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Electromagnetic memory in an expanding universe.

Bachelor Thesis

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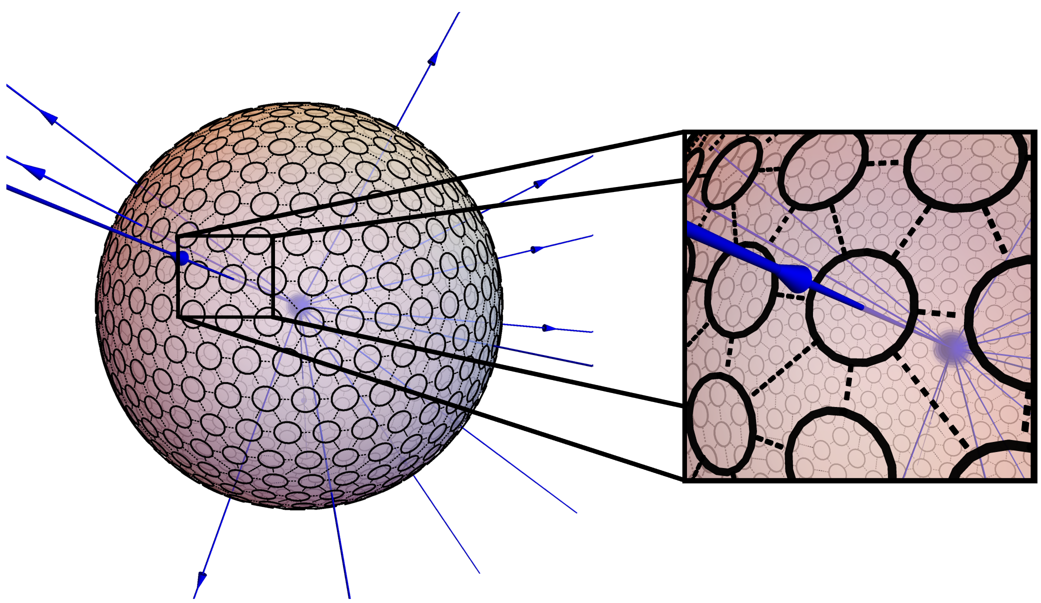
Physics



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Abstract

The memory effect is the total change in fields over infinite time and has been developed in detail for gravitation. For electromagnetism the memory effect is a very recent study. In this paper we analyze and calculate the memory as proposed by Susskind [1]. This memory is embedded in a spherical shell of superconducting nodes with penetration depth λ. Between these nodes a current will flow that measures the memory. This current is proportional to the gradient of the superconductor phase ϕ. We also extend this memory to an expanding universe, since this connection has not yet been made for the electromagnetic memory. As a charge source we take a single charged particle moving in the $\hat{z}$ direction. In Minkowski space we find an explicit expression for the final phase which causes the memory current. For the expanding universe calculation we assumed the universe consists of a single component and undergoes decelerated expansion. We also look at times $t \gg \lambda$ only. Under these assumptions we have also found an explicit expression for the memory effect in the expanding universe. Both solutions oscillate in time, so the current we measure depends on the time of measurement.

The image on the front pages visualises the memory effect we are calculating. An explosion in the center of a spherical shell shoots charged particles away in the radial direction. The spherical shell consists of rings made of a superconducting material. These rings are not connected. As the particles move they alter properties of the rings depending on their trajectory (blue lines). If we now connect the rings after infinite time, electrical currents will flow from one ring to the other. This current is the memory of the particles embedded in the superconductor.

Contents

1 Introduction

Imagine two negatively charged test particles. One particle (particle A) is moving with constant velocity. As particle A is moving with constant velocity it is pushing the other particle (particle B) away causing it to accelerate. Integrated over time this acceleration becomes a change in velocity. When integrating the acceleration over all time, this velocity change is called the memory effect. The memory effect is not necessarily a change in velocity only. To be precise it is the total effect of a particle on an observer long after it has passed. For gravitation this memory is actually a change in displacement instead of velocity.

The memory effect has gained popularity in the past few decades. The gravitational memory effect was first introduced by Zel'dovich and Polnarev in 1974 [2]. In 1991 Christodoulou showed that gravitational waves have a nonlinear memory effect that was not priorly known [3]. The analogy to an electromagnetic memory effect came only much later and was first analyzed by Bieri and Garfinkle in 2013 [4]. They showed that for electromagnetism there is also a memory effect, and also a nonlinear memory effect of the kind that Christodoulou discovered if there are massless charged particles.
In 2015 Susskind proceeded to propose a method of measuring the electromagnetic memory effect [1]. This memory effect is based on a system of superconductors. In a large sphere of superconductors currents will flow in a manner caused by particles moving through the sphere. This flow of currents still happens after infinite time, so it is a memory effect. To approach this we describe the superconductor as a complex scalar field exhibiting spontanuous symmetry breaking. The flow of currents is then determined by the gauge field and the argument of the complex scalar field. It is important that the currents are caused by the gauge field and not by the electromagnetic field. This means that even though at infinite time the electromagnetic field vanishes, the gauge field does not, rather it is a pure gauge. So even in the absence of electric and magnetic fields there can still be a current. This might seem to break gauge symmetry, however this problem is fixed by the complex scalar field as will be shown. To find this flow of currents both the electromagnetic gauge field caused by the particle needs to be determined, as well as the change in fields inside the superconductor itself. In Section 4.3 we analyze this method in detail and in Section 5 we provide an exact calculation. All electromagnetism memory effects in previous literature were calculated in Minkowski spacetime only, whereas for gravitation the link to an expanding universe has been made already. In this paper we generalize Susskind's proposal to an expanding universe as well.

There are multiple reasons the memory effect is interesting. The first is the ability to find new information in existing data. Imagine the two particles from the beginning of this section. Assume that we know what the velocity of particle B was before particle A entered its field of influence. This can be done for example by simply putting particle B at rest at some location ourselves. Then we can measure this same particle again after enough time, to notice that it is moving. The direction and speed with which it is moving tells us something about the behaviour of particle A. For gravitational waves this is more interesting. Let us place two test particles at rest with some distance L between them. After the gravitational wave has passed, the distance between the particles has increased to some $L + \Delta L$. By measuring this distance we find ΔL, which gives us additional properties on the shape of the wave. This in turn gives us information on the source of the wave, for example a collision of stars. Measurements of this memory have been proposed at LIGO [5], which is the gravitational wave detection centre which made the first actual measurement of a wave in 2015. The principle of gravitational memory is shortly explained in Section 3.3.
Memory also has fundamental connections to other theories. This connection is explained in much detail by Strominger [6]. The memory effect has connections to the soft photon theorem and to asymptotic symmetries. The soft photon theorem maintains that during any physical process an infinite amount of photons with zero energy is created. The properties of these photons are correlated to the physical process. The memory effect is linked to the creation and annihilation of these soft photons by a Fourier transformation. Similarly there is a connection between the memory and asymptotic symmetries. Asymptotic symmetries are symmetries of the system at very large distances. This connection provides understanding and meaning to all three theories, and gives the question of whether these kind of connections are valid for other theories as well. For gravitation and electromagnetism these connections exist as shown by [6] [7]. Strominger proposes that for other theories like quantum chromodynamics this connection might still hold, yet this remains to be proven [6]. The connection for electromagnetism will be explained in Section 3.2.

In Section 2.1 we first discuss of the mechanics of a four-dimensional spacetime and the properties of an expanding universe. Because of the difference in the nature of a time dimension and a space dimension there is classicaly unexpected behaviour in a four-dimensional spacetime. This difference arises from a different sign in the metric tensor for time and space components. We also briefly introduce the mechanics of an expanding universe, its properties and its complications. In Section 3 the memory effect itself is explained, along with the earlier work done by Bieri and Garfinkle [4] and Strominger [6]. Proceeding in Section 4 we first explain the principle of superconduction in field theory. This is in order to be able to understand the memory that is analyzed in Section 4.3 and 4.4 after introducing a calculation trick in Section 4.2. In Section 5 we calculate the memory effect in Minkowski space first and afterwards we generalize this calculation to an expanding universe in Section 6. We wrap the paper up by analyzing the similarities and differences between the calculation for an expanding and a static universe.

2 The structure of spacetime

This paper will involve spacetime mechanics of both Minkowski spacetime and an expanding spacetime. In order to calculate these spacetimes accurately analysis into their structure is needed, especially when analysing asymptotic behaviour, which is behaviour for very large radial distance r. Indeed when analysing a spacetime in the asymptotic regime the standard coordinates are not well-defined for massless particles. This can be understood in the following way. Any particles' four-momentum p_μ obeys the following relation $p_\mu p^\mu = -m^2$. This equation holds for any metric. Here the $(-,+,+,+)$ sign convention has been used. Indices are raised and lowered using the metric tensor $p_\mu = g_{\mu\nu}p^\nu$ and a repeated index implies summation. m is the rest mass of the particle. For a massless particle in particular this means $p_\mu p^\mu = 0$. Massless particles thus follow worldlines, trajectories through spacetime, which are lightlike. A lightlike path obeys the equation $\mathrm{d}s^2 = 0$, where $\mathrm{d}s$ is the physical distance in an infinitesimally small interval. Massive particles follow timelike trajectories where $\mathrm{d}s^2 < 0$ since then $-m^2 < 0$. The physical distance between two infinitely close events is given by the metric by

$$\mathrm{d}s^2 = g_{\mu\nu}x^\mu x^\nu. \tag{2.1}$$

Here the four-vector x^ν is the coordinate of the event. An event is a physical happening in spacetime. It can now be anything, for example a pulse of radiation or a quantum excitation. Each event corresponds to a four-dimensional coordinate x^ν. In particular in Minkowski spacetime for a massless particle moving in a constant direction this implies $\mathrm{d}t^2 = \mathrm{d}r^2$ or $t = r + c$. This shows indeed that when taking the limit $t \to \infty$ the distance r goes to infinity *equally* fast. When analyzing asymptotic behaviour the coordinates t and r both approach infinity, making them unsuitable for asymptotic analysis. This invites us to need a different set of coordinates for massive and massless particles. In order to see why this happens and understand these spacetimes altogether we need a method of visualizing spacetimes. Because they are four dimensional we need some method to take two dimensions away without losing information. This can be done using Penrose diagrams.

2.1 Penrose diagrams

This section and Section 2.2 is based on [8] and [9]. Carter-Penrose conformal diagrams, or in short *Penrose diagrams*, are diagrams that display the properties of a 4D spacetime in a 2D plane by compactifying the three spatial dimensions into one. In practice this is done by removing angular dimensions and only taking the distance r into account. In other words they map a (1+3)-space into a (1+1)-diagram. This way each point (excluding boundaries) in a Penrose diagram in fact respresents a S^2 two-sphere. These Penrose diagrams preserve the causal structure of a spacetime, meaning in practice mostly that light cones still have to be at 45° angles, although the mathematics of causal structure is more complex. It is also important that timelike geodesics stay timelike and lightlike geodesics stay lightlike. The most important utilities of Penrose diagrams lie in the fact that all infinities that are embedded in a spacetime, as well as all singularities, are at finite distances in a Penrose diagram. Although not every spacetime can be conformally transformed in such a manner that a proper Penrose diagram can be obtained, for example for four-dimensional anti-de

Sitter spacetime [10], for more simple spacetimes it is possible, for example Minkowski spacetime and the Schwarzschild solution. The Schwarzschild solution is the metric for a spherically symmetric distributed mass at rest at the origin, for example our sun. For an expanding spacetime a general penrose diagram is possible which will be noted in Section 2.3.
Penrose diagrams can be obtained in practice using *conformal transformations*. Conformal transformations are transformations on the metric $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ which leave the coordinates unchanged. The angles between vectors are also preserved. Indeed the generalized angle between two arbitrary four vectors x^ν and y^ν is given by

$$\text{angle} = \frac{g_{\mu\nu} x^\nu y^\mu}{\sqrt{g_{\rho\sigma} x^\rho x^\sigma}\sqrt{g_{\lambda\tau} y^\lambda y^\tau}}. \tag{2.2}$$

If we now transform the metric $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, where Ω is some function, then in this equation the Ω's above cancel the ones below. These conformal transformations are fundamental for Penrose diagrams because they allow us to remove infinities. The goal is to transform the infinite radial and time coordinate to a finite coordinate while preserving the causal structure. The Penrose diagram is then obtained by drawing the range of these new finite coordinates. In fact only the radial and time component are to be drawn. This means that the direction of coordinates in $\mathbb{R}^3$ is removed. In this drawing various infinites are now actually the boundaries of the finite coordinates so they are visible within the diagram. Mostly because of this property Penrose diagrams are very important for analysing asymptotic behaviour at infinites and singularities.

Infinities

Before continuing about Penrose diagrams the different infinities used in this paper need to be defined because of their importance in asymptotic behaviour. For a general spacetime there are different kinds of infinities depending on which coordinate we choose to go to infinity. Behaviour of massless particles also adds a difficulty of t and r going to infinity equally fast so that new coordinates need to be defined. In a short overview these are given by:

- Future timelike infinity, $t \to \infty$, is the region of spacetime in the far future, denoted by i^+. All physically relevant massive particles and fields end up at future timelike infinity [11]. This is important because this means that any theory that uses massive particles or massive fields only needs to take timelike infinity into account.
- Past timelike infinity, same as future timelike except now that $t \to -\infty$, denoted by i^-. This is where all massive particles and fields originate from, massless particles not necessarilly.
- Spacelike infinity, $r \to \infty$, is the region in space where interactions vanish because the distance between all events diverges, denoted by i^0. This infinity cannot be reached in a finite time in Minkowski space.
- Future null infinity, $v = t + r \to \infty$, is the infinity where massless particles arrive after an infinitely long time. Here v is the retarded time. Future null infinity is denoted by $\mathcal{I}^+$.
- Past null infinity, where $u = t - r \to \infty$. Past null infinity is where are past-direction moving massless particles end up after enough time or where future moving massless particles start. Here u is the retarded time. Past null infinity is denoted by $\mathcal{I}^-$.

The null infinities are of special significance in theories involving massless particles. For these particles spatial coordinates are not well defined in the $t \to \infty$ limit because in this limit $r \to \infty$ as well, however u stays finite. To properly analyze the spatial behaviour of masless particles at infinity it is then useful to make an asymptotic expansion for $r \to \infty$ while keeping u constant. The behaviour for massless particles is also important for connecting the memory effect to other theories because the uncoupled gauge field of electromagnetism is massless. For massless fields the same reasoning as for massless particles holds. The memory effect is also usually analyzed in the large r limit, which means that retarded or advanced time need to be used to avoid coordinate problems [6]. This paper will use massive particles only, so for the memory effect only i^+ is important, however the different infinities are relevant for the background information on the memory effect. It is also important to note this difference for massless and massive particles, and to understand why this paper uses future timelike infinity.

2.2 Minkowski spacetime

Probably the most important Penrose diagram is that of Minkowski spacetime. The Minkowski space metric is given in spherical coordinates by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.3}$$

Here $\mathrm{d}\Omega$ is the unit metric on S^2, in spherical coordinates given by $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2$. We transform this metric into the advanced and retarded coordinates giving

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \left(\frac{v-u}{2}\right)^2 \mathrm{d}\Omega^2. \tag{2.4}$$

The coordinates u and v still range from $-\infty$ to ∞ here. These need to be transformed such that their range is finite and $\mathrm{d}\Omega$ does not change. Note that it does not matter if a factor is placed in front of the metric, because this factor can be removed using a rescaling $\mathrm{d}s^2 \to \Omega^2\mathrm{d}s^2$. In other words we can remove such a factor using a conformal transformation. This is allowed for the diagrams to work since no structure has been changed. In order to transform u and v into finite coordinates the trigonometric tangent function is especially useful. Choosing

$$u = \tan U, \qquad v = \tan V, \tag{2.5}$$

such that $-\frac{\pi}{2} < U \leq V < \frac{\pi}{2}$, so the coordinates are finitely bounded. The substitution to tangent functions could not be done on t and r, because then the 45° lightcone rule would be violated. This would mean the causal structure were altered. With these new coordinates the metric becomes

$$\mathrm{d}s^2 = \frac{1}{4\cos^2 U \cos^2 V}(-4\mathrm{d}U\mathrm{d}V + \sin^2(V-U)\mathrm{d}\Omega^2). \tag{2.6}$$

The $4\cos^2 U \cos^2 V$ prefactor can be rescaled into $\mathrm{d}s^2$ by a conformal transformation. Denote the rescaled metric by $\mathrm{d}\tilde{s}^2$. Furthermore transform the new coordinates U and V back to their original form $U = \frac{1}{2}(T-R)$ and $V = \frac{1}{2}(T+R)$ with T and R the new coordinates to give

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\mathrm{d}\Omega^2. \tag{2.7}$$

This equation is of the form of the earlier Minkowski equation with the exception of $r \to \sin R$ in front of the $\mathrm{d}\Omega$ and the important fact that the coordinates T, R are bound in the following way:

$$|T| + R < \pi, \qquad 0 \leq R < \pi. \tag{2.8}$$

Plotting these equations gives a right-pointing triangle which is the Penrose diagram for Minkowski spacetime. This triangle can be mirrored in the $R = 0$ axis to result in a diamond shape, which is also a possible and more neat Penrose diagram of Minkowski spacetime. This can be seen as letting $-\pi < R < \pi$ which means the radial coordinate is not strictly positive anymore, however a negative radial coordinate is effectively a position vector mirrored in the origin or antipodally transformed. Two antipodal points are the two points on a sphere that connect to each other through the origin so $\vec{x}$ has $-\vec{x}$ as its antipodal conjugate. When a particle reaches $r = 0$ its angular direction is also antipodally transformed. In the case of a triangle this transformation is visualized by reflection, whereas in the diamond diagram the particles just keep moving. In this sense the left side of the square diagram can be seen as the antipodal map of the right side.

Figure 1 and 2 give both possible penrose diagrams for Minkowski spacetime. The red lines are lines of constant t and the blue lines are of constant r. The worldline of an arbitrary massive particle and of a lightray have been illustrated. These two diagrams containt the exact same information since they are as explained above antipodally connected. Every point in the diagram for $r \neq 0$ represents a two sphere S^2. For figure 1 the two spheres for certain points are also connected to the other side of the diagram by antipodally mapping the entire sphere. The infinities denoted by $\mathcal{I}^{\pm}_{\pm}$ denote the boundaries of future and past null infinity and will be of importance in Section 3.2.

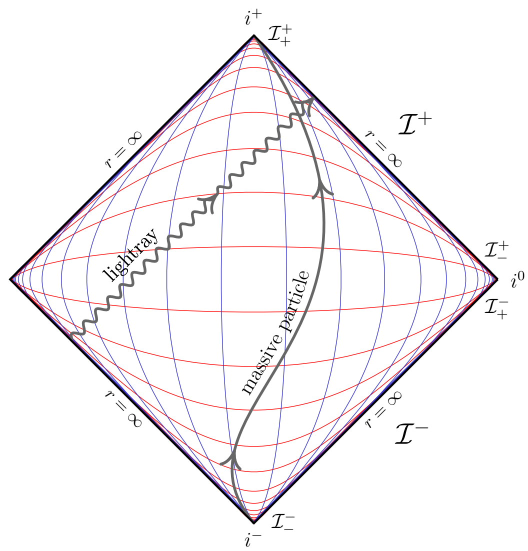

Figure 1: The other valid Minkowski spacetime Penrose diagram extended for $r < 0$ [6].

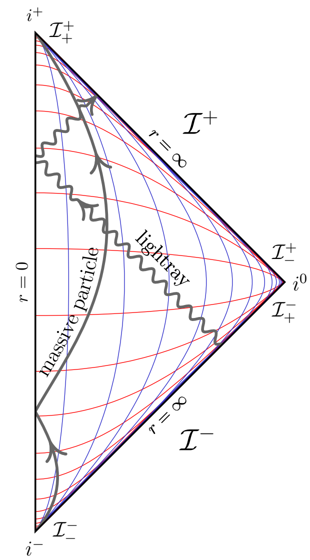

Figure 2: The Penrose diagram as obtained by the described conformal transformations [6].

The worldlines of massive particles have to start at i^- and end up at $i+$ whereas the worldlines of massless particles have to start at $\mathcal{I}^-$ and end up at $\mathcal{I}^+$. This is visualized in the graph. This shows an important difference between massive and massless particles in the way they need to be treated mathematically. When analysing the memory for massive particles the limit to future time infinity needs to be taken whereas when analysing the memory for massless particles the limit to future null infinity needs to be taken. A few other properties of Minkowski spacetime can now also easily be observed. Any timelike observer will eventually (when reaching i^+) see the entire spacetime, so for a timelike observer everything that ever happened in the spacetime can be observed. Furthermore past and future light cones intersect for any two events, meaning that for Minkowski spacetime any two events were in fact causally connected in the past. This is not necessarily true for all spacetimes, for example a closed matter dominated universe [9].

2.3 Expanding universe

The *Friedmann-Lemaître-Robertson-Walker* or FLRW metric is a solution to the Einstein field equations for an *isotropic* and *homogenuous* universe. On a large scale this approximation is correct since galaxies are very small compared to the size of the observable universe. The generalized metric makes no assumption on the flatness of space, however in this paper no curvature is assumed. In this case the metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right). \tag{2.9}$$

Here $a(t)$ is the *scale factor* of cosmology and it is determined by the constituents of the universe using the Friedmann equations. This scale factor is of fundamental importance for how an expanding universe behaves. For our universe the scale factor vanishes at $t = 0$, which corresponds to the big bang. This is not necessarely true for all possible FLRW metric solutions, however for the ones we are discussing this will be true, as will be explained later. This scale factor describes the expansion of space. To proceed the assumption that $a(t)$ increases monotonically is made, so the function is bijective. This allows us to rewrite the metric into the *conformal time* τ which obeys $\mathrm{d}t \to a(\tau)\mathrm{d}\tau$ to give a new metric

$$\mathrm{d}s^2 = a(\tau)^2 \left(-\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right). \tag{2.10}$$

Note that in Minkowski space the conformal time equals the normal time because then $a(t) = 1$. Also in this equation it is clearly visible that for any flat FLRW space where the transformation to conformal time is allowed, the space is conformally equivalent to Minkowski space. This property is useful for understanding the FLRW mechanics as well as for calculational purposes. It will be explained more in Section 4.2. The conformal time might have a new domain due to the transformation. To analyze this domain first the solution for $a(t)$ for a single component universe needs to be found. This function is given by the solution of the Friedmann equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.12}$$

The dot denotes a derivative over time. G is Newtons constant and ρ and p are respectively the energy density and pressure. If the universe consists of a single component, for example only matter or radiation, then the pressure and energy have the following relation

$$p = w\rho. \tag{2.13}$$

where w is a constant specific for each type of energy, $w = 0$ for matter dominated, $w = 1/3$ for radiation dominated and $w = -1$ for dark energy dominated. With this relation the Friedmann equations have an exact solution given by

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(w+1)}} \tag{2.14}$$

Here t_0 is a constant which fixes $a(t_0) = 1$. We can now also see for which values of w there is a big bang, namely those values where $\frac{2}{3(w+1)}$ is larger than zero, whereas if it is smaller than zero the universe actually tends to a singularity instead of coming from one. Our universe actually comes from a big bang and tends to a singularity as $t \to \infty$, but this is due to our universe consisting of more than one component. The domain of the conformal time can be analyzed with this solution. The definition for conformal time is the above transformation integrated:

$$\tau = \int_0^t \frac{1}{a(t)} dt. \tag{2.15}$$

This gives $\tau \in [0, \infty)$ for $w > -1/3$ and $\tau \in [0, \alpha], \alpha < \infty$ for $w < -1/3$. The value $w = -1/3$ is a critical boundary value for which $a \propto t$ which we exclude in this paper. As it turns out in Section 6 the calculations are more difficult when $w < -1/3$ because even though the physical distance of a free moving particle always goes to infinity as $t \to \infty$ (if $a \to \infty$ so in an expanding universe) the coordinate distance might not. To simplify the calculation $w > -1/3$ is now assumed, which corresponds to a *decelerated expanding universe* $\ddot{a} < 0$. In fact the actual assumption that needs to be made is only that $\ddot{a} < 0$ without assuming there is only one component. However toward the end of the paper we need to make a single component ansatz anyway. Assuming $\ddot{a} < 0$ ensures that the conformal time and coordinate distances of free moving observers go to infinite which is important for integration. This will be the system that is worked with in Section 4, 5 and 6. Our current universe is in a cosmological constant, so dark energy, dominated phase meaning this ansatz does not apply to our current state. In the present the universe is in accelerated expansion. However in the past until a few billion years ago the universe was expanding with $\ddot{a} < 0$ when the universe was matter dominated so although this calculation has no connection to our universe right now it does tell us something about the universe in the past.

The concept of distance

In the FLRW metric the concept of distance can be looked at from two different points of view. We will be looking at distances on a hypersurface of constant t, τ only. There is the coordinate or *comoving distance* from the origin x^i and the physical distance $\sqrt{x_i x^i}$. As the scale factor increases the coordinate distance for a particle at rest stays the same. This means that the expansion of space generated by the scale factor has no effect on the coordinates of a particle at rest. Nevertheless the physical distance to the particle does increase since

$$x^2 = x_i x^i = g_{ij} x^i x^j. \tag{2.16}$$

As stated before x^i does not change if a particle is at rest, however g_{ij} does. Since i, j denote the spatial part only $g_{ij} = a^2 \delta_{ij}$. This introduces a scale factor into the physical distance contrary to the comoving distance. This increase in the physical distance proportional to a produces the expansion of space, where the distance between all physical objects increases, even though their coordinates stay the same. This also introduces the problem of wanting some object at a fixed physical distance. In this case the object needs to be confined at some coordinate distance $x^i \propto a^{-1}$ so that the physical distance stays the same. This is possible through the use of non-gravitational forces such as electromagnetism, since these are not taken into account in the ansatz of a homogenuous universe.

Penrose diagram

This flat decelerated expanding FLRW universe also has a Penrose diagram that has been determined before by Kehagias and Riotte [12]. To find this Penrose diagram a transformation to retarded coordinates is used of the form $u = \tau - r$ to give

$$\mathrm{d}s^2 = a^2 \left[-\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \mathrm{d}\Omega^2 \right]. \tag{2.17}$$

This is finitely transformed in the same manner as Minkowski spacetime in section 2.2, however $\tau > 0$ whereas in Minkowski $-\infty < t < \infty$. At $\tau = 0$ there is the Big Bang, so a new type of singularity is expected at $\tau = 0$ and for the rest similarity to Minkowski can be expected. Indeed the Penrose diagram is given by figure 3.

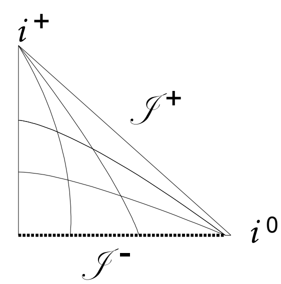

Figure 3: The penrose diagram of a decelerated expanding FLRW universe [12].

The diagram is the same as the Minkowski triangle for $\tau > 0$ with the same future timelike infinity i^+, future null infinity $\mathscr{I}^+ = \mathcal{I}^+$ and spacelike infinity i^0. For $\tau < 0$ the space is undefined and at $\tau = t = 0$ the space has a singularity which is the big bang. For this metric the same fact that massive particles have to end up at i^+ holds, even though they can start anywhere at $\mathscr{I}^-$. Also the basic principles of causality and null behaviour are the same as Minkowski space. This is very important because it means we do not have to account for any causal horizons or unexpected asymptotic behaviour in our system.

2.3.1 Scale factor in conformal time

For the calculation in Section 6 we need the scale factor as a function of conformal time for a single component universe. To do this we solve the Friedmann equation after substituting time to conformal time. This gives for the chain rule

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{1}{a}\frac{\partial}{\partial \tau}. \tag{2.18}$$

We also use the relation $p = w\rho$ again. With this the Friedmann equations can be reduced to a single nonlinear differential equation. Applying equation 2.18 as well gives

$$\frac{\partial}{\partial \tau}\left(\frac{1}{a(\tau)}\frac{\partial a(\tau)}{\partial \tau}\right) = -\frac{1+3w}{2}\frac{1}{a(\tau)^2}\left(\frac{\partial a(\tau)}{\partial \tau}\right)^2. \tag{2.19}$$

This equation has an exact solution given by

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{\frac{2}{1+3w}}. \tag{2.20}$$

Here τ_0 is a constant which chooses the initial condition $a(\tau_0) = 1$. Note that specifically for $w = 1/3$ we have $a \propto \tau$ and for $w = 0$ we have $a \propto \tau^2$.

3 The memory effect

The memory effect can be seen as the total change that a particle has caused to physical measurable properties of an observer long after the particle has passed. In other words even though the particle has no instantaneous effect on the observer anymore the effect that it has had in the past is still measurable. This memory of the particles effects on the observer in the past is called the memory effect. It is classicaly a known effect although the term memory for it is very recent: the change in velocity of a test particle due to some force is also a memory effect! In this case it is the memory of the particle which causes the force field onto the test particle. Gravitational waves similarly leave a memory effect on two test particles. Long after the wave train has passed the particles have gained a permanent displacement, their relative distance has increased. Electromagnetic waves have an effect on test particles too, they cause a permanent displacement of test particles after a wave train has passed, however because of electromagnetic noise this effect is impossible to measure [4]. Electromagnetic waves however have a fundamental difference with respect to gravitational waves: for electromagnetic waves a permanent change in velocity, *a velocity kick*, is allowed, whereas for gravitational waves this is forbidden and only permanent displacements are allowed [4]. In this paper the memory effect is analyzed from a slightly different point of view inspired by a note of Susskind as stated in the introduction [1]. The principle is that a moving charged test particle leaves a change in the phase of a superconductor. This phase can be measured as currents running through the superconductor. Although this is different than described before, this is also a memory effect, because it is the change that a test particle causes to a measurable field, namely the phase of the superconductor (or more precisely the gradient of this phase). This process and theory will be explained in much more detail in Section 4.

3.1 Electromagnetic memory

The memory effect of electromagnetism has been analyzed in detail by Bieri and Garfinkle [4]. This section explains their article in depth to give a quantitative understanding of what the memory effect is. The angular coordinates used are a set of complex stereographic coordinates that is useful for calculations, they are explained in Appendix A. The memory effect derived here is the completely general effect for arbitrary charge distribution in Minkowski spacetime for a test particle close to $\mathcal{I}^+$, so that the limit $r \to \infty$ needs to be taken, and u replaces the time coordinate. First the electric and magnetic fields are asymptotically expanded in spherical coordinates, taking only lowest order terms into account:

$$E_r = \frac{E_r^{(2)}}{r^2} + \dots, \qquad E_z = E_z^{(0)} + \dots, \tag{3.1}$$

$$B_r = \frac{B_r^{(2)}}{r^2} + \dots, \qquad B_z = B_z^{(0)} + \dots, \tag{3.2}$$

$$J^0 = \rho = \frac{\rho^{(2)}}{r^2} + \dots, \qquad J^r = J_r = \frac{J_r^{(2)}}{r^2} + \dots, \qquad \rho^{(2)} = J_r^{(2)}, \tag{3.3}$$

and angular current terms are negligible. Here the carthesian components of the magnetic and electric fields are assumed to go as $\frac{1}{r}$, so the angular parts go as r^0 (this follows from the general tensor transformation). Then it follows from the equations of motion that the radial components behave as $\frac{1}{r^2}$. Now the assumption that the four-current consists of *massless particles* is made to ensure there is a nonlinear effect as well. No massless charged particles have been discovered yet, however there is no proof against their existence either. In the case of massive particles the charge density has to fall off faster than any power of r. Substituting advanced and retarded coordinates gives $J_u = -\frac{1}{2}(J_r + \rho), J_v = \frac{1}{2}(J_r - \rho)$. This means that J_v along with J_z can be ignored, and only $J_u = -J_r^{(2)} r^{-2}$ is of importance. Inserting the expansions into the Maxwell equations and leaving out equations that give identical results later on gives:

$$-\partial_u E_r^{(2)} + D^z E_z^{(0)} = J_r^{(2)}, \tag{3.4}$$

$$-\partial_u B_r^{(2)} + D^z B_z^{(0)} = 0, \tag{3.5}$$

$$\partial_u E_z^{(0)} - \epsilon_z{}^{\bar{z}} \partial_u B_{\bar{z}}^{(0)} = 0. \tag{3.6}$$

Here D_z is the *covariant derivative* on S^2 with metric $\gamma_{z\bar{z}}$. The covariant derivative is a derivative that takes the metric into account. It is defined through the use of Christoffel symbols. The repeated index over z

implies summation over z and $\bar{z}$. The ϵ is the antisymmetric levi-civita tensor on S^2. The last equation can easily be solved by $B_z^{(0)} = E_{\bar{z}}^{(0)}$. This gives a new smaller set of equations:

$$-\partial_u E_r^{(2)} + D^z E_z^{(0)} = J_r^{(2)}, \tag{3.7}$$

$$\partial_u B_r^{(2)} + \epsilon^{z\bar{z}} D_z E_{\bar{z}}^{(0)} = 0. \tag{3.8}$$

Define new quantities $S_z = \int_{-\infty}^{\infty} E_z^{(0)} \mathrm{d}u$ and $Q = \int_{-\infty}^{\infty} J_r^{(2)} \mathrm{d}u$. Integrating the previous equations with respect to u over $\mathbb{R}$ gives new equations in terms of S and Q:

$$D_z S^z = (E_r^{(2)}(\infty) - E_r^{(2)}(-\infty)) + Q, \tag{3.9}$$

$$\epsilon^{z\bar{z}} D_z S_{\bar{z}} = B_r^{(2)}(\infty) - B_r^{(2)}(-\infty). \tag{3.10}$$

Note that integration over u is natural since the asymtotic behaviour near $\mathcal{I}^+$ is analyzed. The velocity change $\Delta v = \int_{-\infty}^{\infty} a \mathrm{d}u$ where the acceleration $a = \frac{qE}{m}$ can be written neatly because of the definition of S_z into

$$\boxed{\Delta v = \frac{q}{mr}|S_z|}. \tag{3.11}$$

Where $|S_z| = \sqrt{2\gamma^{z\bar{z}} S_z S_{\bar{z}}}$. Note that the norm is simply the tensor contraction over the unit S^2 metric. Because of this $|S_z|$ is not dependend on r anymore. This is the explicit formula for the memory effect. Note that this equation is nothing else than the acceleration integrated over all time, where the accerelation is given by Newtons equation. To have quantitative information on Δv, S_z needs to be determined.

Now to calculate S_z only systems are considered that at asymptotic times consist of widely seperated charges moving with constant velocity. From the Liénard Wiechert solution of electromagnetism the radial component of the magnetic field falls away. This follows from the fact that $B_r = (\vec{\beta} \times \vec{E}) \cdot \hat{r}$ and $\vec{E} \propto (r\hat{r} - \vec{\beta} t)$ for a single particle. Some vector calculus shows that $B_r = 0$ and from superposition $B_r = 0$ for many particles as well. This is because these particles are widely seperated so that they do not interact. The Liénard Wiechert solution for the electric field is given by equation 3.18. Now it can be concluded that $B_r^{(2)}(\pm\infty) = 0$ so that:

$$\epsilon^{z\bar{z}} D_z S_{\bar{z}} = 0 \longrightarrow S_z = D_z \Psi. \tag{3.12}$$

With Ψ a real scalar field that now need to be determined. From equation 3.9 it follows that

$$D_z D^z \Psi = (E_r^{(2)}(\infty) - E_r^{(2)}(-\infty)) + Q. \tag{3.13}$$

Now from the maxwell equation $\partial_i E_i = \rho$ it follows from the divergence theorem that

$$\int_V \rho \mathrm{d}V = \int_V \partial_i E_i \mathrm{d}V = \int_{\delta V} E_r \mathrm{d}\Omega, \tag{3.14}$$

and thus also that $\int_{S^2} Q \mathrm{d}\Omega = \int_{S^2} (E_r^{(2)}(-\infty) - E_r^{(2)}(\infty)) \mathrm{d}\Omega$, meaning that the left hand side of equation 3.13 integrated over all solid angles is zero. Let subscript [0] denote the average of a quantity over the two-sphere, then it follows from equation 3.13 that $(E_r^{(2)}(-\infty) - E_r^{(2)}(\infty))_{[0]} = Q_{[0]}$. Adding this to equation 3.13 alongside with a redefinition $\Psi = \Psi_1 + \Psi_2$ gives:

$$D_z D^z \Psi_1 = (E_r^{(2)}(\infty) - E_r^{(2)}(-\infty)) - (E_r^{(2)}(\infty) - E_r^{(2)}(-\infty))_{[0]}, \tag{3.15}$$

$$D_z D^z \Psi_2 = Q - Q_{[0]}, \tag{3.16}$$

and $S_z = S_{1z} + S_{2z} \Longleftrightarrow S_{1z} = D_z \Psi_1, S_{2z} = D_z \Psi_2$. These equations can be solved using multiple methods, where the result will still depend linearly on the sources. Now after determining S_z there are two different kind of memories. The first one belonging to Ψ_1 is the familiar change in velocity obtained by integrating the electrical field. Note that in classical mechanics the integral of the force is simply the change in velocity over some interval times the mass. In the same sense is this memory purely classically propertional to the change in velocity over all time. This is the memory that should be familiar, however there is also a memory analoguous to the nonlinear memory that Christodoulou derived for gravitational waves which is governed by Ψ_2 [4]. This memory fails the classical notion since it assumes lightlike charged particles for which a relativistic theory is necessary. Although no lightlike charged particles have been observed up to date there is no proof that they cannot exist. Equation 3.16 is only dependent on the amount of lightlike particles that have radiated away to future null infinity, so the null kick is only caused by the amount of charge that has radiated away to null infinity. This means that the memory effect is not only dependent on the electromagnetic field generated by the particles, but also on the particles themself. In the assumption that no lightlike charged particles exist the memory is only given by the integrated Lorentz force law.

3.2 The infrared triangle

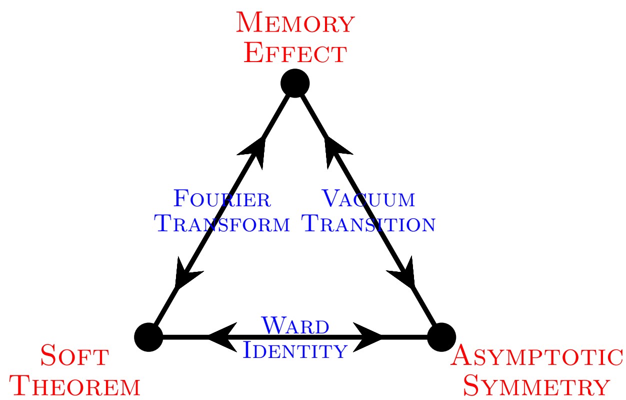

Figure 4: The infrared triangle which visually illustrates the connection between the three theorems [6].

The memory effect appears to have connections to multiple other theories as well. These connections are valid for both electromagnetism and gravity [7]. In his lecture notes Strominger explains the relation between the memory effect, *asymptotic symmetries* and *soft photon theorems* in the infrared limit [6]. The lecture notes are the source of information for this section and Section 3.3. The infrared limit means that low energy situations are analyzed which corresponds to looking at long distance asymptotic behaviour only. The memory effect is related to soft photon theorems through a Fourier transform, and the soft photon theorem is related to asymptotic symmetries through ward identities. This gives some fundamental meaning to all three of these theories. The soft photon theorem charactizes quantum scattering processes when an external massless particle reaches zero energy. From this it also follows that an infinite amount of soft photons will be created as we will see. Ward identities are identities which relate the quantum scattering amplitudes of a scattering process. In this section the connection to asymptotic symmetries and soft photon theorem is explained shortly. Let the system be described by a standard electromagnetism action in Minkowski space:

$$S = \int \left[-\frac{1}{2} F \wedge \star F \right] + S_M. \tag{3.17}$$

with S_M a general matter action so that the equations of motions are given by $\mathrm{d} \star F = \star J$ with $J^\nu = \frac{\delta S_M}{\delta A_\nu}$. The notation is that of differential forms as explained in Appendix B. Let the current be given by n charged noninteracting particles with charge Q_k and velocity β_k. Then the solution of the equations of motion is given by Liénard and Wiechert:

$$F_{it}(\vec{x}, t) = \frac{1}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (\vec{x} - t\vec{\beta}_k)}{|\gamma_k^2 (t - r\hat{x} \cdot \vec{\beta}_k) - t^2 + r^2|}. \tag{3.18}$$

This formula is not *single-valued* and has a *discontinuity* at i^0. This is important since it possesses asymptotic symmetries that follow from this fact. Evaluating the limit of fixed u and $r \to \infty$, and afterwards $u \to -\infty$

(denote this limit by $\mathcal{I}^+_-$) gives for the r component

$$F_{rt}\bigg|_{\mathcal{I}^+_-} = \frac{1}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1-\hat{x}\cdot\vec{B}_k)}, \tag{3.19}$$

wheres when letting $r\to\infty$ with v constant and then $v\to\infty$ afterwards (denoted by $\mathcal{I}^-_+$) gives

$$F_{rt}\bigg|_{\mathcal{I}^-_+} = \frac{1}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1+\hat{x}\cdot\vec{B}_k)}. \tag{3.20}$$

These two are in a general case clearly not equal. This means that the function is not continuous at the infinitely small transition between $\mathcal{I}^-_+$ and $\mathcal{I}^+_-$ (see Figure 1), however they are linked to each other through the *antipodal matching condition*:

$$F_{ru}|_{\mathcal{I}^+_-}(\vec{x},t) = F_{rv}|_{\mathcal{I}^-_+}(-\vec{x},t). \tag{3.21}$$

From this matching condition it follows that there is an infinite amount of conserved charges, one for each ϵ defined by $\epsilon(z,\bar{z})|_{\mathcal{I}^+_-} = \epsilon(z,\bar{z})|_{\mathcal{I}^-_+}$. The charge corresponding to each of these ϵ is given by $Q^+_\epsilon = \int_{\mathcal{I}^+_-}\epsilon\star F$. Using the equations of motion the conserved charge can be written in terms of an integral over the field and an integral over the current:

$$Q^+_\epsilon = \underbrace{-\int_{\mathcal{I}^+_-}\mathrm{d}u\mathrm{d}^2z\left(\partial_z\epsilon F^{(0)}_{u\bar{z}} + \partial_{\bar{z}}\epsilon F^{(0)}_{uz}\right)}_{\text{soft charge } Q_S} + \underbrace{\int_{\mathcal{I}^+_-}\mathrm{d}u\mathrm{d}^2z\epsilon\gamma_{z\bar{z}}J^{(2)}_u}_{\text{hard charge } Q_H}. \tag{3.22}$$

Note that in this equation the tensor field F and current J have been asymptotically expanded similar to Section 3.1. The equation has been split in a soft charge Q_S and a hard charge Q_H. The name soft charge comes from the fact that the left side of equation 3.22 connects with soft photons generated in this system. The soft charge term involves the integral $\int \mathrm{d}uF^{(0)}_{uz}$. Remembering the definition of the Faraday tensor that $F^{(0)}_{uz} = E^{(0)}_z - B^{(0)}_{\bar{z}}$ this integral is quite the same as the one in Section 3.1, so this integral is in fact the *memory effect* of electromagnetism. This means that the soft charge Q_S in equation 3.22 is linearly dependent on the memory effect only. The integral over the charge can also be written as a the limit of a Fourier transform

$$N_z = \lim_{\omega\to 0}\int_{\mathbb{R}} F^{(0)}_{uz}e^{i\omega u}\mathrm{d}u. \tag{3.23}$$

In quantum field theory the electric field can be expressed as a Fourier transform, where the components are given by creation and annihilation operators of photons. More quantitatively speaking for the gauge field (neglecting helicity)

$$A^i = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(a_pe^{ip^\mu x_\mu}\epsilon^i + a^\dagger_pe^{-ip^\mu x_\mu}\epsilon^i\right). \tag{3.24}$$

Here ϵ is a polarization vector and p^μ and x^μ are the four-momentum and coordinate vector respectively. Note that this is simply a Fourier transformation of the gauge field where the creation and annihilation operators a_p and $a^\dagger_p$ are the Fourier components. Similarly if we (inverse) Fourier transform the electric field the Fourier integral gets removed, so for the Fourier transform as given in equation 3.23 the only surviving components are the creation and annihilation operators corresponding to photons with energy $p^0 = \omega$. As such when promoting the integral in equation 3.23 to a quantum operator before taking the limit, it creates and annihiliates photons with energy ω. After taking the limit N_z in fact creates and annihilates photons with *zero energy*. This is the basis of the soft photon theorem, and since the integral over the electric field is the memory effect, the limiting Fourier transform of this memory is the connection to the soft photon theorem. The soft charge Q_s then belongs to soft photons with polarization given by $\partial_{\bar{z}}\epsilon$, meaning there is an infinite amount of soft photons, one for each ϵ. To see this note that ϵ can be chosen as the spherical harmonics $Y_{\ell m}$ so that for each ℓ and m there is an independent function ϵ since the spherical harmonics are orthogonal. Clearly the connection between the memory effect and the soft photon theorem is actually very straightforward in its most basic form.

Asymptotic symmetries

The antipodal matching condition invites us to find asymptotic symmetries. Indeed these asymptotic symmetries are given by the functions $\epsilon(z,\bar{z})$. This symmetry has a corresponding conserved charge Q_ϵ^+ as a generator, of which the explicit form was given in equation 3.22. This charge can now be expressed as a quantum operator that has the following commutator with the quantum gauge field in equation 3.24

$$\left[Q_\epsilon^+, A_z^{(0)}(u, z\bar{z})\right] = i\partial_z\epsilon(z,\bar{z}). \tag{3.25}$$

This means that the conserved charge Q_ϵ^+ generates a gauge transformation with parameter ϵ. The condition $A_z = 0$ is not invariant under these symmetries since Q_ϵ^+ gauge transforms the field. The fact that this condition is not invariant means that the symmetry is spontanuously broken. A spontanuously broken symmetry has the fundamental property that it transforms the vacuum state. For this case it means that if $|0\rangle$ is a vacuum state, then Q_ϵ^+ does not annihilate the vacuum $Q_\epsilon^+|0\rangle \neq |0\rangle$. Rather the vacuum state is transformed

$$Q_\epsilon^+|0\rangle = |v_Q\rangle. \tag{3.26}$$

Here $|v_Q\rangle$ is a new vacuum state different from $|0\rangle$. This means in other words that the charge connected to the asymptotic symmetry generates a *vacuum transition*. In order to connect this to the memory effect we need to look at the definition of the charge Q_ϵ^+ again. As stated by equation 3.22 this charge consisted of a *soft* charge and a *hard* charge. The hard charge Q_H in fact does commute with A_z so the commutation relation in equation 3.25 can be rewritten as

$$\left[Q_S, A_z^{(0)}(u, z\bar{z})\right] = i\partial_z\epsilon(z,\bar{z}). \tag{3.27}$$

This means that the commutator is fully determined by the soft charge Q_S. The term Q_S is linearly dependent on the quantity N_z, however N_z gives us the memory effect. This means that the commutation relation (equation 3.25) is linearly dependent on the same quantity as the memory effect! Similarly the charge Q_S generates the vacuum transition of equation 3.26. This means that the memory effect is directly related to the vacuum transition. More precisely can the memory effect be seen as a difference in vacuum state between late and early times. This difference in vacuum state, the vacuum transition, caused by the memory effect is the same transition as caused by the conserved charge Q_ϵ^+ for some specific ϵ. This finally means that the memory effect is directly related to some asymptotic symmetry with generating function ϵ through the vacuum transition that is caused by both Q_ϵ^+ as well as the memory effect itself. This shows the connection between the memory effect and asymptotic symmetries and completes the two relations with the memory effect in the infrared triangle (figure 4).

We can now conclude that the memory effect is of significant importance given its connection to both the soft photon theorem as well as asymptotic symmetries. This adds both physical and mathematical meaning to all three of these theories. The memory effect is the best observable, so with the help of the memory using this it might be possible to verify the other two theories. The soft photon theorem might seem unphysical, however the memory effect automatically involves the soft theorem, and the memory effect has a more intuitive basis. We will not go into detail into ward identities, since this has little relevance to the memory effect. From these connections we can also conclude different ways of looking at the memory effect. Whereas we defined the memory effect as the change over all time of physically measurable fields we can now define it differently. This can be useful since the new definitions are less ambiguous. We can define the memory effect as the change in soft photon modes after infinite time, but a more useful definition comes from the vacuum transition. It is possible to define the memory effect as being a transition between the vacuum state at $t \to -\infty$ and $t \to \infty$. This provides a strong definition for the memory effect. We have seen that the connection between the memory effect and vacuum transitions is linear so we can make this definition without problem. It is a nice way of looking at the memory effect, because it means that the memory of a particle is actually embedded in the vacuum state of spacetime itself.

3.3 Gravitational memory

The gravitational memory effect has been analyzed more than the electromagnetic memory [6]. This section is based on a review of Strominger on the gravitational memory [13]. We will shortly discuss the memory effect for gravity in order to provide a quick view into the physics of general relativity and the memory effect from another point of view. The gravitional memory effect can hopefully also give some insight in how we can measure the memory effect in the future. Although for gravitational memory the same infrared triangle as in the previous section exists, we will not analyze this triangle any further at this point. In order to understand the memory effect of gravity, the principle of gravity in modern physics needs to be reviewed briefly first. The effect of gravity in general relativity works fundamentally different from Newtonian gravity. Free moving particles follow geodesics in space. Geodesics are paths from some coordinate 1 to 2 that minimize the metric distance ds between these points. Mathematically speaking they are the paths that minimize the action

$$S = c\int_1^2 \mathrm{d}s. \tag{3.28}$$

Here c is a constant that is usually the rest mass m of the particle. Because of this the "force" of gravity is not generated by a presence of mass, but by the metric tensor, since

$$\mathrm{d}s^2 = g_{\mu\nu}x^\mu x^\nu. \tag{3.29}$$

Although the metric tensor is in turn determined by the mass, in addition to any other form of energy and pressure present in the system, the mass dependence is drastically more complex than the Newtonian potential. While the metric tensor has been solved in specific configurations for energy in space, for example for a spherically symmetric mass around the origin, general solutions are impossible. For the memory effect we are again interested in the asymptotic behaviour as $r \to \infty$ again though, so that for an asymptotically flat spacetime up to leading order in r the metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} + 2\frac{m_B}{r}\mathrm{d}u^2 + rC_{zz}\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\mathrm{d}\bar{z}^2 + D^zC_{zz}\mathrm{d}u\mathrm{d}z + D^{\bar{z}}C_{\bar{z}\bar{z}}\mathrm{d}u\mathrm{d}\bar{z}. \tag{3.30}$$

In this metric the quantities m_B and C_{zz} are the Bondi mass aspect, however their exact meaning is not qualitatively relevant for the memory effect in the end. Let us take two test particles at some locations in space.

These particles are free so that they follow geodesics in space. Their seperation is written as $s^{\bar{z}}$. The equation of geodesic deviation, in other words the equation determing the deviation between two geodesics, is now given near $\mathcal{I}^+$ by

$$r^2\gamma_{z\bar{z}}\partial_u^2 s^{\bar{z}} = -R_{uzuz}s^z. \tag{3.31}$$

The tensor R_{uzuz} is the Riemann tensor which is dependent on the metric only. Using the metric in equation 3.30 this tensor can be found to give

$$R_{uzuz} = -\frac{r}{2}\partial_u^2 C_{zz}. \tag{3.32}$$

Imagine that the metric is now governed by some gravitational pulse, so that we can write the difference between quantities before the pulse and after the pulse with a Δ. The above two equations can be integrated twice with respect to u to give

$$\Delta s^{\bar{z}} = \frac{\gamma^{z\bar{z}}}{2r}\Delta C_{zz}s^z. \tag{3.33}$$

This $\Delta s^{\bar{z}}$ means that the displacement between the particles has changed if ΔC_{zz} is nonzero. This is the gravitational memory effect. The particles gain a permanent displacement change when a gravitational wave train has passed them. The change in displacement is also linearly dependent on the displacement before the pulse. This has to be since if the initial displacement was zero contractions of spacetime do not affect the displacement in any way. The equation also needs to hold for a Lorentz transformed frame, so if the

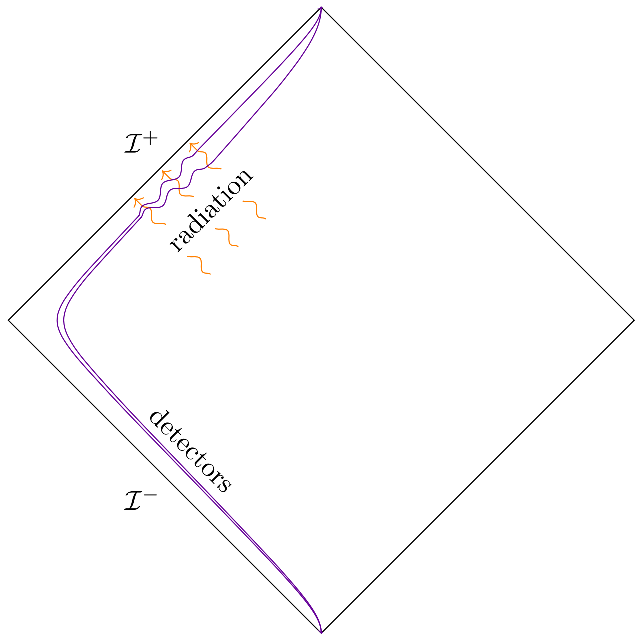

Figure 5: The memory effect on two particles near $\mathcal{I}^+$ after a gravitational wave train has passed. Their relative position has gained a permanent displacement [6].

discplacement is smaller due to Lorentz contraction $s'^{\bar{z}}_{\text{before}} = \lambda s^{\bar{z}}_{\text{before}}$ with $\lambda < 1$ at the beginning, then the final displacement needs to differ by the same amount $s'^{\bar{z}}_{\text{after}} = \lambda s'^{\bar{z}}_{\text{after}}$. The difference between the displacement before and after is now also shifted $\Delta s^{\bar{z}} = \lambda \Delta s^{\bar{z}}$. Assuming the function for the memory effect is Lorentz invariant, the memory effect has to depend linearly on the displacement, so that the λ cancels out. This equation is different from the one for electromagnetism in 3.1 because this equation gives us a displacement change instead of a velocity change. Another important difference is that for the electromagnetic memory the velocity kick was not linearly dependent on the initial velocity. There is however no reason to expect a nonzero kick for a particle at rest, since we can choose a different frame where the particle is not at rest anymore. The displacement however can not be removed through a transformation, it can only be made smaller. The change in ΔC_{zz} obeys a relation to the source given by

$$\Delta C_{zz}(z, \bar{z}) = \frac{4}{\pi} \int \mathrm{d}^2 z' \gamma_{z'\bar{z}'} \frac{\bar{z} - \bar{z}'}{z - z'} \frac{(1 + z'\bar{z})^2}{(1 + z'\bar{z}')(1 + z\bar{z})^3} \left(\int_{u_i}^{u_f} \mathrm{d}u \; T_{uu}(z', \bar{z}') + \Delta m_B \right). \tag{3.34}$$

Here u_i is some time far before the pulse and u_f some time long after the pulse. The stress energy component T_{uu} and the mass difference Δm_B are the source of the gravitational pulse in the $\mathcal{I}^+$ limit. By measuring the initial and final displacement of two test particles the difference $\Delta s^{\bar{z}}$ can be determined. This way we can learn something about T_{uu} and Δm_B by measuring the gravitational memory effect. Due to the nature of gravitational effects lying deep in the field of general relativity we will not be going any further into gravitational memory in this paper.

4 Superconductor Memory

The memory effect can manifest in many different ways. The method used in this paper is using the phase of a superconductor as proposed by Susskind [1]. The principle is given by the fact that when a particle moves through a ring of superconductors, it leaves a permanent change into the phase of these superconducting rings depending on how the particle has moved through it. If at first these superconductors are disconnected and as $t \to \infty$ they are reconnected currents will flow proportional to the gradient of this phase. To explain the mechanics of this process first the principle of superconductivity in a field theoretic framework needs to be analyzed. To do this the action of the superconductor needs to be constructed.

4.1 The action of a superconductor

In a superconductor the standard gauge symmetry of electromagnetism is spontanuously broken. This can be understood using the following reasoning. The magnetic field cannot penetrate very far into the superconductor, the *Meissner effect*, which effectively means that the photons have mass. This in turn means that the gauge field has a mass, given by the effective photon mass. A mass term in the action breaks the gauge symmetry. Indeed $F_{\mu\nu}F^{\mu\nu}$ is gauge invariant but $m^2 A_\mu A^\mu$ is not. This means that inside a superconductor the gauge symmetry is explicitly broken. To make the theory physically valid again the system can be restored by adding a complex scalar field which transforms under the gauge symmetry as well, so that the symmetry is now spontanuously broken instead. As Weinberg explains in [14] the action of a general complex scalar field ψ with spontanuous symmetry breaking is given by

$$S = \int \left(-\mathrm{d}\psi \wedge \star \mathrm{d}\bar{\psi} - V(|\psi|)\right). \tag{4.1}$$

This action is invariant under the *global* symmetry of the form $\psi \to \psi e^{i\alpha}$ where α is a constant. The potential V is to ensure that there is a non-zero stable vacuum solution for the field so that the theory does not vanish. The choice for the potential will be the so-called mexican hat potential given by $V(x) = g(x^2 - m^2/(2g))^2$ as illustrated in Figure 6. This potential is special since shifting ψ as $\psi \to \psi e^{i\alpha}$ does not alter $V(|\psi|)$, so this potential has an infinite amount of minima given by the disk with radius $|\psi|^2 = \frac{m^2}{2g}$. This disk also explains

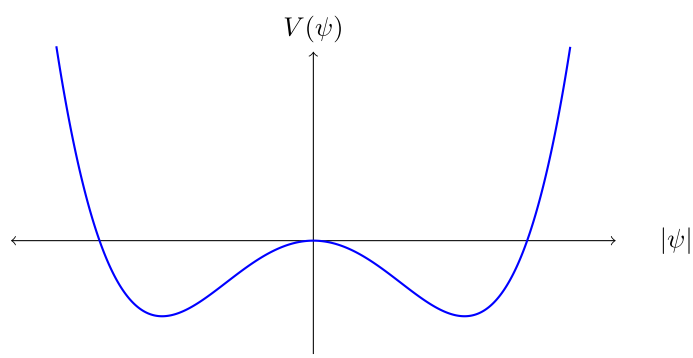

Figure 6: The mexican hat potential [6].

the hidden symmetry breaking in this action. For some vacuum state $|\mathrm{Arg}(\psi) = 0\rangle$ shifting ψ by a complex rotation α alters this vacuum state into $|\mathrm{Arg}(\psi) = \alpha\rangle$ while conserving the potential minimum. We conclude that the vacuum state is not invariant under the global symmetry of the action [6]. A theory involving global spontaneous symmetry breaking involves the appearance of a massless spin-zero boson called the goldstone boson [14]. These bosons are responsible for the symmetry breaking and the field ψ describes them. In Section 3.2 the goldstone bosons of the asymptotic symmetry breaking were in fact the soft photons. For this paper the exact meaning of the goldstone bosons is of no importance though. When splitting the complex field in polar coordinates $\psi = \rho_c e^{ie\phi}$ we find two seperate functions, ρ_c and ϕ. The phase ϕ is called the *goldstone mode* and describes the field of the goldstone bosons. Note that this is not a quantum mechanical wave function, however it does carry information on the behaviour of the bosons. The norm ρ_c can be seen as describing the intensity of the superconductor, in some sense equivalent to the charge density in solid state mechanics. To be more exact ρ_c is equivalent to the fermion ladder operators of a superconductor when deriving the action from a quantummechanical method. In this sense ρ_c is propertional to the density of cooper pairs inside the superconductor [15]. Cooper pairs are bound pairs of electrons that move freely inside a superconductor.

This action is however for a spontaneous symmetry breaking theory only, but in our system there is also an electromagnetic field. To insert the electromagnetic field into the action as well the field action can be added, but the complex scalar field also has to be coupled to the gauge field. Quantummechanically this can be done by changing the derivative d of the scalar action to a *covariant derivative* $D = \mathrm{d} - ieA$ to give, noting that complex conjugation of the derivative is needed as well to ensure the action is real:

$$S = \int \left(-\frac{1}{2} F \wedge \star F - (\mathrm{d} - ieA)\psi \wedge \star (\mathrm{d} + ieA)\bar{\psi} - \star g|\psi|^4 + \star m^2 |\psi|^2 \right) + S_M. \tag{4.2}$$

Note that $F = \mathrm{d}A$ is the Faraday tensor. The S_M inserted is a general matter action which ensures the source of the gauge field. Later there will be given explanation to this term but for now it is not of importance. In this action there is a $U(1)$ symmetry that is now *local* (coordinate dependent) given by

$$\psi \to \psi e^{i\alpha(x)} \tag{4.3}$$
$$A \to A + \mathrm{d}\alpha(x). \tag{4.4}$$

We can now see that the complex scalar field ψ indeed solves the gauge symmetry problem in the beginning of the section. The theory was not invariant under the transformation $A \to A + \mathrm{d}\alpha(x)$, however it is invariant by transforming ψ too. The fact that there is still spontanuous symmetry breaking comes from the fact that the transformation for ψ is non-linearly realized. For a theory with local symmetry breaking, so a theory involving gauge fields, the goldstone bosons are strictly speaking no longer goldstone bosons anymore since they are now massive as is also visible in equation 4.2. This mass comes from the gauge field and the mechanism is known as the Higgs mechanism [14]. Nevertheless the action still stand and it is in fact the action of a superconductor! To see this the action is rewritten into polar coordinates $\psi = \rho_c e^{ie\phi}$ to give

$$S = \int \left(-\frac{1}{2} F \wedge \star F - \mathrm{d}\rho_c \wedge \star \mathrm{d}\rho_c - e^2 \rho_c^2 (\mathrm{d}\phi - A) \wedge \star (\mathrm{d}\phi - A) - \star g \rho_c^4 + \star m^2 \rho_c^2 \right) + S_M. \tag{4.5}$$

In equation 4.5 the dependence on ϕ is only in combination with A as $\mathrm{d}\phi - A$. It is now remarkable to note that we can theoretically transform $A \to A + \mathrm{d}\phi$ removing the phase altogether. This would fix the gauge completely, meaning that effectively inside a superconductor the gauge field is uniquely determined. This is of little practical use however because the Maxwell equations are gauge invariant, and the gauge fixing condition is too complex to insert.

4.1.1 Physical properties

Of course there should come multiple known physical properties of superconductors out of this action. Indeed the most important effects follow rather easily. Weinberg has done this in [14] on which this section is based. Because of the action only depending on $\mathrm{d}\phi - A$ it follows that in the hamiltonian the only term depending on ϕ is given by $(\partial_0\phi - A_0)^2 + |\nabla\phi - \vec{A}|^2$. This means that the energy of the system is at a local minimum exactly when $\partial_\mu\phi = A_\mu$, which holds when looking deep in a superconductor where boundary conditions do not matter. However when the gauge field is pure gauge there are no electric and magnetic fields, in other words any magnetic fields from outside the superconductor are going to be expelled. This is the *Meissner effect* and explains why magnets float on superconductors; the magnetic fields these magnets produce are expelled from the superconductor, so the closer the magnets come to the superconductor the stronger the expelling force is. If the magnet is not too large relative to the superconductor at some point the force of expelling the field is equal to the force of gravity causing the magnet to float. Similarly if the magnetic field is too strong it is energetically favorable to dismiss the superconductivity altogether.
Note that in equation 4.5 the gauge field also appears to have a mass now given by $\rho_c^2 e^2$, which indeed then means that effectively inside a superconductor photons do have mass. For a massive field the potential is roughly given by the Yukawa potential $V \propto e^{-e\rho_c r}/r$ for a field with mass $e\rho_c$. This means that inside the superconductor the electromagnetic field actually decays *exponentially* because the photon mass is not zero. This effect is similar to the Meissner effect in the sense that deep inside a superconductor the electromagnetic field vanishes, which can be understood as photons entering the superconductor gain mass and as such slow down untill they are at rest inside the superconductor.

Secondly the magnetic flux deep inside a superconductor has to be quantized. This follows from the following reasoning: the magnetic field has to be equal to the gradient of ϕ. The magnetic flux can be taken as an integral over a closed loop

$$\Phi_B = \iint_S \vec{B} \cdot \mathrm{d}\vec{S} = \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{\ell} = \oint_{\partial S} \nabla\phi \cdot \mathrm{d}\vec{\ell} = \phi(\text{end}) - \phi(\text{begin}), \tag{4.6}$$

where ∂S is a closed loop, so the integral goes over a closed loop with line element $\mathrm{d}\vec{\ell}$. Because the loop is closed $\phi(\text{end}) = \phi(\text{begin})$ up to a value that does not change the earlier complex scalar field, so an integer multiple of π/e. This gives that the magnetic flux inside the superconductor is also an integer multiple of $\Phi_B = n\pi/e$. Because of this the electric current going through the boundaries of the superconductor that maintains this flux cannot decay smoothly, but also only by integer multiples, which means that there is no ordinary electrical resistance. These properties do indeed show that equation 4.5 accurately describes a superconductor.

Two types of superconductors
In order to proceed with the calculation in section 6 we introduce two typcial length scales inside a superconductor. The first is the correlation length which is given by [14] as

$$\xi = \frac{\sqrt{2}}{m}. \tag{4.7}$$

This length scale is obtained from the equations of motion for ρ_c. It describes the length scale at which small perturbations in the field ρ_c happen. We neglect these perturbations in this paper by assuming ρ_c is stable in the potential minimum of the mexican hat. This approximation is valid if we take the superconductor to be large enough for these perturbations to be insignificant. Another typical length scale which plays a large role in a superconductor is the penetration length which is given by the inverse of the the mass of the gauge field as it was given in equation 4.5 for ρ_c in the potential minimum $\rho_c = m/\sqrt{2g}$. This gives for the penetration length

$$\lambda = \frac{m}{\sqrt{2g}e}. \tag{4.8}$$

The penetration length descibes how far the magnetic field is capable of penetrating a superconductor. As shown in this section for the massive gauge field the potential is given by the yukawa potential. In this potential the field decays exponentially and the typical rate of how fast this decay is, is given by the penetration length. The penetration length is experimentally shown to be of order of 10^{-7}m. The correlation length varies greatly per material. For type I superconductors we have $\xi > \lambda$ whereas for type II superconductors we have $\lambda > \xi$. However superconductors around the boundary $\xi \sim \lambda$ exist, so we take the correlation length to have the same order of magnitude as the penetration length.

4.1.2 Conserved current

We have seen that the action of equation 4.5 was invariant under the new gauge transformation. In polar coordinates this transformation is written as:

$$\begin{aligned} \phi &\rightarrow \phi + \alpha && (4.9)\\ A &\rightarrow A + \mathrm{d}\alpha && (4.10)\\ \alpha &: \mathbb{R}^4 \longmapsto \mathbb{R}. && (4.11) \end{aligned}$$

Following Noether's theorem this symmetry has a corresponding conserved current and conserved charge. This conserved current can be derived by variating the action with respect to the gauge transformation and noting that it does not change, so that $S[\phi+\alpha, A+\mathrm{d}\alpha] = S[\phi, A]$. We now take the special case in which α is constant and note that the action is only dependent on $\mathrm{d}\phi$ and not on ϕ, this gives $S[\phi+\alpha] = S[\phi] \longrightarrow \frac{\delta S}{\delta \phi} = 0$. Indeed this last equation is obeyed by the action. This equation can be rewritten as $S[\phi + \alpha] - S[\phi] = 0$. If we now take α infinitesimally small this equation is simply the definition of a minimized action so the conserved current can be found by minizing the action with respect to ϕ which gives

$$\mathrm{d}\star(-2e^2\rho_c^2(\mathrm{d}\phi - A)) = 0 \tag{4.12}$$

$$\Longrightarrow \boxed{J_C = 2e^2\rho_c^2(\mathrm{d}\phi - A)} \tag{4.13}$$

Here J_C is the conserved current of the system. This means that in a superconductor currents will flow as a gradient of the phase when the gauge field is absent, and when there is a gauge field these currents will also be drawn in the direction of the gauge field. This conserved current equation will be of fundamental importance for the memory that we are calculating because this is the earlier mentioned memory current we would measure. Note that J_C^0 is actually the charge density of this current, which would be the charge density of freely moving cooper pairs. This shows the connection between ρ_c and the charge density, however $\partial_o\phi - A_0$ also plays a role.

4.2 Weyl transformation

In this subsection the action will first be Weyl transformed to provide easier methods for calculation. A *Weyl transformation* is a transformation that changes the metric without shifting the coordinates of the form $\mathrm{d}s^2 \to \Omega^2(x^\nu)\mathrm{d}s^2$. This is also a conformal transformation, however we will not be changing only the metric but also the physical fields. Indeed it is easily visible that the FLRW metric in conformal time is perfect for a Weyl transformation. The metric as in Section 2.3 is given by

$$\mathrm{d}s^2 = a(\tau)^2\left[-\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right]. \tag{4.14}$$

The Weyl transformation $\Omega = 1/a(\tau)$ transforms the metric into the Minkowski metric. Since the Minkowski metric is spatially flat and constant this metric is perfect for quantitative analysis. Along with the redefinition of the metric it is useful to redefine some functions and symbols in the superconductor action as well to maintain the same action. First the choice $\rho_c \to a\rho_c$ is made, secondly we choose A_μ to be constant. The phase ϕ is a scalar which does not change. The matter action S_M is not invariant under the transformation, however only the current $J^\mu_M = \frac{\delta S_M}{\delta A_\mu}$ is important, which we choose invariant in its upper index (so $J^\mu_M \to J^\mu_M$). In a short list this is given by

$$\begin{aligned}
g_{\mu\nu} &\to \frac{1}{a^2}g_{\mu\nu}, &(4.15)\\
A_\mu &\to A_\mu, &(4.16)\\
\rho_c &\to a\rho_c, &(4.17)\\
\phi &\to \phi, &(4.18)\\
J^\mu_M &\to J^\mu_M. &(4.19)
\end{aligned}$$

After the Weyl transformation the metric becomes indeed the Minkowski metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{4.20}$$

since $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. This means that the FLRW spacetime is *conformally equivalent* to the Minkowski spacetime, which means that the causal structure as well as the angles between events are the same for the FLRW metric as for the Minkowski metric as long as this transformation is allowed. Physically this is to be expected since the spacetime was assumed to be isotropic and homogenuous meaning that there can be no angular transformation without breaking the symmetry ansatz of the FLRW metric. Only the spatial distances and time seperations of events (the coordinates with length dimension) change under this transformation. Indeed the FLRW metric as introduced in section 2.3 only scales the physical distances between events, the Weyl transformation removes this scaling. To transform the action we need to transform all seperate parts using the list above. With these rules it is important to note that A^μ does change since $A^\mu = g^{\mu\nu}A_\nu$, this also gives for the Faraday tensor $F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}g^{\mu\tau}g^{\nu\sigma}F_{\tau\sigma}$. Lowering the indices is important for the transformation because now the transformation of $F_{\mu\nu}F^{\mu\nu}\sqrt{-g}$ gives

$$F_{\mu\nu}F^{\mu\nu}\sqrt{-g} = F_{\mu\nu}g^{\mu\tau}g^{\nu\sigma}F_{\tau\sigma}\sqrt{-g} \xrightarrow{\text{Weyl transformation}} F_{\mu\nu}\mathbf{a}^2g^{\mu\tau}\mathbf{a}^2g^{\nu\sigma}F_{\tau\sigma}\frac{\sqrt{-g}}{\mathbf{a}^4} = F_{\mu\nu}F^{\mu\nu}\sqrt{-g}. \tag{4.21}$$

This means that $F_{\mu\nu}F^{\mu\nu}\sqrt{-g}$ is in fact invariant under Weyl transformations or Weyl invariant. In a similar fashion the rest of the transformation can be inserted into the action to give

$$S^{\text{new}} = \int\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{a^2}\partial_\mu(a\rho_c)\partial^\mu(a\rho_c) - e^2\rho_c^2(\partial_\mu\phi - A_\mu)(\partial^\mu\phi - A^\mu) - g\rho_c^4 + \frac{m^2}{a^2}\rho_c^2\right]\sqrt{-g}\mathrm{d}^4x + S_M. \tag{4.22}$$

The action is almost the same as in equation 4.5. Let us analyze the $\partial_\mu(a\rho_c)\partial^\mu(a\rho_c)$ more closely:

$$\frac{1}{a^2}\partial_\mu(a\rho_c)\partial^\mu(a\rho_c) = \partial_\mu\rho_c\partial^\mu\rho_c + 2\frac{1}{a}\rho_c\partial_\mu\rho_c\partial^\mu a + \frac{1}{a^2}\rho_c^2\partial_\mu a\partial^\mu a = \partial_\mu\rho_c\partial^\mu\rho_c - \mathcal{H}\partial_0\rho_c^2 - \rho_c^2\mathcal{H}^2, \tag{4.23}$$

where the (conformal) Hubble parameter $\mathcal{H} = \frac{\dot{a}}{a}$ has been used. This is slightly different from the normal hubble parameter H because the derivative is with respect to conformal time now. The term $\mathcal{H}\partial_0\rho_c^2$ can be rewritten as $\partial_0(\mathcal{H}\rho_c^2) - \dot{\mathcal{H}}\rho_c^2$. Now for the action the total differential $\partial_0(\mathcal{H}\rho_c^2)$ does not matter since it reduces to zero due to vanishing boundary conditions, which means that $\mathcal{H}\partial_0\rho_c^2 \equiv -\dot{\mathcal{H}}\rho_c^2$. This can finally be inserted to give the fundamental action for our calculation

$$S^{\text{new}} = \int\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\rho_c\partial^\mu\rho_c - e^2\rho_c^2(\partial_\mu\phi - A_\mu)(\partial^\mu\phi - A^\mu) - g\rho_c^4 + M^2(\tau)\rho_c^2\right]\sqrt{-g}\mathrm{d}^4x + S_M. \tag{4.24}$$

This action is the same as the general action before the Weyl transformation (equation 4.5) except for the now *time dependent mass* $M^2 = \frac{m^2}{a^2} + \mathcal{H}^2 - \dot{\mathcal{H}}$. This is expected since conformal transformations as explained in Section 2.1 preserves light cones, so the theory of a massless field, for example the gauge field with massless photons, should not change under a conformal transformation. In the case of a superconductor the gauge field is massive, however its mass ρ_c^2 has been rescaled. This gives that only the mass of the complex scalar field transforms. In other words the physical properties of the superconductor change as spacetime expands. This would affect the gauge field indirectly as well but we will assume the superconductor to be small enough for this effect to be negligible. When the superconductor covers some volume V, this volume V expands as space expands, which also means that the density of the superconductor decays as $1/V$. From this we expect the superconductor to decay in strength and after a certain while have no effect anymore. Indeed the mass of the superconductor now decays and as $\tau \to \infty$ the mass $M \to 0$ so when space is maximally expanded the superconductor has no superconducting properties anymore, similar to the volume V extending over all space with zero density. The phase ϕ of the superconductor is invariant which means that when we determine the phase after the Weyl transformation, we also immidiately know the phase from before. Because of invariance of ϕ we will only work with the transformed action.

The mass of the superconductor

To check whether this mass is correct we can transform the action of a massive scalar field which is known to be Weyl or conformally invariant and see if the mass correction is correct. Let us take the following invariant action [16]

$$S = \int\left[-\partial_\mu\psi\partial^\mu\psi - \frac{R}{6}\psi^2\right]\sqrt{-g}\mathrm{d}^4x, \tag{4.25}$$

where ψ is a real scalar field and R is the Ricci scalar. The Ricci scalar transforms under a transformation $g \to \Omega^2 g$ as [16]

$$\tilde{R} = \Omega^{-2}[R - 6g^{\mu\nu}\partial_\mu\partial_\nu\ln\Omega - 6g^{\mu\nu}(\partial_\mu\ln\Omega)(\partial_\nu\ln\Omega)]. \tag{4.26}$$

In the case that $\Omega = \frac{1}{a}$ this is given by

$$\tilde{R} = a^2[R - 6\dot{\mathcal{H}} + 6\mathcal{H}^2]. \tag{4.27}$$

Adding this to the previous lagrangian gives

$$-\partial_\mu\psi\partial^\mu\psi - \frac{R}{6}\psi^2 \Longrightarrow -\frac{1}{a^2}\partial_\mu a\psi\partial^\mu a\psi - \frac{\tilde{R}}{6a^2}\psi^2 = \mathscr{L} + \frac{R}{6}\psi^2 + (\mathcal{H}^2 - \dot{\mathcal{H}})\psi^2 - \psi^2\left[\frac{R}{6} - \dot{\mathcal{H}} + \mathcal{H}^2\right] = \mathscr{L}. \tag{4.28}$$

The theory is indeed conformally invariant. This means that the earlier mass correction obtained by expanding $\frac{1}{a^2}\partial_\mu(a\rho_c)\partial^\mu(a\rho_c)$ is current since it perfectly cancels the Ricci scalar.

4.3 Superconductor memory

The memory of a passing particle can manifest itself as a velocity kick, but it can also manifest itself in the phase ϕ of the superconductor as proposed by Susskind [1]. Assume that at some initial conformal time τ_i with corresponding initial time t_i there is only charge at rest at the origin and the gauge field $A_\mu(\tau_i) = 0$ and the phase $\phi(\tau_i) = 0$. There is no external field. At τ_i there is an explosion so that charged particles move outwards in the radial direction. For the experiment imagine that at some *coordinate distance* $r = R$ which obeys $R \gg \lambda$ there is a large sphere of superconducting nodes that covers all angles. Before $\tau = \tau_i$ the nodes are connected and any currents are discharged to ensure that $\phi(\tau_i) = 0$. At $\tau = \tau_i$ the nodes are disconnected, the particles move through the nodes and change their phase, this is different for each node. In the end, when all particles have left, the nodes are reconnected and currents will flow between them determined by the conserved current (equation 4.12). These currents would discharge very quickly because they affect the gauge field themselves, but would exist long enough to be measurable.

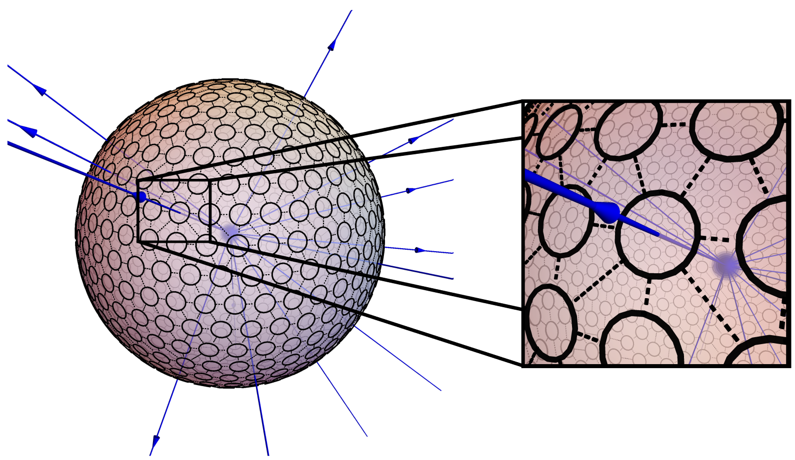

Figure 7: An image illustrating the setup of the system. The large spherical shell is covered in superconducting nodes, illustrated as small rings. The dashed lines are wires connecting the nodes, which are disconnected between $\tau = \tau_i$ and $\tau = \infty$. The particles of the explosion follow the blue trajectories. They move through the rings, and can pass freely without collision this way. As they move through the ring the phase inside the ring and neighbouring rings gets changed. When reconnecting the wires as $\tau \to \infty$ currents will flow along the circular nodes and wires proportional to the gradient of the phase determined by the particles moving through the nodes.

The main principle is that at $\tau = \infty$ the phase ϕ and the gauge field A_μ have changed dependent on the particle movement. To find the memory effect, i.e. the current which would flow at $\tau \to \infty$ the phase and gauge field at $\tau \to \infty$ need to be determined. Let us start from the equations of motion. Variating the action of equation 4.5 to the gauge field gives the equation of motion for the gauge field:

$$\delta S = S[A + \delta A] - S[A] = \int \delta A \wedge (-\mathrm{d} \star F + \star 2e^2 \rho_c^2 (\mathrm{d}\phi - A)) + \delta S_M, \tag{4.29}$$

$$\implies \mathrm{d} \star F = \star J_M + \star J_C, \tag{4.30}$$

where J_M is the current generated by the matter action given by $J_M^\mu = \frac{\delta S_M}{\delta A_\mu}$ and the conserved current from Section 4.1.2 has been identified. This equation is in a complex manner coupled to the superconductor, however the superconducting nodes were disconnected after $\tau = \tau_i$. We can conlude that in the modified Maxwell equation 4.30 the current J_C is of no importance. It is to be noted that of course the charge density J_C^0 would still exist, however we never discussed the fact that the superconductor would also have stationary protons, and be neutral on average. With this in mind we can remove J_C^0 as well. This gives the classical Maxwell equation written in tensor form as

$$\partial_\mu F^{\mu\nu} = -J_M^\nu. \tag{4.31}$$

The charge density $\rho_M = J_M^0$ is singular at $r = 0$ for all $\tau < \tau_i$ and is given by the moving particles for $\tau > \tau_i$ in a fashion which will be explained in Section 5 and 6. We now choose the temporal gauge $A_0 = 0$ which still leaves some gauge freedom, to be precise the freedom to make gauge transformations where α is space dependent only. We have chosen $A(\tau_i) = 0$ however, so that the gauge is now completely fixed. Indeed any space dependent but time independent transformation would remove this initial condition. In this gauge Maxwell's equation for $\nu = 0$ can be written as

$$\partial_0 \nabla \cdot \vec{A} = -\rho_M, \tag{4.32}$$

where the identities $A^\mu = (0, \vec{A})$ and $\nabla = \partial_i$ have been used, as well as the definition $F^{\mu\nu} = 2\partial^{[\mu}A^{\nu]}$ where [...] denotes antisymmetrization. This equation determines the gauge field as it changes because of the particles moving through space. To obtain the gauge field at time infinity this equation can be integrated over all time resulting in

$$\nabla \cdot \vec{A}(\infty) = -Q(\infty), \tag{4.33}$$

where $Q(\tau)$ is defined as all the charge that has passed through a point $\vec{x}$ after a time τ. This means that $Q(\infty)$ is simply all the charge that has passed ever in history, so that $Q(\infty)$ integrated over all solid angle is all charged that has been released at the initial explosion. Note that this assumption could not have been made when $w < -1/3$ because in that case if the radius of the sphere was large enough the particles might not have reached the sphere at all due to the expansion of space going faster than the movement of the particles! At time infinity long after the particles have passed the electric and magnetic field have to vanish, meaning that the gauge field becomes pure gauge $\vec{A} = \nabla\lambda$. We can gauge transform this residual λ away at the cost of altering the phase of the superconducting nodes $\phi \to \phi - \lambda$ and removing the condition $A(\tau_i) = 0$. This condition does not matter anymore however for our measurement of the current at $\tau = \infty$. The gauge field is completely absent now meaning that all memory is embedded in the superconducting phase only. If there was a residual gauge field the effect on the superconductor would not necessarily be a memory, so it is important that it can be transformed away. The gauge field is easily determined this way however the phase also transforms due to the movement of the particles. We can argue that the phase in fact has to transform in a manner linearly dependent on the gauge field because of the gauge transformation. If the phase were to be constant we could make a transformation $\phi \to \alpha(\vec{x})$, removing this assumption.

To find quantitatively how the phase transforms we start again with the definition of the conserved current because this is also the equation of motion. The conserved current equation 4.12 actually states that the divergence over spacetime of the current is zero. In this case we confine the superconductor to some distance $r = R$ meaning that the divergence over the spherical shell only has to be zero. This gives the following equation for the phase in vector notation

$$\partial_0(\rho_c^2 \partial_0 \phi) - \rho_c^2 \Delta_\Omega \phi = -\rho_c^2 \nabla_\Omega \cdot \vec{A}_\Omega. \tag{4.34}$$

The Ω subscript denotes that only the components on S^2 need to be taken, so only the angular components, and Δ is the Laplace operator given by $\Delta = \nabla \cdot \nabla$. Solving this equation for ϕ and taking the $\tau \to \infty$ limit gives the memory phase. When this ϕ has been obtained we also substract the gauge field function λ so that the phase is the only remaining memory. This final phase is then the same in the old FLRW metric since the phase does not change during the Weyl transformation so this finalises the result. The superconductor norm ρ_c will be assumed to be in the potential minimum $\rho_c^2 = \frac{M^2}{2g}$ at all time.

Physical distance

It is possible to confine the superconducting sphere to a fixed physical distance $x = R$ instead. This would mean that the radial *coordinate* would be given by $r = \frac{R}{a}$. This can be inserted into the conserved current equation taken on the spherical shell only by adding the radial part of the divergence as well with $r = \frac{R}{a}$ inserted. This gives

$$\partial_r = -\frac{a}{\mathcal{H}R}\partial_0. \tag{4.35}$$

These radial derivatives can be inserted into equation 4.34 to give an addition time derivative term. This makes the calculation unnecessary more complex however without adding a lot of additional value. Because of this we choose the spherical shell at fixed coordinate distance $r = R$ instead.

4.3.1 The charge distribution

For our setup we take the initial explosion to consist of two particles, one moving in the $\hat{z}$ and one in the $-\hat{z}$ direction in order to conserve momentum. Both these particles are massive and have the same mass so that their initial speed β is identical. They will also have opposite charge q to preserve charge. We assume them to be massive since no massless charged particles have been observed. The particles are assumed noninteracting so that only the solution for the particle moving in the $\hat{z}$ direction is needed. This solution is actually the *fundamental solution* since any other solution can be obtained by a sum of rotated fundamental solutions. Note that for the fundamental solution the azimuthal symmetry dictates that all ϕ coordinate dependence of any function needs to vanish. In this paper we will only determine the fundamental solution since the total solution can be found by letting $\theta \to \pi - \theta$ and $q \to -q$. The fact that in the FLRW spacetime the particles also do not change direction is easily argued from the fact that the spacetime was assumed isotropic and homogenuous. Since these particles have no interaction their paths are described by timelike geodesics. The action of a single charged particle with mass m and charge q is given by

$$S_M = \int \left(-m\sqrt{-g_{\mu\nu}\frac{\partial x_p^\mu}{\partial \lambda}\frac{\partial x_p^\nu}{\partial \lambda}} + qA_\mu \frac{\partial x_p^\mu}{\partial \lambda} \right) \mathrm{d}\lambda, \tag{4.36}$$

where λ is an affine parameter that parametrizes the particles movement. In this action x_p^μ denotes the four-position of the particle and the gauge field A_μ that couples to the particle movement needs to be evaluated at x_p^μ. The coupling of this gauge field is neglected for the movement of the particle, but it is important for the definition of the matter current $J_M = \frac{\delta S_m}{\delta A}$, which can now be derived to be given by

$$J^\mu(x^\nu) = q\int_{\mathbb{R}} \dot{x}_p^\mu \delta^{(4)}(x^\nu - x_p^\nu(\lambda))\mathrm{d}\lambda, \tag{4.37}$$

where $\dot{x}_p^\mu = \frac{\partial x^\mu}{\partial \lambda}$. The solution of the geodesic equation can be inserted into equation 4.37 to obtain the charge current of the problem. These geodesics will need to be treated carefully since the particles are assumed to have an instantanuous velocity kick at $\tau = \tau_i$. This automatically also gives a discontinuity in the charge current and in the fields at $\tau = \tau_i$. This discontinuity might seem unphysical but it is actually only a mathematical method for analyzing the field. The behaviour at the explosion is of no importance, only the characteristics of the particles movements afterwards, so it is of no physical importance if we choose the particles to accelerate for a very short time, or instantly. Mathematically however working with a discontinuity is rather easy. The electric and magnetic fields also recieve an infinite singularity at $\tau = \tau_i$, this is however expected. Any charged particle that undergoes acceleration emits electromagnetic radiation, in this case the acceleration is infinitely large for an infinitely short time, so the radiation peak should have the same behaviour. The derivation of the charge current will be done in detail in Section 5 and 6. This charge current is fundamental for the memory effect since it is the source of the gauge field and the phase of the superconducting shell.

4.4 Symmetry analysis

It is possible for an initial charge Q to be confined at $r = 0$ for any $\tau < \tau_i$ since this does not alter the initial condition $A(\tau_i) = 0$. Such a charge distribution would be spherically symmetric, and since the superconductor is also spherical symmetric, this charge distribution should not be able to have any effect on the final current. We can argue this since if it adds any angular dependence to the phase, this dependence should be broken by rotating the system since the charge distribution is spherically symmetric. Indeed for such an initial charge Q the electric field is simply $\vec{E}_\Omega = 0$ and $E_r \propto \frac{1}{r^2}$. In the temporal gauge $\dot{\vec{A}} = -\vec{E}$ so that it is easily chosen that $\vec{A}_\Omega = 0$ and $A_r \propto \frac{\tau + c}{r^2}$. For this final part we can freely choose $c = -\tau_i$ so that indeed $\vec{A}(\tau_i) = 0$. We now proceed to look at spherical symmetry of the system and its meaning for the result.

In the original note of Susskind he proposes to rewrite equation 4.33 as

$$\nabla_\Omega \cdot \vec{A}_\Omega = -\frac{1}{r^2}\partial_r(r^2 A_r) - Q. \tag{4.38}$$

In other words strip off the radial part of the divergence and take it to the other side. Then he proceeds to assume the integral over the electric field vanishes for large r since the electric field of a particle moving towards the sphere cancels the field as it moves toward the sphere. This field goes as $1/r^2$ since the assumption is true for a flat disk, and the deviation of a disk goes as $1/r^2$. Q goes as $1/r^2$ as well however, so the field integral is not negligible relative to Q. We will prove now that both terms are equally important for the physical behaviour of the system. Starting from stokes theorem on the Maxwell equation

$$\int_{S^3(r)} \nabla \cdot \vec{A} \mathrm{d}V = \int_{S^2} r^2 A_r \mathrm{d}\Omega = -\int_{S^3(r)} Q \mathrm{d}V. \tag{4.39}$$

Here $S^3(r)$ denotes a volume of a sphere with radius r, S^2 is the boundary of this sphere and $\mathrm{d}V$ denotes the volume element of the sphere. Taking the derivative to r on both sides and rewriting gives

$$\int_{S^2} \frac{1}{r^2} \partial_r (r^2 A_r) \mathrm{d}\Omega = -\int_{S^2} Q \mathrm{d}\Omega. \tag{4.40}$$

This result means first of all that the order of r of both $\frac{1}{r^2}\partial_r(r^2 A_r)$ and Q need to be the same, since their integral over all solid angle for arbitrary r needs to be the same. Furthermore this gives the important identity

$$\int_{S^2} \nabla_\Omega \cdot \vec{A}_\Omega \mathrm{d}\Omega = 0. \tag{4.41}$$

Mathematically speaking this result has to be true, since stokes theorem can be used again on equation 4.41 but now S^2 is closed so there is no boundary, meaning the original integral had to vanish as well. Indeed this integration should give zero from a physical perspective as well since the divergence over S^2 should only be nonzero where the field has a source contribution, however any charged particle that generates such a sink in the field at some direction $\hat{r}$ also has to generate it at $-\hat{r}$ because of antipodal symmetry. If some particle moves in the $\hat{r}$ direction the gauge field on S^2 flows toward a sink at $\hat{r}$, however since it flows in this direction it has to originate from some source, which from spherical symmetry easily follows to be $-\hat{r}$. These two singularities cancel each other, and since any field can be expressed as a superposition of many particles the result has to be true for any gauge field as well. An interesting case to look at is the case of complete spherical symmetry of Q, in which case the integrals over all solid angles in equation 4.40 can be removed because the functions have to be direction independent giving

$$\frac{1}{r^2} \partial_r (r^2 A_r) = -Q. \tag{4.42}$$

In this case it follows that $\nabla_\Omega \cdot \vec{A}_\Omega = 0$. Indeed $\vec{A}_\Omega$ should be direction independent as well. This means however that the memory effect vanishes for spherically symmetric charge distributions since the memory is only dependent on $\vec{A}_\Omega$ which was fixed to be zero at $\tau = \tau_i$. This shows why the initial conditions are of little importance as long as they are chosen spherically symmetric. In fact any spherically symmetric charge distribution can be present during the experiment without altering the memory effect, since no interaction between the charged particles was assumed. The charge density ρ_M can then be split in a symmetric and an antisymmetric part. The only contribution to the memory then comes from the antisymmetric part.

5 Minkowski spacetime

5.1 The gauge field

To analyze this memory effect and how it is derived we first proceed in a simpler case where $a(\tau) = 1$, so the Minkowski spacetime. Now the conformal time is simply equal to the normal time $\tau = t$. In this spacetime the electric and magnetic fields generated by this particle have already been determined in the past by Liénard and Wiechert in the Lorentz gauge. We need to transform it to the temporal gauge for determining the phase. We also need to slightly transform the Liénard-Wiechert solution to account for the discontinuity at $t = t_i$. Since in Minkowski spacetime there is no convention that at $t = 0$ there is a big bang, we choose for simplicity for this section only that $t_i = 0$, which means that the explosion and the discontinuity is at $t = 0$. We start with the four-position of the particle which is explicitely

$$x_p^\nu = \begin{pmatrix} t \\ \vec{\beta}(t)\Theta(t) \end{pmatrix}. \tag{5.1}$$

Here $\Theta(x)$ is the Heaviside step function defined as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{5.2}$$

Under this definition the derivative $\Theta'(x) = \delta(x)$ where $\delta(x)$ is the dirac delta function. The four-position (equation 5.1) of the particle generates the following elements of the four current:

$$\rho_M = q\Big(\delta(\vec{x})\Theta(-t) + \delta(\vec{x} - \vec{\beta}t)\Theta(t)\Big), \tag{5.3}$$

$$\vec{j}_M = \vec{\beta}\rho_M. \tag{5.4}$$

Indeed the charge current also suddenly changes at $t = 0$. For the Liénard-Wiechert potentials ψ_{LW} and $\vec{A}_{LW}$ this means that for $t < 0$ the potential is given by a particle at rest at the origin and for $t > 0$ the gauge field is that of a relativistic moving particle. However we need to take causality into account: the gauge field propagates at the speed of light because the photon is massless, so at some distance r the field for the particle at rest is observed untill $t > r$. This finally gives the modified potentials in the Lorentz gauge

$$\psi_{LG} = \frac{q}{4\pi r}\Theta(r - t) + \psi_{LW}\Theta(t - r), \tag{5.5}$$

$$\vec{A}_{LG} = \vec{\beta}\psi_{LW}\Theta(t - r) \tag{5.6}$$

$$\psi_{LW} = \frac{q\gamma}{4\pi}\frac{1}{|\gamma^2(t - \beta r\cos\theta)^2 - t^2 + r^2|^{1/2}}. \tag{5.7}$$

This is the complete gauge field in the Lorentz gauge. The potential ψ_{LW} is the Liénard-Wiechert potential rewritten for a particle moving with $\vec{\beta} = \beta\hat{z}$. This potential needs to be transformed to the temporal gauge in order to obtain the one we need. To see how this happens note the definition of the temporal gauge $A^0 = 0$. This means that in some way A^i has been modified so that A^0 is part of it. Let us choose some α so that $\partial_0\alpha = A_0$. Then this α generates the gauge transformation to the temporal gauge since now

$$A^0 \rightarrow A^0 - \partial^0\alpha = 0, \tag{5.8}$$

$$A^i \rightarrow A^i - \partial^i\alpha. \tag{5.9}$$

This finally means that the vector field $\vec{A}$ in the temporal gauge is given by

$$\vec{A}(t) = \vec{A}_{LG}(t) + \nabla\int_0^t \psi_{LG}(t')\mathrm{d}t'. \tag{5.10}$$

Note that $\vec{A}_{LG}(0) = 0$ so that $\vec{A}(0) = 0$ as well for any r so the initial condition is still satisfied. The fact that the sign of the integral changed to a plus comes from the fact that $A^0 = \psi$ but $\partial^0 = -\partial_0$. We can insert the functions for $\vec{A}_{LG}$ and ψ_{LG} to finally obtain the full gauge field in the temporal gauge. The only component of interest is actually $\vec{A}_\theta$ since $\vec{A}_\phi = 0$ and for the calculation $\vec{A}_r$ is not important. Inserting ψ_{LW}, looking at the theta component only and working out the integral finally gives the complete *temporal gauge potential* that we need

$$\vec{A}_\theta = \frac{q\gamma}{4\pi}\frac{1}{\beta r\sin\theta}\frac{\Theta(t-r)}{|\gamma^2(t-\beta r\cos\theta)^2 - t^2 + r^2|^{1/2}}\left(r - \beta t\cos\theta - \beta^2 r\sin^2\theta\right) - \frac{q\Theta(t-r)}{4\pi\beta\sin\theta}. \tag{5.11}$$

To obtain this the integral had been partially integrated, the fact that $\vec{\beta} = \beta\hat{z}$ and the delta function identity

$$f(a)\Theta(b-a) = \int_{-\infty}^{b} f(x)\delta(x-a)dx \tag{5.12}$$

has been used. Equation 5.11 is the only component we need for determing the phase using equation 4.34. The gauge field still has the Heaviside step function, which is good, since we still need it to obey the same causality principles. This field resembles the standard Liénard-Wiechert solution, but there are angular and velocity dependent terms added which ensure the gauge.

5.1.1 The gauge lambda

As stated in Section 4.3 the gauge field $\vec{A}$ becomes a pure gauge as $t \to \infty$ so that $\vec{A} = \nabla\lambda$. To find λ we now need to solve equation 4.33. First we need to find Q for the given ρ_M. Q was defined as the total charge that has gone through the sphere with initial value $Q(t_i) = Q(0) = 0$ so that

$$Q = \int_0^\infty \rho_M \mathrm{d}t. \tag{5.13}$$

Now ρ_M as it was given in equation 5.3 was in carthesian coordinates. When switching the delta functions to spherical coordinates a problem arises: we cannot just take rewrite the arguments or the fundamental property of the delta function $\int_{\mathbb{R}} \delta(x)\mathrm{d}x = 1$ is not satisfied. When we rewrite $\delta(\vec{x}) = \delta(r)\delta(\theta)\delta(\phi)$ the jacobian of the integral $r^2\sin\theta$ goes to zero so that the integral can never give 1. To have the correct charge density the delta function transform needs to correct for the jacobian. Since the system is in azimuthal symmetry we also want the dirac deltas to be ϕ independent so that also needs to be taken into account. Indeed the correct form of the charge density in spherical coordinates is given by (for $t > 0$)

$$\rho_M = \frac{q}{2\pi r^2\sin\theta}\delta(r-\beta t)\delta(\theta). \tag{5.14}$$

Integrating this over all time is straightforward with the definition of the delta function so that Q is given by

$$Q = \frac{q\delta(\theta)}{2\pi\beta r^2\sin\theta}. \tag{5.15}$$

To find λ we now need to solve

$$\Delta\lambda = -\frac{q\delta(\theta)}{2\pi\beta r^2\sin\theta}. \tag{5.16}$$

This equation is a basic Poisson equation that has a solution using a Green's function given by

$$\lambda = \frac{-1}{4\pi}\int_{\mathbb{R}^3}\frac{Q}{|\vec{r}-\vec{r}'|}\mathrm{d}\vec{r}' = \frac{q}{4\pi\beta}\int_0^\infty \frac{1}{|\vec{r}-r'\hat{z}|}\mathrm{d}r'. \tag{5.17}$$

This integral has a slight problem: it does not converge. This is luckily not very bad since we can shift λ by any constant without breaking a symmetry or changing gauge. Indeed in the local $U(1)$ symmetry of equation 4.9 letting α be constant only alters ϕ, but physically only $\mathrm{d}\phi$ is important. To have a finite result we substract the constant value of $\lambda(r = a, \theta = 0)$, with a some finite nonzero radius, from the original λ, which would be a diverging constant, but we change the order of substraction and taking the limit of the integral to give

$$\lambda = \frac{q}{4\pi\beta} \int_0^\infty \left(\frac{1}{|\vec{r} - r'\hat{z}|} - \frac{1}{|a - r'|} \right) \mathrm{d}r'. \tag{5.18}$$

The indefinite integral is given by

$$\int \left(\frac{1}{|\vec{r} - r'\hat{z}|} - \frac{1}{|a - r'|} \right) \mathrm{d}r' = \log(r' - r\cos\theta + \sqrt{r'^2 + r^2 - 2rr'\cos\theta}) \tag{5.19}$$

$$-\frac{1}{2}\Big((\mathrm{sgn}(r' - a) - 1)\log(a - r') + (\mathrm{sgn}(r' - a) + 1)\log(r' - a) \Big). \tag{5.20}$$

When taking the limit $r' \to \infty$ the integral would have diverged before but now the asymptotic behaviour goes as

$$\int \left(\frac{1}{|\vec{r} - r'\hat{z}|} - \frac{1}{|a - r'|} \right) \mathrm{d}r' \xrightarrow{r' \to \infty} \log(2r') - \log(r') = \log(2) \tag{5.21}$$

so that the limit indeed exists and the renormalization works. The $\log(2)$ that remains can be transformed away again later. The total integral for both boundaries is now given by

$$\lambda = -\frac{q}{4\pi\beta}\log(r(1 - \cos\theta)) + \cancelto{0}{\frac{q}{4\pi\beta}\log(2a)}. \tag{5.22}$$

The right part of the equation was constant, so it can be transformed away without losing information or breaking symmetry as explained above. As such the only remaining term is the left part, which is the final scalar field that generates $\vec{A}$ at $t \to \infty$. This is the scalar field that later needs to be substracted of the phase ϕ to find the memory.

5.2 Phase

For the phase we need to solve equation 4.34 wih the use of the gauge field given in equation 5.11. However since now we are calculating in Minkowski space where $a = 1$ we get $M^2 = m^2$. We assumed ρ_c to be stable in the potential minimum. Looking at the action after the Weyl transform (equation 4.24) this means that

$$\rho_c^2 = \frac{M^2}{2g} = \frac{m^2}{2g} \tag{5.23}$$

which is constant. This means that we can leave ρ_c out of the phase equation so that the phase is now finally determined by

$$\partial_0^2\phi - \Delta_\Omega\phi = -\nabla_\Omega \cdot \vec{A}_\Omega. \tag{5.24}$$

Note that the derivatives ∇_Ω and Δ_Ω are actually only derivates to θ only because of the azimuthal symmetry given by

$$\Delta_\Omega = \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta), \qquad \nabla_\Omega \cdot \vec{A}_\Omega = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\vec{A}_\theta). \tag{5.25}$$

To solve equation 5.24 we expand both functions ϕ and $\nabla_\Omega \cdot \vec{A}_\Omega$ into spherical harmonics. However because the derivatives are with respect to θ only we find $m = 0$ for the spherical harmonics. This means that the functions are independent on the polar angle which indeed should be true. We find that instead we can use a Legendre polynomial expansion

$$\phi(\theta, t) = \sum_{\ell=0}^{\infty} a_\ell(t)P_\ell(\cos\theta), \qquad \nabla_\Omega \cdot \vec{A}_\Omega(\theta, t) = \sum_{\ell=0}^{\infty} b_\ell(t)P_\ell(\cos\theta). \tag{5.26}$$

The Legendre polynomials form an orthogonal basis and are a special case of the spherical harmonics up to a normalization constant. This means that the expansions are valid and fully describe the phase and gauge field as they are required to be smooth with respect to θ on their domain. Inserting this expansion back into equation 5.24 gives

$$\partial_0^2 a_\ell + \omega_\ell^2 a_\ell = -b_\ell. \tag{5.27}$$

Where the fact that $\Delta_\Omega P_\ell = -\frac{\ell(\ell+1)}{r^2} P_\ell$ has been used, or in other words the Legendre polynomials are eigenfunctions of the symmetric spherical laplace operator, and $\omega_\ell^2 = \frac{\ell(\ell+1)}{r^2}$ has been defined. This equation is a simple forced harmonic oscillator without damping with a force. This means that the phase can have waves passing over the surface of the sphere when there is no source. These waves would be periodic and boundary condition dependent. We assume boundary conditions here such that there are no waves, so that the homogenuous solution becomes zero. The homogenuous equation has the two solutions $e^{\pm i\omega_\ell t}$. Using Appendix C we can now solve the inhomogenuous equation using the Green's functions that comes from the homogenuous equation. The process is outlined in the same appendix. The Green's functions that comes from the complex solutions is complex as well, however we need the phase to be real. To solve this we take the real part of the Green's function as well. The Green's function is given by

$$G(t,t') = \frac{1}{2i\omega_\ell}\Big(\Theta(t-t')e^{i\omega_\ell(t-t')} + \Theta(t'-t)e^{-i\omega_\ell(t-t')}\Big), \tag{5.28}$$

$$\mathrm{Re}(G(t,t')) \xrightarrow{t\to\infty} \frac{\sin(\omega_\ell(t-t'))}{2\omega_\ell}\Theta(t-t'). \tag{5.29}$$

Using this Green's function we can write the solution for a_ℓ as

$$a_\ell = -\int\limits_0^t \frac{\sin(\omega_\ell(t-t'))b_\ell(t')}{2\omega_\ell}\mathrm{d}t'. \tag{5.30}$$

This Green's functions is not well defined for $\ell = 0$. For $\ell = 0$ we need to look at equation 5.27 that gives $\partial_0^2 a_0 = -b_0$ for $\ell = 0$. This gives a solution for $\ell = 0$ given by

$$a_0 = -\int\limits_0^t\int\limits_0^{t'} b_0(t'')\mathrm{d}t''\mathrm{d}t' = -\int\limits_0^t (t-t')b_0(t')\mathrm{d}t'. \tag{5.31}$$

The last equality can be checked by partial integration. We can insert this solution for a_ℓ back into the series 5.26 to give

$$\phi = -\int\limits_0^t \left(\sum_{\ell=1}^{\infty}\frac{\sin(\omega_\ell(t-t'))b_\ell(t')}{2\omega_\ell}P_\ell(\cos\theta) + (t-t')b_0(t')\right)\mathrm{d}t'. \tag{5.32}$$

The coefficients b_ℓ are defined as

$$b_\ell = \frac{2\ell+1}{2}\int\limits_0^\pi \sin\theta\nabla_\Omega\cdot\vec{A}_\Omega P_\ell(\cos\theta)\mathrm{d}\theta. \tag{5.33}$$

Here the prefactor comes from the fact that the Legendre polynomials are not properly normalized but instead obey the orthogonality relation

$$\int\limits_{-1}^{1} P_\ell(x)P_{\ell'}(x)\mathrm{d}x = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \tag{5.34}$$

By inserting equation 5.33 the summation over ℓ can be removed and a new function can be introduced giving

$$\phi(t,\theta) = -\int_0^t\int_0^\pi F(t,t',\theta,\theta')\nabla_\Omega\cdot\vec{A}_\Omega(t',\theta')\mathrm{d}t'\sin\theta'\mathrm{d}\theta' \tag{5.35}$$

$$F(t,t',\theta,\theta') = \sum_{\ell=1}^{\infty}\frac{\sin(\omega_\ell(t-t'))}{4\omega_\ell}(2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta')+\frac{1}{2}(t-t'). \tag{5.36}$$

This result explicitely expresses the final phase in terms of the gauge field. This function F is in fact the Green's function of the total operator $\partial_0^2-\Delta_\Omega$ in the $t\to\infty$ limit. Since this is the Green's function we remark that this equation for the phase is in fact valid for any gauge field, not just the gauge field of a single moving particle.

5.3 The fundamental solution

The memory phase as $t\to\infty$ is found by gauge transforming λ away. This means that $\phi\to\phi-\lambda$. Using this we can express the total phase as $t\to\infty$ as

$$\phi = -\lim_{t\to\infty}\int_0^t\int_0^\pi F(t,t',\theta,\theta')\nabla_\Omega\cdot\vec{A}_\Omega(t',\theta')\mathrm{d}t'\sin\theta'\mathrm{d}\theta' - \frac{q}{4\pi\beta}\log(r(1-\cos\theta)), \tag{5.37}$$

$$F(t,t',\theta,\theta') = \sum_{\ell=1}^{\infty}\frac{\sin(\omega_\ell(t-t'))}{4\omega_\ell}(2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta')+\frac{1}{2}(t-t'), \tag{5.38}$$

$$\nabla_\Omega\cdot\vec{A}_\Omega = \frac{q\gamma\Theta(t-r)}{4\pi\beta r\sin\theta}\partial_\theta\left(\frac{r-\beta t\cos\theta-\beta^2 r\sin^2\theta}{|\gamma^2(t-\beta r\cos\theta)^2-t^2+r^2|^{1/2}}\right). \tag{5.39}$$

This set of equations gives the final solution for the memory effect in Minkowski space. If the function F is numerically estimated, the solution for the phase is quickly found using the integrals. This function F however appears to be oscillating around zero towards time infinity. This poses a problem since then the limit is undefined. It is still possible to determine ϕ as a function of time though instead. This integral might be very hard to solve, however we do not necessarily need to solve it. Assume we have measured $\partial_\theta\phi$ from the current. By integrating it we can find ϕ up to an integration constant of no importance. The irrelevance of this constant comes from the fact that it only affects the zero component a_0 of the Legendre expansion, and not the components for $\ell\geq 1$. With the measured phase, we can determine a_1 numerically using equation 5.33. Since the Legendre polynomials are orthonormal this a_1 is unique and can be connected to the a_1 that was determined theoretically using the above equations. With this we can measure the charge q, speed β and radius r or fractions of them. Since this current might depend on time the measurement would depend on the time it was done after the explosion.

We can conclude that in Minkowski space there is indeed a measurable memory. The particle indeed causes a current in the superconductor even after infinite time. This current memory is given analytically by the gradient of equations 5.37 to 5.39. The memory depends on time of measurement, and the particle's charge, speed and direction of movement.

6 FLRW spacetime

6.1 The geodesic equation

In FLRW metric the scale factor is not constant anymore. To find the gauge field we now have to start from the charge current. We calculate this in the FLRW metric before the Weyl transformation

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\tau^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2). \tag{6.1}$$

The charge current was defined as

$$J_M^\nu(x^\nu) = q\int\dot{x}_p^\nu(\lambda)\delta^{(4)}(x^\nu-x_p^\nu(\lambda))\mathrm{d}\lambda. \tag{6.2}$$

before the Weyl transformation where x_p^ν denotes the four position of the particle. To find the four position of the particle we take a look again at the matter action caused by the particle

$$S_M = \int \left(-m\sqrt{-g_{\mu\nu}\dot{x}_p^\nu \dot{x}_p^\mu} + q\dot{x}_{p,\mu}A^\mu\right) \mathrm{d}\lambda. \tag{6.3}$$

Since we assumed that the particle is not affected by any other of the particles of the explosion the gauge field A^μ can be set to zero for calculating the trajectory. This means that the four position is solved by only variating $\sqrt{-g_{\mu\nu}\dot{x}^\nu \dot{x}^\mu}$ in the action. This yields the geodesic equation of the form

$$\frac{\partial^2 x_p^\nu}{\partial\lambda^2} + \Gamma^\nu_{\mu\rho}\frac{\partial x_p^\mu}{\partial\lambda}\frac{\partial x_p^\rho}{\partial\lambda} = 0, \tag{6.4}$$

$$\Gamma^\nu_{\mu\rho} = \frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{6.5}$$

Here $\Gamma^\nu_{\mu\rho}$ are the christoffel symbols and λ is an affine parameter that fully parametrizes the particles worldline. Since we take the particle to start moving in the $\hat{z}$ direction and space is homogenuous and isotropic we can take $x_p^1 = x_p^2 = 0$. The derivative can also be expanded using the chain rule

$$\frac{\partial}{\partial\lambda} = \frac{\partial\tau}{\partial\lambda}\frac{\partial}{\partial\tau} = \dot{x}_p^0\frac{\partial}{\partial\tau} = x_p^0\frac{\partial a}{\partial\tau}\frac{\partial}{\partial a} = x_p^0\mathcal{H}a\frac{\partial}{\partial a} \tag{6.6}$$

where a is the scale factor and $\mathcal{H}$ the Hubble parameter. Rewriting the geodesic in terms of $\dot{x}_p^\nu$ with the constraint on x_p^1, x_p^2 and equation 6.6 gives

$$x_p^0\mathcal{H}a\frac{\partial\dot{x}_p^\nu}{\partial a} + \Gamma^\nu_{00}(\dot{x}_p^0)^2 + \Gamma^\nu_{33}(\dot{x}_p^3)^2 + 2\Gamma^\nu_{03}\dot{x}_p^0\dot{x}_p^3 = 0. \tag{6.7}$$

We now note the general constraint on the four velocity of a massive particle that $g_{\mu\nu}\dot{x}_p^\nu\dot{x}_p^\mu = -1$. This constraint allows us to choose only one component to solve in the geodesic equation, where we choose the component $\nu = 0$. In this case for the christoffel symbols $\Gamma^0_{03} = 0$, $\Gamma^0_{00} = \Gamma^0_{33} = \mathcal{H}$. This finally gives us the total set of equations necesarry to solve for the geodesic

$$\begin{cases} \frac{a}{2}\frac{\partial(\dot{x}_p^0)^2}{\partial a} + 2(\dot{x}_p^0)^2 - \frac{1}{a^2} = 0, \\ (\dot{x}_p^3)^2 = (\dot{x}_p^0)^2 - \frac{1}{a^2}. \end{cases} \tag{6.8}$$

This equation has an easy solution given by

$$(\dot{x}_p^0)^2 = \frac{1}{a^2} + \frac{c}{a^4} \tag{6.9}$$

$$(\dot{x}_p^3)^2 = \frac{c}{a^4}. \tag{6.10}$$

The constant c has to be determined by initial conditions. To this end we assume that at initial time $\tau = \tau_i$ the scale factor is normalized so that $a(\tau_i) = 1$, and furthermore that the particle moves in the $\hat{z}$ direction with velocity β so that the initial four velocity has to be given by a Minkowski four velocity of the form $x_p^0(\tau_i) = \gamma, x_p^3(\tau_i) = \gamma\beta$. Inserting this into the solution for x_p^ν finally gives

$$\dot{x}_p^0 = \frac{1}{a}\sqrt{1 + \frac{\gamma^2\beta^2}{a^2}}, \tag{6.11}$$

$$\dot{x}_p^3 = \frac{\gamma\beta}{a^2}. \tag{6.12}$$

This is the final solution for the four velocity of a massive particle moving along a geodesic. The four position can be found by integrating this result with respect to λ with initial value $x_p^0(\tau_i) = \tau_i$ and $x_p^3(\tau_i) = 0$. To integrate for x_p^3 explicit knowledge of the scale factor is necessary whereas for x_p^0 not, which will be shown below. Note that since the particle is only moving for $\tau > \tau_i$ the four velocity and four position are both stricly positive at all times. Remarkable in the four velocity is that it decays with the scale factor, meaning that any particle moving along a geodesic comes to rest at infinite time. The time velocity, which is also the energy divided by its mass, also decays to zero as the scale factor grows, which is perhaps somewhat odd since the particle also has a rest energy given by m. This comes however from the fact that we use conformal time instead of time, so that the time component of the metric tensor also has a scale factor squared. Indeed if we want to calculate the rest energy of the particle we need to use $g_{\mu\nu}\dot{x}_p^\nu \dot{x}_p^\mu$ which was fixed to be -1, so that the particle indeed still has rest energy m. The spatial components however are the same, so any spatial velocity of a particle along a geodesic in fact does decay as space expands.

The solution of x_p^0
The solution for $x_p^0(\tau)$ is trivial since the 0-component of the four position has to equal time, so if we parametrize the four position as a function of time the zero component becomes time. Indeed if we explicitely integrate

$$x_p^0 = \int \dot{x}_p^0 \mathrm{d}\lambda = \int \dot{x}_p^0 \frac{\partial\lambda}{\partial\tau}\mathrm{d}\tau = \int \dot{x}_p^0 \frac{1}{\dot{x}_p^0}\mathrm{d}\tau = \int \mathrm{d}\tau = \tau + c. \tag{6.13}$$

We can choose $c = 0$ without any loss of generality. This finally means x_p^0 is indeed just equal to τ independent on the scale factor. We can now write $x_p^0 = \tau(\lambda)$ where τ depends on λ if necessary. Moreover we can rather choose to work in a τ basis instead of λ basis so that $x_p^0 = \tau$, $x_p^3(\lambda(\tau)) = x_p^3(\tau)$ and the velocities depending on τ as well.

6.2 The charge current

In order to find the charge current we need to insert the newly found coordinates of the particle into equation 6.2. We can also immediately note that $J_M^1 = J_M^2 = 0$. For the other components the most important part is the integral over the delta function. This is actually four times the delta function with only one coordinate λ, so that it turns into three delta functions when integrated, which connect the coordinates where the charge current is observed. We can take $\delta(x)\delta(y)$ out so that the only necessary term to calculate is given by

$$\int \dot{x}_p^\nu(\lambda)\delta(\tau - x_p^0(\lambda))\delta(z - x_p^3(\lambda))\mathrm{d}\lambda. \tag{6.14}$$

Since a is monotonic the function $\lambda(\tau')$ is bijective, so that we can choose τ' as a new coordinate of integration resulting in

$$\int \frac{\dot{x}_p^\nu(\tau')}{\dot{x}_p^0(\tau')}\delta(\tau - \tau')\delta(z - x_p^3(\lambda(\tau')))\mathrm{d}\tau'. \tag{6.15}$$

This integral is easy to work out using the property of the delta function that $\int_{\mathbb{R}} f(x)\delta(x-a)\mathrm{d}x = f(a)$ to give

$$J_M^\nu(x^\nu) = q\frac{\dot{x}_p^\nu(\tau)}{\dot{x}_p^0(\tau)}\delta(x)\delta(y)\delta(z - x_p^3(\tau)), \tag{6.16}$$

where the particles velocity and location have been rewritten in a τ basis. Also we need to insert again that for $\tau < \tau_i$ the particle is at rest at the origin. Rewriting this in spherical coordinates gives explicitly for both nonzero components

$$J_M^0 = \frac{q\delta(\theta)}{2\pi r^2 \sin\theta}\Big(\Theta(\tau_i - \tau)\delta(r) + \Theta(\tau - \tau_i)\delta(r - x_p^3(\tau))\Big), \tag{6.17}$$

$$J_M^3 = \frac{qv(\tau)\delta(\theta)}{2\pi r^2 \sin\theta}\Theta(\tau - \tau_i)\delta(r - x_p^3(\tau)). \tag{6.18}$$

Here $v(\tau)$ is defined as the *coordinate velocity* $v = \frac{\dot{x}_p^3}{\dot{x}_p^0} = \frac{\partial x_p^3}{\partial \tau}$. Under this definition the coordinate x_p^3 can also be written as the integral of the coordinate velocity $x_p^3 = \int v(\tau)\mathrm{d}\tau$. This is the general charge current inside the FLRW spacetime that generates the gauge field. Note that in section 4.2 the charge current also stays the same after the Weyl transform. This charge current largely looks like the charge current as it was defined in Minkowski spacetime, except that the velocity of the particle now also depends on time. For a general scale factor this equation cannot be simplified further. Although the coordinate velocity v is known exactly using equation 6.11 and 6.12 the location x_p^3 can at this moment only be written as the integral over the velocity

$$x_p^3(\tau) = \int\limits_{\tau_i}^{\tau} \frac{\gamma\beta}{\sqrt{a^2(\tau') + \gamma^2\beta^2}} \mathrm{d}\tau'. \tag{6.19}$$

for $\tau > \tau_i$. This integral cannot be solved without knowledge of the scale factor.

6.3 The gauge field

With the new charge current the gauge field becomes a lot harder to calculate. We will from this point work after the Weyl transform of Section 4.2. Again we calculate the gauge field in the Lorentz gauge and transform it to the temporal gauge later on. In the Lorentz gauge the relevant equations of motion are given by

$$\partial_\mu \partial^\mu A^\nu_{LG} = -J^\nu_M. \tag{6.20}$$

The solution for this equation are given by the general Liénard-Wiechert potentials for an arbitrary moving particle, modified for the discontinuity at $\tau = \tau_i$ to give

$$\psi_{LG} = A^0_{LG} = \frac{q}{4\pi} \frac{\Theta(\tau_r - \tau_i)}{R - v(\tau_r)\hat{z} \cdot \vec{R}} + \Theta(\tau_i - \tau_r)\frac{q}{4\pi r} \qquad \vec{R} = \vec{r} - x_p^3(\tau_r). \tag{6.21}$$

Where τ_r is the *retarded time* that solves $\tau_r - R - \tau = 0$ with constraint $\tau_r < \tau$, and the vector potential is given by $\vec{A}_{LG} = \Theta(\tau_r - \tau_i) v(\tau_r) \psi_{LG} \hat{z}$. The heaviside step function can be rewritten as

$$\Theta(\tau_r - \tau_i) = \Theta(\tau - \tau_i - r). \tag{6.22}$$

To see this we look at the definition of the retarded time when $\tau_r = \tau_i$. This gives $\tau_i + |\vec{r} - x_p^3(\tau_i)| - \tau = 0$. The position of the particle x_p^3 is however zero at τ_i since that is the time of the explosion. This means that the equation reduces to $\tau_i + r - \tau = 0$. It follows that $\tau = \tau_i + r$ is equivalent to $\tau_r = \tau_i$. This proves equation 6.22. We should also expect this from causality. If an event happens at $r = 0$ and spacetime is isotropic and homogenuous the signal should reach an observer at a specific distance r at the same time independent on angle. To find the theta component in the temporal gauge we need to use the same transform used in Section 5.1 to give

$$\vec{A}_\theta = \frac{q v(\tau_r)}{4\pi(R - v(\tau_r)\hat{z} \cdot \vec{R})} (\hat{\theta} \cdot \hat{z}) + \frac{q}{4\pi r} \partial_\theta \int\limits_{\tau_i + r}^{\tau} \left(\frac{1}{R - v(\tau_r)\hat{z} \cdot \vec{R}} \right) \mathrm{d}\tau. \tag{6.23}$$

This is only for $\tau > \tau_i + r$, for $\tau < \tau_i + r$ we have $\vec{A}_\theta = 0$. The integral can be simplified by integrating over τ_r instead of over τ giving

$$\vec{A}_\theta = \frac{q v(\tau_r)}{4\pi(R - v(\tau_r)\hat{z} \cdot \vec{R})} (\hat{\theta} \cdot \hat{z}) + \frac{q}{4\pi r} \partial_\theta \int\limits_{\tau_i}^{\tau_r} \frac{1}{|\vec{r} - \hat{z} x_p^3(\tau')|} \mathrm{d}\tau'. \tag{6.24}$$

Note that the lower integration boundary is actually $\tau_r(\tau_i + r)$, however $\tau_r = \tau_i$ solves the equation $\tau_r - |\vec{r} - \hat{z} x_p^3(\tau_r)| - \tau_i - r = 0$. Indeed inserting $\tau_r = \tau_i$ gives $x_p^3(\tau_i) = 0$, from which the result easily follows. This allows us to say $\tau_r(\tau_i + r) = \tau_i$, which is the new integration boundary.

6.3.1 The gauge lambda

To find the function λ as $\tau \to \infty$ we need to solve the equation

$$\Delta\lambda = -Q. \tag{6.25}$$

Here Q is given by the integral of ρ_M over all time giving

$$Q = \frac{q\delta(\theta)}{2\pi r^2 \sin\theta} \int_{\tau_i}^{\infty} \delta(r - x_p^3(\tau))\mathrm{d}\tau = \frac{q\delta(\theta)}{2\pi r^2 \sin\theta} \int_{\tau_i}^{\infty} \frac{1}{v(\tau)} \delta(\tau - (x_p^3)^{-1}(r)))\mathrm{d}\tau = \frac{q\delta(\theta)}{2\pi r^2 \sin\theta} \frac{1}{v((x_p^3)^{-1}(r))}. \tag{6.26}$$

Here $(x_p^3)^{-1}$ denotes the inverse function. The part $v((x_p^3)^{-1}(r))$ is actually a very intuitive quantity: it is the velocity the particle had when passing through the sphere of radius r. This means that the radius of the sphere is now important for the gauge field at $\tau = \infty$. It is also physically logical that the velocity at the sphere needs to be taken and not the initial velocity for example, since the only directly observable velocity at some radius r is the velocity at that radius. This differs fundamentally from Minkowski space since there the velocity is constant. Denote this velocity $v((x_p^3)^{-1}(r)) = v_s(r)$ where the subscript s stand for "sphere". We can now solve the equation for λ using the same Green's function as in section 5.1.1 giving

$$\lambda = \frac{q}{4\pi} \int_{0}^{\infty} \frac{1}{v_s(r')} \frac{1}{|\vec{r} - \hat{z}r'|} \mathrm{d}r'. \tag{6.27}$$

We can simplify this equation to some degree by substituting $r = x_p^3(\tau)$ resulting in

$$\lambda = \frac{q}{4\pi} \int_{\tau_i}^{\infty} \frac{1}{|\vec{r} - \hat{z}x_p^3(\tau)|} \mathrm{d}\tau. \tag{6.28}$$

We can also now identify this lambda in equation 6.24. Indeed when taking the limit $\tau \to \infty$ A_θ reduces to the gradient of λ, since the left part of equation 6.24 goes to zero as $\tau \to \infty$ so only the right side remains, which is the gradient $\nabla_\theta = \frac{1}{r}\partial_\theta$ of λ as given in equation 6.28.

6.4 Phase

For the phase we need to solve equation 4.34 with the use of the gauge field given in Section 6.3. Now we cannot assume M^2 = constant. We do assume ρ_c to be stable in the potential minimum. Looking at the action after the Weyl transform (equation 4.24) this means that

$$\rho_c^2 = \frac{M^2}{2g} \tag{6.29}$$

which is a function of time only. Using the product rule we can now write the phase equation as

$$\rho_c^2 \partial_0^2 \phi + \partial_0 \rho_c^2 \partial_0 \phi - \rho_c^2 \Delta_\Omega \phi = -\rho_c^2 \nabla_\Omega \cdot \vec{A}_\Omega. \tag{6.30}$$

For this equation the temporal gauge was particularly useful: ρ_c^2 is a function of time, so the product rule would make the right hand side of equation 6.30 more complicated, however choosing $A^0 = 0$ removes the time derivative component altogether. This equation can be rewritten in a more neat way using the fact that $2g$ is constant into

$$\partial_0^2 \phi + \partial_0 \log(M^2) \partial_0 \phi - \Delta_\Omega \phi = -\nabla_\Omega \cdot \vec{A}_\Omega. \tag{6.31}$$

To solve this equation we expand both functions ϕ and $\nabla_\Omega \cdot \vec{A}_\Omega$ into Legendre polynomials again

$$\phi = \sum_{\ell=0}^{\infty} a_\ell(\tau) P_\ell \qquad \nabla_\Omega \cdot \vec{A}_\Omega = \sum_{\ell=0}^{\infty} b_\ell(\tau) P_\ell. \tag{6.32}$$

Inserting this expansion back into equation 6.31 gives

$$\partial_0^2 a_\ell + \partial_0 \log(M^2)\partial_0 a_\ell + \omega_\ell^2 a_\ell = -b_\ell. \tag{6.33}$$

Where the same definition for $\omega_\ell^2 = \frac{\ell(\ell+1)}{r^2}$ has been used. The homogenuous equation is still that of a harmonic oscillator, however now it is damped and the damping term is not constant anymore. What this tells us is that while the phase is still determined by oscillating nodes in the spherical harmonic expansion, the amplitude of these nodes change as space expands. In this way the damping term quantitatively adds the expansion of spacetime into the phase. The homogenuous solution accounts for waves traveling over the spherical surface, but these waves slowly decay and after infinite time are gone, since the spherical shell expands with space, so if the energy of the waves stays the same, this energy is smeared out over an infinite surface after infinite time. Equation 6.33 has no general solution for unknown scale factor a, even the homogenuous equation does not. To proceed at this point a simplification without too much loss of generality is needed. First we let the universe consist of a single component only as described in Section 2.3, which means the scale factor is given by

$$a = \left(\frac{\tau}{\tau_i}\right)^{\frac{2}{1+3w}}. \tag{6.34}$$

Remark that choosing only a single component is not a very unphysical ansatz. As long as spacetime is dominated by a single component even though other components are present during the experiment, there is no problem with removing the other altogether. Only a slight error is introduced which will be of small magnitude depending on the manner of dominance of this component. Under this definition of the scale factor the hubble parameter is given by

$$\mathcal{H} = \frac{2}{1+3w}\frac{1}{\tau}. \tag{6.35}$$

From these equations it is clear that if we choose $\tau_i \gg 1/m$ then $\mathcal{H} \ll \frac{m}{a}$, since around $\tau = \tau_i$ $\mathcal{H} \sim \frac{1}{\tau_i}$ whereas $a \sim 1$. This means that we can choose τ_i very large so that

$$M^2 \approx \frac{m^2}{a^2}. \tag{6.36}$$

Equation 6.36 means that the rate of expansion of space is not important anymore relative to the amount space has expanded already. Indeed we assume spacetime to be in a decelerated expansion, which means that at large time the rate of expansion becomes very small, even though space has expanded by a large amount. We want to know if this assumption makes any physical sense or if we accidentaly remove all physically relevant times. To do this we need the order of magnitude of $1/m$. As shown in equation 4.7 this magnitude is given by the magnitude of the correlation length which we took equal to the magnitude of the penetration length which is 10^{-7}m. In units of time this gives 10^{-15}s which is obtained by dividing by c. This shows us that we can easily take the initial conformal time to be as little as one second while introducing an error of only 10^{-30} because the equation for M^2 actually involves a square. An initial conformal time corresponds to an initial normal time by

$$t_i = \int_0^{\tau_i} a(\tau)\mathrm{d}\tau = \tau_i \frac{1+3w}{3+3w}. \tag{6.37}$$

From this we can conclude that the initial conformal time is the same order of magnitude as the initial normal time. This means that the assumption that $\tau_i \gg 1/m$ is very nice, since we only exclude the very first few nanoseconds after the big bang, which obey very different mechanics anyway. We need to remark that effectively in the FLRW spacetime *after the weyl transformation* the penetration length and correlation length actually change over time since the mass in equation 4.7 and 4.8 becomes time dependent. In the original FLRW spacetime we choose $1/m$ as the correlation length however, so that after the transformation the correlation length is the same at $\tau = \tau_i$ because $a(\tau_i) = 1$. Since m is constant we can rewrite equation 6.33 with the help of equation 6.36 into

$$\partial_0^2 a_\ell - \frac{4}{1+3w}\frac{1}{\tau}\partial_0 a_\ell + \omega_\ell^2 a_\ell = -b_\ell. \tag{6.38}$$

Remark that the prefactor $4/(1+3w)1/\tau = 2\mathcal{H}$. The homogenuous equation is solvable with the help of a Bessel function. Using Appendix C we can again find the Green's function for this problem. The process for this is outlined in Appendix C which gives the following Green's function

$$G(\tau,\tau') = \frac{\pi\tau'}{2}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\Theta(\tau-\tau')J_{\alpha_w}(\omega_\ell\tau')B_{\alpha_w}(\omega_\ell\tau). \tag{6.39}$$

Here J_n is the Bessel function of the first kind, B_n the Bessel function of the second kind and α_w is defined as

$$\alpha_w = \frac{5+3w}{2+6w}. \tag{6.40}$$

Remark that the Bessel functions have known explicit forms for half integer values. If we evaluate equation 6.40 for matter dominated $w=0$ and radiation dominated $w=1/3$ we find respectively $\alpha_w = 5/2$ and $\alpha_w = 3/2$. This means that in these two cases the Bessel functions can actually be removed for explicit analytical counterparts. With this Green's function the solution for the coefficients a_ℓ can now finally be written

$$a_\ell = \frac{-\pi\tau^{\alpha_w}B_{\alpha_w}(\omega_\ell\tau)}{2}\int_{\tau_i}^{\tau}\left(\frac{1}{\tau'^{\alpha_w-1}}J_{\alpha_w}(\omega_\ell\tau')b_\ell(\tau')\right)\mathrm{d}\tau'. \tag{6.41}$$

and the phase can be found by summing this solution over all ℓ. Again the value for $\ell=0$ needs to be taken seperately since $B_n(0)$ is not well defined. For a_0 we find by solving equation 6.38 with $\omega_\ell = 0$ that

$$a_0 = \int_{\tau_i}^{\tau}\tau_1^{\alpha_w-1/2}\int_{\tau_i}^{\tau_1}\tau_2^{1/2-\alpha_w}b_0(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 = \int_{\tau_i}^{\tau}\frac{\tau'}{\alpha_w+1/2}\left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2}-1\right)b_0(\tau')\mathrm{d}\tau'. \tag{6.42}$$

The last equality can again be checked by partial integration. Note that for $\alpha_w = 1/2$ this expression is the same as in Minkowski spacetime, a property that will be analyzed later on. We sum over all ℓ and introduce an integration function similar to Section 5.2. With the definition of b_ℓ we write the equation for the phase as

$$\phi(\tau,\theta) = -\int_{\tau_i}^{\tau}\int_0^{\pi}K_1(\tau,\tau',\theta,\theta')\nabla_\Omega\cdot\vec{A}_\Omega(\tau',\theta')\mathrm{d}\tau'\sin\theta'\mathrm{d}\theta', \tag{6.43}$$

$$\begin{aligned}K_1(\tau,\tau',\theta,\theta') = {} & \frac{\pi\tau'}{4}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\sum_{\ell=1}^{\infty}B_{\alpha_w}(\omega_\ell\tau)J_{\alpha_w}(\omega_\ell\tau')(2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta') \\ & + \frac{\tau'}{2(\alpha_w+1/2)}\left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2}-1\right).\end{aligned} \tag{6.44}$$

Written this way the result looks similar to the Minkowski solution. Now K is the Green's function of the operator $\partial_0(\rho_c^2\partial_0) - \rho_c^2\Delta_\Omega$ in the $\tau\to\infty$ and $\tau_i \gg 1/m$ limit. This function resembles much of the Minkowski space one, but now the time dependent part is replaced by Bessel functions, taking the fact that space is not constant into account. Similar to Minkowski spacetime this integral is valid for any gauge field, not just the single particle one.

6.4.1 Approximating Minkowski spacetime

Let us look at the Green's function K a bit closer now. The arguments for the Bessel function are given by either τ or τ'. We let $\tau\to\infty$ so that $B_{\alpha_w}(\omega_\ell\tau)$ can be asymptotically expanded. Now what can we say about τ'? We know that τ' only runs in the interval $[\tau_i,\tau]$. This means we can asymptotically expand $J_{\alpha_w}(\omega_\ell\tau')$ as well if $\omega_\ell\cdot\tau_i \gg 1$. Remember the definition of $\omega_\ell^2 = 1/r^2(\ell(\ell+1))$. The argument is now large enough for an asymptotic expansion if

$$\tau_i \gg r, \tag{6.45}$$

since the values of ℓ need to be arbitrary. We assumed in the past that $r \gg \lambda$ and $\tau \gg \xi$. Since ξ and λ are of the same order we cannot work with these limits alone. The limit $r \gg \lambda$ is easily achieved however as we can take the sphere to be anything larger than a few meters, so that if r is a few million meters $r/c \sim 1$s. If we now take the sphere to be a few billion years after the big bang, which is where we are now, we would have $\tau_i \sim 10^{16}$s. This shows that this limit is a decent approximation in a system like ours a few billion years ago. The introduced error would be of order 10^{-8} since it goes as a square root as well be shown later. For $\tau_i \gg r$ we can asymptotically expand $J_{\alpha_w}(\omega_\ell \tau')$ too. First we note that as shown in Appendix C another appropriate Green's function is instead given by

$$K = \sum_{\ell=1}^{\infty} \left[\frac{\pi\tau'}{8} \left(\frac{\tau}{\tau'}\right)^{\alpha_w} \Theta(\tau-\tau') \Big(J_{\alpha_w}(\omega_\ell\tau')B_{\alpha_w}(\omega_\ell\tau) - J_{\alpha_w}(\omega_\ell\tau)B_{\alpha_w}(\omega_\ell\tau') \Big) \cdot (2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta') \right] + \frac{\tau'}{2(\alpha_w+1/2)} \left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2} - 1 \right). \tag{6.46}$$

As given by [17] the Bessel functions can be asymptotically expanded in the form

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}), \tag{6.47}$$

$$B_n(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}). \tag{6.48}$$

Applying this expansion to the Green's function K gives us a new Green's functions for the $\tau_i \gg r$ limit up to highest order in τ_i/r by

$$K = \sum_{\ell=1}^{\infty} \frac{\tau'}{4\omega_\ell} \frac{1}{\sqrt{\tau\tau'}} \left(\frac{\tau}{\tau'}\right)^{\alpha_w} \Theta(\tau-\tau') \Big(\cos\left(\omega_\ell\tau' - \frac{\alpha_w\pi}{2} - \frac{\pi}{4}\right) \sin\left(\omega_\ell\tau - \frac{\alpha_w\pi}{2} - \frac{\pi}{4}\right) - \cos\left(\omega_\ell\tau - \frac{\alpha_w\pi}{2} - \frac{\pi}{4}\right) \sin\left(\omega_\ell\tau' - \frac{\alpha_w\pi}{2} - \frac{\pi}{4}\right) \Big) (2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta') + \frac{\tau'}{2(\alpha_w+1/2)} \left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2} - 1 \right). \tag{6.49}$$

Note that the the τ and τ' not inside the cosine and sine can be taken out of the sum. Also the trigonometric identity $\sin(a)\cos(b) - \cos(a)\sin(b) = \sin(a-b)$ is useful at this point. With this we can finally write the Green's function as

$$K = \left(\frac{\tau}{\tau'}\right)^{\alpha_w - \frac{1}{2}} \Theta(\tau-\tau') \sum_{\ell=1}^{\infty} \frac{\sin(\omega_\ell(\tau-\tau'))}{4\omega_\ell} (2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta') + \frac{\tau'}{2(\alpha_w+1/2)} \left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2} - 1 \right). \tag{6.50}$$

Recognise that the sum is actually the same sum as in the Green's function in Minkowski space! The only difference is that the normal time has been replaced by conformal time, this has little consequence however since it is just a change of variables. From this we can also read that FLRW spacetime reduces to Minkowski when $\alpha_w = 1/2$. For the value of $\alpha_w = 1/2$ the Bessel function asymptotic expansions become exact (the $O(x^{-1})$ terms vanish) so the Green's function is not an aproximation anymore. Solving the equation for α_w gives $w \to \infty$. Indeed for $w \to \infty$ the scale factor also turns time independent. From this we can conclude that Minkowski spacetime actually physically corresponds with a vanishing density. If the density does not vanish the pressure blows up to infinity which physically cannot happen. This means that the Minkowski metric is actually only valid in the absence of energy. This is of course not true in our universe, however on small scales the approximation is valid and useful. We can also think about what happens if we keep increasing τ_i. The universe is undergoing a decelerated expansion, so we expect it to become stationary and imitate Minkowski spacetime after some redefining of variables as $\tau \to \infty$. This means that if we let $\tau_i \to \infty$ while keeping the interval of integration nonzero we should approach the Minkowski solution. Let us take the limit of $\tau_i \to \infty$ equally fast as $\tau \to \infty$ so that

$$\frac{\tau}{\tau_i} \to 1. \tag{6.51}$$

This means that the scale factor becomes time independent in this specific limit. Similarly the prefactor $\left(\frac{\tau}{\tau'}\right)^{\alpha_w-1/2} \to 1$ because $\tau' \in [\tau_i, \tau]$ which means $\tau' \to \infty$ equally fast as well by the squeeze theorem. The part for a_0 is a little bit different because it tends to zero as $\tau' \to \tau$, however the first order contribution in its taylor series around $\tau' = \tau$ is given by

$$\frac{\tau'}{2(\alpha_w+1/2)}\left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2}-1\right) \approx \frac{1}{2(\alpha_w+1/2)}\cdot\Big(-(\alpha_w+1/2)(\tau'-\tau)\Big) = \frac{1}{2}(\tau-\tau'). \tag{6.52}$$

We now see that in the $\tau_i \to \infty$ limit, up to highest order in τ' the Green's function K of FLRW spacetime matches exactly that of the Minkowski spacetime function F. This means that in this limit we indeed effectively approach the Minkowski spacetime and the memory effect also becomes identical. It is tricky to find this since the integration domain becomes an empty space. If we now choose to have taken the limits and extend τ a little bit more to obtain an integration domain we can fix this problem. In this case the phase becomes exactly that of Minkowski space. This means that indeed space becomes stationary as $\tau \to \infty$. We can see more importantly that the FLRW solution for the phase becomes a function of τ_i, which should asymptotically converge to the Minkowski solution.

6.5 The fundamental solution in an expanding universe

The total memory effect can now be written as a set of equations

$$\phi(\tau,\theta) = -\lim_{\tau\to\infty}\int_{\tau_i}^{\tau}\int_{S^2} F(\tau,\tau',\hat{r},\hat{r}')\nabla_\Omega\cdot\vec{A}_\Omega(\tau',\hat{r}')\mathrm{d}\tau'\mathrm{d}\Omega' - \frac{q}{4\pi}\int_{\tau_i}^{\infty}\frac{1}{|\vec{r}-\hat{z}x_p^3(\tau)|}\mathrm{d}\tau, \tag{6.53}$$

$$\begin{aligned} K_1(\tau,\tau',\theta,\theta') &= \frac{\pi\tau'}{4}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\sum_{\ell=1}^{\infty}B_{\alpha_w}(\omega_\ell\tau)J_{\alpha_w}(\omega_\ell\tau')(2\ell+1)P_\ell(\cos\theta)P_\ell(\cos\theta') \\ &\quad+\frac{\tau'}{2(\alpha_w+1/2)}\left(\left(\frac{\tau}{\tau'}\right)^{\alpha_w+1/2}-1\right), \end{aligned} \tag{6.54}$$

$$\nabla_\Omega\cdot\vec{A}_\Omega = \frac{q\Theta(\tau-\tau_i-r)}{4\pi r\sin\theta}\partial_\theta\left(\frac{-r\sin^2\theta v(\tau_r)}{(R-v(\tau_r)\hat{z}\cdot\vec{R})}+\partial_\theta\int_{\tau_i}^{\tau_r(\tau)}\frac{1}{|\vec{r}-\hat{z}x_p^3(\tau')|}\mathrm{d}\tau'\right). \tag{6.55}$$

Here x_p^3 is given by equation 6.19. With these equations we have succesfully determined the full solution for the memory effect in FLRW spacetime. Although further analytical progression is not possible at this point, the gauge field can easily numerically be calculated. Similarly if the function K_1 can be numerically estimated the final result is easily integrated. The result resembles that of the Minkowski spacetime, because the Bessel functions are oscillating functions in time, except with decaying amplitude. This means in other words that the periodic sines have been replaced by Bessel functions where the amplitude of oscillation decays which characterizes the expansion of space. The Green's function K_1 also has other time dependences which the Minkowski function K does not have. These characterize not the rate of expansion of space but the amount it has expanded at that time.

We can again look at a single component of the Legendre expansion only. In this case we still measure q, β and r but now we also measure the initial time τ_i which introduces a new observable parameter. Because the initial time affects the memory we can now also measure it. This means we can determine when the explosion happened. We have also seen that the Green's function K_1 asymptotically converges to Minkowski spacetime meaning that if we measure at very large time we find up to leading order the Minkowski solution with a small deviation that goes as $1/\tau_i$. In short we have explicitely calculated the memory effect in FLRW spacetime. It is only different from the Minkowski solution by time-dependent quantities. This is expected because the metric $g_{\mu\nu}$ is also only time-dependent. The current is analytically given by equation 6.53 to 6.55. It depends on time of measurement, the particle's properties and now also on the time of the explosion. A final note is that the current also depends on w. This means that by measuring the current we can find what type of energy currently dominates the universe.

7 Conclusion, Discussion and Outlook

7.1 Conclusion

The aim of this paper was to analyze the electromagnetic memory effect in an expanding universe and provide an explicit calculation for the memory effect as it was proposed by [1]. To do this we first gave a short overview of the necessary knowledge of spacetime mechanics and the FLRW metric. In Section 3 we defined the memory effect, showed prior research and its importance. Moving on in Section 4 we derived the necessary action to proceed with the calculation. The concept of spontanuous symmetry breaking was introduced to explain the derivation of this action. From this action we derived some elementary properties of a superconductor from a field theoretic framework. Also the difference between a type I and type II superconductor was shown. The action exhibited a $U(1)$ symmetry with a connected conserved current that is fundamental for the memory effect. We have shown that the superconductor is mostly invariant under a Weyl transformation from Minkowski spacetime to FLRW spacetime. A remarkable equivalence between the expanding universe and stationary universe is clearly seen in this transformation. In the next section we started to explain the memory effect as proposed by Susskind [1]. We gave an explanation on the system and how we approach calculating it. It was important to note that we assume there are no spatial currents meaning the gauge field was determined by the moving particles only, which we assumed noninteracting. For the calculation we have proven that the divergence of the radial field can not be neglected in a large r approximation, and how spherically symmetric charge distributions do not alter the memory effect.

The original findings of the paper are the exact Weyl transformation and Symmetry analysis (Section 4.2 and 4.4) and the calculation. Other original findings include the some interpretations and minor sections that have no source. To give a familiar approach we first completed the calculation in flat Minkowski spacetime. In this case the gauge field is already known by the Liénard Wiechert solution. Proceeding to calculate the phase with a Green's function method we found an explicit expression for the final memory effect as given in Section 5.3.

Finally we calculate the memory effect in an expanding universe. First we solved the geodesic equation for a free moving particle in the FLRW metric. This result gave us the charge distribution from which we derived the gauge potential. With this gauge potential we calculate the necessary phase in the same way as Minkowski spacetime. In order to proceed we needed to make the ansatz of a single component universe with initial time $\tau_i \gg \xi$ which proved to be very good. With this ansatz we successfully calculated the memory effect in the FLRW metric aswell with solution given in Section 6.5. To conlude the paper we showed that in the appropriate limits the memory effect approaches that of Minkowski space. This is perfectly in line with out earlier ansatz that space is in a decelerated expansion.

7.2 Discussion

We successfully calculated the finaly memory effect analytically, giving an explicit formula in Section 5.3 and 6.5. It is noteworthy that we have also derived Green's functions along the road so that an extension to any gauge field can easily be made. In order to achieve the result in an expanding universe we needed to make a few approximations, namely that of a single component universe and that $\tau_i \gg \xi$. Nevertheless the latter introduced an error in the phase of order of magnitude as small as 10^{-30} being completely irrelevant. The former introduced an error with less clarity on its magnitude, however since w is still arbitrary this error would mostly depend on the duration of the experiment. Since this duration of the experiment is not necessarily very large this error should not pose a problem either. Nevertheless we can also consider the theoretical case of a single component universe in which there is no error at all. In order to check the calculation we did a few methods. The first and easiest was checking all of the equations through a dimensional analysis test which they passed. Secondly some equations could have been derived in a few different ways, for example the gauge field and the gauge λ. We attempted all of these methods and the same result was found. Thirdly we can look at the correspondence principle: the equations have to reduce to familiar equations in the appropriate limits or special cases. This indeed also proved to be the case. Finally all equations that *wolfram mathematica* can do, have been inserted into mathematica to find the same result. Of course these methods could not have been done on all equations, since some equations involve quantities that have no analytical representation.

The calculation might involve a complication yet we have no proofs on this. The calculation is done by solving the conserved current equation 4.34. This equation might however not be completely valid because we implicitly added a constraint that there should be no spatial currents. We have not inserted an explicit constraint, but rather assumed ρ_c to be in the potential minimum and ignored the effect of the current on the gauge field. This makes sure that ρ_c is determined with the constraint, and the gauge field as well. The phase however makes no use of the constraint. If one were to insert the constraint explicitely, it might be found that the equation for the phase that we have used is not completely valid. We have not explicitely checked this. To make sure the calculation is correct or prove that something went wrong a deeper analysis into the constraint would be needed.

For the calculation we made use of the temporal gauge with $A(0) = 0$. We could instead have used $A(\infty) = 0$ or a completely different gauge, for example the Lorentz gauge. This is a choice that we have made. It might be however that the calculation is easier in a different gauge. In the Lorentz gauge for example we have certainty that if $A(\infty) = \nabla\lambda$ then $\lambda = 0$. This follows from the fact that λ needs to obey $\partial_\mu\partial^\mu\lambda = 0$. This equation only has waves as solution, which we set to zero from boundary conditions. Then the change in phase is the only remaining part.

7.3 Outlook

This paper provides quite a lot of opportunity for further research. One of these is mentioned in the discussion. It is possible to attempt and extend this paper to an accelerated universe $w < -1/3$. This would involve a very different approach, since particles never reach an infinite coordinate distance in conformal time, but conformal time is also bound on a finite interval. In fact a conformal time approach would mean conformal time is bound finitely everywhere, providing limits on the allowance of Fourier transforms. Nevertheless it is an interesting case to try and analyze given the fact that we are currently in an accerelated universe. Another possibility is to redo the calculation for the case that the nodes are electromagnetically bound to each other so that $r = R/a$, so a time dependent radius. This only slightly alters the phase equation, but perhaps to unsolvable levels. We have not checked this explicitely. This setup is physically more relevant however because any real spherical shell of superconducting nodes would somehow be constrained so that the nodes have the same physical distance from each other at all times. Additionaly we calculated the memory effect for massive particles, however it is also possible to do the calculation for massless particles. This would be very different because the gauge field is not smooth defined in this case, rather it is singular on the plane tangent to the particles trajectory, and the particles move over null geodesics. This invites us to use null infinity instead of timelike infinity and also allows for analysis into infrared behaviour. Finally as shown in Section 3.2 the memory effect has many relationships to other physical theories. It could be a possibility to try and find these relationships again in this setup. In his original note Susskind did mention that the gauge field equation can be written in a form so that it resembles the soft photon emission theorem, however we did not look into this [1]. The connection has not been priorly investigated in this setup so it is worth looking into.

8 Acknowledgement

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A Stereographic coordinates

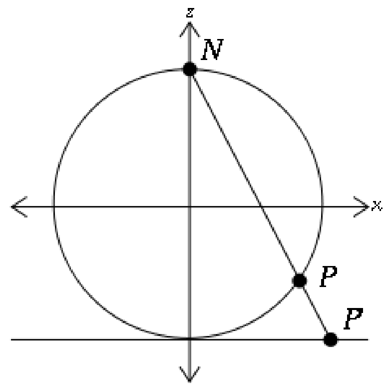

Figure 8: An image illustrating the definition of stereographic coordinates. The angles θ, ϕ of point P are mapped to the coordinates x, y of P' where now the stereographic coordinate $z = x + iy$ [18].

Stereographic coordinates are coordinates that describe a point on a two-sphere using a single complex coordinate. Although the coordinates themselves become more abstract, the equations that require them become a lot easier because of removal of the angles, making all coordinates range from $-\infty$ to $+\infty$ and removing the $\sin^2\theta$ in the standard spherical metric. The concept of stereographical coordinates is obtained in the following way: First the two-sphere of importance is placed on an infinite plane, the south pole of the sphere just touching it. Then you take a line from the north pole to the plane, and the point that it touches on the plane corresponds to the point that it intersects with on the sphere. This process is visualized in Figure 8. The coordinate transformation is bijective so no information is lost and it has the following shape:

$$z = \frac{x^1 + ix^2}{x^3 + r} \qquad z = \frac{\sin\theta(\sin\phi + \cos\phi)}{1 + \cos\theta}, \tag{A.1}$$

with x^1, x^2, x^3 the standard carthesian coordinates. This transformation is reversed by:

$$\phi = \arctan\left(\frac{z - \bar{z}}{i(z + \bar{z})}\right) \qquad \theta = \arccos\left(\frac{1 - z\bar{z}}{1 + z\bar{z}}\right), \tag{A.2}$$

where the arctan technically needs to be the so-called atan2 to correct for the corresponding domain. This definition can be inserted into the Minkowski metric equation to yield:

$$d\Omega^2 = 2\gamma_{z\bar{z}} dz d\bar{z} \qquad \gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}, \tag{A.3}$$

where $\gamma_{z\bar{z}}$ is the metric of the stereographical coordinates on the two-sphere that can be obtained through a standard tensor transformation from spherical to stereographic coordinates. Indeed when $\gamma_{z\bar{z}}$ is written in matrix form

$$\gamma_{z\bar{z}} = \begin{pmatrix} 0 & \frac{2}{(1+z\bar{z})^2} \\ \frac{2}{(1+z\bar{z})^2} & 0 \end{pmatrix}, \tag{A.4}$$

this is the metric tensor that generates dΩ. We define $\gamma_{z\bar{z}}$ as the metric, so that $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0$ and $\gamma_{z\bar{z}}$ given above.

B Differential forms

For this section [19] and [20] have been consulted, and equations B.10-B.21 have been self derived. Differential forms are a compact and efficient method of doing manipulations on integrals and differential equations. A form is technically a mathematical scalar, however it sums over elements that are orthogonal to each other so it containts information in multiple dimensions, in the same way that complex numbers can hold two-dimensional information in a single scalar. A p-form is a form with a p-element basis. A form is defined by contracting a totally antisymmetric tensor with basis elements, so if A is a differential p-form and $A_{\mu_1\cdots\mu_p}$ a tensor A is defined by:

$$A = \frac{1}{p!} A_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}, \tag{B.1}$$

where $\wedge$ denotes the wedge product given by:

$$A \wedge B = \frac{(p+q)!}{p!q!} A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p+q}}, \tag{B.2}$$

where A is a p-form and B a q-form. The [...] brackets denote antisymmetrisation. Furthermore there are two actions defined on p-forms, the hodge star on a p-form A in d dimensions:

$$\star A = \frac{1}{p!} \epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p}} A^{\mu_1\cdots\mu_p} dx^{\nu_1} \wedge ... \wedge dx^{\nu_{d-p}}, \tag{B.3}$$

where ϵ is the levi civita tensor in d dimensions. $\star A$ is a (d-p)-form, meaning that $A \wedge \star A$ is a d-form, so that integrating it is in fact a standard volume integral. For this reason $\star A$ is sometimes called the hodge dual, because it's the dual to A that makes it integrable over all space. The hodge dual also has the following important property for a p-form A on a Lorentz manifold with dimension d:

$$\star\star A = (-1)^{p(q-p)+1} A. \tag{B.4}$$

The other action on p-forms is the exterior derivative defined by:

$$\mathrm{d}A = (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p+1}}, \tag{B.5}$$

which is the normal notion of the derivative on forms. The exterior derivative maintains the following basic property:

$$\mathrm{d}^2 = 0 \tag{B.6}$$

for all forms on which it acts. This easily follows from the antisymmetrization and the fact that the differential operators commute. For a p-form A and q-form B the following property holds:

$$\mathrm{d}(A \wedge B) = (\mathrm{d}A) \wedge B + (-1)^p A \wedge \mathrm{d}B. \tag{B.7}$$

Forms are particularly useful for integrals because of the following definition on d-forms A where d is the amount of dimensions of the system:

$$\int A = \int \sum_{\mu_1\cdots\mu_d} A_{\mu_1\cdots\mu_d} \sqrt{-g}\ d^d x. \tag{B.8}$$

For integrals over forms a generalization of stokes' integral theorem holds:

$$\int_{\partial\Omega} A = \int_{\Omega} \mathrm{d}A, \tag{B.9}$$

for a form A and a manifold Ω with boundary $\partial\Omega$.

Differential forms have the following property as well for two forms p-forms A and B:

$$A \wedge \star B = B \wedge \star A, \tag{B.10}$$

which can easily be seen through the property that

$$(A \wedge \star B)_{\mu_1 \cdots \mu_d} = \frac{1}{p!} A_{\nu_1 \cdots \nu_p} B^{\nu_1 \cdots \nu_p} \epsilon_{\mu_1 \cdots \mu_d}. \tag{B.11}$$

This last identity is equal to the fact that

$$\int A \wedge \star B = \int \frac{1}{p!} A_{\nu_1 \cdots \nu_p} B^{\nu_1 \cdots \nu_p} \sqrt{-g} d^d x. \tag{B.12}$$

A generalization of this equation is given by

$$\star (A \wedge \star B)_{\mu_1 \cdots \mu_{(q-p)}} = \frac{(-1)^{(q-p)(d-q)+1}}{p!} A_{\tau_1 \cdots \tau_p} B^{\tau_1 \cdots \tau_p}{}_{\mu_1 \cdots \mu_{(q-p)}}. \tag{B.13}$$

With A a p form and B a q form, where $q > p$.

Forms in spacetime

For using forms it is important to note how they transform. First it is important to note that in spacetime $d = 4$. Using equation B.13 it follows that:

$$F_{\mu\nu} F^{\mu\nu} = -2 \star (F \wedge \star F), \tag{B.14}$$
$$A_\mu J^\mu = - \star (A \wedge \star J), \tag{B.15}$$
$$\partial_\mu \psi \partial^\mu \psi^* = - \star (\mathrm{d}\psi \wedge \star \mathrm{d}\psi^*), \tag{B.16}$$
$$(\star d \star F)^\nu = -\partial_\mu F^{\mu\nu}, \tag{B.17}$$
$$(\star \star J)^\mu = J^\mu, \tag{B.18}$$
$$\star \star \psi = -\psi. \tag{B.19}$$

Although letters that will be used a lot have been used in the above, these equations hold for any forms that have the same type of tensor structure. When we want to write the lagrangian in form notation, the hodge dual of the above identities needs to be used:

$$\int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} d^4 x = \int \left(-\frac{1}{4} \star F_{\mu\nu} F^{\mu\nu} \right) = \int \left(\frac{1}{2} \star \star (F \wedge \star F) \right) = \int \left(-\frac{1}{2} F \wedge \star F \right). \tag{B.20}$$

It can now finally be shown that

$$\int \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right] \sqrt{-g} d^4 x = \int \left[-\frac{1}{2} F \wedge \star F - A \wedge \star J \right]. \tag{B.21}$$

C Green's function

For solving the inhomogenuous equations arising in the phase calculation having a Green's function is an optimal method, so we need to find the Green's function. This is possible using a method outlined by D. Skinner [21]. For a general second order ordinary inhomogenuous differential equation

$$\mathcal{L}f(t) = \alpha(t)\frac{\partial^2 f(t)}{\partial t^2} + \beta(t)\frac{\partial f(t)}{\partial t} + \gamma(t)f(t) = g(t), \tag{C.1}$$

the Green's function that solves $\mathcal{L}G(t,t') = \delta(t-t')$ automatically gives the full solution for the inhomogenuous equation

$$f(x) = \int_{\mathbb{R}} G(t,t')g(t')\mathrm{d}t'. \tag{C.2}$$

Let f_1, f_2 denote the solution of the homogenuous equation where $g(t) = 0$. With this knowledge we can write the Green's function as

$$G(t,t') = \frac{1}{\alpha(t')W(t')}\Big(\Theta(t'-t)f_1(t)f_2(t') + \Theta(t-t')f_1(t')f_2(t)\Big). \tag{C.3}$$

Here $W(t)$ is the Wronskian given by $W = f_1 f_2' - f_2 f_1'$ where the prime denotes the derivative. Using this definition the Wronskian can also be written as a function of the coefficients of the differential equation immediately

$$W'(t) = -\beta(t)W(t). \tag{C.4}$$

With these identities the Green's functions can be derived. These Green's functions will be defined on the interval $[t_i, \infty] \times [t_i, \infty]$, and since we will be interested in the limit $t \to \infty$ only the part with $\Theta(t-t')$ remains. We will now discuss the two specific cases.

Minkowski spacetime

In Minkowski spacetime we need the Green's function for the equation

$$\partial_0^2 a_{\ell m} + \omega_\ell^2 a_{\ell m} = -b_{\ell m}. \tag{C.5}$$

The homogenuous equation can be rewritten into

$$\partial_0^2 a_{\ell m} = -\omega_\ell^2 a_{\ell m}. \tag{C.6}$$

This equation has two trivial solutions

$$f_1 = e^{-i\omega_\ell t}, \tag{C.7}$$

$$f_2 = e^{i\omega_\ell t}\ . \tag{C.8}$$

With these solutions the Wronskian is clearly given by $W = 2i\omega_\ell$. We can now write the total Green's function since $\alpha = 1$ as

$$G(t,t') = \frac{1}{2i\omega_\ell}\Big(\Theta(t'-t)e^{-i\omega_\ell(t-t')} + \Theta(t-t')e^{i\omega_\ell(t-t')}\Big). \tag{C.9}$$

Using eulers identity $e^{i\omega_\ell(t-t')} = \cos(\omega_\ell(t-t')) + i\sin(\omega_\ell(t-t'))$ the imaginary part can be taken out to give

$$G(t,t') = \frac{\sin(\omega_\ell(t-t'))}{2\omega_\ell}\Big(\Theta(t-t') - \Theta(t'-t)\Big). \tag{C.10}$$

This is finally the total real Green's function for the inhomogenuous equation.

FLRW spacetime

For the FLRW metric equation 6.38 we need to find the Green's function again. First we analyze the homogenuous equation

$$\partial_0^2 a_{\ell m} - \frac{4}{3w+5)}\frac{1}{\tau}\partial_0 a_{\ell m} + \omega_\ell^2 a_{\ell m} = 0. \tag{C.11}$$

This equation has the following homogenuous solutions

$$\begin{aligned} f_1 &= \tau^{\alpha_w} J_{\alpha_w}(\omega_\ell \tau), & \text{(C.12)} \\ f_2 &= \tau^{\alpha_w} B_{\alpha_w}(\omega_\ell \tau), & \text{(C.13)} \end{aligned}$$

where J_n is the Bessel function of the first kind, B_n the Bessel function of the second kind and the constant α_w has been defined as

$$\alpha_w = \frac{9+3w}{10+6w}. \tag{C.14}$$

For the Wronskian of the equation we write

$$\begin{aligned} W &= \tau^{\alpha_w} J_{\alpha_w}(\omega_\ell \tau)\frac{\partial}{\partial \tau}(\tau^{\alpha_w} B_{\alpha_w}(\omega_\ell \tau)) - \tau^{\alpha_w} B_{\alpha_w}(\omega_\ell(\tau))\frac{\partial}{\partial \tau}(\tau^{\alpha_w} J_{\alpha_w}(\omega_\ell \tau)) & \text{(C.15)} \\ &= \tau^{2\alpha_w}\omega_\ell\bigg(J_{\alpha_w}(\omega_\ell\tau)\frac{\partial}{\partial \omega_\ell \tau}B_{\alpha_w}(\omega_\ell\tau) - B_{\alpha_w}(\omega_\ell(\tau))\frac{\partial}{\partial \omega_\ell\tau}J_{\alpha_w}(\omega_\ell\tau)\bigg) & \text{(C.16)} \\ &= \frac{2}{\pi}\tau^{2\alpha_w - 1}. & \text{(C.17)} \end{aligned}$$

With this Wronskian and the homogenuous solutions the Green's function can now be written down as

$$G(\tau,\tau') = \frac{\pi\tau'}{2}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\bigg(\Theta(\tau'-\tau)J_{\alpha_w}(\omega_\ell\tau)B_{\alpha_w}(\omega_\ell\tau') + \Theta(\tau-\tau')J_{\alpha_w}(\omega_\ell\tau')B_{\alpha_w}(\omega_\ell\tau)\bigg). \tag{C.18}$$

Since we are interested in the $\tau \to \infty$ limit only again, we can rewrite the Green's function for this limit as

$$G_1(\tau,\tau') = \frac{\pi\tau'}{2}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\Theta(\tau-\tau')J_{\alpha_w}(\omega_\ell\tau')B_{\alpha_w}(\omega_\ell\tau). \tag{C.19}$$

This is however not the only possible Green's function. Another valid Green's function can be obtained by swapping f_1 and f_2. When swapping this the sign of the wronskian is changed and the argument of the Bessel function. This gives for the second Green's function

$$G_2(\tau,\tau') = -\frac{\pi\tau'}{2}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\Theta(\tau-\tau')J_{\alpha_w}(\omega_\ell\tau)B_{\alpha_w}(\omega_\ell\tau'). \tag{C.20}$$

The Green's function of equation C.11 can now be written as the sum of these two. Indeed if $\mathcal{L}G(\tau,\tau') = \delta(\tau-\tau')$ is the equation for a Green's function, and both G_1 and G_2 are Green's functions, then $\mathcal{L}(\frac{1}{2}(G_1 + G_2)) = \delta(\tau-\tau')$. This finally means that $G = \frac{1}{2}(G_1+G_2)$ so we can write

$$G(\tau,\tau') = \frac{\pi\tau'}{4}\left(\frac{\tau}{\tau'}\right)^{\alpha_w}\Theta(\tau-\tau')\bigg(J_{\alpha_w}(\omega_\ell\tau')B_{\alpha_w}(\omega_\ell\tau) - J_{\alpha_w}(\omega_\ell\tau)B_{\alpha_w}(\omega_\ell\tau')\bigg). \tag{C.21}$$

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