)\mathrm{d}U = \begin{cases}
\exp\big(\frac{1}{2}\sum_{m>0}\frac{1}{m}(p_m^2 - 2p_m) + \sum_{m>0}\frac{1}{2m}p_{2m}\big), & \text{for } SO(2N+1), \\
\exp\big(\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 - \sum_{m>0}\frac{1}{2m}p_{2m}\big), & \text{for } Sp(2N), \\
\exp\big(\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 + \sum_{m>0}\frac{1}{2m}p_{2m}\big), & \text{for } SO(2N).
\end{cases}$$

With Eq. (6.14), this can be written in terms of $\tau_\mp(\mathbf{p}|\mathbf{p}) = \big(e^{\Omega_\pm(\mathfrak{p})}\big)^{-1}$. □

We remark that we may also write Eq. (6.22) for $G(N) = SO(2N+1), Sp(2N)$ and $SO(2N)$ as a sum over Schur polynomials

$$\int_{G(N)} f(U)\mathrm{d}U = \Xi^{-1}_{G(N),N} = \sum_{\ell(\nu)\leq N} s_\nu(x)\mathfrak{s}^\nu_{G(N)}(x). \tag{6.23}$$

This is analogous to the $U(N)$ result, Proposition 6.1.6 with $\lambda = \mu = 0$.

Using the same the method as in the proof of Theorem 6.1.7, we can generalize Proposition 6.1.6 to $D^{\lambda,\mu}_{G(N)}$ as follows.

Proposition 6.1.8. *Let $f(z)$ be as in Proposition 6.1.6, with $x = y$. Let furthermore $\Xi_{G(N),N}$ be defined by Eq.* (6.21)*. Then*

$$D^{\lambda,\mu}_N(f) = \Xi^{-1}_{G(N),N}(x) \sum_{\ell(\sigma)\leq N} b^\lambda_{\mu\sigma} s_\sigma(x)$$

Proof. This is again a simple calculation exploiting Schur orthogonality,

$$\begin{aligned}
D^{\lambda,\mu}_N(f) &= \int_{G(N)} \chi^\lambda_{G(N)}(U^{-1})\chi^\mu_{G(N)}(U)H(x;U)H(x;U^{-1})\mathrm{d}U \\
&= \Xi^{-1}_{G(N),N}(x) \sum_{\ell(\sigma)\leq N} s_\sigma(x) \int_{G(N)} \chi^\lambda_{G(N)}(U^{-1})\chi^\mu_{G(N)}(U)\chi^\sigma_{G(N)}(U)\mathrm{d}U \\
&= \Xi^{-1}_{G(N),N}(x) \sum_{\ell(\sigma)\leq N} s_\sigma(x) \sum_\tau b^\tau_{\mu\sigma} \int_{G(N)} \chi^\lambda_{G(N)}(U^{-1})\chi^\tau_{G(N)}(U)\mathrm{d}U \\
&= \Xi^{-1}_{G(N),N}(x) \sum_{\ell(\sigma)\leq N} b^\lambda_{\mu\sigma} s_\sigma(x)
\end{aligned}$$

Here, the coefficients $b^\nu_{\lambda\mu}$ are defined by the relation [64]

$$\chi^\lambda_{G(U)}(U)\chi^\mu_{G(N)}(U) = \sum_\nu b^\nu_{\lambda\mu}\chi^\nu_{G(N)}(U). \tag{6.24}$$

These exist because the polynomials $\mathfrak{s}_\lambda(x)$, defined by Eq. (6.12), form a complete basis of polynomials in x. In fact, in terms of the Littlewood-Richardson coefficients, these are given by

$$b^\kappa_{\lambda\mu} = \sum_{\sigma,\rho,\tau} c^\lambda_{\sigma\tau} c^\mu_{\rho\tau} c^\kappa_{\sigma\rho}. \tag{6.25}$$

Note that from Eq. (6.3) it follows that $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$, so that also $b^\kappa_{\lambda\mu} = b^\kappa_{\mu\lambda}$, as is required by Eq. (6.25) □

Proposition 6.1.8 together with Theorem 6.1.7 show that any ratio of $D^{\lambda,\mu}_N(f)$ is independent of the specific group $G(N)$, for $G(N) = SO(2N+1), Sp(2N)$ and $SO(2N+1)$. This can be extended to also allow for $G(N) = U(N)$.

Proposition 6.1.9 (Theorem 6 of [10])**.** *The following limit exists for $G(N)$ any of $U(N), Sp(2N), SO(2N)$ or $SO(2N+1)$,*

$$\lim_{N\to\infty} \frac{D^{\lambda,\mu}_{G(N)}(f)}{D_{G(N)}(f)} = \sum_\nu s_{\lambda/\nu}(x)s_{\mu/\nu}(x).$$

Here $f(z) = H(x;z)H(x;z^{-1})$.

Proof. Due to the limit $N \to \infty$ we may replace the sums $\sum_{\ell(\nu)\leq N}$ with $\sum_\nu$.

For $G(N) = U(N)$, this follows directly from Proposition 6.1.6 and the identity

$$\sum_\nu s_{\nu/\lambda}(x)s_{\nu/\mu}(x) = \sum_\nu s_\nu(x)s_\nu(x)\sum_\nu s_{\lambda/\nu}(x)s_{\mu/\nu}(x).$$

Consider now the groups $G(N) = SO(2N+1), Sp(2N)$ and $SO(2N)$. The result follows from

$$\sum_\nu b^\lambda_{\mu\nu}s_\nu(x) = \sum_{\sigma,\rho,\tau,\nu} c^\mu_{\sigma\tau}c^\nu_{\rho\tau}c^\lambda_{\sigma\rho}s_\nu(x) = \sum_{\sigma,\rho,\tau} c^\mu_{\sigma\tau}c^\lambda_{\sigma\rho}s_\rho(x)s_\tau(x) = \sum_\sigma s_{\lambda/\sigma}(x)s_{\mu/\sigma}(x).$$

□

The only nuisance of Proposition 6.1.9 is that it considers the ratio $D^{\lambda,\mu}_{G(N)}(f)/D_{G(N)}(f)$ instead of $D^{\lambda,\mu}_{G(N)}(f)/D_{G(N+1)}(f)$. To fix this, we need a little lemma.

Lemma 6.1.10. *Let $\{x_i\}_{i=1}^\infty$ be a sequence converging to $x \neq 0$. Suppose furthermore that $\{y_i\}_{i=1}^\infty$ is such that the following limit exists*

$$\lim_{i\to\infty}\frac{y_i}{x_i} =: z.$$

Then, the limit $\lim_{i\to\infty} y_i/x_{i+1}$ exists and is equal to z.

Proof. Let $\varepsilon > 0$. Without loss of generality, we may assume $\varepsilon < |x|/2$. Since $\{x_i\}_{i=1}^\infty$ converges to x, there exists an N_1 such that $|x_i - x_j| < \varepsilon^2$ for all $i, j > N$. Because x is nonzero, we can find an N_2 such that $|x_i| > \varepsilon$ for all $i > N_2$. For $i, j > \max(N_1, N_2)$ this results in the estimate

$$\left|\frac{x_i}{x_{i+1}} - 1\right| = \frac{1}{|x_{i+1}|}|x_i - x_{i+1}| < \frac{1}{\varepsilon}\varepsilon^2 = \varepsilon.$$

Finally, we can find an N_3 such that $|y_i/x_i - z| < \varepsilon$ for all $i > N_3$. Let N be the maximum of N_1, N_2 and N_3. Then, for $i > N$ we have the following estimate for $|y_i/x_{i+1} - z|$,

$$\begin{aligned}\left|\frac{y_i}{x_{i+1}} - z\right| &= \left|\left(\frac{x_i}{x_{i+1}} - 1\right)\frac{y_i}{x_i} + \left(\frac{y_i}{x_i} - z\right)\right| \\ &= \left|\left(\frac{x_i}{x_{i+1}} - 1\right)\left(\frac{y_i}{x_i} - z\right) + \left(\frac{x_i}{x_{i+1}} - 1\right)z + \left(\frac{y_i}{x_i} - z\right)\right| \\ &< \varepsilon^2 + |z|\varepsilon + \varepsilon.\end{aligned}$$

Now, $\varepsilon + |z| + 1$ is just a constant, so for any $\varepsilon' > 0$ we can find an $\varepsilon > 0$ such that $\varepsilon^2 + |z|\varepsilon + \varepsilon < \varepsilon'$. The reasoning above gives us an N for which this is satisfied. This concludes the proof. □

Corollary 6.1.11. *The limit*

$$\lim_{N\to\infty}\frac{D^{\lambda,\mu}_{G(N)}(f)}{D_{G(N+1)}(f)}$$

does not depend on the specific group $G(N) = U(N), SO(2N+1), Sp(2N)$ or $SO(2N)$, provided that $\lim_{N\to\infty} D_{G(N)}(f) \neq 0$.

The functions $f(z) = H(x;z)H(y;z)$ considered so far satisfy $f(0) = 1$. All results can easily be generalized to f such that $f(0) = e^{c_0}$, but this involves extra factors of e^{Nc_0}, due to the definition of $f(U) = \prod_{i=1}^N f(z_i)$, with $U \in G(N)$ and z_i, $i = 1, \ldots, N$ the eigenvalues of U.

6.2 Eigenvectors of Toeplitz matrices

Our main tool for calculating eigenvectors is as follows. Consider a Hermitian matrix H with orthonormal basis of eigenvectors $\{v_i\}_{i=1}^N$, with corresponding eigenvalues E_i. Then, the eigenvector-eigenvalue identity [14], referred to us by the Projected Cramer's rule (Corollary 5.4.2), implies

$$\sum_{i=1}^{N} \frac{|v_{i,j}|^2}{E_i - E} = \frac{\det(M_{jj} - E)}{\det(H - E)},$$

where $v_{i,j}$ is the j-th component of v_i. The variable E is any complex number outside the spectrum of H. The matrix M_{jj} is the (j,j)-th minor of H. In the special case of Toeplitz matrices $H = T_N(f)$, this results in

$$\sum_{i=1}^{N} \frac{|v_{i,j}|^2}{E_i - E} = \frac{D_{N-1}^{(1^j),(1^j)}(f - E)}{D_N(f - E)}. \tag{6.26}$$

This can be reduced to an equation for $|v_{i,j}|^2$ by performing a contour integral around a single eigenvalue E_i, that is, taking the residue at E_i.

Using Eq. (6.26) and Corollary 6.1.11, we are now able to state our main result.

Proposition 6.2.1. *Let E_i be an isolated eigenvalue of $T_{G(N)}(f)$, in the limit of $N \to \infty$. Here, $T_{G(N)}(f)$ is the Toeplitz $\pm$ Hankel matrix generated by f, as defined in Corollary 6.1.3, that is*

$$\begin{aligned}
T_{U(N)}(f) &= [d_{j-k}]_{j,k=1}^N, \\
T_{SO(2N+1)}(f) &= [d_{j-k} - d_{j+k-1}]_{j,k=1}^N, \\
T_{Sp(2N)}(f) &= [d_{j-k} - d_{j+k}]_{j,k=1}^N, \\
T_{SO(2N)}(f) &= \frac{1}{2}[d_{j-k} + d_{j+k-2}]_{j,k=1}^N.
\end{aligned}$$

Note that, with this definition, $D_{G(N)}(f) = \det T_{G(N)}(f)$. Then, for $G(N)$ one of $U(N), SO(2N+1)$ or $Sp(2N)$, the absolute values of the eigenvector components do not depend on the group $G(N)$, but only on f and the eigenvalue E_i.

Proof. Note that for these groups $G(N)$, the characteristic polynomial is given by $D_{G(N)}(f - E)$. This is not true for $G(N) = SO(2N)$, because for $j, k = 1$, $d_{j+k-2} = d_0$, so another factor of E will be included on the $(1,1)$-th entry by the Hankel part.

The absolute value of the j-th component of an eigenvector v_i of the matrix $T_{G(N)}(f)$ can be calculated using the residue formula,

$$|v_{i,j}|^2 = \int_{\mathcal{C}_i} \sum_{k=1}^{N} \frac{|v_{k,j}|^2}{E_k - E} \mathrm{d}E = \int_{\mathcal{C}_i} \frac{D_{G(N-1)}^{(1^j),(1^j)}(f - E)}{D_{G(N)}(f - E)} \mathrm{d}E. \tag{6.27}$$

Here, $\mathcal{C}_i$ is a contour in the complex plane around E_i, and around E_i only, crossing none of the other eigenvalues. The right hand side of Eq. (6.27) does not depend on $G(N)$ by Corollary 6.1.11, in the limit $N \to \infty$. □

7 Conclusion

Random matrix theory is not only useful for the description of eigenvalue statistics of quantum systems whose classical counterpart is chaotic, but also for both eigenvalue and eigenvector statistics of intermediate statistics in the transition to chaos. We have studied two important sets of random matrix ensembles, the Gaussian ensembles and the Circular ensembles, and how these arise naturally when assuming time translation invariance. We explored the Coulomb gas analogy, Dyson Brownian motion and the connection with the two-matrix model and the Calogero-Sutherland Hamiltonian. This allowed for the interpretation of Brownian motion on the Weyl chambers.

Next, we discussed the technique of orthogonal polynomials. This technique can be used to derive level spacing distributions, which allowed us to compare Wigner-Dyson statistics to Poisson statistics. These distributions were employed to quantitatively describe systems with intermediate statistics, using, for example, the level compressibility. We studied the inverse participation ratios, and showed how this resulted in the interpretation of wave function multifractality. This part was concluded with a study of methods for calculating these inverse participation ratios, where we introduced the eigenvalue-eigenvector identity.

Finally, in the main part of this thesis, we applied this identity to Toeplitz±Hankel matrices. We first showed integral representations of determinants of Toeplitz±Hankel matrices using Andréief's identity. Then, we derived formulas in terms of Schur polynomials using Newton's identities, the Cauchy identities and Schur orthogonality. We showed, using a simple new proof exploiting relations of Littlewood-Richardson coefficients, that the ratio of two determinants of Toeplitz±Hankel matrices is independent of the integration group. Using the eigenvalue-eigenvector identity, we showed our main result; under some constraints on the coefficients of a Toeplitz and Toeplitz±Hankel, if such Toeplitz and Toeplitz±Hankel matrix have the same eigenvalue, the absolute value of their corresponding eigenvector component is equal.

For further research, it would be interesting to look at applying these results to the translational invariant Rosenzweig-Porter model. Furthermore, one could look at applying Morozov's formula to the two-matrix model, to provide an alternative method to the nonlinear sigma model for calculating IPRs of a near chaotic random matrix ensemble.

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Appendices

A Non-linear sigma model for the Gaussian Unitary Ensemble

Let us apply the non-linear sigma model for the Gaussian Unitary Ensemble and derive the Wigner semicircle law, the Porter Thomas distribution and the inverse participation ratios. For this derivation we follow [45]. The reader could also look in the book by Haake [41, Chapter 6] for more details. As seen in Lemma 3.2.4, we are interested in Greens functions with slightly perturbed energies, $x \to x \pm i\varepsilon$. These will be called advanced and retarded Green's functions. Slightly more general, we also allow for a real energy offset $x \to x \pm (\omega + i\varepsilon)$. Write $G^A(e_j, e_k; x)$ and $G^R(e_j, e_k; x)$ for the advanced and retarded Green's function, that is, $G^{R/A}(e_j, e_k; x) = G(e_j, e_k; x \pm (\omega + i\varepsilon))$. Let $x_\pm$ be $x_\pm := x \pm (\omega + i\varepsilon)$. Write these Green's function as a Gaussian integral using Eq. (5.18),

$$G^{R/A}(e_j, e_k; x) = \frac{\pm i}{Z} \int \mathcal{D}\varphi \mathcal{D}\varphi^* \ \varphi_k \varphi_j^* \exp(iS_\pm[\varphi]),$$

where

$$S_\pm[\varphi] = \mp \sum_{i,j} \varphi_i^* [x_\pm \delta_{ij} - H_{ij}] \varphi_j,$$
$$Z = \det(x_\pm - H)^{-1} = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \exp(iS_\pm[\varphi]).$$

Here, $\mathcal{D}\varphi = \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_N$, where $\varphi_1, \dots, \varphi_N$ are independent complex variables. In the limit of $N \to \infty$, this will become a path integral. We would like to do the averaging first, and then the remaining path integral. Since Z also depends on H, we need to write this expression as a single integral. This is possible with the use of Gaussian integrals over Grassmann variables. The difference with a regular Gaussian integral is that the determinant appears in the numerator,

$$\det(x_\pm - H) = \int \mathcal{D}\mu^* \mathcal{D}\mu \exp(iS_\pm[\mu]),$$

where μ_i and μ_i^*, $i = 1, \dots, N$, are now Grassmann variables. For all i, even μ_i and μ_i^* are independent Grassmann variables, but they do satisfy the additional relation

$$(\mu_i^*)^* = \mu_i.$$

Furthermore, similar to Hermitian conjugation, the conjugation changes the ordering; for Grassmann variables μ, ν, μ^* and ν^*,

$$(\mu\nu)^* = \nu^*\mu^* = -\mu^*\nu^*.$$

Using this notation we can express the Green's function as

$$G^{R/A}(e_j, e_k; x) = \mp i \int \mathcal{D}\psi \ \varphi_k \varphi_j^* \exp(iS_\pm[\psi]),$$

where

$$S_\pm[\psi] = S_\pm[\varphi] + S_\pm[\mu] = \pm \sum_{i,j} \psi_i^\dagger [x_\pm \delta_{i,j} - H_{ij}] \psi_j.$$

Here we have introduced the combined $\psi^\dagger$ and ψ, called *super-vectors*,

$$\psi^\dagger = \begin{pmatrix} \varphi^* & \mu^* \end{pmatrix}, \qquad \psi = \begin{pmatrix} \varphi \\ \mu \end{pmatrix}.$$

Note that $\psi_j^\dagger \psi_i = \varphi_j^* \varphi_j + \mu_j^* \mu_j$. The measure $\mathcal{D}\psi$ is $\mathcal{D}\psi = \mathcal{D}\varphi\mathcal{D}\mu$. The super-symmetry is broken by the term $\psi_k \psi_j^*$. Equivalently, one could write $-\mu_k \mu_j^*$ instead of $\varphi_k \varphi_j^*$ here, which can be seen from Eq. (5.18).

We would now like to do the ensemble average of $\exp\Big(i \sum_{ij} \psi_i^\dagger \psi_j H_{ij}\Big)$. For this, consider the slightly more general case where the entries H_{ij} are independently Gaussian distributed with variance according to the PDF $\exp(-|H_{ij}|^2/A_{ij})$. This would also include the Power-Law Banded Random Matrix ensemble (PBRM). Consider for simplicity the $\beta = 2$ case, so that the variables H_{ij} are complex. For the Gaussian Unitary Ensemble we have $A_{ij} = 1 + \delta_{ij}$.

Let us consider all the terms involving H_{ij} for some $1 \leq i, j \leq N$. The combined term in the exponential is

$$\sum_{ij} \left[\mp i \psi_i^\dagger \psi_j H_{ij} - \frac{1}{A_{ij}} |H_{ij}|^2 \right]$$

For fixed i and j,

$$\begin{aligned} \mp i \psi_i^\dagger \psi_j H_{ij} - \frac{|H_{ij}|^2}{A_{ij}} &= \mp \frac{i}{2} \left(\psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i \right) H_{ij} - \frac{|H_{ij}|^2}{A_{ij}} \\ &= -\frac{1}{A_{ij}} \left(H_{ij} \pm \frac{i}{2} A_{ij} \psi_i^\dagger \psi_j \right) \left(H_{ji} \pm \frac{i}{2} A_{ij} \psi_j^\dagger \psi_i \right) - \frac{A_{ij}}{4} \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i \\ &= -\frac{1}{A_{ij}} |H'_{ij}|^2 - \frac{A_{ij}}{4} \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i, \end{aligned} \tag{.1}$$

where

$$H'_{ij} = H_{ij} \pm \frac{i}{2} \psi_i^\dagger \psi_j.$$

Note that H'_{ij} is still a commuting complex number, and furthermore, $(H'_{ij})^\dagger = H'_{ji}$. For the Gaussian Unitary ensemble, we can therefore just change integration variables from H_{ij} to H'_{ij}. The ensemble average becomes

$$\begin{aligned} \left\langle \exp\left(\mp i \sum_{i,j} \psi_i^\dagger \psi_j \tilde{H}_{ij} \right) \right\rangle &= \int \mathrm{d}H' \exp\left(-\sum_{ij} \frac{1}{A_{ij}} |H'_{ij}| \right) \exp\left(-\sum_{ij} \frac{A_{ij}}{4} \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i \right) \\ &= \exp\left(-\sum_{ij} \frac{A_{ij}}{4} \psi_i^\dagger (\psi_j \psi_j^\dagger) \psi_i \right). \end{aligned} \tag{.2}$$

In the last step, we used that the PDF for $|H'_{ij}|$ is normalized to 1. Let $\tilde{Q}_i$ be the 2×2 *super-matrix*

$$\tilde{Q}_j := \psi_j \psi_j^\dagger = \begin{pmatrix} \varphi_j \varphi_j^* & \varphi_j \mu_j^* \\ \mu_j \varphi_j^* & \mu_j \mu_j^* \end{pmatrix},$$

then Eq. (.2) can be written as

$$\left\langle \exp\left(\mp i \sum_{i,j} \psi_i^\dagger \psi_j \tilde{H}_{ij} \right) \right\rangle = \exp\left(-\sum_{ij} \frac{A_{ij}}{4} \operatorname{tr}_s(\tilde{Q}_i \tilde{Q}_j) \right).$$

The *super-trace* is the trace, but with an extra minus sign for the Grassmann − Grassmann, or Fermion − Fermion part,

$$\operatorname{tr}_s \begin{pmatrix} BB & BF \\ FB & FF \end{pmatrix} := BB - FF.$$

With this definition, the cyclic property of the trace still holds, $\operatorname{tr}_s(AB) = \operatorname{tr}_s(BA)$. Furthermore, we define the *super-determinant* as

$$\det_s \begin{pmatrix} BB & BF \\ FB & FF \end{pmatrix} := \det\big(BB - BF(FF)^{-1}FB\big)/\det(FF), \quad \text{if } (FF)^{-1} \text{ exists.}$$

Note that the ordinary determinant is given by

$$\det\big(BB - BF(FF)^{-1}FB\big) \cdot \det(FF).$$

This definition shares some nice properties with the regular determinant [41, Section 6.3.3],

$$\det_s F = \det_s F^T, \tag{.3a}$$
$$\det_s F_1 F_2 = \det_s F_1 \det_s F_2, \tag{.3b}$$
$$\log \det_s F = \operatorname{tr}_s \log F. \tag{.3c}$$

Note that $\psi_i^\dagger \psi_i = \operatorname{tr}_s(\tilde{Q}_i)$. The expectation values of the Green's functions are therefore,

$$\begin{aligned} \langle G^{R/A}(e_j, e_k; x) \rangle &= \mp i \int \mathcal{D}\psi \varphi_k \varphi_j^* \exp\Big(-F[\tilde{Q}]\Big), \\ F[\tilde{Q}] &= \mp i x_\pm \sum_i \operatorname{tr}_s(\tilde{Q}_i) + \frac{1}{4} \sum_{ij} A_{ij} \operatorname{tr}_s(\tilde{Q}_i \tilde{Q}_j), \quad \tilde{Q}_i = \psi_i \psi_i^\dagger. \end{aligned} \tag{.4}$$

We will be interested, however, in expectation values of combinations of advanced and retarded Green's functions, for example

$$\langle G^R(e_j, e_j; x)^k G^A(e_j, e_j; x)^m \rangle$$

or

$$\langle G^R(e_j, e_j; x) - G^A(e_j, e_j; x) \rangle = -2\pi i \rho(x).$$

A similar procedure as for Eq. (.4) can be applied to conclude that

$$\langle G^R(e_j, e_j; x)^k G^A(e_j, e_j; x)^m \rangle = \frac{i^{m-k}}{k!m!} \int \mathcal{D}\psi^R \mathcal{D}\psi^A (\psi_j^{R*} \psi_j^R)^k (\psi_j^{A*} \psi_j^A)^m e^{-F[\tilde{Q}^A] - F[\tilde{Q}^R]}, \tag{.5}$$

and

$$\rho(x) = \frac{1}{2\pi} \int \mathcal{D}\psi^R \mathcal{D}\psi^A \ (\varphi_j^{R*} \varphi_j^R + \varphi_j^{A*} \varphi_j^A) e^{-F[\tilde{Q}^A] - F[\tilde{Q}^R]}. \tag{.6}$$

Let us therefore combine the fields ψ_i^R and ψ_i^A into a 4-component super-vector Ψ_i,

$$\Psi_i^\dagger = \begin{pmatrix} \varphi_i^{R*} & -\varphi_i^{A^*} & \mu_i^{R*} & -\mu_i^{A^*} \end{pmatrix}, \quad \Psi_i = \begin{pmatrix} \varphi_i^R \\ \varphi_i^A \\ \mu_i^R \\ \mu_i^A \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This ordering of Ψ_i is called *Bose-Fermi notation*. An alternative ordering of retarded wavefunctions appearing first is called *retarded-advanced notation*. Let $\mathcal{D}\Psi = \mathcal{D}\psi^R\mathcal{D}\psi^A$. The quartic term proportional to A_{ij} can be obtained as follows. Group terms proportional to either H_{ij} or H_{ji}. In this case, this is

$$\psi_i^{R\dagger}\psi_j^R - \psi_i^{A\dagger}\psi_j^A = \Psi_i^\dagger\Psi_j \text{for } H_{ij},$$
$$\psi_j^{R\dagger}\psi_i^R - \psi_j^{A\dagger}\psi_i^A = \Psi_j^\dagger\Psi_i \text{for } H_{ji}$$

Hence, we can do the same derivation as Eq. (.1), with ψ replaced by Ψ. The quartic term therefore is

$$\frac{1}{4}\sum_{ij} A_{ij}\Psi_i^\dagger\Psi_j\Psi_j^\dagger\Psi_i = \frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{tr}_s(Q_iQ_j),$$

where $Q_i = \Psi_i\Psi_i^\dagger$. This allows us to write Eq. (.5) and Eq. (.6) as follows

$$\langle (G^R)^k (G^A)^m \rangle = \frac{i^{m-k}}{k!m!}\int \mathcal{D}\Psi\ (\psi_j^{R*}\psi_j^R)^k (\psi_j^{A*}\psi_j^A)^m e^{-F[Q]}, \tag{.7}$$

$$\rho(x) = \frac{1}{2\pi}\int \mathcal{D}\Psi\ (\varphi_j^{R*}\varphi_j^R + \varphi_j^{A*}\varphi_j^A) e^{-F[Q]}, \tag{.8}$$

$$F[Q] = -ix\sum_i \mathrm{tr}_s\, Q_i - i(\omega + i\varepsilon)\sum_i \mathrm{tr}_s(\Lambda Q_i) + \frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{tr}_s(Q_iQ_j). \tag{.9}$$

This is a field theory with a quartic term. Let us concentrate on rewriting $\int \mathcal{D}\Psi e^{-F[Q]}$. The quartic term can be removed at the cost of an extra integral using a *Hubbard-Stratonovich transformation*,

$$e^{-\sum_{ij}\frac{A_{ij}}{4}\mathrm{tr}_s[(\Psi_i\Psi_i^\dagger)(\Psi_j\Psi_j^\dagger)]} = \int \mathcal{D}P e^{-\sum_{ij}A_{ij}^{-1}\mathrm{tr}_s[P_iP_j] + i\sum_i \mathrm{tr}_s[(\Psi_i\Psi_i^\dagger)P_i]}. \tag{.10}$$

We can use Eq. (.3c) to calculate the following Gaussian integral,

$$\int \mathcal{D}\psi\ \exp\Big(\psi_i^\dagger K_i\psi_i\Big) = \exp\bigg(-\sum_i \mathrm{tr}_s[\log K_i]\bigg). \tag{.11}$$

Note that

$$\mathrm{tr}_s(AQ_i) = \mathrm{tr}_s(\Psi_i^\dagger A\Psi_i) = \Psi_i^\dagger A\delta_{ij}\Psi_j,$$

hence it can also be calculated as a Gaussian integral. The result is

$$\int \mathcal{D}\Psi e^{-F[Q]} = \int \mathcal{D}P\mathcal{D}\Psi \exp\bigg(-\sum_{ij} A_{ij}^{-1}\,\mathrm{tr}_s(P_iP_j) + i\sum_i \Psi_i^\dagger(P_i - x - (\omega + i\varepsilon)\Lambda)\Psi_i\bigg) \tag{.12a}$$

$$= \int \mathcal{D}P e^{-F[P]}, \text{where} \tag{.12b}$$

$$F[P] = \sum_{ij} A_{ij}^{-1} \operatorname{tr}_s(P_i P_j) + i \sum_i \operatorname{tr}_s \log(x - P_i + (\omega + i\varepsilon)\Lambda). \tag{.12c}$$

Assumed that ω and ε are small, and furthermore, up to first approximation, that P_i is independent of i, so that we can write $P_i = P_0$. The approximated action is

$$N^{-1} F[P_0] = A_0^{-1} \operatorname{tr}_s P_0^2 + i \operatorname{tr}_s \log(x - P_0).$$

Here $A_i^{-1} := \sum_j A_{ij}^{-1}$ is also assumed to be independent of i, hence $A_i^{-1} = A_0^{-1}$. Let us therefore vary the action $F[P]$ with respect to P_0, to get the following solution of the saddle point equation

$$P_0 I = \frac{1}{2}(x \cdot I + iQ\sqrt{2A_0 I - x^2}).$$

Here, I is the 4×4 identity matrix and Q is any super-matrix such that $Q^2 = 1$ and $\operatorname{tr}_s Q = 0$. We now add variations around the saddle-point minimum to the action, that is, let Φ_i be defined by the following equations

$$P_i = \frac{1}{2}\left(E + i\Phi_i\sqrt{2A_0 - E^2}\right).$$

Then fill this into the action Eq. (.12c) and expand to first order in $\omega + i\varepsilon$,

$$\begin{aligned} F[\Phi] &= -\frac{1}{4}(\sqrt{2A_0 - x^2})^2 \sum_{ij} A_{ij}^{-1} \operatorname{tr}_s(\Phi_i \Phi_j) + i\frac{\sqrt{2A_0 - x^2}}{A_0}(\omega + i\varepsilon) \sum_i \operatorname{tr}_s(\Lambda \Phi_i) \\ &= -\frac{1}{4}(\pi\rho A_0)^2 \sum_{ij} A_{ij}^{-1} \operatorname{tr}_s(\Phi_i \Phi_j) - i\pi\rho(\omega + i\varepsilon) \sum_i \operatorname{tr}_s(\Lambda \Phi_i), \end{aligned}$$

where $\rho := \sqrt{2A_0 - x^2}/A_0$. Here we used that $\operatorname{tr}_s(Q_i) = 0$.

A.1 Spectral density and the Wigner Semicircle

In this section the Wigner Semicircle law is derived. The pre-exponent in Eq. (.8) can be written as a supertrace,

$$(\varphi_j^{R*}\varphi_j^R + \varphi_j^{A*}\varphi_j^A) = \operatorname{tr}_s(\Pi Q)$$

with the projections

$$\Pi = \Pi^R - \Pi^A, \quad \Pi^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi^A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The standard procedure is to add a source term,

$$\delta F[Q, J] = -i \sum_i J_i \operatorname{tr}_s[\Pi Q_j],$$

to the action $F[Q]$, take the derivative with respect to J and then set $J = 0$,

$$\rho(x) = \frac{1}{2\pi i} \frac{\partial}{\partial J_j}\bigg|_{J=0} \int \mathcal{D}\Psi e^{-F[Q] - \delta F[Q,J]}.$$

Note that the new action is the original action with $\omega\Lambda \to \omega\Lambda + J_j\Pi$. The saddle point approximation results in the action Appendix A with $\omega\Lambda$ replaced by $\omega\Lambda + J_j\Pi$. This derivative can now be done easily,

$$\rho(x) = \frac{1}{2}\int \mathcal{D}\Phi \,\mathrm{tr}_s(\Pi\Phi_j)e^{-F[\Phi,\omega=0]}.$$

For the Wigner-Dyson ensemble, $F[\Phi,\omega=0]=0$, $A_{ij}=1$, so $A_0=N$. The semicircle law follows.

A.2 Inverse participation ratios and Porter-Thomas

The supersymmetry method can also be used to calculate IPR's, as done by Fal'ko and Efetov [65]. From Eq. (5.17) it is clear that for the IPR's we must calculate $\langle G^R(x)^l G^A(x)^m\rangle$, which can be expressed as (see Eq. (.5))

$$G^R(x)^l G^A(x)^m = \frac{i^{m-l}}{l!m!}\int \mathcal{D}\Psi(\varphi_j^{R*}\varphi_j^R)^l(\varphi_j^{A*}\varphi_j^A)^m e^{-F[Q]}.$$

Write $\varphi_j^{R*}\varphi_j^R = \mathrm{tr}_s(\Pi^R Q_j)$ and $\varphi_j^{A*}\varphi_j^A = \mathrm{tr}_s(\Pi^A Q_j)$, then with $m=1$ and $l=k-1$ we get

$$\begin{aligned}\langle|\psi_{i,j}|^{2k}\rangle_x &= \frac{1}{2\pi\rho}i^{k-2}\lim_{\varepsilon\to 0}\left[(2\varepsilon)^{k-1}\frac{i^{-k+2}}{(k-1)!}\int \mathcal{D}\Psi \mathrm{tr}_s(\Pi^R Q_j)^{k-1}\,\mathrm{tr}_s(\Pi^A Q_j)e^{-F[Q]}\right]\\ &= \frac{1}{2\pi\rho(k-1)!}\lim_{\varepsilon\to 0}\left[(2\varepsilon)^{k-1}\int \mathcal{D}\Psi \mathrm{tr}_s(\Pi^R Q_j)^{k-1}\,\mathrm{tr}_s(\Pi^A Q_j)e^{-F[Q]}\right],\end{aligned}$$

where,

$$F[Q] = -ix\sum_i \mathrm{tr}_s\, Q_i - i(\omega+i\varepsilon)\sum_i \mathrm{tr}_s(\Lambda Q_i) + \frac{1}{4}\sum_{ij}A_{ij}\,\mathrm{tr}_s(Q_iQ_j).$$

We would like to change from an integral over Ψ to an integral over Q now. In this case however, the fields Q_n are not slowly varying. The remedy is to take combinations of φ_n^* and φ_n which are slowly varying. This will result into $(l+m)!$ possibilities of breaking up the product into slow bi-linear combinations [45, Section XVII]. The result is

$$\langle|\psi_{i,j}|^{2k}|\rangle_x = \frac{k}{2}\lim_{\varepsilon\to 0}\left[(2\pi\rho\varepsilon)^{k-1}\int \mathcal{D}Q\ \mathrm{tr}_s(\Pi^R Q_j)^{k-1}\,\mathrm{tr}_s(\Pi^A Q_j)e^{-F[Q]}\right].$$

Let us define the generating function $Y[Q_j]$ as

$$Y[Q_j] = \int_{Q_i, i\neq j}\mathcal{D}Q e^{-F[Q]}. \tag{.13}$$

This function does not depend on the anti-commuting variables. This integration will result in a factor (-1). To compensate the factor ε^{k-1}, the integral must be large, which can only be done by changing $\lambda\to\varepsilon\lambda$. The supertraces are then proportional to λ^{k-2}, and in this limit the generating function only depends on $\varepsilon\lambda$. Write $s=(2\pi\rho\varepsilon\lambda)$ to include prefactors, then [45, Section XVII]

$$\langle|\psi_{i,j}|^{2k}|\rangle_x = \frac{k(k-1)}{N}\int_0^\infty \mathrm{d}s\ s^{k-2}Y(s).$$

Therefore, the probability density function $\mathcal{P}(|\psi_{i,j}|^2)$ can be obtained as

$$\mathcal{P}(|\psi_{i,j}|^2) = \frac{1}{N}\frac{\partial^2}{\partial s^2}Y(s)\bigg|_{s=|\psi_{i,j}|^2}.$$

In the Wigner-Dyson case for $\beta = 2$, the super-matrices Q_i are fixed, so there is no integration in Eq. (.13), hence

$$\mathcal{P}(|\psi_{i,j}|^2) = Ne^{-N|\psi_{i,j}|^2}.$$

This is called the *Porter-Thomas distribution.*

B Hilbert series

In this section we will present an application of the general theory of Toeplitz ± Hankel determinants for the calculation of Hilbert series. Hilbert series describe the number of gauge invariant terms that can be added to the Lagrangian of some effective field theory, for example the Hilbert $\mathfrak{H}(t)$ series for $U(1)$ is $1 + t + t^2 + \ldots$, which tells us that there is precisely one gauge invariant term of order k, which we know to be $(\varphi^*\varphi)^k$. To calculate the Hilbert series, one needs to integrate the *Plethystic polynomial* over a classical Lie group. This expression is similar to the expressions for calculating Toeplitz±Hankel determinants, as is described in the next subsection.

B.1 Plethystic exponential

In this section we review some basics of the plethystic exponential, based on Appendix A of Hanany and Kalveks [26].

The *Plethystic exponential* (PE) of f is defined as

$$\mathrm{PE}(f(t_1,\ldots,t_N),(t_1,\ldots,t_N)) := \exp\left(\sum_{k=1}^{\infty}\frac{f(t_1^k,\ldots,t_N^k)-f(0)}{k}\right) \tag{.14}$$

One should keep the following example in mind: Let U be an $N\times N$ diagonalizable matrix with eigenvalues $\lambda_1,\ldots,\lambda_N$, then $f(U)$ can naturally be defined as $f(U) = f(x_1,\ldots,x_N)$, in which case $f(U^k) = f(x_1^k,\ldots,x_N^k)$. Furthermore, suppose f is a power series expansion of some $f(t_1,\ldots,t_N)$,

$$f(t_1,\ldots,t_N) = \sum_{n=0}^{\infty}\sum_{i=1}^{N} a_{n_i}t_i^n, \tag{.15}$$

then

$$\mathrm{PE}(f(t_1,\ldots,t_N),(t_1,\ldots,t_N)) = \prod_{n=1}^{\infty}\prod_{i=1}^{N}\frac{1}{(1-t_i^n)^{a_{n_i}}}.$$

The inverse operation is the *Plethystic Logarithm* (PL),

$$\mathrm{PL}(g(t_1,\ldots,t_N),(t_1,\ldots,t_N)) := \sum_{k=1}^{\infty}\frac{1}{k}\mu(k)\log g(t_1^k,\ldots,t_N^k), \tag{.16}$$

where $\mu(k) : \mathbb{Z}_+ \to \{-1, 0, 1\}$ is the Möbius function, defined as

$$\mu(k) = \begin{cases} 0 & \text{if } k \text{ has a squared prime factor,} \\ 1 & \text{if } k \text{ has an even number of distinct non-unital prime factors,} \\ -1 & \text{if } k \text{ has an odd number of distinct non-unital prime factors.} \end{cases}$$

For example $\mu(1) = 1, \mu(2) = -1, \mu(3) = -1, \mu(4) = 0$. The inverse property is expressed by the relation

$$\mathrm{PL}[\mathrm{PE}[f(t_1, \ldots, t_N)]] = f(t_1, \ldots, t_N) = \mathrm{PE}[\mathrm{PL}[f(t_1, \ldots, t_N)]].$$

The Plethystic exponential is symmetrizing, that is 'bosonic'. The 'fermionic' anti-symmetrizing counterpart is the *Fermionic Plethystic Exponential* (PEF),

$$\begin{aligned} \mathrm{PEF}(f(t_1, \ldots, t_N), (t_1, \ldots, t_N)) &:= \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{f(t_1^k, \ldots, t_N^k) - f(0)}{k} \right) \\ &= \prod_{n=1}^{\infty} \prod_{i=1}^{N} (1 + t_i^n)^{a_{n_i}}, \end{aligned} \tag{.17}$$

which has an extra minus sign. The last equality holds if f is defined by Eq. (.15). It's logarithm is called the *Fermionic Plethystic Logarithm* (PLF), and given by

$$\begin{aligned} \mathrm{PLF}(g(t_1, \ldots, t_N), (t_1, \ldots, t_N)) &:= \sum_{m=0}^{\infty} \mathrm{PL}[g(t_1^{2m}, \ldots, t_N^{2m}), (t_1^{2m}, \ldots, t_N^{2m})] \\ &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \mu(k) \log g(t_1^{(2^m)k}, \ldots, t_N^{(2^m)k}). \end{aligned} \tag{.18}$$

Let us present some simple examples;

$$\begin{aligned} \mathrm{PE}[t_1] &= 1/(1 - t_1) = 1 + t_1 + t_1^2 + \ldots, \\ \mathrm{PEF}[t_1] &= 1 + t_1, \\ \mathrm{PE}\left[t \sum_{i=1}^{N} z_i, (t, z_1, \ldots, z_N) \right] &= \exp\left(\sum_{k=1}^{\infty} \frac{p_k(z)}{k} t^k \right) = \prod_{i=1}^{N} \frac{1}{1 - t z_i}, \\ \mathrm{PEF}\left[t \sum_{i=1}^{N} z_i, (t, z_1, \ldots, z_N) \right] &= \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{p_k(z)}{k} t^k \right) = \prod_{i=1}^{N} (1 + t z_i), \end{aligned}$$

where $p_k(x) = x_1^k + \cdots + x_N^k$. Suppose U is an $N \times N$ matrix with eigenvalues $x_1, \ldots, x_N$, then we may also write $p_k(x) = p_k(U) = \mathrm{tr}\big(U^k\big)$.

Remark B.1. Instead of the trace we could consider also other characters χ, and study

$$\begin{aligned} g^G(t, \chi) &:= \mathrm{PE}[t\chi], \\ g_F^G(t, \chi) &:= \mathrm{PEF}[t\chi]. \end{aligned}$$

The integral over G results in the Hilbert Series $\mathfrak{H}$,

$$\mathfrak{H}(\varphi) = \int_G \mathrm{d}\mu \begin{cases} \mathrm{PE}(t\chi(g)) & \text{for a bosonic field,} \\ \mathrm{PEF}(t\chi(g)) & \text{for a fermionic field,} \end{cases}$$

which can be used to count the number of gauge invariant combinations of a field in a certain dimension [66].

Because $\mathrm{PE}[\sum_{j=1}^{M} f_j] = \prod_{j=1}^{M} \mathrm{PE}[f_j]$, we have the following example

$$\mathrm{PE}\left[\left(\sum_{j=1}^{M} t_j\right)\left(\sum_{i=1}^{N} z_i\right)\right] = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(z)}{k} p_k(t)\right) = \prod_{j=1}^{M}\prod_{i=1}^{N} \frac{1}{1-t_j z_i},$$

and a similar identity for PEF. For multiple fields t_j one also has a Hilbert series, where each field t_j transforms in a representation R_j,

$$\mathfrak{H}(\{t_i\}) = \int_G \mathrm{d}\mu \prod_{j=1}^{M} \begin{cases} \mathrm{PE}(t_j \chi_{R_j}(g)) & \text{for a bosonic field } t_j, \\ \mathrm{PEF}(t_j \chi_{R_j}(g)) & \text{for a fermionic field } t_j, \end{cases}$$

For the special case of an infinite number of particles with $R_j = R$ for all j and such that

$$\chi_R(g) = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} x_i^{-1},$$

we get

$$\mathfrak{H}(t_i) = \tau_\cdot(t_i|t_i),$$

where the $\cdot$ indicates either $0, +$ or $-$, depending on the specific group G. Furthermore, suppose the integrand factors, that is

$$\prod_{j=1}^{N} \mathrm{PE}(t_j \chi_{R_j}(g)) = \prod_{i=1}^{N} f(z_i),$$

for some f on the unit circle, then

$$\mathfrak{H}(t_i) = D_{G(N)}(f).$$

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