UTRECHT UnIVERSITY

Master's Thesis

## One-loop quantum backreaction slows down inflation

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#### Abstract

We consider the one-loop quantum backreaction on a spatially flat, homogeneous and isotropic background with a small, positive deceleration parameter driven by a single minimally coupled inflaton field. The one-loop effect is calculated in a fully fixed gauge where the coordinate systems of subsequent constant time hypersurfaces coincide and matter field fluctuations vanish. This gauge has been chosen with an eye on future works in the stochastic approach to inflation. The main result is a nonvanishing backreaction sourced exclusively by scalar interactions, which induces a negative, logarithmic correction to the scale factor; thus, slowing down the universe's growth during inflation. The logarithmic behaviour is consistent with Weinberg's theorem. Curiously, the backreaction is inversely proportional to the first geometric slow-roll parameter; thus, it diverges in the de Sitter limit. Most likely this is a gauge artefact. Future research will have have to reveal whether this dependence survives in other gauges and whether there are any significant late time manifestations of the backreaction.


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## Notation

Here follows a brief summary of the notational conventions adhered to in the present work.

- We will work in natural units, $c=1=\hbar$, although $\hbar$ is written explicitly to indicate the order of the loop expansion.
- In natural units, the reduced Planck mass is defined as $M_{p}^{2}=\left(8 \pi G_{N}\right)^{-1}$.
- The metric signature of the Minkowski metric is: $(-,+,+,+)$.
- A 'dot' represents a partial derivative with respect to coordinate time, $\dot{A} \equiv \frac{1}{c} \frac{\partial}{\partial t} A=\partial_{0} A$. The latter is written in scenarios where the 'dot' is notationally undesirable. For instance in $\partial_{0} \hat{\tilde{\psi}}$.
- A partial derivative with respect to conformal time is denoted by $\partial_{\eta} A \equiv \frac{\partial}{\partial \eta} A$.
- We distinguish between the first geometric slow-roll parameter $\epsilon$, and a small positive constant $\varepsilon$.
- The Einstein summation convention is implied everywhere.
- Greek indices range over all space-time indices, whereas Latin indices are used for spatial indices only. Hence, in $D$-dimensional spacetime, $\mu=0,1,2, \ldots, D-1$, and $i=1,2, \ldots, D-1$.
- The gradient operator is used interchangeably with partial spatial derivatives when the Einstein summation is obvious. For instance, $\nabla^{2} \psi=\partial_{i} \partial^{i} \psi$, or $(\nabla \psi)^{2}=\partial_{i} \psi \partial^{i} \psi$. Similarly the partial operator includes time indices $\partial^{2} \psi=\partial_{\mu} \partial^{\mu} \psi$, or $(\partial \psi)^{2}=\partial_{\mu} \psi \partial^{\mu} \psi$.


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## Chapter 1

## Introduction

Every now and then, I find it hard to comprehend how a world like ours can even exist. No matter what shape it has taken, its mere existence blows my mind. For me, appreciation is ever-present, while apprehension is build step-by-step through experience. I learned that our collective understanding of physics evolves the same way. Physics attempts to make sense of the things in this world at every scale, whether conceivable or inconceivable. Cosmology is a study of extremes on the energy, length and time scale; it attempts to describe the origin and evolution of the universe. Of particular interest is the era of inflation shortly after the big bang during which the universe grew rapidly. Getting the details regarding inflation just right is crucial in order to explain the observed large-scale distribution of galaxies, clusters and voids. Such details include the dynamics of cosmological inhomogeneities, which are ultimately responsible for structure formation. The dynamics of the universe can be studied in the framework of Einstein's theory of general relativity or (extended) versions thereof. However, the quantum nature of primordial fluctuations demands a theory of quantum gravity or, at the very least, an approximation of it.

The theory of inflation has been around since the year 1980, when A.Starobinsky published a generalisation of general relativity as an attempt to include quantum-gravitational effects [1]. Based on his model, he found a universe which starts out in a de Sitter state, meaning it grows exponentially fast. Why would such an inflationary phase be desirable? A model of the universe without inflation has to deal with two fine-tuning problems: the horizon problem and the flatness problem. Both problems require the model to have an absurdly specific initial configuration of inhomogeneities; a set of initial conditions that leads to a universe which is as spatially flat, homogeneous and isotropic as ours is observed to be. Inflation provides an elegant solution to both of these problems. Nowadays, the theory of inflation is well-established within cosmology. In the last forty years a lot of different models of inflation have been suggested. Through precise measurements of cosmological parameters [2], some of these models are disfavoured or even excluded.

Another major success of the theory of inflation is the framework it provides for the formation of large scale structure (LSS) from primordial quantum fluctuations. Inflation stretches quantum fluctuations beyond the Hubble horizon where they freeze and become classical density fluctuations. After inflation, the fluctuations re-enter the Hubble horizon where they act as the seeds for structure formation. A footprint of density fluctuations in the early universe can be found in the cosmic microwave background radiation (CMBR). The inhomogeneities in the CMBR are incredibly tiny, which tells us that the universe was homogeneous and isotropic at large scales. For this reason, perturbation theory is extremely useful when studying these types of cosmological inhomogeneities. Though, there is no unique formulation; there are superfluous degrees of freedom that introduce
coordinate effects. One can distinguish between three kinds of perturbations which evolve separately at linear order: scalar, vectorial and tensorial perturbations. Scalar perturbations are the most important during inflation because they result in the density fluctuations as seen in the CMBR.

The origin of primordial quantum fluctuations can be traced back to the Heisenberg uncertainty principle which allows for spontaneous, minute changes in the vacuum energy. Such a vacuum fluctuation manifests itself as the creation and annihilation of a particle-antiparticle pair. Even though individual pairs can not be detected, the cumulative effect can be significant enough to be observable. Working with quantum effects does come with strings attached. It requires quantization to transition from a classical theory to a quantum theory. In the case of gravity, this is rather difficult because the quantized version of classical gravity, quantum gravity, is poorly understood at this time. At best one can take a perturbative approach. One way of quantizing gravity in that case is by following the same quantization procedure used in the development of quantum mechanics and quantum field theory. Canonical quantization, as it is called, is not straightforward as one has to deal with divergences. Ultraviolet divergences can be dealt with by defining and subtracting the appropriate counterterms, whereas infrared divergences signal that something is deeply unphysical about the calculation. Janssen et al. [3] deal with infrared divergences when deriving the propagator for a massless minimally coupled scalar on a D-dimensional, spatially flat, homogeneous and isotropic background with arbitrary constant deceleration parameter. Their results make the present work possible. Here, we derive the propagators for scalar and tensor metric perturbations in similar fashion and on the same background.

Doing perturbation theory up to quadratic order one is able to describe interactions, which introduce non-Gaussianities in the spectrum of the CMBR [4. Not only that, interactions may affect the large-scale cosmic evolution through a backreaction. If inflation lasts long enough, the energy density of vacuum fluctuations may become high enough such that it influences the dynamics of the universe on large scales. An effective way to deal with a quantum backreaction is to realize that inflation is a macroscopic phenomenon. This means that, for practical purposes, the behaviour of the scalar field is only relevant at large spatial scales because this is the part that is observable in the CMBR and eventually sources the formation of large scale structure. This notion is used in the framework of stochastic inflation to construct an effective classical theory of fluctuations on super-Hubble scales by averaging the contributions from sub-Hubble scales. However, in inflation, sub-Hubble modes are constantly crossing the Hubble horizon. These quantum-fluctuating modes can then be considered as a stochastic source to the equation of motion of the infrared part of the theory. That quantum fluctuations of a scalar can be described by a classical stochastic process was shown by A.Vilenkin [5. When applied to inflation one essentially ends up with a classical Langevin equation for large scale fluctuations. This result was first derived under slow-roll conditions by Starobinsky [6] and was later generalized by Nakao, Nambu, and Sasaki [7] [8] and many others. The beauty of this approach is that it is a classical approximation to infrared quantum gravity, which is a huge simplification.

The original goal of this thesis was to develop a stochastic description of inflationary dynamics for the infrared sector of the theory. For that purpose, we fixed a gauge where the coordinate systems of subsequent constant time hypersurfaces coincide. This gauge is particularly useful in stochastic inflation because the equations simplify. By carefully treating the gauge one finds that there is an additional scalar contribution to the off-diagonal part of the metric. We continue by defining new independent scalar and tensor fields. From there, we derive the classical action expanded up to quadratic order in metric perturbations and find that it agrees with other fully gauge fixed and
gauge invariant formulations [9]. These new fields are quantized according to canonical quantization using the results from [3]. Finally, we calculate the one-loop quantum backreaction on the dynamics of scalar metric fluctuations and find a negative time-dependent correction which is singular in the de Sitter limit. It turns out that, on average, gravitational wave interactions do not contribute to the backreaction at this order and that only scalar interactions are relevant. Unfortunately, the time frame did not allow for a stochastic treatment of the results, so this is left for future work.

## Chapter 2

## Standard model of the universe

History has known many models of the universe. In the earliest models, stars and planets would rotate around the earth in circular orbits, while later models proposed the sun as the centre of the universe. Nowadays, the scientific consensus is that the universe does not a have a preferred position and direction. This notion is made precise in the cosmological principle which is minimally stated as the universe is homogeneous and isotropic. The homogeneity of the universe is to be understood only on large scales, i.e., when the universe is averaged over cells with a diameter of $10^{8}$ to $10^{9}$ light-years. At these scales, the distributions of mass and luminous sources as found by observations are compatible with the cosmology principle [10, [11, [12]. The cosmological principle is crucial in modern cosmology not for its accuracy, but rather, because it provides an enormous simplification such that the parameters of theoretical models might be fitted by astronomical observations.

Often, new perspectives on astronomy and cosmology are accompanied by a new understanding of gravity. The work of Galileo on the concept of relativity permanently shifted the perspective away from geocentric models, while Newton's theory of gravity provided the mathematical tools to understand the motion of planets in the solar system. Expanding on this work, Einstein published the theory of general relativity in 1915, which accounts for several shortcomings of Newtonian gravity while also making a large number of predictions, such as the expansion of the universe and the existence of black holes. General relativity is supported by an overwhelming amount of observational evidence and, as such, is the mathematical foundation of many cosmological models. The quest to understanding gravity continues to the present day with the development of quantum gravity which seeks to unify Einstein's theory of general relativity with quantum mechanics. Undoubtedly, an accurate description of quantum gravity will provide new insights into the workings of the universe.

Today's 'standard' model of the universe is compatible with both the cosmological principle and Einstein's theory of general relativity. The $\Lambda$ CDM-model, as it is called, is the simplest model that provides a reasonably accurate description of the universe. The main purpose of this chapter is to discuss the origin and workings of this model at an introductory level. Section 2.1 conceptualizes homogeneity and isotropy as the FLRW metric and introduces the possibility of a non-trivial scale factor. Sections 2.2 and 2.3 discuss the evolution of the scale factor from Einstein's equations. Tracing back our universe to its origin introduces so-called fine-tuning problems, which are covered in sections 2.4 .1 and 2.4 .2 . A period of rapid expansion, called inflation, is hypothesized as a solution to several fine-tuning problems. In section 2.5 we discuss the several models of inflation. Finally, section 2.6 explains how the standard model of the universe is used to understand large scale structure formation.

### 2.1 Friedmann-Lemaître-Robinson-Walker

The most general homogeneous and isotropic spacetime is described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which, in generic spatial coordinates, is of the form,

$$
\begin{equation*}
d s^{2}=-d t^{2}+a^{2}(t) d \boldsymbol{\Sigma}^{2}, \tag{2.1}
\end{equation*}
$$

where $a(t)$ is known as the scale factor, while $d \Sigma^{2}$ is a space-like line element on a generic spatial hypersurface of uniform curvature. In fact, as Robertson and Walker rigorously proved in 1935, the FLRW metric is unique; it is the only one that describes a homogeneous and isotropic universe. As the FLRW metric describes a spatially isotropic universe, it is more natural to use spherical coordinates. The corresponding line element,

$$
\begin{equation*}
d s^{2}=-d t^{2}+a^{2}(t) \frac{d r^{2}}{1-\kappa r^{2}}+a^{2}(t) r^{2}\left(d \vartheta^{2}+\sin ^{2}(\vartheta) d \varphi^{2}\right), \tag{2.2}
\end{equation*}
$$

is expressed in terms of the comoving spherical coordinates $r, \vartheta, \varphi$ and the spatial curvature parameter $\kappa$. In a positively curved spacetime $\kappa>0$, while $\kappa<0$ in a negatively curved spacetime and, when $\kappa=0$, the spacetime is spatially flat. In this interpretation, $\kappa$ has units of length ${ }^{-2}$ and is equal to the Gaussian curvature at the time when $a(t)=1$. There is a redundancy in the description of the line element $(2.2)$; the metric is invariant under the rescaling: $a \rightarrow \lambda a, r \rightarrow r / \lambda$, $\kappa \rightarrow \lambda^{2} \kappa$. This means that $\lambda$ can be chosen such that $a\left(t_{0}\right)=1$ at the present time $t_{0}$.

The precise interpretation of the scale factor $a(t)$ is most easily illustrated by calculating the physical (or proper) distance between the observer with comoving coordinates $(r, \vartheta, \varphi)$ and the origin,

$$
d_{\text {phys }}(t)=a(t) \chi=a(t) \int_{0}^{r} \frac{d r}{\sqrt{1-\kappa r^{2}}}=a(t) \times \begin{cases}\sqrt{\kappa}^{-1} \sin ^{-1}(\sqrt{\kappa} r) & \kappa>0  \tag{2.3}\\ \sqrt{|\kappa|}{ }^{-1} \sinh ^{-1}(\sqrt{|\kappa|} r) & \kappa<0 \\ r & \kappa=0\end{cases}
$$

where $\chi$ is the comoving distance; the distance between the two points measured along a path defined at the present time. The physical distance scales directly with the scale factor. This means that the scale factor can increase (or decrease) the distance between the two positions. According to the cosmological principle, this holds for any two comoving observers everywhere in the universe. In the presence of a nontrivial scale factor, the FLRW metric is no longer invariant under Lorentz boosts. This means that the universe picks out a preferred rest frame: the comoving coordinates. One can imagine the comoving coordinates as a grid painted on a balloon. As the balloon inflates or deflates, the physical distance between any two grid points changes by a factor $a(t)$. This distance changes at a rate

$$
\begin{equation*}
v_{\text {phys }}=\frac{d}{d t} d_{\text {phys }}(t)=\frac{\dot{a}(t)}{a(t)} d_{\text {phys }}(t)=H(t) d_{\text {phys }}(t) \tag{2.4}
\end{equation*}
$$

where $H(t)$ is known as the Hubble parameter. In the case that $\dot{a}(t)>0$, the spacetime is expanding, while in the case that $\dot{a}(t)<0$ the spacetime is contracting. Note that the interpretation of the scale factor is independent of the curvature parameter $\kappa$.

### 2.2 Expansion of the universe

When Friedmann wrote down the FLRW metric for the first time in 1922, there was no reason to assume that the universe was expanding. Cosmic inflation was only confirmed seven years later by the efforts of Hubble and Lemaitre. They found a linear relation between the recessional velocity of galaxies and their distance from us by observing their redshift. The Hubble-Lemaître law, as it is called, is precisely of the form of equation (2.4), with $H_{0}=H\left(t_{0}\right)(\approx 70(\mathrm{~km} / \mathrm{s}) / \mathrm{Mpc})$; the Hubble parameter at the time of observation $t=t_{0}$ (sometimes referred to as Hubble's constant). With the Hubble parameter determined at the present time, the question naturally arises: what was it in the past?

Friedmann had already begun to answer this question when he derived his equations for the evolution of the scale factor from Einstein's field equations. Einstein's field equations describe the relation between the geometry of spacetime and the distribution of matter within it,

$$
\begin{equation*}
G_{\mu \nu}-g_{\mu \nu} \Lambda=8 \pi G T_{\mu \nu}, \tag{2.5}
\end{equation*}
$$

where $G_{\mu \nu}$ is the Einstein curvature tensor, $g_{\mu \nu}$ is the metric tensor, $\Lambda$ is the cosmological constant, and $T_{\mu \nu}$ is the stress-energy tensor of matter fields. The Einstein curvature tensor is derived uniquely from the geometry of spacetime, while the stress-energy tensor contains information about the density and flux of energy and momentum. The cosmological constant, denoted by $\Lambda$, is the energy of the vacuum that exists throughout the entire universe. At first, the importance of the cosmological constant was overlooked. However, as will be discussed later, it is actually quite important as it is part of where the $\Lambda$ CDM model got its name.

The Friedmann equations are derived based on the assumption that the universe can be modelled as a perfect fluid; a fluid without shear stresses, viscosity and heat conduction. This assumption is mostly predicated on the cosmological principle which requires the universe to be homogeneous and isotropic. A perfect fluid is characterized by two quantities, namely the energy density $\rho$ and the pressure $\mathcal{P}$. In the language of general relativity the expression for the stress-energy tensor of a perfect fluid is

$$
\begin{equation*}
T_{\mu \nu}=(\rho+\mathcal{P}) u_{\mu} u_{\nu}-g_{\mu \nu} \mathcal{P}, \tag{2.6}
\end{equation*}
$$

where $u_{\mu}$ is the four-velocity vector field of the fluid. In the rest (comoving) frame of the fluid, in which $u_{\mu}=\delta_{\mu}^{0}$, the components of the stress-energy tensor reduce to

$$
\begin{equation*}
T_{00}=\rho, \quad T_{0 i}=0, \quad T_{i j}=g_{i j} \mathcal{P} . \tag{2.7}
\end{equation*}
$$

The components of the Einstein curvature tensor can be derived from the metric in equation (2.2),

$$
\begin{equation*}
G_{00}=3\left[\left(\frac{\dot{a}}{a}\right)^{2}+\frac{\kappa}{a^{2}}\right], \quad G_{0 i}=0, \quad G_{i j}=g_{i j}\left[2 \frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^{2}+\frac{\kappa}{a^{2}}\right], \tag{2.8}
\end{equation*}
$$

and together with equations (2.5) and (2.7) they yield the Friedmann equations,

$$
\begin{align*}
& H^{2} \equiv\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3} \rho+\frac{\Lambda}{3}-\frac{\kappa}{a^{2}},  \tag{2.9}\\
& \dot{H}+H^{2} \equiv \frac{\ddot{a}}{a}=-\frac{4 \pi G}{3}(\rho+3 \mathcal{P})+\frac{\Lambda}{3} . \tag{2.10}
\end{align*}
$$

This set of equations tells us the history and fate of a universe defined by the set of parameters $(\rho, \mathcal{P}, \kappa, \Lambda)$. One particularly interesting case would be that of a vanishing vacuum energy $(\Lambda=0)$. Then, if $\rho+3 \mathcal{P}>0$, the expansion of the universe would be decelerating ( $\ddot{a}<0$ ). The condition, $\rho+3 \mathcal{P}>0$, is known as the strong energy condition and is satisfied by all conventional matter. In this scenario, our universe, which is expanding today, as we know from Hubble's law, has been expanding at every point in the past. Naturally, this means that the expansion must have started at some moment in the past with $a=0$. When the scale factor vanishes, the mass of the universe is contained in a single point: a singularity. First proposed by Lemaitre in 1931, the theory of a universe expanding from a very high density initial state became known as the big bang theory. The theory provides an upper bound on the age of universe by using the Hubble constant to trace back the slowest possible expansion (as shown in figure 2.1),

$$
\begin{equation*}
t_{0}-t_{\mathrm{BB}}<H_{0}^{-1}=13.799 \pm 0.021 \text { billion years. } \tag{2.11}
\end{equation*}
$$

Nowadays, it is well established that the vacuum energy is nonzero which has significant implications on the expansion history of the universe.


Figure 2.1: The upper bound on the age of the universe. The dashed line is the slowest possible expansion based on the Hubble constant. The solid line depicts an arbitrary evolution of the scale factor which starts at $t_{\mathrm{BB}}$. Image source: [13]

### 2.3 Expansion history of the universe

In order to uncover the expansion history of our universe, one must learn what the set of parameters $(\rho, \mathcal{P}, \kappa, \Lambda)$ looked like in the past. The cosmological constant does not change with time, while the other parameters are generally time-dependent. A powerful tool to constrain this time evolution is in the form of conservation laws. One such conservation law can be obtained from the Friedmann equations, namely

$$
\begin{align*}
\dot{\rho} & =-3 H(\rho+\mathcal{P})  \tag{2.12}\\
& =-3 H \rho(1+\omega), \quad \text { with } \omega=\mathcal{P} / \rho .
\end{align*}
$$

This is the expression for the conservation of energy in the cosmological setting, where $\omega$ is the equation of state parameter. Equation (2.12) can be solved to yield

$$
\begin{equation*}
\rho_{\omega}(t)=\frac{\rho_{\omega, 0}}{a^{3(1+\omega)}}, \tag{2.13}
\end{equation*}
$$

where $\rho_{\omega, 0}=\rho_{\omega}\left(t_{0}\right)$. The scaling properties of the energy density depend solely on the equation of state parameter. One can classify the matter of the universe based on this parameter; an overview is

| Major category | Minor category | $\omega$ | $\rho(t)$ | $\Omega_{i, 0}$ |
| :---: | :---: | :---: | :---: | :---: |
| Matter $(\mathrm{m})$ | Baryonic matter $(\mathrm{b})$ | 0 | $a^{-3}$ | $0.0493 \pm 0.0007$ |
|  | Dark matter $(\mathrm{dm})$ | 0 | $a^{-3}$ | $0.2645 \pm 0.0058$ |
| Radiation (r) | Neutrinos $(\nu)$ | $1 / 3$ | $a^{-4}$ | $<0.0021$ |
|  | Photons $(\gamma)$ | $1 / 3$ | $a^{-4}$ | $\sim 10^{-5}$ |
| Dark energy (de) | Vacuum energy $(\Lambda)$ | -1 | $a^{0}$ | $0.6847 \pm 0.0073$ |

Table 2.1: An overview of the matter content of the universe categorised based on the equation of state - The equation of state parameter - The scaling of the energy density - Values of the density parameters as obtained by the Planck 2018 collaboration [2].
given in table 2.1. The total energy density is the sum of the energy densities of all fluids contained in the universe

$$
\begin{equation*}
\rho=\sum_{i} \rho_{i}=\rho_{\mathrm{b}}+\rho_{\mathrm{dm}}+\rho_{\nu}+\rho_{\gamma}+\rho_{\Lambda}+\ldots, \tag{2.14}
\end{equation*}
$$

where the dots represent components that are too small to be relevant for large-scale dynamics of the universe. The Friedmann equation (2.9) can be rewritten in terms of dimensionless parameters

$$
\begin{equation*}
1=\Omega_{\mathrm{b}}+\Omega_{\mathrm{dm}}+\Omega_{\nu}+\Omega_{\gamma}+\cdots+\Omega_{\kappa}+\Omega_{\Lambda}, \tag{2.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega_{i}=\frac{\rho_{i}}{\rho_{\mathrm{cr}}}, \quad \rho_{\mathrm{cr}}=\frac{3}{8 \pi G} H^{2}, \quad \Omega_{\Lambda}=\frac{\Lambda}{3 H^{2}}, \quad \Omega_{\kappa}=-\frac{\kappa}{a^{2} H^{2}} . \tag{2.16}
\end{equation*}
$$

A universe with density $\rho_{\text {cr }}$ is spatially flat. The values of the dimensionless parameters today as obtained by observations can be found in table 2.1. Subsequently, the curvature term can be obtained from equation (2.15)

$$
\begin{equation*}
\Omega_{\kappa, 0} \equiv 1-\Omega_{\mathrm{b}, 0}-\Omega_{\mathrm{dm}, 0}-\Omega_{\nu, 0}-\Omega_{\gamma, 0}-\Omega_{\Lambda}-\cdots=-0.0027 \pm 0.0039 \longrightarrow\left|\Omega_{\kappa, 0}\right|<0.01 . \tag{2.17}
\end{equation*}
$$

Amazingly, the total energy density today is very close to the critical density resulting in a universe that is nearly spatially flat ( $\kappa \approx 0$ ). This coincidence is called the flatness problem and it is one of the fine-tuning problems without which the evolution of the universe would have proceeded very differently and life might not have been possible. This, and other, fine-tuning problems will be discussed in section 2.4.

Nowadays, the three largest energy contributions are from: dark energy (68.5\%), cold dark matter $(26.5 \%)$, and baryonic matter ( $4.9 \%$ ). However, in the history of the universe, there have been eras in which different types of matter were dominant. One can retrace the energy contributions at different times using the scaling of the energy densities in table 2.1 the result is shown in figure 2.2. The dominant energy contribution has changed twice in the history of the universe. Initially, the relativistic components such as photons and neutrinos represented the majority of the energy. At $a_{\mathrm{eq}} \simeq 3 \cdot 10^{-4}$ (or redshift $\left.z_{\mathrm{eq}} \simeq 3230\right)$ this changed, and matter became the dominant energy source. Only recently, at $a_{\Lambda} \simeq 0.7(z \simeq 0.4)$, the universe entered a dark energy-dominated era.

Using the Friedmann equation (2.9), one can derive the evolution of the scale factor as a function of time in flat, single-component universes. Single-component universes can serve as approximations to the universe away from moments of energy equality. In accordance with the discussion in section 2.2, the rate of expansion of a flat universe decelerates in matter- and radiation-dominated universes. Whereas in a flat universe described exclusively by a positive cosmological constant the


Figure 2.2: A plot of the energy densities in the expansion history of the universe. The energy densities of radiation (red line), matter (blue line), dark energy (green line) scale differently with the scale factor. Therefore, the energy composition of the universe is constantly changing. The vertical dotted lines show the moments of energy equality between matter-radiation and matterdark energy. As such they mark the boundaries of the different eras. Image source: [14]
scale factor grows exponentially with time. Such a universe is characterized as a de Sitter universe. While a de Sitter universe expands exponentially on spatially flat sections, the same is only true at late times for curved sections of spacetime.

One can extend this analysis to multi-component universes. For example, in a flat universe dominated by matter and a cosmological constant, the scale factor is

$$
\begin{equation*}
a(t)=\left(\frac{\Omega_{m}}{8 \pi G \Omega_{\Lambda}}\right)^{1 / 3} \sinh ^{2 / 3}\left(\frac{\sqrt{3 \Lambda} t}{2}\right) . \tag{2.18}
\end{equation*}
$$

This solution to the Friedmann equation describes our universe quite well. At late times the scale factor increases exponentially ( $a \sim e^{H t}$ ) as expected, while at early times the scale factor increases by the same power law as in a matter-dominated universe ( $a \sim t^{2 / 3}$ ). Unlike the de Sitter universe, the solution in 2.18) has a singularity at $t=0$, which is consistent with the paradigm of the big bang theory.

One can go another step further. In the present age of data processing, the Friedmann equation (2.15) can be integrated numerically which allows for more complicated models. The simplest parametrization that describes our universe well is the $\Lambda$ CDM model which is based on six parameters. The values of these parameters are not predicted by theory, but rather derived from

| Description | Symbol | value |
| :---: | :---: | :---: |
| Baryon density parameter | $\Omega_{b} h^{2}$ | $0.02242 \pm 0.00014$ |
| Dark matter density parameter | $\Omega_{c} h^{2}$ | $0.11933 \pm 0.00091$ |
| Size of the sounds horizon | $100 \theta_{\mathrm{MC}}$ | $1.04092 \pm 0.00031$ |
| Scalar spectral index | $n_{s}$ | $0.9649 \pm 0.0042 \mathrm{~km} / \mathrm{s} / \mathrm{Mpc}$ |
| Amplitude of primordial curvature fluctuations | $\ln \left(10^{10} A_{s}\right)$ | $3.044 \pm 0.014$ |
| Reionization optical depth | $\tau_{e}$ | $0.66 \pm 0.012$ |

Table 2.2: The six base parameters of the $\Lambda$ CDM model and their values as obtained by the Planck 2018 collaboration [2]
observations. An overview of the parameters and their values are given in table 2.2 . The $\Lambda$ CDM model presupposes that certain other parameters, such as the total energy density, are fixed. Natural extensions of the $\Lambda$ CDM model allow one or more of these fixed parameters to vary in an attempt to remedy some of the shortcomings of the model; usually at the cost of certainty.

### 2.4 Inflation

Using the $\Lambda$ CDM model one can quite confidently trace the universe back to when it was about one picosecond old; when the four fundamental forces took their present form. However, some questions remain unanswered. For example, why is the universe almost spatially flat? Or, why is the universe homogeneous and isotropic on large scales? That last one is particularly hard to answer given that the theory of gravitation is attractive and, thus, inhomogeneities are expected to grow. In order to explain both of these queries, the universe must have started out in an incredibly fine-tuned initial state. That is to say, a miracle took place. For most physicists this would be too much of a coincidence, to not be a problem. In this section, we introduce the concept of inflation; a short period of rapid expansion in the first picosecond of the universe. The theory of inflation resolves the aforementioned issues, as well as others, rather successfully.

### 2.4.1 The flatness problem

The first query is known as the flatness problem; a universe as flat as ours is improbable. In order to quantify the flatness problem, recall the Friedmann equation 2.15 and the definition of the curvature density parameter 2.16

$$
\begin{equation*}
1=\Omega_{\mathrm{b}}+\Omega_{\mathrm{dm}}+\Omega_{\nu}+\Omega_{\gamma}+\cdots+\Omega_{\kappa}+\Omega_{\Lambda}, \quad \Omega_{\kappa}=-\frac{\kappa}{a^{2} H^{2}} \tag{2.19}
\end{equation*}
$$

The value of the curvature density parameter today is constraint by observation, $\left|\Omega_{\kappa, 0}\right|<0.01$. Using the expansion history of the universe discussed in section 2.3 one can trace its value back to the early universe. A sufficiently accurate approximation can be made by modelling the matterand radiation-dominated eras as single-component universes. In this case, the curvature density parameter evolves as

$$
\begin{array}{ll}
\Omega_{\kappa}(a)=\Omega_{\mathrm{tot}}(a)-1 \sim a & (\text { matter era }) \\
\Omega_{\kappa}(a)=\Omega_{\mathrm{tot}}(a)-1 \sim a^{2} & \text { (radiation era) } \tag{2.21}
\end{array}
$$

As a result, at the time of matter-radiation equality $\left|\Omega_{\kappa}\right| \leq 10^{-6}$, while at the beginning of radiation era $\left|\Omega_{\kappa}\right| \leq 10^{-30}$. Hence, the universe was even more flat in the past. This level of flatness is considered a fine-tuning problem. Fine-tuning may not technically be a problem as it is physically possible, nevertheless we would like an explanation for this phenomena if possible.

One such explanation is provided by adding a phase of de Sitter inflation before the beginning of the radiation era. During de Sitter inflation, the curvature density parameter evolves as

$$
\begin{equation*}
\Omega_{\kappa}(a)=\Omega_{\mathrm{tot}}(a)-1 \sim a^{-2} \quad(\text { de Sitter inflation }) . \tag{2.22}
\end{equation*}
$$

The inflationary phase drives the universe to be flat. If inflation last sufficiently long, the spatial curvature can be become as small as needed. An equivalent argument based on geometry can be stated as follows: any small region of a smooth, possibly curved, manifold will become flat when enlarged by a sufficient amount. Thus, our universe could have started with practically any non-zero curvature before inflation drove it to be as flat as it would need to be at the start of the radiation era. Hence, fine-tuning of the curvature parameter is no longer needed and the flatness problem is solved.

### 2.4.2 The horizon problem

As mentioned in the beginning of this chapter, the cosmological principle of large scale homogeneity and isotropy is satisfied by our universe. The strongest footprint of the cosmological principle in the early universe is found in the oldest electromagnetic radiation: the cosmic microwave background radiation (CMBR). The CMBR is a remnant of the hot big bang, i.e., when the universe was a hot plasma in an approximate thermal equilibrium. As the universe expanded and cooled down, the matter became less dense and photons were able to move freely, whereas before photons were constantly scattered by uncombined electrons and protons. Said photons have been travelling through space ever since. The CMBR that is reaching us now has a very precise spectrum of wavelengths, that of a black-body with temperature $2.725 \pm 0.001 \mathrm{~K}$. Figure 2.3 shows the anisotropies of the CMBR, which are at the level of $\delta T \sim 30 \mu \mathrm{~K}$. This level of isotropy represents another fine-tuning


Figure 2.3: A map of the temperature fluctuations of the cosmic microwave background radiation (CMBR) based on the data of the Wilkinson Microwave Anisotropy Probe (WMAP). Image source: [15]
problem. According to the $\Lambda$ CDM-model as described earlier in this chapter, different regions of the cosmic microwave background were out of causal contact at the time the CMBR was created. Following the evolution of the universe described in this chapter, one would be able to observe $\sim 10^{3}$ of these regions nowadays. Here is the crux of the horizon problem: how can $\sim 10^{3}$ causally disconnected regions have the same temperature up to an accuracy of $\delta T / T \sim 10^{-5}$ ?

The horizon problem is nicely formulated with the help of conformal diagrams, also known as Penrose diagrams. The goal of a conformal diagram is to show the causal structure of spacetime. These diagrams are obtained by applying a particularly clever coordinate transformation on ordinary space-time diagrams. An infinite spacetime is mapped to a finite representation. In such a representation, the light cones are portrayed at $45^{\circ}$ and each point represents a 2 -sphere. Figure 2.4 a shows a conformal diagram of our universe without inflation. Two points in the CMB that are sufficiently separated are shown to have been out of causal contact since the beginning of the universe at $\tau_{i}=0$. The light cones do not overlap. This means that these two points of the sky have never been able to interact. Therefore, it is strange that they are observed to be at equal temperature. In reality, there are a lot of regions that are causally disconnected in this way. The way inflation solves this issue is by adding a whole section to the conformal diagram between $\tau_{i}=-\infty$

(a) A conformal diagram of the universe without inflation. At the time of the CMBR, the two pink regions are outside of causal contact. Taking into account the evolution of the universe, one finds that these regions have been out of causal contact since the hot big bang at $\tau_{i}=0$.

(b) A conformal diagram of the universe with inflation. Before the hot big bang at $\tau=0$, the universe underwent a phase of rapid expansion in the first picosecond of the universe. In the conformal diagram, inflation occurs between $\tau_{i}=-\infty$ and $\tau=0$ allowing the pink regions to have causally communicated in the past.

Figure 2.4: Figures from [16]
and $\tau=0$. Assuming inflation lasts long enough, the light cones of the previous two regions now overlap in the past. With inflation in place, it makes sense that the temperature distribution of the CMBR is homogeneous and isotropic on large scales because the universe reached thermal equilibrium at some time during inflation. In this way, inflation provides an elegant solution to the horizon problem.

### 2.5 Inflationary models

The previous discussion seems to suggest that the universe has gone through an era of inflation. Unfortunately, there is no direct evidence that supports this suggestion. One can only test the predictions of inflation against observations. Certainly, the observed spatial flatness of the universe and the large scale isotropy and homogeneity count towards indirect evidence. Although unverified, the theory of inflation is a vital part of the current understanding of the universe. Therefore, we would like a comprehensive model of inflation. Since this is an active area of research, there is a variety of models, some have more merit than others.

The first model of inflation was published by Starobinsky in 1980 [1]. His model is based on a generalisation of Einstein's theory of general relativity to include curvature squared corrections. The action for this particular model is,

$$
\begin{equation*}
S_{S}=\frac{1}{2} \int d^{4} x \sqrt{-g}\left\{M_{p}^{2} R+\frac{R^{2}}{6 M^{2}}\right\}, \tag{2.23}
\end{equation*}
$$

where $g$ is the metric determinant, $M_{p}$ is the reduced Planck mass, $R$ is the Ricci scalar, and $M$ is a constant. Starobinsky wanted to include quantum-gravitational effects to see how they might change the beginning of the universe. The first quantum corrections to Einstein's equations manifest themselves as curvature squared corrections to the Einstein-Hilbert action. The addition of $R^{2}$ leads to an effective scalar, called the scalaron, when curvatures are large, i.e., when the universe is small. This led Starobinsky to a picture of the universe which is initially in a de Sitter state, meaning that flat spatial sections are expanding exponentially fast. Later, Chibisov and Mukhanov found that the spectrum of scalar cosmological perturbations in Starobinsky's model is nearly scale invariant [17], which still agrees with current observations of anisotropies in the CMBR. Nowadays, the Starobinsky model and its descendants are categorized as $f(R)$ modified gravity theories.

Another way to model inflation is by using the vacuum energy of a scalar field. Imagine a universe dominated by the energy of a scalar field $\Phi(t, \vec{x})$, known as the inflaton field. The dynamics of such a scalar field in a curved spacetime is described by the following action

$$
\begin{equation*}
S=S_{E H}+S_{\Phi}=\int d^{4} x \sqrt{-g}\left\{\frac{M_{p}^{2}}{2} R-\frac{1}{2} g^{\mu \nu} \partial_{\mu} \Phi \partial_{\nu} \Phi-V(\Phi)\right\} \tag{2.24}
\end{equation*}
$$

where $V(\Phi)$ denotes the potential of the scalar field, which describes its self-interaction. In this particular action the scalar field is minimally coupled to gravity for the sake of simplicity. For the same reason, the cosmological constant, $\Lambda$, is set to zero. The equation of motion for the scalar field is obtained by varying the action with respect to $\Phi$,

$$
\begin{equation*}
\frac{\delta S}{\delta \Phi}=\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} \partial^{\mu} \Phi\right)-V^{\prime}(\Phi)=0 \tag{2.25}
\end{equation*}
$$

where $V^{\prime}(\Phi)=d V(\Phi) / d \Phi$. The effect of the scalar field on the surrounding spacetime and vice versa is governed by the Einstein equation

$$
\begin{equation*}
G_{\mu \nu}=8 \pi G T_{\mu \nu}^{\Phi} \tag{2.26}
\end{equation*}
$$

where the energy-momentum tensor is

$$
\begin{equation*}
T_{\mu \nu}^{\Phi}=\partial_{\mu} \Phi \partial_{\nu} \Phi-g_{\mu \nu}\left(\frac{1}{2} \partial^{\sigma} \Phi \partial_{\sigma} \Phi+V(\Phi)\right) \tag{2.27}
\end{equation*}
$$

From this point on, assume a spatially homogeneous scalar field $\Phi(t, \vec{x})=\phi(t)$ and the FLRW metric as in equation 2.1). The energy-momentum tensor reduces to that of a perfect fluid with energy density $\rho_{\phi}$ and pressure $\mathcal{P}_{\phi}$ given by

$$
\begin{align*}
\rho_{\phi} & =\frac{1}{2} \dot{\phi}^{2}+V(\phi),  \tag{2.28}\\
\mathcal{P}_{\phi} & =\frac{1}{2} \dot{\phi}^{2}-V(\phi) . \tag{2.29}
\end{align*}
$$

Additionally, Einstein's equations yield the Friedmann equations as discussed in section 2.2

$$
\begin{align*}
& H^{2} \equiv\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3} \rho_{\phi}  \tag{2.30}\\
& \dot{H}+H^{2} \equiv \frac{\ddot{a}}{a}=-\frac{4 \pi G}{3}\left(\rho_{\phi}+3 \mathcal{P}_{\phi}\right) \tag{2.31}
\end{align*}
$$

where the spatial curvature is neglected because inflation will flatten the universe immensely. Finally, the equation of motion for the scalar field reduces to

$$
\begin{equation*}
\ddot{\phi}+3 H \dot{\phi}+V^{\prime}(\phi)=0 . \tag{2.32}
\end{equation*}
$$

The previous five equations fully determine the dynamics of the inflaton field, although they are not all linearly independent. From these equations it is not immediately clear what happens to the scale factor; whether or not the universe is expanding and at what rate. To this end, examine the equation of state parameter introduced in section 2.3 .

$$
\begin{equation*}
\omega_{\phi}=\frac{\mathcal{P}_{\phi}}{\rho_{\phi}}=\frac{\frac{1}{2} \dot{\phi}^{2}-V(\phi)}{\frac{1}{2} \dot{\phi}^{2}+V(\phi)} . \tag{2.33}
\end{equation*}
$$

A universe dominated by a scalar will exhibit accelerated expansion only if $\omega_{\phi}<-1 / 3$. It is clear that the scalar potential is of some importance in this matter.

In 1981, Alan Guth proposed a potential that would lead to inflation. In his original work [19], he elaborated on the practicality of inflation when it comes to solving the fine-tuning problems. He continued by describing inflation as a delayed first order phase transition. The scalar potential he proposed is shown in figure 2.5a. Initially, the inflaton is stuck in a local minimum of the potential, and then tunnels through the potential barrier towards the true minimum. Guth realized that this approach to inflation is inherently flawed. The transition to the true vacuum occurs by chance and, therefore, can not occur everywhere simultaneously. In an exponentially expanding universe, it is impossible for the entire space to tunnel to the true vacuum. In fact, the amount of space in the false vacuum increases because the expansion of space is much faster than the tunnelling process.

Therefore, the inflationary scenario Guth proposed is unending. This is known as the 'graceful exit problem'.

In order to solve the graceful exit problem, one might consider the potential in figure 2.5b, introduced by Andrei Linde and by Albrecht and Steinhardt in 1982. The potential is very flat around the origin, i.e., $V^{\prime}(\phi) \approx 0$. This way, the inflaton rolls slowly down the potential into the true vacuum, avoiding the need for tunnelling. The benefit of this potential is that inflation naturally ends when the inflaton reaches the end of the flat section, thus solving the graceful exit problem. The drawback is that inflation generated in this way generally does not last long enough to solve the flatness- and horizon problem. Only when the potential is extremely flat or when the initial conditions are fine-tuned such that the scalar field starts at $\phi \approx 0$ and $\dot{\phi} \approx 0$, but $\phi \neq 0$. Certainly such fine-tuning conditions are undesirable.

One model that attempts to deal with these fine-tuning conditions is the model of chaotic inflation, also proposed by Andrei Linde. The model relies on either of the following potentials,

$$
\begin{equation*}
V(\phi)=\frac{1}{2} m^{2} \phi^{2}, \quad V(\phi)=\frac{1}{4!} \lambda \phi^{4} \tag{2.34}
\end{equation*}
$$

which are relatively simple; $m$ and $\lambda$ are constants. Using incorrect arguments, Linde argues that it is natural for the scalar field to start far away from the origin, such that $V(\phi) \gg \dot{\phi}^{2}$, which is required to start inflation. Not only is a revision of the initial scale of inflation required, but also the naturalness of the potential itself can be called into question. In order to reconcile the predictions of chaotic inflation with large scale CMB anisotropies, one has to fine-tune the constant in the potential to a very small number. Furthermore, chaotic inflation based on the quartic potential has been disfavoured by observations based on the relatively large deviation from scale invariance.

The above discussion implies that there exist some conditions for inflation: the potential must be sufficiently flat such that the potential energy dominates the kinetic energy and $\ddot{\phi}$ must be small such that inflation lasts long enough. These requirements are known as the slow-roll conditions,

$$
\begin{equation*}
\dot{\phi}^{2} \ll V(\phi), \quad \ddot{\phi} \ll 3 H \dot{\phi}, V^{\prime}(\phi) \tag{2.35}
\end{equation*}
$$


(a) The scalar potential proposed by Guth which gives rise to inflation by means of a first order phase transition. The idea is that the inflaton field would tunnel through the potential barrier into the true vacuum. However, the tunnelling process is too slow and, thus, the resulting inflationary period is endless.

(b) An alternative proposal for the scalar potential from an inflationary model called 'new inflation'. In this case, the potential is extremely flat around the origin. The necessity for such a flat potential inadvertently leads to fine-tuning conditions, called slow-roll conditions.

Figure 2.5: Figures from [18]

It is mathematically more desirable to formulate these conditions in terms of dimensionless quantities, which are known as the slow-roll parameters,

$$
\begin{align*}
\epsilon_{V} & \equiv \frac{1}{2} M_{p}^{2}\left(\frac{V^{\prime}}{V}\right)^{2}  \tag{2.36}\\
\eta_{V} & \equiv M_{p}^{2} \frac{V^{\prime \prime}}{V} \tag{2.37}
\end{align*}
$$

In terms of these parameters, the slow-roll conditions are simply $\epsilon_{V}, \eta_{V} \ll 1$. In the slow-roll regime, the Friedmann equations become

$$
\begin{align*}
& H^{2}=\frac{8 \pi G}{3} V(\phi),  \tag{2.38}\\
& H^{2}+\dot{H}=\frac{8 \pi G}{3} V(\phi) . \tag{2.39}
\end{align*}
$$

These equations immediately imply that $\dot{H} \approx 0$, which represents an almost exponentially expanding universe known as a 'quasi-de Sitter' universe. Inflation ends when the slow-roll parameters become of order unity. The duration of inflation is expressed in terms of the number of e-foldings,

$$
\begin{equation*}
N=\int_{t}^{t_{e n d}} H\left(t^{\prime}\right) d t^{\prime}=\ln \left(\frac{a_{\text {end }}}{a}\right) \tag{2.40}
\end{equation*}
$$

The flatness problem and the horizon problem require inflation to last at least 60 e-foldings, which is satisfied in most inflationary scenarios. After inflation has ended, the inflaton enters an oscillatory regime after which it will decay. The energy of the inflaton potential will be released and take the form of a thermalized gas of Standard Model particles in a process called 'reheating'.

### 2.6 Evolution of scales and large scale structure formation

A comprehensive model of the universe explains all that can be observed within it. The most palpable observation is the presence of structure in our universe. Structure is the collective name for planets, solar systems, galaxies, galaxy clusters and many other compact objects in space. Thus, structure can be found at many different scales. A big success of the $\Lambda$ CDM model is the prediction of structure formation on the largest scales. The presence of galaxy clusters and voids is said to originate from quantum fluctuations in the early universe. Due to the evolution of the universe, in particular the rapid expansion of inflation, these tiny density fluctuations grow to be larger and larger. An early imprint of these density fluctuations is found in the temperature spectrum of the cosmic microwave background radiation emitted at the end of the radiation era (see figure 2.3). These density fluctuations form the seeds of large scale structure, in that, after the decoupling of photons, baryons could freely move and, due to the attractive nature of gravity, over-densities accumulated more and more matter which eventually lead to the formation of galaxies.

There are several important scales relevant to structure formation. The most important scales are the Hubble radius $\left(R_{H}\right)$ and the physical scale of perturbations ( $\lambda_{\text {phys }}$ ). While the latter evolves proportional to the scale factor, the former scales as one over the Hubble parameter,

$$
\begin{equation*}
R_{H} \propto \frac{1}{H}, \quad \lambda_{\text {phys }} \propto a \tag{2.41}
\end{equation*}
$$

The Hubble radius is interpreted as the distance to the Hubble limit. Beyond the Hubble limit, the expansion of the universe is faster than the speed of light. Therefore, any two regions separated by a distance greater than the Hubble radius are outside of causal contact with each other. Figure 2.6 shows the evolution of the aforementioned scales in inflation, radiation era, and matter era. During inflation, the Hubble radius is constant while the physical scale of perturbations grows with the expansion rate of the universe. Inevitably, the physical scale of the perturbations is going to outgrow the Hubble radius. The first horizon crossing, as it is called, occurs at $a_{1 \mathrm{x}}$ in figure 2.6. Hence, one can classify a scale $R$ as sub-Hubble ( $R \ll R_{H}$ ), or super-Hubble ( $R \gg R_{H}$ ). At sub-Hubble scales the amplitude of vacuum fluctuations for any field evolves inversely proportional to the scale, i.e., $\left.\delta \varphi\right|_{R} \propto 1 / R$ for $R \ll R_{H}$. At super-Hubble scales, the behaviour of fluctuations is not the same for all fields. One distinguishes between conformally coupled fields, which do not couple to gravity when appropriately rescaled, and nonconformally coupled fields. The latter category will be the most relevant as it includes minimally coupled scalar fields, gravitons and massive fermionicand gauge fields. The amplitude of fluctuations for nonconformally coupled fields remains constant at super-Hubble scales while for conformally coupled fields it scales as $1 / R$. This means that, at the second horizon crossing $a_{2 \mathrm{x}}$, there are large-scale fluctuations whose amplitudes have not been suppressed by the universe's expansion. In particular, this is true for scalar metric fluctuations which manifest as gravitational potential wells upon re-entering the horizon. These gravitational wells pull in surrounding matter, starting an accumulation which will eventually result in galaxy clusters and, on the other hand, voids.

There is one important ingredient which was not explicitly mentioned in the above discussion: dark matter. Although it is included in the term 'matter', a specific remark is in order. Dark matter is an important puzzle piece in understanding the evolution of the universe and, thus, structure formation. It is estimated that dark matter accounts for $85 \%$ of matter in the universe. Dark matter is different from baryonic matter in that it only interacts with gravity. Other than that, it


Figure 2.6: The evolution of scales in a simplified history of the universe. The blue line tracks the Hubble radius and the red line tracks the physical scale of perturbations in time through inflation, radiation era, and matter era. The marks on the time axis, $a_{1 \mathrm{x}}$ and $a_{2 \mathrm{x}}$, denote the first and second horizon crossing of the physical scale.
is a mysterious form of matter; its composition is unknown. However, because it does not interact with radiation it can move freely in the radiation era. Therefore, it starts to accumulate earlier than baryonic matter. It is quite certain that without dark matter structure formation would not have been able to progress to its current state.

Finally, a proper discussion of perturbations and large scale structure formation requires quantization of the relevant fields. Chapter 4 contains such, and a treatment of quantum fluctuations.

## Chapter 3

## Cosmological perturbations

In reality, the inflationary scenario is more complicated than the homogeneous picture sketched in section 2.5. The universe is not homogeneous at small scales. Quantum fluctuations in the inflaton field generated during inflation are claimed to be responsible for the temperature anisotropies of the cosmic microwave background radiation. Not only that, these primordial (quantum) fluctuations eventually led to the formation of the large scale structure of the universe (as discussed in section 2.6). This chapter provides a classical treatment of perturbations at both linear and quadratic order. In section 3.1, the background field method will be introduced and the perturbations will be categorized according to the scalar-vector-tensor decomposition. Sections 3.2 and 3.3 discuss the scientific jargon concerning gauges. Then, section 3.4 covers the main principles from earlier sections applied to classical linear perturbation theory of gravity by expanding Einstein's equations. Quadratic perturbation theory is treated in the framework of the action formalism, introduced in section 3.5. We start by following existing literature in deriving the quadratic action in the comoving gauge. With an eye on future calculations, we introduce a different gauge, named the stacked gauge, in section 3.6. A gauge transformation is performed in order to obtain the action in the desired gauge. Finally, in section 3.7, the exact equations of motion are derived and expanded up to quadratic order in perturbations. This quadratic equation of motion will be used in chapter 5 to calculate the one-loop quantum backreaction to the tadpole equation.

### 3.1 Scalar-vector-tensor decomposition

Cosmological perturbations are usually studied by splitting the field into a homogeneous classical background and inhomogeneous (quantum) fluctuations

$$
\begin{align*}
& g_{\mu \nu}(t, \vec{x})=g_{\mu \nu}^{b}(t)+\delta g_{\mu \nu}(t, \vec{x})  \tag{3.1}\\
& \Phi(t, \vec{x})=\phi(t)+\varphi(t, \vec{x}), \tag{3.2}
\end{align*}
$$

where the background fields are denoted by $g_{\mu \nu}^{b}, \phi$, and the fluctuations are denoted by $\delta g_{\mu \nu}$ and $\varphi$. This method is based on the assumption that the fluctuations are small compared to the background, i.e., $\delta g_{\mu \nu} \ll 1$ and $\varphi \ll \phi$. There is one caveat to the background field method: the above decomposition is far from unique. This is on account of the fact that there are more degrees of freedom than required to describe the underlying physics. As a result, not all degrees of freedom are physical. A more technical discussion of gauge freedom and gauge transformations will take place in section 3.2.

It is common practice to categorize perturbations based on their transformation under spatial rotation. Each of these components evolves separately (at linear order) which simplifies the study of cosmological perturbations significantly. There are three categories: scalar, vectorial, and tensorial perturbations. Their classification is based on helicity. The effect of helicity can be shown as follows. Let $\delta X$ be a generic perturbation, then its Fourier form is

$$
\begin{equation*}
\delta X_{\vec{k}}(t)=\int d^{3} x e^{i \vec{k} \vec{x}} \delta X(t, \vec{x}) . \tag{3.3}
\end{equation*}
$$

Performing a rotation by an angle $\theta$ around the wave vector $\vec{k}$ will demonstrate the essential property for massless perturbations,

$$
\begin{equation*}
\delta X_{\vec{k}}(t) \rightarrow e^{i m \theta} \delta X_{\vec{k}}, \tag{3.4}
\end{equation*}
$$

where $m$ is called the helicity. Scalar, vectorial and tensorial perturbations have helicity $m=$ $0, \pm 1, \pm 2$ respectively. The decomposition based on this property is called the scalar-vector-tensor decomposition, or SVT decomposition for short.

Let us apply the SVT decomposition to the metric fluctuation in equation (3.1). One can rewrite the metric up to first order in perturbations,

$$
\begin{align*}
d s^{2} & =g_{\mu \nu} d x^{\mu} d x^{\nu} \\
& =-(1+2 \Xi) d t^{2}-2 a B_{i} d t d x^{i}+a^{2}\left[(1-2 \Psi) \delta_{i j}+2 S_{i j}\right] d x^{i} d x^{j}, \tag{3.5}
\end{align*}
$$

where the background metric is taken to be $g_{\mu \nu}^{b}=\operatorname{diag}\left(-1, a^{2}(t), a^{2}(t), a^{2}(t)\right)$. One can break down the above quantities further by explicitly separating them into scalar, vectorial and tensorial degrees of freedom

$$
\begin{array}{ll}
B_{i}=\partial_{i} B+B_{i}^{T}, & \text { with } \partial^{i} B_{i}^{T}=0, \\
S_{i j}=2 \partial_{i} \partial_{j} E+2_{(i} F_{j)}+h_{i j}^{T T} & \text { with } \partial^{i} F_{i}=0, \text { and } h_{i i}^{T T}=0=\partial_{i} h_{i j}^{T T} \tag{3.7}
\end{array}
$$

Some of these fields are restricted by constraints, e.g., $B_{i}^{T}, F_{i}$, and $h_{i j}^{T T}$. The following is an overview of the degrees of freedom and their category.

- 4 scalars: $\Xi, \Psi, B, E \quad$ i.e. 4 scalar degrees of freedom,
- 2 vectors: $B_{i}^{T}, F_{i} \quad$ i.e. 4 vectorial degrees of freedom,
- 1 tensor: $h_{i j}^{T T} \quad$ i.e. 2 tensorial degrees of freedom.

In total this accounts for the ten independent components of the symmetric $4 \times 4$ metric perturbation $\delta g_{\mu \nu}$. As mentioned, not all of these are physical degrees of freedom. The scalar and vector components will transform under a general coordinate transformation. Consequently, there exists a transformation after which either the scalar or vector perturbation is zero. Therefore, these perturbations should not be interpreted as physical. On the other hand, the tensor components remain invariant under such a transformation. The symmetric tensor $h_{i j}^{T T}$ has six independent components and satisfies the four constraints in equation (3.7). The remaining two independent tensorial degrees of freedom of $h_{i j}^{T T}$ are interpreted as the two polarisations of gravitational waves: the plus $(+)$ and cross $(\times)$ polarization. Gravitational waves carry energy in the form of gravitational radiation. They can be generated by the acceleration of massive objects, such as in a binary black hole. To some level, they are also generated during inflation and as such they will be relevant in the remainder of this thesis. Similarly, scalar perturbations are important during inflation as they are generated by the vacuum fluctuation of the inflaton field. Vector perturbations, in comparison, are generally considered irrelevant during inflation and, thus, will be disregarded in this thesis.

### 3.2 Gauge transformations

The theory of general relativity is invariant under diffeomorphisms; the metric field $g_{\mu \nu}$ may be changed at will by performing a general coordinate transformation. Such an operation will not change the underlying physics, since it is simply a relabelling that leaves the underlying theory unchanged. However, this can give rise to ambiguities; certain perturbations might vanish by performing a particular transformation. Thus, it is crucial to distinguish between the true dynamical information and the information related to the coordinate system. Generally, there are two ways to avoid the clash between the smaller number of variables describing the dynamics and the redundant variables needed to ensure the correct transformation properties: gauge fixing and gauge invariant variables.

Fixing a gauge corresponds to establishing the slicing and threading of spacetime. Slicing the spacetime refers to the choice of the equal time hypersurfaces. Threading the spacetime then defines the spatial coordinates on each spacelike hypersurface. The most intuitive way to illustrate the workings of this slicing and threading is at the hand of the ADM-formalism. Performing an ADM-decomposition of the metric comes down to rewriting it as follows,

$$
\begin{equation*}
d s^{2}=-N^{2} d t^{2}+g_{i j}\left(d x^{i}+N^{i} d t\right)\left(d x^{j}+N^{j} d t\right), \tag{3.8}
\end{equation*}
$$

where $N=N(t, \vec{x})$ is called the lapse function and $N_{i}=N_{i}(t, \vec{x})$ is called the shift vector. The shift vector takes into account the movement of the observer on the three-space while the lapse function captures the fact that the evolution of coordinate time is generally not homogeneous throughout spacetime. Neither are physical effects, rather, they are coordinate effects. Figure 3.1 shows an image of the physical interpretation of the lapse function and shift vector between two space-like hypersurfaces. In the ADM-formalism, the coordinate-dependent information is nicely separated from the true physical dynamics. Fixing a gauge comes down to solving the constraint equations satisfied by the lapse function and shift vector, which then defines the slicing and threading of spacetime.


Figure 3.1: The changes in coordinate system between two space-like hypersurfaces $\Sigma_{t}$ and $\Sigma_{t+d t}$ as a result of the lapse function and shift vector. The shift vector $N_{i}$ defines the movement of the coordinate system on the hypersurfaces; if $N_{i}=0$, the hypersurfaces are stacked on top of each other perfectly. The lapse function $N$ describes the local evolution of coordinate time; a non-uniform lapse function will cause the hypersurface to contort along the time axis.

As already mentioned, there are specific gauges in which perturbations vanish. Two examples are the comoving gauge and the zero-curvature gauge. In the comoving gauge, the slicing of spacetime is chosen orthogonal to comoving observers such that the observer does not witness any perturbations in the scalar field, i.e., $\varphi=0$. In the zero-curvature gauge, the trace over the spatial components of the metric vanishes, i.e., $\operatorname{Tr}\left[g_{i j}\right]=0$. One can go from one gauge to another by performing a gauge transformation. Consider an infinitesimal coordinate transformation,

$$
\begin{equation*}
x^{\mu} \rightarrow \tilde{x}^{\mu}=x^{\mu}+\xi^{\mu}(x), \tag{3.9}
\end{equation*}
$$

where $\xi^{\mu}$ is some infinitesimal vector. The scalar field and the metric field transform under this coordinate transformation as follows

$$
\begin{align*}
& \Phi(x) \rightarrow \tilde{\Phi}(\tilde{x})=\Phi(x),  \tag{3.10}\\
& g_{\mu \nu}(x) \rightarrow \tilde{g}_{\mu \nu}(\tilde{x})=\frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\nu}} g_{\rho \sigma}(x) . \tag{3.11}
\end{align*}
$$

Expanding up to linear order in $\xi^{\mu}$, the above equations become

$$
\begin{align*}
& \tilde{\Phi}(x)=\Phi(x)-\xi^{\mu} \partial_{\mu} \Phi(x),  \tag{3.12}\\
& \tilde{g}_{\mu \nu}(x)=g_{\mu \nu}(x)-\nabla_{\mu} \xi_{\nu}(x)-\nabla_{\nu} \xi_{\mu}(x)=g_{\mu \nu}(x)-\mathcal{L}_{\xi} g_{\mu \nu}(x), \tag{3.13}
\end{align*}
$$

where $\nabla_{\mu}$ is the covariant derivative and $\mathcal{L}_{\xi}$ is the Lie derivative in the direction of $\xi$. The above transformations form the group of infinitesimal coordinate transformations. The arbitrary vector $\xi^{\mu}$ has four independent components. In principle, one could set four out of the eleven fields to zero with a single gauge transformation such that only dynamical fields remain.

### 3.3 Gauge-invariant variables

The definition of a gauge transformation in section 3.2 also immediately defines what it means for a field to be gauge-invariant: it is identical in all reference frames. Constructing a gauge-invariant formalism is typically very challenging. In practice, one does not specify the hypersurfaces of constant time as one does when fixing a gauge. Instead, the choice of hypersurfaces is left arbitrary and one defines infinitesimally invariant variables on them. When constructing gauge-invariant variables out of the metric perturbation it is important to distinguish between the scalar, vectorial and tensorial degrees of freedom as discussed in section 3.1. Recall that the metric perturbation can be decomposed in four scalars $\Xi, \Psi, B, E$, two vectors $B_{i}^{T}, F_{i}$ and one tensor $h_{i j}^{T T}$,

$$
\begin{align*}
d s^{2} & =g_{\mu \nu} d x^{\mu} d x^{\nu} \\
& =a^{2}(\eta)\left\{-(1+2 \Xi) d \eta^{2}-2 B_{i} d \eta d x^{i}+\left[(1-2 \Psi) \delta_{i j}+2 S_{i j}\right] d x^{i} d x^{j}\right\}, \tag{3.14}
\end{align*}
$$

where $d \eta=a^{-1} d t$ defines conformal time. According to the transformation law in equation (3.11), the components of the metric perturbation transform as follows

$$
\begin{align*}
& \Xi \rightarrow \tilde{\Xi}=\Xi+\frac{a^{\prime}}{a} \xi^{0}+\xi^{0 \prime},  \tag{3.15}\\
& B \rightarrow \tilde{B}=B+\xi^{0}+\xi^{\prime},  \tag{3.16}\\
& B_{i}^{T} \rightarrow \tilde{B}_{i}^{T}=B_{i}^{T}-\frac{a^{\prime}}{a} \xi_{i}^{T}+\xi_{i}^{T \prime}  \tag{3.17}\\
& \Psi \rightarrow \tilde{\Psi}=\Psi-\frac{a^{\prime}}{a} \xi^{0},  \tag{3.18}\\
& E \rightarrow \tilde{E}=E-\xi,  \tag{3.19}\\
& F_{i} \rightarrow \tilde{F}_{i}=F_{i}-\frac{\xi_{i}^{T}}{a^{2}},  \tag{3.20}\\
& h_{i j}^{T T} \rightarrow \tilde{h}_{i j}^{T T}=h_{i j}^{T T}, \tag{3.21}
\end{align*}
$$

where $a^{\prime}=\partial_{\eta} a$ is the partial derivative with respect to conformal time, and the shift vector $\xi^{i}$ is broken into a transverse and longitudinal part

$$
\begin{equation*}
\xi^{i}=\xi^{i T}+\delta^{i j} \partial_{j} \xi, \quad \text { with } \partial_{i} \xi_{i}^{T}=0 . \tag{3.22}
\end{equation*}
$$

Only in the special case that the infinitesimal transformation is purely longitudinal ( $\xi_{i}^{T}=0$ ), the scalar nature of the metric perturbations is preserved. This shows that not all four scalar fields can be set to zero by a gauge transformation; only the $\xi^{0}$ and $\xi$ components can be used in gauge fixing the scalars. By looking at the transformations of the four scalar components $\Xi, \Psi, B, E$ as above, one construct the following gauge-invariant quantities

$$
\begin{align*}
& \Xi_{B}=\Xi-\frac{1}{a}\left[\left(B+E^{\prime}\right) a\right]^{\prime},  \tag{3.23}\\
& \Psi_{B}=\psi+\frac{a^{\prime}}{a}\left[B+E^{\prime}\right], \tag{3.24}
\end{align*}
$$

which are called the Bardeen potentials. It must be noted that in Bardeen's original notation [20] they are defined slightly different, partly because of the different metric signature. Another important remark is that the Bardeen potentials are only gauge-invariant up to linear order in infinitesimal perturbations; to study perturbations to higher order one would have to find a new set of variables that are gauge-invariant up to the required order. Finally, there is an infinite amount of gauge-invariant quantities. For example, any linear combinations of Bardeen potentials is again gauge-invariant up to first order. The Bardeen potentials themselves are linearly independent.

Although the gauge-invariant formalism provides a complete description of the physics, it could still be beneficial to consider specific gauges. In passing, we note the following two gauges that are prominent in the literature,

- The synchronous gauge: $\Xi=0, B=0$,
- The longitudinal gauge: $B=E=0$, which implies that $\Xi_{B}=\Xi$, and $\Psi_{B}=\Psi$.

In the synchronous gauge, there is still some residual gauge freedom after fixing $\Xi=0, B=0$. The resulting unphysical gauge modes tend to make physical interpretation challenging in this gauge. Conversely, the coordinates in the longitudinal gauge are fully fixed and the Bardeen potentials reduce to the two gravitational potentials in equation (3.14).

### 3.4 Linear perturbation theory

The previous sections have prepared us enough to attempt linear perturbation theory of gravity. In this section, the focus will be on deriving the dynamical equations describing the evolution of classical perturbations. These equations are obtained by expanding the Einstein equations up to linear order. It was shown [21] that, in FLRW universes, the linearized Einstein equations can be considered as linearizations of the exact solutions to the full nonlinear equations. This ensures that linear perturbation theory is mathematically well-defined. This section will closely follow the discussion in [22].

To set the scene, consider the metric in equation (3.14), where the background metric is chosen to be the flat FLRW metric in conformal time. The metric perturbation will be treated as classical. One can also split the Einstein tensor and energy-momentum tensor into a background and fluctuations,

$$
\begin{equation*}
G_{\nu}^{\mu}=\bar{G}_{\nu}^{\mu}+\delta G_{\nu}^{\mu}, \quad T_{\nu}^{\mu}=\bar{T}_{\nu}^{\mu}+\delta T_{\nu}^{\mu} \tag{3.25}
\end{equation*}
$$

As a result, the Einstein equation itself decomposes into a background equation and an equation of motion for small fluctuations. The components of the background Einstein tensor were already derived in section 2.2. The components of the perturbed Einstein tensor can be calculated directly from the metric in equation (3.14). It is a rather tedious calculation, the results can be found in [22].

The next step is to construct a gauge-invariant formulation of the perturbation equation. First of all, $\delta G_{\nu}^{\mu}$ and $\delta T_{\nu}^{\mu}$ are not gauge-invariant. The components of the Einstein tensor transform as follows under the coordinate transformation in equation (3.9)

$$
\begin{align*}
& \delta G_{0}^{0} \rightarrow \tilde{\delta} G_{0}^{0}=\delta G_{0}^{0}+\left(\bar{G}_{0}^{0}\right)^{\prime} \xi^{0},  \tag{3.26}\\
& \delta G_{i}^{0} \rightarrow \tilde{\delta} G_{i}^{0}=\delta G_{i}^{0}+\left(\bar{G}_{i}^{0}-\frac{1}{3} \bar{G}_{k}^{k}\right) \xi_{\mid i}^{0},  \tag{3.27}\\
& \delta G_{i}^{j} \rightarrow \tilde{\delta} G_{i}^{j}=\delta G_{i}^{j}+\left(\bar{G}_{i}^{j}\right)^{\prime} \xi^{0}, \tag{3.28}
\end{align*}
$$

where $\xi_{\mid i}^{0}$ denotes the covariant derivative. $\delta T_{\nu}^{\mu}$ undergoes the same gauge transformation with $\delta G_{\mu}^{\nu} \rightarrow \delta T_{\mu}^{\nu}$. It is quite straightforward to construct gauge-invariant quantities using the transformation laws (3.15) to (3.21),

$$
\begin{align*}
\delta G_{0}^{(\mathrm{gi}) 0} & =\delta G_{0}^{0}-\left(\bar{G}_{0}^{0}\right)^{\prime}\left(B+E^{\prime}\right),  \tag{3.29}\\
\delta G_{i}^{(\mathrm{gi}) 0} & =\delta \bar{G}_{i}^{0}-\left(\bar{G}_{i}^{0}-\frac{1}{3} \bar{G}_{k}^{k}\right) \partial_{i}\left(B+E^{\prime}\right),  \tag{3.30}\\
\delta G_{i}^{(\mathrm{gi})}{ }^{2} & =\delta G_{i}^{j}-\left(\bar{G}_{i}^{j}\right)^{\prime}\left(B+E^{\prime}\right) . \tag{3.31}
\end{align*}
$$

The gauge-invariant quantities are analogous for $\delta T_{\nu}^{\mu}$. The Einstein equation for small fluctuations can now be written in a manifestly gauge-invariant way

$$
\begin{equation*}
\delta G_{\nu}^{(\mathrm{gi}) \mu}=8 \pi G \delta T_{\nu}^{(\mathrm{gi}) \mu}, \tag{3.32}
\end{equation*}
$$

This dynamical equation for classical cosmological perturbations can be put to good use within the inflationary paradigm discussed in section 2.5. Recall that the energy-momentum tensor is given by equation (2.27),

$$
\begin{equation*}
T_{\mu \nu}^{\Phi}=\partial_{\mu} \Phi \partial_{\nu} \Phi-g_{\mu \nu}\left(\frac{1}{2} \partial^{\sigma} \Phi \partial_{\sigma} \Phi+V(\Phi)\right) \tag{3.33}
\end{equation*}
$$

Applying the background field method to the scalar field, one can find an explicit expression for the perturbed energy-momentum tensor. The background tensor $\bar{T}_{\nu}^{\mu}$ is one of a perfect fluid (equation (2.6), while $\delta T_{\nu}^{\mu}$ is linear in both matter and metric fluctuations: $\varphi$, and $\delta g_{\nu}^{\mu}$. The eventual gaugeinvariant expression $\delta T^{(\mathrm{gi})}{ }_{\nu}^{\mu}$ can be found in [22]. Equation (3.32) can be expressed in a manifestly gauge-invariant way by writing it in terms of the Bardeen potentials defined in equations (3.23) and (3.24). One of the equations tells us that the Bardeen potentials coincide $\Xi_{B}=\Psi_{B}$. By taking appropriate combinations of the remaining equations, one can retrieve the following second order partial differential equation expressed in terms of $\Xi_{B}$,

$$
\begin{equation*}
\Xi_{B}^{\prime \prime}+2\left(\mathcal{H}-\frac{\phi^{\prime \prime}}{\phi^{\prime}}\right) \Xi_{B}^{\prime}-\nabla^{2} \Xi_{B}+2\left(\mathcal{H}^{\prime}-\mathcal{H} \frac{\phi^{\prime \prime}}{\phi^{\prime}}\right) \Xi_{B}=0 \tag{3.34}
\end{equation*}
$$

where the prime indicates differentiation with respect to conformal time, and $\mathcal{H}=a^{\prime} / a$. This equation describes the evolution of cosmological perturbations in a universe dominated by a single scalar field. Another useful interpretation of the above equation is obtained by introducing the following variable expressed in coordinate time,

$$
\begin{equation*}
\zeta=\frac{2}{3} \frac{\left(H^{-1} \dot{\Xi}_{B}+\Xi_{B}\right)}{1+\omega}+\Xi_{B} \tag{3.35}
\end{equation*}
$$

where $\omega$ is the equation of state parameter. Equation (3.34) can be rewritten as a conservation law for $\zeta$,

$$
\begin{equation*}
\frac{3}{2} \dot{\zeta} H(1+\omega)=\ddot{\Xi}_{B}+\left(H-2 \frac{\ddot{\phi}}{\dot{\phi}}\right) \dot{\Xi}_{B}+2\left(\dot{H}-H \frac{\ddot{\phi}}{\dot{\phi}}\right) \Xi_{B}=\frac{1}{a^{2}} \nabla^{2} \Xi_{B} \tag{3.36}
\end{equation*}
$$

expressed in coordinate time. On super-Hubble scales, one has $\left|\partial_{i}\right|^{2} \ll(a H)^{2}$, i.e., the spatial gradient $\nabla^{2} \Xi_{B}$ is negligible. Thus, by virtue of equation (3.34) the right-hand side vanishes and $\zeta$ is conserved. It was pointed out by Lyth [23] that, in the co-moving gauge, $\zeta$ can be interpreted as a curvature perturbation, thus showing that curvature perturbations freeze at super-horizon scales.

One has to be careful when using the above equations within the inflationary scenario. The assumptions of the background field method must remain satisfied, i.e., $\delta g_{\mu \nu} \ll 1, \varphi \ll \phi$. If not, linear perturbation theory breaks down and higher order terms in perturbations become relevant. Of course, the CMBR indicates that perturbations are indeed small, $\delta T / T \sim 10^{-5}$. In addition, the perturbations are nearly Gaussian, such that the contributions from interactions are not large either.

### 3.5 Action formalism

This section is dedicated to deriving an action for the study of (quantum) fluctuations in an expanding universe. This section will closely follow the treatment of the four-dimensional case in [24].

The inflationary model of choice is the single scalar field inflationary model introduced in section 2.5. However, in this section, spacetime will be generalized to $D$ dimensions for the benefit of dimensional regularisation. The action is

$$
\begin{equation*}
S=S_{E H}+S_{\phi}=\int d^{D} x \sqrt{-g}\left\{\frac{M_{p}^{2}}{2} R^{(D)}-\frac{1}{2} g^{\mu \nu} \partial_{\mu} \Phi \partial_{\nu} \Phi-V(\Phi)\right\}, \tag{3.37}
\end{equation*}
$$

where $R^{(D)}$ is the $D$-dimensional Ricci scalar. General slow-roll conditions apply. We proceed by applying the ADM decomposition introduced in section 3.2,

$$
\begin{equation*}
d s^{2}=-N^{2} d t^{2}+g_{i j}\left(d x^{i}+N^{i} d t\right)\left(d x^{j}+N^{j} d t\right) \tag{3.38}
\end{equation*}
$$

where $N$ is the lapse function and $N_{i}$ is the shift vector. The action can be rewritten in terms of these functions [9] (up to some boundary terms),

$$
\begin{equation*}
S=\int d^{D} x \sqrt{g}\left\{\frac{N}{16 \pi G}\left(R+K_{i j} K^{i j}-K^{2}\right)+\frac{1}{2 N}\left(\partial_{0} \Phi-N^{i} \partial_{i} \Phi\right)^{2}-\frac{N}{2} g^{i j} \partial_{i} \Phi \partial_{j} \Phi-N V(\Phi)\right\} \tag{3.39}
\end{equation*}
$$

where $R$ is the ( $D-1$ )-dimensional Ricci scalar, and

$$
\begin{equation*}
K_{i j}=-\frac{1}{2 N}\left(\frac{\partial g_{i j}}{\partial t}-\nabla_{i} N_{j}-\nabla_{j} N_{i}\right) \tag{3.40}
\end{equation*}
$$

is called the extrinsic curvature and its trace is denoted by $K=K_{i}^{i}$. The eigenvalues of $K_{i}^{j}(x)$ are the $(D-1)$ radii of curvature as measured in the surrounding $D$-dimensional space at some point on the hypersurface. In other words, the extrinsic curvature defines the infinitesimal time evolution of the normals to the hypersurface. In the context of the above action, $N$ and $N_{i}$ are considered Lagrange multipliers. This means that the equations obtained from varying the action with respect to $N$ and $N_{i}$ are, in fact, constraint equations. They are often referred to as the Hamiltonian constraint and the momentum constraint respectively,

$$
\begin{align*}
\frac{\delta S}{\delta N} & =0 \Rightarrow \frac{1}{16 \pi G}\left(R-K_{i j} K^{i j}+K^{2}\right)-\frac{\dot{\Phi}^{2}}{2 N^{2}}-V(\Phi)=0  \tag{3.41}\\
\frac{\delta S}{\delta N^{j}} & =0 \Rightarrow \nabla_{i}\left[K_{j}^{i}-\delta_{j}^{i} K\right]=0 \tag{3.42}
\end{align*}
$$

These constraint equations describe the conservation of energy and momentum of a system with a matter field $\Phi$ and a geometry $g_{\mu \nu}$. The strategy is to solve these conservation equations, thus obtaining expressions for $N$ and $N_{i}$, and to plug the results back into the action before expanding up to quadratic order in perturbations. It is sufficient to calculate $N$ and $N_{i}$ up to linear order in perturbations since second order terms would multiply the Friedmann (background) equations in $D$ dimensions,

$$
\begin{align*}
& H^{2}=\frac{16 \pi G}{(D-1)(D-2)}\left(\frac{1}{2} \dot{\phi}^{2}+V\right)  \tag{3.43}\\
& \dot{H}+H^{2}=\frac{16 \pi G}{(D-2)} V \tag{3.44}
\end{align*}
$$

which are satisfied. We will execute the aforementioned strategy in the comoving gauge,

$$
\begin{equation*}
\varphi=0, \quad g_{i j}=a^{2}(t)\left[(1+2 \zeta(t, \vec{x})) \delta_{i j}+\gamma_{i j}(t, \vec{x})\right], \quad \partial^{i} \gamma_{i j}=0, \quad \gamma_{i i}=0, \tag{3.45}
\end{equation*}
$$

where the matter fluctuations vanish, i.e., $\varphi=0$, and $\zeta$ and $\gamma_{i j}$ are the degrees of freedom describing metric fluctuations. In this gauge, the constraint equations are solved up to first order by

$$
\begin{equation*}
N=1+\frac{\dot{\zeta}}{H}, \quad N^{i}=\partial^{i} \Omega, \quad \Omega=-a^{-2} H^{-1} \zeta+\chi, \quad \nabla^{2} \chi=\epsilon \dot{\zeta}, \quad \epsilon=\frac{16 \pi G}{2(D-2)} \frac{\dot{\phi}^{2}}{H^{2}} \tag{3.46}
\end{equation*}
$$

where $\epsilon$ is the first geometric slow-roll parameter. At this point it is clear to see that four out of the five scalar degrees of freedom have been determined: two degrees of freedom ( $\varphi$ and $\gamma_{i i}$ ) are
set to zero by gauge-fixing, while the other two ( $N$ and $\partial_{i} N^{i}$ ) are physical constraints which are obtained by solving equations (3.41) and (3.42). This leaves precisely one physical scalar degree of freedom while the two remaining components of $\gamma_{i j}$ constitute the physical tensorial degrees of freedom. Plugging the above solution back into the action, the scalar part of the action becomes

$$
\begin{align*}
S_{\zeta}= & \int d^{D} x a^{(D-1)} e^{(D-1) \zeta}\left\{\frac { 1 } { 1 6 \pi G } ( 1 + \frac { \dot { \zeta } } { H } ) \left[-2(D-2) a^{-2} e^{-2 \zeta} \nabla^{2} \zeta\right.\right. \\
& \left.\left.-(D-2)(D-3) a^{-2} e^{-2 \zeta}(\nabla \zeta)^{2}-(D-1)(D-2) H^{2}-16 \pi G V\right]+\frac{1}{(1+\dot{\zeta} / H)} \frac{\dot{\phi}^{2}}{2}\right\}, \tag{3.47}
\end{align*}
$$

up to a total derivative which is linear in $\Omega$. By means of partial integration and the background equations in (3.43) and (3.44), the action reduces to

$$
\begin{equation*}
S_{\zeta}=\int d^{D} x a^{(D-1)} \frac{2(D-2) \epsilon}{16 \pi G}\left\{\frac{1}{2} \dot{\zeta}^{2}-\frac{1}{2} a^{-2}(\nabla \zeta)^{2}\right\} . \tag{3.48}
\end{equation*}
$$

Equation (3.48) is the action for scalar metric perturbations up to quadratic order in the comoving gauge. Calculating the tensor part of the action is relatively straightforward. The total action in the comoving gauge is

$$
\begin{equation*}
S=\int d^{D} x \frac{a^{(D-1)}}{16 \pi G}\left\{2(D-2) \epsilon\left[\frac{1}{2} \dot{\zeta}^{2}-\frac{1}{2} a^{-2}(\nabla \zeta)^{2}\right]+\left[\frac{1}{4} \dot{\gamma}_{i j} \dot{\gamma}^{i j}-\frac{1}{4} a^{-2} \partial_{k} \gamma_{i j} \partial^{k} \gamma^{i j}\right]\right\} . \tag{3.49}
\end{equation*}
$$

It can be quite useful to consider different gauges. For example, the zero-curvature gauge, in which $\zeta=0$ and $\varphi(t, \vec{x}) \neq 0$, is particularly useful when doing slow-roll computations. Similarly, in the stochastic approach to inflation, it is useful to consider a gauge in which $N_{i}=0$. In the next section, we will define such a gauge and derive the corresponding quadratic action by means of a gauge transformation from the comoving gauge. The resulting action will later be used to quantize the dynamical scalar field in section 4.3.

### 3.6 Stacked gauge

The shift vector $N_{i}$ describes the change of the spatial coordinates on consecutive space-like hypersurfaces. In a gauge where $N_{i}=0$, these coordinate systems do not shift relative to one another. One could say that they are stacked on top of each other perfectly.

In this section, we will construct a fully fixed gauge in which $N_{i}=0$. As a start, define the following gauge which will be referred to as the stacked gauge,

$$
\begin{gather*}
\varphi=0, \quad g_{i j}=a^{2}(t)\left((1+2 \psi(t, \vec{x})) \delta_{i j}+\Psi_{i j}(t, \vec{x})\right), \quad \partial^{i} \Psi_{i j}=\partial_{j} v(t, \vec{x}), \quad \Psi_{i i}=0,  \tag{3.50}\\
N(t, x)=1+n(t, x), \quad N^{i}(t, x)=0, \tag{3.51}
\end{gather*}
$$

where the spatial part of the metric $g_{i j}$ is split into a traceless tensorial part, $\Psi_{i j}(t, \vec{x})$ and a trace part $\psi(t, \vec{x})$. As opposed to the comoving gauge, $\Psi_{i j}$ is not, a priori, transverse. Additionally, the scalar fields $\psi(t, \vec{x}), v(t, \vec{x})$, and $n(t, \vec{x})$ are not independent of each other. Their precise relations, as well as the underlying dynamical scalar, will be uncovered when performing a gauge transformation from the comoving gauge.

Performing a linear gauge transformation comes down to shifting the coordinates by a vector $\xi^{\mu}$,

$$
\begin{equation*}
x^{\mu} \rightarrow \tilde{x}^{\mu}=x^{\mu}+\xi^{\mu}(x), \tag{3.52}
\end{equation*}
$$

which changes the scalar field and metric field as follows,

$$
\begin{align*}
& \tilde{\Phi}(x)=\Phi(x)-\xi^{\mu} \partial_{\mu} \Phi(x),  \tag{3.53}\\
& \tilde{g}_{\mu \nu}(x)=g_{\mu \nu}(x)-\nabla_{\mu} \xi_{\nu}(x)-\nabla_{\nu} \xi_{\mu}(x) . \tag{3.54}
\end{align*}
$$

Before performing the transformation it is comprehensive to state the original gauge and the final gauge. The original gauge is the comoving gauge in equation (3.46) where matter perturbations vanish ( $\varphi=0$ ) and the $D$-dimensional metric can be compactly written up to linear order in perturbations as

$$
g_{\mu \nu}(t, \vec{x})=\left(\begin{array}{cc}
-1-2 \dot{\zeta} / H & a^{2} \partial_{i} \Omega  \tag{3.55}\\
a^{2} \partial_{j} \Omega & a^{2}\left((1+2 \zeta) \delta_{i j}+\gamma_{i j}\right)
\end{array}\right) .
$$

In the stacked gauge, $\tilde{g}_{0 i}=\tilde{N}_{i}=0$ as well as $\tilde{\varphi}=0$. These conditions are the constraints that will fix the gauge transformation. For the sake of clarity, one can write the metric in the stacked gauge as follows,

$$
\tilde{g}_{\mu \nu}(t, \vec{x})=\left(\begin{array}{cc}
-1-2 n(t, \vec{x}) & 0  \tag{3.56}\\
0 & a^{2}\left((1+2 \psi(t, \vec{x})) \delta_{i j}+\Psi_{i j}(t, \vec{x})\right)
\end{array}\right),
$$

Between these two gauges, the matter field does not transform, i.e., $\tilde{\Phi}(x)=\Phi(x)$. Consequently, equation (3.53) implies that $\xi^{0}=0$. With this in mind, the condition $\tilde{g}_{0 i}=0$ implies that

$$
\begin{equation*}
\partial_{0} \xi_{i}-2 H \xi_{i}=a^{2} \partial_{i} \Omega . \tag{3.57}
\end{equation*}
$$

It will be convenient to denote $\xi^{i}=\partial^{i} \lambda$ such that the previous equation reduces to $\partial_{i} \dot{\lambda}=\partial_{i} \Omega$. This fully fixes the gauge transformation. Thus, each of the functions in the final gauge can be derived from equation (3.54),

$$
\begin{align*}
& n(t, \vec{x})=\dot{\zeta}(t, \vec{x}) / H(t),  \tag{3.58}\\
& \partial_{i} \dot{\lambda}(t, \vec{x})=\partial_{i} \Omega(t, \vec{x}),  \tag{3.59}\\
& \psi(t, \vec{x})=\zeta(t, \vec{x})-\frac{1}{D-1} \nabla^{2} \lambda(t, \vec{x}),  \tag{3.60}\\
& \Psi_{i j}(t, \vec{x})=\gamma_{i j}(t, \vec{x})-2 \partial_{i} \partial_{j} \lambda(t, \vec{x})+\frac{2}{D-1} \delta_{i j} \nabla^{2} \lambda(t, \vec{x}),  \tag{3.61}\\
& \partial_{j} v(t, \vec{x})=-\frac{2(D-2)}{(D-1)} \partial_{j} \nabla^{2} \lambda(t, \vec{x}) . \tag{3.62}
\end{align*}
$$

The first important thing to note is that $\Psi_{i j}$ is not transverse as opposed to $\gamma_{i j}$. Another important point is that $\lambda$ is not independent of $\zeta$, namely

$$
\begin{equation*}
\nabla^{2} \dot{\lambda}(t, \vec{x})=\nabla^{2} \Omega(t, \vec{x})=\epsilon \dot{\zeta}(t, \vec{x})-a^{-2} H^{-1} \nabla^{2} \zeta(t, \vec{x}) . \tag{3.63}
\end{equation*}
$$

From this relation and equation 3.60, it immediately follows that

$$
\begin{equation*}
(D-1) \dot{\psi}(t, \vec{x})=(D-1-\epsilon) \dot{\zeta}(t, \vec{x})+a^{-2} H^{-1} \nabla^{2} \zeta(t, \vec{x}) . \tag{3.64}
\end{equation*}
$$

We would like to use the above relation as well as equation (3.61) to derive from the action (3.49) the corresponding action in the stacked gauge. However, it is hard to invert this equation, i.e., to express
$\zeta$ explicitly in terms of $\psi$. Therefore we resort to deriving the action in the stacked gauge separately and performing the above gauge transformation to show it is consistent with the action in equation (3.49). The details of the derivation as well as the gauge transformation can be found in appendix $A$.

The action up to quadratic order in perturbations in the stacked gauge is (equation A.30)

$$
\begin{array}{r}
S=\int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{-(D-1)(D-2) \dot{\psi}^{2}+a^{-2}\left[(D-2)(D-3)(\nabla \psi)^{2}+(D-3) \psi \nabla^{2} v-\frac{1}{2} \Psi_{i j} \partial^{i} \partial^{j} v\right]\right. \\
+\frac{1}{4}\left[\left(\dot{\Psi}_{i j}\right)^{2}-a^{-2}\left(\nabla \Psi_{i j}\right)^{2}\right]-n\left[-2(D-1)(D-2) \dot{\psi} H+a^{-2}\left(2(D-2) \nabla^{2} \psi-\nabla^{2} v\right)\right] \\
\left.-n^{2} H^{2}(D-2)(D-1-\epsilon)\right\}, \tag{3.65}
\end{array}
$$

which describes both scalar and tensor perturbations. Using the relations in equations (3.58) to (3.62), it is straightforward to perform the transformation into the co-moving gauge and retrieve action (3.49) (see appendix A). Action (3.49) is quite elegant in that there is precisely one scalar field. Conversely, the action (3.65) contains three scalar fields. Of course, these three fields are related by the identities (3.58) to (3.62) such that there is still only one independent scalar degree of freedom. Fortunately, the previous gauge transformation precisely tells us how to express the action in terms of a single scalar field. Define the following two fields,

$$
\begin{align*}
& \tilde{\psi}(t, \vec{x})=\psi(t, \vec{x})-\frac{1}{2(D-2)} v(t, \vec{x}),  \tag{3.66}\\
& \tilde{\Psi}_{i j}(t, \vec{x})=\Psi_{i j}(t, \vec{x})-\frac{D-1}{D-2} \frac{\partial_{i} \partial_{j}}{\nabla^{2}} v(t, \vec{x})+\frac{1}{D-2} \delta_{i j} v(t, \vec{x}), \tag{3.67}
\end{align*}
$$

where $\tilde{\Psi}_{i j}$ is transverse. In terms of these fields, the action (3.65) becomes

$$
\begin{equation*}
S=\int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{2(D-2) \epsilon\left[\frac{1}{2} \dot{\tilde{\psi}}^{2}-\frac{1}{2} a^{-2}(\nabla \tilde{\psi})^{2}\right]+\left[\frac{1}{4} \dot{\tilde{\Psi}}_{i j} \dot{\tilde{\Psi}}^{i j}-\frac{1}{4} a^{-2} \partial_{k} \tilde{\Psi}_{i j} \partial^{k} \tilde{\Psi}^{i j}\right]\right\} . \tag{3.68}
\end{equation*}
$$

Although the above action seems very similar to action (3.49), the fields that appear are not the same. In the stacked gauge, the field $\tilde{\psi}$ includes both the scalar perturbations as well as the nontransverse part of the tensor perturbations while the transverse part is described by $\tilde{\Psi}_{i j}$. Whereas in the comoving gauge, the space-like hypersurfaces are shifted such that the tensor perturbations are transverse and traceless, and the scalar perturbations remain as the single dynamical scalar degree of freedom.

### 3.7 Quadratic equations of motion

The action (3.68) is expanded up to quadratic order in perturbations and, from it, we were already able to derive the linear equations of motion for both the scalar and tensor fields. However, we are interested in higher order contributions, specifically quadratic order contributions to the scalar equation of motion. One way of going about this would be to derive the action up to cubic order in perturbations, as is done in [24] for two different gauges, but that goes beyond the goal of this thesis. Another, easier, way would be to return to the ADM action in equation (3.39) and derive a general equation of motion before expanding it up to quadratic order. This section will explore the
latter approach.
For convenience, let us restate the ADM action

$$
\begin{equation*}
S=\int d^{D} x \sqrt{g}\left\{\frac{N}{16 \pi G}\left(R+K_{i j} K^{i j}-K^{2}\right)+\frac{1}{2 N}\left(\partial_{0} \Phi-N^{i} \partial_{i} \Phi\right)^{2}-\frac{N}{2} g^{i j} \partial_{i} \Phi \partial_{j} \Phi-N V(\Phi)\right\} . \tag{3.69}
\end{equation*}
$$

The dynamical equations are derived by varying the action with respect to the metric $g^{a b}$. The result is

$$
\begin{align*}
\frac{\delta S}{\delta g^{a b}}=0 \Rightarrow & -\nabla_{a} \nabla_{b} N+g_{a b} g^{i j} \nabla_{i} \nabla_{j} N+N\left[R_{a b}-\frac{1}{2} g_{a b} R\right]-g_{i a} g_{j b} \partial_{0} K^{i j}+N K K_{a b} \\
& -K_{a b} \nabla_{i} N^{i}+2 N K_{i a} K_{j b} g^{i j}-\frac{N}{2} g_{a b} K_{i j} K^{i j}+g_{a b} \partial_{0} K  \tag{3.70}\\
& +g_{a b} K \nabla_{i} N^{i}-K \nabla_{a} N_{b}-K \nabla_{b} N_{a}-\frac{N}{2} g_{a b} K^{2}-\frac{\sqrt{g} N}{2} S_{a b}=0,
\end{align*}
$$

where

$$
\begin{equation*}
S_{i j}=\partial_{i} \Phi \partial_{j} \Phi+g_{i j}\left(\frac{1}{2} \Pi_{\Phi}^{2}-\frac{1}{2} \partial_{i} \Phi \partial^{i} \Phi-V(\Phi)\right), \quad \Pi_{\Phi}=\frac{1}{N}\left(\partial_{0} \Phi-N^{i} \partial_{i} \Phi\right) . \tag{3.71}
\end{equation*}
$$

Seeing as we are interested in expanding the dynamical equation in a gauge where $N_{i}=0$, we can already set $N_{i}=0=N^{i}$. One can separate the scalar and tensor dynamical equations by respectively taking the trace, and traceless part of equation (3.70),

$$
\begin{align*}
& \partial_{0} K=-\partial_{i} \partial^{i} N+N\left[\frac{(D-1)}{2(D-2)} \bar{K}_{i j} \bar{K}^{i j}+\frac{1}{2} K^{2}+\frac{(D-3)}{2(D-2)} R+\frac{8 \pi G S}{(D-2)}\right],  \tag{3.72}\\
& \partial_{0} \bar{K}_{b}^{a}=-\partial^{a} \partial_{b} N+\frac{1}{(D-1)} \delta_{b}^{a} \partial_{i} \partial^{i} N+N\left[K \bar{K}_{b}^{a}+\bar{R}_{b}^{a}-8 \pi G \bar{S}_{b}^{a}\right], \tag{3.73}
\end{align*}
$$

where $\bar{K}_{i j}=K_{i j}-\frac{1}{D-1} K g_{i j}$ denotes the traceless part of the extrinsic curvature. The above equations agree with the results as stated in [25] for four-dimensional spacetimes.

The next step is to expand equation (3.72) in the stacked gauge,

$$
\begin{equation*}
\Phi=\phi(t), \quad g_{i j}=a(t)^{2}\left((1+2 \psi(t, \vec{x})) \delta_{i j}+\Psi_{i j}(t, \vec{x})\right), \quad \partial^{i} \Psi_{i j}=\partial_{j} v(t, \vec{x}), \quad \Psi_{i i}=0 . \tag{3.74}
\end{equation*}
$$

Defining the fields $\tilde{\psi}$ and $\tilde{\Psi}_{i j}$ as in equations (3.66) and rescaling $v$ such that $v=-\frac{D-2}{D-1} h$, leads to the following identities,

$$
\begin{align*}
& \tilde{\psi}(t, \vec{x})=\psi(t, \vec{x})+\frac{1}{2(D-1)} h(t, \vec{x}),  \tag{3.75}\\
& \tilde{\Psi}_{i j}(t, \vec{x})=\Psi_{i j}(t, \vec{x})-\left(\frac{\delta_{i j}}{D-1}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}}\right) h(t, \vec{x}),  \tag{3.76}\\
& \dot{h}(t, \vec{x})=-2 H^{-1} a^{-2} \nabla^{2} \tilde{\psi}(t, \vec{x})+2 \epsilon \dot{\tilde{\psi}}(t, \vec{x}) . \tag{3.77}
\end{align*}
$$

Since the goal is to expand equation (3.72) up to quadratic order, the energy constraint should be solved to quadratic order as well. The energy constraint can be derived by varying (3.69) with respect to $N$,

$$
\begin{equation*}
R-\bar{K}_{i j} \bar{K}^{i j}+\frac{(D-2)}{(D-1)} K^{2}-16 \pi G \varepsilon=0 . \tag{3.78}
\end{equation*}
$$

It is solved by the following lapse function

$$
\begin{align*}
& N(t, \vec{x})=1+n(t, \vec{x})+\tilde{n}^{(2)}(t, \vec{x}), \quad n(t, \vec{x})=\frac{\dot{\tilde{\psi}}(t, \vec{x})}{H},  \tag{3.79}\\
& \tilde{n}^{(2)}= \frac{1}{2(D-2)(D-1-\epsilon) H^{2}}\left(R^{(2)}-\frac{1}{4}\left(\left(\dot{\tilde{\Psi}}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} \dot{h}\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)-\frac{\dot{h}^{2}}{D-1}\right)\right. \\
&+\frac{D-2}{D-1}\left(-(D-1)^{2} \dot{\tilde{\psi}}^{2}+(D-1) \dot{h} \dot{\tilde{\psi}}+\frac{1}{4} \dot{h}^{2}-(D-1) H\left(\tilde{\Psi}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} h\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)\right) \\
&\left.-3(D-2) \epsilon \dot{\tilde{\psi}}^{2}\right) . \tag{3.80}
\end{align*}
$$

Here, $R^{(2)}$ are the quadratic terms in the expansion of the Ricci scalar in the gauge (3.74),

$$
\begin{align*}
& R^{(2)}=a^{-2}\left\{\frac{1}{2} \tilde{\Psi}_{i j} \nabla^{2} \tilde{\Psi}^{i j}+\frac{1}{4} \partial_{k} \tilde{\Psi}_{i j} \partial^{k} \tilde{\Psi}^{i j}-\frac{1}{2} \partial_{k} \tilde{\Psi}_{i j} \partial^{i} \tilde{\Psi}^{j k}+4(D-2)\left(\tilde{\psi} \nabla^{2} \tilde{\psi}\right)-(D-2)(D-3)(\nabla \tilde{\psi})^{2}\right. \\
& -(D-3)\left(\partial^{i} \tilde{\psi} \partial_{i} h\right)-\frac{1}{2}(\nabla h)^{2}-\frac{1}{2}\left(\frac{\partial_{i} \partial_{j}}{\nabla^{2}} h \partial^{i} \partial^{j} h\right)-\frac{1}{4}\left(\frac{\partial_{i} \partial_{j} \partial_{k}}{\nabla^{2}} h\right)^{2}-2(D-2)\left(\frac{\partial_{i} \partial_{j}}{\nabla^{2}} h \partial^{i} \partial^{j} \tilde{\psi}\right) \\
& \left.+\frac{1}{2}\left(\frac{\partial_{i} \partial_{j}}{\nabla^{2}} h \nabla^{2} \tilde{\Psi}^{i j}\right)+\frac{3}{2} \tilde{\Psi}_{i j} \partial^{i} \partial^{j} h-\frac{1}{2}\left(\frac{\partial_{i} \partial_{j} \partial_{k}}{\nabla^{2}} h \partial^{k} \tilde{\Psi}^{i j}\right)+2(D-2) \tilde{\Psi}_{i j} \partial^{i} \partial^{j} \tilde{\psi}\right\} . \tag{3.81}
\end{align*}
$$

When all is said and done, the scalar dynamical equation expanded up to quadratic order is

$$
\begin{align*}
& \epsilon \ddot{\tilde{\psi}}+(D-1) H \epsilon \dot{\tilde{\psi}}-\epsilon a^{-2} \nabla^{2} \tilde{\psi}=-\partial_{0}\left[(D-1) H \tilde{n}^{(2)}-\frac{1}{2} \frac{\dot{h} \dot{\tilde{\psi}}}{H}+\frac{1}{2}\left(\tilde{\Psi}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} h\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)\right] \\
& -a^{-2}\left[\nabla^{2} \tilde{n}^{(2)}-2 H^{-1} \tilde{\psi} \nabla^{2} \dot{\tilde{\psi}}-H^{-1}\left(\tilde{\Psi}^{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} h\right) \partial_{i} \partial_{j} \dot{\tilde{\psi}}\right]+2 H^{-1} a^{-2} \dot{\tilde{\psi}} \nabla^{2} \tilde{\psi}-\frac{2}{D-2} R^{(2)} \\
& +\frac{D-1}{4(D-2)}\left[\left(\dot{\tilde{\Psi}}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} \dot{h}\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)-\frac{\dot{h}^{2}}{D-1}\right]+(D-1) \epsilon H\left[\frac{\dot{\tilde{\psi}}^{2}}{H^{2}}-\tilde{n}^{(2)}\right] . \tag{3.82}
\end{align*}
$$

This equation will be extremely important when calculating the one-loop backreaction to the dynamics of scalar perturbations, which is the topic of chapter 5. In the upcoming chapter, the fields $\tilde{\psi}$ and $\tilde{\Psi}_{i j}$ will be quantized according to the canonical quantization procedure, thus turning equation (3.82) into an equation of (quantum) operators.

## Chapter 4

## Quantum fluctuations

Although the classical approach to perturbation theory developed in chapter 3 seems rather complete, it does not incorporate any quantum effects, which are crucial to describe large scale structure formation. Our discussion has been completely classical. We have treated the metric- and matter field fluctuations as classical fields while they are, in fact, quantum fields. So far, the assumption has been made that these fluctuations simply exist without the need to understand their origin. It will not be enough to take the classical equations of motion and try to interpret them quantummechanically; it is impossible to normalize the modes in linear classical physics [26]. A different treatment of quantum fluctuations must be build from the ground up. This chapter will discuss the quantum nature of the fluctuations as well as the tools that will be needed to deal with them within the framework of quantum field theory. Among these tools are the Feynman propagator as well as the Wightman functions and the process of canonical quantization (or second quantization). This chapter is structured as follows. Section 4.1 contains a rudimentary discussion of quantum effects as well as some remarks on quantum field theory as a whole and how it pertains to gravity. Section 4.2 will introduce the aforementioned tools at the hand of a simple example. Subsequently, these tools will be adjusted and used in section 4.3 to quantize the scalar and tensor metric fluctuations.

### 4.1 On quantum field theory

Quantum fluctuations are nothing but a temporary change in the vacuum energy at a point in space. The origin of quantum fluctuations is typically ascribed to the Heisenberg uncertainty principle which is a quantum effect. A vacuum fluctuation manifests itself as the creation and annihilation of a particle-antiparticle pair. This is possible because of the inherent uncertainty between the canonically conjugate variables time and energy. The creation and annihilation of these virtual particles needs to happen within a small enough time frame in order for energy to be conserved. Although these particle pairs can not be detected, sometimes their cumulative effect is observable. A nice example is the Casimir effect [27].

Within the context of inflation, quantum fluctuations are immensely important. Take, for example, the inflaton field $\phi(t)$. As discussed in section 2.5, inflation lasts as long as the potential energy dominates the inflaton field's kinetic energy. Essentially, the inflaton field acts like a clock counting down to the end of inflation. Fluctuations in the inflaton field will cause inflation to last longer or shorter in different regions of space. When inflation ends, the different expansion histories will show up as density perturbations which eventually lead to the formation of large scale structure (see section 2.6). There are other, more complicated, examples of quantum effects. Unfortunately, little


Figure 4.1: The relation between important physical theories. In general, field theories deal with a large number of degrees of freedom such as fields, or when operating in the thermodynamics limit $N \rightarrow \infty$. While the non-relativistic field theories do not provide a fully accurate description of nature they provide a good enough approximation when velocities are small compared to the speed of light.
is understood about their influence on the evolution of the universe. Often times such quantum effects are neglected in cosmological models, even though they might turn out to be significant.

Quantum field theory (QFT) offers a framework to deal with a wide variety of quantum problems. Actually, quantum field theory is an overarching term that is used to describe theories that provide a description of nature in terms of local quantum fields. Examples of quantum field theories are: quantum electrodynamics (QED), which describes how light and matter interact; quantum chromodynamics (QCD), which is a theory of the strong interactions between quarks and gluons; and the electro-weak theory, which provides a combined description of the weak interaction and electromagnetism. Together, these three theories are combined in the standard model of particle physics. One way of looking at quantum field theory is as an extension of quantum mechanics (QM), classical field theory (CFT) and even special relativity (SRT) at the same time. The relation between the aforementioned physical theories is shown diagrammatically in figure 4.1. There are many ways in which quantum field theory is a worthwhile extension of quantum mechanics. While quantum mechanics usually deals with a finite (small) number of degrees of freedom, quantum field theory generally deals with a large number (or infinite) degrees of freedom. In fact, fields such as the electromagnetic field are impossible to deal with in the framework of quantum mechanics. Also, quantum mechanics preserves the number of particles which is observationally incorrect; particles can be created or destroyed by interactions. Therefore, processes that have a variable number of degrees of freedom, such as scattering processes, are best studied within quantum field theory. Finally, the inconsistency of quantum mechanics with the locality principle of special relativity can only be resolved by a field theory such as QFT.

There are quantum field theories for three out of the four fundamental forces. Yet, there is no commonly accepted way of quantizing gravity. An intuitive reason as to why quantizing gravity is hard compared to the other forces is because it is inextricably linked with the curvature of spacetime by general relativity. Spacetime is more fundamental than any interaction occurring within it. Therefore, a theory of quantum gravity would have to be consistent with both quantum mechanics and general relativity. The most notable candidates for such a theory are canonical quantum gravity, loop quantum gravity, and string theory. String theory attempts to tackle quantization of gravity
(and other forces) at a fundamental level by changing the building blocks from point-particles into one-dimensional strings. On the other hand, canonical quantum gravity builds upon quantum field theory without changing it. In this thesis, it will assumed that perturbative quantum gravity applies and we will utilize standard canonical quantization to implement it. Then, we will study the one-loop quantum effect of the gravitational tadpole on the background spacetime in inflation.

### 4.2 Canonical quantization and propagators

The canonical quantization of classical fields has a lot of overlap with the analogous procedure in quantum mechanics. In this subsection, we will discuss relevant concepts in terms of fields and only occasionally highlight key differences with the quantum mechanical scenario. A pedestrian introduction to non-equilibrium quantum field theory can be found in [28]. Here, we will elaborate only on the details relevant to the remainder of this thesis.

It will be comprehensive to complement the discussion with an example. Perhaps, the most simple example is that of a scalar field $\Phi(t, \vec{x})$ with a canonical kinetic term,

$$
\begin{equation*}
S[\Phi]=\int d^{D-1} x \sqrt{-g}\left\{-\frac{1}{2} g^{\mu \nu} \partial_{\mu} \Phi \partial_{\nu} \Phi-\frac{1}{2} m^{2} \Phi^{2}-V_{\mathrm{int}}(\Phi)\right\}, \tag{4.1}
\end{equation*}
$$

where $m$ is the field's mass and $V_{\mathrm{int}}(\Phi)$ is some interaction potential. Associated with the scalar field $\Phi(t, \vec{x})$, is the conjugate field

$$
\begin{equation*}
\Pi(t, \vec{x})=\frac{\delta S}{\delta \dot{\Phi}(t, \vec{x})}=-\sqrt{-g} g^{0 \mu} \partial_{\mu} \Phi(t, \vec{x}) . \tag{4.2}
\end{equation*}
$$

The scalar field $\Phi$ and its conjugate field $\Pi$ are directly analogous to the canonical coordinate $q$ and its generalized momentum $p$ in classical mechanics. They even satisfy similar Poisson brackets: $\left\{\Phi(t, \vec{x}), \Pi\left(t, \vec{x}^{\prime}\right)\right\}=\delta^{D-1}\left(\vec{x}-\vec{x}^{\prime}\right)$. In both quantum mechanics and quantum field theory quantization proceeds by treating the canonical variables as operators and imposing the following commutation relations

$$
\begin{align*}
& {\left[\hat{\Phi}(t, \vec{x}), \hat{\Pi}\left(t, \vec{x}^{\prime}\right)\right]=i \hbar \delta^{D-1}\left(\vec{x}-\vec{x}^{\prime}\right),}  \tag{4.3}\\
& {\left[\hat{\Phi}(t, \vec{x}), \hat{\Phi}\left(t, \vec{x}^{\prime}\right)\right]=0}  \tag{4.4}\\
& {\left[\hat{\Pi}(t, \vec{x}), \hat{\Pi}\left(t, \vec{x}^{\prime}\right)\right]=0} \tag{4.5}
\end{align*}
$$

It is not clear that these equal-time commutation relations are Lorentz invariant, but there exists an equivalent set of commutation relations which is manifestly covariant. Finally, one can derive the equations of motion by varying the action with respect to $\Phi(t, \vec{x})$

$$
\begin{equation*}
\frac{\delta S}{\delta \Phi}=0 \Rightarrow \square \hat{\Phi}-m^{2} \hat{\Phi}-\frac{d V_{\mathrm{int}}(\hat{\Phi})}{d \Phi}=0, \quad \square=g^{\mu \nu} \nabla_{\mu} \nabla_{\nu}=\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu \nu} \partial_{\nu} \tag{4.6}
\end{equation*}
$$

where $\square$ is the d'Alembertian operator in a general spacetime $g_{\mu \nu}$. If the metric is space-time dependent, then $\hat{\Phi}$ is non-trivially coupled to a space-time dependent background through the d'Alembertian. As you can imagine, this could get complicated quickly. Such problems are treated by non-equilibrium quantum field theory.

For simplicity's sake, we will consider a space-time independent Minkowski background $\eta_{\mu \nu}$ and a vanishing interaction potential ( $V_{\text {int }}=0$ ). In this simplified scenario, the equation of motion reduces to

$$
\begin{equation*}
\left(\partial^{2}-m^{2}\right) \hat{\Phi}=0, \quad \partial^{2}=\eta^{\mu \nu} \partial_{\mu} \partial_{\nu}=-\partial_{t}^{2}+\nabla^{2} \tag{4.7}
\end{equation*}
$$

which is better known as the Klein-Gordon equation. An important distinction has to be made concerning the operator-valued field $\hat{\Phi}(t, \vec{x})$ in quantum field theory and the wave function $\psi(t, \vec{x})$ in quantum mechanics, i.e., the quantum mechanical state in position representation. The wave function in quantum mechanics is acted upon by operators, while, in quantum field theory, $\hat{\Phi}(t, \vec{x})$ itself acts on a space of states. Also, the space-time dependence in quantum field theory is contained in the operator-valued fields, while in quantum mechanics the same information is contained in states (in terms of probabilities for position measurement). Thus, the wave function is not the quantum mechanical analogue to the operator-valued field of quantum field theory.

In early attempts to reconcile special relativity and quantum mechanics, the Klein-Gordon equation was derived by squaring the Schrödinger equation, an already quantized equation. Afterward, the solutions to the derived Klein-Gordon equation would be considered as classical fields, which can be quantized 'again' by the above procedure. This is sometimes referred to as second quantization, which is often criticized for being a misnomer. This is because, in the second instance, the field that is being quantized is not the single particle wave function from relativistic quantum mechanics, but rather a classical field that was not previously quantized.

Leaving the previous remarks for what they are, let us define the following operator (in the Heisenberg picture),

$$
\begin{equation*}
\hat{\mathcal{O}}_{h}(x)=T\left[\hat{\Phi}(x) \hat{\Phi}\left(x^{\prime}\right)\right]=\theta\left(t-t^{\prime}\right) \hat{\Phi}(x) \hat{\Phi}\left(x^{\prime}\right)+\theta\left(t^{\prime}-t\right) \hat{\Phi}\left(x^{\prime}\right) \hat{\Phi}(x) \tag{4.8}
\end{equation*}
$$

where we used the shorthand $x=(t, \vec{x})$ and $T$ is the time-ordering operator. The expectation value of the above operator is called the Feynman propagator $i \Delta_{F}\left(x, x^{\prime}\right)$ and it satisfies the following equation of motion

$$
\begin{equation*}
\left(\partial^{2}-m^{2}\right) i \Delta_{F}\left(x, x^{\prime}\right)=i \delta^{D-1}\left(x-x^{\prime}\right), \tag{4.9}
\end{equation*}
$$

with suitable boundary conditions. The Feynman propagator can be further decomposed in terms of the Wightman functions $i \Delta^{+}\left(x, x^{\prime}\right)$ and $i \Delta^{-}\left(x, x^{\prime}\right)$ which satisfy the homogeneous equation of motion

$$
\begin{align*}
& i \Delta_{F}\left(x, x^{\prime}\right)=\theta\left(t-t^{\prime}\right) i \Delta^{+}\left(x, x^{\prime}\right)+\theta\left(t^{\prime}-t\right) i \Delta^{-}\left(x, x^{\prime}\right)  \tag{4.10}\\
& i \Delta^{+}\left(x, x^{\prime}\right)=\left\langle\hat{\Phi}(x) \hat{\Phi}\left(x^{\prime}\right)\right\rangle, \quad i \Delta^{-}\left(x, x^{\prime}\right)=\left\langle\hat{\Phi}\left(x^{\prime}\right) \hat{\Phi}(x)\right\rangle  \tag{4.11}\\
& \left(\partial^{2}-m^{2}\right) i \Delta^{ \pm}\left(x, x^{\prime}\right)=0, \quad i \Delta^{ \pm}\left(x, x^{\prime}\right)=i \Delta^{\mp}\left(x^{\prime}, x\right) \tag{4.12}
\end{align*}
$$

The main use of the Feynman propagator and the Wightman functions is in time-dependent problems in quantum field theory, specifically in perturbative calculations as the Feynman propagator is a major component of the Feynman rules of non-equilibrium field theories. One other way of looking at the Feynman propagator is as the inverse of the differential operator $\left(\partial^{2}-m^{2}\right)$ or Green's function (of the Klein-Gordon equation) as it is often called.

In order to obtain an explicit expression for the Feynman propagator, we ought to solve equation (4.7). To do this we make the following mode expansion,

$$
\begin{equation*}
\hat{\Phi}(x)=\int \frac{d^{D-1} k}{(2 \pi)^{D-1}}\left(\Phi(k, t) e^{i \vec{k} \cdot \vec{x}} \hat{a}(\vec{k})+\Phi^{*}(k, t) e^{-i \vec{k} \cdot \vec{x}} \hat{a}^{\dagger}(\vec{k})\right), \tag{4.13}
\end{equation*}
$$

where $k \equiv \sqrt{\vec{k} \cdot \vec{k}}$ and $\hat{a}^{\dagger}(\vec{k})$ and $\hat{a}(\vec{k})$ are the creation and annihilation operators. Similar to quantum mechanics, they obey the following commutation relations

$$
\begin{equation*}
\left[\hat{a}(\vec{k}), \hat{a}^{\dagger}\left(\vec{k}^{\prime}\right)\right]=(2 \pi)^{D-1} \delta^{D-1}\left(\vec{k}-\vec{k}^{\prime}\right), \quad\left[\hat{a}(\vec{k}), \hat{a}\left(\vec{k}^{\prime}\right)\right]=0=\left[\hat{a}^{\dagger}(\vec{k}), \hat{a}^{\dagger}\left(\vec{k}^{\prime}\right)\right] . \tag{4.14}
\end{equation*}
$$

The state $|0\rangle$, which is annihilated by $\hat{\alpha}(\vec{k})$, is the vacuum state of the theory. By inserting the above mode expansion in the commutation relation of the canonical variables in equation 4.3) one can derive the following normalisation condition for the mode functions,

$$
\begin{equation*}
W\left[\Phi(t, k), \Phi^{*}(t, k)\right] \equiv \Phi(t, k) \partial_{0} \Phi^{*}(t, k)-\Phi^{*}(t, k) \partial_{0} \Phi(t, k)=i \hbar, \tag{4.15}
\end{equation*}
$$

where $W\left[\Phi, \Phi^{*}\right]$ is called the Wronskian. Inserting the mode expansion into equation 4.7) gives the equations of motion for the mode functions,

$$
\begin{equation*}
\left(\partial_{t}^{2}+k^{2}+m^{2}\right) \Phi(t, \vec{k})=0, \quad\left(\partial_{t}^{2}+k^{2}+m^{2}\right) \Phi^{*}(t, \vec{k})=0 \tag{4.16}
\end{equation*}
$$

which have the following normalized solutions

$$
\begin{equation*}
\Phi(t, k)=\sqrt{\frac{\hbar}{2 \omega}} e^{-i \omega t}, \quad \Phi^{*}(t, k)=\sqrt{\frac{\hbar}{2 \omega}} e^{i \omega t} \tag{4.17}
\end{equation*}
$$

where $\omega=\sqrt{k^{2}+m^{2}}$. Note that $\Phi$ and $\Phi^{*}$ are related by complex conjugation.
Let us now calculate the positive and negative frequency Wightman functions in the vacuum state $|0\rangle$. For the sake of simplicity, we treat the case when $D=4$. The expectation values in equation (4.11) reduce to
$i \Delta^{+}\left(x, x^{\prime}\right)=\langle 0| \hat{\Phi}(x) \hat{\Phi}\left(x^{\prime}\right)|0\rangle=\int \frac{d^{3} k}{(2 \pi)^{3}} e^{i \vec{k} \cdot\left(\vec{x}-\vec{x}^{\prime}\right)} \Phi(k, t) \Phi^{*}\left(k, t^{\prime}\right)=\frac{\hbar}{4 \pi^{2} r} \int_{0}^{\infty} d k \frac{k}{\omega} \sin (k r) e^{-i \omega\left(t-t^{\prime}\right)}$
$i \Delta^{-}\left(x, x^{\prime}\right)=\langle 0| \hat{\Phi}(x) \hat{\Phi}\left(x^{\prime}\right)|0\rangle=\int \frac{d^{3} k}{(2 \pi)^{3}} e^{-i \vec{k} \cdot\left(\vec{x}-x^{\prime}\right)} \Phi^{*}(k, t) \Phi\left(k, t^{\prime}\right)=\frac{\hbar}{4 \pi^{2} r} \int_{0}^{\infty} d k \frac{k}{\omega} \sin (k r) e^{i \omega\left(t-t^{\prime}\right)}$
where the final integrals are expressed in terms of spherical coordinates with $r \equiv\left\|\vec{x}-\vec{x}^{\prime}\right\|$ and it was used that

$$
\begin{align*}
& \langle 0| \hat{a}(\vec{k}) \hat{a}\left(\vec{k}^{\prime}\right)|0\rangle=0, \quad\langle 0| \hat{a}^{\dagger}(\vec{k}) \hat{a}^{\dagger}\left(\vec{k}^{\prime}\right)|0\rangle=0, \quad\langle 0| \hat{a}^{\dagger}(\vec{k}) \hat{a}\left(\vec{k}^{\prime}\right)|0\rangle=0,  \tag{4.20}\\
& \langle 0| \hat{a}(\vec{k}) \hat{a}^{\dagger}\left(\vec{k}^{\prime}\right)|0\rangle=\langle 0|\left[\hat{a}(\vec{k}), \hat{a}^{\dagger}\left(\vec{k}^{\prime}\right)\right]+\hat{a}^{\dagger}(\vec{k}) \hat{a}\left(\vec{k}^{\prime}\right)|0\rangle=(2 \pi)^{3} \delta^{3}\left(\vec{k}-\vec{k}^{\prime}\right) . \tag{4.21}
\end{align*}
$$

The final expressions in equation (4.18) and (4.19) are divergent integrals. Even though they are divergent, they can be evaluated by means of analytic continuation. The essence of this technique is to extent the domain of the function. In the case of the Wightman functions, one usually extents the domain by allowing the time variables to be complex. This helps because the integrals are convergent for certain complex times. In fact, it is sufficient to add a small imaginary part to $t-t^{\prime}$, such that

$$
\begin{equation*}
i \Delta^{ \pm}\left(x, x^{\prime}\right)=\frac{\hbar}{4 \pi^{2} r} \int_{0}^{\infty} d k \frac{k}{\omega} \sin (k r) e^{\mp i \omega\left(t-t^{\prime} \mp i \varepsilon\right)}, \tag{4.22}
\end{equation*}
$$

where $\varepsilon>0$, but small. The above equation can still be difficult to evaluate, so for simplicity's sake we consider $m=0$, in which case

$$
\begin{align*}
& i \Delta^{ \pm}\left(x, x^{\prime}\right)=-\frac{\hbar}{4 \pi^{2}} \frac{1}{\left(t-t^{\prime} \mp i \varepsilon\right)^{2}-r^{2}}  \tag{4.23}\\
& i \Delta_{F}\left(x, x^{\prime}\right)=-\frac{\hbar}{4 \pi^{2}} \frac{1}{\left(\left|t-t^{\prime}\right|-i \varepsilon\right)^{2}-r^{2}}=-\frac{\hbar}{4 \pi^{2}} \frac{1}{\Delta x_{F}^{2}} \tag{4.24}
\end{align*}
$$

The Feynman propagator in the last line is expressed in terms of the invariant distance $\Delta x_{F}=$ $\left(\left|t-t^{\prime}\right|-i \varepsilon\right)^{2}-\left\|\vec{x}-\vec{x}^{\prime}\right\|^{2}$. The final expression can now be used in the study of interacting scalar particles, for instance in scattering processes. Of course, one would need to consider an action with a non-trivial interaction term. Then, one could write down the appropriate Feynman rules for the corresponding theory in which the above expression plays a major role. The procedure that was used here to obtain the Feynman propagator will also be used in the next section to find the scalar and graviton propagator of the metric perturbations, all be it in less detail.

### 4.3 Quantization of metric perturbations

Recall the action from equation (3.68),

$$
\begin{equation*}
S=\int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{2(D-2) \epsilon\left[\frac{1}{2} \dot{\tilde{\psi}}^{2}-\frac{1}{2} a^{-2}(\nabla \tilde{\psi})^{2}\right]+\left[\frac{1}{4} \dot{\tilde{\Psi}}_{i j} \dot{\tilde{\Psi}}^{i j}-\frac{1}{4} a^{-2} \partial_{k} \tilde{\Psi}_{i j} \partial^{k} \tilde{\Psi}^{i j}\right]\right\} \tag{4.25}
\end{equation*}
$$

with $\tilde{\psi}$ a scalar field and $\tilde{\Psi}_{i j}$ a traceless tranverse tensor field. Analogous to the example in the previous section, one can derive propagators for both of these fields, although it may be more difficult. Actually, a great deal of the calculation overlaps with the work already done by Janssen et al. [3] where they derived the Feynman propagator for a minimally coupled scalar field on D-dimensional, spatially flat, homogeneous and isotropic background with arbitrary constant deceleration parameter. For a comprehensive discussion we refer to [3]. Relevant remarks will be made when necessary.

As already hinted to, we need to assume one more simplification, namely a constant deceleration parameter $q(t) \equiv-1+\epsilon(t)$. Even though the Hubble parameter may change many orders of magnitude during the universe's evolution, the deceleration parameter is believed to only vary from roughly -1 to +1 during the era of radiation domination. Therefore, there is definitely some relevance to studying a constant deceleration universe. For such a universe, which is spatially flat, homogeneous and isotropic, the Hubble parameter $H(t)$ and the expansion parameter $a(t)$ can be derived from the definition of the first geometric slow-roll parameter $\epsilon$,

$$
\begin{align*}
& \epsilon \equiv-\frac{\dot{H}}{H^{2}} \Rightarrow H(t)=\frac{H_{0}}{1+\epsilon H_{0} t},  \tag{4.26}\\
& H \equiv \frac{\dot{a}}{a} \Rightarrow a(t)=\left[1+\epsilon H_{0} t\right]^{\frac{1}{\epsilon}}, \tag{4.27}
\end{align*}
$$

where $H_{0} \equiv H(0)$ and $a(0)=1$. The discussion in this section will be easier in conformal time $\eta$, defined as $d \eta=d t / a(t)$. Therefore, we will state the above expressions in conformal time as well as some useful relations

$$
\begin{align*}
& a(\eta)=\frac{1}{\left[-(1-\epsilon) H_{0} \eta\right]^{\frac{1}{1-\epsilon}}}, \quad H(\eta)=\frac{H_{0}}{\left[-(1-\epsilon) H_{0} \eta\right]^{\frac{-\epsilon}{1-\epsilon}}},  \tag{4.28}\\
& H a=\frac{-1}{(1-\epsilon) \eta}, \quad \partial_{\eta} a=H a^{2}, \quad \partial_{\eta} H=-\epsilon H^{2} a . \tag{4.29}
\end{align*}
$$

The universe is accelerating if $0 \leq \epsilon<1$, in which case the conformal time is negative, i.e., $-\infty<\eta<0$, and decelerating if $\epsilon>1$ when conformal time will be positive. Here, we will focus on the former scenario. In fact, $\epsilon$ will be considered a very small, positive number.

### 4.3.1 Scalar propagator

This subsection includes a derivation of the scalar propagator from the action 4.25) and a discussion on the choice of vacuum. The process of quantization carries over from the previous section.

First, derive the equation of motion for the scalar field $\tilde{\psi}(t, \vec{x})$ and the conjugate field $\Pi_{\tilde{\psi}}(t, \vec{x})$ from equation (4.25)

$$
\begin{align*}
& \frac{\delta S}{\delta \tilde{\psi}}=0 \Rightarrow \epsilon \ddot{\tilde{\psi}}+\dot{\epsilon} \dot{\tilde{\psi}}+(D-1) \epsilon H \dot{\tilde{\psi}}-a^{-2} \epsilon \nabla^{2} \tilde{\psi}=0  \tag{4.30}\\
& \Pi_{\tilde{\psi}}=\frac{\delta S}{\delta \partial_{0} \tilde{\psi}}=\frac{2(D-2) \epsilon}{16 \pi G} a^{D-1} \partial_{0} \tilde{\psi}=\frac{2(D-2) \epsilon}{16 \pi G} a^{D-2} \partial_{\eta} \tilde{\psi} \tag{4.31}
\end{align*}
$$

The equation of motion in conformal time can be written in a nice way by rescaling the scalar field by a factor of $a^{D / 2-1}$,

$$
\begin{equation*}
\left(\partial_{\eta}^{2}-\nabla^{2}-\frac{\nu^{2}-1 / 4}{\eta^{2}}\right)\left(a^{D / 2-1} \tilde{\psi}(\eta, \vec{x})\right)=0, \quad \nu=\frac{D-1-\epsilon}{2(1-\epsilon)} . \tag{4.32}
\end{equation*}
$$

The canonical fields are quantized according to the following quantization rules,

$$
\begin{align*}
& {\left[\hat{\tilde{\psi}}(\eta, \vec{x}), \hat{\Pi}_{\tilde{\psi}}\left(\eta, \vec{x}^{\prime}\right)\right]=i \hbar \delta^{D-1}\left(\vec{x}-\overrightarrow{x^{\prime}}\right),}  \tag{4.33}\\
& {\left[\hat{\tilde{\psi}}(\eta, \vec{x}), \hat{\tilde{\psi}}\left(\eta, \vec{x}^{\prime}\right)\right]=0,}  \tag{4.34}\\
& {\left[\hat{\Pi}_{\tilde{\psi}}(\eta, \vec{x}), \hat{\Pi}_{\tilde{\psi}}\left(\eta, \vec{x}^{\prime}\right)\right]=0 .} \tag{4.35}
\end{align*}
$$

The quantized fields can be expressed as mode expansions,

$$
\begin{align*}
\hat{\tilde{\psi}}(\eta, \vec{x}) & =\int \frac{d^{D-1} k}{(2 \pi)^{D-1}}\left\{\tilde{\psi}(\eta, k) e^{i \vec{k} \vec{x}} \hat{a}(\vec{k})+\tilde{\psi}^{*}(\eta, k) e^{-i \vec{k} \vec{x}} \hat{a}^{\dagger}(\vec{k})\right\},  \tag{4.36}\\
\hat{\Pi}_{\tilde{\psi}}(\eta, \vec{x}) & =\int \frac{d^{D-1} k}{(2 \pi)^{D-1}}\left\{\Pi_{\tilde{\psi}}(\eta, k) e^{i \vec{k} \vec{x}} \hat{a}(\vec{k})+\Pi_{\tilde{\psi}}^{*}(\eta, k) e^{-i \vec{k} \vec{x}} \hat{a}^{\dagger}(\vec{k})\right\}, \tag{4.37}
\end{align*}
$$

where the creation and annihilation operators $\hat{\alpha}^{\dagger}(\vec{k})$ and $\hat{\alpha}(\vec{k})$ obey the usual commutation relations in equation (4.14). The mode functions obey the following equations of motion,

$$
\begin{equation*}
\left(\partial_{\eta}^{2}+k^{2}-\frac{\nu^{2}-1 / 4}{\eta^{2}}\right)\left(a^{D / 2-1} \tilde{\psi}(\eta, k)\right)=0, \quad\left(\partial_{\eta}^{2}+k^{2}-\frac{\nu^{2}-1 / 4}{\eta^{2}}\right)\left(a^{D / 2-1} \tilde{\psi}^{*}(\eta, k)\right)=0 . \tag{4.38}
\end{equation*}
$$

These equations have simple solutions in terms of modified Bessel functions. See, for instance, equation (8.491.5) in [29].

$$
\begin{gather*}
\tilde{\psi}(\eta, k)=C a^{1-D / 2} \sqrt{-\eta} H_{\nu}^{(1)}(-k \eta)  \tag{4.39}\\
\tilde{\psi}^{*}(\eta, k)=C^{*} a^{1-D / 2} \sqrt{-\eta} H_{\nu}^{(2)}(-k \eta) . \tag{4.40}
\end{gather*}
$$

The modified Bessel functions of the second kind are related by complex conjugation, so the two mode functions are indeed complex conjugates of each other. The mode functions $\Pi_{\tilde{\psi}}(\eta, k)$ and $\Pi_{\tilde{\psi}}^{*}(\eta, k)$ can be derived from equation (4.31), but their explicit expressions won't be needed. Recall that the normalisation of the mode functions follows from the commutation rules in equation (4.33), which imply

$$
\begin{equation*}
\left[\hat{\tilde{\psi}}(\eta, \vec{x}), \hat{\Pi}_{\tilde{\psi}}\left(\eta, \vec{x}^{\prime}\right)\right]=i \hbar \delta^{D-1}\left(\vec{x}-\vec{x}^{\prime}\right) \Rightarrow\left[\tilde{\psi}(\eta, k) \Pi_{\tilde{\psi}}^{*}(\eta, k)-\Pi_{\tilde{\psi}}(\eta, k) \tilde{\psi}^{*}(\eta, k)\right]=i \hbar . \tag{4.41}
\end{equation*}
$$

By using equation 4.31, this can be rewritten as a condition on the Wronskian,

$$
\begin{equation*}
W_{\eta}\left[\psi(\eta, k), \psi^{*}(\eta, k)\right] \equiv \tilde{\psi}(\eta, k) \partial_{\eta} \tilde{\psi}^{*}(\eta, k)-\tilde{\psi}^{*}(\eta, k) \partial_{\eta} \tilde{\psi}(\eta, k)=\frac{i \hbar}{\epsilon} \frac{16 \pi G}{2(D-2)} a^{2-D} . \tag{4.42}
\end{equation*}
$$

On the other hand, the Wronskian calculated from the mode functions in equations (4.39) and (4.40) is (see equation (8.477.1) in [29])

$$
\begin{equation*}
W_{\eta}\left[\psi(\eta, k), \psi^{*}(\eta, k)\right]=\frac{4 i}{\pi}|C|^{2} a^{2-D} \tag{4.43}
\end{equation*}
$$

Equating the previous equations immediately gives an expression for the normalisation constant

$$
\begin{equation*}
|C|^{2}=\frac{16 \pi^{2} G \hbar}{8(D-2) \epsilon} . \tag{4.44}
\end{equation*}
$$

Since the normalized solutions have been found, the next step is to derive the Wightman functions. However, the choice of the vacuum is bit more involved in curved spacetime than it is in Minkowski space. Whereas before the vacuum could be uniquely fixed by virtue of the Poincaré symmetry of Minkowski space, there is no such luxury in expanding FLRW universes. In this case, the most common choice of vacuum is called the Bunch-Davies vacuum, denoted by $|\Omega\rangle$. A simple way to define this particular choice of vacuum starts by noticing that the solutions for the mode functions in equations (4.39) and (4.40) are not uniquely determined. In fact, any linear combination of the normalized solutions will do, i.e.,

$$
\begin{equation*}
\tilde{\psi}_{\mathrm{gen}}(\eta, k)=\alpha(k) \tilde{\psi}(\eta, k)+\beta(k) \tilde{\psi}^{*}(\eta, k), \tag{4.45}
\end{equation*}
$$

as long as $|\alpha(k)|^{2}-|\beta(k)|^{2}=1$, which is required by the Wronskian condition. This opens up a variety of choices, which come down to picking $\alpha(k)$ and $\beta(k)$. A special choice is the Bunch-Davies vacuum where one picks $\alpha(k)=1$ and $\beta(k)=0$ for all $\vec{k}$ such that $\tilde{\psi}_{\text {gen }}=\tilde{\psi}$. This choice of vacuum is usually motivated by two arguments. First of all, one can look at the short-wavelength (or UV) modes for which $k / a \gg H$. In the distant past, i.e., in the limit $\eta \rightarrow-\infty$, modes become more and more UV as can be seen from the first relation in equation (4.29). At very short wavelengths the curvature of the universe should not be relevant. Thus, it is expected that, in the limit $\eta \rightarrow-\infty$, the mode function should behave as the positive-frequency mode function in Minkowski space. This is precisely the case in the Bunch-Davies vacuum,

$$
\begin{equation*}
\tilde{\psi}_{\mathrm{gen}} \sim \frac{1}{\sqrt{2 k}} e^{-i k \eta}, \quad \text { as } \eta \rightarrow-\infty \tag{4.46}
\end{equation*}
$$

One can compare this behaviour with equation (4.17). The second argument is based on the fact that the energy of field excitations is proportional to $\mathcal{E}(k, \eta) \propto|\alpha(k)|^{2}+|\beta(k)|^{2}$, which is minimized
by the Bunch-Davies vacuum. There is one caveat to this argument: it is only valid in the UV regime. The situation is more complicated in the IR regime $(k / a \ll H)$ because the above arguments do not justify the choice of the Bunch-Davies at long wavelengths where the field couples strongly to gravity. In fact, the Bunch-Davies vacuum will turn out to be unphysical in the deep infrared since the propagator contains infrared divergences. This is dealt with meticulously in [3]. In the meantime, we will calculate the Wightman functions and the propagator with respect to the Bunch-Davies vacuum by following [3], where they remove the unphysical nature of the infrared divergences by performing a simple IR cut-off regularisation of the deep IR of the state, which amounts to removing the modes below some IR cutoff scale $k_{0}$.

The Wightman functions can be expressed as follows,

$$
\begin{equation*}
i \Delta^{ \pm}\left(x, x^{\prime}\right)=\langle\Omega| \tilde{\psi}(x) \tilde{\psi}^{*}\left(x^{\prime}\right)|\Omega\rangle=\int \frac{d^{D-1} k}{(2 \pi)^{D-1}} i \Delta^{ \pm}\left(k, \eta, \eta^{\prime}\right) e^{ \pm i \vec{k} \cdot\left(\vec{x}-\vec{x}^{\prime}\right)} \tag{4.47}
\end{equation*}
$$

where

$$
\begin{equation*}
i \Delta^{+}\left(k, \eta, \eta^{\prime}\right)=\frac{16 \pi^{2} G \hbar}{8(D-2) \epsilon} \sqrt{\eta \eta^{\prime}}\left(a a^{\prime}\right)^{1-D / 2} H_{\nu}^{(1)}(-k \eta) H_{\nu}^{(2)}\left(-k \eta^{\prime}\right), \quad i \Delta^{+}\left(k, \eta, \eta^{\prime}\right)=i \Delta^{-}\left(k, \eta^{\prime}, \eta\right) \tag{4.4.4}
\end{equation*}
$$

Note that the Fourier transforms of the Wightman functions are related by exchange of times $\eta \leftrightarrow \eta^{\prime}$. One can write down the expression for the Feynman propagator at the hand of equation 4.10,

$$
\begin{align*}
i \Delta_{\infty}\left(x, x^{\prime}\right)= & \frac{16 \pi^{2} G \hbar}{8(D-2) \epsilon} \sqrt{\eta \eta^{\prime}}\left(a a^{\prime}\right)^{1-D / 2} \int \frac{d^{D-1} k}{(2 \pi)^{D-1}} e^{i \vec{k}\left(\vec{x}-\vec{x}^{\prime}\right)}  \tag{4.49}\\
& \times\left\{\theta\left(\eta-\eta^{\prime}\right) H_{\nu}^{(1)}(-k \eta) H_{\nu}^{(2)}\left(-k \eta^{\prime}\right)+\theta\left(\eta^{\prime}-\eta\right) H^{(1)}\left(-k \eta^{\prime}\right) H^{(2)}(-k \eta)\right\} .
\end{align*}
$$

The propagator's subscript denotes that the propagator is constructed assuming that the spatial manifold is $\mathbb{R}^{D-1}$, which is a non-compact space. This assumption is responsible for the nonphysicality of the propagator and, thus, the infrared divergences. The solution put forth in [30, is a more physical regularization of the infrared. The universe is placed in a large comoving box, whose comoving size is $\mathrm{L} \sim 2 \pi / k_{0}$. One can show that in the limit when the size of the comoving box goes to infinity (and hence $k_{0} \rightarrow 0$ ), all terms with negative powers of $L$ that appear after quantization can be equivalently reproduced by imposing a simple infrared cut-off where $k_{0} \sim L^{-1}$. The corrections this method entails are derived in [3. Here we will state the final result,

$$
\begin{align*}
i \Delta_{\tilde{\psi}}\left(x, x^{\prime}\right) & =\frac{16 \pi G \hbar}{2(D-2) \epsilon} \frac{\left[(1-\epsilon)^{2} H H^{\prime}\right]^{D / 2-1}}{(4 \pi)^{D / 2}}\left\{\Gamma(D / 2-1)\left(\frac{4}{y}\right)^{D / 2-1}+\frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)}\right. \\
& \times\left[\frac{\Gamma(3 / 2+\nu) \Gamma(3 / 2-\nu)}{\Gamma(3-D / 2)}\left(\frac{4}{y}\right)^{D / 2-2}+\sum_{n=1}^{\infty}\left(\frac{\Gamma(3 / 2+\nu+n) \Gamma(3 / 2-\nu+n)}{\Gamma(3-D / 2+n)(n+1)!}\left(\frac{y}{4}\right)^{n-D / 2+2}\right.\right. \\
& \left.\left.-\frac{\Gamma\left(\frac{D-1}{2}+\nu+n\right) \Gamma\left(\frac{D-1}{2}-\nu+n\right)}{\Gamma(D / 2+n) n!}\left(\frac{y}{4}\right)^{n}\right)\right]+\frac{2(1-\epsilon) \Gamma(2 \nu) \Gamma(\nu)}{\epsilon(D-2) \Gamma\left(\frac{D-1}{2}\right) \Gamma(1 / 2+\nu)} \\
& \left.\times\left[\left(\frac{1}{k_{0}^{2} \eta \eta^{\prime}}\right)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}}+\frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(\nu)} \frac{\Gamma(1-D / 2)}{\Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D-1}{2}+\nu\right) \Gamma\left(\frac{D-1}{2}-\nu\right)}{\Gamma(2 \nu)} \frac{\epsilon(D-2)}{2(1-\epsilon)}\right]\right\} . \tag{4.50}
\end{align*}
$$

In the above expression, $k_{0}$ is an infrared cut-off introduced at the level of the mode expansion and
$y$ is defined similar to the invariant distance from the previous section, namely as

$$
\begin{equation*}
y\left(x ; x^{\prime}\right)=\frac{\left\|\vec{x}-\vec{x}^{\prime}\right\|^{2}-\left(\left|\eta-\eta^{\prime}\right|-i \varepsilon\right)^{2}}{\eta \eta^{\prime}} . \tag{4.51}
\end{equation*}
$$

Scalar gravitational perturbations are very light and their wave function is very broad when $\epsilon$ is very small. This is because the wave function of the canonical momentum becomes sharply peaked around zero (see equation (4.31). In the limit $\epsilon \rightarrow 0$, the de Sitter symmetry gets broken, and the wave function will exhibit a secular growth meaning that it grows linearly with time (or logarithmically with the scale factor) during inflation. If there were no interactions, the growth of the wave function would continue without end. The main reason for studying interactions in inflation is motivated by this observation. The scalar propagator in equation (4.50) will be used in section 5.2 to calculate the contribution of scalar interactions to the one-loop quantum backreaction.

### 4.3.2 Graviton propagator

The derivation of the graviton propagator is more or less identical to the derivation of the scalar propagator, with the exception of the quantization rules. This section is limited to include only important results and short remarks when necessary.

The first important result is the equation of motion for the tensor field written in conformal time,

$$
\begin{equation*}
\left(\partial_{\eta}^{2}-\frac{(D-2)}{(1-\epsilon) \eta} \partial_{\eta}-\nabla^{2}\right) \tilde{\Psi}_{i j}(\eta, \vec{x})=0 . \tag{4.52}
\end{equation*}
$$

The tensor field $\tilde{\Psi}_{i j}$ and its conjugate field $\Pi_{\tilde{\Psi}}^{i j}=(32 \pi G)^{-1} a^{D-1} \dot{\tilde{\Psi}}^{i j}$ are quantized according to

$$
\begin{align*}
& {\left[\hat{\tilde{\Psi}}_{i j}(\eta, \vec{x}), \hat{\Pi}_{\tilde{\Psi}}^{k l}\left(\eta, \vec{x}^{\prime}\right)\right]=\frac{i \hbar}{2}\left(P_{i k} P_{j l}+P_{i l} P_{j k}-\frac{2}{(D-2)} P_{i j} P_{k l}\right) \delta^{(D-1)}\left(\vec{x}-\vec{x}^{\prime}\right),}  \tag{4.53}\\
& {\left[\hat{\tilde{\Psi}}_{i j}(\eta, \vec{x}), \hat{\tilde{\Psi}}_{k l}\left(\eta, \vec{x}^{\prime}\right)\right]=0}  \tag{4.54}\\
& {\left[\hat{\Pi}_{\tilde{\Psi}}^{i j}(\eta, \vec{x}), \hat{\Pi}_{\tilde{\Psi}}^{k l}\left(\eta, \vec{x}^{\prime}\right)\right]=0} \tag{4.55}
\end{align*}
$$

where $P_{i j}=\delta_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}}$ is a transverse projector. The fact that the right-hand side of equation (4.53) is expressed in terms of these projectors, ensures that the tensor field and its conjugate field are transverse and traceless. Next, we ought to make a mode expansion,

$$
\begin{equation*}
\hat{\Psi}_{i j}(\eta, \vec{x})=\int \frac{d^{D-1} k}{(2 \pi)^{D-1}} \sum_{p}\left\{e^{i \vec{k} \vec{x}} \epsilon_{i j}^{p}(\vec{k}) \Psi(\eta, k) \hat{a}_{p}(\vec{k})+e^{-i \vec{k} \vec{x}} \epsilon_{i j}^{p *}(\vec{k}) \Psi^{*}(\eta, k) \hat{a}_{p}^{\dagger}(\vec{k})\right\}, \tag{4.56}
\end{equation*}
$$

where the sum runs over all graviton polarisation $p$ and $\epsilon_{i j}^{p}(\vec{k})$ is the polarisation tensor which satisfies

$$
\begin{equation*}
\sum_{p} \sum_{i j} \epsilon_{i j}^{p}(\vec{k}) \epsilon_{i j}^{p^{\prime}}(\vec{k})=\sum_{p} \delta_{p p^{\prime}}=\frac{1}{2} D(D-3) . \tag{4.57}
\end{equation*}
$$

The final equality signifies that there are $\frac{1}{2} D(D-3)$ physical graviton polarizations. The creation and annihilation operators also have a polarisation index. Operators for different polarizations commute, i.e.,

$$
\begin{equation*}
\left[\hat{a}_{p}(\vec{k}), \hat{a}_{p^{\prime}}^{\dagger}(\vec{k})\right]=(2 \pi)^{D-1} \delta_{p p^{\prime}} \delta^{D-1}\left(\vec{k}-\vec{k}^{\prime}\right) . \tag{4.58}
\end{equation*}
$$

From here, we skip directly to the next important result; the normalized solutions for the mode functions are

$$
\begin{equation*}
\Psi(\eta, k)=\sqrt{8 \pi^{2} G \hbar} a^{1-\frac{D}{2}} \sqrt{-\eta} H_{\nu}^{(1)}(-k \eta), \quad \Psi^{*}(\eta, k)=\sqrt{8 \pi^{2} G \hbar} a^{1-\frac{D}{2}} \sqrt{-\eta} H_{\nu}^{(2)}(-k \eta) . \tag{4.59}
\end{equation*}
$$

Up to a (normalisation) constant these are identical to the mode functions for the scalar field. This means that the propagator is also identical up to a prefactor. For the sake of completeness, we state the full graviton propagator

$$
\begin{align*}
i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right) & =32 \pi G \hbar \frac{\left[(1-\epsilon)^{2} H H^{\prime}\right]^{D / 2-1}}{(4 \pi)^{D / 2}}\left\{\Gamma(D / 2-1)\left(\frac{4}{y}\right)^{D / 2-1}+\frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)}\right. \\
& \times\left[\frac{\Gamma(3 / 2+\nu) \Gamma(3 / 2-\nu)}{\Gamma(3-D / 2)}\left(\frac{4}{y}\right)^{D / 2-2}+\sum_{n=1}^{\infty}\left(\frac{\Gamma(3 / 2+\nu+n) \Gamma(3 / 2-\nu+n)}{\Gamma(3-D / 2+n)(n+1)!}\left(\frac{y}{4}\right)^{n-D / 2+2}\right.\right. \\
& \left.\left.-\frac{\Gamma\left(\frac{D-1}{2}+\nu+n\right) \Gamma\left(\frac{D-1}{2}-\nu+n\right)}{\Gamma(D / 2+n) n!}\left(\frac{y}{4}\right)^{n}\right)\right]+\frac{2(1-\epsilon) \Gamma(2 \nu) \Gamma(\nu)}{\epsilon(D-2) \Gamma\left(\frac{D-1}{2}\right) \Gamma(1 / 2+\nu)} \\
& \times\left[\left(\frac{1}{k_{0}^{2} \eta \eta^{\prime}}\right)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}}+\frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(\nu)} \frac{\Gamma(1-D / 2)}{\Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D-1}{2}+\nu\right) \Gamma\left(\frac{D-1}{2}-\nu\right) \epsilon(D-2)}{\Gamma(2 \nu)} \frac{\epsilon(D-\epsilon)}{2(1-\epsilon)}\right] . \tag{4.60}
\end{align*}
$$

## Chapter 5

## One-loop quantum backreaction

For a while, since the early 2000s, there has been disagreement within the scientific community about the effect of inhomogeneities in matter and geometry on the average cosmic evolution. Such an effect is commonly known as a 'backreaction'. An intuitive example of a backreaction is when a massive particle curves the surrounding space. This typically has a small effect on its own dynamics and that of other nearby particles. Including such effects is often pivotal when constructing a self-consistent theory. One can reformulate the question of 'backreaction' in terms relevant to cosmology: does the presence of inhomogeneities influence the dynamics of the background in a significant way? The answer to this question has been subject to controversy. Some conclude that backreaction effects could replace dark energy as the source for the recently observed accelerated expansion of the universe, while others say they are completely negligible. And, of course, there is a middle ground. For a review, and references to recent research, we refer to [31]. The fact of the matter remains that, in the present age of precision cosmology, even small effects could be relevant.

In this chapter, we aim to calculate the one-loop expression for the backreaction of scalar and tensor metric fluctuations onto the scalar equation of motion. This comes down to taking the vacuum expectation value of the right-hand side of equation (3.82). As mentioned in the previous chapters, we perform the calculation on a FLRW background with a constant deceleration parameter. Specifically, we take $\epsilon$ to be a small, positive constant $(\epsilon \ll 1)$ such that we can neglect higher order terms in $\epsilon$. The backreaction introduces a condensate which we remove by shifting the scalar operator. Finally, we can draw conclusions about the influence of the backreaction on the background geometry. Section 5.1 will cover the underlying theory of functional methods, whereas section 5.2 contains the actual calculation of the one-loop quantum backreaction.

### 5.1 On functional methods

This section aims to provide the reader with some information on functional methods that underlie the calculations done in the remainder of this chapter. This includes a path integral approach to field theory and the diagrammatic representation of n-point functions in terms of Feynman diagrams. Notes on different kinds of Feynman diagrams are included as well. By covering the underlying theory, it is easier to point out where approximations will be made in the future.

The path integral approach to quantum field theory offers an alternative to the operator formalism of canonical quantization covered in section 4.1. Although both concepts are equivalent within quantum field theory, the path integral approach is more convenient when it comes to describ-
ing diagrammatic methods or stochastic processes. Path integrals are used to calculate transition amplitudes which are crucial when calculating the probability that a quantum process will occur. Often times, a quantum process, such as the movement of a (quantum) particle from A to B , can be realized in many different ways. This is in direct contrast to the classical notion of a unique trajectory. Therefore, a path integral takes into account all possible 'paths' a system may take in between its initial and final states and assigns them a fixed weight, but differing phase.

One of the most important path integrals is the partition functional,

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \phi e^{\frac{i}{\hbar} S[\phi]+\int d^{4} x J(x) \phi(x)}, \tag{5.1}
\end{equation*}
$$

where $\phi(x)$ is a general field, $S[\phi]$ is the corresponding classical action functional and $J(x)$ is an unspecified source. The source is mainly there to enable a mathematical trick that allows us to calculate the n-point correlation functions for $\phi$. The partition functional is closely related to the partition function in statistical mechanics in that it enables access to all the information about the system if solved exactly. For instance, one can derive the n-point functions by means of functional derivatives acting on the auxiliary source $J$,

$$
\begin{align*}
\langle 0| T\left[\phi\left(x_{1}\right) \ldots \phi\left(x_{n}\right)\right]|0\rangle & \equiv \int \mathcal{D} \phi \phi\left(x_{1}\right) \ldots \phi\left(x_{n}\right) e^{\frac{i}{\hbar} S[\phi]} \\
& =\left.\left(\frac{\hbar}{i}\right)^{n} \frac{\delta^{n} Z[J]}{\delta J\left(x_{1}\right) \ldots \delta J\left(x_{n}\right)}\right|_{J=0} . \tag{5.2}
\end{align*}
$$

A user friendly way of writing the n-point functions is diagrammatically in terms of Feynman diagrams. The specific rules for drawing Feynman diagrams depend on the interaction Lagrangian of the theory. The partition functional is simply proportional to the sum over all possible Feynman diagrams in the presence of a non-zero source $J$.

The linked cluster theorem [32] states that it is not necessary to calculate all Feynman diagrams to determine the partition functional; the connected diagrams are enough. As a result, one can define another functional, called the generating functional,

$$
\begin{equation*}
W[J] \equiv \frac{i}{\hbar} \ln (Z[J]), \tag{5.3}
\end{equation*}
$$

which is determined by connected diagrams only. The generating functional generates so-called connected n-point functions which are closely related to the above n-point functions, but the vacuum expectation values are normalized. For example, the first functional derivative with respect to $J$ is simply the normalized vacuum expectation value of the field $\phi$, while the second functional derivative is known as the connected two-point function,

$$
\begin{equation*}
\frac{\hbar}{i} \frac{\delta^{2} W[J]}{\delta J\left(x_{1}\right) \delta J\left(x_{2}\right)}=\left[\frac{\langle 0| T\left[\phi\left(x_{1}\right) \phi\left(x_{2}\right)\right]|0\rangle}{\langle 0 \mid 0\rangle}-\frac{\langle 0| \phi\left(x_{1}\right)|0\rangle\langle 0| \phi\left(x_{2}\right)|0\rangle}{\langle 0 \mid 0\rangle^{2}}\right], \tag{5.4}
\end{equation*}
$$

which is simply the two-point function as in equation (5.2) with the disconnected parts cancelled by the second term. Therefore, it consists only of connected diagrams. The same is true for all other (connected) n-point functions derived in this way. The connected two-point function is often referred to as the full propagator, or connected two-point Green's function.

Apart from being connected or disconnected, Feynman diagrams can be categorized in multiple ways. Here we will highlight some of these categories. First of all, there are vacuum (sub)diagrams. A vacuum (sub)diagram is a (part of a) diagram that is not connected to any external legs. They appear, for instance, in the normalization $\langle 0 \mid 0\rangle$; the sum of all vacuum diagrams. This makes it that they do not contribute when calculating normalized correlation functions. Second of all, a tadpole (sub)diagram is a (part of a) diagram that can be disconnected from all external legs by cutting precisely one line. Tadpole diagrams contribute to the vacuum expectation value, or 1-point function, of the field. A simplified diagrammatic representation of the vacuum expectation value is shown in figure 5.1. In most situations the vacuum expectation value of a field vanishes, or else one can work with a shifted field defined by $\hat{\chi}(x) \equiv \hat{\phi}(x)-\langle\Omega| \hat{\phi}(x)|\Omega\rangle$. In such a formulation, the tadpole diagrams would cancel out.


Figure 5.1: The above is a simplified depiction of the sum over all tadpole diagrams. The circle indicates a sum over all connected diagrams without external legs besides the one at $x_{1}$. Cutting the line would separate the connected diagrams from the external leg.

Another type of Feynman diagrams is the one-particle irreducible, or 1-PI, diagram. A 1-PI diagram is essentially a super-connected diagram in the sense that it can not be separated into two pieces by removing a single internal line (a line between two internal vertices). For this reason they are called one-particle irreducible. An additional convention to the definition of 1-PI diagrams is to not consider external legs, i.e., to consider them as amputated. As such, 1-PI diagrams can be considered as the building blocks to connected diagrams. This will become more apparent later. Similar to the generating functional for connected n-point functions one can find a functional which generates proper n-point (1PI) vertices made up of 1-PI diagrams,

$$
\begin{equation*}
\Gamma[\bar{\phi}] \equiv W[J]-\int d^{4} x J \bar{\phi}, \tag{5.5}
\end{equation*}
$$

where $J[\bar{\phi}]$ is the inverse of

$$
\begin{equation*}
\bar{\phi}[J]=\frac{\delta W[J]}{\delta J} . \tag{5.6}
\end{equation*}
$$

$\Gamma[\bar{\phi}]$ is referred to as the (1-PI) effective action and is precisely the Legendre transform of the generating functional. As can be seen from equation (5.6), $\bar{\phi}$ is essentially the vacuum expectation value of the field $\phi$. The way one can generate proper $n$-point vertices is by taking functional derivatives with respect to $\bar{\phi}$. As an example, take the first functional derivative,

$$
\begin{equation*}
\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}(x)}=-J(x) . \tag{5.7}
\end{equation*}
$$

The above equation is an equation of motion for the vacuum expectation value of the field in the presence of an external source. It replaces the classical field equation $\delta S[\phi] / \delta \phi=-J$ in the classical limit $\bar{\phi} \rightarrow \phi$. As such, the effective action can be seen as the classical action corrected to include quantum effects.


Figure 5.2: A graphical representation of equation (5.9) in terms of Feynman diagrams. The left hand-side is the sum over all connected diagrams with exactly two external lines, i.e., the connected two-point function. The proper two-point vertex is represented by the middle circle on the righthand side.

Another interesting example is the second functional derivative of the effective action,

$$
\begin{equation*}
\frac{\delta^{2} \Gamma[\bar{\phi}]}{\delta \bar{\phi}\left(x_{1}\right) \delta \bar{\phi}\left(x_{2}\right)}=-\frac{\delta J\left(x_{1}\right)}{\delta \bar{\phi}\left(x_{2}\right)}=\left[-\frac{\delta \bar{\phi}\left(x_{2}\right)}{\delta J\left(x_{1}\right)}\right]^{-1}=\left[-\frac{\delta^{2} W[J]}{\delta J\left(x_{1}\right) \delta J\left(x_{2}\right)}\right]^{-1}=i \Pi^{-1}\left(x_{1}, x_{2}\right), \tag{5.8}
\end{equation*}
$$

where $\Pi^{-1}\left(x_{1}, x_{2}\right)$ is the inverse of the connected two-point function. The above relation can be rewritten more intuitively as

$$
\begin{equation*}
\Pi\left(x_{1}, x_{2}\right)\left(\frac{1}{i} \frac{\delta^{2} \Gamma[\bar{\phi}]}{\delta \bar{\phi}\left(x_{2}\right) \bar{\phi}\left(x_{3}\right)}\right) \Pi\left(x_{3}, x_{4}\right)=\Pi\left(x_{1}, x_{4}\right), \tag{5.9}
\end{equation*}
$$

which is depicted in terms of Feynman diagrams in figure 5.2. This implicitly shows how to construct the connected two-point function from the proper two-point (1-PI) vertex. Similar relations hold for the other n-point functions. Alternatively, one can find a more explicit expression for the connected n-point functions in terms of 1-PI diagrams by an argument based on diagrams. Any connected diagram can be constructed by stringing together a sufficient number of 1-PI diagrams. This is to be understood as follows. Any connected diagram that is not one-particle irreducible, has at least one internal line such that, when cut, the diagram falls into two disconnected components. If one is able to identify all such internal lines, the remainder of the diagram must consist of oneparticle irreducible subdiagrams. Thus, the whole diagram is just a series of (smaller) 1-PI diagrams connected by single lines. This argument simplifies calculations once more by reducing the number of diagrams that ought to be calculated; in principle, calculating the 1-PI diagrams is sufficient. Consider, for instance, the connected two-point function $\Pi\left(x_{1}, x_{2}\right)$ and let $\Pi_{0}\left(x_{1}, x_{2}\right)$ denote the free (Feynman) propagator and $\Sigma\left(x_{1}, x_{2}\right)$ the sum over all 1-PI diagrams with two amputated external legs, then

$$
\begin{equation*}
\Pi=\Pi_{0}+\Pi_{0} \Sigma \Pi_{0}+\Pi_{0} \Sigma \Pi_{0} \Sigma \Pi_{0}+\cdots=\Pi_{0}\left(\sum_{n=0}^{\infty}\left(\Sigma \Pi_{0}\right)^{n}\right) \tag{5.10}
\end{equation*}
$$

This relation is referred to as the Schwinger-Dyson equation and is drawn in terms of diagrams in figure 5.3. The equation shows that the connected two-point function (or full propagator) can be


Figure 5.3: The Dyson equation in terms of Feynman diagrams. The full propagator on the lefthand side of the equation is represented by the sum over all connected diagrams with two external legs. The single line denotes the free propagator and $\Sigma$ is the sum over all 1-PI diagrams with two amputated external legs. By flanking $\Sigma$ with two free propagators one is able to construct connected diagrams.
expressed as a geometric series with common ratio $\Sigma \Pi_{0}$. Sometimes, this can be used to find an exact expression for the full propagator. Other times, it is used as the start of an approximation scheme.

Often, approximations are important since it is generally hard to find an exact expression for the effective action. First and foremost, one can expand the effective action in terms of the (non-local) proper n-point functions in a functional Taylor series,

$$
\begin{equation*}
\Gamma[\bar{\phi}]=\sum_{n=0}^{\infty} \frac{1}{n!} \int d x_{1} \cdots d x_{n} \Gamma_{n}\left(x_{1}, \ldots, x_{n}\right) \bar{\phi}\left(x_{1}\right) \ldots \bar{\phi}\left(x_{n}\right), \tag{5.11}
\end{equation*}
$$

where $\Gamma_{n}\left(x_{1}, \ldots, x_{n}\right)$ denotes the proper n-point vertex,

$$
\begin{equation*}
\Gamma_{n}\left(x_{1}, \ldots, x_{n}\right) \equiv\left(\frac{1}{i}\right)^{n-1} \frac{\delta^{n} \Gamma[\bar{\phi}]}{\delta \bar{\phi}\left(x_{1}\right) \ldots \delta \bar{\phi}\left(x_{n}\right)} . \tag{5.12}
\end{equation*}
$$

Diagrammatically, this can be understood as follows. The diagrams which are part of the effective action can be rearranged by the number of external legs as is shown in figure 5.4. The first term are the (1-PI) vacuum diagrams which do not couple to the field. Then, the (1-PI) tadpole diagrams couple to the field linearly. The diagrams that couple quadratically to the field contribute to what is called the self-energy. The self-energy is interpreted as the portion of the particle's energy that is gained or lost due the interaction with its own environment. One can continue to include terms that couple to the field to a higher degree. However, the series is usually truncated at a specific order.

An additional expansion of the proper n-point vertices in terms of 1-PI Feynman diagrams gives a concise way of doing perturbation theory. This is because the Feynman rules generally restrict the number of internal vertices one can have with respect to the internal edges; which inevitably leads to loops in connected diagrams. Subsequently, one can order the diagrams by the number of independent loops, which is called a loop expansion. The lowest order in the expansion are diagrams without loops, called tree level diagrams, and they usually represent the classical theory. Then, oneloop diagrams express the first quantum corrections to the classical theory and so on. Only when the expansion is truncated at some finite loop order, does the calculation become perturbative. It can be shown that the number of loops in a diagram is directly related to the power of $\hbar$ in the prefactor [33]. Therefore, $\hbar$ can be used as a bookkeeping device to keep track of the order in the perturbation. Note that this does not require $\hbar$ to be small. It does solidify the claim that the tree level diagrams represent the classical level since these are the last terms that remain in the classical limit $\hbar \rightarrow 0$.


Figure 5.4: A representation of the ordering of diagrams based on the number of external legs in the effective action where $\Gamma_{n}$ denotes the proper n-point vertex expressed in terms of Feynman diagrams.

### 5.2 One-loop quantum backreaction to the tadpole equation

Using the terminology introduced in the previous section, one can interpret the quadratic dynamical equation derived in equation (3.82). By the quantization procedure highlighted in section 4.3 . equation (3.82) is in fact an equation in terms of quantum operators. Taking its vacuum expectation value (VEV) with respect to the Bunch-Davies vacuum results in the following equation,

$$
\begin{equation*}
\epsilon\left[\partial_{0}^{2}+(D-1) H \partial_{0}-a^{-2} \nabla^{2}\right]\langle\Omega| \hat{\tilde{\psi}}(x)|\Omega\rangle=\langle\Omega| \hat{T}_{i}^{i(1)}(x)|\Omega\rangle, \tag{5.13}
\end{equation*}
$$

where the right-hand side of equation (3.82) is compactly written as $T_{i}^{i(1)}(x)$, which can be seen as a source generated by the terms quadratic in metric perturbations. Recall that equation (5.13) is obtained by expanding the trace of the Einstein equation in the stacked gauge up to quadratic order in perturbations. Therefore, the right-hand side of equation (5.13) acts as an effective stress-energy tensor. The relevance of the (1) in the exponent is to identify a one-loop expression, i.e., a first quantum correction. The influence of the interactions described by $T_{i}^{i(1)}(x)$ on the dynamics of the VEV of $\tilde{\psi}$ is what is understood as a backreaction. If the source were not there, the equation would be trivial. However, since there is a nontrivial source, the VEV of the field gets (quantum) corrected at the one-loop level. As discussed in the previous section, one way of deriving the new VEV at this level is by summing all one-loop tadpole diagrams (see figure 5.5). However, evaluating these diagrams would require complete knowledge of the Feynman rules, in particular the expressions for the vertex operators. These could be read off from the cubic Lagrangian in a straightforward manner. However, as motivated in section 3.7, we took a different approach where calculating the cubic Lagrangian is inessential. Instead, the aim will be to calculate the right-hand side of equation (5.13) and see how this affects the VEV of the field through its equation of motion. This alternative approach only requires knowledge of the (free) scalar and graviton propagators, which were derived in section 4.3.


Figure 5.5: One-loop tadpole diagrams that contribute to the VEV of the scalar field $\tilde{\psi}(x)$. The solid lines represent the scalar propagator, while the squiggly line indicates a graviton propagator. The vertex in the right-most diagram is a counterterm vertex which is present to cancel out divergences from the other diagrams.

Calculating the expectation value of $\hat{T}_{i}^{i(1)}(x)$ is a considerable undertaking since there are many independent terms. Recall that,

$$
\begin{align*}
T_{i}^{i(1)}= & -\partial_{0}\left[(D-1) H \tilde{n}^{(2)}-\frac{1}{2} \frac{\dot{h} \dot{\tilde{\psi}}}{H}+\frac{1}{2}\left(\tilde{\Psi}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} h\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)\right]-a^{-2}\left[\nabla^{2} \tilde{n}^{(2)}\right. \\
& \left.-2 H^{-1} \tilde{\psi} \nabla^{2} \dot{\tilde{\psi}}-H^{-1}\left(\tilde{\Psi}^{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} h\right) \partial_{i} \partial_{j} \dot{\tilde{\psi}}\right]+\frac{D-1}{4(D-2)}\left[\left(\dot{\tilde{\Psi}}^{i j}-\frac{\partial^{i} \partial^{j}}{\nabla^{2}} \dot{h}\right)\left(\dot{\tilde{\Psi}}_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}} \dot{h}\right)\right. \\
& \left.-\frac{\dot{h}^{2}}{D-1}\right]-\frac{2}{D-2} R^{(2)}+2 H^{-1} a^{-2} \dot{\tilde{\psi}} \nabla^{2} \tilde{\psi}+(D-1) \epsilon H\left[\frac{\dot{\tilde{\psi}}^{2}}{H^{2}}-\tilde{n}^{(2)}\right] . \tag{5.14}
\end{align*}
$$

where the $x$-dependence of the fields is left implicit. Additionally, $\tilde{n}^{(2)}$ and $R^{(2)}$ are as in equation (3.80) and (3.81) respectively. Fortunately, there are some ways the expression simplifies when taking the expectation value. First of all, the scalar and tensor perturbations of the metric evolve independently at linear order. This can immediately be seen from the quadratic action (3.68). Also, the independent evolution of scalar and tensor components motivated the scalar-vector-tensor decomposition in the first place. As a result, any term combining scalar and tensor perturbations does not contribute to the expectation value at this order. This means, for instance,

$$
\begin{equation*}
\langle\Omega| \hat{\tilde{\Psi}}_{i j} \frac{\partial_{i} \partial_{j}}{\nabla^{2}} \hat{h}|\Omega\rangle=0 \tag{5.15}
\end{equation*}
$$

Another way the expectation value may simplify is by considering the properties of the free scalar and graviton propagators in equations (4.50) and 4.60) respectively. In both cases, the spacedependence is present only in the form of the absolute distance $\left\|\vec{x}-\vec{x}^{\prime}\right\|$. The same is true for any and all derivatives of the propagators. This makes it so that, at coincidence, the propagators become space-independent. As a result, terms in equation (5.14) which are total spatial gradients do not contribute to the vacuum expectation value. At the same time one can use this fact to move around spatial derivatives. For instance,

$$
\begin{align*}
\langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle & =\partial^{i}\langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \partial_{i} \hat{\tilde{\psi}}(x)|\Omega\rangle-\langle\Omega| \partial^{i} \partial_{0} \hat{\tilde{\psi}}(x) \partial_{i} \hat{\tilde{\psi}}(x)|\Omega\rangle \\
& =\partial^{i}\left[\partial_{0} \partial_{i}^{\prime} i \Delta_{\tilde{\psi}}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}-\partial^{i}\langle\Omega| \partial^{i} \partial_{0} \hat{\tilde{\psi}}(x) \hat{\tilde{\psi}}(x)|\Omega\rangle+\langle\Omega| \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x) \hat{\tilde{\psi}}(x)|\Omega\rangle \\
& =0-0+\langle\Omega| \nabla^{2} \partial_{0} \tilde{\tilde{\psi}}(x) \tilde{\tilde{\psi}}(x)|\Omega\rangle \tag{5.16}
\end{align*}
$$

which is obtained by applying the product rule twice. One essentially moves the spatial derivatives as in partial integration; obtaining a minus sign each time. Do note that the same is not true for time derivatives.

One final approximation we will make is to only keep terms up to lowest order in the first geometric slow-roll parameter $\epsilon$. Since $\epsilon$ will be taken to be small, this is a reasonable approximation. An immediate consequence is that one can approximate equation (3.77) as

$$
\begin{equation*}
\partial_{0} \hat{h}(x) \approx-2 H^{-1} a^{-2} \nabla^{2} \hat{\tilde{\psi}}(x) . \tag{5.17}
\end{equation*}
$$

Taking into account the above remarks, the right-hand side of equation (5.13) reduces to

$$
\begin{align*}
\langle\Omega| \hat{T}_{i}^{i(1)}(x)|\Omega\rangle= & -a^{-2} \frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \nabla^{2} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle}{4(D-2) H}+\frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0}^{2} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle}{4(D-2) H}+\frac{D-1}{4(D-2)}\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle \\
& +\frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0} \hat{\tilde{\Psi}}^{i j}+\hat{\tilde{\Psi}}_{i j} \partial_{0}^{2} \hat{\tilde{\Psi}}}{} \hat{\tilde{T}}^{i j}|\Omega\rangle \\
2(D-2) & \frac{D-3}{2 H} a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}} \nabla^{2} \hat{\tilde{\psi}}|\Omega\rangle \\
& +\frac{D^{2}-4 D+5}{2(D-2)(D-1)} a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}} \nabla^{2} \hat{h}|\Omega\rangle-a^{-2} \frac{\langle\Omega| \partial_{0} \hat{h} \nabla^{2} \hat{h}|\Omega\rangle}{4(D-2) H} \\
& +\frac{D-1}{2(D-2) H} a^{-2}\langle\Omega| \partial_{0} \hat{h} \nabla^{2} \hat{\tilde{\psi}}|\Omega\rangle-\frac{\langle\Omega| \partial_{0}^{2} \hat{h} \partial_{0} \hat{\tilde{\psi}}-\partial_{0} \hat{h} \partial_{0}^{2} \hat{\tilde{\psi}}|\Omega\rangle}{2(D-2) H}  \tag{5.18}\\
& +\frac{\langle\Omega| \partial_{0}^{2} \hat{h} \partial_{0} \hat{h}|\Omega\rangle}{2(D-1)(D-2) H}+\frac{\langle\Omega| \hat{h} \partial_{0}^{2} \hat{h}|\Omega\rangle}{2(D-2)}+\frac{D}{4(D-2)}\langle\Omega| \partial_{0} \hat{h} \partial_{0} \hat{h}|\Omega\rangle+\mathcal{O}(\epsilon) .
\end{align*}
$$

This leaves us only to evaluate each of the above independent terms which is done in appendix B. It turns out that the tensor terms cancel each other, such that the total tensor contribution vanishes. This is quite a remarkable result and it means that the backreaction is sourced solely by scalar interactions. In order to evaluate the remaining scalar terms one has to deal with some tough integral expressions. An example of which is worked out in appendix C. Inevitably, one has to resort to evaluating these integrals in the late-time limit by means of an asymptotic expansion. After all is said and done, equation (5.18) can be approximated by,

$$
\begin{equation*}
\langle\Omega| \hat{T}_{i}^{i(1)}(x)|\Omega\rangle \approx-\frac{2175}{32 \pi} \frac{\hbar H_{0}^{4} G}{\epsilon}\left(\frac{1}{\epsilon}+4 \log (a(t))+\text { constant terms }\right), \tag{5.19}
\end{equation*}
$$

in four space-time dimensions. Higher order terms in $\epsilon$ were neglected in the approximation. Additionally, the constant terms in equation (5.19) remain undetermined because calculations are already quite involved at this order in $\epsilon$. In the following calculations, these undetermined constant terms, and any other constant terms at the same order, will be ignored.

To figure out precisely the effect of scalar interactions, one can split the scalar field into a homogeneous part and a one-loop backreaction,

$$
\begin{equation*}
\hat{\tilde{\psi}}(t, \vec{x})=\hat{\tilde{\psi}}_{0}(t, \vec{x})+B^{(1)}(t), \tag{5.20}
\end{equation*}
$$

such that,

$$
\begin{align*}
& {\left[\partial_{0}^{2}+(D-1) H \partial_{0}-a^{-2} \nabla^{2}\right] \hat{\tilde{\psi}}_{0}(t, \vec{x})=0,}  \tag{5.21}\\
& {\left[\partial_{0}^{2}+(D-1) H \partial_{0}-a^{-2} \nabla^{2}\right] B^{(1)}(t)=-\frac{2175}{32 \pi} \frac{\hbar H_{0}^{4} G}{\epsilon^{2}}\left(\frac{1}{\epsilon}+4 \log (a(t))\right) .} \tag{5.22}
\end{align*}
$$

If there were no interactions, the scalar dynamics would be described in the form of $\hat{\tilde{\psi}}_{0}(t, \vec{x})$. However, since interactions have a non-trivial effect, the scalar operator gets quantum corrected at the one-loop level by $B^{(1)}(t)$. One can calculate $B^{(1)}(t)$ by solving equation (5.22) up to the appropriate order in $\epsilon$,

$$
\begin{equation*}
B^{(1)}(t)=-\frac{2175}{32 \pi} \frac{\hbar H_{0}^{4} G}{\epsilon^{3}}\left(\log (a(t))+\epsilon \log (a(t))^{2}\right) . \tag{5.23}
\end{equation*}
$$

As a side remark, the constant terms in equation (5.19) would contribute to $B^{(1)}(t)$ as a term proportional to $\epsilon^{-2} \log (a)$ with an undetermined prefactor (which does not depend on $\epsilon$ ).

One curiosity that immediately jumps out is that the one-loop backreaction has a singularity at $\epsilon=0$, so that it diverges in the de Sitter limit. It is not clear at the moment whether this behaviour is a gauge artefact. Certainly, the effective action is gauge-dependent through its dependence on the propagators. However, it has not been proven that any vacuum expectation value derived from it, e.g. equation (5.19), is gauge-dependent as well. Another striking feature of equation (5.23) is the time-dependence which appears as a logarithm of the scale factor. As a result, the average effect of interactions will increase during inflation. Finally, the minus sign indicates that this is a negative shift of the scalar operator and, thus, a negative shift on the VEV of the spatial part of the metric,

$$
\begin{equation*}
\langle\Omega| \hat{g}_{i j}(x)|\Omega\rangle=g_{i j}^{\mathrm{cl}}(x)-\frac{2175}{32 \pi} \frac{\hbar H_{0}^{4} G}{\epsilon^{3}} a(t)^{2}\left(\log (a(t))+\epsilon \log (a(t))^{2}\right) \delta_{i j}+\mathcal{O}\left(\hbar^{2}\right), \tag{5.24}
\end{equation*}
$$

where $g_{i j}^{\mathrm{cl}}(x)$ is the classical (background) metric. The principle effect of this negative shift is that it slows down the expansion of the universe during inflation. There is some intuitive understanding to be had as to why the interactions affect the expansion of the universe in this way by considering the nature of the interactions. Since the gravitational force is attractive, gravitational interactions lower the energy of a system. Therefore, it makes sense that the effective stress-energy tensor is negative at one-loop order (equation (5.19) and that the first quantum corrections contribute negatively to the universe's expansion rate during inflation.

## Chapter 6

## Conclusion and outlook

I would have gladly continued this research towards a stochastic description of inflation, but time did not allow for that. Nevertheless, the results presented here could be valuable to future research in this direction. In the stochastic approach one makes a distinction between long-wavelength and short-wavelength behaviour. Fluctuations are assumed to become classical when growing larger than the Hubble radius. One can solve the constraint equations and derive the stochastic dynamical equations at long wavelengths by using Hamilton-Jacobi theory. In this long-wavelength approximation one only keeps terms up to leading order in spatial gradients because at scales larger than the Hubble radius the dynamics is dominated by time derivatives. For instance, this is done in [34]. When the wavelength of a growing sub-Hubble mode inevitably exceeds the Hubble radius, it acts as a kick to the long-wavelength dynamics, in essence changing the field's initial conditions. Each mode that crosses the Hubble horizon has a random phase and, thus, one elegantly ends up with a classical long-wavelength equation sourced by a random noise term. The random noise term for scalar fluctuations is determined from its two-point function. In this thesis, we have shown how to calculate the propagator for scalar metric fluctuations by solving the linearized equations for the mode functions using the results from [3]. One can calculate higher order corrections to the two-point function by using the Swinger-Keldysh (or in-in) formalism which provides a way to study non-Gaussian contributions to the noise term without requiring knowledge of suitable late time states. One could also calculate the one-loop corrected scale factor from the results obtained in this thesis and see how it influences the long-wavelength equations directly by rederiving the Hamilton-Jacobi equation.

An important remark is that one should be mindful when working with a gauge where the shift vector vanishes, $N_{i}=0$. This is a gauge that is often used in the stochastic framework, because it simplifies the equations. As such, D.S. Salopek and J.R. Bond [34, [35] fix a gauge where

$$
\begin{equation*}
d s^{2}=-N^{2}(t, \vec{x}) d t^{2}+e^{2 \alpha(t, \vec{x})} h_{i j}(\vec{x}) d x^{i} d x^{j}, \tag{6.1}
\end{equation*}
$$

as they study a system of real, minimally coupled scalar fields $\Phi_{j}, j=1, \ldots, k$, which self-interact through a potential $V\left(\Phi_{j}\right)$. Gravitational radiation is said to be dynamically unimportant relative to the scalar field at long wavelengths, hence the time dependence of the three metric $g_{i j}(t, \vec{x})$ can be expressed by the field $\alpha(t, \vec{x})=\ln (a(t, \vec{x}))$. In the case of a single matter field ( $\mathrm{k}=1$ ), there would be three scalar fields in this gauge: $N(t, \vec{x}), \alpha(t, \vec{x}), \Phi(t, \vec{x})$. The authors derive a set of longwavelength equations from the energy and momentum constraints of GR by neglecting second-order spatial gradients,

$$
\begin{equation*}
\left(\frac{d H}{d \Phi}\right)^{2}=\frac{3}{2} H^{2}-\frac{1}{2} V(\Phi), \quad H(t, \vec{x}) \equiv \frac{\dot{\alpha}(t, \vec{x})}{N(t, \vec{x})}, \quad H(t, \vec{x})=H(\Phi(t, \vec{x})), \tag{6.2}
\end{equation*}
$$

where the local expansion parameter $H$ is purely determined by its dependence on the scalar field $\Phi$. The first of the equations is the Hamilton-Jacobi equation which relates the scalar fields. This is only enough to constrain one scalar field in this gauge, leaving two independent scalar fields. One might be tempted to use this freedom to conveniently choose the constant time hypersurfaces, like the authors did. However, by carefully treating this gauge one can derive an additional condition which will constrain another scalar field. For instance, by performing a linear gauge transformation from, e.g., the comoving gauge, one finds that, at linear order,

$$
\begin{equation*}
\dot{\bar{\alpha}}(t) n(t, \vec{x})-\delta \dot{\alpha}(t, \vec{x})=\frac{1}{2} \dot{\bar{\phi}}(t) \delta \phi(t, \vec{x}), \tag{6.3}
\end{equation*}
$$

where the scalar fields have been split into a homogeneous background $(\bar{\alpha}, \bar{\phi})$ and a space dependent perturbation $(\delta \alpha, \delta \phi)$, likewise $N(t, \vec{x})=1+n(t, \vec{x})$. Equation (6.3) signifies that there is an additional constraint relating the three scalar fields. This means that: either one is not free to choose the constant time hypersurfaces; or there is another scalar field 'hidden' within the spatial part of the metric and one is not able to extract the time dependence into a single scalar field as is done in equation (6.1). An instance of the latter is seen in the stacked gauge treated in this thesis (equation (3.50), where the tensor metric perturbations are non-transverse. In this case, one can not neglect the tensorial part of the metric by the argument that gravitational radiation is dynamically insignificant because it contains an additional scalar field which has to be taken into account. The reason why the stacked gauge is fully fixed is because matter field fluctuations are set to zero. If one were to allow for arbitrary matter field fluctuations, then one could choose the constant time hypersurfaces instead. By carefully treating this type of gauge, one may be able to shed some light on how to properly incorporate gravitational fluctuations in the stochastic framework.

The main result of this thesis is the one-loop quantum backreaction on a D-dimensional, spatially flat, homogeneous and isotropic background with arbitrary constant deceleration parameter (equation (5.23). The result is inversely proportional to the first geometric slow roll parameter cubed. This may be a gauge artefact and it would be interesting to see if this effect survives in other gauges. Seeing as the result is time-dependent it might have some effect on the duration of inflation. Perhaps the backreaction even becomes dominant enough to end inflation. It might also be worthwhile to consider if there is any significant late time effects originating from this backreaction. This would require an extension beyond inflation. For instance, one could use a sudden matching approximation in which inflation is immediately followed by an era of radiation domination and a subsequent era of matter domination. This procedure is elaborated on in [36], in the case of a massless minimally coupled scalar field. Whether a significant effect can occur from a backreaction has been subject to controversy. Some papers claim that backreaction effects could potentially replace dark energy as the source for the recent accelerated expansion of the universe [37] [38], while others claim it is completely negligible [39] [40. Such conclusions are model-dependent and future research will have to point out whether backreaction effects are needed to explain, for instance, non-Gaussianities in the CMB.

The fact of the matter remains that until a full theory of quantum gravity is developed, we have to try to make sense of the origin and evolution of our universe by probing the edges of our understanding. Whether that might be through perturbation theory or through the non-perturbative ADS/CFT correspondence, I am excited to see what the future holds.

## Appendix A

## Derivation of the quadratic action

This appendix contains the derivation of the action (3.65) up to quadratic order in perturbations in the stacked gauge (equation (3.50). Also, we perform a gauge transformation to the comoving gauge and find the action (3.49). The derivation of the action uses the canonical form of the ADM action as developed in 9 .

From [9], the action for a minimally coupled scalar field in canonical form is

$$
\begin{equation*}
S=\int d^{D} x\left\{p^{i j} \partial_{0} g_{i j}+p_{\Phi} \partial_{0} \Phi-N \mathcal{H}-N_{i} \mathcal{H}^{i}\right\} \tag{A.1}
\end{equation*}
$$

where $N$ and $N^{i}$ are the lapse function and shift vector and the Hamiltonian $\mathcal{H}$ and the momentum density $\mathcal{H}^{i}$ are

$$
\begin{align*}
& \mathcal{H}=-\sqrt{g} R+\frac{1}{\sqrt{g}}\left[p^{i j} g_{i k} g_{j l} p^{k l}-\frac{1}{D-2} p^{2}\right]+\frac{1}{\sqrt{g}} \frac{p_{\Phi}^{2}}{2}+\frac{\sqrt{g}}{2} g^{i j} \partial_{i} \Phi \partial_{j} \Phi+\sqrt{g} V(\Phi),  \tag{A.2}\\
& \mathcal{H}^{i}=\partial^{i} \Phi p_{\Phi}-2 \nabla_{j} p^{i j} . \tag{A.3}
\end{align*}
$$

The canonical momenta $p^{i j}$ and $p_{\Phi}$, conjugate to the metric $g_{i j}$ and the matter field $\Phi$, are

$$
\begin{align*}
p^{i j} & =\sqrt{g}\left(K g^{i j}-K^{i j}\right) \\
& =-\sqrt{g}\left(\frac{1}{2 N} g^{i j} g^{k l} \partial_{0} g_{k l}+\frac{1}{2 N} \partial_{0} g^{i j}\right),  \tag{A.4}\\
p_{\Phi} & =\frac{\sqrt{g}}{N}\left(\partial_{0} \Phi\right) . \tag{A.5}
\end{align*}
$$

Next, we follow [9] in that we perform the following split into background and fluctuations,

$$
\begin{align*}
p^{i j} & =\frac{\mathcal{P}(t)}{2(D-1) a(t)}\left(\delta^{i j}+\pi^{i j}(t, \vec{x})\right),  \tag{A.6}\\
p_{\Phi} & =\mathcal{P}_{\phi}(t)\left(1+\pi_{\varphi}(t, \vec{x})\right),  \tag{A.7}\\
g_{i j} & =a(t)^{2}\left(\delta_{i j}+h_{i j}(t, \vec{x})\right),  \tag{A.8}\\
\Phi & =\phi(t)+\varphi(t, \vec{x}),  \tag{A.9}\\
N & =\bar{N}(t)+n(t, \vec{x}) . \tag{A.10}
\end{align*}
$$

Later we will identify each of these fields with the fields present in the stacked gauge. From the action A.11, one can derive the background Friedmann and field equations,

$$
\begin{align*}
& \ddot{\phi}+(D-1) H \dot{\phi}+V^{\prime}(\phi)=0,  \tag{A.11}\\
& H^{2}=\frac{1}{(D-1)(D-2)}\left[\frac{1}{2} \dot{\phi}^{2}+V(\phi)\right],  \tag{A.12}\\
& \dot{H}=\frac{-1}{2(D-2)} \dot{\phi}^{2}, \tag{A.13}
\end{align*}
$$

where the dot represents a time reparametrization invariant derivative $\dot{a} \equiv \bar{N}^{-1} d a / d t$. The action (A.1) can be expanded up to quadratic order in perturbations, although it is quite some work. We quote the result from [9] with an additional $n^{2}$ term,

$$
\begin{array}{r}
S^{(2)}=\int d^{D} x\left\{\frac{\mathcal{P}}{2(D-1)}\left(2 \pi^{i j} h_{i j} \partial_{0} a+a \pi^{i j} \partial_{0} h_{i j}\right)+\mathcal{P}_{\phi} \pi_{\varphi} \partial_{0} \varphi-\bar{N} \mathcal{H}^{(2)}-n \mathcal{H}^{(1)}-N_{i} \mathcal{H}^{i(1)}\right. \\
\left.-n^{2}(D-2)(D-1-\epsilon)\right\} \tag{A.14}
\end{array}
$$

where $\epsilon \equiv-\dot{H} H^{-2}$ is the first geometric slow-roll parameter, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are the Hamiltonian up to first- and second order in perturbations respectively,

$$
\begin{align*}
\mathcal{H}^{(1)}= & -a^{D-3}\left[\partial_{i} \partial_{j} h^{i j}-\nabla^{2} h\right]-\frac{1}{a^{D-3}} \frac{\mathcal{P}^{2}}{2(D-1)^{2}(D-2)}\left[\pi^{i j} \delta_{i j}-\frac{1}{4}(D-5) h\right]  \tag{A.15}\\
& +\frac{1}{a^{D-1}} \frac{p_{\Phi}^{2}}{2}\left[2 \pi_{\varphi}-\frac{1}{2} h\right]+a^{D-1}\left[\frac{1}{2} h V+V^{\prime} \varphi\right], \\
\mathcal{H}^{(2)}= & -a^{D-3}\left[-\frac{1}{4} h \nabla^{2} h+\frac{1}{2} h \partial^{i} \partial^{j} h_{i j}-\frac{1}{2} h_{i j} \partial^{i} \partial^{l} h_{j l}+\frac{1}{4} h^{i j} \nabla^{2} h_{i j}\right] \\
& +\frac{\mathcal{P}^{2}}{4(D-1)^{2} a^{D-3}}\left[\frac{1}{2} \pi^{i j} A_{i j k l} \pi^{k l}+\frac{\pi^{i j}}{D-2}\left(2(D-3) h_{i j}-h \delta_{i j}\right)+h_{i}^{j} h_{j}^{i}-\frac{D-1}{D-2}\left(\frac{1}{4} h_{i}^{j} h_{j}^{i}+\frac{1}{8} h^{2}\right)\right] \\
& +\frac{\mathcal{P}_{\phi}^{2}}{2 a^{D-1}}\left[\pi_{\varphi}^{2}+\frac{1}{4} h_{i}^{j} h_{j}^{i}+\frac{1}{8} h^{2}-h \pi_{\varphi}\right]+\frac{a^{D-3}}{2} \partial^{i} \varphi \partial_{i} \varphi \\
& +a^{D-1}\left[\left(-\frac{1}{4} h_{i}^{j} h_{j}^{i}+\frac{h^{2}}{8}\right) V+\frac{1}{2} h \varphi V^{\prime}+\frac{1}{2} V^{\prime \prime} \varphi^{2}\right], \tag{A.16}
\end{align*}
$$

where $h=h_{i j} \delta^{i j}$, and $A_{i j k l}=\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}-\frac{2}{D-2} \delta_{i j} \delta_{k l}$. The momentum density $\mathcal{H}^{i}$ is only relevant up to linear order since the shift function $N^{i}$ is a pure fluctuation, i.e., the background value vanishes,

$$
\begin{equation*}
\mathcal{H}^{i(1)}=\partial^{i} \varphi \frac{1}{a^{2}} \mathcal{P}_{\phi}-\frac{\mathcal{P}}{(D-1) a}\left(\partial_{j} \pi^{i j}+\partial_{j} h^{i j}-\frac{1}{2} \partial^{i} h\right) . \tag{A.17}
\end{equation*}
$$

## A. 1 Fixing the stacked gauge

In the stacked gauge 3.50,

$$
\begin{equation*}
\Phi=\phi(t), \quad g_{i j}=a(t)^{2}\left((1+2 \psi(t, \vec{x})) \delta_{i j}+\Psi_{i j}(t, \vec{x})\right), \quad \partial^{i} \Psi_{i j}=\partial_{j} v(t, \vec{x}), \quad \Psi_{i i}=0, \tag{A.18}
\end{equation*}
$$

the fields in A.6) to A.10) take the following form

$$
\begin{align*}
& \bar{N}(t)=0,  \tag{A.19}\\
& N^{i}(t, \vec{x})=0,  \tag{A.20}\\
& g_{i j}(t, \vec{x})=a(t)^{2}(1+2 \psi(t, \vec{x})) \delta_{i j}+a(t)^{2} \Psi_{i j}(t, \vec{x}),  \tag{A.21}\\
& h_{i j}(t, \vec{x})=2 \psi \delta_{i j}(t, \vec{x})+\Psi_{i j}(t, \vec{x}),  \tag{A.22}\\
& \mathcal{P}(t)=-2(D-1)(D-2) a(t)^{D-2} H(t),  \tag{A.23}\\
& \pi^{i j}(t, \vec{x})=(D-3) \psi(t, \vec{x}) \delta^{i j}+\frac{\dot{\psi}(t, \vec{x})}{H(t)} \delta^{i j}-\Psi^{i j}(t, \vec{x})-\frac{\dot{\Psi}^{i j}(t, \vec{x})}{2(D-2) H(t)},  \tag{A.24}\\
& \mathcal{P}_{\varphi}(t)=a(t)^{D-1} \dot{\phi}(t),  \tag{A.25}\\
& \pi_{\phi}(t, \vec{x})=(D-1) \psi(t) . \tag{A.26}
\end{align*}
$$

The Hamiltonian up to first order becomes

$$
\begin{equation*}
\mathcal{H}^{(1)}=-2(D-1)(D-2) \dot{\psi} H+a^{-2}\left(2(D-2) \nabla^{2} \psi-\partial^{i} \partial^{j} \Psi_{i j}\right), \tag{A.27}
\end{equation*}
$$

while the Hamiltonian up to quadratic order is expressed in terms of the above fields as follows,

$$
\begin{align*}
\mathcal{H}^{(2)}= & -a^{D-3}\left[-(D-2)(D-3) \psi \nabla^{2} \psi-\Psi_{i j} \partial^{i} \partial^{j} \psi-\frac{1}{2} \psi_{i j} \partial^{i} \partial^{k} \Psi_{j k}+\frac{1}{4} \Psi_{i j} \nabla^{2} \Psi^{i j}\right] \\
& +a^{D-1}\left\{\frac{1}{2}(D-1)(D-2)(D-3)(D-9) \psi^{2} H^{2}-4(D-1)(D-2) \psi \dot{\psi} H\right.  \tag{A.28}\\
& -(D-1)(D-2) \dot{\psi}^{2}-\frac{1}{4}(D-2)(D-9) \Psi_{i j} \Psi^{i j} H^{2}+\Psi_{i j} \dot{\Psi}^{i j} H+\frac{1}{4} \dot{\Psi}_{i j} \dot{\Psi}^{i j} \\
& \left.-\frac{1}{4}(D-1)(D-3) \psi \dot{\phi}^{2}+\frac{1}{8} \Psi_{i j} \Psi^{i j} \dot{\phi}^{2}+\left(\frac{1}{2}(D-1)(D-3) \psi^{2}-\frac{1}{4} \Psi_{i j} \Psi^{i j}\right) V\right\} .
\end{align*}
$$

The final ingredient is the kinetic term which reduces to

$$
\begin{align*}
& \int d^{D} x\left\{\frac{\mathcal{P}}{2(D-1)}\left(2 \pi^{i j} h_{i j} \partial_{0} a+a \pi^{i j} \partial_{0} h_{i j}\right)+\mathcal{P}_{\phi} \pi_{\varphi} \partial_{0} \varphi\right\}= \\
& \int d^{D} x \bar{N} a^{D-1}\left\{\left[-4(D-1)(D-2)(D-3) \psi^{2} H^{2}-2(D-1)^{2}(D-2) \psi \dot{\psi} H-2(D-1)(D-2) \dot{\psi}^{2}\right]\right. \\
& \left.+\left[2(D-2) \Psi_{i j} \Psi^{i j} H^{2}+(D-1) \Psi_{i j} \dot{\Psi}^{i j} H+\frac{1}{2} \dot{\Psi}_{i j} \dot{\Psi}^{i j}\right]\right\} . \tag{A.29}
\end{align*}
$$

Plugging the above expressions into the action (A.14) results in the following quadratic action,

$$
\begin{align*}
S= & \int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{-(D-1)(D-2) \dot{\psi}^{2}+\frac{1}{4}\left[\left(\dot{\Psi}_{i j}\right)^{2}-a^{-2} \partial_{k} \Psi_{i j} \partial^{k} \Psi^{i j}\right]\right. \\
& -a^{-2}\left[(D-2)(D-3) \psi \nabla^{2} \psi-(D-3) \psi \partial^{i} \partial^{j} \Psi_{i j}+\frac{1}{2} \Psi_{i j} \partial^{i} \partial^{k} \Psi_{j k}\right]  \tag{A.30}\\
& -n\left[-2(D-1)(D-2) \dot{\psi} H+a^{-2}\left(2(D-2) \nabla^{2} \psi-\partial^{i} \partial^{j} \Psi_{i j}\right)\right] \\
& \left.-n^{2} H^{2}(D-2)(D-1-\epsilon)\right\} .
\end{align*}
$$

## A. 2 Gauge transformation to the comoving gauge

In order to verify the validity of the above action for the stacked gauge we will perform the gaugetransformation to the comoving gauge as derived in section 3.6. The result is precisely action (3.49). The fields in the two gauges are related as follows

$$
\begin{align*}
& \Psi_{i j}(t, \vec{x})=\gamma_{i j}(t, \vec{x})+\left(\frac{\delta_{i j}}{D-1}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}}\right) h(t, \vec{x}), \quad \partial_{i} \gamma_{i j}=0, \quad \tilde{\gamma}_{i i}=0  \tag{A.31}\\
& \psi(t, \vec{x})=\zeta(t, \vec{x})-\frac{1}{2(D-1)} h(t, \vec{x}), \tag{A.32}
\end{align*}
$$

where $h=-\frac{(D-1)}{D-2} v$. In terms of these new fields one has

$$
\begin{align*}
& \dot{h}(t, \vec{x})=-2 H(t)^{-1} a(t)^{-2} \nabla^{2} \zeta(t, \vec{x})+2 \epsilon \dot{\zeta}(t, \vec{x}),  \tag{A.33}\\
& n(t, \vec{x})=\frac{\dot{\zeta}(t, \vec{x})}{H(t)} . \tag{A.34}
\end{align*}
$$

Filling these expression back into the action, results in

$$
\begin{align*}
S= & \int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{(D-2) \epsilon \dot{\zeta}^{2}-\frac{D-2}{4(D-1)} \dot{h}^{2}+a^{-2}\left[-2(D-2) H^{-1} \dot{\zeta} \nabla^{2} \zeta+\frac{D-2}{4(D-1)}(\nabla h)^{2}\right.\right. \\
& \left.\left.+(D-2)(D-3)(\nabla \zeta)^{2}\right]+\frac{1}{4}\left[\dot{\gamma}_{i j} \dot{\gamma}^{i j}+\frac{D-2}{D-1} \dot{h}^{2}\right]-\frac{1}{4} a^{-2}\left[\partial_{k} \gamma_{i j} \partial^{k} \gamma^{i j}+\frac{D-2}{D-1}(\nabla h)^{2}\right]\right\} . \tag{A.35}
\end{align*}
$$

As is obvious, all $h$-terms cancel. One final partial integration gives

$$
\begin{equation*}
S=\int d^{D} x \frac{a^{D-1}}{16 \pi G}\left\{2(D-2) \epsilon\left[\frac{1}{2} \dot{\zeta}^{2}-\frac{1}{2} a^{-2}(\nabla \zeta)^{2}\right]+\left[\frac{1}{4} \dot{\gamma}_{i j} \dot{\gamma}^{i j}-\frac{1}{4} a^{-2} \partial_{k} \gamma_{i j} \partial^{k} \gamma^{i j}\right]\right\} . \tag{A.36}
\end{equation*}
$$

This is precisely the action derived in section 3.5. One can also check that the above action is in agreement with the gauge-invariant action for a minimally coupled scalar field derived in 9 .

## Appendix B

## Calculation of the one-loop backreaction

This appendix is dedicated to the derivation of the individual expectation values in equation (5.18) that constitute the one-loop backreaction. The first geometric slow roll parameter is taken to be small, $\epsilon \ll 1$. The free scalar and graviton propagators in equations 4.50 and 4.60 are identical to the propagator from [3] up to a normalisation constant, i.e.,

$$
\begin{equation*}
i \Delta_{\tilde{\psi}}\left(x, x^{\prime}\right)=\frac{1}{2(D-2) \epsilon} i \Delta\left(x, x^{\prime}\right), \quad i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right)=2 i \Delta\left(x, x^{\prime}\right) \tag{B.1}
\end{equation*}
$$

where

$$
\begin{align*}
i \Delta\left(x, x^{\prime}\right) & =16 \pi G \hbar \frac{\left[(1-\epsilon)^{2} H H^{\prime}\right]^{D / 2-1}}{(4 \pi)^{D / 2}}\left\{\Gamma(D / 2-1)\left(\frac{4}{y}\right)^{D / 2-1}+\frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)}\right. \\
& \times\left[\frac{\Gamma(3 / 2+\nu) \Gamma(3 / 2-\nu)}{\Gamma(3-D / 2)}\left(\frac{4}{y}\right)^{D / 2-2}+\sum_{n=1}^{\infty}\left(\frac{\Gamma(3 / 2+\nu+n) \Gamma(3 / 2-\nu+n)}{\Gamma(3-D / 2+n)(n+1)!}\left(\frac{y}{4}\right)^{n-D / 2+2}\right.\right. \\
& \left.\left.-\frac{\Gamma\left(\frac{D-1}{2}+\nu+n\right) \Gamma\left(\frac{D-1}{2}-\nu+n\right)}{\Gamma(D / 2+n) n!}\left(\frac{y}{4}\right)^{n}\right)\right]+\frac{2(1-\epsilon) \Gamma(2 \nu) \Gamma(\nu)}{\epsilon(D-2) \Gamma\left(\frac{D-1}{2}\right)}\left[\left(\frac{1}{k_{0}^{2} \eta \eta^{\prime}}\right)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}}\right. \\
& \left.\left.+\frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(\nu)} \frac{\Gamma(1-D / 2)}{\Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D-1}{2}+\nu\right) \Gamma\left(\frac{D-1}{2}-\nu\right)}{\Gamma(2 \nu)} \frac{\epsilon(D-2)}{2(1-\epsilon)}\right]\right\} \tag{B.2}
\end{align*}
$$

with $\nu=\frac{D-1-\epsilon}{2(1-\epsilon)}$, and $y$ the invariant distance

$$
\begin{equation*}
y\left(x ; x^{\prime}\right)=\frac{\left\|\vec{x}-\vec{x}^{\prime}\right\|^{2}-\left(\left|\eta-\eta^{\prime}\right|-i \varepsilon\right)^{2}}{\eta \eta^{\prime}} \tag{B.3}
\end{equation*}
$$

Several simplifications are in order. First and foremost, the $D$ dependent powers of $y$ can be automatically subtracted in dimensional regularisation. Hence, they will not be relevant in the calculation of expectation values. Second of all, the time dependence of the penultimate term cancels with the time dependence in the prefactor, thus it contributes as a constant. Since each term in equation 5.18 contains a derivative, constants will not be relevant either. We can conclude that the part of the propagator $i \Delta\left(x, x^{\prime}\right)$ relevant for the calculation of expectation values is

$$
\begin{equation*}
\left[(1-\epsilon)^{2} H H^{\prime}\right]^{D / 2-1} \sum_{n=0}^{\infty} C_{n} y^{n} \tag{B.4}
\end{equation*}
$$

where $16 \pi G$ and $\hbar$ are set to 1 to avoid cluttering the equations later on. In addition, the constants in the sum are,

$$
\begin{align*}
& C_{0}=\frac{1}{(4 \pi)^{D / 2}} \frac{\Gamma(1-D / 2) \Gamma\left(\frac{D-1}{2}+\nu\right) \Gamma\left(\frac{D-1}{2}-\nu\right)}{\Gamma(1 / 2-\nu)}  \tag{B.5}\\
& C_{n}=-\frac{1}{4^{n}} \frac{1}{(4 \pi)^{D / 2}} \frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D-1}{2}+\nu+n\right) \Gamma\left(\frac{D-1}{2}-\nu+n\right)}{\Gamma(D / 2+n) n!}, \quad \text { for } n \geq 1 . \tag{B.6}
\end{align*}
$$

The final term in equation (B.2) was absorbed into the sum by defining $C_{0}$ as above.
Recall that we are working in a universe with constant deceleration parameter. The expressions for the scale factor and Hubble parameter as well as some useful relations between them are summarized in equations (4.28) and 4.29). Most notably, one has $H a(1-\epsilon)=-\eta^{-1}$. In the upcoming calculations, the following identities will be helpful

$$
\begin{align*}
& \left.\partial_{\mu} y\left(x, x^{\prime}\right)\right|_{x \rightarrow x^{\prime}}=0,  \tag{B.7}\\
& \left.\partial_{\mu} \partial_{\nu} y\left(x, x^{\prime}\right)\right|_{x \rightarrow x^{\prime}}=-\left.\partial_{\mu} \partial_{\nu}^{\prime} y\left(x, x^{\prime}\right)\right|_{x \rightarrow x^{\prime}}=\frac{2}{\eta^{2}} \eta_{\mu \nu}=2(1-\epsilon)^{2} H^{2} a^{2} \eta_{\mu \nu},  \tag{B.8}\\
& {\left[(1-\epsilon)^{2} H H^{\prime}\right]^{D / 2-1}=\left((1-\epsilon) H_{0}\right)^{\frac{D-2}{(1-\epsilon)}}\left(\eta \eta^{\prime}\right)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} .} \tag{B.9}
\end{align*}
$$

The last identity makes it clear that any and all time derivatives acting on the prefactor (i.e. not on $y^{n}$ ) will give rise to an overall factor of $\epsilon$. These terms will be disregarded as they are higher order terms in the $\epsilon$ expansion.

In calculating the contributions from the individual expectation values in equation (5.18) we will distinguish between tensor terms as well as scalar terms. It turns out that the sum of contributions originating from tensor terms will vanish, while the net contribution from scalar terms scales as $\log (a)$.

## B. 1 Tensor contributions

For clarity's sake, the sum we will calculate in this section is

$$
\begin{align*}
B_{\tilde{\Psi}}= & -a^{-2} \frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \nabla^{2} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle}{4(D-2) H}+\frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0}^{2} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle}{4(D-2) H}+\frac{D-1}{4(D-2)}\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle  \tag{B.10}\\
& +\frac{\langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j} \partial_{0} \hat{\tilde{\Psi}}^{i j}+\hat{\tilde{\Psi}}_{i j} \partial_{0}^{2} \hat{\tilde{\Psi}}^{i j}|\Omega\rangle}{2(D-2)} .
\end{align*}
$$

Each of these terms can be calculated by considering the fundamental relations

$$
\begin{align*}
\langle\Omega| \hat{\tilde{\Psi}}_{i j}(x) \hat{\tilde{\Psi}}_{k l}\left(x^{\prime}\right)|\Omega\rangle & =\frac{1}{2}\left(P_{i k} P_{j l}+P_{i l} P_{j k}-\frac{2}{D-2} P_{i j} P_{k l}\right)\left(i \Delta\left(x, x^{\prime}\right)\right),  \tag{B.11}\\
\langle\Omega| \hat{\tilde{\Psi}}_{i j}(x) \hat{\tilde{\Psi}}_{i j}\left(x^{\prime}\right)|\Omega\rangle & =\frac{D(D-3)}{2}\left(i \Delta\left(x, x^{\prime}\right)\right) \tag{B.12}
\end{align*}
$$

where, similar to the quantization procedure in section 4.3 .2 , the transverse projectors $P_{i j}=\delta_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}}$ ensure that the tracelessness and transversality of the fields is satisfied. Using this relation and the
expression for the propagator in equation (B.2) one finds up to lowest order in $\epsilon$,

$$
\begin{align*}
& \langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j}(x) \partial_{0} \hat{\tilde{\Psi}}^{i j}(x)|\Omega\rangle=\frac{D(D-3)}{2}\left[a^{-2} \partial_{\eta} \partial_{\eta}^{\prime} i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=2 D(D-3) C_{1} H_{0}^{D},  \tag{B.13}\\
& \langle\Omega| \hat{\tilde{\Psi}}_{i j}(x) \partial_{0}^{2} \hat{\tilde{\Psi}}^{i j}(x)|\Omega\rangle=\frac{D(D-3)}{2}\left[a^{-2}\left(\partial_{\eta}^{2}-\mathcal{H} \partial_{\eta}\right) i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=-2 D(D-3) C_{1} H_{0}^{D},  \tag{B.14}\\
& \langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j}(x) \partial_{0}^{2} \hat{\tilde{\Psi}}^{i j}(x)|\Omega\rangle=\frac{D(D-3)}{2}\left[a^{-2} a^{\prime-1}\left(\partial_{\eta}^{2}-\mathcal{H} \partial_{\eta}\right) \partial_{\eta^{\prime}} i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=0,  \tag{B.15}\\
& \langle\Omega| \partial_{0} \hat{\tilde{\Psi}}_{i j}(x) \nabla^{2} \hat{\tilde{\Psi}}^{i j}(x)|\Omega\rangle=\frac{D(D-3)}{2}\left[a^{-1} \partial_{\eta} \nabla^{\prime 2} i \Delta_{\tilde{\Psi}}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=-2 D(D-3)(D-1) a^{2} C_{1} H_{0}^{D+1}, \tag{B.16}
\end{align*}
$$

where $\mathcal{H} \equiv a^{-1} \partial_{\eta} a$. From these expressions it is clear that the second and final term in equation (B.10) vanish, while the first and third term cancel each other. The conclusion is that $B_{\Psi}$ is zero at this order. This is a remarkable result: up to one-loop order the expected influence of tensor metric perturbations on the dynamics of scalar metric perturbations is small as it is at least suppressed by $\epsilon$ in a universe with $\epsilon \ll 1$.

## B. 2 Scalar contributions

The contribution of scalar terms to the backreaction (5.18) is

$$
\begin{align*}
B_{\tilde{\psi}}= & -\frac{D-3}{2 H} a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}} \nabla^{2} \hat{\tilde{\psi}}|\Omega\rangle+\frac{D^{2}-4 D+5}{2(D-2)(D-1)} a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}} \nabla^{2} \hat{h}|\Omega\rangle-a^{-2} \frac{\langle\Omega| \partial_{0} \hat{h} \nabla^{2} \hat{h}|\Omega\rangle}{4(D-2) H} \\
& +\frac{D-1}{2(D-2) H} a^{-2}\langle\Omega| \partial_{0} \hat{h} \nabla^{2} \hat{\tilde{\psi}}|\Omega\rangle-\frac{\langle\Omega| \partial_{0}^{2} \hat{h} \partial_{0} \hat{\tilde{\psi}}-\partial_{0} \hat{h} \partial_{0}^{2} \hat{\tilde{\psi}}|\Omega\rangle}{2(D-2) H}+\frac{\langle\Omega| \partial_{0}^{2} \hat{h} \partial_{0} \hat{h}|\Omega\rangle}{2(D-1)(D-2) H} \\
& +\frac{\langle\Omega| \hat{h} \partial_{0}^{2} \hat{h}|\Omega\rangle}{2(D-2)}+\frac{D}{4(D-2)}\langle\Omega| \partial_{0} \hat{h} \partial_{0} \hat{h}|\Omega\rangle . \tag{B.17}
\end{align*}
$$

Some of these terms are readily calculated (up to lowest order in $\epsilon$ ),

$$
\begin{align*}
& \langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle=a^{-1}\left[\partial_{\eta} \nabla^{\prime 2} i \Delta_{\psi}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=\frac{D-1}{(D-2) \epsilon} C_{1} a^{2} H_{0}^{D+1},  \tag{B.18}\\
& \langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle=\left[\nabla^{2} \nabla^{\prime 2} i \Delta_{\psi}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=\frac{4(D-1)(D+1)}{(D-2) \epsilon} C_{2} a^{4} H_{0}^{D+2},  \tag{B.19}\\
& \langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x)|\Omega\rangle=a^{-2}\left[\partial_{\eta} \partial_{\eta^{\prime}} \nabla^{\prime 2} i \Delta_{\psi}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=\frac{D-1}{(D-2) \epsilon}\left(C_{1}+4 C_{2}\right) a^{2} H_{0}^{D+2},  \tag{B.20}\\
& \langle\Omega| \partial_{0}^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle=a^{-2}\left[\left(\partial_{\eta}^{2}-\mathcal{H} \partial_{\eta}\right) \nabla^{\prime 2} i \Delta_{\psi}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=\frac{(D-1)}{(D-2) \epsilon}\left(C_{1}-2 C_{2}\right) a^{2} H_{0}^{D+2}, \\
& \langle\Omega| \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle=a^{-1}\left[\partial_{\eta} \nabla^{2} \nabla^{\prime 2} i \Delta_{\psi}\left(x, x^{\prime}\right)\right]_{x \rightarrow x^{\prime}}=\frac{8(D-1)(D+1)}{(D-2) \epsilon} C_{2} a^{4} H_{0}^{D+3}, \tag{B.21}
\end{align*}
$$

$$
\begin{align*}
& \langle\Omega| \partial_{0}^{2} \hat{h}(x) \partial_{0} \hat{\tilde{\psi}}(x)|\Omega\rangle=\left\{-\frac{2}{H} a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x)|\Omega\rangle+4 a^{-2}\langle\Omega| \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle\right\} \\
& =2 \frac{D-1}{(D-2) \epsilon} H_{0}^{D+1}\left(C_{1}-4 C_{2}\right),  \tag{B.23}\\
& \langle\Omega| \partial_{0} \hat{h}(x) \partial_{0}^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle=\left\{-\frac{2}{H} a^{-2}\langle\Omega| \partial_{0}^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle\right\}=-2 \frac{D-1}{(D-2) \epsilon} H_{0}^{D+1}\left(C_{1}-2 C_{2}\right),  \tag{B.24}\\
& \langle\Omega| \partial_{0} \hat{h}(x) \partial_{0} \hat{h}(x)|\Omega\rangle=\left\{\frac{4}{H^{2}} a^{-4}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle\right\}=\frac{16(D-1)(D+1)}{(D-2) \epsilon} H_{0}^{D} C_{2},  \tag{B.25}\\
& \langle\Omega| \partial_{0} \hat{h}(x) \nabla^{2} \tilde{\psi}(x)|\Omega\rangle=\left\{-\frac{2}{H} a^{-2}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle\right\}=-\frac{8(D-1)(D+1)}{(D-2) \epsilon} H_{0}^{D+1} C_{2},  \tag{B.26}\\
& \langle\Omega| \partial_{0}^{2} \hat{h}(x) \partial_{0} \hat{h}(x)|\Omega\rangle=\left\{\frac{4}{H^{2}} a^{-4}\langle\Omega| \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle-\frac{8}{H} a^{-4}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{\tilde{\psi}}(x)|\Omega\rangle\right\}=0, \tag{B.27}
\end{align*}
$$

where it was used that

$$
\begin{equation*}
\partial_{0} \hat{h}(x)=-2 H^{-1} a^{-2} \nabla^{2} \hat{\tilde{\psi}}(x) . \tag{B.28}
\end{equation*}
$$

All of these terms are constants divided by $\epsilon$. As will be shown shortly, these constants are not as relevant as the contributions coming from the three remaining terms on the first row of equation (B.17). Calculating these remaining expectation values is a bit more involved since we can not directly use the relation (B.28). Instead, we will have to introduce an integral to be able to substitute an expression for $\partial_{\eta} h(\eta, \vec{x})$. For instance, one has

$$
\begin{align*}
\frac{a^{-2}}{H}\langle\Omega| \partial_{0} \hat{h}(x) \nabla^{2} \hat{h}(x)|\Omega\rangle & =-\frac{2}{H^{2}} a^{-4}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(x) \nabla^{2} \hat{h}(x)|\Omega\rangle \\
& =\int_{\eta_{0}}^{\eta} d \eta^{\prime}\left\{-\frac{2 a^{-4}}{H^{2}}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(\eta, \vec{x}) \nabla^{2} \partial_{\eta}^{\prime} \hat{h}\left(\eta^{\prime}, \vec{x}\right)|\Omega\rangle\right\} \\
& =\int_{\eta_{0}}^{\eta} d \eta^{\prime}\left\{\frac{4 a^{-4} a^{\prime-1}}{H^{2} H^{\prime}}\langle\Omega| \nabla^{2} \hat{\tilde{\psi}}(\eta, \vec{x}) \nabla^{2} \nabla^{2} \hat{\tilde{\psi}}\left(\eta^{\prime}, \vec{x}\right)|\Omega\rangle\right\} \\
& =\int_{\eta_{0}}^{\eta} d \eta^{\prime}\left\{\frac{4 a^{-4} a^{\prime-1}}{H^{2} H^{\prime}}\left[\nabla^{2} \nabla^{\prime 2} \nabla^{\prime 2} i \Delta_{\tilde{\psi}}\left(x, x^{\prime}\right)\right]_{\vec{x} \rightarrow \vec{x}}\right\} \\
& =\int_{\eta_{0}}^{\eta} d \eta^{\prime}\left\{\frac{4 a^{-4} a^{\prime-1}}{H^{2} H^{\prime}} \frac{\left.\left[(1-\epsilon)^{2} H H^{\prime}\right]\right]^{D / 2-1}}{2(D-2) \epsilon} \sum_{n=0}^{\infty} C_{n}\left[\nabla^{2} \nabla^{\prime 2} \nabla^{\prime 2} y\left(x, x^{\prime}\right)^{n}\right]_{\vec{x} \rightarrow \vec{x}^{\prime}}\right\} . \tag{B.29}
\end{align*}
$$

The way the spatial derivatives act on $y$, can be derived by combinatorics. Using rule (B.8), one can show that

$$
\begin{align*}
{\left[\nabla^{2} \nabla^{\prime 2} \nabla^{\prime 2} y^{n}\right]_{\vec{x} \rightarrow \vec{x}^{\prime}} } & =\left.8\left((D-1)^{3}+6(D-1)^{2}+8(D-1)\right) n(n-1)(n-2) H^{3} H^{\prime 3} a^{3} a^{\prime 3} y^{n-3}\right|_{\vec{x} \rightarrow \vec{x}^{\prime}} \\
& =\left.8(D-1)(D+1)(D+3) n(n-1)(n-2) H^{3} H^{\prime 3} a^{3} a^{\prime 3} y^{n-3}\right|_{\vec{x} \rightarrow \vec{x}^{\prime}} \tag{B.30}
\end{align*}
$$

Substituting this into equation (B.29) and rewriting gives,

$$
\begin{align*}
\frac{a^{-2}}{H}\langle\Omega| \partial_{0} \hat{h}(x) \nabla^{2} \hat{h}(x)|\Omega\rangle= & \frac{32(D-1)(D+1)(D+3)}{2(D-2) \epsilon}(1-\epsilon)^{D+4} a^{-1} H^{D / 2} \\
& \times \int_{\eta_{0}}^{\eta} d \eta^{\prime}\left\{\left.a^{\prime 2} H^{\prime D / 2+1} \sum_{n=3}^{\infty} C_{n} \frac{\Gamma(n+1)}{\Gamma(n-2)} y^{n-3}\right|_{\vec{x} \rightarrow \vec{x}^{\prime}}\right\} . \tag{B.31}
\end{align*}
$$

In order to calculate this integral, let's introduce the dimensionless quantity $x=\eta^{\prime} / \eta$. One has

$$
\begin{align*}
& \left.y\left(x, x^{\prime}\right)\right|_{\vec{x} \rightarrow \vec{x}^{\prime}}=-\frac{(1-x)^{2}}{x},  \tag{B.32}\\
& a=a^{\prime} x^{\frac{1}{1-\epsilon}}  \tag{B.33}\\
& H=H^{\prime} x^{\frac{-\epsilon}{1-\epsilon}},  \tag{B.34}\\
& d \eta^{\prime}=\eta d x=-\frac{1}{(1-\epsilon) H a} d x . \tag{B.35}
\end{align*}
$$

Substituting the above identities gives

$$
\begin{align*}
\frac{a^{-2}}{H}\langle\Omega| \partial_{0} \hat{h}(x) \nabla^{2} \hat{h}(x)|\Omega\rangle= & -\frac{32(D-1)(D+1)(D+3)}{2(D-2) \epsilon}(1-\epsilon)^{D+3} H^{D} \\
& \times \sum_{n=0}^{\infty}(-1)^{n} C_{n+3} \frac{\Gamma(n+4)}{\Gamma(n+1)} \int_{x_{0}}^{1} d x\left\{(1-x)^{2 n} x^{-n+\alpha-1}\right\} \tag{B.36}
\end{align*}
$$

where $\alpha=\frac{D \epsilon-2}{2(1-\epsilon)}$. One can state similar expressions for the remaining terms,

$$
\begin{array}{r}
\langle\Omega| \hat{h}(x) \partial_{0}^{2} \hat{h}(x)|\Omega\rangle=\frac{32(D-1)(D+1)}{2(D-2) \epsilon}(1-\epsilon)^{D+2} H^{D} \times \int_{x_{0}}^{1} d x\left\{\sum_{n=0}^{\infty}(-1)^{n} C_{n+3}\right. \\
\left.\frac{\Gamma(n+4)}{\Gamma(n+1)}(1-x)^{2 n} x^{-n+\alpha}\left(-\frac{1}{x}+x\right)\right\}, \\
a^{-2}\langle\Omega| \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x) \hat{h}(x)|\Omega\rangle=\frac{32(D-1)(D+1)}{2(D-2) \epsilon}(1-\epsilon)^{D+2} H^{D} \times \int_{x_{0}}^{1} d x\left\{\sum_{n=0}^{\infty}(-1)^{n} C_{n+3}\right.  \tag{B.38}\\
\left.\frac{\Gamma(n+4)}{\Gamma(n+1)}(1-x)^{2 n} x^{-n+\alpha}\left(-\frac{1}{x}+x\right)-2 \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)}(1-x)^{2 n} x^{-n+\alpha}\right\} .
\end{array}
$$

These terms contain three unique integral expressions. Each of them can be calculated in the late time limit $\left(x_{0} \rightarrow \infty\right)$ in $D=4$ with the help of hypergeometric functions and asymptotic expansions. An example is worked out in appendix C. The other terms follow analogously. One finds,

$$
\begin{align*}
& \frac{a^{-2}}{H}\langle\Omega| \partial_{0} \hat{h}(x) \nabla^{2} \hat{h}(x)|\Omega\rangle \approx \frac{1575}{32 \pi^{2}} \frac{H_{0}^{4} M_{p}^{-2} \hbar^{2}}{\epsilon}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right),  \tag{B.39}\\
& \langle\Omega| \hat{h}(x) \partial_{0}^{2} \hat{h}(x)|\Omega\rangle \approx \frac{225}{32 \pi^{2}} \frac{H_{0}^{4} M_{p}^{-2} \hbar^{2}}{\epsilon}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right),  \tag{B.40}\\
& a^{-2}\langle\Omega| \nabla^{2} \partial_{0} \hat{\tilde{\psi}}(x) \hat{h}(x)|\Omega\rangle \approx-\frac{315}{32 \pi^{2}} \frac{H_{0}^{4} M_{p}^{-2} \hbar^{2}}{\epsilon}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right), \tag{B.41}
\end{align*}
$$

where constant terms and higher order terms in $\epsilon$ were neglected in the approximation. Using the above approximations, one finds for the scalar contribution to the backreaction,

$$
\begin{equation*}
B_{\tilde{\psi}} \approx-\frac{2175}{256 \pi^{2}} \frac{H_{0}^{4} M_{p}^{-2} \hbar^{2}}{\epsilon}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right) . \tag{B.42}
\end{equation*}
$$

## Appendix C

## Late time limit and asymptotic expansion

As an example we will show how to deal with (and expand) the final term of equation (B.38) in the late time limit $x_{0} \rightarrow \infty$,

$$
\begin{equation*}
I=H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} \int_{x_{0}}^{1} d x(1-x)^{2 n} x^{-n+\alpha} . \tag{C.1}
\end{equation*}
$$

The prefactor of $H^{D}$ will be important in the expansion around $\epsilon=0$. The integral can be written as the sum of two incomplete beta functions,

$$
\begin{equation*}
I=H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)}\left[B(1,-n+\alpha+1,2 n+1)-B\left(x_{0},-n+\alpha+1,2 n+1\right)\right] \tag{C.2}
\end{equation*}
$$

where

$$
\begin{equation*}
B(x, a, b) \equiv \int_{0}^{x} t^{a-1}(1-t)^{b-1} d t=\frac{x^{a}}{a}{ }_{2} F_{1}(a, 1-b, a+1, x), \quad B(1, a, b)=\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} . \tag{C.3}
\end{equation*}
$$

In order to continue the calculation in an orderly fashion we will treat each beta function separately. The second identity for the beta function in equation (C.3) allows us to rewrite the first part of $I$ as

$$
\begin{equation*}
I_{1}=H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} \frac{\Gamma(\alpha+1-n) \Gamma(2 n+1)}{\Gamma(\alpha+2+n)} . \tag{C.4}
\end{equation*}
$$

The following identities will be helpful,

$$
\begin{align*}
& \Gamma(\alpha+1-n)=\frac{(-1)^{n} \pi}{\sin (\alpha \pi) \Gamma(n-\alpha)},  \tag{C.5}\\
& \Gamma(2 n+1)=4^{n} \frac{\Gamma(n+1 / 2) \Gamma(n+1)}{\sqrt{\pi}},  \tag{C.6}\\
& \frac{\sqrt{\pi}}{\sin (\alpha \pi)} \frac{\Gamma(1 / 2)}{\Gamma(-\alpha) \Gamma(\alpha+2)}=-\frac{\alpha \Gamma(\alpha)}{\Gamma(\alpha+2)}=-\frac{1}{1+\alpha},  \tag{C.7}\\
& C_{n+2}=-\frac{1}{4^{n+2}} \frac{1}{(4 \pi)^{D / 2}} \frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D+3}{2}+\nu+n\right) \Gamma\left(\frac{D+3}{2}-\nu+n\right)}{\Gamma(D / 2+2+n)(n+2)!} . \tag{C.8}
\end{align*}
$$

These expressions will help us rewrite equation (C.4) in terms of a hypergeometric function,

$$
\begin{align*}
I_{1}= & {\left[\frac{1}{4^{2}(4 \pi)^{D / 2}} \frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)} \frac{\Gamma\left(\frac{D+3}{2}+\nu\right) \Gamma\left(\frac{D+3}{2}-\nu\right)}{\Gamma(D / 2+2)}\right] }  \tag{C.9}\\
& \times \frac{H^{D}}{1+\alpha}{ }_{4} F_{3}\left(\frac{1}{2}, 1, \frac{D+3}{2}+\nu, \frac{D+3}{2}-\nu ; \frac{D}{2}+2,-\alpha, \alpha+2 ; 1\right) .
\end{align*}
$$

The generalized hypergeometric function is defined as,

$$
\begin{equation*}
{ }_{p} F_{q}\left(a_{1}, a_{2}, \cdots, a_{p} ; b_{1}, b_{2}, \cdots, b_{q} ; z\right)=\sum_{n=0}^{\infty} \frac{\left(a_{1}\right)_{n} \cdots\left(a_{p}\right)_{n}}{\left(b_{1}\right)_{n} \cdots\left(b_{q}\right)_{n}} \frac{z^{n}}{n!}, \quad(a)_{n} \equiv \frac{\Gamma(a+n)}{\Gamma(a)} . \tag{C.10}
\end{equation*}
$$

The generalized hypergeometric function has a branch point at $z=1$. Nevertheless, by means of analytic continuation along any path in the complex plane that avoids the branch points at 1 and infinity, it can often still be evaluated. For instance, the hypergeometric function in equation (C.9) can be evaluated when $D=4$, in which case it is only singular at

- $\nu=-5 / 2,-3 / 2,-1 / 2,1 / 2,3 / 2,5 / 2$,
- $\alpha=$ integer.

The expression for the hypergeometric function ${ }_{4} F_{3}$ with general $\alpha$ and $\nu$ in $D=4$ is,

$$
\begin{align*}
& { }_{4} F_{3}\left(\frac{1}{2}, 1, \frac{D+3}{2}+\nu, \frac{D+3}{2}-\nu ; \frac{D}{2}+2,-\alpha, \alpha+2 ; 1\right)= \\
& \frac{8(1+\alpha)^{2}\left(512 \alpha^{3}+128 \alpha^{4}-16 \alpha^{2}\left(-13+20 \nu^{2}\right)-32 \alpha\left(19+20 \nu^{2}\right)+15\left(9-40 \nu^{2}+16 \nu^{4}\right)\right)}{5(-5+2 \nu)(-3+2 \nu)(-1+2 \nu)(1+2 \nu)(3+2 \nu)(5+2 \nu)} \\
& \quad+\frac{65536(1+\alpha) \pi \alpha\left(-6+\alpha+4 \alpha^{2}+\alpha^{3}\right)}{5(-5+2 \nu)(-3+2 \nu)(-1+2 \nu)(1+2 \nu)(3+2 \nu)(5+2 \nu)} \\
& \quad \times\left[\frac{\Gamma(-3-\alpha) \Gamma(-1+\alpha)}{\Gamma\left(-\frac{5}{4}-\frac{\alpha}{2}-\frac{\nu}{2}\right) \Gamma\left(-\frac{1}{4}+\frac{\alpha}{2}-\frac{\nu}{2}\right) \Gamma\left(-\frac{5}{4}-\frac{\alpha}{2}+\frac{\nu}{2}\right) \Gamma\left(-\frac{1}{4}+\frac{\alpha}{2}+\frac{\nu}{2}\right)}\right] \tag{C.11}
\end{align*}
$$

If one takes $\alpha$ and $\nu$ as we defined them, i.e.,

$$
\begin{equation*}
\alpha=\frac{D \epsilon-2}{2(1-\epsilon)}, \quad \nu=\frac{D-1-\epsilon}{2(1-\epsilon)}, \tag{C.12}
\end{equation*}
$$

one finds that, in $D=4$,

$$
\begin{equation*}
-\frac{5}{4}-\frac{\alpha}{2}+\frac{\nu}{2}=0 \tag{C.13}
\end{equation*}
$$

As a result the second term in equation (C.11) is suppressed by $\Gamma(0)^{-1}$, thus it vanishes. On the other hand, the first term is not singular around $\epsilon=0$. Keeping terms up to first order in $\epsilon$,

$$
\begin{equation*}
\frac{1}{\alpha+1}{ }_{4} F_{3}\left(\frac{1}{2}, 1, \frac{D+3}{2}+\nu, \frac{D+3}{2}-\nu ; \frac{D}{2}+2,-\alpha, \alpha+2 ; 1\right)=\frac{6-6 \epsilon}{-5+10 \epsilon}=-\frac{6}{5}-\frac{6}{5} \epsilon+\mathcal{O}\left(\epsilon^{2}\right) . \tag{C.14}
\end{equation*}
$$

Hence, $I_{1}$ will contribute as a constant at the lowest order in $\epsilon$.

Next, let's consider the remaining beta function in equation (C.2),

$$
\begin{align*}
I_{x_{0}} & \equiv-H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} B\left(x_{0},-n+\alpha+1,2 n+1\right) . \\
& =-H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} \frac{x_{0}^{\alpha+1} x_{0}^{-n}}{\alpha+1-n}{ }_{2} F_{1}\left(\alpha+1-n,-2 n, \alpha+2-n, x_{0}\right) . \tag{C.15}
\end{align*}
$$

One can perform a linear transformation on the above hypergeometric function. For instance, equation (15.10.25) from [41,

$$
\begin{align*}
{ }_{2} F_{1}(\alpha, \beta ; \gamma ; z)= & \frac{\Gamma(\gamma) \Gamma(\beta-\alpha)}{\Gamma(\beta) \Gamma(\gamma-\alpha)}(-z)^{-\alpha}{ }_{2} F_{1}\left(\alpha, \alpha+1-\gamma ; \alpha+1-\beta ; \frac{1}{z}\right) \\
& +\frac{\Gamma(\gamma) \Gamma(\alpha-\beta)}{\Gamma(\alpha) \Gamma(\gamma-\beta)}(-z)^{-\beta}{ }_{2} F_{1}\left(\beta, \beta+1-\gamma ; \beta+1-\alpha ; \frac{1}{z}\right), \tag{C.16}
\end{align*}
$$

directly applied to equation (C.15) gives

$$
\begin{align*}
{ }_{2} F_{1}(\alpha+1-n, & \left.-2 n, \alpha+2-n ; x_{0}\right)= \\
& \frac{\Gamma(\alpha+2-n) \Gamma(-n-\alpha-1)}{\Gamma(-2 n) \Gamma(1)}\left(-x_{0}\right)^{n-\alpha-1}{ }_{2} F_{1}\left(\alpha+1-n, 0, \alpha+2-n ; x_{0}^{-1}\right) \\
& +\frac{\Gamma(\alpha+2-n) \Gamma(\alpha+1+n)}{\Gamma(\alpha+1-n) \Gamma(\alpha+2+n)}\left(-x_{0}\right)^{2 n}{ }_{2} F_{1}\left(-2 n,-n-\alpha-1 ;-n-\alpha ; x_{0}^{-1}\right) . \tag{C.17}
\end{align*}
$$

The first term is suppressed by $\Gamma(-2 n)^{-1}$, while the second term can be represented as a finite series because its first argument is a negative integer,

$$
\begin{equation*}
{ }_{2} F_{1}\left(-2 n,-n-\alpha-1 ;-n-\alpha ; x_{0}^{-1}\right)=\sum_{k=0}^{2 n}(-1)^{k}\binom{2 n}{k} \frac{\alpha+1+n}{\alpha+1+n-k} x_{0}^{-k} . \tag{C.18}
\end{equation*}
$$

The result of the linear transformation of equation (C.15) is,

$$
\begin{equation*}
I_{x_{0}}=-H^{D} \sum_{n=0}^{\infty} \sum_{k=0}^{2 n}(-1)^{n+k} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} \frac{\Gamma(2 n+1)}{\Gamma(k+1) \Gamma(2 n-k+1)} \frac{x_{0}^{\alpha+1} x_{0}^{n-k}}{\alpha+1+n-k} . \tag{C.19}
\end{equation*}
$$

The above expression is simply a (double) sum over powers of $x_{0}$. Since we are interested in the limit $x_{0} \rightarrow \infty$, we can approximate the second sum by keeping only the highest powers of $x_{0}$ for a specific $n$,

$$
\begin{align*}
I_{x_{0}} & =-H^{D} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} x_{0}^{\alpha+1}\left(\frac{x_{0}^{n}}{\alpha+1+n}-2 n \frac{x_{0}^{n-1}}{\alpha+n}+\left(2 n^{2}-n\right) \frac{x_{0}^{n-2}}{\alpha-1+n}+\ldots\right), \\
& \approx-H^{D} \sum_{n=0}^{\infty}(-1)^{n}\left(C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)}+2 C_{n+3} \frac{\Gamma(n+4)}{\Gamma(n+1)}\right) \frac{x_{0}^{\alpha+1+n}}{\alpha+1+n} . \tag{C.20}
\end{align*}
$$

Equivalently, one can write the above in terms of hypergeometric functions,

$$
\begin{align*}
I_{x_{0}}= & \frac{1}{4^{2}} \frac{1}{(4 \pi)^{D / 2}} \frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)} H^{D} \\
& \times\left\{\frac{\Gamma\left(\frac{D+3}{2}+\nu\right) \Gamma\left(\frac{D+3}{2}-\nu\right)}{\Gamma(D / 2+2)(\alpha+1)} x_{0}^{\alpha+1}{ }_{3} F_{2}\left(\frac{D+3}{2}+\nu, \frac{D+3}{2}-\nu, \alpha+1 ; D / 2+2, \alpha+2 ;-\frac{x_{0}}{4}\right)\right. \\
& \left.+\frac{1}{2} \frac{\Gamma\left(\frac{D+5}{2}+\nu\right) \Gamma\left(\frac{D+5}{2}-\nu\right)}{\Gamma(D / 2+3)(\alpha+1)} x_{0}^{\alpha+1}{ }_{3} F_{2}\left(\frac{D+5}{2}+\nu, \frac{D+5}{2}-\nu, \alpha+1 ; D / 2+3, \alpha+2 ;-\frac{x_{0}}{4}\right)\right\} . \tag{C.21}
\end{align*}
$$

At this point we would like to perform an asymptotic expansion in the variable $x_{0}$. This this end, define the following formal infinite series (equation 16.11.2 from [41]),

$$
\begin{equation*}
H_{p, q}(z)=\sum_{m=1}^{p} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \Gamma\left(a_{m}+k\right)\left(\frac{\prod_{l=1, l \neq m}^{p} \Gamma\left(a_{l}-a_{m}-k\right)}{\prod_{l=1}^{q} \Gamma\left(b_{l}-a_{m}-k\right)}\right) z^{-a_{m}-k} \tag{C.22}
\end{equation*}
$$

where $a_{l}, b_{l}$ are real or complex parameters. In the case that $|z|>1, H_{q+1, q}(z)$ converges and one can make an asymptotic expansion (equation 16.11.6 from [41]),

$$
\begin{equation*}
\left(\frac{\prod_{l=1}^{q+1} \Gamma\left(a_{l}\right)}{\prod_{l=1}^{q} \Gamma\left(b_{l}\right)}\right)_{q+1} F_{q}\left(a_{1}, \ldots, a_{q+1} ; b_{1}, \ldots, b_{q} ; z\right)=H_{q+1, q}(-z) . \tag{C.23}
\end{equation*}
$$

Here, we apply the asymptotic expansion to the hypergeometric functions in equation (C.21). For instance,

$$
\begin{align*}
& \frac{\Gamma\left(\frac{D+3}{2}+\nu\right) \Gamma\left(\frac{D+3}{2}-\nu\right)}{\Gamma(D / 2+2)(\alpha+1)} x_{0}^{\alpha+1}{ }_{3} F_{2}\left(\frac{D+3}{2}+\nu, \frac{D+3}{2}-\nu, \alpha+1 ; D / 2+2, \alpha+2 ;-x_{0} / 4\right)= \\
& \sum_{k=0}^{\infty}\left(-\frac{x_{0}}{4}\right)^{-k} \frac{x_{0}^{\alpha+1}}{k!}\left\{\left(\frac{x_{0}}{4}\right)^{-\frac{D+3}{2}-\nu} \frac{\Gamma\left(\frac{D+3}{2}+\nu+k\right) \Gamma(-2 \nu-k) \Gamma\left(\alpha+1-\frac{D+3}{2}-\nu-k\right)}{\Gamma\left(\alpha+2-\frac{D+3}{2}-\nu-k\right) \Gamma\left(D / 2+2-\frac{D+3}{2}-\nu-k\right)}\right. \\
& +\left(\frac{x_{0}}{4}\right)^{-\alpha-1} \frac{\Gamma(\alpha+1+k) \Gamma\left(\frac{D+3}{2}+\nu-\alpha-1-k\right) \Gamma\left(\frac{D+3}{2}-\nu-\alpha-1-k\right)}{\Gamma(1-k) \Gamma(D / 2+2-\alpha-1-k)}  \tag{C.24}\\
& \left.\quad+\left(\frac{x_{0}}{4}\right)^{-\frac{D+3}{2}+\nu} \frac{\Gamma\left(\frac{D+3}{2}-\nu+k\right) \Gamma(2 \nu-k) \Gamma\left(\alpha+1-\frac{D+3}{2}+\nu-k\right)}{\Gamma\left(D / 2+2-\frac{D+3}{2}+\nu-k\right) \Gamma\left(\alpha+2-\frac{D+3}{2}+\nu-k\right)}\right\} .
\end{align*}
$$

The result of the asymptotic expansion is, once again, a sum over powers of $x_{0}$. Although this time, the powers are non-positive. The highest order term is a constant with respect to $x_{0}$; all other terms are suppressed. Expanding the other hypergeometric function in equation (C.21) in the same way, and only keeping the constant term, gives a simplified expression for $I_{x_{0}}$,

$$
\begin{align*}
I_{x_{0}} & =-\frac{1}{4^{2}} \frac{1}{(4 \pi)^{D / 2}} \frac{\Gamma(D / 2-1) \Gamma(2-D / 2)}{\Gamma(1 / 2+\nu) \Gamma(1 / 2-\nu)} H^{D} 4^{\alpha+1} \Gamma(\alpha+1) \\
& \times\left\{\frac{\Gamma\left(\frac{D+3}{2}+\nu-\alpha-1\right) \Gamma\left(\frac{D+3}{2}-\nu-\alpha-1\right)}{\Gamma(D / 2+2-\alpha-1)}+\frac{1}{2} \frac{\Gamma\left(\frac{D+5}{2}+\nu-\alpha-1\right) \Gamma\left(\frac{D+5}{2}-\nu-\alpha-1\right)}{\Gamma(D / 2+3-\alpha-1)}\right\} . \tag{C.25}
\end{align*}
$$

At last, one can expand around $\epsilon=0$. The most important part of the expansion comes from

$$
\begin{equation*}
H^{D} 4^{\alpha+1} \Gamma(\alpha+1)=\frac{2}{D-2} H_{0}^{D}\left(\frac{1}{\epsilon}+D \log \left(-H_{0} \eta\right)\right)+\text { constant terms }+\mathcal{O}(\epsilon) \tag{C.26}
\end{equation*}
$$

such that $I_{x_{0}}$ becomes, in $D=4$,

$$
\begin{equation*}
I_{x_{0}} \approx \frac{9}{256 \pi^{2}} H_{0}^{4}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right) \tag{C.27}
\end{equation*}
$$

As $I_{1}$ is simply a constant, equation (C.27) is also the leading order behaviour for the original integral expression in equation (C.1).

One can apply the same techniques to find expression for other, closely related integral expressions,

$$
\begin{align*}
& H^{4} \sum_{n=0}^{\infty}(-1)^{n} C_{n+2} \frac{\Gamma(n+3)}{\Gamma(n+1)} \int_{x_{0}}^{1} d x(1-x)^{2 n} x^{-n+\alpha} \approx \frac{9}{256 \pi^{2}} H_{0}^{4}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right),  \tag{C.28}\\
& H^{4} \sum_{n=0}^{\infty}(-1)^{n} C_{n+3} \frac{\Gamma(n+4)}{\Gamma(n+1)} \int_{x_{0}}^{1} d x(1-x)^{2 n} x^{-n+\alpha-1} \approx-\frac{15}{512 \pi^{2}} H_{0}^{4}\left(\frac{1}{\epsilon}+4 \log \left(-H_{0} \eta\right)\right),  \tag{C.29}\\
& H^{4} \sum_{n=0}^{\infty}(-1)^{n} C_{n+3} \frac{\Gamma(n+4)}{\Gamma(n+1)} \int_{x_{0}}^{1} d x(1-x)^{2 n} x^{-n+\alpha+1} \approx \frac{9}{256 \pi^{2}} H_{0}^{4}\left(1+\epsilon \log \left(-H_{0} \eta\right)\right), \tag{C.30}
\end{align*}
$$

where, in the approximations, constant terms and higher order terms in the $\epsilon$ expansion are neglected.

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