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Scalar particle production by gravitational waves

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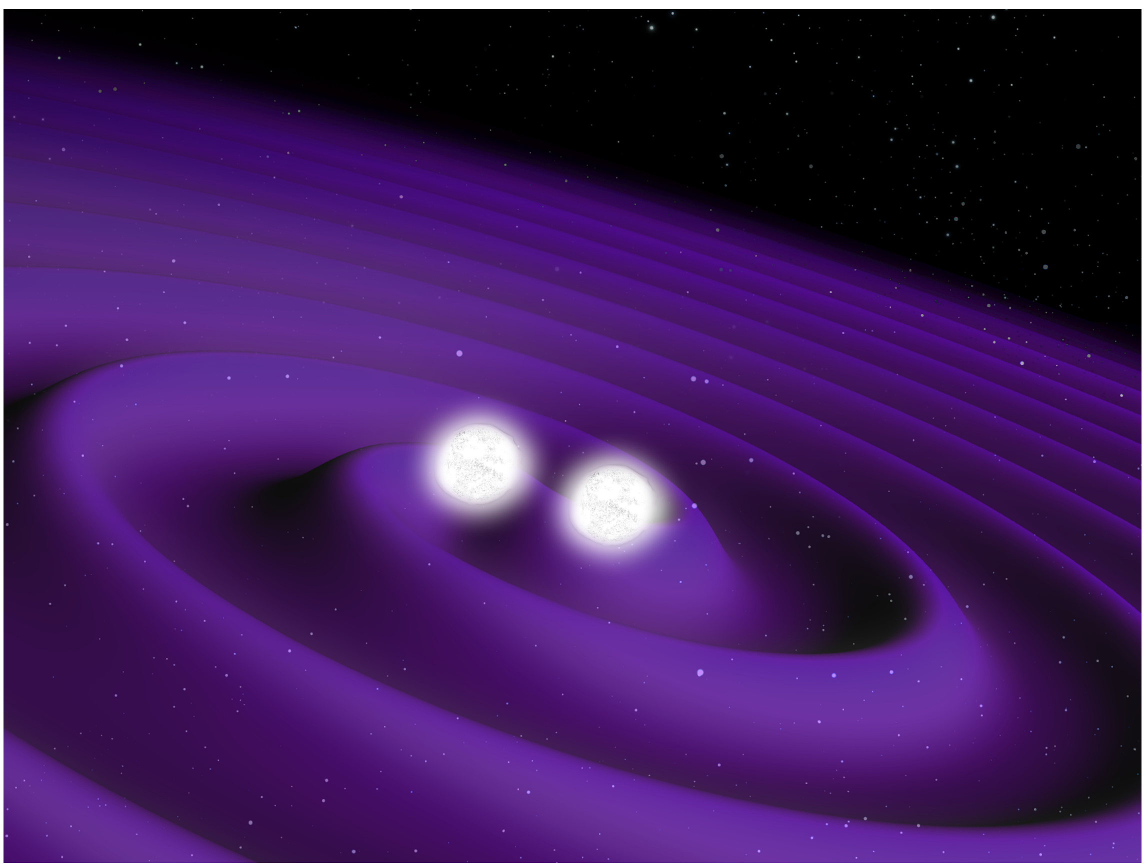

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Contents

1 Abstract

We compute the expected number of particles generated by a plane gravitational wave, and show that it is possible for such a wave to generate particles out of the vacuum. We compute this number both classically and using canonical quantization. The normal procedure for canonical quantization states that the field and its conjugate momentum commute, which is not possible. We show two ways one can change this approach to produce results consistent with the classical analysis. However, the plane wave we use gives infinite particle creation density. This result is probably due to a plane gravitational wave spanning all of space not being physically possible. We suggest how the latter can be fixed, which would make it possible to compute the particle creation number for a realistic wave.

2 Definitions and notation

(Four-)vectors and indices A letter with an arrow indicates a normal vector (i. e. not a four-vector). For example, we use $\vec{x}$ for the position vector. For a four-vector, we write the letter of the vector with an index, such as x^μ for a point in space-time. Greek indices (predominantly α, β, μ, ν) take values 0 to 3, so we will use them for four-vectors. Latin letters (predominantly i, j) take values 1 to 3, so excluding the 0-th component.

As space and time are not independent, we combine them into a four-vector

$$x^\mu \equiv (ct, \vec{x}) = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

and

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

where ∇ is an ordinary gradient. Note that a lower-index means differentiating with respect to an upper-index.

For integrating over space-time, we write

$$d^4x \equiv dx^0 dx^1 dx^2 dx^3$$

In contrast to most physics textbooks, we shall write integrals as $\int f(x)\, dx$ instead of $\int dx\, f(x)$.

Metric For the flat space metric, we use

$$\eta_{\mu\nu} = (-, +, +, +) = \mathrm{diag}(-1, 1, 1, 1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is used by most authors in general relativity. We use $g_{\mu\nu}$ as a general metric. As gravitational waves require the metric to be non-flat, we will only use η for the trivial case.

Indices are raised and lowered using the metric, for example $x_\mu = g_{\mu\nu}x^\nu$. Repeated indices are summed over, as usual. The same raising and lowering rule applies to any other four-vector or tensor. We denote the trace of the metric by $g_{\mu\mu}$, summing over μ (even though μ appears twice as a lower index here).

We denote g to be $\det[g_{\mu\nu}]$, so the determinant of the metric. If we are not discussing the determinant but the metric itself, we will always write the indices. That is, $g_{\mu\nu}$ refers to the metric and g to its determinant.

3 Physical explanation of gravitational waves

3.1 Special and general relativity

In Newtonian mechanics, there is a notion of *distance*. This can be written as follows. Suppose that we have two points, separated by a vector (dx, dy, dz) in some inertial coordinate frame[1]. For every inertial observer, the quantity

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

is the same. Any rotation of our coordinate frame does not affect this quantity: furthermore, if we change the speed of our coordinate frame, the distances dx, dy, dz are unchanged. Time is considered independent: it is the same for every observer.

However, in the Michelson-Morley experiment, it was discovered that every inertial observer measures the exact same light speed, regardless of the direction in which the light travels. This is not possible in Newtonian mechanics: according to that theory, we should measure a change in speed of light if the observer changes speed. This paradox required a massive change in theory: time and distance are now related, and relative to the observer. This leads us to Einstein's theory of special relativity. In this theory, there is a new invariant measure which is the same for all inertial frames:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

This equation relates time and space measurements. Note that while in Newtonian physics we had $ds^2 > 0$ for any two distinct points, for this measure the sign can be positive or negative. If $ds^2 > 0$, we call the path between the two points *spacelike*: it is impossible for an observer to travel between two such points, as it would require exceeding the speed of light. Similarly if $ds^2 < 0$ we call it *timelike*: such trajectories can be travelled by an observer. And if $ds^2 = 0$ we call it *lightlike*, which means that this path can only be traversed by an object that moves with the speed of light.

This notion of distance invites us to combine space and time into one vector which is a point in *spacetime*. This is usually written in one of the following, equivalent ways:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{x})$$

Here the index μ, being a Greek letter, runs from 0 to 3. Note that using the c, all four components of the vector have dimension distance. Now x^μ is called a *four-vector*. We also introduce the *flat metric* $\eta_{\mu\nu}$ as

$$\eta_{\mu\nu} \equiv \mathrm{diag}(-1, +1, +1, +1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This way, equation (2) can be written as

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu} \tag{3}$$

Note that we use the *summation convention* here: indices that are repeated are summed over, unless indicated otherwise. This is simply shorthand notation: without it, our equations would be a lot longer.

Now we look at gravity from this perspective. Due to the Sun's gravity, the Earth does not fly away, and instead orbits the sun. But how fast does the effect of gravity travel? In Newtonian mechanics, this would be a silly question: there is no travel time (or, equivalently, the 'speed of gravity' is infinite). However, in special relativity, this is not possible: no observable signal can travel at a speed faster than light. So if the Sun would suddenly start moving away, the Earth would only notice it after a 8 minute delay for the signal to travel[2].

[1] An inertial frame is a frame in which there are no fictitious forces: this means that a body in the frame on which no forces act does not accelerate.

[2] In this thought-experiment we made several assumptions that we cannot motivate now, such as that gravitational waves would actually travel at the speed of light. The point is that we need a new theory to describe gravity, and I am trying to underline a few of the core principles of it.

This paradox is not easily solved, and it requires the theory of *general relativity*. In this theory, we no longer consider space-time to be flat: in equation (3) we replace the Minkowski metric $\eta_{\mu\nu}$ to a general metric $g_{\mu\nu}$. What we know about this general metric is that we may assume it to be symmetric, as the terms $g_{\mu\nu}$ and $g_{\nu\mu}$ both give us $dx^\mu dx^\nu$. Furthermore, as it turns out, $g_{\mu\nu}$ always has three positive and one negative eigenvalue, which gives it a negative determinant denoted by g. The metric $g_{\mu\nu}$ can be viewed as a generalization of the gravitational potential in flat spacetime, and we will use it to describe gravitational waves.

3.1.1 Lorentz transformations

Now that we have introduced the notion of distance between points in flat space-time, we may wonder what types of transformations leave this distance invariant. Clearly translations do so, as they do not change dx^μ for any μ. So we may wonder if there are more. The only other type of linear transformation besides the translation is the matrix multiplication by a constant matrix. So let's denote a matrix by Λ^ν_μ, so that

$$x'_\mu = \Lambda^\nu_\mu x_\nu$$

where x' is our new vector. Let's see if this changes the length of our vector:

$$\Delta s'^2 = \eta_{\mu\nu}\Delta x'^\mu \Delta x'^\nu = \eta_{\mu\nu}\Lambda^\mu_\alpha x^\alpha \Lambda^\nu_\beta x^\beta$$

and that means that the distance is preserved if

$$\eta_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta} \tag{4}$$

In matrix notation, this corresponds to

$$\Lambda^T \eta \Lambda = \eta$$

Now the solutions to this equations are called *Lorentz transformations*, and together they turn out to form a group under matrix multiplication called the *Lorentz group*. They can be split up into two categories. The first one is *rotations*, which only affect the position components. For example, a rotation of the x and y axis is given by

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The parameter θ determines the angle of the rotation, and an angle $\theta = 0$ corresponds to no rotation. Note that we may restrict our angle θ to the interval $[0, 2\pi)$, as a rotation over an angle of 2π corresponds to no rotation.
In 3-dimensional space (so not space-time), all matrices satisfying (4) are rotation matrices, and hence generated by rotations around x, y and z-axis. However, this is not the case for space-time, as the metric has a -1 for the time component.
The equivalent for rotations that involve time is called *boost*, and if we do a boost involving time and the x-component of space, it looks like

$$\Lambda = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One major difference between boosts and rotations is that here ϕ runs from $-\infty$ to ∞. Furthermore, the elements of the matrix Λ will generally become much larger than 1. Again, an angle of $\phi = 0$ corresponds to the identity transformation.
A general Lorentz transformation (so without a translation) can now be made by composition of rotations and boosts. For the rotation, we have three degrees of freedom: 2 in choosing the direction, 1 in choosing the angle. For the boost we have three degrees again: one for each component. So, we have six degrees of freedom in total for a general Lorentz transformation. Note that this means that if we take the three basic rotation matrices (around the x, y, z-axis) and the three basic boosts (involving the x, y, z-component), the six form a basis for all Lorentz transformations, in a sense that all transformations are a product of them.
Now that we have introduced Lorentz transformations, let's move on to an actual gravitational wave.

3.2 Choosing a frame

General relativity is difficult to study as it has a huge symmetry group: every Lorentz transformation has to leave the theory invariant. This symmetry is called *Gauge symmetry*[3]. As a consequence of this, it is difficult to understand what is physically relevant and what is not, as a lot of things can be 'gauged away'. For example, it is always possible to make space locally flat around a point in a small enough environment (with only second order or higher terms). However, such an approximation is not very useful, as it throws us back to special relativity (as the metric becomes flat). On top of that, it is not valid in a very large region, especially during GW's.
The simplest possible gravitational wave arises when we consider an otherwise flat space-time, that has a very small perturbation h. We write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. We use the Gauge symmetry to make the background space flat. This clearly breaks a part of the Gauge symmetry, but as it turns out there is still symmetry left. If we let $\Lambda^\mu{}_\nu$ be a random Lorentz transformation, then we have

$$\begin{aligned} g'_{\alpha\beta} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} \\ &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\eta_{\mu\nu} + h_{\mu\nu}) \\ &= \eta_{\alpha\beta} + \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu} \end{aligned}$$

So,

$$h'_{\alpha\beta} = g'_{\alpha\beta} - \eta_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu}$$

Thus, h transforms as a tensor! Any Lorentz transformation that does not change the condition $|h_{\mu\nu}| \ll 1$ can be applied without changing the theory. Recall that every Lorentz transformation can be split into a rotation and a boost part. We note that a rotation cannot spoil the condition $|h_{\mu\nu}| \ll 1$, as a rotation does not change the length of a vector. However, a boost can change it, so boosts are only allowed if they do not increase $|h_{\mu\nu}|$ too much. We will use this freedom to simplify h even further.

As we are in linearised theory, we will raise and lower indices with $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$, as any additional terms arising from replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$ are automatically higher-order terms. We define

$$h = \eta^{\mu\nu} h_{\mu\nu}, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Next, we claim that it is possible to choose the *De Donder gauge* such that

$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

To show this, we consider a coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu$$

Now, we want to know the change in the metric $g_{\mu\nu}$ due to this. We have

$$\begin{aligned} g'_{\mu\nu}(x') &= g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= \frac{\partial}{\partial x'^\mu}(x'^\alpha - \xi^\alpha) \frac{\partial}{\partial x'^\nu}(x'^\beta - \xi^\beta) g_{\alpha\beta}(x) \\ &= (\delta^\alpha_\mu - \partial'_\mu \xi^\alpha)(\delta^\beta_\nu - \partial'_\nu \xi^\beta) g_{\alpha\beta}(x) \\ &= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta}(x) - \delta^\alpha_\mu \partial_\nu \xi^\beta g_{\alpha\beta}(x) - \partial_\mu \xi^\alpha \delta^\beta_\nu g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) - \partial_\nu \xi^\beta g_{\mu\beta} - \partial_\mu \xi^\alpha \delta_{\alpha\nu} \\ &= g_{\mu\nu}(x) - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \end{aligned}$$

Note that in the first expression, the primes are on the lower indices. The reason is that we can view the lower-index $h_{\mu\nu}$ as transforming 2 vectors into a scalar (as it has two lower indices and no upper indices).

[3] Please note that Gauge symmetry is also used in other fields of physics, for mathematical transformations that leave the underlying physics invariant

We start with two vectors in the primed coordinate system. We can either directly do the transformation $h'_{\mu\nu}(x')$ to get the scalar, but we can (equivalently) first transform both vectors into the unprimed coordinate system, and then apply $h_{\mu\nu}(x)$ to get the same scalar. The reason both scalars are the same is as both are h turning the two vectors into a scalar: it is just that one way is direct and the other one takes a side road. In the third line, we turned the primed derivatives $\partial'_\mu \xi^\alpha$ into unprimed $\partial_\mu \xi^\alpha$. Up to first order, these two are equal. Proof:

$$\begin{aligned}
\partial'_\mu \xi^\alpha - \partial_\mu \xi^\alpha &= \frac{\partial \xi^\alpha}{\partial x'^\mu} - \frac{\partial \xi^\alpha}{\partial x^\mu} \\
&= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial \xi^\alpha}{\partial x^\nu} - \frac{\partial \xi^\alpha}{\partial x^\mu} \\
&= \left(\delta^\nu_\mu - \frac{\partial \xi^\nu}{\partial x'^\mu}\right)\frac{\partial \xi^\alpha}{\partial x^\nu} - \frac{\partial \xi^\alpha}{\partial x^\mu} \\
&= -\frac{\partial \xi^\nu}{\partial x'^\mu}\frac{\partial \xi^\alpha}{\partial x^\nu} + \delta^\nu_\mu \frac{\partial \xi^\alpha}{\partial x^\nu} - \frac{\partial \xi^\alpha}{\partial x^\mu}
\end{aligned}$$

Now, the first term is second order and thus it drops, while the second and third term cancel. So, the two terms are indeed equal up to first order. In a similar fashion, we can show that up to first order $g'_{\mu\nu}(x') = g'_{\mu\nu}(x)$:

$$\begin{aligned}
g'_{\mu\nu}(x') - g'_{\mu\nu}(x) &= g'_{\mu\nu}(x+\xi) - g'_{\mu\nu}(x) \\
&= g'_{\mu\nu}(x) + \xi^\alpha \partial_\alpha g'_{\mu\nu}(x) - g'_{\mu\nu}(x) \\
&= \xi^\alpha \partial_\alpha g'_{\mu\nu}(x)
\end{aligned}$$

In the second line, we used that ξ^α is small compared to x^α, so we can use the calculus identity $f(x+dx) = f(x) + f'(x)dx$ in four dimensions.
As a result, we get that

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

and thus

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \tag{5}$$

So now we know how $h_{\mu\nu}$ changes under a coordinate transformation. Next, we want to know how $\bar{h}_{\mu\nu}$ changes under the same coordinate transformation. So we compute

$$\begin{aligned}
\bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}\bar{h}'_{\alpha\beta}) \\
&= h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}(h_{\alpha\beta} - (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha))) \\
&= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}\partial_\alpha \xi_\beta + \eta^{\alpha\beta}\partial_\beta \xi_\alpha) \\
&= \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha \xi^\alpha + \partial_\beta \xi^\beta) \\
&= \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha)
\end{aligned}$$

Therefore

$$\begin{aligned}
(\partial^\nu \bar{h}_{\mu\nu})' &= \partial^\nu(\bar{h}'_{\mu\nu}) \\
&= \partial^\nu \bar{h}_{\mu\nu} - (\partial_\mu \partial^\nu \xi_\nu + \partial_\nu \partial^\nu \xi_\mu - \partial^\nu \eta_{\mu\nu}\partial_\alpha \xi^\alpha) \\
&= \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu + \partial_\mu \partial^\alpha \xi_\alpha - \partial_\mu \partial^\nu \xi_\nu \\
&= \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu
\end{aligned}$$

Here we used that when an index is summed over, we can raise one while lowering the other:

$$a_\mu b^\mu = \eta^{\mu\nu} a_\mu b_\nu = a^\nu b_\nu$$

and just replace the dummy index ν by μ. Furthermore, we used the definition of the d'Alembert operator $\Box = \partial_\mu \partial^\mu$. So what we want to do now, is choose ξ such that $\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}$. As it turns out this is *always* possible, as the d'Alembert operator is invertible[4]. So, we choose ξ such that $\partial^\nu \bar{h}_{\mu\nu}$ vanishes. In this case, Einstein's equations reduce to

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{6}$$

This is a result of the Einstein equations. However, proving it requires an enormous amount of additional theory which is not relevant for the remainder of this thesis, so I am omitting it. If you wish to see a derivation, see [2].
Note that this equation implies $\partial^\nu T_{\mu\nu} = 0$, which is the conservation of energy and momentum in linearised theory.

3.3 The TT Gauge

The simplest example of a gravitational wave is when we are outside of the source, which implies $T_{\mu\nu} = 0$. Then (6) becomes $\Box \bar{h}_{\mu\nu} = 0$. Note that this equation implies that gravitational waves travel by the speed of light, as $\Box = \nabla^2 - 1/c^2 \partial_t^2$. We already set the gauge such that $\partial^\nu \bar{h}_{\mu\nu} = 0$, and we are now going to set it such that $h_{00} = 0$.
To do this, we need another coordinate transformation $x^\mu \to x^\mu + \xi^\mu$. In order not to spoil $\partial^\nu \bar{h}_{\mu\nu} = 0$, we demand that $\Box \xi_\mu = 0$. Now if we define

$$\xi_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha$$

the transformation $x^\mu \to x^\mu + \xi^\mu$ corresponds to $\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \xi_{\mu\nu}$.
Now we wish to simplify $h_{\mu\nu}$ by choosing a proper ξ^μ. However, we cannot carelessly set components to zero, as we don't know if it is possible to choose functions ξ^μ to attain that. So the first thing we will do is note that the trace $\xi \equiv \eta^{\mu\nu} \xi_{\mu\nu}$ is equal to

$$\eta^{\mu\nu} \xi_{\mu\nu} = -2\partial_0 \xi_0 + 2\partial_i \xi_i - 2\partial_\alpha \xi^\alpha = -4\partial_0 \xi_0$$

The first two terms we simply wrote out, note that the index i only takes values $1, 2, 3$ here! Now for ξ_0 we choose a function such that $-4\partial_0 \xi_0 = \bar{h}_{\mu\nu}$. Now after the transformation, $\bar{h}$ will be zero, *regardless* of our choice for the other three components of ξ^μ. Next, we note that

$$\xi_{i0} = \partial_i \xi_0 + \partial_0 \xi_i$$

as the flat metric $\eta_{i0} = 0$. Now we can choose each ξ_i independently such that

$$\partial_i \xi_0 + \partial_0 \xi_i = h_{i0}$$

This way, we get that after the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ the new $\bar{h}_{\mu\nu}$ has the following properties. Due to our choice of ξ_0 we have $\bar{h} = 0$, and hence $\bar{h}_{\mu\nu} = h_{\mu\nu}$. So we will leave out the bar from now on, and work with $\partial^\nu h_{\mu\nu} = 0$. We also have $h_{0i} = 0$, and hence $\partial^i h_{0i} = 0$ reduces to $\partial^0 h_{00} = 0$. So, h_{00} is time-independent.
Now we note that

$$ds^2 = (-1 + h_{00}) dx^0 dx^0 + (\eta_{ij} + h_{ij}) dx^i dx^j$$

with i, j running over spatial components only. So the h_{00} component, being time independent, is only relevant for how fast time is running, corresponding to the value of the time-independent Newtonian potential of the source. Effectively, it corresponds to the gravitational time dilation. As we wish to only examine the wave, we may just as well ignore this, and set $h_{00} = 0$. Note that the traceless-condition $\eta^{\mu\nu} h_{\mu\nu} = 0$ now reduces to ${h^i}_i = 0$
At this point, we have the following properties:

[4]For proof and more information on the d'Alembert operator, see [1].

$$\boxed{h_{0\mu} = 0, \qquad h^i{}_i = 0, \qquad \partial^i h_{ij} = 0}$$

This is called the transverse-traceless Gauge, or TT gauge.
To solve the problem for this case, we assume a plane wave:

$$h_{ij} = e_{ij}(\vec{k})e^{ik^\mu x_\mu}$$

where we have the wave 4-vector[5] $k^\mu = (\omega/c, \vec{k})$. Now note that

$$0 = \partial^j h_{ij} = k^j e_{ij}(\vec{k})e^{ik^\mu x_\mu}$$

So, $k^j e_{ij} = 0$. If we assume a wave travelling in the z-direction for simplicity we get $\vec{k} = k_z\hat{z}$. So the summation over j collapses into

$$k^3 e_{i3} = 0$$

implying that $e_{i3} = 0$ for every i. If we define $e_{11} = h_+$ and $e_{12} = h_\times$, then the traceless condition gives us that $e_{22} = -h_+$ and $e_{21} = h_\times$. Next, we have to remember that h_{ij} is real, so we have to take the real part of $e^{ik^\mu x_\mu}$, which is $\cos(k^\mu x_\mu) = \cos(k^0 \cdot ct - k_z z)$. As by definition[6] $\omega = c|\vec{k}| = ck_z$, this simplifies to $\cos(\omega(t - z/c))$. So we obtain

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(\omega(t - z/c))$$

If we take indices a, b running only over 1 and 2, this can be shortened into

$$h_{ab} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos(\omega(t - z/c)) \tag{7}$$

In this form, we note several important results. First of all, a gravitational wave cannot change the metric in a direction parallel to its propagation: only directions perpendicular to it are affected. Second, we can distinguish between two types of polarization: + and ×, corresponding to two different types of waves. A wave with a + polarization does not add a coupling: it only changes the coefficients of dx^2 and dy^2, by adding a plane wave to them. Equivalently, it only affects directions longitudinally. Only a wave with a cross polarization induces a non-zero $dxdy$ term: such a wave, instead, affects directions transversally.
The deformation induced by such a gravitational wave can be viewed as follows.

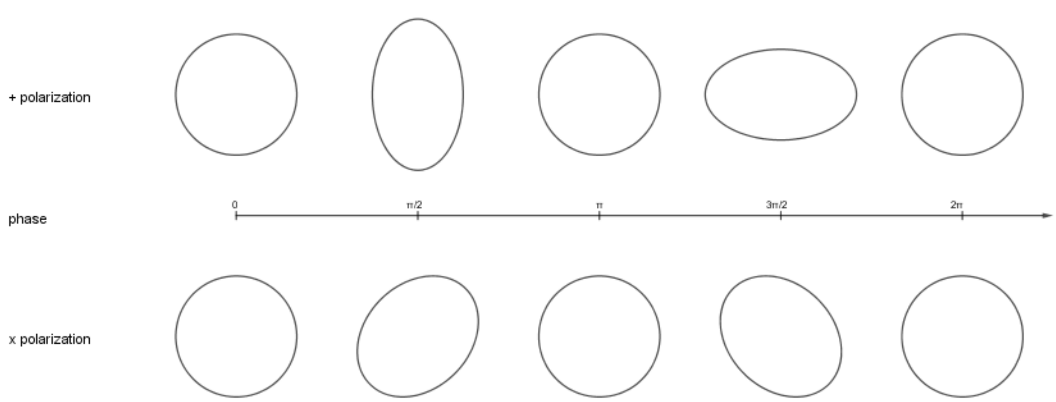

Figure 1: Deformation of a circle under the influence of a gravitational wave.

[5] Note that $k^\mu x_\mu$ is actually $\vec{k} \cdot \vec{x} - \omega t$, so this is the same as for a classical wave.
[6] As we chose the wave to travel in the $+z$-direction, k_z is positive by definition. So we do not have to write $|k_z|$ here.

There are several things to say about this picture. First of all we assumed that a wave is either $+$ polarized or $\times$ polarized, but typically it is a combination of both. Secondly, the movement is *greatly* exaggerated. A typical value for h_+ or $h_\times$ is $\lesssim 10^{-21}$. Finally, we see that $+$ polarization and $\times$ polarization can be transformed into each other by a rotation of $45°$.
If we consider a wave with only $+$ polarization, the change in metric becomes

$$h_{ab} = h_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos(\omega(t - z/c)) \tag{8}$$

We will analyse particle creation by this wave in the next section.

4 Scalar fields in gravitational waves

Only at this point, we are ready to formulate our research question. We view a plane, + polarized gravitational wave in the TT gauge, and we wish to know if it is possible for this wave to generate particles from the vacuum. We will use two different methods to compute the particle generation: one method is by explicitly solving the equation of motion for a massless scalar field in a plane gravitational wave. The second method will be using canonical commutation relations on a scalar field.
Furthermore, we set the speed of light c to be dimensionless 1. This means that we set $1\text{ s} = 3\cdot 10^8$ m. This is a frequently used mathematical tool to simplify equations. Other constants, such as G, are frequently also set to be 1, but we won't do that (as we don't use G that often). If we know what units a number should have, we can actually restore any c's by comparing units.

4.1 General theory and the fundamental equation

All fields that we consider in classical field theory transform as tensors. A particularly interesting tensor is a *scalar*, i. e. a tensor without indices. The reason for this, is that as it has no indices, it is invariant under coordinate transformations. Hence it is a Lorentz invariant.
It is well known that the simplest action for a relativistic real scalar field in a flat metric is given by

$$S_{\text{flat}}[\phi] = \int -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\, d^4x \tag{9}$$

To see this, we have to look at what terms can occur in an expansion. Suppose that we have *any* Lagrangian. Make an expansion in terms of ϕ and derivatives of ϕ. Now as there can be no distinction between coordinates at this point, we have that whenever a derivative to a coordinate μ occurs, there must be an equal derivative to all other coordinates. This implies that we can write all derivatives using the operators ∂_μ and ∂^μ.
From practical experience[7] we know that we only have to consider terms up until order 4. For the order, every ϕ adds an order 1, and so does every derivative. For a real field ϕ, this would give us the expansion

$$\mathcal{L} = \underbrace{A\phi}_{\text{1st order}} + \underbrace{B\phi^2}_{\text{2nd order}} + \underbrace{C\phi^3 + D\partial_\mu\partial^\mu\phi}_{\text{3rd order}} + \underbrace{E\phi^4 + F\partial_\mu\phi\partial^\mu\phi + G\phi\partial_\mu\partial^\mu\phi}_{\text{4th order}}$$

The constants A to G here are just letters, they do not correspond to their usual meanings (so G is not the gravitational constant here). Later we will change them to different letters to avoid confusion.
Note that terms such as $\partial_\mu\phi$ are not allowed in this expansion, as they are not scalar terms. Hence, only these terms are left. The term $\partial_\mu\partial^\mu(\phi^2)$ can be simplified into $2\partial_\mu\phi\partial^\mu\phi$ using the product rule.
We can use $\mathbb{Z}_2$-symmetry to assume that our action is invariant under the transformation $\phi \to -\phi$. Therefore, the three odd terms (ϕ, ϕ^3 and $\partial_\mu\partial^\mu\phi$ drop).
Furthermore, we note that the terms $\partial_\mu\phi\partial^\mu\phi$ and $\phi\partial_\mu\partial^\mu\phi$ are equivalent. This is because the relevant quantity is the action $S = \int \mathcal{L}\, d^4x$, and the we can use partial integration to turn a $\partial_\mu\phi\partial^\mu\phi$ term into a $\partial_\mu\phi\partial^\mu\phi$ term[8].
So, in the end we are left with three terms: $\phi^2, \phi^4, \partial_\mu\phi\partial^\mu\phi$. The coefficient of the term $\partial_\mu\phi\partial^\mu\phi$ is usually set to $-1/2$ by rescaling ϕ to match this condition[9]. The coefficient of ϕ^2 is usually taken to be $-m^2/2$, and the coefficient of ϕ^4 is called $-\Lambda/4$. So our Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\Lambda}{4}\phi^4$$

For the simplest possible non-trivial system, we set Λ to zero. This way, we obtain the Lagrangian already mentioned in (9).
Now for a general gravitational background, note that all indices in the Lagrangian are summed over, which should be as the action has only one component. To obtain the action, we integrate the Lagrangian. However,

[7] There is a deeper reason behind this, but it goes beyond the scope of this thesis.
[8] More rigorously, we could integrate over a very large sphere, perform partial integration, and then throw away the boundary term as ϕ vanishes at infinity.
[9] Authors who use the metric $\eta = (-,+,+,+)$ set this constant to $1/2$ instead

integrating it over d^4x is incorrect, as d^4x may change under coordinate transformations, and hence may not be the same for all points in spacetime. To solve this, we add a factor $\sqrt{-g}$ to our action:

$$S[\phi] = \int \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}|\phi|^2 \right) \sqrt{-g}\, d^4x \tag{10}$$

where g is the determinant of the matrix $g_{\mu\nu}$. As it turns out, $\sqrt{-g}\, d^4x$ is invariant under coordinate transformations[10]. From this, we will derive the equation of motion for ϕ. The integrand is

$$\mathcal{L} = \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \right) \sqrt{-g}$$

and hence, the Euler-Lagrange equations give us that[11]

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\ -m^2 \phi \sqrt{-g} &= \partial_\mu \left(-(\partial^\mu \phi)\sqrt{-g} \right) \\ &= -\frac{1}{2}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \\ &= -\frac{1}{2}\left(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu \right) \phi \end{aligned}$$

In the last equation, I wrote the expression using an operator. Now this reduces to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\left(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu \right) \phi = m^2 \phi \tag{11}$$

Note that we reduced the original equation of motion to an eigenvalue problem here. For a flat metric (hence $g_{\mu\nu} = \eta_{\mu\nu}$) the operator on the left hand side reduces to the flat-space d'Alembertian. We will denote this operator by $\Box$, which depends on the metric.

Since gravitational waves travel at the speed of light, they are described by a massless scalar field, i. e. $m = 0$. Hence, we obtain the equation

$$\boxed{\frac{1}{\sqrt{-g}}\left(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu \right) \phi = 0} \tag{12}$$

as the fundamental equation for a massless scalar field.

[10] See section 7.1 for a proof.

[11] Please note that the derivation of the Euler-Lagrange equations does not change under a coordinate transformation, hence the equations themselves are invariant.

5 Solving the equation classically

At this point we want to solve (12) for a plane, + polarized gravitational wave in the TT gauge, travelling in the z direction. However, solving it is rather hard in the standard Cartesian coordinate system. So, we first introduce a new coordinate system called *light-cone coordinates* to simplify our equation, and then we move on to solving it. Other researchers attempted separation of variables, such as in [3]: however, the results obtained in that paper do not hold for a real scalar field (see Sections 8.4 and 8.5). We will use Fourier transformation instead, and we compute the particle creation from the solutions we obtain.

5.1 Light front coordinates

Recall that the wave we are considering (as described in section 4) is a plane, + polarized gravitational wave in the TT gauge. We already computed $h_{\mu\nu}$ for this case in Section 3.3, and since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we have that

$$g_{\mu\nu} = \mathrm{diag}(-1, 1 + h_+ \cos(\omega(z-t)), 1 - h_+ \cos(\omega(z-t)), 1) \tag{13}$$

and we have to solve the equation

$$\frac{1}{\sqrt{-g}} \left(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu\right) \phi = 0$$

Now as the wave travels in the z-direction, it would make sense to introduce[12] a coordinate $u = z - t$ and $v = z + t$. Then, we have

$$du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt = dz - dt, \qquad dv = \frac{\partial v}{\partial z} dz + \frac{\partial v}{\partial t} dt = dz + dt \tag{14}$$

giving us that $dudv = -dt^2 + dz^2$. So, we obtain

$$ds^2 = dudv + (1 + h_+ \cos(\omega u))dx^2 + (1 - h_+ \cos(\omega u))dy^2$$

To simplify this, set $c = \cos(\omega u)$ and $s = \sin(\omega u)$. Then[13] we get

$$ds^2 = dudv + (1 + h_+ \cos(\omega u))dx^2 + (1 - h_+ \cos(\omega u))dy^2 \tag{15}$$

As $g_{\mu\nu}$ must be symmetric, we obtain $g_{uv} = g_{vu} = 1/2$ and $g_{xx} = 1 + h_+ \cos(\omega u)$ and $g_{yy} = 1 - h_+ \cos(\omega u)$ while all other components of g are zero, so that the metric satisfies $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Next, we need $\sqrt{-g}$ in the new coordinate system. We have

$$g = \det(g_{\mu\nu}) = -\frac{1}{2} \cdot \frac{1}{2} \cdot (1 + h_+ c) \cdot (1 - h_+ c)$$

This is easily seen as it is the only non-vanishing product in the definition of the determinant of $g_{\mu\nu}$. Hence, we obtain

$$\sqrt{-g} = \frac{1}{2} \sqrt{1 - h_+^2 c^2}$$

Now, we wish to compute the d'Alembertian. To compute it, we need the upper-index components $g^{\mu\nu}$, and so we need to invert $g_{\mu\nu}$. Recall that $g_{\mu\nu}$ was given by

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 + h_+ c & 0 \\ 0 & 0 & 0 & 1 - h_+ c \end{pmatrix}$$

so the inverse $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1+h_+c} & 0 \\ 0 & 0 & 0 & \frac{1}{1-h_+c} \end{pmatrix} \tag{16}$$

[12] Many authors define light-cone coordinates as $u = (t - z)/\sqrt{2}$, $v = (t + z)/\sqrt{2}$: however, this differs only in a sign and a constant factor.

[13] This is why I set the speed of light to be 1: otherwise it would confuse with this c.

If we substitute this into (12) while eliminating all terms for which $g^{\mu\nu}$ vanishes (as all of them are identically zero), we obtain

$$\frac{2}{\sqrt{1-h_+^2c^2}}\Bigg[\partial_x\frac{1}{1+h_+c}\cdot\frac{1}{2}\sqrt{1-h_+^2c^2}\partial_x+\partial_y\frac{1}{1-h_+c}\cdot\frac{1}{2}\sqrt{1-h_+^2c^2}\partial_y$$
$$+\partial_u 2\cdot\frac{1}{2}\sqrt{1-h_+^2c^2}\partial_v+\partial_v 2\cdot\frac{1}{2}\sqrt{1-h_+^2c^2}\partial_u\Bigg]\phi=0 \quad (17)$$

Now only in the third term does the first of the two differentials act on the part in between. We compute

$$\partial_u 2\cdot\frac{1}{2}\sqrt{1-h_+^2c^2}=\frac{1}{2\sqrt{1-h_+^2c^2}}\cdot -2h_+^2c\cdot -s\cdot\omega=\frac{h_+^2cs\omega}{\sqrt{1-h_+^2c^2}}$$

Now we can simplify to (note that we have to use the product rule on the third term, so this term splits into two terms!)

$$\left[\frac{1}{1+h_+c}\partial_x^2+\frac{1}{1-h_+c}\partial_y^2+\frac{2h_+^2cs\omega}{1-h_+^2c^2}\partial_v+2\partial_u\partial_v+2\partial_v\partial_u\right]\phi=0$$

Recall that all we computed here is simply $\Box\phi$ in our new coordinate system. This gives us that

$$\boxed{\Box=\frac{\partial_x^2}{1+h_+c}+\frac{\partial_y^2}{1-h_+c}+4\partial_u\partial_v+\frac{2h_+^2cs\omega}{1-h_+^2c^2}\partial_v} \quad (18)$$

is the d'Alembertian in our new coordinate system.

5.2 Fourier transformation

One way of solving the equation is by attempting to find a general solution using Fourier transforms. Instead of looking for a specific class of solutions, this method has the advantage that it gives the general solution to the equation. As ϕ is usually considered in position space, we are going to transform it to phase space. However, a transformation into ordinary phase space will be rather hopeless, as the resulting differential equation is going to be second-order, with non-constant coefficients. So, we will use a different type of phase space than usual. We choose our position and momentum vectors to be

$$\vec{x}=(x,y,v),\quad \vec{k}=(k_x,k_y,k_v)$$

and we view $\phi=\phi(\vec{x},u)$, and in wave space $\phi_{\vec{k}}(u)$.
So, the Fourier transformation is

$$\phi(\vec{x},u)=\int\frac{1}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\,d^3k \quad (19)$$

Taking the d'Alembertian, we obtain

$$\begin{aligned}\Box\phi(\vec{x},u)&=\Box\int\frac{1}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\,d^3k\\&=\int\frac{1}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\left[\frac{-k_x^2}{1+h_+c}\phi_{\vec{k}}(u)+\frac{-k_y^2}{1-h_+c}\phi_{\vec{k}}(u)+4ik_v\frac{\partial\phi_{\vec{k}}(u)}{\partial u}+\frac{2h_+^2cs\omega\cdot ik_v}{1-h_+^2c^2}\phi_{\vec{k}}(u)\right]d^3k\end{aligned}$$

In order for this to be zero, we set the integrand to zero. So, we have

$$4ik_v\frac{\partial\phi_{\vec{k}}(u)}{\partial u}=\frac{k_x^2}{1+h_+c}\phi_{\vec{k}}(u)+\frac{k_y^2}{1-h_+c}\phi_{\vec{k}}(u)-\frac{2h_+^2cs\omega\cdot ik_v}{1-h_+^2c^2}\phi_{\vec{k}}(u) \quad (20)$$

This differential equation is separable, so we can solve it by integration. We get

$$\int\frac{\partial\phi_{\vec{k}}(u)}{\phi_{\vec{k}}(u)}=\frac{1}{4ik_v}\int\frac{k_x^2}{1+h_+c}+\frac{k_y^2}{1-h_+c}-\frac{2ih_+^2cs\omega k_v}{1-h_+^2c^2}\,du$$

We will now evaluate this integral. The left hand side is simply a logarithmic function. The two leftmost terms in the right hand side are standard integrals. The rightmost term can be evaluated using the substitution $a = c = \cos(\omega u)$, so that $da = -\omega \sin(\omega u)\, du = -\omega s\, du$ so that

$$\int -\frac{2ih_+^2 cs\omega k_v}{1-h_+^2c^2}\, du = ik_v \int \frac{2h_+^2 a}{1-h_+^2a^2} da = -ik_v \log(1-h_+^2a^2) = -ik_v \log(1-h_+^2c^2)$$

The other two terms evaluate into

$$\int \frac{k_x^2}{1+h_+c} du = \frac{2k_x^2}{\omega\sqrt{1-h_+^2}} \arctan\left(\sqrt{\frac{1-h_+}{1+h_+}} \tan\left(\frac{\omega u}{2}\right)\right)$$

and

$$\int \frac{k_y^2}{1-h_+c} du = \frac{2k_y^2}{\omega\sqrt{1-h_+^2}} \arctan\left(\sqrt{\frac{1+h_+}{1-h_+}} \tan\left(\frac{\omega u}{2}\right)\right)$$

Plugging everything in, we obtain

$$\phi_{\vec{k}}(u) = \frac{A_{\vec{k}}}{\sqrt[4]{1-h_+^2c^2}} \exp\left[\frac{1}{2ik_v\omega\sqrt{1-h_+^2}} \left\{k_x^2 \arctan\left(\sqrt{\frac{1-h_+}{1+h_+}} \tan\left(\frac{\omega u}{2}\right)\right) + k_y^2 \arctan\left(\sqrt{\frac{1+h_+}{1-h_+}} \tan\left(\frac{\omega u}{2}\right)\right)\right\}\right] \tag{21}$$

Here, $A_{\vec{k}}$ depends on k_x, k_y and k_v: note that none of these derivatives occur in our differential equation, so we cannot compute $A_{\vec{k}}$ any further at this point. However, we are not interested in that. Note that the magnitude of $\phi_{\vec{k}}(u)$ does not depend on the exponential, as this is purely complex! So, we have that

$$|\phi_{\vec{k}}(u)|^2 = \frac{|A_{\vec{k}}|^2}{\sqrt{1-h_+^2c^2}} \tag{22}$$

At this point, we can apply the formula for particle creation:

$$|\phi_{\vec{k}}(u)|^2 = (1+2n)|\phi_{\vec{k}}(u)|^2_{h_+=0} \tag{23}$$

That is, the relative increase in $|\phi_{\vec{k}}(u)|^2$ that we get when we turn on the gravitational wave is proportional to the expected particle creation number n.
The number n is dimensionless here: it corresponds to the particle creation density in the phase space. So if we integrate n over momenta space, we get the particle density in position space, and if we integrate that over all of position space, we get the total number of particles created. Please note that this does not happen in a specified time interval. This is because the gravitational wave here is a plane wave. Introducing it leads to a creation of particles, however, leaving it on does not create more particles: it just maintains the particles that are already created by turning the field on[14]. It is similar to saying that even though the wave itself carries energy, maintaining it does not require adding new energy. This is because we are in the idealized situation of a plane wave that covers all of space, and hence we don't need to add any new perturbations to maintain the wave: it will simply propagate[15].
The number n in may, in principle, depend on $\vec{k}$ and u. Now, let's compute this density. We have that

$$\frac{|A_{\vec{k}}|^2}{\sqrt{1-h_+^2c^2}} = (1+2n)|A_{\vec{k}}|^2$$

[14]This is sloppy language, as n is the expected value of the number of particles created, and not the actual number. However, this sentence would be rather confusing if I replaced everything by an expectation value.

[15]Furthermore, we assume that there is nothing to stop the wave.

Assuming $|A_{\vec{k}}|^2 \neq 0$ (which is true for any non-trivial solution) we have

$$1 + 2n = \frac{1}{\sqrt{1 - h_+^2 c^2}}$$

$$1 + 2n \simeq 1 + \frac{1}{2} h_+^2 c^2$$

$$\boxed{n \simeq \frac{h_+^2 c^2}{4}} \tag{24}$$

This means that there is indeed a non-zero number of particles being generated! We will further interpret this number n in Section 7.2.

6 Quantum analysis

After the classical approach of simply solving the differential equation, we now move on to canonical quantization, and we attempt to derive a similar result from this. This means that we do not compute the function $\phi_{\vec{k}}(u)$, and instead we view ϕ as an operator $\hat{\phi}(\vec{x}, u)$, and then use a Fourier transformation. However, as it turns out, this standard approach turns out to fail here, and we attempt to solve it by making changes.
Unless indicated otherwise, we will use light-cone coordinates, so we have $\vec{x} = (x, y, v)$ and u as our coordinates.

6.1 Initial approach

In quantum mechanics, we turn physical quantities into operators: for example, the discrete position and momentum turn into[16]

$$\hat{x} \to x, \quad \hat{p} \to -i\hbar\partial_x$$

This gives us the *canonical commutation relations*

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}$$

where δ_{ij} is a Kronecker delta.
However, we do not have discrete momenta and position here. To replace the momentum, we have the field $\hat{\phi}$. Corresponding to every field, we have a canonical momentum $\hat{\pi}$. This is usually defined as

$$\pi(\vec{x}, t) = \frac{\delta S}{\delta(\partial_t \phi)} \tag{25}$$

However, in our case, we are working with a coordinate frame (light-cone coordinates) that does not explicitly use the coordinate t (although it does use it implicitly). So instead, we will be using the coordinate u as our time coordinate. To show that this is possible, consider the hypersurface Σ_u that is linearly spanned by the coordinates x, y and v in 4-dimensional space. Now u is clearly a normal to this surface at every point. However, the metric (14) gives us that this normal has length zero. So, the hypersurface Σ_u is lightlike, and as such a surface must be lightlike or spacelike, we may conclude that u can be chosen as a proper time coordinate.
So, we have a momentum

$$\pi(\vec{x}, t) = \frac{\delta S}{\delta(\partial_u \phi)} \tag{26}$$

The Lagrangian can be written as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \\
&= -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 \\
&= -2\partial_u\phi\partial_v\phi - \frac{1}{2}\left(\frac{1}{1+h_+c}(\partial_x\phi)^2 + \frac{1}{1-h_+c}(\partial_y\phi)^2\right) - \frac{m^2}{2}\phi^2
\end{aligned}$$

and therefore

$$S[\phi] = \int \left(-2\partial_u\phi\partial_v\phi - \frac{1}{2}\left(\frac{1}{1+h_+c}(\partial_x\phi)^2 + \frac{1}{1-h_+c}(\partial_y\phi)^2\right) - \frac{m^2}{2}\phi^2\right)\sqrt{-g}\, d^4x \tag{27}$$

So, we obtain

$$\boxed{\pi(\vec{x}, u) = -2\partial_v\phi\sqrt{-g}} \tag{28}$$

[16]Actually, $\hat{x}$ and $\hat{p}$ are pure operators: this is just the representation of these operators in position space. They take different forms in, for example, momentum space.

The canonical commutation relation becomes[17]

$$[\hat{\phi}(\vec{x},u),\hat{\pi}(\vec{x'},u)] = i\hbar\delta^3(\vec{x}-\vec{x'}) \tag{29}$$

Now let's move on to Fourier transformations.
The Fourier transform of the operator $\hat{\phi}$ is given by

$$\hat{\phi}(\vec{x},u) = \int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}}\phi^\star_{\vec{k}}(u)\hat{a}^\dagger_{\vec{k}}\right\} d^3k \tag{30}$$

Here, $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ are ladder operators. Similarly, the Fourier transform of π is given by

$$\hat{\pi}(\vec{x'},u) = \int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k'}\cdot\vec{x'}}\pi_{\vec{k'}}(u)\hat{a}_{\vec{k'}} + e^{-i\vec{k'}\cdot\vec{x'}}\pi^\star_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k'}}\right\} d^3k'$$

We used primes in the last equation as we are going to compute the commutator. The first term of this commutator is

$$\hat{\phi}(\vec{x},u)\hat{\pi}(\vec{x'},u) = \int \frac{1}{(2\pi)^3}\int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}}\phi^\star_{\vec{k}}(u)\hat{a}^\dagger_{\vec{k}}\right\}\left\{e^{i\vec{k'}\cdot\vec{x'}}\pi_{\vec{k'}}(u)\hat{a}_{\vec{k'}} + e^{-i\vec{k'}\cdot\vec{x'}}\pi^\star_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k'}}\right\} d^3k d^3k'$$

which is equal to

$$\begin{aligned}= \iint \frac{1}{(2\pi)^6}\Big\{ & e^{i(\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi_{\vec{k'}}(u)\hat{a}_{\vec{k}}\hat{a}_{\vec{k'}} + e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k'}} + \\ & e^{i(-\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k'}} + e^{i(-\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{\vec{k'}}\Big\} d^3k d^3k'\end{aligned} \tag{31}$$

Similarly, the other commutator term is equal to

$$\begin{aligned}\hat{\pi}(\vec{x},u)\hat{\phi}(\vec{x'},u) = \iint \frac{1}{(2\pi)^6}\Big\{ & e^{i(\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi_{\vec{k'}}(u)\hat{a}_{\vec{k'}}\hat{a}_{\vec{k}} + e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k'}}\hat{a}_{\vec{k}} + \\ & e^{i(-\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi_{\vec{k'}}(u)\hat{a}_{\vec{k'}}\hat{a}^\dagger_{\vec{k}} + e^{i(-\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)\hat{a}^\dagger_{\vec{k'}}\hat{a}^\dagger_{\vec{k}}\Big\} d^3k d^3k'\end{aligned} \tag{32}$$

Note that we used the fact that $\phi_{\vec{k}}(u)$ and $\pi_{\vec{k}}(u)$ are just numbers, and hence we may switch them with the operators. Now to compute the commutator, we need the commutation relations:

$$[\hat{a}_{\vec{k}},\hat{a}_{\vec{k'}}] = 0, \quad [\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{k'}}] = (2\pi)^3\delta^3(\vec{k}-\vec{k'}) \tag{33}$$

For the commutator, we subtract (32) from (31). The first terms in both expressions cancel when subtracted, as we obtain a commutator of the form $[\hat{a}_{\vec{k}},\hat{a}_{\vec{k'}}]$ which is zero. Similarly, the fourth terms cancel. So we are left with

$$[\hat{\phi}(\vec{x},u),\hat{\pi}(\vec{x'},u)] = \iint \frac{1}{(2\pi)^6}\left\{e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{k'}}] + e^{i(-\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi_{\vec{k'}}(u)[a^\dagger_{\vec{k}},a_{\vec{k'}}]\right\} d^3k d^3k'$$

Filling in (33) we obtain

$$= \iint \frac{1}{(2\pi)^6}\left\{e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi^\star_{\vec{k'}}(u)(2\pi)^3\delta^3(\vec{k}-\vec{k'}) + e^{i(-\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi_{\vec{k'}}(u)\cdot -(2\pi)^3\delta^3(\vec{k}-\vec{k'})\right\} d^3k d^3k'$$

and getting rid of the Dirac delta

$$= \iint \frac{1}{(2\pi)^3}\left\{e^{i(\vec{k}\cdot\vec{x}-\vec{k}\cdot\vec{x'})}\phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - e^{i(-\vec{k}\cdot\vec{x}+\vec{k}\cdot\vec{x'})}\phi^\star_{\vec{k}}(u)\pi_{\vec{k}}(u)\right\} d^3k$$

[17] This relation is a copy of the corresponding relation in Cartesian coordinates: we assume it to be unchanged. Note that it can be viewed as a continuous version of the discrete commutation relations, where the Kronecker delta becomes a Dirac delta

$$= \iint \frac{1}{(2\pi)^3} \left\{ e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi^\star_{\vec{k}}(u)\pi_{\vec{k}}(u) \right\} d^3k$$

Now at this point, we have two terms left, we wish to combine them into one term. To do this, we invert the direction of $\vec{k}$ in the second term:

$$= \iint \frac{1}{(2\pi)^3} \left\{ e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi^\star_{-\vec{k}}(u)\pi_{-\vec{k}}(u) \right\} d^3k$$

$$= \iint \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left\{ \phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - \phi^\star_{-\vec{k}}(u)\pi_{-\vec{k}}(u) \right\} d^3k$$

Using the standard integral

$$\int \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} d^3k = \delta^3(\vec{x})$$

we obtain the very important result that

$$\boxed{\phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - \phi^\star_{-\vec{k}}(u)\pi_{-\vec{k}}(u) = i\hbar} \tag{34}$$

Now if we look at equation (20), we see that if we turn $\vec{k}$ into $-\vec{k}$ then all coefficients turn into complex conjugates. Therefore, $\phi^\star_{\vec{k}}$ becomes a solution if we turn $\vec{k}$ into $-\vec{k}$. Since this differential equation is first-order, there is only one linearly independent solution, so this *must* be the solution. This is also reflected in the solution (21). So, we have

$$\phi_{-\vec{k}} = \phi^\star_{\vec{k}} \tag{35}$$

However, using (26), we obtain that $\pi_{-\vec{k}} = \pi^\star_{\vec{k}}$, and hence equation (34) reduces to $0 = i\hbar$. This is clearly a contradiction. What has gone wrong? In the next sections, we discuss several ways to resolve this paradox. The first one is by attempting to stop the minus sign from conjugating our field and momentum, so that (34) no longer automatically reduces to zero. The second approach is by imposing a new commutation relation coupling $a_{\vec{k}}$ and $a_{-\vec{k}}$. A third attempt at solving it is making $\vec{k} \to -\vec{k}$ not correspond to complex conjugation but to a phase shift. However, this fails to resolve the problem, so I added it as an appendix, Section 8.2.

6.2 Eliminating conjugation

The main issue that we get from (34) is that getting rid of the minus sign also conjugates both our functions. This is not a property of light-cone coordinates, but rather a property of this wave. For example, let's look at the same wave in Cartesian coordinates. The d'Alembertian becomes

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{-g}} \left(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu\right)\phi \\
&= \frac{1}{\sqrt{1-h_+^2c^2}} \left(\partial_x \sqrt{1-h_+^2c^2} \frac{1}{1+h_+c} \partial_x + \partial_y \sqrt{1-h_+^2c^2} \frac{1}{1-h_+c} \partial_y + \partial_z \sqrt{1-h_+^2c^2} \partial_z - \partial_t \sqrt{1-h_+^2c^2} \partial_t \right) \\
&= \frac{\partial_x^2}{1+h_+c} + \frac{\partial_y^2}{1-h_+c} + \partial_z^2 + \frac{kh_+^2cs}{1-h_+^2c^2}\partial_z - \partial_t^2 + \frac{\omega h_+^2cs}{1-h_+^2c^2}\partial_t
\end{aligned}$$

and if we take the d'Alembertian of a Fourier transform in Cartesian coordinates, we obtain

$$\begin{aligned}
\Box\phi &= \Box \int \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \phi_{\vec{k}}(t)\, d^3k \\
&= \int \frac{1}{(2\pi)^3} \left[\left\{ \frac{-k_x^2}{1+h_+c} + \frac{-k_y^2}{1-h_+c} + -k_z^2 + \frac{i\omega h_+^2csk_z}{1-h_+^2c^2} \right\} \phi_{\vec{k}}(t) + \frac{\omega h_+^2cs}{1-h_+^2c^2} \frac{\partial\phi_{\vec{k}}(t)}{\partial t} - \frac{\partial^2\phi_{\vec{k}}(t)}{\partial t^2} \right] e^{i\vec{k}\cdot\vec{x}}\, d^3k
\end{aligned}$$

so we obtain the differential equation[18]

$$\left\{\frac{-k_x^2}{1+h_+c}+\frac{-k_y^2}{1-h_+c}+-k_z^2+\frac{i\omega h_+^2 csk_z}{1-h_+^2c^2}\right\}\phi_{\vec{k}}(t)+\frac{\omega h_+^2 cs}{1-h_+^2c^2}\frac{\partial\phi_{\vec{k}}(t)}{\partial t}-\frac{\partial^2\phi_{\vec{k}}(t)}{\partial t^2}=0 \tag{36}$$

This gives us a second order differential equation for $\phi_{\vec{k}}(t)$, so we should get two linearly independent solutions. In light-cone coordinates, we only had one. So how did that happen?
We note that the d'Alembertian in light-cone coordinates (18) is actually second-order, so its solution should have two degrees of freedom. However, due to the Fourier transformation used, every ∂_v derivative simply becomes ik_v. This is why we lose a degree of freedom. So perhaps, we should not be Fourier transforming over v. To do this, we should Fourier transform the coordinates x and y only: we eliminate the transformation over v. This idea is further supported by the fact that the complex conjugation is not due to flipping the sign in k_x and k_y, but due to flipping the sign in k_v (in the case of (20) and (21)) or k_z (in the case of (36)). This way, we no longer have to do the conjugation. We will do this in Section 6.2.1.
We note that if we use the transformation $\vec{k}\to-\vec{k}$, then all coefficients turn into their complex conjugates. This means that $\phi_{-\vec{k}}=\phi_{\vec{k}}^\star$ also works here: hence this is *not* a property of the light-cone coordinate system. However, it is not the only solution: in this case, we have two solutions, as our equation is second order. In contrast, in light-cone coordinates, it is the only solution. In the case of a flat metric (so in the limit $h_+\to 0$), we have that the two solutions in this coordinate system turn out to be complex conjugates, and therefore we can set $\phi_{-\vec{k}}=\phi_{\vec{k}}$. We will take a closer look at this limit case in Section 8.3.

6.2.1 Two-dimensional transform

Set

$$\hat{\phi}(\vec{x},u,v)=\int\frac{1}{(2\pi)^2}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u,v)\hat{a}_{\vec{k}}+e^{-i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}^\star(u,v)\hat{a}_{\vec{k}}^\dagger\right\}d^2k$$

Here, $\vec{x}=(x,y)$ and $\vec{k}=(k_x,k_y)$. Note how all cubes become squares. Similarly to in the formerly used transform, $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$ are ladder operators: however, this time, they are two-dimensional. So the commutation relation (33) becomes

$$[\hat{a}_{\vec{k}},\hat{a}_{\vec{k'}}^\dagger]=(2\pi)^2\delta^2(\vec{k}-\vec{k'}) \tag{37}$$

Similarly, we set

$$\hat{\pi}(\vec{x'},u,v)=\int\frac{1}{(2\pi)^2}\left\{e^{i\vec{k'}\cdot\vec{x'}}\pi_{\vec{k'}}(u,v)\hat{a}_{\vec{k'}}+e^{-i\vec{k'}\cdot\vec{x'}}\pi_{\vec{k'}}^\star(u,v)\hat{a}_{\vec{k'}}^\dagger\right\}d^2k'$$

With a similar derivation to in Section 6.1, we derive that

$$\phi_{\vec{k}}(u,v)\pi_{\vec{k}}^\star(u,v)-\phi_{-\vec{k}}^\star(u,v)\pi_{-\vec{k}}(u,v)=i\hbar$$

This time, however, we note that we can get rid of the minus signs:

$$\phi_{\vec{k}}(u,v)\pi_{\vec{k}}^\star(u,v)-\phi_{\vec{k}}^\star(u,v)\pi_{\vec{k}}(u,v)=i\hbar \tag{38}$$

So now, we need to determine $\phi_{\vec{k}}(u,v)$. Let's consider the Klein-Gordon equation: four this we consider the Fourier transform[19]

$$\phi(\vec{x},u,v)=\int\frac{1}{(2\pi)^2}e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u,v)\,d^2k$$

[18] At this point, we can also see why light-cone coordinates are necessary for this problem. This equation in Cartesian coordinates is a second-order differential equation with non-constant coefficients, since c depends on t. This means it is highly unlikely that we can solve it analytically. Furthermore, there is no point in numerical analysis, as we would have to do it for all constants k_x,k_y,k_z and z: all of these are in the coefficients. Therefore, this equation is far more difficult to deal with than the version in light-cone coordinates, Equation (20).

[19] You may wonder if it is a problem that we only consider one term here, and not both. The reason it is justified, is that the two coefficients are each other's complex conjugate, and hence both result in the exact same equation up to conjugation. And as $\Box$ does not act on $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$, the only difference is in the conjugation, so there is no real point in considering both terms.

and so

$$\Box\phi(\vec{x},u,v) = \Box\int \frac{1}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \phi_{\vec{k}}(u,v)\, d^2k$$

$$= \int \frac{1}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \left[-\left(\frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c} \right) \phi_{\vec{k}}(u,v) + 4\partial_u\partial_v\phi_{\vec{k}}(u,v) + \frac{2h_+^2 cs\omega}{1-h_+^2c^2}\partial_v\phi_{\vec{k}}(u,v) \right] d^2k$$

We set the integrand to zero:

$$4\partial_u\partial_v\phi_{\vec{k}}(u,v) + \frac{2h_+^2 cs\omega}{1-h_+^2c^2}\partial_v\phi_{\vec{k}}(u,v) = \left(\frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c} \right) \phi_{\vec{k}}(u,v) \tag{39}$$

To solve this, we use separation of variables:

$$\phi_{\vec{k}}(u,v) = \chi(u)\psi(v)$$

Technically, χ and ψ depend on $\vec{k}$, but we leave that out for convenience. Plugging this in:

$$4\frac{\partial\chi}{\partial u}\frac{\partial\psi}{\partial v} + \frac{2h_+^2 cs\omega}{1-h_+^2c^2}\frac{\partial\psi}{\partial v}\chi = \left(\frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c} \right)\chi\psi$$

$$\frac{1}{\psi}\frac{\partial\psi}{\partial v}\left(4\frac{1}{\chi}\frac{\partial\chi}{\partial u} + \frac{2h_+^2 cs\omega}{1-h_+^2c^2} \right) = \frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c}$$

$$\frac{1}{\psi}\frac{\partial\psi}{\partial v} = \left(\frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c} \right)\left(4\frac{1}{\chi}\frac{\partial\chi}{\partial u} + \frac{2h_+^2 cs\omega}{1-h_+^2c^2} \right)^{-1}$$

And as the right hand side does not depend on v, we have that ψ is a plane wave or an exponential. However, an exponential would blow up at infinity, which means that we are dealing with a wave. So, we set

$$\psi(v) = e^{ik_v v}$$

where this time, we define k_v to be the exponential. Setting

$$f(u) \equiv \frac{2h_+^2 cs\omega}{1-h_+^2c^2}, \quad g(u) \equiv \frac{k_x^2}{1+h_+c} + \frac{k_y^2}{1-h_+c}$$

we obtain the equation

$$ik_v\left(4\frac{1}{\chi}\frac{\partial\chi}{\partial u} + f(u) \right) = g(u)$$

$$\frac{\partial\chi}{\partial u} = \frac{1}{4}\chi(-f(u) + g(u)/ik_v)$$

Using u as a time coordinate, (28) gives us

$$\pi_{\vec{k}}(u,v) = -2\partial_v\phi_{\vec{k}}(u,v)\sqrt{-g} = -2ik_v\phi\sqrt{-g}$$

so (38) reduces to

$$\phi\cdot 2ik_v\phi^\star\sqrt{-g} - \phi^\star\cdot -2ik_v\phi\sqrt{-g} = i\hbar$$

$$4k_v|\phi|^2\sqrt{-g} = \hbar$$

and since $\sqrt{-g} = \frac{1}{2}\sqrt{1-h_+^2c^2}$ we obtain

$$\boxed{|\phi|^2 = \frac{\hbar}{2k_v\sqrt{1-h_+^2c^2}}} \tag{40}$$

Now as for particle creation, we have that

$$|\phi|^2 \propto \frac{1}{\sqrt{1-h_+^2c^2}} \tag{41}$$

just like in (21). So we obtain the exact same relation as before! This resolves our paradox.
We can actually also take the momentum to be as in (25), and then we have

$$\pi = \sqrt{-g}\partial_t\phi = \sqrt{-g}\left(\frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v}\right)\phi = \sqrt{-g}(-\partial_u + \partial_v)\phi$$

Note that

$$\frac{\partial \phi}{\partial u} = \psi\frac{\partial \chi}{\partial u} = \frac{1}{4}\phi\big(-f(u) + g(u)/ik_v)\big)$$

This gives us

$$\pi = \sqrt{-g}\big(ik_v\phi - \frac{1}{4}\phi(-f(u) + g(u)/ik_v)\big) = \sqrt{-g}\phi\big(ik_v - \frac{1}{4}(-f(u) + g(u)/ik_v)\big)$$

So the commutator becomes

$$\begin{aligned}\phi\pi^\star - \phi^\star\pi &= \sqrt{-g}\phi\phi^\star\big(-ik_v + f(u)/4 + g(u)/4ik_v - ik_v - f(u)/4 + g(u)/4ik_v\big)\\ &= \sqrt{-g}|\phi|^2\big(-2ik_v - ig(u)/2k_v\big)\end{aligned}$$

so

$$\boxed{|\phi|^2 = -\frac{\hbar}{k_v + g(u)/4k_v}\frac{1}{\sqrt{1-h_+^2c^2}}} \tag{42}$$

which means that we obtain the same dependence as in (41), even though our field ϕ has changed in magnitude. This change has to do with a different definition of our conjugate momentum. Note that the coefficient has opposite sign with respect to k_v: this means that k_v is negative, while it was positive in (40).

6.3 Different commutation relations

For this section, we once again set $\vec{x} = (x, y, v)$ and $\vec{k} = (k_x, k_y, k_v)$.
Another idea to resolve the problem is by changing the commutation relations between the raising and lowering operators. We considered $a_{\vec{k}}$ and $a_{-\vec{k}}$ to be unrelated before. But if we consider them to be related, we actually turn out to obtain the desired result. Suppose

$$[a_{\vec{k}}, a_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k} + \vec{k}') \tag{43}$$

Although this might seem random at first glance, it is also suggested by working out the canonical commutator in terms of the ladder operators, as we will show now. We can turn the field into

$$\begin{aligned}\hat{\phi}(\vec{x}, u) &= \int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}}\phi^\star_{\vec{k}}(u)\hat{a}^\dagger_{\vec{k}}\right\} d^3k\\ &= \int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}_{\vec{k}} + e^{i\vec{k}\cdot\vec{x}}\phi^\star_{-\vec{k}}(u)\hat{a}^\dagger_{-\vec{k}}\right\} d^3k\\ &= \int \frac{1}{(2\pi)^3}\left\{e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}_{\vec{k}} + e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)\hat{a}^\dagger_{-\vec{k}}\right\} d^3k\\ &= \int \frac{1}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}(u)(\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}})\, d^3k\end{aligned}$$

where in the second line, we inverted the direction of $\vec{k}$ for the second term, and in the third line we used equation (35). Similarly,

$$\hat{\pi}(\vec{x'}, u) = \int \frac{1}{(2\pi)^3} e^{i\vec{k'}\cdot\vec{x'}} \pi_{\vec{k'}}(u)(\hat{a}_{\vec{k'}} + \hat{a}^\dagger_{-\vec{k'}})\, d^3k'$$

So for the commutator, we obtain

$$[\hat{\phi}(\vec{x}, u), \hat{\pi}(\vec{x'}, u)] = \iint \frac{1}{(2\pi)^6} e^{i(\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})} \phi_{\vec{k}}(u)\pi_{\vec{k'}}(u)[\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}, \hat{a}_{\vec{k'}} + \hat{a}^\dagger_{-\vec{k'}}]\, d^3k d^3k' \tag{44}$$

so let's take a look at the *ladder commutator*

$$[\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}, \hat{a}_{\vec{k'}} + \hat{a}^\dagger_{-\vec{k'}}] \tag{45}$$

If we have $k = k'$ then this commutator is clearly 0, since both sides are equal. So assuming that $a_{\vec{k}}$ and $a_{\vec{k'}}$ commute if $\vec{k}$ is not $\pm\vec{k'}$, then we must have $\vec{k'} = -\vec{k}$ for any non-vanishing term. In that case, (45) reduces to

$$\begin{aligned}
[\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}, \hat{a}_{-\vec{k}} + \hat{a}^\dagger_{\vec{k}}] &= [\hat{a}_{\vec{k}}, \hat{a}_{-\vec{k}}] + [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] + [\hat{a}^\dagger_{-\vec{k}}, \hat{a}_{-\vec{k}}] + [\hat{a}^\dagger_{-\vec{k}}, \hat{a}^\dagger_{\vec{k}}] \\
&= [\hat{a}_{\vec{k}}, \hat{a}_{-\vec{k}}] + (2\pi)^3\delta^3(0) - (2\pi)^3\delta^3(0) + [\hat{a}^\dagger_{-\vec{k}}, \hat{a}^\dagger_{\vec{k}}] \\
&= [\hat{a}_{\vec{k}}, \hat{a}_{-\vec{k}}] + ([\hat{a}_{\vec{k}}, \hat{a}_{-\vec{k}}])^\dagger
\end{aligned}$$

And this is clearly zero if $a_{\vec{k}}$ and $a_{-\vec{k}}$ commute. This is why it makes sense to suppose that they do not commute! So assuming (43), we get

$$= [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] + ([\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}])^\dagger = (2\pi)^3\delta^3(0) + (2\pi)^3\delta^3(0) = 2(2\pi)^3\delta^3(0)$$

which means that the ladder commutator becomes

$$[\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}, \hat{a}_{\vec{k'}} + \hat{a}^\dagger_{-\vec{k'}}] = 2(2\pi)^3\delta^3(\vec{k} + \vec{k'}) \tag{46}$$

since we already noted that it is zero if $\vec{k'} \neq -\vec{k}$.
So, equation (44) becomes

$$\begin{aligned}
[\hat{\phi}(\vec{x}, u), \hat{\pi}(\vec{x'}, u)] &= \iint \frac{1}{(2\pi)^6} e^{i(\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})} \phi_{\vec{k}}(u)\pi_{\vec{k'}}(u)[\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}, \hat{a}_{\vec{k'}} + \hat{a}^\dagger_{-\vec{k'}}]\, d^3k d^3k' \\
&= \iint \frac{1}{(2\pi)^6} e^{i(\vec{k}\cdot\vec{x}+\vec{k'}\cdot\vec{x'})} \phi_{\vec{k}}(u)\pi_{\vec{k'}}(u) 2(2\pi)^3\delta^3(\vec{k} + \vec{k'})\, d^3k d^3k' \\
&= \iint \frac{1}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x}-\vec{k}\cdot\vec{x'})} 2\phi_{\vec{k}}(u)\pi_{-\vec{k}}(u)\, d^3k \\
&= \iint \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} 2\phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u)\, d^3k
\end{aligned}$$

And this means that

$$2\phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) = i\hbar$$

and using (28) we obtain

$$2\phi_{\vec{k}}(u) \cdot 2ik_v\phi^\star_{\vec{k}}(u)\sqrt{-g} = i\hbar$$

$$|\phi_{\vec{k}}(u)|^2 = \frac{\hbar}{2k_v\sqrt{1 - h_+^2c^2}}$$

again resulting in our familiar dependence on h_+. Note that this is actually exactly the same as (40).
The problem with this solution is that, although it produces an outcome consistent with the other methods we used, I could not find any physical motivation for coupling $\hat{a}_{\vec{k}}$ and $\hat{a}_{-\vec{k}}$. Another issue imposed by this technique is what happens at $\vec{k} = 0$. We will discuss these results further in Section 7.1.

7 Discussion and conclusions

We have computed the expectation value for the number of particles created by a plane gravitational wave in two different ways: using a Fourier transformation and then explicitly solving the equation of motion, and by performing canonical quantization and then using the commutation relations. From both methods, we derived the same particle creation density. In the case of quantum analysis, however, we had to make several fundamental changes, as a standard procedure resulted in a physically unacceptable result. So we are first going to discuss the methods and changes used in Section 6. We may also wonder how many particles are *actually* created by this wave, and if this is measurable: we will do this in Section 7.2. Finally, we compare this to earlier research and give several suggestions for further research on this problem.

7.1 Changes in quantum analysis

In the quantum analysis, the initial approach using Fourier transformation and canonical commutation relations for raising and lowering operators failed: the outcome was that $\hat{\phi}$ and $\hat{\pi}$ commute, which is not possible. So we tried solving it in several different ways. One of them was phase shifting, but that turned out to still give incorrect results: therefore I moved it to Appendix 8.2. The two methods that did work in the end, were limiting our Fourier transform to only the x and y component, and imposing a new commutation relation.
Limiting the Fourier transform makes sense as flipping the sign of k_x or k_y does not change our equations for $\phi_{\vec{k}}(u)$, and hence it does not change $\phi_{\vec{k}}(u)$. However, one may wonder why it is that this calculation, which is very similar to the one in Section 6.1, is *expected* on beforehand to produce results that are any different. The only thing that we are doing differently, is not Fourier transforming over v. This means that there would be something fundamentally wrong with our Fourier transformation over v, as otherwise it should produce the same results. In other words, the initial transformation (30) is either incorrect, or we interpreted it incorrectly. This probably has to do with the fact that we are using light-cone coordinates, and that there are different rules for Fourier transformation over light-cone coordinates than over Cartesian coordinates. It is likely that this has to do with the fact that the coefficient v is lightlike, whereas all coefficients that we usually transform over are spacelike. It is likely that eliminating v from the Fourier transform avoids this problem, and therefore it does produce the correct outcome.

As for the additional commutation relation, we knew that something went wrong in our initial Fourier transformation, and this could explain it. However, it has several problems with it. First of all, we may wonder what happens if $\vec{k} = 0$: this states that

$$0 = [a_0, a_0] = [a_0, a_{-0}] \neq 0$$

which is clearly not possible. This contradiction can only be resolved by excluding $\vec{k} = 0$ as a possible value for $\vec{k}$. A potential justification for excluding this is that for a massless particle, a value $\vec{k} = 0$ would mean that it has no net energy (as it is already massless and has no energy as a wave either). For example, a photon with $\vec{k} = 0$ does not exist, as it would have zero energy: therefore it is in no way distinguishable from vacuum.
But even with this out of the way, I have no justification for imposing this commutation relation other than that it produces the correct outcome. So I am unsure if this is what went wrong in our Fourier transformation.

7.2 Particle creation

In wave space, the particle creation density is equal to

$$n = \frac{h_+^2}{4} \cos^2(\omega u)$$

So, n only depends on u. Note that $n \geq 0$, which is obviously true as the wave is assumed to travel through a vacuum, and therefore it is not possible for it to generate a negative number of particles. Furthermore, it depends on the phase of the wave. If $\omega u = \pi/2 + \ell \cdot \pi$ with ℓ an integer, then we have $c = 0$, so there is no particle creation. At these points, however, the metric (in x, y, z, t-coordinates) turns into

$$g_{\mu\nu} = \mathrm{diag}(-1, 1 + h_+ c, 1 - h_+ c, 1) = \mathrm{diag}(-1, 1, 1, 1)$$

so the metric actually becomes flat at these points. This result is rather expected: in Minkowski space, we do not typically observe spontaneous particle creation[20].

The particle creation does not specify how many particles are generated in an interval of time. The reason for this is that while there are particles generated by turning the field on, there are not *more* particles generated by leaving it on: the wave merely maintains the already created particles.

Although the *phase* depends on ω, we note that the magnitude $h_+^2/4$ of n does not depend on ω, and so it does not depend on the frequency of the gravitational wave itself. Apparently it only depends on the magnitude of the gravitational wave h_+, and specifically on h_+^2. All of these results are similar to those obtained in classical theory: the energy or intensity of a wave is proportional to the square of its amplitude. The fact that it does not depend on ω may appear as surprising, however it can be explained as follows. The quantity h_+ is actually dimensionless, and hence it is not really an analogue to a classical amplitude, which has dimension length. It only describes how large the perturbation in the metric is. Aside from this, dimensional analysis makes it impossible for n to depend on ω, as n must be a dimensionless quantity. Note that this cannot be restored by multiplication with the speed of light, as that would require the unit of ω to be m s^{-1} and not s^{-1}.

As for detection, we want to know what the actual number of particles generated will be.

If we integrate n over all of wave space, the resulting integral diverges as n is constant with respect to $\vec{k}$. There was no *a priori* reason for this, however it just turned out to be true. This means that infinitely many particles would be generated, which is not physical. There are several possible explanations for this.

One explanation is that the wave we are considering is not physical. Typical gravitational waves are not flat but spherical, and more importantly, they do not span all of space. One way of dealing with this is by considering a superposition of waves into a wave package: we let

$$h_+(\omega) = h_+ \exp\left[\frac{-(\omega-\omega_0)^2}{2(\Delta\omega)^2}\right] \tag{47}$$

However, this way, the equation would be far more difficult to solve, as it means that our metric becomes an integral over all frequencies ϕ. This means that our metric would become very complicated, and we would have to linearise it and then solve the equations up to first order. This would be an idea for further research.

Another explanation is that the particles that are generated, actually affect the metric themselves. In our calculation, we did not take this effect, known as *gravitational backreaction* into account. If this were taken into account, the integral should become convergent. Another way to make the integral converge would be to use renormalisation of the integral.

Finally, in our approach, we made k run from zero to infinity, as we found no reason to exclude high-frequency particles. One could, of course, exclude such particles, and that would make the integral converge. However, this requires a fundamental change of the theory, in such a way that it prohibits high-frequency particles. The action would have to be turned into

$$S[\phi] = \int \exp\left(-\frac{\Box}{\Lambda^2}\right)\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}|\phi|^2\right)\sqrt{-g}\, d^4x \tag{48}$$

This extra factor in the Lagrangian excludes particles that have $k^2 > \Lambda^2$. However, this would require us to redo the entire theory, as it also effects the equation of motion (11).

7.3 Comparison to other research

If we compare our results to those obtained in [3], we see several important differences. This paper used a complex scalar field, which had norm equal to

$$\varphi^\star\varphi \approx \frac{A^2h_+^4}{4} + \frac{3A^2h_+^6}{8}\cos(2Ku) \tag{49}$$

Now this result is in position space which is curious for various reasons. First, it is not a density, so it is not the number of particles detected per unit volume (something that is not considered in this paper). As we are considering a plane gravitational wave spanning all of space, the total number of particles generated should

[20] Note that the metric is not actually flat, as the Riemann curvature tensor is non-zero at these points.

always be infinite, and not finite like here. Secondly, this goes to zero if we set the field to zero. This is interesting as it appears to violate equation (23): it would give us an infinite particle creation density. Third, it depends on the constant A which is a constant factor in φ. According to the paper, this is a normalization constant, however this constant is not computed, and it is not clarified with respect to which equation it normalizes φ.

7.4 Further research

The first thing we could do is do more research on Fourier transformation in light-cone coordinates, and see what exactly goes wrong in Section 6.1. Then, we can also confirm if imposing the new commutation relation in Section 6.3 is actually a solution.

As our integral to compute particle production does not converge, the best way to continue is to make the integral converge. As stated before, this can be done by choosing a physically realistic wave, by taking gravitational backreaction into effect, or by applying renormalisation.

Another suggestion for further research is that one could try to relate this to the Unruh effect. This effect states that the notion of vacuum depends on the path of the observer. If one observer is accelerating and a second observer is inertial, the first observer may see the vacuum of the second observer as a state with particles. This effect of particle creation could simply be a special version of this more general effect.

Finally, an idea for further research is to investigate what types of particles are typically generated. In this thesis, we only focused on massless scalar particle creation in general: this means photons, gluons and gravitons. We might wonder if all three types are actually generated, and if so, in what ratio.

8 Appendices

8.1 Proof of invariant measure

We will show that $\sqrt{-g}\, d^4x$ is constant. For this, we will follow [4]. Consider a change in the coordinate system δx, so that $x' = x + \delta x$. Now, we have

$$\begin{aligned}
\delta(d^4x) &= (d^4x)' - d^4x \\
&= d(x^0 + \delta x^0)d(x^1 + \delta x^1)d(x^2 + \delta x^2)d(x^3 + \delta x^3) - d^4x \\
&= dx^0dx^1dx^2dx^3(1 + \frac{\partial \delta x^\mu}{\partial x^\mu}) - d^4x \\
&= d^4x(1 + \frac{\partial \delta x^\mu}{\partial x^\mu}) - d^4x \\
&= d^4x\frac{\partial \delta x^\mu}{\partial x^\mu}
\end{aligned}$$

In the third line, we threw away any second or higher order terms, and we wrote the other terms using summation: if you're not convinced of this right away, write it out. So, we have $\delta(d^4x) = d^4x\partial_\mu \delta x^\mu$. Now on the other hand, we have

$$\delta(\sqrt{-g}) = \frac{1}{2\sqrt{-g}}\delta(-g)$$

by the chain rule. Now recall that $-g = -\det(g_{\mu\nu})$. We know that $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ for any matrix A, and applying this to $\log(-g_{\mu\nu})$ we obtain

$$\begin{aligned}
\delta(-g) &= -\delta\det(g_{\mu\nu}) \\
&= -\delta\left(\det(\exp(\log(g_{\mu\nu})))\right) \\
&= -\delta\left(\exp(\mathrm{Tr}(\log(g_{\mu\nu})))\right) \\
&= -\exp(\mathrm{Tr}(\log(g_{\mu\nu})))\delta(\mathrm{Tr}(\log(g_{\mu\nu}))) \\
&= -\det(\exp(\log(g_{\mu\nu})))\mathrm{Tr}(\delta(\log(g_{\mu\nu}))) \\
&= -g\mathrm{Tr}((g_{\mu\nu})^{-1}\delta g_{\nu\rho}) \\
&= -g\mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\rho}) \\
&= -g(g^{\mu\nu}\delta g_{\nu\mu}) \\
&= -gg^{\mu\nu}\delta g_{\mu\nu}
\end{aligned}$$

In the sixth line, we used the chain rule for matrix differentiation, and we used an index twice to obtain matrix multiplication. We raised the indices to get an inverse matrix. Simplifying on:

$$\begin{aligned}
g'_{\mu\nu} &= \left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\right)g_{\alpha\beta} \\
&= \left(\frac{\partial(x'^\alpha - \delta x^\alpha)}{\partial x'^\mu}\frac{\partial(x'^\beta - \delta x^\beta)}{\partial x'^\nu}\right)g_{\alpha\beta} \\
&= \left(\delta^\alpha_\mu - \frac{\partial \delta x^\alpha}{\partial x'^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial \delta x^\beta}{\partial x'^\nu}\right)g_{\alpha\beta} \\
&= \left(\delta^\alpha_\mu - \frac{\partial \delta x^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial \delta x^\beta}{\partial x^\nu}\right)g_{\alpha\beta} \\
&= \delta^\alpha_\mu\delta^\beta_\nu g_{\alpha\beta} - \delta^\alpha_\mu\frac{\partial \delta x^\beta}{\partial x^\nu}g_{\alpha\beta} - \delta^\beta_\nu\frac{\partial \delta x^\alpha}{\partial x^\mu}g_{\alpha\beta} \\
&= g_{\mu\nu} - \frac{\partial \delta x^\beta}{\partial x^\nu}g_{\mu\beta} - \frac{\partial \delta x^\alpha}{\partial x^\mu}g_{\alpha\nu}
\end{aligned}$$

In the sixth line, we simply dropped the primes, as any contribution that they give is second-order or higher. So, we obtain the result

$$\delta g_{\mu\nu} = -\frac{\partial \delta x^\beta}{\partial x^\nu}g_{\mu\beta} - \frac{\partial \delta x^\alpha}{\partial x^\mu}g_{\alpha\nu}$$

and therefore

$$\begin{aligned}
g^{\mu\nu}\delta g_{\mu\nu} &= -\frac{\partial \delta x^\beta}{\partial x^\nu} g^{\mu\nu} g_{\mu\beta} - \frac{\partial \delta x^\alpha}{\partial x^\mu} g^{\mu\nu} g_{\alpha\nu} \\
&= -\frac{\partial \delta x^\beta}{\partial x^\nu} \delta^\nu_\beta - \frac{\partial \delta x^\alpha}{\partial x^\mu} \delta^\mu_\alpha \\
&= -2\frac{\partial \delta x^\mu}{\partial x^\mu}
\end{aligned}$$

where in the second equation we used that upper and lower index metric are each others inverse, so we are left with a Kronecker delta. Finally, we get

$$\begin{aligned}
\delta(\sqrt{-g}\, d^4x) &= \delta(\sqrt{-g})d^4x + \sqrt{-g}\delta(d^4x) \\
&= \frac{1}{2\sqrt{-g}}\delta(-g)d^4x + \sqrt{-g}\frac{\partial \delta x^\mu}{\partial x^\mu} d^4x \\
&= \frac{1}{2\sqrt{-g}} - g g^{\mu\nu}\delta g_{\mu\nu} d^4x + \sqrt{-g}\frac{\partial \delta x^\mu}{\partial x^\mu} d^4x \\
&= \frac{1}{2}\sqrt{-g} \cdot -2\frac{\partial \delta x^\mu}{\partial x^\mu} + \sqrt{-g}\frac{\partial \delta x^\mu}{\partial x^\mu} d^4x \\
&= -\sqrt{-g}\frac{\partial \delta x^\mu}{\partial x^\mu} d^4x + \sqrt{-g}\frac{\partial \delta x^\mu}{\partial x^\mu} d^4x \\
&= 0
\end{aligned}$$

So $\sqrt{-g}\, d^4x$ is indeed invariant under a general coordinate transformation, as desired.

8.2 Shifting phase

In this section, we try solving the problem of $\hat{\phi}(\vec{x}, u)$ and $\hat{\pi}(\vec{x}, u)$ commuting as follows. Instead of letting the transformation $\vec{k} \to -\vec{k}$ conjugate $\phi_{\vec{k}}(u)$, we attempt to let it be a phase shift. However, as we will see, this will not solve the problem.
Recall that

$$\phi_{\vec{k}}(u, v) = e^{ik_v v}\chi(u) \tag{50}$$

Note that $k_v \to -k_v$ now does not lead to complex conjugation, but to shifting by a factor $e^{-2ik_v v}$. So, instead of using conjugation, we set (with a three-dimensional vector $\vec{k} = (k_x, k_y, k_v)$)

$$\phi_{-\vec{k}}(u) = e^{-2ik_v v}\phi_{\vec{k}}(u) \tag{51}$$

So let's apply this: we have (due to (50))

$$\pi_{\vec{k}}(u) = -2\partial_v \phi\sqrt{-g} = -2 \cdot ik_v\phi_{\vec{k}}(u)\sqrt{-g}$$

and hence the commutation relation (34) turns into

$$\begin{aligned}
i\hbar &= \phi_{\vec{k}}(u) \cdot 2ik_v\phi^\star_{\vec{k}}(u)\sqrt{-g} - (e^{-2ik_v v}\phi_{\vec{k}}(u))^\star \cdot 2ik_v\phi_{-\vec{k}}(u)\sqrt{-g} \\
&= 2ik_v|\phi_{\vec{k}}(u)|^2\sqrt{-g} - e^{2ik_v v}\phi^\star_{\vec{k}}(u) \cdot 2ik_v e^{-2ik_v v}\phi_{\vec{k}}(u)\sqrt{-g} \\
&= 0
\end{aligned}$$

Note that in the first line, the minus sign in $\pi_{-\vec{k}}(u)$ cancels, as the k_v becomes $-k_v$. So this once again leads us to a contradiction and does not resolve the problem.
Another attempt to get rid of the conjugation is by transforming the Fourier transform in Cartesian coordi-

nates into one in light-cone coordinates. Note that we have $z = (u + v)/2$, so we have

$$\begin{aligned}
\phi &= \int \frac{1}{(2\pi)^3} e^{i(k_x x + k_y y + k_z z)} \phi_{\vec{k}}(t)\, d^3k \\
&= \int \frac{1}{(2\pi)^3} e^{i(k_x x + k_y y + k_z (u+v)/2)} \phi_{\vec{k}}(t)\, d^3k \\
&= \int \frac{1}{(2\pi)^3} e^{i(k_x x + k_y y + k_z v/2)} e^{ik_z u/2} \phi_{\vec{k}}(t)\, d^3k
\end{aligned}$$

and we note that this looks a lot like the light-like Fourier transform (19), but now with an additional factor $e^{ik_z u/2}$ in the $\phi_{\vec{k}}(t)$. This suggests that

$$\phi_{\vec{k}}(u) = e^{ik_z u/2} \phi_{\vec{k}}(t)$$

where on the left the $\vec{k} = (k_x, k_y, k_v)$ and on the right $\vec{k} = (k_x, k_y, k_z)$. This would imply that

$$\phi_{-\vec{k}}(u) = e^{-ik_z u} \phi^\star_{\vec{k}}(u) \tag{52}$$

And this gives us

$$\begin{aligned}
i\hbar &= \phi_{\vec{k}}(u) \cdot 2ik_v \phi^\star_{\vec{k}}(u) \sqrt{-g} - (e^{-ik_z u} \phi^\star_{\vec{k}}(u))^\star \cdot 2ik_v \phi_{-\vec{k}}(u) \sqrt{-g} \\
&= 2ik_v |\phi_{\vec{k}}(u)|^2 \sqrt{-g} - e^{ik_z u} \phi_{\vec{k}}(u) \cdot 2ik_v e^{-ik_z u} \phi^\star_{\vec{k}}(u) \sqrt{-g} \\
&= 0
\end{aligned}$$

producing the same contradiction we found earlier. Note that removing the conjugate on the right hand side in 52 does not help: the outcome is still zero.
All in all, this approach does not resolve the problem of the 'commutator' (34) being zero.

8.3 Limit of no wave

In this section, we discuss what happens in the limit $h_+ \to 0$. We will study this in Cartesian coordinates, for simplicity. Equation (36) turns into

$$(-k_x^2 - k_y^2 - k_z^2)\phi_{\vec{k}}(t) - \frac{\partial^2 \phi_{\vec{k}}(t)}{\partial t^2} = 0 \tag{53}$$

Setting $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, this turns into

$$\partial_t^2 \phi_{\vec{k}}(t) + k^2 \phi_{\vec{k}}(t) = 0$$

so we have

$$\phi_{\vec{k}}(t) = Ae^{ikt} + Be^{-ikt}$$

Note that the transformation $\vec{k} \to -\vec{k}$ does not change this function!
We have

$$\pi_{\vec{k}}(t) = \partial_t \phi_{\vec{k}}(t) = Aike^{ikt} - Bike^{-ikt}$$

and therefore, the commutator (34) (note that the derivation of this in Cartesian coordinates is the exact same as in light-cone coordinates) becomes

$$\begin{aligned}
i\hbar &= \phi_{\vec{k}}(u)\pi^\star_{\vec{k}}(u) - \phi^\star_{-\vec{k}}(u)\pi_{-\vec{k}}(u) \\
&= (Ae^{ikt} + Be^{-ikt}) \cdot (-A^\star ike^{-ikt} + B^\star ike^{ikt}) - (A^\star e^{-ikt} - B^\star e^{ikt}) \cdot (Aike^{ikt} - Bike^{-ikt}) \\
&= -ik|A|^2 + ik|B|^2 + AB^\star ike^{2ikt} - A^\star Bike^{-2ikt} - ik|A|^2 + ik|B|^2 - AB^\star ike^{2ikt} + A^\star Bike^{-2ikt} \\
&= -2ik|A|^2 + 2ik|B|^2
\end{aligned}$$

hence

$$|B|^2 - |A|^2 = \frac{\hbar}{2k} \tag{54}$$

and since the energy is proportional to $E \sim |A|^2 + |B|^2$, the minimum energy is obtained when $A = 0$, so we get

$$|\phi_{\vec{k}}(t)|^2 = |Be^{-ikt}|^2 = |B|^2 = \frac{\hbar}{2k} \tag{55}$$

In the limit $h_+ \to 0$, however, our solution (40) becomes

$$|\phi|^2 = \frac{\hbar}{2k_v}$$

This solution is exactly the same, as (55) but now with k replaced by k_v. It does not really make sense for k_v to equal k, as it would mean that k_v depends on k_x and k_y.
However, this result was obtained with $\pi = \partial_u \phi$. If we switch to $\pi = \partial_t \phi$, we get equation (42) which reduces to

$$|\phi|^2 = -\frac{4\hbar k_v}{4k_v^4 + k_x^2 + k_y^2}$$

This looks a lot more similar to equation (55), so let's solve for k_v in terms of k_x, k_y and k_z:

$$\begin{aligned} -\frac{4\hbar k_v}{4k_v^2 + k_x^2 + k_y^2} &= \frac{\hbar}{2k} \\ -8k_v k &= 4k_v^2 + k_x^2 + k_y^2 \\ 4k_v^2 + 8k_v k + k^2 &= k_z^2 \\ 4(k_v + k)^2 &= k_z^2 + 3k^2 \\ k_v &= -k \pm \frac{1}{2}\sqrt{k_z^2 + 3k^2} \end{aligned}$$

Although this is a solution, it again depends on k_x and k_y, which was not expected to happen. The most logical explanation is that k_v was not originally defined as a wave number here, but rather as an exponential coefficient, which we assumed corresponds to the wave number. So it might actually not correspond to a wave number after all.

8.4 Separation of variables

In this appendix, we attempt to solve the Klein-Gordon equation (12) using separation of variables. This is a very frequently used approach to such a differential equation, and it was done in [3]. However, this computation had several fundamental flaws in it. First of all, a complex scalar field was used, instead of a real field. This does not make sense, since we are in classical field theory, and we are using a real metric. Secondly, the coefficients used for g_{xx} and g_{yy} are the squares of what we obtained for a plane gravitational wave. This means that the trace of the metric (which is $g_{\mu\mu}$) is no longer equal to 2, but now equal to $2 + 2h_+^2 c^2$. However, the trace of $g_{\mu\nu}$ should not be changed by a gravitational wave.
One might think that these are silly objections. We could just take the real part of the complex field, and the perturbation in the trace of $g_{\mu\nu}$ is second order in h_+ and hence negligible. However, we claim that if we replace the metric by a real metric and remove the squares in the coefficients, we no longer obtain the results of the original calculations[21].
So let's start redoing the calculation. Recall that we want to solve the Klein-Gordon equation for a massless field $\Box\phi = 0$, so we get

$$\left(\frac{\partial_x^2}{1 + h_+ c} + \frac{\partial_y^2}{1 - h_+ c} + 4\partial_u \partial_v + \frac{2h_+^2 cs\omega}{1 - h_+^2 c^2} \right) \phi = 0$$

[21] Actually, if we do not remove the squares, we obtain similar results: see Section 8.5.

which we can simplify into

$$\left[(1-h_+c)\partial_x^2 + (1+h_+c)\partial_y^2 + 2h_+^2cs\omega\partial_v + 4(1-h_+^2c^2)\partial_u\partial_v\right]\phi = 0 \tag{56}$$

So let's solve this. First, due to the so far unbroken symmetry in the x and y axis, we require that

$$(\partial_x^2 - \partial_y^2)\phi = 0 \tag{57}$$

and this way we get

$$\left[\partial_x^2 + \partial_y^2 + 2h_+^2cs\omega\partial_v + 4(1-h_+^2c^2)\partial_u\partial_v\right]\phi = 0 \tag{58}$$

Now we apply the same separation of variables as has been used in [3]. So, we set

$$\phi = X(x)Y(y)U(u)V(v)$$

and this separates into

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + 2h_+^2cs\omega\frac{1}{V}\frac{\partial V}{\partial v} + 4(1-h_+^2c^2)\frac{1}{UV}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} = 0 \tag{59}$$

Now the leftmost term depends on x only. So this term is constant, call it C. The remaining differential equation is well known: its solutions depend on the value of the constant C. If $C > 0$, then write $C = D^2$, then the general solution is

$$X(x) = Ae^{Dx} + Be^{-Dx}$$

However, this solution is physically unacceptable, as it blows up as x goes to ∞ or $-\infty$. Note that one of the terms may be zero, but then it will still blow up in one of the limits: if both terms are zero we are left with the trivial solution $\phi = 0$. So, this case can be eliminated.

Hence, $C \leq 0$. If $C = 0$, the solutions are linear functions: however, any non-constant linear function will blow up in the limit $x \to \infty$, and is therefore not acceptable. So in that case, $X(x)$ is constant. Now if we assume $C < 0$, the solution is a plane wave. Any constant factor can be absorbed into U and V. Any phase can be absorbed by shifting the origin. So, we get

$$X(x) = \cos(p_x x)$$

Note that the constant solution can be written this way by setting $p_x = 0$, so all physical solutions can be written this way. The same argument goes for Y:

$$Y(y) = \cos(p_y y)$$

Note that (57) implies that $p_x = p_y$, so we will only use p_x from now on. Equation (59) reduces to

$$2h_+^2cs\omega\frac{1}{V}\frac{\partial V}{\partial v} + 4(1-h_+^2c^2)\frac{1}{UV}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} = p_x^2 \tag{60}$$

We try to solve this for $V(v)$:

$$\frac{1}{V}\frac{\partial V}{\partial v}\left(2h_+^2cs\omega + 4(1-h_+^2c^2)\frac{1}{U}\frac{\partial U}{\partial u}\right) = p_x^2$$

$$\frac{1}{V}\frac{\partial V}{\partial v} = \frac{p_x^2}{2h_+^2cs\omega + 4(1-h_+^2c^2)\frac{1}{U}\frac{\partial U}{\partial u}}$$

Now as the right hand side depends on u only, it is constant in v. So let's call it D. Then we get

$$\frac{1}{V}\frac{\partial V}{\partial v} = D$$

and if $D \neq 0$ the only solution to this equation is

$$V = e^{Dv}$$

where we absorbed a constant factor into V. However, this solution is physically unacceptable, as it blows up at $+\infty$ or $-\infty$ (which one depends on the sign of D). So we must have $D = 0$. But then $p_x^2 = 0$, and we get that X, Y and V are all constant. If we plug all of this in, we obtain $0 = 0$, so $U(u)$ can be any random function. So the only solutions that we obtain are those with $\phi(x^\mu) = \phi(u)$, which are trivial solutions. We conclude that when correcting the two mistakes, we no longer obtain the non-trivial solutions mentioned in [3] using separation of variables.

8.5 Separation of variables with squared coefficients

In this appendix we will use the originally proposed metric in [3], and show that it only admits trivial solutions if we use a real scalar field.
Write $a = 1 + h_+c$ and $b = 1 - h_+c$, then our metric is

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & b^2 \end{pmatrix}$$

Note that this metric gives us $g = -\frac{1}{2} \cdot \frac{1}{2} \cdot a^2 \cdot b^2 = -\frac{1}{4}a^2b^2$, so $\sqrt{-g} = \frac{1}{2}ab$. The inverse $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1/a^2 & 0 \\ 0 & 0 & 0 & 1/b^2 \end{pmatrix}$$

Plugging this into (12) we obtain

$$\frac{2}{ab}\left(\partial_x \frac{1}{a^2} \cdot \frac{1}{2}ab\partial_x + \partial_y \frac{1}{b^2} \cdot \frac{1}{2}ab\partial_y + \partial_u 2 \cdot \frac{1}{2}ab\partial_v + \partial_v 2 \cdot \frac{1}{2}ab\partial_u\right)\phi = 0 \tag{61}$$

Now we use the chain rule to simplify this. Recall that a and b depend on u only, so only the term with ∂_u on the left splits into two terms with the product rule, for the other terms we can simply pull the factor out. We obtain

$$\left(\frac{1}{a^2}\partial_x^2 + \frac{1}{b^2}\partial_y^2 + \frac{2}{ab}\partial_u(ab)\partial_v + 4\partial_u\partial_v\right)\phi = 0$$

and after multiplying by a^2b^2 we get

$$\left(b^2\partial_x^2 + a^2\partial_y^2 + 2ab\partial_u(ab)\partial_v + 4a^2b^2\partial_u\partial_v\right)\phi = 0$$

Recall that the x and y-directions are physically identical at this point, so we require $(\partial_x^2 - \partial_y^2)\phi = 0$. At this point, we plug in our functions a and b: this gives us

$$\Big((1 - 2h_+\cos(ku) + h_+^2\cos^2(ku))\partial_x^2 + (1 + 2h_+\cos(ku) + h_+^2\cos^2(ku))\partial_y^2 + \\ 2(1 - h_+^2\cos^2(ku))\partial_u(1 - h_+^2\cos^2(ku))\partial_v + 4(1 - h_+^2\cos^2(ku))^2\partial_u\partial_v\Big)\phi = 0 \tag{62}$$

We can simplify this as follows:

$$\Big((1 + h_+^2\cos^2(ku))\partial_x^2 + (1 + h_+^2\cos^2(ku))\partial_y^2 + 2(1 - h_+^2\cos^2(ku)) \cdot 2h_+^2 k\sin(ku)\cos(ku)\partial_v + \\ 4(1 - h_+^2\cos^2(ku))^2\partial_u\partial_v + 2h_+\cos^2(ku)(\partial_y^2 - \partial_x^2)\Big)\phi = 0 \tag{63}$$

Now the last term cancels due to our requirement that $(\partial_x^2 - \partial_y^2)\phi = 0$.
To shorten this long equation, define

$$F(u) = (1 - h_+^2\cos^2(ku))^2, \qquad G(u) = h_+^2(1 - h_+^2\cos^2(ku))\sin(ku)\cos(ku), \qquad H(u) = 1 + h_+^2\cos^2(ku)$$

This way, we obtain

$$\left[H(u)(\partial_x^2 + \partial_y^2) + 4kG(u)\partial_v + 4F(u)\partial_u\partial_v\right]\phi = 0 \tag{64}$$

Next, we make our familiar separation of variables $\phi = X(x)Y(y)U(u)V(v)$. If we divide the equation we obtain by that substitution by $X(x)Y(y)U(u)V(v)H(u)$, we get a term that depends on X only, and a term that depends on Y only:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + 4k\frac{G(u)}{H(u)}\frac{1}{V}\frac{\partial V}{\partial v} + 4\frac{F(u)}{H(u)}\frac{1}{UV}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} = 0$$

Now for X, we obtain $\partial_x^2 X = cX$ with c a separation constant. Similar to the argumentation in section 8.4, we have $c \leq 0$ for any relevant solutions. Hence, we write $c = -p^2$. As $(\partial_x^2 - \partial_y^2)\phi = 0$, we have $\partial_y^2\phi = -p^2\phi$. So we get

$$X(x) = \cos(px), \qquad Y(y) = \cos(py)$$

Again, any phase is set to zero by moving the origin. If this worries you, you can just add a phase and it won't affect any further calculations.
With this, our equation becomes

$$-2p^2 H(u) + 4kG(u)\frac{1}{V}\frac{\partial V}{\partial v} + 4F(u)\frac{1}{UV}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} = 0$$

Separating out V:

$$\frac{1}{V}\frac{\partial V}{\partial v}\left(4kG(u) + 4F(u)\frac{1}{U}\frac{\partial U}{\partial u}\right) = 2p^2 H(u)$$

and we see that the V part here actually has to be constant. The only solution to this differential equation is an exponential function, which is physically unacceptable, as it blows up as v goes to ∞ or $-\infty$ (one of the two, which one depends on the sign of the part between the large brackets). The only way out of this is if $p = 0$. But then X, Y and V are all constant, and we are again left with only trivial solutions $\phi = U(u)$.

If we compare our computation to the original computation in [3], we note that the machine fails when we try to compute $V(v)$. In the paper, a complex field was used, so we could use a complex exponential to solve the differential equation for V. However, this is obviously not allowed here. It was probably thought that if we replaced the complex field by a real field, the complex exponentials would become sines and cosines. However, as it turns out, this does not happen.

8.6 Bibliography

References

[1] Ruxun Liu, D'ALEMBERT FORMULA AND THE DIRECT OR INVERSE PROBLEMS FOR WAVE EQUATION, Acta Mathematica Scientia, Volume 10, Issue 1, 1990, Pages 47-56, ISSN 0252-9602, https://doi.org/10.1016/S0252-9602(18)30378-3.

[2] Maggiomore, Michele. (2007). Gravitational waves. Volume 1: Theory and Experiments. Oxford University Press.

[3] Jones, Preston & McDougall, Patrick & Ragsdale, Michael & Singleton, Douglas. (2017). Scalar field vacuum expectation value induced by gravitational wave background. Physics Letters B. 10.1016/j.physletb.2018.04.055.

[4] Arutyunov, Gleb & Stoof, Henk. (2015). Classical field theory. Utrecht University.

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