Utrecht University
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Master's thesis

Grothendieck constructions in higher category theory

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Introduction

Throughout recent years, higher category theory has become increasingly important in modern mathematics. Higher category theory is a further abstraction of category theory, which is well-suited to study mathematical objects up to homotopy. Hence, the development of higher category theory has blurred the lines between on one hand, category theory, and on the other hand, homotopy theory. For instance, diagrams in (weak) higher categories do not have to commute strictly like in classical category theory, but merely up to homotopy. This entails that in order to define a diagram in a higher category, one has to provide homotopy coherence data. To encode this behavior, higher categories are equipped with a notion of n-morphisms where n is a natural number that might be arbitrarily large, or is bounded by some other natural number. In such a category, $(n+1)$-morphisms provide a way to relate n-morphisms. Weak higher categories have relaxed notions of composition, where $(n+1)$-morphisms witness compositions of n-morphisms and composites are no longer unique but only up to an invertible higher morphism. Consequently, it makes no sense to speak of strict associativity laws within a higher category. Instead, associativity of composition holds up to an invertible higher morphism in such a weak higher category. We will be studying weak higher categories which have morphisms in arbitrary degree and have the property that all morphisms of degrees $n > 1$ are invertible. These are called $(\infty, 1)$-categories and we will just refer to them as ∞-categories. These ∞-categories may be modelled using certain simplicial categories (i.e. categories enriched over simplicial sets). One can then take 'free' resolutions of ordinary categories (see Appendix A) to obtain a good notion of (homotopy coherent) diagrams in this model. Unfortunately, this model has some serious drawbacks and is too rigid to nicely develop the theory of ∞-categories. However, it is still a useful model as ∞-categories are often incarnated as simplicial categories in practice.

In this thesis, we will make use of the model for ∞-categories developed by André Joyal and Jacob Lurie using simplicial sets. The work *Higher Topos Theory* [Lur09] by Lurie contains a good introduction to this material. The third chapter of Lurie's work is devoted to developing an ∞-categorical version of the Grothendieck construction from classical category theory (see Chapter 4 for a review of this construction), and showing that this construction gives rise to a *straightening-unstraightening equivalence*. This equivalence entails that we can describe diagrams in $\mathbf{Cat}_\infty$, the ∞-category of (small) ∞-categories, using certain fibrations which are easier to understand. The following statement is the shadow of a more precise statement that has been shown by Lurie, and we will give a different account of later:

Theorem 0.0.1. *Let $\mathcal{C}$ be an ∞-category. Then there exists an adjoint equivalence of ∞-categories*

$$\mathbf{coCart}(\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty).$$

The left adjoint is called the *straightening* or *rectification* functor and the right adjoint is called the *unstraightening* functor (in this thesis, we adhere to the convention that the arrow corresponding to the left adjoint points to the right). Here $\mathbf{coCart}(\mathcal{C})$ denotes the ∞-category of coCartesian fibrations. These fibrations are higher categorical generalizations of Grothendieck opfibrations. The straightening-unstraightening equivalence plays an important role in Lurie's approach to the theory ∞-categories.

However, its importance is not limited to the foundations of higher category theory. For instance, a monoidal structure on an ∞-category $\mathcal{C}$ may be defined as being a certain simplicial object $\mathcal{C}^\otimes_\bullet$ in $\mathbf{Cat}_\infty$ (i.e. a functor $\Delta^{\mathrm{op}} \to \mathbf{Cat}_\infty$) which satisfies satisfies so-called *Segal conditions* and comes with a categorical equivalence $\mathcal{C}^\otimes_1 \simeq \mathcal{C}$. This is a direct generalization of the notion of monoidal structures in classical category theory (in particular, a monoidal structure on an ∞-category induces a monoidal structure on its homotopy category). In light of the theorem stated above, we may equivalently define a monoidal structure on $\mathcal{C}$ as being a coCartesian fibration $\mathcal{C}^\otimes \to \Delta^{\mathrm{op}}$ with the properties that its fiber $\mathcal{C}^\otimes \times_{\Delta^{\mathrm{op}}} \{[1]\}$ is equivalent to $\mathcal{C}$ and its straightening satisfies the Segal conditions. This is the preferred way to define monoidal ∞-categories, as writing down an explicit simplicial object in $\mathbf{Cat}_\infty$ is hard in practice (one has to write down a lot of coherence data!). Note that we did not distinguish here in notation between the nerve $N\Delta$ of Δ and the category Δ which is of course justified as the nerve functor is fully faithful. However, for sake of clarity, we will make this distinction in notation for the remaining of this thesis.

The main objective of this thesis is to present an alternative proof of the straightening-unstraightening equivalence. The approach is conceptual, and has been inspired by the ideas found in the articles of Heuts and Moerdijk [HM15] and Stevenson [Ste17]. In these articles, a few versions of the straightening-unstraightening equivalence for left fibrations are proved. We will use similar approaches to prove the general version for coCartesian fibrations. Along the way, we also develop new tools and produce applications of the developed theory, which cannot be found in the current literature. Moreover, we will also provide alternative proofs of a handful of results found in [Lur09].

The aim of this thesis is to be self-contained, however, we will assume that the reader is familiar with the contents of Chapters 1 and 2 of [Lur09]. The thesis contains 4 chapters, and 2 appendices:

- Chapter 1 contains material on the coCartesian model structure. In particular, it contains a short review of the material in Section 3.1 of [Lur09] relevant to this thesis. More importantly, this chapter contains a section on minimal coCartesian fibrations. This material is new (in particular, it differs from the notion of minimality introduced in [Ngu18]), and will be used to demonstrate a homotopy descent property for coCartesian fibrations. This descent property is used multiple times throughout the thesis. For instance, it will be used to show that the coCartesian model structure is homotopy invariant. The chapter ends with a first attempt at straightening coCartesian fibrations.

- Chapter 2 is devoted to proving the straightening-unstraightening equivalence in case that the base is given by the nerve of a (1-)category. It ends with a few applications. For instance, a marked version of Quillen's theorem A is proven. We also present a new proof of the fact that every coCartesian fibration is a categorical fibrations, and show that coCartesian fibrations may be extended along trivial cofibrations in $\mathbf{sSet}_{\mathrm{Joyal}}$.
- In Chapter 3, we study coCartesian fibrations and marked simplicial diagrams on localizations of respectively ∞-categories and simplicial categories.
- The objective of Chapter 4 is to prove the straightening-unstraightening equivalence in full generality. We will use the results of Chapter 3 to reduce the problem to the case that the base is given by the nerve of a 1-category. We then appeal to the results of Chapter 2.
- Appendices A and B contain material on simplicial computads and model categories respectively. We will refer to this material when needed.

Throughout this thesis, we will make use of the homotopy coherent nerve adjunction

$$\mathfrak{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat} : N,$$

which relates the two models of $(\infty, 1)$-categories via simplicial sets and simplicial categories. Recall that the left adjoint was defined as a left Kan extension of the restricted functor

$$\Delta \to \mathbf{Cat} \xrightarrow{FU_\bullet(-)} \mathbf{sCat}$$

along the Yoneda embedding $\Delta \to \mathbf{sSet}$. A quick recollection of the functor $FU_\bullet(-)$, which associates to every 1-category a certain 'free resolution', is given in Appendix A. This resolution is a simplicial computad (i.e. a cofibrant object in the Bergner model structure on simplicial categories) where all strict composites in the 1-category are now 'unstrictified'. For our convenience, we have reindexed the structure maps in our definition of $FU_\bullet(-)$. It coincides with an opped version of the free resolution functor which is dominant in the current literature. That means that the mapping complexes of $\mathfrak{C}S$ are opposites of the mapping complexes of $\widetilde{\mathfrak{C}}S$, where $\widetilde{\mathfrak{C}}$ is the left adjoint of the homotopy coherent nerve functor in [Lur09]. In particular, the mapping complexes of these two incarnations of S as simplicial category carry (naturally) the same homotopy type. The geometric description of $\mathfrak{C}\Delta^n$ is now given as follows. We may describe the mapping complexes $\mathfrak{C}\Delta^n(i,j)$ by the nerve of the partially ordered set P_{ij} of subsets of $\{0,\dots,n\}$ bounded by and containing i and j, with ordering given by the superset relation $(\supset)$. The composition maps

$$\mathfrak{C}\Delta^n(j,k) \times \mathfrak{C}\Delta^n(i,j) \to \mathfrak{C}\Delta^n(i,k)$$

may now be identified with the nerve of the map of posets $P_{jk} \times P_{ij} \to P_{ik}$ given by taking unions. Finally, we emphasize that we will not use the fact that the pair $(\mathfrak{C}, N)$ forms a Quillen equivalence when **sSet** is endowed with the Joyal model structure. Instead, we will use the fact that $\mathfrak{C}N\mathcal{C}$ and $FU_\bullet\mathcal{C}$ are naturally isomorphic for every 1-category $\mathcal{C}$. This is shown in [Rie14, Theorem 16.4.7] using the theory of *necklaces*. The counit $\mathfrak{C}N\mathcal{C} \to \mathcal{C}$ may then be identified with the map $FU_\bullet\mathcal{C} \to \mathcal{C}$, which is readily seen to be a DK-equivalence (see also Appendix A).

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CHAPTER 1

The coCartesian model structure

The purpose of this chapter is to review the coCartesian model structure, and prove some new results of this model structure which will be of use later in the thesis. The underlying ∞-category of this model structure is the appropriate ∞-category of coCartesian fibrations. This fact will be of use later, since this allows us to approach the straightening-unstraightening problem using model categorical techniques.

1.1 The coCartesian model structure

Recall that the coCartesian model category is defined on over categories of the category of marked simplicial sets. Marked simplicial sets are simplicial sets with marked edges. These markings will be used to track coCartesian edges. More precisely, marked simplicial sets are defined as follows:

Definition 1.1.1. A marked simplicial set is pair (X, Σ) consisting of a simplicial set X and a collection Σ of edges of X, containing the degenerate edges. These marked simplicial sets fit into a category $\mathbf{sSet}^+$, where a map of marked simplicial sets $(X, \Sigma_X) \to (Y, \Sigma_Y)$ is given by map of simplicial sets $X \to Y$ which carries the marked edges Σ_X of X into the marked edges Σ_Y of Y.

This category defined above is sufficiently nice: it admits all small colimits, limits and is (locally) cartesian closed as well. A convenient way to establish these facts, is to show that $\mathbf{sSet}^+$ is a reflective subcategory of a category of presheaves. We may adjoin an element e to the simplex category Δ together with two unique maps $[1] \to e$ and $e \to [0]$ which factor the unique map $[1] \to [0]$. This yields a category Δ^+. Then it is readily verified that the obvious inclusion functor $i : \mathbf{sSet}^+ \to \mathbf{Set}^{(\Delta^+)^{\mathrm{op}}}$ given by the formulas $i(X, \Sigma)_n = X_n$ and $i(X, \Sigma)_e = \Sigma$, admits a left adjoint.

There are of course canonical ways of adding markings to a simplicial set X. Namely, we can only mark the edges of X which are degenerate – we will denote this marked simplicial set by $X^\flat$ – or we can mark all edges of X, yielding a marked simplicial set $X^\sharp$. These constructions are clearly functorial, and they fit in adjunctions. Namely, given a marked simplicial set X, we can forget its markings and obtain its underlying simplicial set $X_\flat$. Furthermore, we can consider the *marked core* of X. This is the simplicial subset $X_\sharp$ of X consisting of the simplices of X whose edges are marked. It is readily verified that we have two adjunctions

$$\mathbf{sSet} \underset{(-)_\flat}{\overset{(-)^\flat}{\rightleftarrows}} \mathbf{sSet}^+ \quad (-)^\flat \dashv (-)_\flat, \qquad \mathbf{sSet} \underset{(-)_\sharp}{\overset{(-)^\sharp}{\rightleftarrows}} \mathbf{sSet}^+ \quad (-)_\sharp \dashv (-)^\sharp.$$

Definition 1.1.2. Let S be a simplicial set. The category $(\mathbf{sSet}^+)_{/S}$ of marked simplicial sets over S is defined to be the over category $(\mathbf{sSet}^+)_{/S^\sharp}$.

We consider the category $(\mathbf{sSet}^+)_{/S}$ to be enriched over simplicial sets as follows. For marked simplicial sets X, Y over S, the hom-objects are denoted by $\mathrm{Map}_S^\sharp(X,Y)$ and given by

$$\mathrm{Map}_S^\sharp(X,Y) := (Y^X \times_{(S^\sharp)^X} \Delta^0)_\sharp.$$

Recall that $(\mathbf{sSet}^+)_{/S}$ inherits a tensoring over simplicial sets as follows. Given a marked simplicial set X over S, and a simplicial set A, the underlying marked simplicial set of $A \otimes X = A^\sharp \times X$ is the cartesian product $A^\sharp \times X$. The projection map is the composite $A \times X \to X \to S$. It is readily verified that this indeed defines a tensoring over simplicial sets, as there are isomorphisms

$$\mathbf{sSet}(A, \mathrm{Map}_S^\sharp(X,Y)) \cong (\mathbf{sSet}^+)_{/S}(A^\sharp \times X, Y)$$

natural in A, X and Y.

Theorem 1.1.3 (Corollary 3.1.4.4 of [Lur09])**.** *There exists a combinatorial simplicial model structure on* $(\mathbf{sSet}^+)_{/S}$ *such that:*

(i) the cofibrations are the maps whose underlying map of simplicial sets are monomorphisms,
(ii) the fibrant objects are given by marked simplicial sets $X^\natural$, where $X^\natural$ denotes the marked simplicial set associated to a coCartesian fibration $X \to S$ whose marked edges are its coCartesian edges.

The weak equivalences in this model category are called coCartesian equivalences.

The following should be clear:

Proposition 1.1.4. *The coCartesian model structure has a set of generating cofibrations given by the inclusions* $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$, $(\Delta^1)^\flat \to (\Delta^1)^\sharp$.

Similarly to the role that left anodynes play in the covariant model structure, there is an important class of trivial cofibrations in the coCartesian model structure:

Definition 1.1.5. The class of marked anodyne maps in $(\mathbf{sSet}^+)_{/S}$ is the smallest weakly saturated class of maps containing:

(i) the inner horn inclusions $(\Lambda^n_i)^\flat \to (\Delta^n)^\flat$, $0 < i < n$,
(ii) the inclusion

$$(\Lambda^n_0)^\flat \bigcup_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \to (\Delta^n)^\flat \bigcup_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp$$

for $n > 1$ and the inclusion $(\Lambda^1_0)^\flat \to (\Delta^1)^\sharp$,
(iii) the inclusion

$$(\Lambda^2_1)^\sharp \bigcup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp,$$

(iv) the map $K^\flat \to K^\sharp$ for every Kan complex K.

Proposition 1.1.6 ([Lur09, Section 3.1])**.** *The marked anodynes have the following properties:*

(i) every marked anodyne is a trivial cofibration,
(ii) suppose that $X \to Y$ is a marked anodyne map, then for any cofibration $A \to B$, the induced map

$$A \times Y \bigcup_{A \times X} B \times X \to B \times Y$$

is again marked anodyne.

Proposition 1.1.7. *A map $p : X \to Y^\natural$ of marked simplicial sets over S is a fibration if and only if p has the right lifting property with respect to the marked anodyne maps.*

Proof. Let us show the non-trivial assertion. Suppose that p has the right lifting property w.r.t. marked anodyne maps. In view of [Lur09, Proposition 3.1.1.6], we deduce that X is fibrant. Hence we may replace X by a marked simplicial set $X^\natural$ associated to a coCartesian fibration $X \to S$. We must demonstrate that for a trivial cofibration $i : A \to B$ in $(\mathbf{sSet}^+)_{/S}$, the following square

$$\begin{array}{ccc} A & \xrightarrow{f} & X^\natural \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y^\natural \end{array}$$

admits a filler. Since $X^\natural$ is fibrant, there exists a an extension $h : B \to X^\natural$ of f. Note that the map $\mathrm{Map}^\sharp_S(B, Y^\natural) \to \mathrm{Map}^\sharp_S(A, Y^\natural)$ induced by i, is a trivial Kan fibration since the coCartesian model structure is simplicial. Consequently, the fiber

$$\mathrm{Map}^\sharp_S(B, Y^\natural) \times_{\mathrm{Map}^\sharp_S(A, Y^\natural)} \{pf\}$$

is contractible. Hence, there exists an edge $H : \Delta^1 \to \mathrm{Map}^\sharp(B, Y^\natural)$ connecting ph and g such that $H|_{(\Delta^1)^\sharp \times A} = pf\,\mathrm{pr}_A$. By assumption, the following diagram admits a filler

$$\begin{array}{ccc} (\Delta^1)^\sharp \times A \cup_{\{0\}\times A} \{0\} \times B & \xrightarrow{f\,\mathrm{pr}_A \cup h} & X^\natural \\ \downarrow & & \downarrow p \\ (\Delta^1)^\sharp \times B & \xrightarrow{H} & Y^\natural. \end{array}$$

Restricting the filler to $B \times \{1\}$, we obtain a solution to the original lifting property. □

Corollary 1.1.8. *Every trivial cofibration in* $(\mathbf{sSet}^+)_{/S}$ *with fibrant codomain is a marked anodyne.*

Proof. This follows from a standard argument. Suppose that $i : X \to Y^\natural$ trivial cofibration in $(\mathbf{sSet}^+)_{/S}$. Then we may factor i as a marked anodyne $X \to Z$ followed by a map $Z \to Y^\natural$ which has the right lifting property with marked anodynes. Then Z is again fibrant and the map $Z \to Y^\natural$ is a trivial fibration. Then a lifting argument shows that the map $X \to Y^\natural$ is a retract of the marked anodyne $X \to Z$. Hence i is marked anodyne as well. □

Corollary 1.1.9. *Let* $\mathcal{M}$ *be a model category, and suppose that* $F : (\mathbf{sSet}^+)_{/S} \to \mathcal{M}$ *is a left adjoint. Then* F *is left Quillen precisely when* F *carries cofibrations to cofibrations and marked anodyne maps to trivial cofibrations.*

Proof. Let us show the non-trivial implication. In view of Lemma B.0.1, it suffices to show that the right adjoint of F preserves fibrations between fibrant objects. Recall that the fibrant objects of $(\mathbf{sSet}^+)_{/S}$ are the coCartesian fibrations, which may be characterized as the objects having the right lifting property with respect to the marked anodynes. Consequently, an adjointness argument and an application of Proposition 1.1.7 show that the right adjoint of F preserves fibrant objects. Similarly, one shows that the right adjoint of F carries fibrations between fibrant objects to fibrations in $(\mathbf{sSet}^+)_{/S}$. □

Observe that Proposition 1.1.7 yields a characterization of fibrations between fibrant objects in the coCartesian model structure. There is a convenient characterization of coCartesian equivalences between fibrant objects as well:

Proposition 1.1.10 (Proposition 3.1.3.5 of [Lur09])**.** *Let* $p : X^\natural \to Y^\natural$ *be a map of marked simplicial sets over* S*. Then* p *is a coCartesian equivalence if and only if the map* p *descends to a coCartesian equivalence* $X^\natural_s \to Y^\natural_s$ *(equivalently, the underlying map of* ∞*-categories is a categorical equivalence) on fibers, for every vertex* s *of* S*.*

The generators of the marked anodynes may be replaced by other generators. Replacements can be found in Section 3.1.1 of HTT. We will highlight a few replacements, which will be of use later.

Proposition 1.1.11 ([Lur09, Corollary 3.1.1.8])**.** *Let* A *be the pushout of the cospan*

$$\Delta^1 \xleftarrow{s_1 \sqcup s_0} \Delta^2 \sqcup \Delta^2 \xrightarrow{d_2 \sqcup d_1} \Delta^3.$$

The maps (iv) in Definition 1.1.5 may be replaced by the inclusion (iv') $A^\flat \to (A,\Sigma)$ where Σ is the set of degenerate edges of A alongside with the image of the edge $\Delta^{\{0,1\}} \subset \Delta^3$ in A.

There are also very useful replacements for the generators (ii) and (iii) of Definition 1.1.5, which were introduced in the PhD thesis of Nguyen [Ngu18].

Definition 1.1.12. The class of cellular marked anodyne maps is the smallest weakly saturated class of maps containing the maps

$$(\Delta^1)^\sharp \times A \bigcup_{\{0\}\times A} \{0\} \times B \to (\Delta^1)^\sharp \times B,$$

induced by cofibrations $A \to B$ of marked simplicial sets.

Proposition 1.1.13. *The following assertions are true:*

(i) any cellular marked anodyne map is marked anodyne,
(ii) any $\sharp$-marked left anodyne map is cellular marked anodyne,
(iii) one may replace generators (ii) and (iii) of Definition 1.1.5 by (generators of) cellular marked anodynes.

Proof. The first assertion follows from the fact that the inclusion $\{0\} \to (\Delta^1)^\sharp$ is marked anodyne. The second assertion follows from the fact that the left anodynes are generated by inclusions of the form $\Delta^1\times\partial\Delta^n\cup_{\{0\}\times\partial\Delta^n}\{0\}\times\Delta^n \to \Delta^1\times\Delta^n$ (see [Lur09, Proposition 2.1.2.6]). The final assertion remains to be shown. Denote the smallest weakly saturated class of maps containing the maps (i), (iv) of Definition 1.1.5 by W. In view of assertion (i), W is contained in the marked anodynes. Conversely, generators of type Definition 1.1.5(ii) are contained in W on account of [Lur09, Proposition 3.1.1.5]. Hence it remains to show that the map $(\Lambda^2_1)^\sharp \bigcup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ is contained in W. This follows directly from the observation that Corollary 1.1.8 continues to hold for maps in W instead of marked anodynes. □

Corollary 1.1.14. *The adjunction*

$$(-)^\sharp : \mathbf{sSet}_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/S} : (-)_\sharp,$$

is a Quillen adjunction when the category on the left is endowed with the covariant model structure.

Proposition 1.1.15. *Let $i : X \to Y$ be a marked deformation retract. That is, i admits a retraction $r : Y \to X$ in $\mathbf{sSet}^+$ such that that there there there exists a homotopy $H : (\Delta^1)^\sharp \times Y \to Y$ of marked simplicial sets, relative to X, such that $H|_{\{0\}\times Y} = ir$ and $H_{\{1\}\times Y} = \mathrm{id}_Y$. Then i is cellular marked anodyne.*

Proof. This follows from a standard argument: the diagram

$$\begin{array}{ccccc} X & \longrightarrow & (\Delta^1)^\sharp \times X \cup_{\{0\}\times X} \{0\} \times Y & \xrightarrow{\mathrm{pr}_X \cup r} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & (\Delta^1)^\sharp \times Y & \xrightarrow{H} & Y \end{array}$$

witnesses $X \to Y$ as a retract of a cellular marked anodyne map. □

Proposition 1.1.16 (Theorem 3.2.4 of [Ngu18])**.** *Consider the following pullback square of marked simplicial sets*

$$\begin{array}{ccc} X \times_B A & \xrightarrow{j} & X \\ \downarrow & & \downarrow p \\ A & \xrightarrow{i} & B. \end{array}$$

Suppose that the map $p^{\mathrm{op}} : X^{\mathrm{op}} \to B^{\mathrm{op}}$ has the right lifting property with respect to cellular marked anodynes. Then j is marked anodyne if i is cellular marked anodyne.

Proof. We follow the proof of Nguyen. Let W be the class of monomorphisms $i : A \to B$ such that for any map $X^{\mathrm{op}} \to B^{\mathrm{op}}$ which has the right lifting property w.r.t. cellular marked anodynes, the statement holds. It is clear that W is weakly saturated. Furthermore, the class W has the right cancellation property. To wit, suppose that $i : A \to B$ and $j : B \to C$ have the property that ji and i are in W. Let $X^{\mathrm{op}} \to C^{\mathrm{op}}$ be a fibration. We obtain the following commutative diagram

$$\begin{array}{ccccc} X \times_C A & \longrightarrow & X \times_C B & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{i} & B & \xrightarrow{j} & C \end{array}$$

On account of the pasting law, the left square is a pullback square. Hence the top left horizontal arrow is marked anodyne. The composite of the two arrows in the top row is also marked anodyne by assumption. Consequently, since the marked anodynes have the right cancellation property[1], the top right horizontal arrow is marked anodyne. Hence j is also in W.

Thus it suffices to check the statement for generators of cellular marked anodynes. Moreover, since W is right cancellative, we may reduce checking the statement for the case that $A = \{0\} \times C$, $B = (\Delta^1)^\sharp \times C$ and the map i is the obvious inclusion. Here C is any marked simplicial set. Note that i admits a retraction $r : B \to A$ which comes with a homotopy $H : (\Delta^1)^\sharp \times B \to B$ such that $H|_{\{0\}\times B} = ir$ and $H|_{\{1\}\times B} = \mathrm{id}_B$. By assumption, the following square admits a filler

$$\begin{array}{ccc} (\Delta^1)^\sharp \times (X \times_B A) \bigcup_{\{1\}\times(X\times_B A)} \{1\} \times X & \xrightarrow{j\,\mathrm{pr}_{X\times_B A} \cup \mathrm{id}_X} & X \\ \downarrow & & \downarrow p \\ (\Delta^1)^\sharp \times X & \xrightarrow{H(\mathrm{id}_{(\Delta^1)^\sharp} \times p)} & B \end{array}$$

[1]This follows from a standard argument. There is a more general notion of a coCartesian model structure defined on arbitrary marked simplicial sets (see for instance [Ngu18, Section 3.1] or [Lur17, Appendix B]). Suppose that a composition of maps $X \to Y \to Z$ is marked anodyne, and the map $X \to Y$ is marked anodyne. Then the map $Y \to Z$ must be a trivial cofibration in $(\mathbf{sSet})_{/Z}$ as well, since trivial cofibrations have the right cancellation property. Since the codomain Z is fibrant in $(\mathbf{sSet})_{/Z}$, the desired result follows from a more general form of Corollary 1.1.8.

Thus the desired result follows from Proposition 1.1.15. □

The dual form of this proposition has a very useful implication:

Corollary 1.1.17. *Suppose that $X \to T$ is a coCartesian fibration. Consider the following pullback square*

$$\begin{array}{ccc} X^{\natural} \times_{T^{\sharp}} S^{\sharp} & \xrightarrow{j} & X^{\natural} \\ \downarrow & & \downarrow \\ S^{\sharp} & \xrightarrow{i^{\sharp}} & T^{\sharp} \end{array}$$

Then j is a trivial cofibration in $\mathbf{sSet}^+ = (\mathbf{sSet}^+)_{/\Delta^0}$ when i is right anodyne.

Proof. Under the assumption that i is right anodyne, the opposite j^{op} of the map j is marked anodyne on account of Proposition 1.1.16. Thus the result follows from the fact that the Cartesian model structure and the coCartesian model structure on $\mathbf{sSet}^+$ coincide (see [Lur09, Remark 3.1.4.6]). □

1.2 Minimal coCartesian fibrations

In this section we develop an analogue of minimal ∞-categories (see [Lur09, Section 2.3.3]) for coCartesian fibrations. The work in this section generalizes the known results for minimal ∞-categories proven by Lurie.

Suppose that $p : X \to S$ is a coCartesian fibration, and we are given the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{q} & S, \end{array}$$

where the left arrow is a cofibration. Then we call two fillers $f, g : B \rightrightarrows X$ homotopic relative A over S if there is exists an edge connecting f and g in the fiber of the map

$$\mathrm{Map}_S^{\sharp}(B^{\flat}, X^{\natural}) \to \mathrm{Map}_S^{\sharp}(A^{\flat}, X^{\natural})$$

above h. Notice that this fiber is a Kan complex. By adjunction, this precisely corresponds to the data of a homotopy $H : (\Delta^1)^{\sharp} \times B^{\flat} \to X^{\natural}$ such that $H|_{(\Delta^1)^{\sharp} \times A^{\flat}} = h\mathrm{pr}_{A^{\flat}}$. Equivalently, this is a homotopy

$$\begin{array}{ccc} \Delta^1 \times B & \xrightarrow{H} & X \\ & {\scriptstyle q\mathrm{pr}_B} \searrow \quad \swarrow {\scriptstyle p} & \\ & S & \end{array}$$

such that $H|_{\Delta^1 \times A} = h\mathrm{pr}_A$ and $H|_{\Delta^1 \times \{b\}}$ is a p-coCartesian edge for all vertices b of B. Minimal coCartesian fibrations will have the property that any two maps which are homotopic relative to A, coincide. We start with the following observation:

Proposition 1.2.1. *Let $X \to S$ be a coCartesian fibration. The following assertions are equivalent:*

(i) for any cofibration $A \to B$ of marked simplicial sets, every fiber F of the induced Kan fibration $\mathrm{Map}^\sharp_S(B, X^\natural) \to \mathrm{Map}^\sharp_S(A, X^\natural)$ has the property $(*)$ that the map $F_0 \to \pi_0 F$ is a bijection,
(ii) any two n-simplices which are homotopic relative to $\partial\Delta^n$ agree,
(iii) for any cofibration $A \to B$ of simplicial sets, any two maps $f, g : B \rightrightarrows X$ which are homotopic relative to A agree.

Proof. We will show the only non-trivial implication (ii) $\Rightarrow$ (i). Note that the fibers of the Kan fibration $\mathrm{Map}^\sharp_S((\Delta^1)^\sharp, X^\natural) \to \mathrm{Map}^\sharp_S((\Delta^1)^\flat, X^\natural)$ have property $(*)$. Therefore, it suffices to show that the class W of monomorphisms $f : A \to B$ in $\mathbf{sSet}^+$ which have the property that the fibers of $\mathrm{Map}^\sharp_S(f, X^\natural)$ have property $(*)$, are weakly saturated. This fact is readily established. □

Definition 1.2.2. A coCartesian fibration $X \to S$ which meets any of the equivalent conditions of Proposition 1.2.1 is called a minimal coCartesian fibration.

Remark 1.2.3. The above definition recovers the notion of minimal ∞-categories in the case that $S = \Delta^0$. More precisely, suppose that $\mathcal{C}$ is an ∞-category, then the map $\mathcal{C} \to \Delta^0$ is a minimal coCartesian fibration if and only if $\mathcal{C}$ is a minimal ∞-category.

The known results [Lur09, Section 2.3.3] and [Joy, Chapter 9] for minimal ∞-categories and minimal left fibrations generalize to minimal coCartesian fibrations. We have the following result:

Proposition 1.2.4. *Let $f : X^\natural \to Y^\natural$ be a coCartesian equivalence of coCartesian fibrations over S. Then the following statements are true:*

(i) if $X^\natural$ is minimal then f is a trivial cofibration,
(ii) if $Y^\natural$ is minimal then f is a trivial fibration,
(iii) if $X^\natural$ and $Y^\natural$ are both minimal then f is an isomorphism.

Proof. Observe that (iii) follows from (i) and (ii). Consequently, it suffices to show (i) and (ii).

Let us commence by proving (i). We have to show that the underlying map f of simplicial sets is monic. Since f is a homotopy equivalence, there exists a map $g : Y^\natural \to X^\natural$ and a homotopy $H : (\Delta^1)^\sharp \times X^\natural \to X^\natural$ such that $H|_{\{0\}\times X^\natural} = gf$ and $H|_{\{1\}\times X^\natural} = \mathrm{id}_{X^\natural}$. We must show that for any two n-simplices $\sigma, \tau : \Delta^n \rightrightarrows X$ we have $f\sigma = f\tau$ precisely when $\sigma = \tau$. We proceed by induction on the dimension n. There is nothing to be shown for the case $n = -1$. Let $n \geq 0$ and suppose that the claim holds for $n-1$. Since $f\sigma|_{\partial\Delta^n} = f\tau|_{\partial\Delta^n}$, the induction hypothesis asserts that $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$. We may view σ, τ as maps $(\Delta^n)^\flat \to X^\natural$. The restrictions $H(\mathrm{id}_{(\Delta^1)^\sharp}\times\sigma)$ and $H(\mathrm{id}_{(\Delta^1)^\sharp}\times\tau)$ of H glue to a map $(\Lambda^2_0)^\sharp\times(\Delta^n)^\flat \to X^\natural$ since $gf\sigma = gf\tau$. Furthermore, we have a map $(\Delta^2)^\sharp \times (\partial\Delta^n)^\flat \to X^\natural$ given by

$H(s_1^\sharp \times \sigma|_{(\partial\Delta^n)^\flat}) = H(s_1^\sharp \times \tau|_{(\partial\Delta^n)^\flat})$. The constructed maps make the following diagram commute:

$$\begin{array}{ccc} (\Delta^2)^\sharp \times (\partial\Delta^n)^\flat \bigcup_{(\Lambda^2_0)^\sharp \times (\partial\Delta^n)^\flat} (\Lambda^2_0)^\sharp \times (\Delta^n)^\flat & \longrightarrow & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^2)^\sharp \times (\Delta^n)^\flat & \longrightarrow & S^\sharp. \end{array}$$

Note that the left vertical map is a trivial cofibration in $(\mathbf{sSet}^+)_{/S}$, hence this diagram admits a filler $H' : (\Delta^2)^\sharp \times (\Delta^n)^\flat \to X^\natural$. The restriction $(\Delta^{\{1,2\}})^\sharp \times (\Delta^n)^\flat \to X^\natural$ shows that σ and τ are homotopic relative to $\partial\Delta^n$. Thus it follows from minimality of $X^\natural$ that $\sigma = \tau$.

Finally, we prove statement (ii). We need to solve the following lifting problem:

$$\begin{array}{ccc} A & \xrightarrow{h} & X^\natural \\ \downarrow & & \downarrow f \\ B & \xrightarrow{q} & Y^\natural. \end{array}$$

In view of Lemma B.0.3, the induced map

$$\mathrm{Map}^\sharp_S(B, X^\natural) \times_{\mathrm{Map}^\sharp_S(A, X^\natural)} \{h\} \to \mathrm{Map}^\sharp_S(B, Y^\natural) \times_{\mathrm{Map}^\sharp_S(A, Y^\natural)} \{fh\}$$

is surjective on π_0. Thus the path component of q gets hit, and this path component only consists of q on account of the minimality of $Y^\natural$. □

So far, we have only seen the fundamental properties of minimal coCartesian fibrations. But like any notion of minimal objects in a model category, we expect that any coCartesian admits a *minimal model*. Furthermore, we would like that these minimal models are unique.

Definition 1.2.5. A minimal model for a coCartesian fibration $X^\natural$ is a trivial cofibration $M^\natural \to X^\natural$ where the domain $M^\natural$ is a minimal coCartesian fibration.

We conclude this section by presenting two propositions which take care of the unicity and existence of minimal models.

Proposition 1.2.6. *Let $i : M^\natural \to X^\natural$ be a minimal model for $X^\natural$. Then the following statements are true:*

(i) the map i admits a retraction $r : X^\natural \to M^\natural$, which is a trivial fibration,
(ii) for any other minimal model $j : N^\natural \to X^\natural$, there exists an isomorphism $N^\natural \xrightarrow{\cong} M^\natural$.

Proof. Since i is a trivial cofibration, it admits a retraction r, which is a coCartesian equivalence. In view of Proposition 1.2.4, this retraction is a trivial fibration. To prove (ii), we compose j with r to obtain a weak equivalence $f : N^\natural \to M^\natural$. It follows from Proposition 1.2.4 that f is an isomorphism. □

Proposition 1.2.7. *Every coCartesian fibration $X^\natural$ admits a minimal model.*

Proof. We proceed similarly as Lurie's proof of [Lur09, Proposition 2.3.3.8]. Let $\simeq_n$ be the equivalence relation on n-simplices of X such that $\sigma \simeq_n \tau$ if and only if σ, τ are homotopic relative to $\partial\Delta^n$. For any $n \in \mathbb{N}$, choose a set of representatives $R_n \subset X_n$ of $\simeq_n$ such that the degenerate n-simplices are contained in R_n (this is possible since every equivalence class of $\simeq_n$ contains at most 1 degenerate simplex). Let X' be the largest simplicial subset of X such that $\sigma \in M_n$ precisely when $\sigma(\partial\Delta^n) \subset M_{n-1}$ and $\sigma \in R_n$. The marked edges Σ of X' are such that $e \in M_1$ is marked if and only if $e \in X_1$ is marked (i.e. coCartesian). To show that (M, Σ) is a minimal model for $X^\natural$, it is sufficient to show that (M, Σ) is a deformation retract of $X^\natural$. Indeed, then (M, Σ) is fibrant in $(\mathbf{sSet}^+)_{/S}$ and it is minimal by construction.

Let $Y \subset X^\natural$ a marked simplicial subset. For the sake of this proof, we call a homotopy $H : (\Delta^1)^\sharp \times Y \to Y$ relative to $(M, \Sigma) \cap Y$ a Y-homotopy when H is satisfies $H|_{\{0\}\times Y} = \mathrm{id}_Y$ and $H(\{1\}\times Y) \subset (M, \Sigma)$. We must cook up a $X^\natural$-homotopy. Proceeding by skeletal induction, we deduce that it suffices to show the following. Suppose that $Y \subset X^\natural$ and a Y-homotopy H are given. Then there exists an extension of H to an Y'-homotopy, where Y' is either the pushout $Y' = Y \bigcup_{(\partial\Delta^n)^\flat} (\Delta^n)^\flat$ with $\sigma : (\Delta^n)^\flat \to X^\natural$ a non-degenerate n-simplex, or $Y' = Y \bigcup_{(\partial\Delta^1)^\flat} (\Delta^1)^\sharp$ where $\sigma : (\Delta^1)^\sharp \to X^\natural$ is a non-degenerate marked edge of $X^\natural$.

Let us first show that this can be done when Y' is a pushout of the first form. If σ factors through (M, Σ), then the homotopy trivially extends to a Y'-homotopy. Suppose that σ does not factor through (M, Σ). Consider the following commutative diagram

$$\begin{array}{ccc} \{0\} \times (\Delta^n)^\flat \bigcup_{\{0\}\times(\partial\Delta^n)^\flat} (\Delta^1)^\sharp \times (\partial\Delta^n)^\flat & \xrightarrow{\sigma \cup H(\mathrm{id}_{(\Delta^1)^\sharp} \times \sigma|_{(\partial\Delta^n)^\flat})} & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^1)^\sharp \times (\Delta^n)^\flat & \longrightarrow & S^\sharp. \end{array}$$

This filler admits a filler h providing a homotopy between σ and a n-simplex τ. Note that $\tau|_{(\partial\Delta^n)^\flat}$ factors through $(M, \Sigma) \subset X^\natural$. Take a homotopy $h' : (\Delta^1)^\sharp \times (\Delta^n)^\flat \to X^\natural$ exhibiting $\tau \simeq_n \tau'$ for a representative $\tau' \in R_n$. Then it follows that $\tau' \in M_n$. Consider now the diagram

$$\begin{array}{ccc} (\Lambda^2_1)^\sharp \times (\Delta^n)^\flat \bigcup_{(\Lambda^2_1)^\sharp\times(\partial\Delta^n)^\flat} (\Delta^2)^\sharp \times (\partial\Delta^n)^\flat & \xrightarrow{h \cup h' \cup H(s_1^\sharp \times \sigma|_{(\partial\Delta^n)^\flat})} & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^2)^\sharp \times (\Delta^n)^\flat & \longrightarrow & S^\sharp. \end{array}$$

Take a filler h'' of this diagram. We extend H to Y' by setting $H|_{(\Delta^1)^\sharp\times(\Delta^n)^\flat} := h''|_{(\Delta^{\{0,2\}})^\sharp\times(\Delta^n)^\sharp)}$. It is clear that this extension is again a Y'-homotopy.

Finally, to handle the case that $Y' = Y \bigcup_{(\partial\Delta^1)^\flat} (\Delta^1)^\sharp$, we can perform the same construction by taking $n = 1$ and replacing $(\Delta^n)^\flat$ by $(\Delta^1)^\sharp$ in the above. $\square$

1.3 Change of base

Let $f : S \to T$ be a map of simplicial sets. Then this map induces a base change adjunction

$$f_! : (\mathbf{sSet}^+)_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/T} : f^*.$$

Here the functor $f_!$ sends a marked simplicial set $X \to S^\sharp$ to $X \to S^\sharp \to T^\sharp$. The right adjoint f^* of $f_!$ is given by pullback along f.

Proposition 1.3.1. *The pair $(f_!, f^*)$ is a simplicial and Quillen adjunction.*

Proof. It is clear that $f_!$ preserves cofibrations and marked anodyne maps. Hence $f_!$ is left Quillen on account of Corollary 1.1.9. It is also clear that for a simplicial set A and a marked simplicial set X over S, we have a $f_!(A^\sharp \times X^\natural) = A^\sharp \times f_! X^\natural$. Hence $(f_!, f^*)$ is simplicial. □

In favourable circumstances, the adjunction is even a Quillen equivalence. We will come back to this later. The following observation is useful.

Proposition 1.3.2. *The right Quillen functor f^* preserves minimal coCartesian fibrations. Moreover, if $M^\natural \to X^\natural$ is a minimal model for a coCartesian fibration $X^\natural$, then $f^*M^\natural \to f^*X^\natural$ is a minimal model for $f^*X^\natural$.*

Proof. Let us start by demonstrating the first assertion. Suppose that $M^\natural$ is a minimal coCartesian fibration. For any cofibration $A \to B$ of marked simplicial sets over S, we have a commutative square

$$\begin{array}{ccc} \mathrm{Map}^\sharp_S(f_!B, M^\natural) & \xrightarrow{\cong} & \mathrm{Map}^\sharp_S(B, f^*M^\natural) \\ \downarrow & & \downarrow \\ \mathrm{Map}^\sharp_S(f_!A, M^\natural) & \xrightarrow{\cong} & \mathrm{Map}^\sharp_S(A, f^*M^\natural) \end{array}$$

since the adjunction $(f_!, f^*)$ is simplicial. The fibers of the Kan fibration on the left have property $(*)$ of Proposition 1.2.1, hence fibers of the right Kan fibration have property $(*)$ as well. Thus $f^*M^\natural$ is minimal as well.

Let $i : M^\natural \to X^\natural$ be a minimal model for $X^\natural$. Then i admits a retraction $r : X^\natural \to M^\natural$ which is a trivial fibration on account of Proposition 1.2.6. Then f^*r is again a retraction of f^*i and again a trivial fibration since f^* is right Quillen. Hence f^*i is a trivial cofibration. Note that $f^*M^\natural$ is a minimal coCartesian in view of the first part of the proposition. Hence f^*i is a minimal model for $f^*X^\natural$ as desired. □

Using this proposition, we can readily prove the homotopy descent property of coCartesian fibrations.

Proposition 1.3.3. *Suppose that we are given the following diagram*

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xleftarrow{f_1} & S_0 & \xrightarrow{f_2} & S_2, \end{array}$$

such that the vertical arrows are coCartesian fibrations and at least one of the f_i's is monic. Furthermore, assume that the maps $X_0^\natural \to f_i^* X_i^\natural$ are coCartesian equivalences. Then there exists a coCartesian fibration

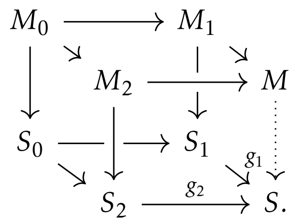
$$X \to S_1 \bigcup_{S_0} S_2$$

such that the pullback of this coCartesian fibration to S_i is coCartesian equivalent to $X_i^\natural$ for $i = 1, 2$.

Proof. Let us commence by picking minimal models $j_i : M_i^\natural \to X_i^\natural$ for each $X_i^\natural$ with retracts $r_i : X_i^\natural \to M_i^\natural$. These retracts are automatically trivial fibrations. Observe that the compositions

$$M_0^\natural \xrightarrow{i_0} X_0^\natural \to f_i^* X_i^\natural \xrightarrow{f_i^* r_i} f_i^* M_i^\natural$$

are coCartesian equivalences between minimal coCartesian fibrations on account of Proposition 1.3.2. Consequently, the composite is an isomorphism. Thus M_0 is the strict pullback of M_i's along f_i. Consider the following commutative cube,



Here the top and bottom faces are pushouts. The faces containing the arrow $M_0 \to S_0$ are pullbacks by the above. Hence the squares containing the induced arrow $M \to S$ are pullback squares as well. Hence $M \to S$ pulls back to $M_1 \to S_1$ and $M_2 \to S_2$. Since these maps are coCartesian, $M \to S$ is coCartesian as well. We have coCartesian equivalences

$$X_i^\natural \xrightarrow{j_i} M_i^\natural \cong g_i^* M^\natural$$

hence $M^\natural$ is the desired coCartesian fibration. □

We will use this homotopy descent property to partly prove the aforementioned *homotopy invariance of the coCartesian model structure*:

Theorem 1.3.4. *Suppose that f is a categorical equivalence, then the base change adjunction*

$$f_! : (\mathbf{sSet}^+)_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/T} : f^*.$$

is a Quillen equivalence.

We will see that the proof of this theorem can be reduced to checking that Theorem 1.3.4 holds for inner horn inclusions. We postpone the proof of the latter to the next chapter because with the machinery developed there, we will be able to give an efficient demonstration of this fact.

Lemma 1.3.5. *If Theorem 1.3.4 holds for the inner horn inclusions, then it holds for all inner anodynes.*

Proof. Let W be the class of monomorphisms such that Theorem 1.3.4 holds. We will demonstrate that this class is weakly saturated. It then follows W contains in the inner anodynes precisely when W contains the inner horn inclusions, thereby proving the statement. Observe that W is certainly closed under retracts, since Quillen equivalences are closed under retracts. Thus it remains to show that W is closed under pushouts and transfinite compositions.

Suppose that we have the following pushout square

$$\begin{array}{ccc} S_0 & \xrightarrow{i} & S_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S_1 & \xrightarrow{j} & S \end{array}$$

and suppose that i is contained in W. We need to show that j lies in W as well. Let X be coCartesian fibration over S_1. Then $f^*X^\natural$ is a coCartesian fibration over S_0. Since $\mathbf{R}i^*$ is assumed to be essentially surjective, there exists a coCartesian fibration Y over S_2 and a coCartesian equivalence $f^*X^\natural \to i^*Y^\natural$. It now follows from Proposition 1.3.3 that there exists a coCartesian fibration Z on the pushout S which comes with a coCartesian equivalence $X^\natural \to j^*Z^\natural$. This entails that $\mathbf{R}j^*$ is essentially surjective as well. Next, we demonstrate that $\mathbf{R}j^*$ is fully faithful. I.e. for X a coCartesian fibration on S, we should show that the counit $j_!j^*X^\natural \to X^\natural$ is a coCartesian equivalence. But this map is a pushout of the map $g_!i_!i^*g^*X^\natural \to g_!g^*X^\natural$ (this is the strict Mather cube lemma) and the latter is trivial cofibration since $\mathbf{R}i^*$ is fully faithful and $g_!$ preserves trivial cofibrations. Hence the map $j_!j^*X^\natural \to X^\natural$ is a trivial cofibration, as desired.

Finally, we prove that the class W is also closed under transfinite compositions. It is clear that W is closed under finite compositions, since the composite of two Quillen equivalences is again a Quillen equivalence. Consequently, it suffices to show the following: let α be a limit ordinal and suppose that we have compatible maps $i_\beta : S_0 \to S_\beta$ in W for every $\beta < \alpha$. Then the map

$$i : S_0 \to \varinjlim_{\beta<\alpha} S_\beta =: S_\alpha$$

lies again W. It is clear that the right derived functor $\mathbf{R}i^*$ is again essentially surjective. Denote the inclusion $S_\beta \to S_\alpha$ by j_β. Then we have natural isomorphisms

$$\begin{aligned} \mathbf{sSet}(\Delta^n, \varprojlim_{\beta<\alpha} \mathrm{Map}^\sharp_{S_\beta}(j_\beta^*X, j_\beta^*Y)) &= \varprojlim_{\beta<\alpha} \mathbf{sSet}(\Delta^n, \mathrm{Map}^\sharp_{S_\beta}(j_\beta^*X, j_\beta^*Y)) \\ &\cong \varprojlim_{\beta<\alpha} (\mathbf{sSet}^+)_{/S_\beta}((\Delta^n)^\sharp \times j_\beta^*X, j_\beta^*Y) \\ &\cong \mathbf{sSet}(\Delta^n, \mathrm{Map}^\sharp_{S_\alpha}(X, Y)). \end{aligned}$$

This entails that the function complex $\mathrm{Map}^{\sharp}_{S_\alpha}(X,Y)$ may be computed as the inverse limit $\varprojlim_{\beta<\alpha} \mathrm{Map}^{\sharp}_{S_\beta}(j_\beta^* X, j_\beta^* Y)$. In view of this, and the fact that trivial fibrations are stable under taking inverse limits of this form, we deduce that the counit map $i_! i^* X^\natural \to X^\natural$ is a trivial cofibration when $j_\beta^* i_! i^* X^\natural \to j_\beta^* X^\natural$ is a trivial cofibration for every $\beta < \alpha$. But this latter map is precisely the counit map $(i_\beta)_! i_\beta^* j_\beta^* X^\natural \to j_\beta^* X^\natural$, which is a trivial cofibration by assumption. □

Lemma 1.3.6. *Suppose that Theorem 1.3.4 holds for inner anodynes, then it holds for categorical equivalences f.*

Proof. Suppose that $f : S \to T$ is a categorical equivalence. Using the small objects argument twice, we obtain a commutative square

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ S_f & \longrightarrow & T_f \end{array}$$

such that the vertical maps are inner anodynes and S_f and T_f are ∞-categories. By the 2-out-of-3 property, the bottom arrow is also a categorical equivalence. In view of the assumption and the fact that Quillen equivalences satisfy the 2-out-of-3 property, the top arrow gives rise to a Quillen equivalence, if and only if the bottom arrow gives rise to a Quillen equivalence. Brown's principle now asserts that it suffices to check the statement in case that f is a trivial fibration.

In this case, the induced map $S^\sharp \to T^\sharp$ in $(\mathbf{sSet}^+)_{/T}$ is a trivial fibration. Consequently, $f_!$ sends fibrant objects to fibrant objects. Combining this with the observation that f is surjective on vertices, we deduce that $\mathbf{L}f_!$ is conservative. It is readily verified that the components of the counit are weak equivalences in $(\mathbf{sSet}^+)_{/T}$. Thus the pair $(f_!, f^*)$ is a Quillen equivalence, as desired. □

There is yet another adjunction related to change of basis. Since the category of marked simplicial sets is locally cartesian, the pullback functor $f^* : (\mathbf{sSet}^+)_{/T} \to (\mathbf{sSet}^+)_{/S}$ admits a right adjoint, which we will denote by f_*. However, the adjunction

$$f^* : (\mathbf{sSet}^+)_{/T} \rightleftarrows (\mathbf{sSet}^+)_{/S} : f_*.$$

is not a Quillen adjunction in general. As an application of the theory developed in Chapter 2, we will later show that in a particular situation, this is the case:

Proposition 1.3.7 (Theorem 3.2.6 of [Ngu18])**.** *The adjunction (f^*, f_*) is Quillen for any right fibration $f : S \to T$.*

1.4 A first step towards straightening

The goal of this thesis is to show how coCartesian fibrations $X \to S$ relate to simplicial functors $\mathfrak{C}S \to \mathbf{sSet}^+$. In this section, we would like to give a first attempt at this: a first approximation. We will show that any coCartesian fibration $X \to \mathcal{C}$ over a ∞-category $\mathcal{C}$ gives rise to a functor $\operatorname{Ho}\mathcal{C} \to \operatorname{Ho}(\mathbf{sSet}^+)$.

Let $X \to S$ be a coCartesian fibration and let $f : s \to t$ be an edge in the base S. Then we can always lift such an edge to a functor $f_! : X_s \to X_t$ by taking a filler H of the following diagram

$$\begin{array}{ccc} \{0\} \times X_s^\natural & \longrightarrow & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^1)^\sharp \times X_s^\natural & \xrightarrow{f^\sharp \mathrm{pr}_{(\Delta^1)^\sharp}} & S^\sharp, \end{array}$$

and restricting the filler H to $\{1\}\times X_s^\natural$. Note that this the obtained lift $f_! : X_s^\natural \to X_t^\natural$ is unique up to homotopy in the Kan complex $\operatorname{Map}^\sharp(X_s^\natural, X_t^\natural)$. Namely, for any other filler H' obtained in this way, there exists a homotopy

$$(\Delta^1)^\sharp \times (\Delta^1)^\sharp \times X_s^\natural \to X^\natural$$

connecting H and H', since the map

$$\operatorname{Map}_S^\sharp((\Delta^1)^\sharp \times X_s^\natural, X^\natural) \to \operatorname{Map}_S^\sharp(\{0\} \times X_s^\natural, X^\natural)$$

is a trivial Kan fibration. This homotopy restricts to a homotopy $(\Delta^1)^\sharp \times \{1\} \times X_s^\natural \to X_t^\natural$ in $\mathbf{sSet}^+$ connecting the two lifts.

Proposition 1.4.1. *Let $\mathcal{C}$ be an ∞-category and $X^\natural$ a coCartesian fibration on $\mathcal{C}$, then the above construction gives rise to a well-defined functor*

$$F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}(\mathbf{sSet}^+)$$

given on objects by $Fc := X_c^\natural$ and on morphisms by $F[f] := [f_!]$.

Proof. It remains to show that F is compatible with compositions. Let $f : x \to y$ and $g : y \to z$ be two maps in $\mathcal{C}$ and suppose that $\sigma : \Delta^2 \to \mathcal{C}$ is a 2-simplex witnessing the composition of f and g. Let $H : (\Delta^1)^\sharp \times X_x^\natural \to X^\natural$ and $H' : (\Delta^1)^\sharp \times X_y^\natural \to X^\natural$ be the fillers giving rise to lifts $f_!$ and $g_!$ respectively. The following square admits a filler

$$\begin{array}{ccc} (\Lambda_1^2)^\sharp \times X_x^\natural & \xrightarrow{H \cup H'(\mathrm{id}_{(\Delta^1)^\sharp} \times f_!)} & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^2)^\sharp \times X_x^\natural & \xrightarrow{\sigma^\sharp \mathrm{pr}_{(\Delta^2)^\sharp}} & S^\sharp, \end{array}$$

which we will denote by G. This entails that the restriction of G to $\{2\} \times X_x^\natural$ defines a lift $(gf)_!$ of the composite gf. But this is precisely $g_! f_!$, as desired. □

We will show that every coCartesian fibration $X^\natural$ admits a rectification

$$F : \mathfrak{C}\mathcal{C} \to (\mathbf{sSet}^+)^\circ,$$

which also has the following property:

Proposition 1.4.2. *The rectification of a coCartesian fibration is naturally isomorphic to the functor of Proposition 1.4.1 after taking the* π_0 *of this map.*

CHAPTER 2

An elementary account of rectification

In this chapter, we will prove the straightening-unstraightening equivalence in the case that the base is a 1-category. In this particular case, the proof turns out to be elementary and we have a rectification functor that admits an easy description. A fortiori, this rectification functor produces strict diagrams on $\mathcal{C}$, whereas the general rectification functor (defined in Chapter 4) produces homotopy coherent diagrams on $\mathcal{C}$. Concretely, given a marked simplicial set X over $N\mathcal{C}$, we define its rectification $r_!X : \mathcal{C} \to \mathbf{sSet}^+$ by the formula

$$(r_!X)c := X \times_{N\mathcal{C}^\sharp} N\mathcal{C}^\sharp_{/c}.$$

This construction is functorial, and gives rise to the desired rectification functor

$$r_! : (\mathbf{sSet}^+)_{/N\mathcal{C}} \to (\mathbf{sSet}^+)^{\mathcal{C}}.$$

The goal of this chapter is to show that this functor is part of a Quillen equivalence. Since the model structure on the left and the right have underlying ∞-categories $\mathbf{coCart}(N\mathcal{C})$ and $\mathrm{Fun}(N\mathcal{C}, \mathbf{Cat}_\infty)$ respectively, this Quillen equivalence models the ∞-categorical Grothendieck construction: an adjoint equivalence

$$\mathbf{coCart}(N\mathcal{C}) \rightleftarrows \mathrm{Fun}(N\mathcal{C}, \mathbf{Cat}_\infty).$$

The account of this fact we present here, is a generalization of the proof of the straightening-unstraightening equivalence in the case of left fibrations given by Heuts and Moerdijk in [HM15].

2.1 The rectification functor

We commence by analyzing the rectification functor $r_!$, and establishing the fact that this functor a left Quillen functor. Note that the rectification functor preserves colimits. Thus by general non-sense, it admits a right adjoint. We will give a computation of this right adjoint shortly.

Remark 2.1.1. Forgetting the markings in the definition of $r_!$, we obtain a functor $\mathbf{sSet}_{/\mathcal{C}} \to (\mathbf{sSet})^{\mathcal{C}}$, which we will denote by $r_!$ again. This is the rectification functor in [HM15]. This functor is compatible with our rectification functor in the following way. Note that the left adjoint functors $(-)^{\flat/\sharp} : \mathbf{sSet} \to \mathbf{sSet}^+$

induce left adjoint functors $(-)^{\flat/\sharp}: \mathbf{sSet}^{\mathcal{C}} \to (\mathbf{sSet}^+)^{\mathcal{C}}$ (defined pointwise). Then for any n-simplex $\Delta^n \to \mathcal{C}$, we have

$$r_!(\Delta^n)^{\flat/\sharp} = (r_!\Delta^n)^{\flat/\sharp}.$$

This is clear for the $\sharp$-marked case. In the $\flat$-marked case, we have that the marked edges of the pullback $r_!(\Delta^n)^\flat c = (\Delta^n)^\flat \times_{(N\mathcal{C})^\sharp} (N\mathcal{C})^\sharp$ are given by pairs (e, e') with e a degenerate edge of Δ^n (i.e. an identity map), and e' any edge of $N\mathcal{C}_{/c}$ such that the projections of e and e' are equal in $N\mathcal{C}$. But this means that the projection of e' to $N\mathcal{C}$ is degenerate as well. Since the forgetful functor $\mathcal{C}_{/c} \to \mathcal{C}$ is faithful, e' must be an identity map. Thus e' is a degenerate edge as well.

Proposition 2.1.2. *The rectification functor has the following properties:*

(i) the rectification is natural, i.e., for any functor $f : \mathcal{C} \to \mathcal{D}$, the following square commutes

$$\begin{array}{ccc} (\mathbf{sSet}^+)_{/N\mathcal{C}} & \longrightarrow & (\mathbf{sSet}^+)^{\mathcal{C}} \\ \downarrow{\scriptstyle (Nf)_!} & & \downarrow{\scriptstyle f_!} \\ (\mathbf{sSet}^+)_{/N\mathcal{D}} & \longrightarrow & (\mathbf{sSet}^+)^{\mathcal{D}} \end{array}$$

up to natural isomorphism,

(ii) the rectification of a simplex $\Delta^n \to N\mathcal{C}$ is described by the coend

$$r_!(\Delta^n)^{\flat/\sharp} = \int^{i\in[n]} (\Delta^i)^{\flat/\sharp} \times \mathcal{C}(c_i, -).$$

Proof. Part (i) can be deduced directly from the definition, or is readily seen using description (ii). In order to show part (ii), it suffices to prove the statement in the unmarked case in view of Remark 2.1.1.

Note that the Yoneda lemma asserts that we have isomorphisms

$$\begin{aligned}(\mathbf{sSet}^+)^{\mathcal{C}}(A \times \mathcal{C}(c,-), F) &\cong (\mathbf{sSet}^+)^{\mathcal{C}}(\mathcal{C}(c,-), F^A) \\ &\cong \mathbf{Set}^{\mathcal{C}}(\mathcal{C}(c,-), \mathbf{sSet}^+(\Delta^0, F(-)^A)) \cong \mathbf{sSet}^+(A, Fc)\end{aligned}$$

natural in $A \in \mathbf{sSet}^+$, $c \in \mathcal{C}$ and $F \in (\mathbf{sSet}^+)^{\mathcal{C}}$. Thus we obtain canonical maps $\Delta^i \times \mathcal{C}(c_i, -) \to r_!\Delta^n$, determined by the i-simplices $c_0 \to \cdots \to c_i \to c_i$ in $(r_!\Delta^n)c_i$. It is readily verified that for $i \le j$, the square

$$\begin{array}{ccc} \Delta^i \times \mathcal{C}(c_j,-) & \longrightarrow & \Delta^j \times \mathcal{C}(c_j,-) \\ \downarrow & & \downarrow \\ \Delta^i \times \mathcal{C}(c_i,-) & \longrightarrow & r_!\Delta^n \end{array}$$

commutes. Let $F : \mathcal{C} \to \mathbf{sSet}$ be a diagram accompanied by commuting squares

$$\begin{array}{ccc} \Delta^i \times \mathcal{C}(c_j,-) & \longrightarrow & \Delta^j \times \mathcal{C}(c_j,-) \\ \downarrow & & \downarrow{\scriptstyle \lambda_j} \\ \Delta^i \times \mathcal{C}(c_i,-) & \xrightarrow{\lambda_i} & F \end{array}$$

for $i \le j$. Then we define map $\eta : r_!\Delta^n \to F$ as follows. Let $\sigma := c_{k_0} \to \cdots \to c_{k_i} \xrightarrow{f} c$ be a i-simplex of $\Delta^n \times_{N\mathcal{C}} N\mathcal{C}_{/c}$. Then $k : [i] \to [k_i] : j \mapsto k_j$ is order preserving. We set $\eta_c(\sigma) := \lambda_{k_i}(k, f)$. It is readily verified that this defines a map η_c of simplicial sets, which assemble together to the desired map η, which is compatible with the squares above, and uniquely determined by this compatibility. □

Corollary 2.1.3. *Let $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n$ be a n-simplex of $N\mathcal{C}$. Consider a marked simplicial diagram F on $\mathcal{C}$. Then there is a natural bijection*

$$(\mathbf{sSet}^+)^{\mathcal{C}}(r_!(\Delta^n)^\flat, F) \xrightarrow{\cong} \left\{ (x_i)_{i=0}^n \in \prod_{i=0}^n (Fc_i)_i \mid Ff_i(x_{i-1}) = d_i x_i \right\}$$

given by sending a natural transformation $\eta : r_!(\Delta^n)^\flat \to F$ to the tuple $(\eta_{c_i}(c_0 \to c_1 \cdots \to c_i \to c_i))$. Similarly, there is a natural bijection

$$(\mathbf{sSet}^+)^{\mathcal{C}}(r_!(\Delta^n)^\sharp, F) \xrightarrow{\cong} \left\{ (x_i)_{i=0}^n \in \prod_{i=0}^n (Fc_i)_{\sharp,i} \mid Ff_i(x_{i-1}) = d_i x_i \right\},$$

given by sending a map $\eta : r_!(\Delta^n)^\sharp \to F$ to the tuple $(\eta_{c_i}(c_0 \to c_1 \to \cdots \to c_i \to c_i))$.

The computations above allow us to write down an explicit formula for the relative nerve functor r^*. Namely, for a diagram $F : \mathcal{C} \to \mathbf{sSet}^+$, we define

$$(r^*F)_n := \{(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n, (x_i)_{i=0}^n) \mid Ff_i(x_{i-1}) = d_i x_i\}.$$

The marked edges of r^*F are the pairs $(c_0 \to c_1, (x_0, x_1))$ such that x_1 is marked in Fc_1. The face and degeneracy maps act on a n-simplex $\sigma = (c_0 \to c_1 \to \cdots \to c_n, (x_i)_{i=0}^n)$ as follows:

$$d_i\sigma = (c_0 \to \cdots \to \widehat{c_i} \to \cdots \to c_n, (x_0, \ldots, x_{i-1}, \widehat{x_i}, d_i x_{i+1}, \ldots, d_i x_n)),$$
$$s_i\sigma = (c_0 \to \cdots \to c_i \to c_i \to \cdots \to c_n, (x_0, \ldots, x_i, s_i x_i, s_i x_{i+1}, \ldots, s_i x_n)).$$

The components of the counit assocated to the adjunction $(r_!, r^*)$ are now given as follows. A n-simplex of $(r_! r^* F)c$ is a tuple

$$(((c_0 \to \cdots \to c_i \to c_i, (x_i)), c_0 \to \cdots \to c_n \to c)$$

and the counit $r_! r^* F \to F$ carries such a n-simplex to the n-simplex $F(c_n \to c)x_n$ of Fc. Similarly, the components of the unit of the adjunction $(r_!, r^*)$ are given as follows. For a n-simplex x of $X \in (\mathbf{sSet}^+)_{/N\mathcal{C}}$ over $c_0 \to \cdots \to c_n$, the unit $X \to r^* r_! X$ sends x to the n-simplex

$$(c_0 \to \cdots \to c_n, (([i] \to [n])^* x, c_0 \to \cdots \to c_i \to c_i))$$

of $r^* r_! X$. Here the map $[i] \to [n]$ denotes the inclusion.

Remark 2.1.4. In the light of the above, we see that the relative nerve functor has the following convenient property. Consider a diagram $F : \mathcal{C} \to \mathbf{sSet}^+$. Taking the relative nerve of F and passing to fibers, we obtain an isomorphism

$$(r^*F)_c \cong F(c)$$

of marked simplicial sets natural in F.

The remaining of this section is devoted to showing that the pair $(r_!, r^*)$ is a Quillen adjunction. We will make use of Corollary 1.1.9.

Proposition 2.1.5. *The functor $r_!$ preserves cofibrations.*

Proof. In view of Proposition 1.1.4, it suffices to check this for the generating cofibrations $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$, $(\Delta^1)^\flat \to (\Delta^1)^\sharp$. The image of the latter cofibration is readily seen to be a cofibration in $(\mathbf{sSet}^+)^{\mathcal{C}}$. It remains to show that $r_!(\partial\Delta^n)^\flat \to r_!(\Delta^n)^\flat$ is a trivial cofibration.

Let $p : F \to G$ be a trivial fibration in $(\mathbf{sSet}^+)^{\mathcal{C}}$. Then we have to demonstrate that the following square admits a diagonal lift

$$\begin{array}{ccc} r_!(\partial\Delta^n)^\flat & \longrightarrow & F \\ \downarrow & & \downarrow p \\ r_!(\Delta^n)^\flat & \longrightarrow & G. \end{array}$$

By Corollary 2.1.3, the bottom arrow corresponds to simplices $z_0, \ldots, z_n$ with $z_i \in (Fc_i)_i$ and $Gf_i(z_{i-1}) = d_i z_i$. Note that $r_!(\partial\Delta^n)^\flat = \mathrm{coeq}(\coprod_{0\le i<j\le n} r_!(\Delta^{n-2})^\flat \rightrightarrows \coprod_{i=0}^n r_!(\Delta^{n-1})^\flat)$. In view of this and Corollary 2.1.3, the top arrow is given by a suitable family of simplices $(\xi_0^i, \cdots, \widehat{\xi_i^i}, \cdots, \xi_n^i)$, $0 \le i \le n$, such that

$$p_{c_k}(\xi_k^i) = \begin{cases} z_k & \text{if } k < i \\ d_i z_k & \text{if } k > i. \end{cases}$$

Furthermore, we have the compatibility relations

$$\xi_k^i = \xi_k^j \text{ if } k < i, \quad \xi_k^i = d_i \xi_k^j \text{ if } i < k < j, \quad d_{j-1}\xi_k^i = d_i \xi_k^j \text{ if } j < k$$

for $i < j$ and $0 \le k \le n$. It is readily seen that the simplices $\xi_n^0, \ldots, \xi_n^{n-1}, Ff_n(\xi_{n-1}^n)$ define a map $(\partial\Delta^n)^\flat \to Fc_n$ making the following diagram commute

$$\begin{array}{ccc} (\partial\Delta^n)^\flat & \longrightarrow & Fc_n \\ \downarrow & & \downarrow p_{c_n} \\ (\Delta^n)^\flat & \xrightarrow{z_n} & Gc_n. \end{array}$$

Since p_{c_n} is a trivial fibration in $\mathbf{sSet}^+$, this square admits a filler $x_n \in F(c_n)_n$. The simplices $\xi_0^n, \ldots, \xi_{n-1}^n, x_n$ define the desired lifting $r_!(\Delta^n)^\flat \to F$. □

Proposition 2.1.6. *The functor $r_!$ carries marked anodyne maps to trivial (projective) cofibrations in $(\mathbf{sSet}^+)^{\mathcal{C}}$.*

Proof. This is a straightforward check. It is sufficient to check the statement for all the generators (i)-(iii) in Definition 1.1.5 and generator (iv') of Proposition 1.1.11). For generator (i), this is shown similarly to Proposition 2.1.5.

Generator (iii). We have to show that for any fibration $F \to G$, the following square admits a filler,

$$\begin{array}{ccc} r_!(\Lambda^2_1)^\sharp \bigcup_{r_!(\Lambda^2_1)^\flat} (\Delta^2)^\flat & \longrightarrow & F \\ \downarrow & & \downarrow \\ r_!(\Delta^2)^\sharp & \longrightarrow & G. \end{array}$$

This entails that we have simplices x_0, x_1, x_2 with $x_i \in (Fc_i)_i$ and $Ff_i(x_{i-1}) = d_i x_i$, such that x_1 and $d_0 x_2$ are marked. Furthermore, the images $p_{c_i}(x_i)$ have marked edges. We must show that the edges of x_2 are marked. Since Ff_1 is a map of marked simplicial sets, $d_2 x_2$ is marked as well. All in all, the edges of the 0^{th} and the 2^{nd} face of x_2 are marked and all edges of $p_{c_2}(x_2)$ are marked. Since $(\Lambda^2_1)^\sharp \bigcup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ is marked anodyne and p_{c_2} is a fibration, it follows that all edges of x_2 are marked.

Generators (ii). This case is handled similarly to generator (iii).

Generator (iv'). Let $A \to N\mathcal{C}$ be a map of simplicial sets. Such a map corresponds to a 3-simplex

$$c \xrightarrow{f} d \xrightarrow{f^{-1}} c \xrightarrow{f} d$$

of $N\mathcal{C}$, where $f : c \to d$ is an isomorphism in $\mathcal{C}$. We have to show that for any fibration $F \to G$, the following square admits a filler:

$$\begin{array}{ccc} r_! A^\flat & \longrightarrow & F \\ \downarrow & & \downarrow \\ r_!(A, \mathcal{E}) = r_! A^\flat \bigcup_{r_!(\Delta^{\{0,1\}})^\flat} r_!(\Delta^{\{0,1\}})^\sharp & \longrightarrow & G. \end{array}$$

The top arrow corresponds to simplices x_0, x_1, x_2, x_3 with $x_0 \in (Fc)_0$, $x_1 \in (Fd)_1$, $x_2 \in (Fc)_2$ and $x_3 \in (Fd)_3$ such that $Ff(x_0) = d_1 x_1$, $Ff^{-1}(x_1) = d_2 x_2$, $Ff(x_2) = d_3 x_3$ obeying the relations $s_0 x_0 = d_1 x_2$, $s_0 x_1 = d_1 x_3$ and $s_1 x_1 = d_2 x_3$. Furthermore, the above diagram tells us that $p_d(x_1)$ is a marked edge of Gd. The map $r_!(\Delta^{\{0,1\}})^\flat \to r_! A^\flat$ induced by the projection of the edge $\Delta^{\{0,1\}} \subset \Delta^3$ in A, correspond to the simplices x_0, x_1. Consequently, it suffices to show that x_1 is a marked edge of Fd. Observe that $x_3|_{\Delta^{\{0,1\}}} = x_1$ and that the 3-simplex x_3 descends to a map $A \to Fd$ of simplicial sets. Similarly, the simplex $p_d(x_3)$ descends to a map $A \to Gd$ of simplicial sets. Because $p_d(x_1)$ is marked, this gives rise to a map $(A, \mathcal{E}) \to Gd$ of marked simplicial sets. Since $A^\flat \to (A, \mathcal{E})$ is marked anodyne and p_d is a fibration, we deduce that x_1 is marked. □

Corollary 2.1.7. *The adjunction $(r_!, r^*)$ is a simplicial Quillen adjunction.*

Proof. It is readily verified that we have canonical isomorphisms $r_!(A \times X) \cong A \times r_! X$ natural in $A \in \mathbf{sSet}^+$ and $X \in (\mathbf{sSet}^+)_{/N\mathcal{C}}$. Taking $\sharp$-marked simplicial sets for A, we deduce that $(r_!, r^*)$ is a simplicial adjunction. The fact that this adjunction is a Quillen adjunction follows from combining Proposition 2.1.5, Proposition 2.1.6 and Corollary 1.1.9. □

We will show that the adjunction $(r_!, r^*)$ is Quillen equivalence in the next section. This will then give an affirmative answer to Proposition 1.4.2:

Proposition 2.1.8. *Let $F : \mathcal{C} \to \mathbf{sSet}^+$ be a fibrant diagram. The naive rectification of r^*F is naturally isomorphic to $\pi_0 F$.*

Proof. Let $f : c \to d$ be an arrow in $\mathcal{C}$. Then the obvious homotopy

$$H : (\Delta^1)^\sharp \times (r^*F)_c \to r^*F : (\alpha, (x_i)) \mapsto (f\alpha, (F(c \to f(\alpha_i))x_i)).$$

witnesses Ff as a lift of f under the identifications of Remark 2.1.4. □

2.2 Homotopy colimits and rectification

In Section 3.3.4 of [Lur09], Lurie discusses the relation between homotopy colimits in $\mathbf{sSet}^+$ (i.e. colimits in $\mathbf{Cat}_\infty$) and the right adjoint r^*: namely, the underlying marked simplicial set of r^*F is precisely the homotopy colimit of the diagram F. We will exploit this relationship to give a quick proof of the straightening-unstraightening equivalence. Simultaneously, we will obtain an independent demonstration of Corollary 3.3.4.3 of [Lur09] for diagrams with a 1-category as a domain. We postpone the proof of this corollary in full generality to Proposition 4.4.1, where we exploit the results obtained in this section.

Recall that the category $\mathbf{sSet}^+$ is a combinatorial, simplicial model category, hence the bar construction $B(*, \mathcal{C}, F)$ models the homotopy colimit of a diagram $F : \mathcal{C} \to \mathbf{sSet}^+$. The bar construction is computed as geometric realization of a particular simplicial object, and may be computed in this case as

$$B(*, \mathcal{C}, F) = \operatorname{diag}^+\left(\coprod_{c_0 \in \mathcal{C}} Fc_0 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \coprod_{c_0 \to c_1 \in N\mathcal{C}_1} Fc_0 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \cdots \right).$$

Here $\operatorname{diag}^+(-)$ denotes the functor that associates to a simplicial object $X_\bullet$ in $\mathbf{sSet}^+$, its marked diagonal. The underlying simplicial set of $\operatorname{diag}^+(X_\bullet)$ is the ordinary diagonal of the underlying bisimplicial set of $X_\bullet$. The marked edges are precisely the marked edges of X_1. This bar construction gives rise to a functor

$$h_! : (\mathbf{sSet}^+)^{\mathcal{C}} \to (\mathbf{sSet}^+)_{/N\mathcal{C}} : F \mapsto (B(*, \mathcal{C}, F) \to B(*, \mathcal{C}, *) = (N\mathcal{C})^\sharp),$$

which a right adjoint h^*, and fits in a Quillen adjunction:

Proposition 2.2.1. *The adjunction $(h_!, h^*)$ is a simplicial Quillen adjunction.*

Proof. It is readily verified that $h_!$ commutes with tensoring by marked simplicial sets. In light of the properties of the projective model structure, we have to show that for any cofibration $i : A \to B$ in $\mathbf{sSet}^+$, the induced map $h_!(A \times \mathcal{C}(c,-)) \to h_!(B \times \mathcal{C}(c,-))$ is a cofibration, which is trivial if i is trivial. Note that this map is isomorphic to the map $A \times h_!\mathcal{C}(c,-) \to B \times h_!\mathcal{C}(c,-)$. Thus the desired result follows from [Lur09, Corollary 3.1.4.3]. □

The goal of this section is to show that after deriving the rectification functor $r_!$ and the homotopy colimit functor $h_!$, these functors fit in an adjoint equivalence of homotopy categories. More precisely, we will prove the main theorem:

Theorem 2.2.2. *The Quillen adjunctions $(r_!, r^*)$ and $(h_!, h^*)$ are Quillen equivalences. Moreover, the derived functors $\mathbf{L}r_!$ and $\mathbf{L}h_!$ are inverse equivalences.*

We will pursue the same strategy as Heuts and Moerdijk in [HM15]. The proof of the main theorem may be split in two parts:

Lemma 2.2.3. *There exists a 2-cell $r_!h_! \Rightarrow \mathrm{id}_{(\mathbf{sSet}^+)^{\mathcal{C}}}$ whose components are weak equivalences for (projectively) cofibrant diagrams.*

Lemma 2.2.4. *There exists a zig-zag of 2-cells between $h_!r_!$ and $\mathrm{id}_{(\mathbf{sSet}^+)_{/N\mathcal{C}}}$ whose components are coCartesian equivalences.*

Let us start by proving Lemma 2.2.3. Let $F : \mathcal{C} \to \mathbf{sSet}^+$ be a diagram. Note that the n-simplices of $(r_!h_!F)c$ are given by pairs $(\xi, c_0 \to c_1 \to \cdots \to c_n \to c)$ with $\xi \in (Fc_0)_n$. An edge is marked precisely when the component ξ is marked in Fc_0. We now define a map $\epsilon_c : (r_!h_!F)c \to Fc$ by setting

$$\epsilon_c(\xi, c_0 \to \cdots \to c_n \to c) := F(c_0 \to \cdots \to c_n \to c)(\xi).$$

It is clear that this map is compatible with the degeneracy-maps, right and inner face maps. Furthermore, the following computation for a n-simplex $\sigma = (\xi, c_0 \to \cdots \to c)$,

$$\begin{aligned}\epsilon_c(d_0\sigma) &= \epsilon_c(F(c_0 \to c_1)(d_0\xi), c_1 \to \cdots \to c) = F(c_1 \to \cdots \to c_n \to c)F(c_0 \to c_1)(d_0\xi)\\ &= F(c_0 \to \cdots \to c)(d_0\xi) = d_0(\epsilon_c\sigma)\end{aligned}$$

shows that the map is compatible with all structure maps. Moreover, ϵ_c respects the markings, hence we have defined a map of simplicial sets. It is readily verified that these maps constitute a natural map $\epsilon_F : r_!h_!F \to F$.

Proof of Lemma 2.2.3. We show that the 2-cell ϵ has the desired property. The projective model structure $(\mathbf{sSet}^+)^{\mathcal{C}}$ is left proper and combinatorial as $\mathbf{sSet}^+$ already has these properties. A class of generating cofibrations is given by $A \times \mathcal{C}(x,-) \to B \times \mathcal{C}(x,-)$, where $A \to B$ is a cofibration of marked simplicial set, and $c \in \mathcal{C}$. Consequently, Lemma B.0.2 asserts that it suffices to check that ϵ_F is a weak equivalence for $F = A \times \mathcal{C}(x,-)$, where A is some marked simplicial set. It is readily verified that the map ϵ_F is isomorphic to the map $A \times \epsilon_{\mathcal{C}(x,-)}$, thus we may further reduce checking the statement to diagrams of the form $F = \mathcal{C}(x,-)$.

In this case, the underlying map of simplicial sets of $(\epsilon_F)_c : (r_! h_! F)c \to Fc$ may be identified with the nerve of the functor $r : (\mathcal{C}_{/c})_{x/} \to \mathcal{C}(x,c)$, sending a factorization $x \to \xi \to c$ to its composite $x \to c$ (here $\mathcal{C}(x,c)$ is viewed as a discrete category). This functor is a retraction of the inclusion $i : \mathcal{C}(x,c) \to (\mathcal{C}_{/c})_{x/}$ which sends an arrow $f : x \to c$ to

$$x \xrightarrow{\mathrm{id}_x} x \xrightarrow{f} c.$$

Since all edges of $(r_! h_! F)c$ and Fc are marked, and the model structure on $\mathbf{sSet}^+$ is simplicial, it suffices to show that Ni is a trivial cofibration in the Kan-Quillen model structure. This is clear, as there exists an obvious natural transformation $ir \Rightarrow \mathrm{id}_{(\mathcal{C}_{/c})_{x/}}$. □

We shift our attention to the proof of Lemma 2.2.3. For any marked simplicial set X over $N\mathcal{C}$, we define the marked simplicial set LX by the pullback square

$$\begin{array}{ccc} LX & \longrightarrow & \mathrm{Fun}(\Delta^1, N\mathcal{C})^\sharp \\ \downarrow & & \downarrow \\ X & \longrightarrow & N\mathcal{C}^\sharp, \end{array}$$

where the right arrow corresponds to evaluation at the vertex 0. We may consider LX as a marked simplicial set over $N\mathcal{C}$ by taking the projection map to be induced by evaluation at 1. This construction defines a functor

$$L : (\mathbf{sSet}^+)_{/N\mathcal{C}} \to (\mathbf{sSet}^+)_{/N\mathcal{C}}.$$

Note that we have a natural inclusion map $i : X \to LX$ induced by the constant path map $N\mathcal{C}^\sharp \to \mathrm{Fun}(\Delta^1, N\mathcal{C})^\sharp$.

Proposition 2.2.5. *The inclusion $i : X \to LX$ is a trivial cofibration.*

Proof. On account of Proposition 1.1.15, it is sufficient to show that there exists a homotopy $(\Delta^1)^\sharp \times LX \to LX$ (not necessarily over $N\mathcal{C}$) relative to X, such that $H|_{\{0\}\times LX} = ir$ and $H|_{\{1\}\times LX} = \mathrm{id}_{LX}$, where r is the projection $LX \to X$. The unique map of ordered sets $[1]\times[1] \to [1]$ which sends $(1,0)$ to 0 and $(1,1)$ to 1, induces a map $\Delta^1 \times \Delta^1 \to \Delta^1$, which in turn gives rise to a map $\mathrm{Fun}(\Delta^1, N\mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, N\mathcal{C})$. By adjunction, this corresponds to a map $\Delta^1 \times \mathrm{Fun}(\Delta^1, N\mathcal{C}) \to \mathrm{Fun}(\Delta^1, N\mathcal{C})$. Applying the $\sharp$-functor to this map, and combining the resulting map with the projection $(\Delta^1)^\sharp \times X \to X$ yields the desired homotopy H. □

Finally, we show that there exists a natural trivial cofibration $h_! r_! X \to LX$ for every marked simplicial set $p : X \to N\mathcal{C}^\sharp$ over $N\mathcal{C}$. A quick inspection shows that $h_! r_! X$ has n-simplices of the form $(\xi, p(\xi)_n \to c_0 \to c_1 \to \cdots \to c_n)$ with $\xi \in X_n$. An edge is marked precisely when the component ξ is marked in X. Sending such a n-simplex to ξ, we obtain a map $h_! r_! X \to X$ of marked simplicial sets. Furthermore, sending a n-simplex $(\xi, p(\xi)_n \to c_0 \to \cdots \to c_n)$ to the n-simplex

$$\begin{array}{ccccccc} p(\xi)_0 & \longrightarrow & p(\xi)_1 & \longrightarrow & \cdots & \longrightarrow & p(\xi)_n \\ \downarrow & & \downarrow & & & & \downarrow \\ c_0 & \longrightarrow & c_1 & \longrightarrow & \cdots & \longrightarrow & c_n \end{array}$$

of $\mathrm{Fun}(\Delta^1, N\mathcal{C})^\sharp = (N\mathcal{C}^{[1]})^\sharp$, we obtain a map $h_! r_! X \to \mathrm{Fun}(\Delta^1, N\mathcal{C})$. The two maps exhibited above assemble to the desired natural monomorphism $j_X : h_! r_! X \to LX$ over $N\mathcal{C}$.

Remark 2.2.6. Alternatively, the natural transformation $j : h_! r_! \Rightarrow L$ may be defined to be the unique natural transformation such that the map

$$h_! r_! (\Delta^n)^{\flat/\sharp} = h_! \int^{i \in [n]} (\Delta^i)^{\flat/\sharp} \times \mathcal{C}(c_i, -) = \int^{i \in [n]} (\Delta^i)^{\flat/\sharp} \times N\mathcal{C}^\sharp_{c_i/} \to L(\Delta^n)^{\flat/\sharp}$$

is induced by the maps $(\Delta^i)^{\flat/\sharp} \times N\mathcal{C}^\sharp_{c_i/} \to L(\Delta^n)^{\flat/\sharp}$ whose underlying map of simplicial sets corresponds to the nerve of the composition functor

$$[i] \times \mathcal{C}_{c_i/} \to [n] \times_{\mathcal{C}} \mathcal{C}^{[1]} : (k, c_i \to x) \mapsto (k, c_k \to c_i \to x).$$

Lemma 2.2.7. *Let $X \to Y$ be a coCartesian equivalence. Then j_X is a coCartesian equivalence if and only if j_Y is a coCartesian equivalence.*

Proof. Note that the following diagram commutes

$$\begin{array}{ccccc} X & \longrightarrow & LX & \longleftarrow & h_! r_! X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & LY & \longleftarrow & h_! r_! Y. \end{array}$$

Since $h_!$ and $r_!$ are left Quillen, the right vertical map is again a coCartesian equivalence. We already demonstrated that the vertical maps in the left square are coCartesian equivalences. Applying the 2-out-of-3 property twice, we deduce that the the two right horizontal arrows are coCartesian equivalences precisely when one of them is a coCartesian equivalence. □

Proposition 2.2.8. *For any marked simplicial set X over $N\mathcal{C}$, the component j_X is a trivial cofibration.*

Proof. Recall that the X admits a skeletal filtration. By a similar homotopy pushout argument as in Lemma B.0.2 we see that it suffices to show the statement for $X = (\Delta^n)^\flat$ and $X = (\Delta^1)^\sharp$. Note that $\{0\} \to (\Delta^1)^\sharp$ is marked anodyne, hence we may reduce to the case that $X = (\Delta^n)^\flat$ on account of Lemma 2.2.7. Furthermore, we recall that the n-spine inclusion

$$\Delta^{\{0,1\}} \bigcup_{\{1\}} \Delta^{\{1,2\}} \bigcup_{\{2\}} \cdots \bigcup_{\{n-1\}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

is inner anodyne (see [Joy, Proposition 2.13]). Consequently, $\flat$-marking this map yields a marked anodyne map. Thus it suffices to show that j is a coCartesian equivalence for $\flat$-marked n-spines over $N\mathcal{C}$. Such a spine is an iterated pushouts of $(\Delta^1)^\flat$'s, which all are homotopy pushouts. Proceeding inductively, we deduce that it suffices to handle the cases that $n = 0, 1$.

We will demonstrate that we can reduce the case $n = 1$ to the case that $n = 0$. Let $X = (\Delta^1)^\flat$ be an unmarked edge over $N\mathcal{C}$ corresponding to an arrow $f : x \to y$ in $\mathcal{C}$. There is a zig-zag of marked anodynes

$$X \cong (\Delta^{\{0,2\}})^\flat \xrightarrow{\sim} (\Delta^2)^\flat \bigcup_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \xleftarrow{\sim} (\Lambda^2_1)^\flat \bigcup_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp =: Y.$$

Here the marked simplicial set over $N\mathcal{C}$ in the middle witnesses f as the composition of f and id_y. It is thus sufficient to check that the component j_Y is a weak equivalence. Note that Y is the homotopy pushout of the cospan

$$(\Delta^{\{0,1\}})^\sharp \leftarrow \{1\} \to (\Delta^{\{1,2\}})^\flat.$$

The edge $(\Delta^{\{1,2\}})^\flat$ of Y corresponds to id_y. Hence, it may be written as the tensor product of the marked simplicial set $(\Delta^1)^\flat$ with the point y over $N\mathcal{C}$. Since the functors L, $h_!$ and $r_!$, and the natural transformation j are compatible with tensoring marked simplicial sets, and the model structure on $(\mathbf{sSet}^+)_{/N\mathcal{C}}$ is a model category enriched over marked simplicial sets, we deduce that $j_{(\Delta^{\{1,2\}})^\flat}$ is a trivial cofibration when j_{Δ^0} is. We have already seen that showing that $j_{(\Delta^{\{0,1\}})^\sharp}$ is a trivial cofibration may be reduced to this case as well. All in all, we deduce that it is sufficient to check the statement for $X = \Delta^0$.

For $X = \Delta^0$ corresponding to an object c of $\mathcal{C}$, the inclusion j_{Δ^0} may be identified with the $\sharp$-marked nerve of the functor

$$\mathcal{C}_{c/} \to [0] \times_{\mathcal{C}} \mathcal{C}^{[1]} : (c \to x) \mapsto (c \to x)$$

over $\mathcal{C}$. It is clear that this functor is an isomorphism of categories, thus j_{Δ^0} is an isomorphism as well. □

2.3 Application: the homotopy invariance of the coCartesian model structure

In this section we will tactfully exploit the straightening and unstraightening equivalence to finish the proof of Theorem 1.3.4, as promised.

Proof of Theorem 1.3.4. In view of Lemma 1.3.5 and Lemma 1.3.6, it suffices to prove the theorem for an inner horn inclusion $i : \Lambda^n_k \to \Delta^n$. Note that i induces a bijection on vertices, hence a map $X^\natural \to Y^\natural$ in $(\mathbf{sSet}^+)_{/(\Delta^n)}$ is a coCartesian equivalence precisely when $i^*X^\natural \to i^*Y^\natural$ is a coCartesian equivalence on account of Proposition 1.1.10. Thus $\mathbf{R}i^*$ is conservative. Hence it remains to show that for any marked simplicial set X over Λ^n_k, the unit $X \to i^*(i_!X)_f$ is a weak equivalence. Here $(i_!X)_f$ denotes a fibrant replacement for $i_!X$. We may assume that $X = X^\natural$ is a coCartesian fibration over Λ^n_k.

Pick a projectively fibrant diagram $F : [n] \to \mathbf{sSet}^+$ and a weak equivalence $r_! i_! X^\natural \to F$. In view of Theorem 2.2.2, the adjoint map $i_! X^\natural \to r^* F$ is a coCartesian equivalence. We must demonstrate that the adjoint map $X^\natural \to i^* r^* F$ is a coCartesian equivalence. Note that this map factors as $X^\natural \to i^* r^* r_! i_! X^\natural \to i^* r^* F$. Passing to fibers, it suffices to check that the map

$$X^\natural_m \to (r^* r_! i_! X^\natural)_m \to (r^* F)_m$$

is a coCartesian equivalence in $\mathbf{sSet}^+$ for any vertex m of Λ^n_k on account of Proposition 1.1.10. The map on the right is isomorphic to the map $(r_! i_! X^\natural)(m) \to F(m)$, which is a coCartesian equivalence by assumption. The map on the left is isomorphic to the inclusion of fiber

$$(i_! X^\natural)_m \to i_! X^\natural \times_{(\Delta^n)^\sharp} (\Delta^{\{0,\ldots,m\}})^\sharp.$$

This map corresponds to the map $X^\natural_m \to X^\natural \times_{(\Lambda^n_k)^\sharp} (\Delta^{\{0,\ldots,m\}})^\sharp$ if $m < n$, and to the map $X^\natural_m \to X^\natural$ if $m = n$. Thus the result follows from applying Corollary 1.1.17 to the right anodyne maps $\{m\} \to \Delta^{\{0,\ldots,m\}}$ and $\{n\} \to \Lambda^n_k$. □

2.4 Application: categorical fibrations and coCartesian fibrations

Recall that the categorical fibrations are the fibrations in the Joyal model structure. These fibrations are hard to understand, however whenever the base of the fibration happens to be an ∞-category, these fibrations are precisely the isofibrations:

Definition 2.4.1. An isofibration $X \to S$ is an inner fibration which has the right lifting property with respect to the inclusion $\{0\} \to J$.

It readily follows that any coCartesian fibration is an isofibration. Consequently, we deduce that coCartesian fibrations on ∞-categories are categorical fibrations. This result can be extended to any base:

Proposition 2.4.2. *Every coCartesian fibration is a categorical fibration.*

This was already shown by Lurie in [Lur09, Proposition 3.3.1.7]. We will give a different proof of this fact. The statement will readily follow from the fact that coCartesian fibrations can be extended along (in particular) inner anodynes:

Proposition 2.4.3. *Let $i : S \to T$ be a trivial cofibration of simplicial sets in the Joyal model structure on simplicial sets. Then every coCartesian fibration $X \to S$ extends to a coCartesian fibration on T. I.e., there exists a coCartesian fibration $Y \to T$ accompanied by a pullback square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & T. \end{array}$$

Proof. In view of the theory of minimal coCartesian fibrations, the fact that $\mathbf{R}i^*$ is essentially surjective (see Theorem 1.3.4) and the observation that i^* preserves minimality (this is Proposition 1.3.2), we can find minimal coCartesian fibrations $M \to S$ and $N \to T$, an isomorphism $i^*N^\natural \to M^\natural$ and a trivial fibration $X^\natural \to M^\natural$. We now define Y by the pullback square

$$\begin{array}{ccc} Y & \longrightarrow & i_*X^\natural \\ \downarrow & & \downarrow \\ N^\natural & \longrightarrow & i_*M^\natural \end{array}$$

in $(\mathbf{sSet}^+)_{/T}$. Note that the left adjoint i^* of i_* preserves cofibrations. Hence the map $i_*X^\natural \to i_*M^\natural$ is again a trivial fibration. It follows that the map $Y \to N^\natural$ is a trivial fibration. Thus in particular, Y is fibrant. Thus we may write Y as a marked simplicial set $Y^\natural$ over T corresponding to a coCartesian fibration $Y \to T$.

Consider now the commutative diagram

$$\begin{array}{ccccc} i^*Y^\natural & \longrightarrow & i^*i_*X^\natural & \longrightarrow & X^\natural \\ \downarrow & & \downarrow & & \downarrow \\ i^*N^\natural & \longrightarrow & i^*i_*M^\natural & \longrightarrow & M^\natural \end{array}$$

obtained by applying i^* to the first pullback square, and using the naturality of the counit of (i^*, i_*) on the right. Then the left square is a pullback square since i^* is a right adjoint. We claim that the right square is a pullback square as well. Namely, since i is a cofibration, the functor $i_!$ is fully faithful. As the tuple $(i_!, i^*, i_*)$ is an adjoint triple, it follows that i_* is fully faithful as well. Thus the counit of (i^*, i_*) is an isomorphism, from which the claim follows. All in all, we deduce that the outer square is a pullback square. Since the arrow $i^*N^\natural \to M^\natural$ is an isomorphism, the arrow $i^*Y^\natural \to X^\natural$ is an isomorphism as well. This entails that the underlying map $Y \to T$ of $Y^\natural$ is the desired coCartesian fibration. □

Proof of Proposition 2.4.2. The proof is now elementary. Let $X \to S$ be a coCartesian fibration. Then there exists an inner anodyne map $S \to \mathcal{C}$ such that $\mathcal{C}$ is an ∞-category. On account of Proposition 2.4.3, there exists a coCartesian fibration $Y \to \mathcal{C}$ and a pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{C}. \end{array}$$

In particular, the map $Y \to \mathcal{C}$ is an isofibration and hence a categorical fibration since $\mathcal{C}$ is an ∞-category. It follows that the map $X \to S$ is a categorical fibration as well. □

Corollary 2.4.4. *The adjunction*

$$(-)^\flat : \mathbf{sSet}_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/S} : (-)_\flat$$

is a Quillen adjunction, where $\mathbf{sSet}_{/S}$ *is endowed with the model structure induced by the Joyal model structure.*

Proof. This is shown by Lurie in [Lur09, Proposition 3.1.5.3]. □

Remark 2.4.5. Passing to underlying ∞-categories, we obtain a free-forgetful adjunction

$$\mathbf{L}(-)^\flat : (\mathbf{Cat}_\infty)_{/\mathcal{C}} \rightleftarrows \mathbf{coCart}(\mathcal{C}) : \mathbf{R}(-)_\flat =: U$$

for any ∞-category $\mathcal{C}$. We may identify the left adjoint $\mathbf{L}(-)^\flat$ with the the functor F that carries an ∞-category $\mathcal{D}$ over $\mathcal{C}$ to the free coCartesian fibration $F(\mathcal{D})$ on $\mathcal{C}$ (as introduced in [GHN17]). As a fibrant object of $(\mathbf{sSet}^+)_{/\mathcal{C}}$, this free coCartesian fibration $F(\mathcal{D})^\natural$ is decribed by the pullback square

$$\begin{array}{ccc} F(\mathcal{D})^\natural & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C})^\sharp \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ (\mathcal{D}, \Sigma) & \longrightarrow & \mathcal{C}^\sharp \end{array}$$

of marked simplicial sets, with the projection map to $\mathcal{C}^\sharp$ being induced by evaluation in 1. Here Σ denotes the collection of equivalences in $\mathcal{D}$. This indeed defines a fibrant object in view of [Lur09, Corollary 2.4.7.12]. To see that we can identify these functors, we note that there is a natural transformation

$$\begin{array}{ccccc} & & \xrightarrow{\qquad (-)^\flat \qquad} & & \\ (\mathbf{sSet}_{/\mathcal{C}})_f & & \Downarrow & & (\mathbf{sSet}^+)_{/\mathcal{C}}, \\ & \searrow^{F} & & \nearrow & \\ & & ((\mathbf{sSet}^+)_{/\mathcal{C}})_f & & \end{array}$$

whose components are marked anodynes. Namely, the inclusion $\mathcal{D}^\flat \to (\mathcal{D}, \Sigma)$ is marked anodyne as every map in Σ may be extended to the Kan complex J, and the natural inclusion $(\mathcal{D}, \Sigma) \to F(\mathcal{D})^\natural$ is marked anodyne (see the proof of Proposition 2.2.5). Passing to the nerves and localizing along weak equivalences (we will give a recollection of localizations of ∞-categories in the next chapter), we obtain an invertible 1-cell

$$\begin{array}{ccccc} & \nearrow^{\mathbf{L}(-)^\flat} & \mathbf{coCart}(\mathcal{C}) & \searrow & \\ (\mathbf{Cat}_\infty)_{/\mathcal{C}} & & \Downarrow \simeq & & N((\mathbf{sSet}^+)_{/\mathcal{C}})[W^{-1}], \\ & \searrow_{F} & \mathbf{coCart}(\mathcal{C}) & \nearrow & \end{array}$$

in the appropriate functor ∞-category. Here W denotes the (large) set of weak equivalences of $(\mathbf{sSet}^+)_{/\mathcal{C}}$ and we realized $\mathbf{coCart}(\mathcal{C})$ as the localization of the full subcategory of $(\mathbf{sSet}^+)_{/\mathcal{C}}$ spanned by its fibrant objects, w.r.t. the weak equivalences between fibrants. The inclusion $\mathbf{coCart}(\mathcal{C}) \to N((\mathbf{sSet}^+)_{/\mathcal{C}})[W^{-1}]$ admits a retraction, yielding the desired equivalence $\mathbf{L}(-)^\flat \simeq F$. In particular, we deduce that the pair (F, U) forms an adjunction of ∞-categories. This is proven in a more intrinsic fashion by Gepner, Haugseng and Nikolaus in [GHN17, Theorem 4.5].

We will exploit the fact that the pair $((-)^\flat, (-)_\flat)$ forms a Quillen adjunction in the proof of Proposition 1.3.7, which we were still indebted to the reader:

Proof of Proposition 1.3.7. We more or less follow the proof of this fact by Nguyen. It is clear that f^* preserves cofibrations. Moreover, it follows from Proposition 1.1.16 that f^* carries cellular marked anodynes to marked anodynes. Thus it remains to check that f^* carries maps of the form Definition 1.1.5(i), (iv) to trivial cofibrations.

Let us commence by checking the assertion for generator (iv). To this end, we may assume that T is a Kan complex and it suffices to check that $T^\flat \times_{T^\sharp} S^\sharp \to S^\sharp$ is a trivial cofibration in $(\mathbf{sSet}^+)_{/T}$. Note that S is again a Kan complex, and that the underlying simplicial set of $T^\flat \times_{T^\sharp} S^\sharp$ is given by S. Hence, it suffices to show that $i : (S, \Sigma) \to S^\sharp$ is marked anodyne for any Kan complex S and any set Σ of markings on S. Note that any edge of S can be extended to J. Hence, the map i may be written as a pushout along (multiple copies of) $J^\flat \to J^\sharp$, from which it follows that i is marked anodyne.

Finally, we must check that the claim holds for a generator of the form $(\Lambda^n_k)^\flat \to (\Delta^n)^\flat$. We may assume that $T = (\Delta^n)^\sharp$ and we should check that the map $(\Lambda^n_k)^\flat \times_{(\Delta^n)^\sharp} S^\sharp \to (\Delta^n)^\flat \times_{(\Delta^n)^\sharp} S^\sharp$ is a trivial cofibration in $(\mathbf{sSet}^+)_{/\Delta^n}$. Note that this map is precisely the map $(\Lambda^n_k \times_{\Delta^n} S, \Sigma) \to (S, \Sigma)$, where Σ are the equivalences of S. The following square commutes

$$\begin{array}{ccc} (\Lambda^n_k \times_{\Delta^n} S)^\flat & \longrightarrow & S^\flat \\ \downarrow & & \downarrow \\ (\Lambda^n_k \times_{\Delta^n} S, \Sigma) & \longrightarrow & (S, \Sigma), \end{array}$$

and both vertical arrows are pushouts along possibly multiple copies of the map $J^\flat \to J^\sharp$. Hence it suffices to show that the top arrow is a coCartesian equivalence in $(\mathbf{sSet}^+)_{/\Delta^n}$. In view of Corollary 2.4.4, it is enough to show that the underlying map $\Lambda^n_k \times_{\Delta^n} S \to S$ is a categorical equivalence, and this follows from [HM15, Lemma 7.3]. □

2.5 Application: a marked version of Quillen's theorem A

As a final application of all the machinery developed untill now, we shall prove the following version of Quillen's theorem A:

Theorem 2.5.1. *Let $f : X \to Y$ be a map of marked simplicial sets over an ∞-category $\mathcal{C}$. Then f is a coCartesian equivalence if and only if the induced map*

$$X \times_{\mathcal{C}^\sharp} \mathcal{C}^\sharp_{/c} \to Y \times_{\mathcal{C}^\sharp} \mathcal{C}^\sharp_{/c}$$

is a coCartesian equivalence of marked simplicial sets for every object c of $\mathcal{C}$.

Proof. The 'only if' statement follows from the fact that we have functors

$$(\mathbf{sSet}^+)_{/\mathcal{C}} \to (\mathbf{sSet}^+)_{/\mathcal{C}_{/c}} \to \mathbf{sSet}^+.$$

The functor on the right is induced by the terminal map and left Quillen. The functor on the left is given by pullback along the right fibration $\mathcal{C}_{/c} \to \mathcal{C}$, which is left Quillen on account of Proposition 1.3.7.

Let us proceed to show the remaining assertion. On account of what we have shown above, we may replace X and Y by $X^\natural$ and $Y^\natural$ where $X, Y \to \mathcal{C}$ are coCartesian fibrations. Then it suffices to show that f descends to a coCartesian equivalence on fibers $X^\natural_c \to Y^\natural_c$ for all objects c of $\mathcal{C}$. This follows from the fact that the fiber $Z^\natural_c$ has the same homotopy type of the 'thickened' fiber $Z^\natural \times_{\mathcal{C}^\sharp} \mathcal{C}^\sharp_{/c}$ for any coCartesian fibration $Z \to \mathcal{C}$. Namely, we have a pullback square

$$\begin{array}{ccc} Z^\natural_c & \longrightarrow & Z^\natural \times_{\mathcal{C}^\sharp} \mathcal{C}^\sharp_{/c} \\ \downarrow & & \downarrow \\ \{c\} & \longrightarrow & \mathcal{C}^\sharp_{/c}, \end{array}$$

and the (natural) top arrow is a coCartesian equivalence in view of Corollary 1.1.17 and the fact that the map $\{c\} \to \mathcal{C}_{/c}$ is final thus right anodyne. ☐

It is readily verified that the above theorem recovers the following versions of Quillen's theorem A found in the literature.

Corollary 2.5.2 (Proposition G of [HM15])**.** *Let $f : X \to Y$ be a map of simplicial sets over an ∞-category $\mathcal{C}$. Then f is a covariant equivalence if and only if the induced map*

$$X \times_{\mathcal{C}} \mathcal{C}_{/c} \to Y \times_{\mathcal{C}} \mathcal{C}_{/c}$$

is a weak homotopy equivalence for every object c of $\mathcal{C}$.

Corollary 2.5.3 (Theorem 4.1.3.1 of [Lur09])**.** *Let $f : X \to \mathcal{C}$ be a map of simplicial sets, where $\mathcal{C}$ is an ∞-category. Then f is an initial map if and only if the simplicial set $X \times_{\mathcal{C}} \mathcal{C}_{/c}$ is weakly contractible for every object c of $\mathcal{C}$.*

CHAPTER 3

Localizing bases

In this chapter, we will study the relation between coCartesian fibrations on ∞-categories and the coCartesian fibrations on their localizations. We will also recall the similar results for simplicial functors valued in $\mathbf{sSet}^+$ (or in general, taking values in any simplicial model category). We will see that in both cases, the corresponding model categories of the objects on localizations can be described by a left Bousfield localization of the model categories of the objects on the original ∞-category or simplicial category.

The motivation for this pursuit is a result due to Joyal (see Theorem 4.3.2), which states that any ∞-category $\mathcal{C}$ can be obtained as a localization of a 1-category $\mathcal{D}$. The rectification functors for 1-categories defined in Chapter 2 can be extended to arbitrary ∞-categories. The following square of ∞-functors

$$\begin{array}{ccc} \mathbf{coCart}(\mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathbf{Cat}_\infty) \\ \downarrow & & \downarrow \\ \mathbf{coCart}(\mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty) \end{array}$$

commutes up to equivalence in $\mathrm{Fun}(\mathbf{coCart}(\mathcal{D}), \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty))$. We will use this fact to conclude that the bottom functor is also part of an adjoint equivalence. We will pursue this idea by modelling the above square as left Quillen functors of the appropriate model categories. The idea of this approach is due to Stevenson, who used this strategy in [Ste17] to prove the straightening and unstraightening equivalence for left fibrations.

3.1 Localizations of ∞-categories

Recall that given a category $\mathcal{C}$ and a subcategory W of $\mathcal{C}$, we can consider the localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ of $\mathcal{C}$. This localization $\mathcal{C}[W^{-1}]$ might not be a locally small category in the universe we are working in. However, if W happens to be a small subcategory, the localization may be constructed by taking the pushout of the inclusion $W \to \mathcal{C}$ along the groupoid completion $W \to W[W^{-1}]$ of W, where $(-)[(-)^{-1}]$ is the left adjoint to the inclusion $\mathbf{Grpd} \to \mathbf{Cat}$. Here $\mathbf{Cat}$ denotes the category of small categories and $\mathbf{Grpd}$ denotes the full subcategory of $\mathbf{Cat}$ spanned by the small groupoids. It follows directly from this description of

$\mathcal{C}[W^{-1}]$ that the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is (strictly) initial among the functors $\mathcal{C} \to \mathcal{D}$ which carries the maps in W to isomorphisms in $\mathcal{D}$.

A suitable analog for this universal property in the ∞-categorical setting will characterize localizations of ∞-categories:

Definition 3.1.1. Let $(\mathcal{C}, W)$ be a pair of ∞-categories. A localization of $\mathcal{C}$ by W is a functor $\mathcal{C} \to \mathcal{D}$ with the property that for any ∞-category $\mathcal{E}$, the induced functor $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ factors through the subcategory $\mathrm{Fun}_W(\mathcal{C}, \mathcal{E}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{E})$,

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_W(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

such that the first arrow is a categorical equivalence. Here $\mathrm{Fun}_W(\mathcal{C}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by the maps $\mathcal{C} \to \mathcal{E}$ which maps the maps in W to equivalences in $\mathcal{E}$. Equivalently, $\mathrm{Fun}_W(\mathcal{C}, \mathcal{E})$ is the ∞-category defined by the pullback square

$$\begin{array}{ccc} \mathrm{Fun}_W(\mathcal{C}, \mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Map}^\flat(W^\sharp, \mathcal{E}^\natural) & \longrightarrow & \mathrm{Fun}(W, \mathcal{E}). \end{array}$$

We will establish the existence of localizations shortly. The following proposition asserts that localizations are unique up to categorical equivalence (in fact, up to unique isomorphism in $\mathrm{Ho}(\mathbf{sSet}_{\mathrm{Joyal}})$):

Proposition 3.1.2. *Let $(\mathcal{C}, W)$ be a pair of ∞-categories. A localization $f : \mathcal{C} \to \mathcal{D}$ witnesses $\mathcal{D}$ as the representation of the presheaf*

$$\pi_0(\mathrm{Fun}_W(\mathcal{C}, -)^\natural)_\sharp : \mathrm{Ho}(\mathbf{sSet}_{\mathrm{Joyal}}) \to \mathbf{Set}.$$

Proof. We have to show that for any ∞-category $\mathcal{E}$, the map

$$\mathrm{Map}^\sharp(\mathcal{D}^\natural, \mathcal{E}^\natural) \to (\mathrm{Fun}_W(\mathcal{C}, \mathcal{E})^\natural)_\sharp$$

induced by f descends to an isomorphism on π_0. But this follows directly from the fact that f gives rise to a weak homotopy equivalence

$$\mathrm{Map}^\sharp(\Delta^0, \mathrm{Fun}(\mathcal{D}, \mathcal{E})^\natural) \to \mathrm{Map}^\sharp(\Delta^0, \mathrm{Fun}_W(\mathcal{C}, \mathcal{E})^\natural)$$

since f is a localization, and this map is isomorphic to the map above. $\square$

Proposition 3.1.3. *Let $(\mathcal{C}, W)$ be a pair of ∞-categories. Suppose that Σ is a set of arrows in W which span the 1-truncation* $\mathrm{Ho}\, W$. *Then any functor $\mathcal{C} \to \mathcal{D}$ of ∞-categories that fits in a homotopy pushout square*

$$\begin{array}{ccc} \Sigma \times \Delta^1 & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \Sigma \times J & \longrightarrow & \mathcal{D}, \end{array}$$

is a localization of $\mathcal{C}$ by W.

Proof. Denote the pushout $\Sigma \times J \bigcup_{\Sigma\times\Delta^1} W$ by $\widetilde{W}$. On account of the pasting law, the map $\mathcal{C} \to \mathcal{D}$ fits in a homotopy pushout square

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \widetilde{W} & \longrightarrow & \mathcal{D}. \end{array}$$

Consequently, for any ∞-category $\mathcal{E}$, we obtain a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{D},\mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\widetilde{W},\mathcal{E}) & \longrightarrow & \mathrm{Fun}(W,\mathcal{E}). \end{array}$$

Note that $\mathrm{Fun}(\widetilde{W},\mathcal{E}) = \mathrm{Map}^\flat(\widetilde{W}^\sharp,\mathcal{E}^\natural)$. This follows from the fact that any arrow $f : \Delta^1 \to \mathcal{E}$ that factors over $\widetilde{W}$, is an equivalence in $\mathcal{E}$. Namely, the truncation functor $\mathrm{Ho}(-)$ carries homotopy pushout squares of ∞-categories to homotopy pushouts of categories, hence $\pi_0\mathfrak{C}\widetilde{W}$ is equivalent to the pushout of

$$\Sigma \times J \leftarrow \Sigma \times [1] \to \mathrm{Ho}\, W.$$

But this is precisely $\mathrm{Ho}\, W[\Sigma^{-1}] = \mathrm{Ho}\, W[(\mathrm{Ho}\, W)^{-1}]$. Consequently, the map $\mathrm{Ho} f : [1] \to \mathrm{Ho}\,\mathcal{E}$ factors through a groupoid. Thus f is an equivalence.

Note that the map $W \to \widetilde{W}$ is a pushout of a trivial Kan-Quillen cofibration, hence again a trivial Kan-Quillen cofibration. After taking sharps of this map, we obtain a trivial cofibration in $\mathbf{sSet}^+$. Thus the induced map

$$\mathrm{Fun}(\widetilde{W},\mathcal{E}) = \mathrm{Map}^\flat(\widetilde{W}^\sharp,\mathcal{E}^\natural) \to \mathrm{Map}^\flat(W^\sharp,\mathcal{E}^\natural)$$

is a categorical equivalence. This in turn shows that the map $\mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$ factors through the subcategory $\mathrm{Fun}_W(\mathcal{C},\mathcal{E})$ and, moreover, since the pullback square defining $\mathrm{Fun}_W(\mathcal{C},\mathcal{E})$ is a homotopy pullback square, it follows that the map $\mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}_W(\mathcal{C},\mathcal{E})$ is a categorical equivalence. □

For once and for all, we will fix a functorial localization

$$(-)[(-)^{-1}] : \mathbf{Pairs}(\mathbf{sSet}^\circ_{\mathrm{Joyal}}) \to (\mathbf{sSet}^\circ_{\mathrm{Joyal}})^{[1]}.$$

It is defined as follows. We define a functor $L(-,-)$ valued in simplicial sets by the strict pushouts

$$L(\mathcal{C},W) := \mathcal{C} \bigcup_W LW, \quad LW := W \bigcup_{\mathrm{sk}_1 W\times\Delta^1} \mathrm{sk}_1 W \times J.$$

The small objects argument asserts that there exists a natural transformation $\mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ such that $S \to RS$ is inner anodyne and RS is an ∞-category for all simplicial sets S. We now set

$$\mathcal{C}[W^{-1}] := RL(\mathcal{C},W).$$

It follows from Proposition 3.1.3 that this defines a localization of $\mathcal{C}$ by W.

3.2 Localizations of simplicial categories

We will give a quick overview of the theory of localizations of simplicial categories. This theory is mainly due to Dwyer and Kan; they wrote a series of papers in the eighties on localization procedures for simplicial categories. We will conclude this section by discussing how the localization of simplicial categories relate to the localization of simplicial sets.

Let **sCat** denote the simplicial category of small simplicial categories. Denote the subcategory of small simplicial groupoids by **sGrpd**. By general nonsense, the inclusion functor **sGrpd** $\subset$ **sCat** admits a left adjoint, which we will denote by $(-)[(-)^{-1}]$.

Definition 3.2.1. Let $(\mathcal{C}, W)$ be a pair of simplicial categories. Then the Dwyer-Kan localization $\mathcal{C}[W^{-1}]$ of the pair $(\mathcal{C}, W)$ is defined by the pushout square

$$\begin{array}{ccc} W & \longrightarrow & W[W^{-1}] \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[W^{-1}]. \end{array}$$

The following assertion asserts that the localization of ∞-categories defined in the previous section, coincides with the localization of simplicial categories.

Proposition 3.2.2. *Let $(\mathcal{C}, W)$ be a pair of ∞-categories. Then the simplicial categories $\mathfrak{C}\mathcal{C}[W^{-1}]$ and $\mathfrak{C}\mathcal{C}[(\mathfrak{C}W)^{-1}]$ are naturally DK-equivalent over $\mathfrak{C}\mathcal{C}$.*

Proof. Since the map $\mathcal{C} \to \mathcal{C}[W^{-1}]$ factors as

$$\mathcal{C} \to L(\mathcal{C}, W) \to \mathcal{C}[W^{-1}],$$

and the latter arrow is a categorical equivalence, it is sufficient to show that $\mathfrak{C}L(\mathcal{C}, W)$ and $\mathfrak{C}\mathcal{C}[(\mathfrak{C}W)^{-1}]$ are naturally DK-equivalent over $\mathfrak{C}\mathcal{C}$. In view of the definition of Dwyer-Kan localizations, and the left properness of **sCat**, we deduce that it suffices to exhibit a natural DK-equivalence between $\mathfrak{C}LW$ and $\mathfrak{C}W[(\mathfrak{C}W)^{-1}]$.

We proceed as follows. Consider the following commutative square

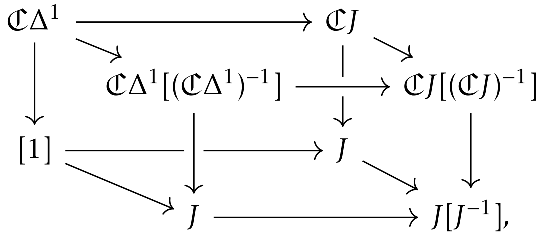
$$\begin{array}{ccc} \mathfrak{C}W & \longrightarrow & \mathfrak{C}LW \\ \downarrow & & \downarrow \\ \mathfrak{C}W[(\mathfrak{C}W)^{-1}] & \longrightarrow & \mathfrak{C}LW[(\mathfrak{C}LW^{-1}]. \end{array}$$

The category $\pi_0\mathfrak{C}LW$ is a groupoid (see the proof of Proposition 3.1.3). It follows from this observation and Proposition 10.4 of [DK80] that the right functor is a DK-equivalence. It remains to show that the bottom map is a DK-equivalence. Since $\mathfrak{C}$ and $(-)[(-)^{-1}]$ are left adjoints, we obtain the following pushout square

$$\begin{array}{ccc} \mathrm{sk}_1 W \times \mathfrak{C}\Delta^1[(\mathfrak{C}\Delta^1)^{-1}] & \longrightarrow & \mathfrak{C}W[(\mathfrak{C}W)^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{sk}_1 W \times \mathfrak{C}J[(\mathfrak{C}J)^{-1}] & \longrightarrow & \mathfrak{C}LW[(\mathfrak{C}LW)^{-1}]. \end{array}$$

It thus suffices to show that the map $\mathfrak{C}\Delta^1[(\mathfrak{C}\Delta^1)^{-1}] \to \mathfrak{C}J[(\mathfrak{C}J)^{-1}]$ is a trivial cofibration in the model structure on simplicial groupoids (see [DK84]).

This follows from a straightforward argument. Namely, we have the following commutative cube



where the vertical arrows of the back face are the natural DK-equivalences $\mathfrak{C}N(-) \to (-)$, and the front face is obtained by applying the $(-)[(-)^{-1}]$ functor to the back face. Note that the arrow $\mathfrak{C}\Delta^1 \to [1]$ is an isomorphism. Hence the arrow $\mathfrak{C}\Delta^1[(\mathfrak{C}\Delta^1)^{-1}] \to J$ is an isomorphism as well. The map $\mathfrak{C}J \to \mathfrak{C}J[(\mathfrak{C}J)^{-1}]$ is a DK-equivalence because J is a Kan complex. Note that the map $J \to J[J^{-1}]$ is an equivalence of categories, thus in particular a DK-equivalence. Since the map $\mathfrak{C}J \to J$ is a DK-equivalence as well, it follows from applying the 2-out-of-3 property twice that the map $\mathfrak{C}\Delta^1[(\mathfrak{C}\Delta^1)^{-1}] \to \mathfrak{C}J[(\mathfrak{C}J)^{-1}]$ is a DK-equivalence.

To check that the map is a cofibration in **sGrpd** as well, it suffices to prove the stronger statement that the functor $(-)[(-)^{-1}] : \mathbf{sCat} \to \mathbf{sGrpd}$ preserves cofibrations. It suffices to check this for generators of relative subcomputads (see Definition A.0.2), and this readily follows from [DK84, Proposition 2.9]. □

3.3 coCartesian fibrations on localizations

In this section, we will study coCartesian fibrations on localizations of ∞-categories. More generally, we fix a simplicial set S, a set of edges $\Sigma \subset S_1$, and consider coCartesian fibrations on the (strict) pushout

$$\begin{array}{ccc} \Sigma \times \Delta^1 & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle i} \\ \Sigma \times J & \longrightarrow & S[\Sigma^{-1}]. \end{array}$$

Recall that for any coCartesian fibration $X \to S$, we can construct lifts lying above edges of S (see Section 1.4). The following observation will turn out to be crucial:

Proposition 3.3.1. *Let $f : s \to t$ be an edge of S. The following assertions are equivalent:*

(i) any lift $f_! : X_s \to X_t$ is a categorical equivalence,
(ii) there exists a lift $f_! : X_s \to X_t$ which is a categorical equivalence,

(iii) for any set I of generating cofibrations for **sSet**$^+$*, the maps induced by the cofibrations $A \to B$ in I and the edge $f : \Delta^1 \to S$*

$$(\Delta^1)^\sharp \times A \bigcup_{\{1\}\times A} \{1\} \times B \to (\Delta^1)^\sharp \times B$$

give rise to a homotopy equivalences on mapping complexes after applying $\mathrm{Map}^\sharp_S(-, X^\natural)$.

Proof. That (i) and (ii) are equivalent is clear, since any two lifts of f are homotopic in $\mathrm{Map}^\sharp(X^\natural_s, X^\natural_t)$.

To see that (iii) is equivalent to (ii), we recall that a lift $f_!$ is determined by a homotopy $H : (\Delta^1)^\sharp \times X^\natural_s \to X^\natural$ which restricts to the inclusion $X^\natural_s \to X^\natural$ on $\{0\} \times X^\natural_s$. The lift $f_!$ gives rise to a section of the trivial Kan fibration

$$\mathrm{Map}^\sharp_S((\Delta^1)^\sharp \times A, X^\natural) \to \mathrm{Map}^\sharp_S(\{0\} \times A, X^\natural) \cong \mathrm{Map}^\sharp(A, X^\natural_s)$$

for any marked simplicial set A. Namely, we send a n-simplex $A \times (\Delta^n)^\sharp \to X^\natural_s$ to the composite $(\Delta^1)^\sharp \times A \times (\Delta^n)^\sharp \to (\Delta^1)^\sharp \times X^\natural_s \to X^\natural$. It is readily verified that the composition

$$\mathrm{Map}^\sharp_S(\{0\} \times A, X^\natural) \to \mathrm{Map}^\sharp_S((\Delta^1)^\sharp \times A, X^\natural) \to \mathrm{Map}^\sharp(\{1\} \times A, X^\natural)$$

is isomorphic to the map $\mathrm{Map}^\sharp(A, X^\natural_s) \to \mathrm{Map}^\sharp(A, X^\natural_t)$ induced by $f_!$. For any cofibration $i : A \to B$ in I, the following square commutes

$$\begin{array}{ccc}
\mathrm{Map}^\sharp_S(\{0\} \times B, X^\natural) & \longrightarrow & \mathrm{Map}^\sharp(\{0\} \times A, X^\natural) \times_{\mathrm{Map}^\sharp_S(\{1\}\times A, X^\natural)} \mathrm{Map}^\sharp_S(\{1\} \times B, X^\natural) \\
\downarrow & & \downarrow \\
\mathrm{Map}^\sharp_S((\Delta^1)^\sharp \times B, X^\natural) & \longrightarrow & \mathrm{Map}^\sharp((\Delta^1)^\sharp \times A, X^\natural) \times_{\mathrm{Map}^\sharp_S(\{1\}\times A, X^\natural)} \mathrm{Map}^\sharp_S(\{1\} \times B, X^\natural).
\end{array}$$

Note that the vertical arrows are homotopy equivalences, since the pullbacks occuring on the right are homotopy pullbacks. The top arrow is isomorphic to the map

$$i \pitchfork f_! : \mathrm{Map}^\sharp(B, X^\natural_s) \to \mathrm{Map}^\sharp(A, X^\natural_s) \times_{\mathrm{Map}^\sharp(B, X^\natural_s)} \mathrm{Map}^\sharp(B, X^\natural_t).$$

Hence the result follows from Lemma B.0.4. □

Definition 3.3.2. We define the simplicial model category of Σ-local coCartesian fibrations

$$L_\Sigma(\mathbf{sSet}^+)_{/S}$$

to be the left Bousfield localization of $(\mathbf{sSet}^+)_{/S}$ along the maps in Proposition 3.3.1(ii) with f ranging in Σ. The (fibrant) objects that are local with respect to these maps, will be called Σ-local. If $(\mathcal{C}, W)$ is a pair of ∞-categories, then we agree to write

$$L_W(\mathbf{sSet}^+)_{/\mathcal{C}} := L_{\mathrm{sk}_1 W}(\mathbf{sSet}^+)_{/\mathcal{C}}.$$

The coCartesian model structure is compatible with localizations in the following sense:

Theorem 3.3.3. *The base change Quillen adjunction* $(i_!, i^*)$ *descends to a Quillen equivalence*

$$i_! : L_\Sigma(\mathbf{sSet}^+)_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/S[\Sigma^{-1}]} : i^*.$$

Corollary 3.3.4. *Suppose that* $f : \mathcal{C} \to \mathcal{D}$ *is a functor witnessing* $\mathcal{D}$ *as the localization of a pair of* ∞*-categories* $(\mathcal{C}, W)$*. Then the base change Quillen adjunction* $(f_!, f^*)$ *descends to a Quillen equivalence*

$$f_! : L_W(\mathbf{sSet}^+)_{/\mathcal{C}} \rightleftarrows (\mathbf{sSet}^+)_{/\mathcal{D}} : f^*.$$

Proof. In view of Proposition 3.1.2, it suffices to show the statement for the particular model $\mathcal{D} = \mathcal{C}[W^{-1}]$ of the localization. In this case, f factors as $\mathcal{C} \to L(\mathcal{C}, W) \to \mathcal{C}[W^{-1}]$, where the latter map is inner anodyne. Thus an application of Theorem 3.3.3 and Theorem 1.3.4 gives the desired result. □

We shift our attention to the proof of Theorem 3.3.3. The localization $S[\Sigma^{-1}]$ is constructed as a pushout along the localization

$$j : \Delta^1 \to J.$$

Consequently, the adjunction $(i_!, i^*)$ can be understood by looking at the base change $(j_!, j^*)$.

Lemma 3.3.5. *Let* $X \to \Delta^1$ *be a coCartesian fibration. The following assertions are equivalent:*

(i) $X^\natural$ *is* Δ^1_1*-local,*
(ii) $X^\natural$ *is classified by a coCartesian equivalence in* $\mathbf{sSet}^+$*,*
(iii) $X^\natural$ *lies in the essential image of the right derived functor*

$$\mathbf{R}r^*\mathbf{R}j^* : \mathrm{Ho}((\mathbf{sSet}^+)^J) \to \mathrm{Ho}((\mathbf{sSet}^+)_{/\Delta^1}),$$

(iv) $X^\natural$ *lies in the essential image of the right derived functor*

$$\mathbf{R}j^* : \mathrm{Ho}((\mathbf{sSet}^+)_{/J}) \to \mathrm{Ho}((\mathbf{sSet}^+)_{/\Delta^1}).$$

Proof. During this proof, we denote the unique non-trivial arrow in $[1]$ by f. It is clear that (iii) and (iv) are equivalent because the following diagram of right Quillen functors commutes up to natural isomorphism

$$\begin{array}{ccc} (\mathbf{sSet}^+)^J & \xrightarrow{j^*} & (\mathbf{sSet}^+)^{[1]} \\ {\scriptstyle r^*}\downarrow & & \downarrow{\scriptstyle r^*} \\ (\mathbf{sSet}^+)_{/J} & \xrightarrow{j^*} & (\mathbf{sSet}^+)_{/\Delta^1}. \end{array}$$

Let us thus commence by showing that (i) and (ii) are equivalent. In view of Theorem 2.2.2, there exists a fibrant diagram $F : [1] \to \mathbf{sSet}^+$ and a coCartesian equivalence $X^\natural \to r^*F$. The functor Ff is a lift of f on account of Proposition 2.1.8. Thus Ff is an equivalence of marked simplicial sets precisely when $X^\natural$ is Δ^1_1-local.

It remains to show that (ii) and (iii) are equivalent. To this end, it suffices to show that any coCartesian equivalence between fibrants in $\mathbf{sSet}^+$ lies in the essential image of the right derived functor $\mathbf{R}j^*$. Let $F : [1] \to \mathbf{sSet}^+$ be a fibrant diagram corresponding to a coCartesian equivalence. Without loss of generality, we may assume that Ff is a trivial fibration of marked simplicial sets. Let $M(1) \to F(1)$ be a minimal model for $F(1)$. Then the pullback $M(1) \times_{F(1)} F(0)$ is again fibrant in $\mathbf{sSet}^+$ (i.e. an ∞-category). Hence we can pick a minimal model $M(0) \to M(1) \times_{F(1)} F(0)$. Let $Mf : M(0) \to M(1)$ be the arrow making the following diagram commute

$$\begin{array}{ccccc} M(0) & \longrightarrow & M(1)\times_{F(1)} F(0) & \longrightarrow & F(0) \\ {\scriptstyle Mf}\downarrow & & \downarrow & & \downarrow{\scriptstyle Ff} \\ M(1) & \xrightarrow{\mathrm{id}_{M(1)}} & M(1) & \longrightarrow & F(1). \end{array}$$

Then Mf is a coCartesian equivalence, hence an isomorphism by minimalty. Thus Mf defines a fibrant diagram $M : J \to \mathbf{sSet}^+$. The outer square of the above diagram witnesses an isomorphism $j^*M \cong F$ in $\mathrm{Ho}((\mathbf{sSet}^+)^{[1]})$, as desired. □

Remark 3.3.6. The fact that (ii) and (iii) are equivalent may also be proven using Theorem 3.4.3.

Proposition 3.3.7. *Let $X \to S$ be a coCartesian fibration. The following assertions are equivalent:*

(i) $X^\natural$ is Σ-local,
(ii) $X^\natural$ lies in the essential image of the right derived functor

$$\mathbf{R}i^* : \mathrm{Ho}((\mathbf{sSet}^+)_{/S[\Sigma^{-1}]}) \to \mathrm{Ho}((\mathbf{sSet}^+)_{/S}).$$

Proof. Let us start by showing that (i) implies (ii). Denote the map $\Sigma \times \Delta^1 \to S$ by p. Note that the pullback $p^*X^\natural$ of $X^\natural$ along p is $(\Sigma \times \Delta^1_1)$-local. Consequently, Lemma 3.3.5 asserts that there exists a coCartesian fibration $Y \to \Sigma \times J$ with a coCartesian equivalence $p^*X^\natural \to (\Sigma \times j)^*Y^\natural$. Using the homotopy descent property for coCartesian fibrations (see Proposition 1.3.3), we thus deduce that there exists a coCartesian equivalence $Z \to S[\Sigma^{-1}]$ with a coCartesian equivalence $X^\natural \to i^*Z^\natural$ as desired.

Suppose now that (ii) holds. Then there exists a coCartesian fibration $Y \to S[\Sigma^{-1}]$ and a coCartesian equivalence $X^\natural \to i^*Y^\natural$. It suffices to show that $f^*X^\natural$ is Δ^1_1-local for any edge f of Σ. Note that the map $f^*X^\natural \to f^*i^*Y^\natural = (if)^*Y^\natural$ is again a coCartesian equivalence. By definition of $S[\Sigma^{-1}]$, the map if factors through the localization $j : \Delta^1 \to J$. Hence a last application of Lemma 3.3.5 yields the desired result. □

Proof of Theorem 3.3.3. On account of Proposition 3.3.7, the base change adjunction descends to a Quillen adjunction

$$i_! : L_\Sigma(\mathbf{sSet}^+)_{/S} \rightleftarrows (\mathbf{sSet}^+)_{/S[\Sigma^{-1}]} : i^*.$$

The same proposition asserts that its right derived functor is essentially surjective. It remains to show that $\mathbf{R}i^*$ is essentially surjective. Since the class of cofibrations k for which $\mathbf{R}k^*$ is fully faithful, is closed under pushouts (see the proof of Lemma 1.3.5), it suffices to show the claim in the case that i is the inclusion $j : \Delta^1 \to J$.

We can appeal to the straightening and unstraightening equivalence to deduce that it suffices to show that the right derived functor of $j^* : (\mathbf{sSet}^+)^J \to (\mathbf{sSet}^+)^{[1]}$ is fully faithful, which will follow from Theorem 3.4.3. We will not do this here, instead we give a direct proof. Let $X \to J$ be a coCartesian fibration. Then we must show that the top arrow in the following pullback square

$$\begin{array}{ccc} X^\natural \times_{J^\sharp} (\Delta^1)^\sharp & \longrightarrow & X^\natural \\ \downarrow & & \downarrow \\ (\Delta^1)^\sharp & \longrightarrow & J^\sharp \end{array}$$

is a trivial cofibration in $(\mathbf{sSet}^+)_{/J}$. Since j is right anodyne, it follows from Corollary 1.1.17 that the top arrow a trivial cofibration in $\mathbf{sSet}^+$. As J is a Kan complex, the marked edges of $X^\natural$ are the equivalences in the ∞-category X. Hence $X^\natural$ is fibrant in $\mathbf{sSet}^+$. Thus the arrow $X^\natural \times_{J^\sharp} (\Delta^1)^\sharp \to X^\natural$ is a fortiori marked anodyne. □

Corollary 3.3.8. *Let $f : \mathcal{C} \to \mathcal{D}$ be a localization of a pair of ∞-categories $(\mathcal{C}, W)$. Then the functor f is initial and final.*

Proof. We only show the fact that f is initial, as the other statement is dual. To show that f is initial, we must demonstrate that f, viewed as a map in $\mathbf{sSet}_{/D}$, is a covariant equivalence, or equivalently, that $f^\sharp : \mathcal{C}^\sharp \to \mathcal{D}^\sharp$ is a coCartesian equivalence in $(\mathbf{sSet}^+)_{/\mathcal{D}}$. But this is precisely the map $f_!\mathcal{C}^\sharp \to \mathcal{D}^\sharp$, and since the adjoint map $\mathcal{C}^\sharp \to f^*\mathcal{D}^\sharp = \mathcal{C}^\sharp$ is the identity, the result follows from Corollary 3.3.4. □

3.4 Marked simplicial diagrams over localizations

There are results analogous to Theorem 3.3.3 and its Corollary 3.3.4 for marked simplicial diagrams over localizations of simplicial categories. These results are due to Dwyer and Kan. Suppose that we have a simplicial functor $f : \mathcal{C} \to \mathcal{D}$ and a simplicial model category $\mathcal{M}$, then f gives rise to a (simplicial) Quillen adjunction

$$f_! : \mathcal{M}^\mathcal{D} \rightleftarrows \mathcal{M}^\mathcal{C} : f^*.$$

The simplicial functor categories are endowed with the projective model structure [Lur09, Section A.3.3]. This section is devoted to the study of this adjunction whenever f is a localization functor via Bousfield localizations. We will work under the assumption that $\mathcal{M}$ is combinatorial. We commence by observing the following analog of Proposition 3.3.1:

Proposition 3.4.1. *Let $F : \mathcal{C} \to \mathcal{M}$ be a projective diagram. Suppose that $f : x \to y$ is a map in W (i.e. a 0-arrow). Then the following assertions are equivalent:*

(i) the map $Fx \to Fy$ is a weak equivalence,
(ii) for any set I of generating cofibrations for $\mathcal{M}$, the maps induced by the cofibrations $A \to B$ in I and f

$$\mathcal{C}(x,-)\otimes A \bigcup_{C(y,-)\otimes A} \mathcal{C}(y,-)\otimes B \to \mathcal{C}(x,-)\otimes B$$

give rise to a homotopy equivalence on mapping complexes after applying $\mathrm{Map}(-,F)$.

Proof. This follows from Lemma B.0.4 and the observation that we have isomorphisms

$$\mathrm{Map}(\mathcal{C}(z,-)\otimes A, F) \cong \mathrm{Map}(\mathcal{C}(z,-), \mathrm{Map}(A,F)) \cong \mathrm{Map}(A, Fz)$$

natural in $z \in \mathcal{C}$ and $A \in \mathcal{M}$. □

Definition 3.4.2. Let $(\mathcal{C}, W)$ be a pair of simplicial categories. Then we define the simplicial model category of W-local $\mathcal{C}$-diagrams

$$L_W \mathcal{M}^{\mathcal{C}}$$

to be the left Bousfield localization of $\mathcal{M}^{\mathcal{C}}$ along the maps in Proposition 3.4.1(ii). The objects that are local with respect to these maps, will be called W-local.

In this setting, we can formulate the following results which are analogs of Theorem 3.3.3 and Corollary 3.3.4:

Theorem 3.4.3. *Let $(\mathcal{C}, W)$ be a pair of simplicial categories and consider the localization functor $f : \mathcal{C} \to \mathcal{C}[W^{-1}]$. Suppose that the inclusion $W \to \mathcal{C}$ is a cofibration between simplicial computads (see Appendix A) in* **sCat**. *Then the base change Quillen adjunction $(f_!, f^*)$ descends to a Quillen equivalence*

$$f_! : L_W \mathcal{M}^{\mathcal{C}} \rightleftarrows \mathcal{M}^{\mathcal{C}[W^{-1}]} : f^*.$$

Corollary 3.4.4. *Suppose that $f : \mathcal{C} \to \mathcal{D}$ is a functor witnessing $\mathcal{D}$ as the localization of a pair of ∞-categories $(\mathcal{C}, W)$. Then the Quillen adjunction $((\mathfrak{C}f)_!, (\mathfrak{C}f)^*)$ descends to a Quillen equivalence*

$$(\mathfrak{C}f)_! : L_{\mathfrak{C}W} \mathcal{M}^{\mathfrak{C}\mathcal{C}} \rightleftarrows \mathcal{M}^{\mathfrak{C}\mathcal{D}} : (\mathfrak{C}f)^*.$$

Proof. It suffices to prove the statement for the particular model $\mathcal{D} = \mathcal{C}[W^{-1}]$. In this case, the desired result follows from combining Theorem 3.4.3, Proposition 3.2.2 and the homotopy invariance of simplicial functor categories (see [Lur09, Proposition A.3.3.6]). □

Theorem 3.4.3 is a slightly sharpened version of results proven in [DK87]. The main result we will use from this paper, is the following:

Lemma 3.4.5 ([DK87])**.** *Theorem 3.4.3 holds when the inclusion $W \to \mathcal{C}$ is a morphism of simplicial computads.*

This is shown at the end of the proof of Theorem 2.2 in [DK87]. Note that their setup slightly differs from the above. Namely, they use their Lemma 4.3 and a diagonal argument to prove the assertion for the simplicial computadic resultions $(FU_\bullet\mathcal{C}, FU_\bullet W)$ of an arbitrary pair of simplicial categories $(\mathcal{C}, W)$.

Proof of Theorem 3.4.3. On account of the small objects argument, the inclusion $W \to \mathcal{C}$ factors as $W \to \widetilde{\mathcal{C}} \to \mathcal{C}$ such that $W \to \widetilde{\mathcal{C}}$ is a relative simplicial computad and $\widetilde{\mathcal{C}} \to \mathcal{C}$ a trivial fibration in **sCat**. Since $W \to \mathcal{C}$ is a cofibration, there exists a map $\mathcal{C} \to \widetilde{\mathcal{C}}$ witnessing $W \to \mathcal{C}$ as the retract of the map $W \to \widetilde{\mathcal{C}}$. This also means that $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is a retract of $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}[W^{-1}]$. Consequently, we obtain a diagram of left Quillen functors

$$\begin{array}{ccccc} L_W\mathcal{M}^{\mathcal{C}} & \longrightarrow & L_W\mathcal{M}^{\widetilde{\mathcal{C}}} & \longrightarrow & L_W\mathcal{M}^{\mathcal{C}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^{\mathcal{C}[W^{-1}]} & \longrightarrow & \mathcal{M}^{\widetilde{\mathcal{C}}[W^{-1}]} & \longrightarrow & \mathcal{M}^{\mathcal{C}[W^{-1}]}, \end{array}$$

commuting up to natural isomorphism, such that both rows compose to the identity functors up to natural isomorphism. Note that the inclusion $W \to \widetilde{C}$ is a morphism in **sCptd** on account of Proposition A.0.3, hence the middle arrow is part of a Quillen equivalence in view of Lemma 3.4.5. We conclude that the outer vertical functors are also part of a Quillen equivalence as desired. □

CHAPTER 4

Rectification in the general case

We almost have arrived at the point where we developed sufficient theory to efficiently prove the straightening and unstraightening equivalence for coCartesian fibrations over arbitrary simplicial sets. We start by constructing the general rectification adjunction $(r_!, r^*)$.

4.1 From the classical to the ∞-categorical Grothendieck construction

Recall that in classical category theory, the unstraightening $\int F$ of a functor $F : \mathcal{C} \to \mathbf{Cat}$ is defined as follows. The underlying category $\int F$ has as objects pairs (c, x) consisting of an object $c \in \mathcal{C}$ and an object $x \in Fc$. The data of a morphism $(c, x) \to (c', x')$ consists of a map $c \to c'$ in $\mathcal{C}$ accompanied by a a map $\eta : (Ff)x \to x'$ in Fc. The following point of view is very useful, and will guide us in the generalization. For $0 \le k \le 2$, we define a thickened 2-category $T[k]$ for the category $[k]$, with same objects but hom-categories given by and hom-categories given by

$$T[k](i,j) := \begin{cases} [0] & \text{if } i = j, \\ [1]^{j-i-1} & \text{if } i < j, \\ \varnothing & \text{else.} \end{cases}$$

In the case that $k = 2$, the composition map $T[2](1,2) \times T[2](0,1) \to T[2](0,2)$ is defined to have image $0 \in [1]$. The image of the 2-category $T[k]$ under the change of enriching category functor induced by the nerve functor, coincides with $FU_\bullet[k] \cong \mathfrak{C}\Delta^k$. We may now describe the category $\int F$ as follows:

- the objects of $\int F$ are pairs (c, x) containing a point $c : T[0] \to \mathcal{C}$ and a cone $x : T[0]^{\triangleleft} = T[1] \to \mathbf{Cat}$ such that $x(0) = *$ and $x(1) = Fc$, visualized as follows:

$$\begin{array}{c} * \\ \downarrow \\ x(1) \\ \\ c \end{array}$$

- the morphisms of $\int F$ are pairs (f,x) containing a map $f : T[1] \to \mathcal{C}$ and a cone $\eta : T[1]^{\triangleleft} = T[2] \to \mathbf{Cat}$ such that $x(0) = *$, $x|_{T\{1,2\}} = Ff$, visually represented as:

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ x(1) & \longrightarrow & x(2) \\ & & \\ f(0) & \longrightarrow & f(1). \end{array}$$

Thus the classical Grothendieck construction is entirely described by 'weak' cones lying over morphisms and points in the category. We would like to proceed in the same way to define the straightening of a simplicial functor $\mathfrak{C}S \to \mathbf{sSet}^+$.

Definition 4.1.1. Suppose that $G : FU_\bullet I \to \mathcal{C}$ is a homotopy coherent diagram valued in a simplicial category $\mathcal{C}$. Then a cone over G with apex $c \in \mathcal{C}$ is a homotopy coherent diagram $\bar{G} : FU_\bullet I^{\triangleleft} \to \mathcal{C}$ such that $\bar{G}(\triangleleft) = c$ and $\bar{G}|_{FU_\bullet I} = G$; where $\triangleleft$ denotes the adjoined object in $I^{\triangleleft}$.

Remark 4.1.2. If $I = [n]$, then a homotopy coherent diagram over I is a simplicial functor $G : FU_\bullet I \cong \mathfrak{C}\Delta^n \to \mathcal{C}$. A cone over G with apex $c \in C$ is a simplicial functor $\bar{G} : \mathfrak{C}\Delta^{n+1} \to \mathcal{C}$ such that $\bar{G}(0) = c$ and $\bar{G}|_{\mathfrak{C}\Delta^{\{1,\dots,n\}}} = G$.

The above notion of cones coincides with cones weighted by the functor $FU_\bullet I^{\triangleleft}(\triangleleft, -) : FU_\bullet I \to \mathbf{sSet}$:

Proposition 4.1.3. *Let $G : FU_\bullet I \to \mathcal{C}$ be a simplicial functor. Then there is a bijection between homotopy coherent cones over G with apex $c \in \mathcal{C}$ and (simplicial) natural transformations $FU_\bullet I^{\triangleleft}(\triangleleft, -) \Rightarrow \mathcal{C}(c, G-)$.*

Proof. Given a homotopy coherent cone $\bar{G} : FU_\bullet I^{\triangleleft} \to \mathcal{C}$ over G with apex $c \in \mathcal{C}$, the maps

$$FU_\bullet I^{\triangleleft}(\triangleleft, i) \xrightarrow{\bar{G}_{\triangleleft,i}} \mathcal{C}(c, Gi)$$

constitute an enriched natural transformation. There is an inverse to this assignment given as follows. Given a natural transformation $\eta : FU_\bullet I^{\triangleleft}(\triangleleft, -) \Rightarrow \mathcal{C}(c, G-)$, we may define $\bar{G}$ on objects by setting $\bar{G}(\triangleleft) := c$ and $\bar{G}i := Gi$ for $i \in I$. There is no choice for the maps $\bar{G}_{\triangleleft,\triangleleft}$ and $\bar{G}_{i,\triangleleft}$ for $i \in I$. We set $\bar{G}_{i,j} := G_{i,j}$, $\bar{G}_{\triangleleft,i} := \eta_i$ for $i, j \in I$. It follows from the enriched naturality of η that $\bar{G}$ defines a simplicial functor. □

Let $F : \mathfrak{C}S \to \mathbf{sSet}^+$ be a simplicial functor. Then we define the unstraightening $r^*F \in (\mathbf{sSet}^+)_{/S}$ as follows. The n-simplices of r^*F are given by the set

$$\{(s,x) \mid s \in S_n, x \text{ is a cone over } F\mathfrak{C}s : \mathfrak{C}\Delta^n \to \mathbf{sSet} \text{ with apex } \Delta^0\}.$$

The structure maps are the obvious ones. Namely, for $\alpha : [m] \to [n]$, we set $\alpha^*(s,x) := (\alpha^* s, \alpha^* x)$, where $\alpha^* x$ is defined to be the composite

$$\mathfrak{C}\Delta^{m+1} \xrightarrow{\mathfrak{C}(\Delta^0 * \alpha)} \mathfrak{C}\Delta^{n+1} \xrightarrow{x} \mathbf{sSet}.$$

We still have to designate markings to r^*F: an edge $(s,x) \in r^*F$ is marked precisely when the edge $\Delta^1 \cong \mathfrak{C}\Delta^2(0,2) \to \mathrm{Map}(x(0),x(2)) = Fs_1$ is marked. This defines a functor

$$r^* : (\mathbf{sSet}^+)^{\mathfrak{C}S} \to (\mathbf{sSet}^+)_{/S}.$$

as follows. Given a map $f : F \to G$ (i.e. a simplicial natural transformation), we have a map $r^*F \to r^*G$ of marked simplicial sets over S which sends a pair (s,x) to (s, f_*x). Here $f_*x : \mathfrak{C}\Delta^{n+1} \to \mathbf{sSet}$ denotes the cone over $G\mathfrak{C}s$ corresponding to the natural transformation

$$\mathfrak{C}\Delta^{n+1}(0,-) \xrightarrow{x} F\mathfrak{C}s \xrightarrow{f\cdot\mathfrak{C}s} G\mathfrak{C}s.$$

The unstraightening functor admits a left adjoint $r_!$, which we can readily compute using the above description. For $X \in (\mathbf{sSet}^+)_{/S}$, we define the simplicial category CX by the following pushout square

$$\begin{array}{ccc} \mathfrak{C}X & \longrightarrow & \mathfrak{C}X^{\triangleleft} \\ \downarrow & & \downarrow \\ \mathfrak{C}S & \longrightarrow & CX. \end{array}$$

This gives rise to a weight

$$CX(\triangleleft,-) : \mathfrak{C}S \to CX \xrightarrow{CX(\triangleleft,-)} \mathbf{sSet}.$$

We may extend this weight to a functor valued in $\mathbf{sSet}^+$ as follows. Let $f : \Delta^1 \to X$ be an edge lying over an edge $f' : s \to t$ of S. The edge f gives rise to a map $\mathfrak{C}(\Delta^0 * f) : \mathfrak{C}\Delta^2 \to \mathfrak{C}X^{\triangleleft}$ which in turn gives rise to a homotopy coherent cone $\widetilde{f} : \mathfrak{C}\Delta^2 \to CX$ over $\mathfrak{C}s$ with apex Δ^0. Let s be an object of S. We now define the set of marked edges $\Sigma_X(s)$ of $CX(\triangleleft, s)$ to consists of the edges of the form

$$\Delta^1 \cong \mathfrak{C}\Delta^2(0,2) \xrightarrow{\widetilde{f}_{0,2}} CX(\triangleleft, s') \xrightarrow{g_*} CX(\triangleleft, s),$$

where f is a marked edge in X lying above an edge $s'' \to s'$ of S, and g an arrow $s' \to s$ in $\mathfrak{C}S$. This yields the extension

$$r_!X : \mathfrak{C}S \to \mathbf{sSet}^+ : s \mapsto (CX(\triangleleft, s), \Sigma_X(s))$$

we were chasing. The above construction fits into a functor

$$r_! : (\mathbf{sSet}^+)_{/S} \to (\mathbf{sSet}^+)^{\mathfrak{C}S}$$

which is the desired left adjoint of r^*:

Proposition 4.1.4. *The pair $(r_!, r^*)$ is a Quillen adjunction.*

Proof. We have to exhibit a natural isomorphism

$$(\mathbf{sSet}^+)^{\mathfrak{C}S}(r_!X, F) \to (\mathbf{sSet}^+)_{/S}(X, r^*F).$$

Since both sides are compatible with colimits in X, it suffices to show that there are natural isomorphisms for $X = (\Delta^n)^\flat$ and $X = (\Delta^1)^\sharp$.

Denote the n-simplex of S that lies under X by s. Forgetting about the markings, it is readily verified that there is a bijection

$$\mathbf{sSet}^{\mathfrak{C}S}(r_!\Delta^n, F) \to (r^*F \times_S \{s\})_n = \mathbf{sSet}_{/S}(\Delta^n, r^*F).$$

obtained by sending a natural transformation $C\Delta^n(\triangleleft, -) \Rightarrow F$ to the composite

$$\mathfrak{C}\Delta^{n+1}(0, -) \Rightarrow C\Delta^n(\triangleleft, \mathfrak{C}s(-)) \Rightarrow F\mathfrak{C}s.$$

Thus it remains to check that this bijection carries a map $f : r_!(\Delta^1)^\sharp \to F$ to a marked edge of r^*F. To this end we must check that the composite map

$$\Delta^1 \cong \mathfrak{C}\Delta^2(0,2) \to C(\Delta^1)^\sharp(\triangleleft, s_1) \to Fs_1$$

is marked, and this readily follows from the definition above, and the fact that the map $C(\Delta^1)^\sharp(\triangleleft, s_1) \to Fs_1$ preserves markings.

To show that $(r_!, r^*)$ is a Quillen adjunction, one proceeds in a similar fashion as in Chapter 2. We will not show this here, instead, we refer the reader to Section 3.2.1 of [Lur09] for a demonstration of this fact. □

The rectification functor defined above, has very similar properties to the rectification functor for 1-categories defined in Chapter 2, as the next proposition asserts. In fact, we will show that for 1-categories, these functors are the same up to equivalence.

Proposition 4.1.5. *The rectification functor has the following properties:*

(i) the rectification is natural, i.e., for any map $f : S \to T$ of simplicial sets, the following square commutes

$$\begin{array}{ccc} (\mathbf{sSet}^+)_{/S} & \longrightarrow & (\mathbf{sSet}^+)^{\mathfrak{C}S} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle (\mathfrak{C}f)_!} \\ (\mathbf{sSet}^+)_{/T} & \longrightarrow & (\mathbf{sSet}^+)^{\mathfrak{C}T} \end{array}$$

up to natural isomorphism,

(ii) the rectification of a simplex $\Delta^n \to S$ is described by the (enriched) coend

$$r_!(\Delta^n)^{\flat/\sharp} = \int^{i \in (\mathfrak{C}\Delta^n)^{\flat/\sharp}} \mathfrak{C}\Delta^{n+1}(0, i+1)^{\flat/\sharp} \times \mathfrak{C}S(s_i, -)^{\flat/\sharp}.$$

Proof. Part (i) follows directly from the coend expression (ii). Let us show part (ii). We have isomorphisms

$$\begin{aligned} \mathbf{sSet}^{\mathfrak{C}S}(r_!\Delta^n, F) &\cong \mathbf{sSet}_{/S}(\Delta^n, F) = (\textstyle\int_{i \in \mathfrak{C}\Delta^n} \mathrm{Map}(\mathfrak{C}\Delta^{n+1}(0, i+1), Fs_i))_0 \\ &= (\textstyle\int_{i \in \mathfrak{C}\Delta^n} \mathrm{Map}(\mathfrak{C}\Delta^{n+1}(0, i+1) \times \mathfrak{C}S(s_i, -), F))_0 \\ &= \mathbf{sSet}^{\mathfrak{C}S}(\textstyle\int^{i \in \mathfrak{C}\Delta^n} \mathfrak{C}\Delta^{n+1}(0, i+1) \times \mathfrak{C}S(s_i, -), F) \end{aligned}$$

natural in $F : \mathfrak{C}S \to \mathbf{sSet}$. Thus the result now follows from the Yoneda lemma, and the fact that $(r_!\Delta^n)^{\flat/\sharp} = r_!(\Delta^n)^{\flat/\sharp}$. □

Remark 4.1.6. Unlike the rectification functor for 1-categories, this rectification functor is not simplicial. However, there exists a comparison map

$$r_!(A \times X) \to A \times r_! X$$

natural in $A \in \mathbf{sSet}^+$ and $X \in (\mathbf{sSet}^+)_{/S}$, which turns out to be a weak equivalence (see [Lur09, Corollary 3.2.1.15]).

4.2 Comparing the rectification functors for 1-categories

For 1-categories, the rectification functors defined in Chapter 2 coincide with the rectification functors define above in the following sense. Let $\mathcal{C}$ be a 1-category. Then there is a natural DK-equivalence $\epsilon : \mathfrak{C}N\mathcal{C} \cong FU_\bullet\mathcal{C} \to \mathcal{C}$. We will show that there exists a 2-cell η (which is even natural in $\mathcal{C}$) fitting in the following diagram

$$\begin{array}{ccccc} & \overset{r_!}{\nearrow} & (\mathbf{sSet}^+)^{\mathfrak{C}N\mathcal{C}} & \overset{\epsilon_!}{\searrow} & \\ (\mathbf{sSet}^+)_{/N\mathcal{C}} & & \Downarrow \eta & & (\mathbf{sSet}^+)^{\mathcal{C}}, \\ & & \underset{r_!}{\longrightarrow} & & \end{array}$$

whose components are weak equivalences.

Using the description of rectification functors by coends, this is a straightforward procedure. Let Δ^n be a n-simplex over $c_0 \to c_1 \to \cdots \to c_n$. Note that $\epsilon_! r_! \Delta^n$ is given by the coend

$$\epsilon_! r_! \Delta^n = \int^{i \in \mathfrak{C}\Delta^n} \mathfrak{C}\Delta^{n+1}(0, i+1) \times \mathcal{C}(c_i, -).$$

We have a natural map $(\Delta^1)^i \cong \Delta^{n+1}(0, i+1) \to \Delta^i$ defined as follows. If we use the geometric description of $\Delta^{n+1}(0, i+1)$, i.e. we realize it as the nerve $NP_{0,i+1}$ where $P_{0,i+1}$ denotes the poset of subsets of $[n+1]$ containing $0, i+1$ ordered by $\supset$, then the natural map is defined to be the nerve of the functor $P_{0,i+1} \to [i] : S \mapsto \min(S \setminus \{0\}) - 1$. Alongside with the DK-equivalence $\mathfrak{C}\Delta^n \to [n]$ this gives rise to a map between coends

$$\epsilon_! r_! \Delta^n = \int^{i \in \mathfrak{C}\Delta^n} \mathfrak{C}\Delta^{n+1}(0, i+1) \times \mathcal{C}(c_i, -) \to \int^{i \in [n]} \Delta^i \times \mathcal{C}(c_i, -) = r_! \Delta^n.$$

Definition 4.2.1. The natural transformation η is the unique natural transformation such that $\eta_{(\Delta^n)^{\flat/\sharp}}$ is the $\flat/\sharp$-marked map above.

Proposition 4.2.2. *The components of η are weak equivalences.*

Proof. Since the rectification functors and $\epsilon_!$ are left Quillen, we may reduce to checking the statement for $X = (\Delta^1)^\flat$ and $X = \Delta^0$. In the case that $n = 0, 1$, the DK-equivalence $\mathfrak{C}\Delta^n \to [n]$ and the natural maps $\mathfrak{C}\Delta^{n+1}(0, i+1) \to \Delta^i$ are isomorphisms. Thus the induced maps $\eta_{(\Delta^n)^\flat}$ are isomorphisms, thus in particular weak equivalences, as desired. □

Remark 4.2.3. Using the (unmarked or alternatively, by attaching ♯-markings to the simplicial sets in the upcoming discussion) version of this comparison, one recovers the classical fact that for any ∞-category $\mathcal{C}$, there exists a natural isomorphism

$$\mathcal{C}_{x/} \times_{\mathcal{C}} \{y\} \cong \mathrm{Map}_{\mathcal{C}}(x, y)$$

in the homotopy category of spaces. Namely, one can apply the above result to the left fibration $\mathcal{C}_{x/} \to \mathcal{C}$. This left fibration classifies the functor $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}$ (or equivalently, the fibrant simplicial functor

$$\mathfrak{C}\mathcal{C} \to \mathcal{F} \xrightarrow{\mathcal{F}(x,-)} \mathbf{sSet},$$

where $\mathcal{F}$ is a fibrant replacement of $\mathfrak{C}\mathcal{C}$). This is not too hard to show explictly using the formula for the rectification (e.g. see the proof of [Lur09, Proposition 2.2.4.1]). Thus pulling back to an object $y \in \mathcal{C}$, we obtain a homotopy equivalence

$$\mathcal{C}_{x/} \times_{\mathcal{C}} \{y\} \to r^* \mathrm{Map}_{\mathcal{C}}(x, y),$$

since pullback and unstraightening commute. Thus the adjoint map

$$r_!(\mathcal{C}_{x/} \times_{\mathcal{C}} \{y\}) \to \mathrm{Map}_{\mathcal{C}}(x, y)$$

is again a weak homotopy equivalence. Since the rectification functor of Chapter 2 is isomorphic to the identity functor, we obtain a natural zig-zag of weak homotopy equivalences

$$\mathcal{C}_{x/} \times_{\mathcal{C}} \{y\} \leftarrow r_!(\mathcal{C}_{x/} \times_{\mathcal{C}} \{y\}) \to \mathrm{Map}_{\mathcal{C}}(x, y),$$

in light of the comparison above.

4.3 Proof of the equivalence

We are now at the end of the journey we embarked on in Chapter 3. We will prove the following theorem:

Theorem 4.3.1. *For any simplicial set S, the rectification adjunction*

$$r_! : (\mathbf{sSet}^+)_{/S} \rightleftarrows (\mathbf{sSet}^+)^{\mathfrak{C}S} : r^*$$

is a Quillen equivalence.

As outlined in the introduction of Chapter 3, in order to prove the straightening-unstraightening equivalence in the general setting, we will make use of the fact that any ∞-category is the localization of a 1-category. This is known as *Joyal's delocalization theorem*:

Theorem 4.3.2. *Let $\mathcal{C}$ be an ∞-category. Then the final vertex map*

$$N\Delta_{/\mathcal{C}} \to \mathcal{C}$$

is a localization of $N\Delta_{/\mathcal{C}}$ at the nerve of the subcategory spanned by the maps $\Delta^n \to \Delta^m$ in $\Delta_{/\mathcal{C}}$ which preserve final vertices.

Proof. A proof of this theorem may be found in [Ste17, Theorem 1.3]. □

Lemma 4.3.3. *Let* $(\mathcal{C}, W)$ *be a pair of* ∞*-categories. Then the Quillen adjunction* $(r_!, r^*)$ *for* $\mathcal{C}$ *descends to a Quillen adjunction*

$$r_! : L_W(\mathbf{sSet}^+)_{/\mathcal{C}} \rightleftarrows L_{\mathfrak{C}W}(\mathbf{sSet}^+)^{\mathfrak{C}\mathcal{C}} : r^*.$$

Moreover, this localized Quillen adjunction is a Quillen equivalence when the unlocalized Quillen adjunction is.

Proof. On account of [Hir03, Theorem 3.3.20], it suffices to show the following claim: for any map f in W, the image under $r_!$ of an induced map as Proposition 3.3.1(iii) is weakly equivalent to a map of the form Proposition 3.4.1(ii) associated to $\mathfrak{C}f$. Using the naturality of the rectification functors with respect to the map $f : \Delta^1 \to \mathcal{C}$, we deduce that it suffices to show the claim in the case that $\mathcal{C} = \Delta^1$ and $W = \Delta^1$.

In this case, the rectification functor is equivalent to the rectification functor of Chapter 2 on account of Proposition 4.2.2. Hence it suffices to show the claim for the latter rectification functor $r_!$. Since this rectification functor $r_!$ is compatible with tensoring marked simplicial sets, it suffices to show that $r_!$ carries the map

$$\{1\} \to (\Delta^1)^\sharp$$

in $(\mathbf{sSet}^+)_{/\Delta^1}$ to an arrow that is weakly equivalent to the map

$$[1](1,-) \to [1](0,-),$$

and this follows from the fact that the following triangle

$$\begin{array}{ccc} r_!\{1\} \cong [1](1,-) & \longrightarrow & [1](0,-) \cong r_!\{0\} \\ & \searrow & \downarrow \\ & & r_!(\Delta^1)^\sharp \end{array}$$

commutes up to homotopy in $(\mathbf{sSet}^+)^{[1]}$, and the arrow on the right is a trivial cofibration. □

Proof of Theorem 4.3.1. Suppose first that S is given by the nerve of a 1-category $\mathcal{C}$. Since the map $\mathfrak{C}N\mathcal{C} \to \mathcal{C}$ is a DK-equivalence, we obtain the desired result from Proposition 4.2.2 and Theorem 2.2.2.

Consider now the general case. On account of the homotopy invariance of coCartesian model structures and simplicial functor categories, it suffices to show the statement in case that S is an ∞-category $\mathcal{C}$. Moreover, in view of Joyal's delocalization theorem and the preceeding case, we may henceforth assume that there exists a functor $f : \mathcal{D} \to \mathcal{C}$ which witnesses $\mathcal{C}$ as a localization of a pair of ∞-categories $(\mathcal{D}, W)$ and that the theorem holds for the base $\mathcal{D}$. We then obtain the following square of left Quillen functors

$$\begin{array}{ccc} L_W(\mathbf{sSet}^+)_{/\mathcal{D}} & \xrightarrow{r_!} & L_{\mathfrak{C}W}(\mathbf{sSet}^+)^{\mathfrak{C}\mathcal{D}} \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle (\mathfrak{C}f)_!} \\ (\mathbf{sSet}^+)_{/\mathcal{C}} & \xrightarrow{r_!} & (\mathbf{sSet}^+)^{\mathfrak{C}\mathcal{C}} \end{array}$$

which commutes up to natural isomorphism. The vertical arrows are part of Quillen equivalences in view of Corollary 3.3.4 and Corollary 3.4.4. By the assumption on $\mathcal{D}$ and the fact that the rectification functors are compatible with localizations (see Lemma 4.3.3), it follows that the top arrow is part of a Quillen equivalence. Hence the bottom arrow is part of a Quillen equivalence as well. □

Corollary 4.3.4. *Suppose that $F : \mathfrak{C}\mathcal{C} \to \mathbf{sSet}^+$ is a fibrant diagram classified by a coCartesian fibration $X \to \mathcal{C}$. Then there exists a natural isomorphism*

$$\mathrm{Ho}(\mathcal{C}) \; \overset{\pi_0 F}{\underset{\Downarrow}{\rightrightarrows}} \; \mathrm{Ho}(\mathbf{sSet}^+),$$

relating the rectification $r_! X^\natural \simeq F$ of $X^\natural$ and the naive rectification of $X^\natural$ defined in Section 1.4.

Proof. It suffices to show the following claim: for any fibrant diagram $F : \mathfrak{C}\mathcal{C} \to \mathbf{sSet}^+$, the naive rectification of r^*F is naturally isomorphic to $\pi_0 F$. Considering a similar square of functors as in the proof of Theorem 4.3.1, but now taking the right Quillen functors, one easily deduces the fact that if a functor $\mathcal{D} \to \mathcal{C}$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$, and the statement holds for $\mathcal{D}$ then it holds for $\mathcal{C}$ as well. Thus in view of Joyal's delocalization theorem, it suffices to show the statement in case that $\mathcal{C}$ is given by the nerve of a 1-category. This case may be handled using Proposition 4.2.2 and Proposition 2.1.8. □

Remark 4.3.5. Note that the corollary above can be stated in a more intrinsic fashion. Namely, if $X \to \mathcal{C}$ is a coCartesian fibration classifying a functor $f : \mathcal{C} \to \mathbf{Cat}_\infty$, then the above result asserts that the functor $\mathrm{Ho}(f)$ is naturally isomorphic to the naive straightening of X.

4.4 Application: colimits in the ∞-category of ∞-categories

We conclude this chapter by proving that colimits in $\mathbf{Cat}_\infty$ (realized as the homotopy coherent nerve of $(\mathbf{sSet}^+)^\circ$) can be computed using coCartesian fibrations. This was already briefly mentioned at the start of Section 2.2.

Proposition 4.4.1. *Let $p : I \to \mathbf{Cat}_\infty$ be a diagram in $\mathbf{Cat}_\infty$ which classifies a coCartesian fibration $X \to I$. Then the colimit of p and $X^\natural$ are weakly equivalent in $\mathbf{sSet}^+$.*

Proof. We start by observing the following. Suppose that $f : J \to I$ is a final map of simplicial sets. Then, if the claim holds for J then it holds for I as well. Namely, we have $\operatorname{colim}(f^*p) = \operatorname{colim}(p)$. Thus $\operatorname{colim}(p)$ and $f^*X^\natural$ are weakly equivalent in $\mathbf{sSet}^+$. Since f is final, it readily follows from Corollary 1.1.17 that the map $f^*X^\natural \to X^\natural$ is a coCartesian equivalence of marked simplicial sets. Using Joyal's delocalization theorem, and the fact that localization functors are final, we may assume that I is given by the nerve of a 1-category $\mathcal{C}$.

We may then assume that p is given by the homotopy coherent nerve of a (fibrant) diagram $P : \mathcal{C} \to \mathbf{sSet}^+$ using a similar argument as in [Lur09, Corollary 4.2.4.7]. Using the naturality of the counit map of the homotopy coherent nerve adjunction, one now deduces that $X^\natural$ is classified by the composite functor $\mathfrak{C}N\mathcal{C} \to \mathcal{C} \to \mathbf{sSet}^+$. In view of the comparison of rectification functors (see Proposition 4.2.2), we deduce that the 1-functor P classifies $X^\natural$. It now follows from Theorem 2.2.2 that $X^\natural$ and $h_!P$ are weakly equivalent. Thus their underlying marked simplicial sets are weakly equivalent as well. Recall that the underlying marked simplicial set of $h_!P$ is a model for the homotopy colimit of P by construction, which is precisely the colimit of p, thus we have obtained the desired result. □

APPENDIX A

Simplicial computads

We will make extensive use of the notion of simplicial computads. These are called free simplicial categories in the terminology of Dwyer and Kan. Simplicial computads are the simplicial version of computads. Recall that a computad is a category that lies in the image of the inclusion

$$F : \mathbf{Graph} \to \mathbf{Cat}.$$

Here **Graph** denotes the category of reflexive directed graphs. The non-identity edges of the corresponding graph of a computad, are called the atomic arrows. Every non-identity arrow of a computad can be uniquely decomposed as the composition of finitely many atomic arrows.

Definition A.0.1. A simplicial category $\mathcal{C}$ is called a simplicial computad if:

(i) every category $\mathcal{C}_n$ is a computad, whose atomic arrows we refer to as the atomic n-arrows,
(ii) degeneracies of atomic arrows are again atomic.

A morphism $\mathcal{C} \to \mathcal{D}$ of simplicial computads is a simplicial functor which carries every atomic arrow of $\mathcal{C}$ to an atomic arrow or identity in $\mathcal{D}$. Equivalently, the full subcategory **sCptd** of simplicial computads is defined by the following pullback square

$$\begin{array}{ccc} \mathbf{sCptd} & \longrightarrow & \mathbf{sCat} \\ \downarrow & & \downarrow \\ \mathbf{Graph}^{\Delta^{\mathrm{op}}_{\mathrm{epi}}} & \longrightarrow & \mathbf{Cat}^{\Delta^{\mathrm{op}}_{\mathrm{epi}}}. \end{array}$$

Here the bottom arrow is induced by F.

We will show that simplicial computads are cofibrant simplicial categories. In fact, more is true: every cofibrant object of **sCat** is a simplicial computad (see [Rie14, Section 16.2]).

Definition A.0.2. A relative simplicial computad is a simplicial functor $\mathcal{C} \to \mathcal{D}$ that can be written as a countable composition of coproducts of:

(i) the functor $\varnothing \to [0]$,

(ii) the standard simplicial subcomputad inclusions $[1](\partial\Delta^n) \to [1](\Delta^n)$.

The class of relative simplicial computads coincides with the cellular cofibrations in **sCat**. The following observation (posed as an exercise in [RV]) is useful:

Proposition A.0.3. *Let $i : \mathcal{C} \to \mathcal{D}$ be an inclusion of simplicial categories, and suppose that $\mathcal{C}$ is a simplicial computad. Then the following are equivalent:*

(i) the inclusion i is a morphism of simplicial computads,
(ii) the inclusion i is a relative simplicial computad.

Proof. To check that (ii) implies (i), we observe that the injective simplicial computad morphisms are closed under countable composition (since this holds for **Graph**). Hence it suffices to check that any standard simplicial subcomputad inclusion is a morphism of simplicial computads, and this is clear.

It remains to check that (i) implies (ii). We first adjoin all objects of $\mathcal{D}$ to $\mathcal{C}$ that are missing in $\mathcal{C}$ via map Definition A.0.2(i), yielding a simplicial category which we denote by $\mathrm{sk}_{-1}\mathcal{D}\bigcup_{\mathrm{sk}_{-1}\mathcal{C}}\mathcal{C}$. The functor $\mathrm{sk}_n : \mathbf{sSet} \to \mathbf{sSet}$ induces a functor $\mathrm{sk}_n : \mathbf{sCat} \to \mathbf{sCat}$. We now claim that we have pushout squares

$$\begin{array}{ccc} \coprod_{\Sigma_n}[1](\partial\Delta^n) & \longrightarrow & \mathrm{sk}_{n-1}\mathcal{D}\bigcup_{\mathrm{sk}_{n-1}\mathcal{C}}\mathcal{C} \\ \downarrow & & \downarrow \\ \coprod_{\Sigma_n}[1](\Delta^n) & \longrightarrow & \mathrm{sk}_n\mathcal{D}\bigcup_{\mathrm{sk}_n\mathcal{C}}\mathcal{C} \end{array}$$

for $n \geq 0$. Here Σ_n denotes the set of non-degenerate atomic n-arrows of $\mathcal{D}$ which are missing in $\mathcal{C}$. The bottom arrows are induced by the map $f : \Delta^n \to \mathcal{D}(x,y)$, for $f : x \to y$ in Σ_n. The restriction of this map to $\partial\Delta^n$ factors through $\mathrm{sk}_{n-1}\mathcal{D}$, giving the top arrows. To see that the above square is pushout square, we observe that level wise, we have squares

$$\begin{array}{ccccc} \coprod_{\Sigma_n}[1](\varnothing) & \longrightarrow & \coprod_{\Sigma_n}[1](\partial\Delta^n_m) & \longrightarrow & (\mathrm{sk}_{n-1}\mathcal{C})_m\bigcup_{(\mathrm{sk}_{n-1}\mathcal{C})_m}\mathcal{C}_m \\ \downarrow & & \downarrow & & \downarrow \\ \coprod_{f\in\Sigma_n}[1](S_{f,m}) & \longrightarrow & \coprod_{\Sigma_n}[1](\Delta^n_m) & \longrightarrow & (\mathrm{sk}_n\mathcal{D})_m\bigcup_{(\mathrm{sk}_n\mathcal{C})_m}\mathcal{C}_m. \end{array}$$

Here $S_{f,m}$ denotes the set of m-simplices which are either f or degeneracies of f, for $f \in \Sigma_n$. The left square is a pushout, hence we just have to check that the outer square is a pushout. This follows from the fact that the atomic m-arrows in $\mathrm{sk}_n\mathcal{D}\bigcup_{\mathrm{sk}_n\mathcal{C}}\mathcal{C}$ missing in $\mathrm{sk}_{n-1}\mathcal{D}\bigcup_{\mathrm{sk}_{n-1}\mathcal{C}}\mathcal{C}$ are precisely given by the $S_{f,m}$'s. Finally, we note that the map $\mathcal{C} \to \operatorname{colim}\mathrm{sk}_n\mathcal{D}\bigcup_{\mathrm{sk}_n\mathcal{C}}\mathcal{C}$ is precisely the inclusion $\mathcal{C} \to \mathcal{D}$ and this concludes the proof. □

Corollary A.0.4. *A simplicial category $\mathcal{C}$ is a simplicial computad precisely when $\varnothing \to \mathcal{C}$ is a relative simplicial computad.*

By general nonsense, the functor F admits a right adjoint U. This right adjoint fits in a adjunction

$$F : \mathbf{Graph} \rightleftarrows \mathbf{Cat} : U.$$

This adjunctions comes with a counit $\epsilon : FU \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ and an unit $\eta : \mathrm{id}_{\mathbf{Graph}} \Rightarrow UF$. Set $\delta := F\eta U$. The triple (FU, δ, ϵ) satifies the comonadic equations $\epsilon(FU\epsilon) = \epsilon(\epsilon FU)$, $(FU\delta)\delta = (\delta FU)\delta$, $(FU\epsilon)\delta = \mathrm{id}_{\mathbf{Cat}} = (\epsilon FU)\delta$. For a category $\mathcal{C}$, we now define a simplicial computad $FU_\bullet\mathcal{C}$ as follows. We set $FU_n\mathcal{C} := (FU)^{n+1}\mathcal{C}$ and define the face and degeneracy maps by

$$\begin{aligned} d_i &:= ((FU)^{n-i}\eta(FU)^i)_{\mathcal{C}} : FU_n\mathcal{C} \to FU_{n-1}\mathcal{C}, \\ s_i &:= ((FU)^{n-i}\delta(FU)^i)_{\mathcal{C}} : FU_n\mathcal{C} \to FU_{n+1}\mathcal{C}. \end{aligned}$$

It is readily verified that this defines a simplicial computad. Viewing $\mathcal{C}$ as a discrete simplicial category, we note that there is a natural functor $FU_\bullet\mathcal{C} \to \mathcal{C}$, which has the following property:

Proposition A.0.5 (Proposition 2.6 of [DK80])**.** *The simplicial functor $FU_\bullet\mathcal{C} \to \mathcal{C}$ is a DK-equivalence.*

The construction above may be extended to simplicial categories. We will show that there exists a bisimplicial category $FU_\bullet\mathcal{C}$ associated to a simplicial category $\mathcal{C}$ such thats its *diagonal* $\mathrm{diag}\, FU_\bullet$ is a simplicial computad, which comes with a natural DK-equivalence $\mathrm{diag}\, FU_\bullet\mathcal{C} \to \mathcal{C}$.

Recall that the category **ssCat** of bisimplicial categories is the category of small categories enriched over the category **ssSet** of bisimplicial sets. Like simplicial categories, we may view bisimplicial categories as functors $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to$ **Cat** such that, levelwise, all categories have the same set of objects and such that all face and degeneracy maps act as the identity on objects. Recall that we have a diagonal functor $\mathrm{diag} : \mathbf{ssSet} \to \mathbf{sSet}$. This gives rise to a diagonal functor

$$\mathrm{diag} : \mathbf{ssCat} \to \mathbf{sCat}.$$

Alternatively, the action of this functor on a bisimplicial category $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to$ **Cat** may be described as precomposing with the diagonal $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$. The following result should not come as surprise:

Proposition A.0.6. *Suppose that $f : \mathcal{C} \to \mathcal{D}$ is a map of bisimplicial categories such that $\mathcal{C}_{n,\bullet} \to \mathcal{D}_{n,\bullet}$ is a DK-equivalence for every n. Then $\mathrm{diag}\, f : \mathrm{diag}\,\mathcal{C} \to \mathrm{diag}\,\mathcal{D}$ is a DK-equivalence.*

Proof. For $x, y \in \mathcal{C}$, the map $\mathcal{C}_{n,\bullet}(x,y) \to \mathcal{D}_{n,\bullet}(fx, fy)$ is a weak homotopy equivalence of simplicial sets. Hence the map $\mathrm{diag}\,\mathcal{C}(x,y) \to \mathrm{diag}\,\mathcal{D}(fx, fy)$ is weak homotopy equivalence as well. The map $\pi_0\,\mathrm{diag}\, f$ is essentially surjective as well. Namely, for every object $y \in D$, there exists a $x \in \mathcal{C}$ such that fx and y are isomorphic in $\mathcal{D}_{1,\bullet}$ by assumption. This entails that there exist 0-arrows $\gamma : fx \to y$, $\delta : y \to fx$, a 1-arrow $H : fx \to fx$ connecting $\gamma\delta$ and id_y, and a

1-arrow $H' : fx \to fx$ connecting $\delta\gamma$ and id_{fx} in $\mathcal{D}_{1,\bullet}$. It is then readily verified that H, H' are 1-arrows in $\operatorname{diag}\mathcal{D}$ which respectively connect $d_0\gamma d_0\delta$ and id_y and $d_0\delta d_0\gamma$ and id_{fx}. Thus γ, δ are inverse isomorphisms between fx and y in $\pi_0 \operatorname{diag}\mathcal{D}$. □

Definition A.0.7. The simplicial computadic resolution $FU_\bullet\mathcal{C}$ of a simplicial category $\mathcal{C}$ is the bisimplicial category defined by

$$FU_{n,m}\mathcal{C} := FU_n\mathcal{C}_m.$$

Proposition A.0.8. *The simplicial computadic resolution $FU_\bullet(-) : \mathbf{sCat} \to \mathbf{ssCat}$ has the following properties:*

- *(i) The composite functor* $\operatorname{diag} FU_\bullet(-)$ *factors through the inclusion* $\mathbf{sCptd} \to \mathbf{sCat}$.
- *(ii) The natural map* $\operatorname{diag} FU_\bullet\mathcal{C} \to \mathcal{C}$ *is a DK-equivalence for any simplicial category* $\mathcal{C}$.

Proof. Part (i) is readily verified and part (ii) follows from Proposition A.0.5 and Proposition A.0.6. □

APPENDIX B

Model categorical tools

This appendix contains a few isolated lemma's on model categories, which are referred to in the main content of this thesis as the need arises.

Lemma B.0.1. *Let $\mathcal{M}$ be a model category. Then a cofibration is a trivial cofibration if and only if it has the left lifting property with respect to fibrations between fibrant objects.*

Consequently, a left adjoint functor $F : \mathcal{M} \to \mathcal{M}'$ of model categories, is left Quillen precisely when F preserves cofibrations and the right adjoint of F preserves fibrations between fibrant objects.

Proof. The second statement readily follows from the first. Let us show the first assertion. Clearly, every trivial cofibration has the left lifting property with respect to fibrations between fibrant objects. Conversely, suppose that $i : x \to y$ is a cofibration in $\mathcal{M}$ which has the left lifting property w.r.t. fibrations between fibrant objects. Let y_f be a fibrant replacement for y. This replacement comes with a trivial cofibration $y \to y_f$. The composite $x \to y \to y_f$ factors as a trivial cofibration $x \to x_f$ followed by a fibration $p : x_f \to y_f$. The situation is described by the following commutative diagram

$$\begin{array}{ccccc} x & \overset{\sim}{\rightarrowtail} & x_f & & \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \\ y & \overset{\sim}{\rightarrowtail} & y_f & \twoheadrightarrow & *. \end{array}$$

Thus by assumption, this square admits a diagonal filler $f : y \to x_f$. Note that the composites fi and pf are weak equivalences, thus f is invertible in the homotopy category of $\mathcal{M}$. Consequently, f must be a weak equivalence. By the 2-out-of-3 property, it follows that i is a weak equivalence as well. □

Lemma B.0.2. *Let $\mathcal{M}, \mathcal{M}'$ be two model categories and assume that $\mathcal{M}'$ is left proper and combinatorial, and that $\mathcal{M}$ admits a generating set I of cofibrations. Suppose that $\eta : F \Rightarrow G$ is a natural transformation between two left Quillen functors $F, G : \mathcal{M} \rightrightarrows \mathcal{M}'$. Then η_x is a weak equivalence for every cofibrant object x if and only if η_x is a weak equivalence for any object x that is either a domain or codomain of a morphism in I.*

Proof. We prove the non-trivial implication. Assume that η_x is a weak equivalence for any object x that is a domain or a codomain of a map in I. Let x be a cofibrant object of $\mathcal{M}$. In light of of the small objects argument, we see that x is a retract of an I-cell complex. Since weak equivalences are closed under retracts, we may as well assume that x is an I-cell complex. This entails that x is the colimit of a transfinite sequence $x_\bullet : \alpha \to \mathcal{M}$ in I, for some ordinal α with $x_0 = \emptyset$. Thus $x_\bullet$ is a projectively cofibrant diagram. Since F, G are left Quillen, $Fx_\bullet$ and $Gx_\bullet$ are projectively cofibrant and their colimits are given by Fx and Gx respectively. The colimit functor is left Quillen with respect to the projective model structure on $(\mathcal{M}')^\alpha$, hence it suffices to show that that the 2-cell $Fx_\bullet \Rightarrow Gx_\bullet$ is compromised of weak equivalences.

Using an induction argument, we deduce that it is sufficient to demonstrate that $\eta_{x_{\beta+1}}$ is a weak equivalence whenever η_{x_β} is a weak equivalence, for every ordinal β. The object $x_{\beta+1}$ is obtained by attaching a set of I-cells $(f_i : a_i \to b_i)$ to x_β. By assumption, all vertical maps of the following commutative diagram are weak equivalences:

$$\begin{array}{ccccc} Fx_\beta & \longleftarrow & \coprod Fa_i & \longrightarrow & \coprod Fb_i \\ \downarrow & & \downarrow & & \downarrow \\ Gx_\beta & \longleftarrow & \coprod Ga_i & \longrightarrow & \coprod Gb_i. \end{array}$$

The objects $Fx_{\beta+1}$ and $Gx_{\beta+1}$ are pushouts of respectively the top and bottom row. These are also homotopy pushouts since $\mathcal{M}'$ is left proper and the two right horizontal arrows are cofibrations. Hence the map $\eta_{x_{\beta+1}} : Fx_{\beta+1} \to Gx_{\beta+1}$ is a weak equivalence as well. □

Lemma B.0.3. *Let $\mathcal{M}$ be a simplicial model category, and $f : x \to y$ a weak equivalence of fibrant objects. Suppose that $a \to b$ is a cofibration. Then for any map $g : a \to x$, the induced map*

$$\mathrm{Map}(b,x) \times_{\mathrm{Map}(a,x)} \{g\} \to \mathrm{Map}(a,x) \times_{\mathrm{Map}(b,x)} \{fg\}$$

is a homotopy equivalence of simplicial sets.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Map}(b,x) & \longrightarrow & \mathrm{Map}(b,y) \\ \downarrow & & \downarrow \\ \mathrm{Map}(a,x) & \longrightarrow & \mathrm{Map}(a,y). \end{array}$$

The two vertical arrows are fibrations, and the horizontal arrows are homotopy equivalences between Kan complexes. It follows that fibers of the two vertical arrows are defined by homotopy pullback squares, hence the two horizontal arrows induce homotopy equivalences on fibers. □

Lemma B.0.4. *Let $\mathcal{M}$ be a simplicial model category and $f : x \to y$ be a map of fibrant objects in $\mathcal{M}$. Then the following statements are equivalent:*

(i) the map f is a weak equivalence,
(ii) for any cofibration $i : a \to b$, the induced map

$$i \pitchfork f : \mathrm{Map}(b,x) \to \mathrm{Map}(a,x) \times_{\mathrm{Map}(a,y)} \mathrm{Map}(b,y)$$

is a homotopy equivalence of simplicial sets.

Proof. We may factor the map f as trivial cofibration $j : x \to z$ followed by a fibration $g : z \to y$. Let $i : a \to b$ be a cofibration. Then the following square commutes

$$\begin{array}{ccc} \mathrm{Map}(b,x) & \xrightarrow{i \pitchfork f} & \mathrm{Map}(a,x) \times_{\mathrm{Map}(a,y)} \mathrm{Map}(b,y) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_* \times_{\mathrm{Map}(a,y)} \mathrm{Map}(b,y)} \\ \mathrm{Map}(b,z) & \xrightarrow{i \pitchfork g} & \mathrm{Map}(a,z) \times_{\mathrm{Map}(a,y)} \mathrm{Map}(b,y). \end{array}$$

Note that the left and right arrows are homotopy equivalences. Moreover, the bottom arrow is a fibration as the model structure on $\mathcal{M}$ is simplicial. All in all, we deduce that it suffices to show that g is a trivial fibration if and only if the map $i \pitchfork g$ is a trivial fibration for any cofibration $i : a \to b$ in $\mathcal{M}$. If g is a trivial fibration, then $i \pitchfork g$ is a trivial fibration since $\mathcal{M}$ is simplicial. Conversely, suppose that $i \pitchfork g$ is a trivial fibration for every cofibration $i : a \to b$. Then in particular, the map $i \pitchfork g$ is surjective on vertices. Since this holds for any cofibration i, this entails that g is trivial fibration. □

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