Seiberg-Witten Theory and the Intersection Form

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Introduction

FOUR-MANIFOLDS live in a strange world. Very much like three-manifolds, their dimension is high enough so that the classical tools used to study curves and surfaces fall short. On the other hand, in the smooth realm, their dimension is too low, and there is not "enough room" to apply advanced techniques of techniques of differential topology to them. Historically, this meant that for the greater part of the 20th century, there was little knowledge of four-manifolds, relative to their higher- and lower-dimensional counterparts.

In the early 80's, Freedman proved that a fundamental tool for studying manifolds of dimension five and greater could *still* be applied for four-manifolds, but ignoring the smooth structure. With these ideas, he proved a complete classification of simply-connected *topological* four-manifolds in terms of their *intersection form*.

Intuitively, the intersection form is a bilinear form over the integers which describes the intersection numbers of embedded surfaces of a four-manifold. Freedman [Fre82] proved that every nondegenerate, symmetric bilinear form over the integers is the intersection form of exactly one or two simply-connected *topological* manifolds. In the case where an intersection form corresponds to two manifolds, *at least* one of them is non-smoothable. This tells us that in order to classify simply-connected four-manifolds up to homeomorphism, it suffices to understand all nondegenerate symmetric bilinear forms; and fortunately we have a *partial* classification of all of them, due to Serre. It also tells us that the is a huge amount of topological four-manifolds which do not admit any smooth structure.

Does this classification hold up in the smooth realm?

The answer is an astounding *no*. By examining the moduli space of anti-self-dual SU(2) connections, Donaldson [Don83] showed that all the *definite* intersection forms of *smooth* manifolds are diagonal. This shows that in the realm of smooth manifolds, the power of the intersection form is severely limited. This study of the anti-self-dual connections on manifolds is known as *Donaldson theory*, and throughout the 80s it was used as a powerful tool for studying smooth four-manifolds.

In the late 80's, Witten [Wit88] showed that Donaldson theory could be understood from the point of view of the high-energy limit of a (mysterious) supersymmetric quantum field theory. This proved useful when in the early 90's, Seiberg and Witten [SW94a; SW94b] found a way to obtain its low-energy limit. For mathematicians, this returned a set of equations defined for *spinors* on manifolds, and the study of their moduli space, known as Seiberg-Witten theory [Wit94], turned out to be another excellent tool for the study of four-manifolds.

In this work, we present a proof of Donaldson's theorem on the intersection form using the Seiberg-Witten moduli space, which is attributed to Kronheimer and Elkies. Our intention is for this work to be a reasonable introduction to both Seiberg-Witten theory and the algebraic topology of the intersection form, aimed at masters' students.

This work is organized as follows. In Chapter 1, we present most of the preliminary notions in algebraic topology, gauge theory, differential geometry, and analysis that is needed to discuss the construction of the Seiberg-Witten moduli space, and how it determines the intersection form. In Chapter 2, we introduce the notion of Spin and Spin^c structures, which are *the* fundamental objects of the Seiberg-Witten equations. We try to be thorough and for the most part, we don't limit ourselves to the case of dimension four. This is to show that many of the properties of spinors on four-manifolds are general ideas that are not specific to dimension four. In Chapter 3 we discuss the construction of the Seiberg-Witten moduli space, and show that it is generically smooth, finite-dimensional, oriented, and compact. This is the most technically demanding chapter, and it requires some results from the analysis of PDE and K-theory that we only state or lightly sketch. Finally, in Chapter 4, we introduce the intersection form, study some of its properties. We close the work with the proof of Donaldson's theorem, using Seiberg-Witten theory.

Of course, due to time and space constraints, there is a lot that had to be left out. Seiberg-Witten theory is broad and intersects with many areas of mathematics. Some great references for Seiberg-witten theory are [Don96; EF97; Kla13; Mar99; Nic00; Nab05; Sal99; Moo96; Mor96]. Our main references are [Nic00; Sal99; Mor96; Sco05].

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CHAPTER 1

Preliminaries

In this chapter we review the basic language of algebraic topology and differential geometry that is needed to discuss Seiberg-Witten theory and the Donaldson theorem. In the differential realm, will assume that the reader is familiar with basic notions and tools of smooth manifolds, de Rham cohomology, vector bundles and some Riemannian geometry. In algebraic topology, we will assume familiarity with singular and cellular homology.

1.1 Gauge theory

Gauge theory is the heart of Seiberg-Witten theory[1]. The name *gauge theory* comes from physics, where gauge fields are a particular class of fields that have many degrees of freedom. Mathematically, the study of gauge fields is done in terms of principal bundles and other structures associated to them. In this section, we will quickly review the basic notions of principal bundles, connections and curvature, and "matter fields". Great comprehensive references are [Nab11a], [Nab11b] and [KN96].

1.1.1 Principal bundles, connections and curvature

The basic objects of gauge theory are *principal bundles*.

Definition 1.1.1 (Principal G-bundle).
Let M be a smooth manifold and G be a Lie group. A **principal G-bundle** *over M is a smooth manifold P and a smooth surjective map $\pi : P \to M$, along with a right action of G on M, satisfying:*

1. *The action of G preserves the fibers: $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$.*
2. *P is a* fiber bundle *with typical fiber G, and the trivialization can be chosen G-equivariant: For every $x \in M$ there is a neighborhood U of x and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ satisfying $\mathrm{pr}_1 \circ \Psi = \pi$ and*
$$\Psi(p \cdot g) = \Psi(p) \cdot g,$$

[1] And Donaldson theory too.

where the action of G on $U \times G$ is right multiplication on the second component.

We write this as $G \hookrightarrow P \xrightarrow{\pi} M$.

The main example that we will work with is the frame bundle of a vector bundle.

Example 1.1.2 (Frame bundles).
Let $E \to M$ be a $\mathbb{K}$-vector bundle of rank k. For each $x \in M$, let $\mathrm{Fr}_x(E)$ be the set of *frames* of E_x, i.e. the set of bases of E_x. The union of all such $\mathrm{Fr}_x(E)$ gives us the *frame bundle*

$$\mathrm{Fr}(E) = \bigsqcup_{x \in M} \mathrm{Fr}_x(E).$$

There is a natural $\mathrm{GL}(k, \mathbb{K})$ action on the fibers: if $e = \{e_i\}$ is a frame of E_x, then for any $A \in \mathrm{GL}(k, \mathbb{K})$ there is another frame $e' = \{e'_i\}$ given by

$$e'_i = \sum_j A_{j,i} e_j$$

. This is a free action which is transitive on the fibers. A trivialization of E determines both a smooth structure on $\mathrm{Fr}(E)$ and a trivialization of it as a principal $\mathrm{GL}(k, \mathbb{K})$-bundle.

If E is a real Riemannian vector bundle, this process can be repeated to obtain the *orthogonal* frame bundle $\mathrm{O}(E)$, which is a principal $\mathrm{O}(k)$-bundle over M.

Similarly, if E is a complex Hermitian bundle, we can obtain the *unitary* frame bundle $\mathrm{U}(E)$, which is a principal $\mathrm{U}(k)$-bundle over M.

An important fact of principal bundles is that local trivializations are uniquely determined by local sections, and vice-versa. Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle and $\{U_i\}_{i \in I}$ a trivializing cover with trivializations $\Psi_i : \pi^{-1}U_i \to U_i \times G$. For each i, there is an associated local section $s_i : U_i \to \pi^{-1}(U_i)$ given as $s_i(x) = \Psi_i^{-1}(x, e)$, where $e \in G$ is the identity. Conversely, every local section $s : U \to \pi^{-1}(U)$ determines a trivialization $\Psi : \pi^{-1}(U) \to U \times G$, precisely in such a way that $\Psi(s(x)) = (x, e)$ for all $x \in U$. In the physics literature, the sections s_i are called *local gauges*.

Given the trivializing cover $\{U_i\}_{i \in I}$ with trivializations Ψ_i, there are smooth functions $g_{ij} : U_i \cap U_j \to G$, called *transition functions* or *gauge transitions* such that

$$(\Psi_j \circ \Psi_i^{-1})(x, h) = (x, h \cdot g_{ji}(x))$$

for all $(x, h) \in U_i \cap U_j \times G$. These gauge transitions are related to the local gauges by

$$s_j(x) = s_i(x) \cdot g_{ij}(x).$$

Definition 1.1.3 (Connection on a principal bundle).
Let G be a Lie group with Lie algebra $\mathfrak{g}$. A connection *on a principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$ is a Lie algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the following:*

1. *ω is Ad_g-equivariant: for all $g \in G$, let $R_g : P \to P$ be the action $R_g(p) = p \cdot g$. Then*

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega.$$

2. *For all $p \in P$, consider the infinitesimal action $a_p : \mathfrak{g} \to T_pP$ given as $a_p(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \cdot \exp(t\xi)$. Then ω is a left inverse of this action:*

$$\omega_p(a_p(\xi)) = \xi$$

for all $\xi \in \mathfrak{g}$ and $p \in P$.

A connection $\omega \in \Omega^1(P, \mathfrak{g})$ determines a splitting of the tangent space T_pP at every point p, given by

$$T_pP = \ker T_p\pi \oplus \ker \omega_p.$$

The subspace $\ker \omega_p \subset T_pP$ is called the *horizontal* space determined by ω. This decomposition is compatible with the group action, in the sense that for all $g \in G$, $(R_g)_* \ker \omega_p = \ker \omega_{p\cdot g}$. Every vector $X \in T_pP$ can be written as $X^V + X^H$, and we say that $X^V \in \ker T_p\pi$ is the *vertical* component and $X^H \in \ker \omega_p$ is the *horizontal* component.

Given a trivializing cover $\{U_i\}_{i\in I}$ with local gauges $s_i : U_j \to P$, we call the pullbacks $\mathcal{A}_i = s_i^*\omega \in \Omega^1(U_i, \mathfrak{g})$ the *local gauge potentials* associated to ω.

Proposition 1.1.4 (Transformation of local gauge potentials).
Let ω be a connection on the principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$, $\{U_i\}_{i\in I}$ a trivializing cover with local gauges $s_i : U_i \to P$, and local gauge potentials $\mathcal{A}_i = s_i^* U_i$. Then in the intersections $U_i \cap U_j$, the local potentials are related by

$$\mathcal{A}_j = \mathrm{Ad}_{g_{ij}^{-1}} \circ \mathcal{A}_i + g_{ij}^*\Theta,$$

where $g_{ij} : U_i \cap U_j \to G$ are the transition functions and $\Theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form, given by the differential of left multiplication: $\Theta_g = T_g L_g^{-1}$.

In particular, if G is a matrix Lie group, then the transformation law becomes

$$\mathcal{A}_j = g_{ij}^{-1} \mathcal{A}_i g_{ij} + g_{ij}^{-1}\, \mathrm{d}g_{ij}\,.$$

Conversely, given a collection of 1-forms defined locally on M which satisfy these transformation law, there is a unique way to "glue" them to obtain a connection on P:

Proposition 1.1.5 (Global connection from local potentials).
Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle, and $\{U_i\}_{i\in I}$ a trivializing cover with local gauges $s_i : U_i \to P$. Suppose that there is a collection of $\mathfrak{g}$-valued 1-forms $\mathcal{A}_i \in \Omega^1(U_i, \mathfrak{g})$ satisfying the transformation law

$$\mathcal{A}_j = \mathrm{Ad}_{g_{ij}^{-1}} \circ \mathcal{A}_i + g_{ij}^*\Theta.$$

Then there exists a unique connection $\omega \in \Omega^1(P, \mathfrak{g})$ such that $\mathcal{A}_i = s_i^*\omega$.

With a connection $\omega \in \Omega^1(P, \mathfrak{g})$, we have an associated **exterior covariant derivative** $\mathrm{d}^\omega : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g})$ given as the *horizontal* part of the de Rham differential: for any $\alpha \in \Omega^k(P, \mathfrak{g})$, define

$$\mathrm{d}^\omega \alpha(X_0, \dots, X_k) = \mathrm{d}\alpha\,(X_0^H, \dots, X_k^H).$$

Definition 1.1.6 (Curvature of a connection).
Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection. The **curvature** *of ω is the 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ given by*

$$\Omega := \mathrm{d}^\omega \omega.$$

This definition is not very useful in practice, and we can find an alternative expression for Ω. For this we will need to define the bracket[2] of valued forms. For every $\alpha \in \Omega^k(P, \mathfrak{g})$ and $\beta \in \Omega^l(P, \mathfrak{g})$, choosing a basis $\{e_a\}$ of $\mathfrak{g}$ we can write $\alpha = \sum_a \alpha^a e_a$ and $\beta = \sum_b \beta^b e_b$, where $\alpha_a \in \Omega^k(P)$ and $\beta_b \in \Omega^l(P)$. We define $[\alpha, \beta] \in \Omega^{k+l}(P, \mathfrak{g})$ as

$$[\alpha, \beta] = \sum_{a,b} \alpha^a \wedge \beta^b [e_a, e_b].$$

It can be shown that this is a good definition, independent from the chosen basis of $\mathfrak{g}$. With this bracket of valued forms we can show:

Proposition 1.1.7 (Cartan structure equation).
Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection and Ω its curvature. Then

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega].$$

Given connection ω with curvature Ω and a trivializing cover $\{U_i\}_{i\in I}$ with associated sections $s_i : U_i \to P$, we define the *local field strengths* $\mathcal{F}_i := s_i^* \Omega \in \Omega^2(U_i, \mathfrak{g})$. Their behavior in the overlaps is much simpler than for the local potentials.

Proposition 1.1.8 (Transformation of local field strengths).
Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection with curvature Ω and a trivializing cover $\{U_i\}_{i\in I}$ with associated sections $s_i : U_i \to P$. The local field strengths $\mathcal{F}_i := s_i^* \Omega \in \Omega^2(U_i, \mathfrak{g})$ are related in the overlaps $U_i \cap U_j$ by the transformation law

$$\mathcal{F}_j = \mathrm{Ad}_{g_{ij}^{-1}} \circ \mathcal{F}_i,$$

where $g_{ij} : U_i \cap U_j \to G$ are the transition functions of the trivialization.

1.1.2 Associated bundles and matter fields

Let $G \hookrightarrow P \overset{\pi}{\to} M$ be a principal bundle, V a vector space and $\rho : G \to \mathrm{GL}(V)$ a representation. The **associated bundle** to P and the representation ρ is the quotient

$$P \times_\rho V = {}^{P \times V}\!/_{G},$$

where the action of G on $P \times V$ is given by

$$(p, v) \cdot g = (p \cdot g, \rho(g^{-1})(v)).$$

The set $P \times_\rho V$ can be endowed with a smooth structure such that the map $\pi_\rho : P \times_\rho V \to M$ given as $\pi_\rho[p, v] = \pi(p)$ is a surjective submersion. We denote elements of $P \times_\rho V$ as $[p, v]$ with $p \in P, v \in V$.

[2] Also called a *twisted wedge product*, which is probably a better name.

Consider a cover $\{U_i\}_{i\in I}$ of M which trivializes P, with local gauges $s_i : U_i \to P$. These determine a trivialization of $P\times_\rho V$ as

$$\begin{aligned} U_i \times V &\to \pi_\rho^{-1}(U_i) \\ (x,v) &\mapsto [s_i(x), v]. \end{aligned}$$

This exhibits $P\times_\rho V$ as a vector bundle over M.

Consider a section $\Psi : M \to P\times_\rho V$. Then we can write $\Psi(x) = [p, \psi(p)]$, where $\psi : P \to V$ and $\pi(p) = x$. For this to be well-defined and consistent, ψ must be ρ-equivariant:

$$\psi(p\cdot g) = \rho(g)^{-1}\psi(p).$$

This determines ψ uniquely, and conversely, Ψ is uniquely determined by ψ. In the physics literature, the sections of $P\times_\rho V$ are called *matter fields*.

Given a connection ω on P, we obtain a connection $\nabla^\omega : \Gamma(P\times_\rho V)\times \mathfrak{X}(M) \to \Gamma(P\times_\rho V)$ as follows: For a section $\Psi : M \to P\times_\rho V$, which can be written as $\Psi(p) = [p, \psi(p)]$, and a vector field $X \in \mathfrak{X}(M)$,we define

$$(\nabla^\omega_X)\Psi(x) = [p, (\mathrm{d}\psi)_p(\tilde{X}) + T_e\rho(\omega_p(\tilde{X}))\psi(p)],$$

where $\tilde{X} \in T_pP$ is a lift of X, i.e. $T_p\pi(\tilde{X}) = X$. We call ∇^ω the *covariant derivative* associated to ω. If $\{U_i\}_{i\in I}$ is a cover that trivializes P with local gauges $s_i : U_i \to P$, we obtain local representations of Ψ in terms of maps $\psi_i : U_i \to V$, such that $\Psi(x) = [s_i(x), \psi_i(x)]$. Then the covariant derivative has the local expression

$$(\nabla^\omega_X)\Psi(x) = [s_i(x), (\mathrm{d}\psi_i)_x(X) + T_e\rho(\mathcal{A}_{i,x}(X))\psi_i(x)].$$

Conversely, given a $\mathbb{K}$-vector bundle $E \to M$ of rank k and a connection ∇ on it, we can obtain a connection ω^∇ on the frame bundle $\mathrm{Fr}(E)$. Let $e_1, \dots, e_k : U \to E$ be a local frame of M. For every fixed e_i, we can think of ∇e_i as a map $\nabla e_i : \mathfrak{X} \to \Gamma(E)$. We write

$$\nabla e_i = \sum_j \Gamma_i^j e_i,$$

where $\Gamma_i^j : \mathfrak{X}(M) \to \mathbb{C}^\infty(M,\mathbb{K})$ are C^∞-linear. We collect them together as matrices to obtain $\mathfrak{gl}(k,\mathbb{K})$-valued 1-forms $\Gamma \in \Omega^1(U, \mathfrak{gl}(k,\mathbb{K}))$. It is straightforward to show that under a change of frame, the forms Γ transform according to the rule of Proposition 1.1.5, and therefore glue to a global connection $\omega^\nabla \in \Omega^1(\mathrm{Fr}(E), \mathfrak{gl}(k,\mathbb{K}))$.

1.1.3 The space of connections

Let $\omega_2, \omega_2 \in \Omega^1(P,\mathfrak{g})$ be connections on a principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$, and write $\Delta = \omega_2 - \omega_1$. This difference satisfies the following properties:

1. Δ is *horizontal*: If $T_p\pi(X) = 0$, then there is a $\xi \in \mathfrak{g}$ satisfying $a_p(\xi) = X$. Therefore

$$\Delta_p(X) = \omega_{2,p}(a_p(\xi)) - \omega_{1,p}(a_p(\xi)) = \xi - \xi = 0.$$

2. Δ is Ad-equivariant:

$$R_g^*\Delta = \mathrm{Ad}_{g^{-1}} \circ \Delta.$$

We say that Δ is an Ad-**tensorial** or **basic**[3] form, and we denote the space of all such basic 1-forms by $\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})$.

Conversely, given a connection ω and an Ad-tensorial 1-form $\Delta \in \Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})$, necessarily $\omega + \Delta$ is also a connection. We conclude:

Proposition 1.1.9 (Connections form an affine space).
Let $\mathrm{Conn}(P) \subset \Omega^1(P,\mathfrak{g})$ *be the set of connections of the principal bundle* $G \hookrightarrow P \to M$*. Then* $\mathrm{Conn}(P)$ *is an affine space modelled on the vector space of* Ad*-tensorial* 1*-forms* $\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})$.

Let's specialize to the case where $G = \mathrm{U}(1)$, so $\mathfrak{g} = i\mathbb{R}$. Since $\mathrm{U}(1)$ is abelian, the adjoint representation is the trivial one, and thus Ad-equivariance is the same as *invariance* under the action of $\mathrm{U}(1)$. Let $\omega, \omega' \in \mathrm{Conn}(P)$ and let $\Delta = \omega' - \omega$. Consider a trivializing cover $\{U_i\}$ of P with local gauges $s_i : U_i \to P$, which determines local potentials $\mathcal{A}_i = s_i^*\omega$ and $\mathcal{A}' = s_i^*\omega'$. Writing $\Delta_i = s_i^*\Delta \in \Omega^1(U_i, i\mathbb{R})$ for the difference of the local gauge potentials, we find that in the overlaps,

$$\Delta_j = \mathcal{A}'_j - \mathcal{A}_j = \mathcal{A}'_i + g_{ij}^{-1}\,\mathrm{d}g_{ij} - \mathcal{A}'_i - g_{ij}^{-1}\,\mathrm{d}g_{ij} = \mathcal{A}'_i - \mathcal{A}_i = \Delta_i.$$

This tells us that the collection $\{\Delta_i\}_{i\in I}$ glues to a one-form *on the base space* $\delta \in \Omega^1(M, i\mathbb{R})$, which satisfies $\pi^*\delta = \Delta$. Conversely, given a one-form δ, the pullback $\Delta = \pi^*\delta$ is a $\mathrm{U}(1)$-invariant 1-form. With this, we conclude that $\Omega^1_{\mathrm{Ad}}(P, i\mathbb{R}) \cong \Omega^1(M, i\mathbb{R})$, and we often abuse the notation and do not distinguish between either.

1.2 Algebraic topology

The main tools we will need from algebraic topology are Stiefel-Whitney classes, Chern classes, and of course, the *intersection form* of four-manifolds. In this section we will quickly review the background that is necessary for stating and proving Donaldson's theorem on the intersection form. For this, we will have to use cohomology with coefficients in an abelian group (specifically, $\mathbb{Z}_2$ and $\mathbb{Z}$). The golden reference for this is Hatcher's Algebraic Topology[4] [Hat02]. We will also follow [Bre97].

1.2.1 Cohomology and cups

Let G be an abelian group[5], and let $(C_\bullet, \partial)$ be a chain complex of free abelian groups. There is a canonically defined **cochain complex** (relative to G) $(C^\bullet_G, \delta)$ given as

$$C^k_G := \mathrm{Hom}(C_k, G)$$

with coboundary operators

$$\delta : C^k_G \to C^{k+1}_G,$$

given as $\delta(\varphi) = \partial^*\varphi = \varphi \circ \partial$. It is straightforward to show that $(C^\bullet_G, \delta)$ is a complex as well, and therefore we can define the **cohomology** groups

$$H^k(C, G) := {}^{\textstyle \ker(\delta : C^k \to C^{k+1})}\big/_{\textstyle \mathrm{im}(\delta : C^{k-1} \to C^k)}.$$

[3] This is a general definition. Let V a vector space and $\rho : G \to \mathrm{GL}(V)$ a representation. We say that a V-valued form $\alpha \in \Omega^k(P, V)$ is **basic** or ρ-**tensorial** if α is *horizontal*: $\alpha_p(X_1, \dots, X_k) = 0$ if $T_p\pi(X_j) = 0$ for *one* of the X_j; and if α is ρ-equivariant: $R_g^*\alpha = \rho(g^{-1}) \circ \alpha$.

[4] It's grown on me, you know? I appreciate it now after years of disliking it.

[5] In practice we will only use $G = \mathbb{Z}$ or $\mathbb{Z}_2$.

How do we understand elements of $H^k(C,G)$? We might be temped to think that $H^k(C,G)$ is precisely $\mathrm{Hom}(H_k(C),G)$, but this is not exactly so. However, we can still interpret an element $\hat{\varphi} \in H^k(C,G)$ as a morphism $H_k(C) \to G$ as follows: Let $\varphi \in \ker\delta \subset C^k$ be a representative of $\hat{\varphi}$. This means that for all $\alpha \in C_{k+1}$,

$$(\delta\varphi)(\alpha) = \varphi(\partial\alpha) = 0,$$

and thus φ vanishes on $\mathrm{im}\,\partial \subseteq C_k$. Therefore φ, when restricted to $\ker\partial \subseteq C_k$, descends to a morphism in the quotient

$$\overline{\varphi} : H_k \to G.$$

The morphism $\overline{\varphi}$ is independent of the representative of $\hat{\varphi}$, precisely since we restrict to $\ker\partial$ before descending to the quotient. Specifically, let $\psi \in C^{k-1}$. Then

$$\varphi|_{\ker\partial} = (\varphi + \delta\psi)|_{\ker\partial},$$

and so $\overline{\varphi} = \overline{\varphi + \delta\psi}$. In conclusion, we have a well-defined morphism

$$\begin{aligned} \hbar : H^k(C,G) &\to \mathrm{Hom}(H_k(C),G) \\ \hat{\varphi} &\mapsto \hbar(\varphi) = \overline{\varphi}. \end{aligned}$$

If $\hbar$ were an isomorphism, all would be great. As we will see, $\hbar$ is surjective, but alas, it is generally not injective[6].

Asking about the surjectivity of $\hbar$ is asking if every homomorphism $\psi : H_k(C) \to G$ is given as $\overline{\varphi}$ for some $\varphi : C_k \to G$ which satisfies that $\delta\varphi = 0$. We *almost* have such a homomorphism. Given ψ, we can find a homomorphism $\psi_0 : \ker\partial \to G$, simply by pre-composing ψ with the quotient $\ker\partial \to H_k(C)$:

$$\psi_0(\sigma) = \psi(\hat{\sigma}),$$

where $\hat{\sigma} \in H_k(C)$ is the class of σ. Naturally, ψ_0 vanishes on $\mathrm{im}\,\partial$. However, it is only defined in $\ker\partial$. Can we extend it to the entirety of C_k? The answer is *yes* since C_k (and, consequently, $\ker\partial$ and $\mathrm{im}\,\partial$) is free: The sequence

$$0 \longrightarrow \ker\partial \hookrightarrow C_k \xrightarrow{\partial} \mathrm{im}\,\partial \longrightarrow 0$$

is a short exact sequence of *free* abelian groups, and therefore it splits[7]. This means that we can find a retraction $r : C_k \to \ker\partial$ such that $r(\sigma) = \sigma$ for all $\sigma \in \ker\partial$. Thus, we can extend ψ_0 to $\varphi : C_k \to G$ as

$$\varphi = \psi_0 \circ r.$$

By construction, we have that $\overline{\varphi} = \psi$, and so we have shown that $\hbar$ is surjective. The following example shows that $\hbar$ is not an isomorphism.

Example 1.2.1 (Non-isomorphism of $\hbar$).
Consider the following complex of free abelian groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow 0.$$

[6] Like my mom says, *de eso tan bueno no dan tanto*.

[7] Mini-proof: Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of free abelian groups, with $f : A \to B$. Let $\mathcal{B} \subset B$ be a set of generators of B, and define $r : B \to A$ as follows: for all $\beta \in \mathcal{B}$, define $r(\beta) = f^{-1}(\beta)$ if $\beta \in \mathrm{im}(f)$, and $r(\beta) = 0$ otherwise. Extend r by "linearity" to all of B. By injectivity of f, this map is well-defined, and it satisfies $r \circ f = \mathrm{id}_A$.

Here, n denotes multiplication by n. The homology of this complex is

$$H_0 = \mathbb{Z}_n, \\ H_1 = 0.$$

When we take the dual with respect to $\mathbb{Z}$, we obtain the cochain complex

$$0 \longleftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \longleftarrow 0.$$

This follows from the fact that a homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is uniquely determined by the value $f(1)$. The cohomology of the cochain complex is

$$H^0 = 0, \\ H^1 = \mathbb{Z}_n.$$

Since the only homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is the trivial one, we indeed have that $H^0 \cong \mathrm{Hom}(H_0, \mathbb{Z})$. However, certainly $\mathbb{Z}_m = H^1 \ncong \mathrm{Hom}(H_1, \mathbb{Z}) = 0$. Therefore $\hbar : H^k(C, G) \to \mathrm{Hom}(H_k(C), G)$ is, in general, *not* an isomorphism.

Even though $\hbar$ is not an isomorphism, we can still use it to define a way in which a cohomology class can be evaluated on homology classes:

Definition 1.2.2 (Pairing of cohomology and homology).
We define a pairing *of cohomology and homology* $\langle \bullet, \bullet \rangle : H^k(C, G) \otimes H_k(C) \to G$ *as*

$$\langle \varphi, \sigma \rangle = \hbar(\varphi)(\sigma).$$

Now that we know that $\hbar$ is not injective, it is but natural to ask oneself, "what is its kernel"? The answer is given to us by the Universal Coefficients Theorem, but first we must define[8] the (first) Ext group associated to an abelian group.

Definition 1.2.3 (Ext(H, G)).
Let H be an abelian group. Take a set of generators $\{h_\alpha\}_{\alpha \in \Lambda}$ of H and let F_0 be the free abelian group on $\{h_\alpha\}_{\alpha \in \Lambda}$. We have a natural morphism $f_0 : F_0 \to H$, which sends each generator of F_0 to its counterpart in H. Let $F_1 = \ker f_0$, so that we have a short exact sequence

$$0 \longrightarrow F_1 \xhookrightarrow{\iota} F_0 \xrightarrow{f_0} H \longrightarrow 0,$$

which we call a free *resolution of H. Let G be an abelian group, and apply the functor* $\mathrm{Hom}(-, G)$ *to this sequence to obtain a sequence*

$$0 \longleftarrow \mathrm{Hom}(F_1, G) \xleftarrow{\iota^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \longleftarrow 0\ ,$$

[8] This definition we present is specialized to the case of free abelian groups and differs a bit from the standard, more general definition.

which is not necessarily exact. The (first) Ext *group is defined as the cohomology in degree* 1 *of this sequence:*

$$\operatorname{Ext}(H,G) = \operatorname{coker} \iota^*.$$

It can be shown that $\operatorname{Ext}(H,G)$ is independent of the free resolution of H. Furthermore, it can be shown that $\operatorname{Ext}(H,G)$ satisfies the following properties [see Hat02, p. 195]:

- $\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H,G) \oplus \operatorname{Ext}(H',G)$,
- $\operatorname{Ext}(H,G) = 0$ if H is free,
- $\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

With these results, we see that if H is finitely generated, then by the fundamental theorem of finitely generated abelian groups, we can write

$$H \cong \mathbb{Z}^k \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$$

And therefore,

$$\operatorname{Ext}(H,\mathbb{Z}) \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \cong \operatorname{Torsion}(H).$$

Now we can present the Universal Coefficients Theorem[Hat02, Theorem 3.2]:

Theorem 1.2.4 (Universal Coefficients for Cohomology).
Let $C_\bullet$ be a chain complex of free abelian groups and G an abelian group. Then there is a split *exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{k-1}(C),G) \longrightarrow H^n(C,G) \xrightarrow{h} \operatorname{Hom}(H_n(C),G) \longrightarrow 0 \ .$$

In particular,

$$H^n(C,G) \cong \operatorname{Hom}(H_n(C),G) \oplus \operatorname{Ext}(H_{n-1}(C),G).$$

This theorem tells us that the cohomology groups are uniquely determined by the homology groups, albeit in a non-trivial way.

All this algebraic theory is meaningful to us when we apply it to the (singular, simplicial, CW)[9] homology groups over a topological space X, of course.

Taking cohomology is a *contravariant* functor: given a continuous map $f : X \to Y$, we have an induced chain map $f_* : C_\bullet(X) \to C_\bullet(X)$, and therefore an induced cochain map $f^* : C^\bullet(Y,G) \to C^\bullet(X,G)$ given by precomposition, $f^*\varphi = \varphi \circ f_*$. This induced map commutes with the coboundary operator, and so it descends to an induced map in cohomology

$$f^* : H^\bullet(Y,G) \to H^\bullet(X,G),$$

which we call the **pullback** by f. This assignment satisfies

$$(g \circ f)^* = f^* \circ g^*$$

for continuous maps $f : X \to Y$ and $g : Y \to Z$.

[9] Since our interest is in manifolds, that beautiful realm where all your topological dreams are true, the choice between these three is essentially irrelevant.

One of the biggest differences between homology and cohomology is that we have a (sort of) natural operation that turns the entire collection of cohomology groups of a space into a ring. This is the *cup product*, defined as follows:

Let X be a topological space and G a ring (in practice, $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{R}$). For $\varphi \in C^k(X,R)$ and $\psi \in C^l(X,R)$, define $\varphi \smile \psi \in C^{k+l}(X,R)$ as

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+l}]}), \tag{1.1}$$

for every singular $(k+l)$-simplex $\sigma : \Delta^{k+l} \to X$. Here $[v_0,\dots,v_j]$ denotes the convex hull of the points $v_0,\dots,v_j$ with $v_0,\dots,v_{k+l}$ the canonical basis of $\mathbb{R}^{k+l+1}$, and $\Delta^{k+l} = [v_0,\dots,v_{k+l}]$. It is a straightforward exercise in tedious bookkeeping to show that

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

In particular, $\delta(\varphi \smile \psi) = 0$ if $\delta\varphi = 0$ and $\delta\psi = 0$. Therefore the cup product descends to cohomology:

$$\smile : H^k(X,R) \times H^l(X,R) \to H^{k+l}(X,R).$$

In fact, the defining formula in equation Equation (1.1) defines relative cap products:

$$H^k(X;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R),$$
$$H^k(X,A;R) \times H^l(X;R) \to H^{k+l}(X,A;R),$$
$$H^k(X,A;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R).$$

This cup product behaves well with pullbacks: for a map $f : X \to Y$, the pullback satisfies

$$f^*(\varphi \smile \psi) = f^*\varphi \smile f^*\psi.$$

Furthermore, if R is commutative, we can show that

$$\varphi \smile \psi = (-1)^{kl} \psi \smile \varphi.$$

If this is reminiscent of the wedge product in differential forms, it's because it basically is the same thing! Under the isomorphism of (smooth) singular cohomology with real coefficients with de Rham cohomology, cup products pass over to wedge products[10].

1.2.2 Orientations and caps

From here on we specialize to the case of topological manifolds. Here we will introduce an *algebraic* notion of orientation, and with it the *Poincaré duality*. Again, the golden standard here is [Hat02, Section 3.3].

> **Remark.** Throughout this section we will consider (co)homology with *integer* coefficients.

First, let's tackle orientation. Let M be a topological manifold of dimension n. For any point $x \in M$, there is a neighborhood $U \subseteq M$ which is homeomorphic to $\mathbb{R}^n$. Consider the relative homology groups with *integer* coefficients $H_k(M, M - \{x\})$. By excision (namely, excising $M - U$), we have

$$H_k(M, M - \{x\}) \cong H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{\tilde{x}\}),$$

[10] An indirect proof can be found in [BT95]. Specifically, [BT95, Theorem 14.28] exhibits a ring isomorphism to *Čech* cohomology. Then in [BT95, p. 192] they explain that Čech cohomology is isomorphic to singular cohomology.

where $\tilde{x} \in \mathbb{R}^n$ is the image under the homeomorphism $U \cong \mathbb{R}^n$. The isomorphism $H_k(M, M - \{x\}) \cong H_k(U, U - \{x\})$ tells us that $H_k(M, M - \{x\})$ depends only on the local topology around x. We call these the *local homology groups* at x.

Now we consider the (beginning of the) long, exact sequence induced by the pair $(\mathbb{R}^n, \mathbb{R}^n - \{\tilde{x}\})$:

$$0 \longrightarrow H_n(\mathbb{R}^n - \{\tilde{x}\}) \longrightarrow H_n(\mathbb{R}^n) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{\tilde{x}\}) \longrightarrow H_{n-1}(\mathbb{R}^n - \{\tilde{x}\}) \longrightarrow \cdots ,$$

We have $H_k(\mathbb{R}^n) = 0$ if $k \neq 0$, and since $\mathbb{R}^n - \{\tilde{x}\}$ retracts to S^{n-1}, we have $H_k(\mathbb{R}^n - \{\tilde{x}\}) \cong H_k(S^{n-1}) = \mathbb{Z}$ if $k = 0$ or $k = n - 1$, and it is zero otherwise. Therefore this sequence reduces to

$$0 \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{\tilde{x}\}) \longrightarrow \mathbb{Z} \longrightarrow 0 ,$$

which implies that

$$H_n(M, M - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{\tilde{x}\}) \cong \mathbb{Z}.$$

This holds for *all* points $x \in M$!

A **local orientation** at $x \in M$ is a choice of a generator $\mu_x \in H_n(M, M - \{x\})$. Suppose that we have a collection of local orientations $\{\mu_x\}_{x \in M}$. Take $x_0 \in M$ and consider an open neighborhood $U \subseteq M$ which is homeomorphic to $\mathbb{R}^n$. We can find a (closed) subset $B \subset U$ containing x_0 which is homeomorphic to a closed ball in $\mathbb{R}^n$. By the same procedure we did above, we find that $H_n(M, M - B) \cong \mathbb{Z}$. For every $x \in B$, the inclusion $(M, M - B) \subseteq (M, M - \{x\})$ induces a morphism

$$H_n(M, M - B) \to H_n(M, M - \{x\}).$$

We say that the family of orientations $\{\mu_x\}_{x \in M}$ is **locally consistent** at x_0 if for all $x \in B$, every μ_x is the image of a single chosen generator of $H_n(M, M - B)$ under this morphism.

Definition 1.2.5 (Orientation of a manifold).
An orientation of a manifold is a collection $\{\mu_x\}_{x \in M}$ of local orientations which is locally consistent at every point $x \in M$. If M admits an orientation, we say that it is orientable.

We have the following result on the orientability of a *closed* manifold:

Theorem 1.2.6 (Orientability of closed manifolds).
Let M is a closed, connected manifold of dimension n. If M is oriented, then for all $x \in M$, the morphism arising from the long exact sequence of the pair $(M, M - \{x\})$,

$$H_n(M) \to H_n(M, M - \{x\})$$

is an isomorphism.

For a proof, see [Hat02, pp. 236-237]. A generator of $H_n(M)$ whose image is a generator of $H_n(M, M - \{x\})$ for all $x \in M$ is called a **fundamental class** of M. We write it as $[M] \in H_n(M)$. Conversely, the existence of a fundamental class implies the existence of an orientation of M.

Let M be a closed, connected, oriented n-dimensional manifold. Given $\varphi \in C^k(X, \mathbb{Z})$ and a singular simplex $\sigma : \Delta^l \to X$, define $\varphi \frown \sigma : C_{l-k}(X)$ as

$$\varphi \frown \sigma = \varphi(\sigma|_{[v_0, \dots, v_l]})\sigma|_{[v_l, \dots, v_k]}.$$

Extending this to $C_l(X)$, we obtain a map $\bullet \frown \bullet : C^k(X,\mathbb{Z}) \times C_l(X) \to C_{l-k}(X)$. Once again, it is a straightforward exercise to show that $\frown$ behaves well with the boundary operator:

$$\partial(\varphi \frown \sigma) = (-1)^{l+1}(\delta\varphi \frown \sigma - \varphi \frown \partial\sigma).$$

Therefore, this operation descends to cohomology and homology, and we call it the **cap product**:

$$\bullet \frown \bullet : H^k(X,\mathbb{Z}) \times H_l(X) \to H_{k-l}(X).$$

The cap and cup products enjoy a *duality* of sorts: for all $\varphi \in H^{k-l}(X,\mathbb{Z})$, $\psi \in H^l(X,\mathbb{Z})$ and $\sigma \in H_k(X)$, we have

$$\langle \varphi, \psi \frown \sigma \rangle = \langle \psi \smile \varphi, \sigma \rangle .$$

Let $[M]$ be the (a) fundamental class of M. Then for every $\varphi \in H^k(M,\mathbb{Z})$, we have an element $\varphi \frown [M] \in H_{n-k}(M)$. The map $H^k(M,\mathbb{Z}) \to H_{n-k}(M)$ is an isomorphism:

Theorem 1.2.7 (Poincaré Duality).
Let M be a closed, connected, oriented n-dimensional manifold and $[M]$ a fundamental class. Then the map $H^k(M,\mathbb{Z}) \to H_{n-k}(M,\mathbb{Z})$ given by

$$\varphi \mapsto \varphi \frown [M]$$

is an isomorphism.

The proof of this theorem is a little bit laborious. The curious reader can check [Hat02, pp. 245-249].

Suppose that M is a compact manifold with boundary. We say that M is orientable if $M - \partial M$ is orientable. In this case, there exists a *unique* class $[M] \in H_n(M, \partial M)$ which restricts to the choice of generator $\mu_x \in H_n(M, M - \{x\})$ for all $x \in M - \partial M$ [see Hat02, Lemma 3.27]. We call $[M]$ the fundamental class of M. Similarly to manifolds without boundary, we have a version of Poincaré duality for manifolds with boundary [Hat02, Theorem 3.43]:

Theorem 1.2.8 (Poincaré Duality for manifolds with boundary).
Let M be a compact manifold with boundary. Then the map $H^k(M,\partial M) \to H_{n-k}(M)$ given by

$$\varphi \mapsto \varphi \frown [M]$$

is an isomorphism.

1.2.3 The intersection pairing

Let M be a closed, connected n-dimensional topological manifold with an orientation over $\mathbb{Z}$. We define the **intersection pairing** $H^k(M,\mathbb{Z}) \times H^{n-k}(M,\mathbb{Z}) \to \mathbb{Z}$ as

$$\varphi \bullet \psi := \langle \varphi \smile \psi, [M] \rangle$$

where $\smile$ is the cup product and $[M]$ is the fundamental class of M, induced by its orientation. This pairing vanishes on the torsion subgroups $T^m \subseteq H^m(M,\mathbb{Z})$, and thus it descends to quotients $H^k(M,\mathbb{Z})/T^k$ and $H^{n-k}(M,\mathbb{Z})/T^{n-k}$. Once we have quotiented the torsion out, the intersection pairing is non-degenerate.

Proposition 1.2.9 (Intersection pairing is non-degenerate).
For each $\varphi \in H^k(M,\mathbb{Z})/T^k$ *and* $\psi \in H^{n-k}(M,\mathbb{Z})/T^{n-k}$, *the maps*

$$\begin{aligned} H^k(M,\mathbb{Z})/T^k &\to \mathrm{Hom}(H^{n-k}(M,\mathbb{Z})/T^{n-k},\mathbb{Z}) \\ [\varphi] &\mapsto \varphi \bullet (-) \end{aligned}$$

and

$$\begin{aligned} H^{n-k}(M,\mathbb{Z})/T^{n-k} &\to \mathrm{Hom}(H^k(M,\mathbb{Z})/T^k,\mathbb{Z}) \\ [\psi] &\mapsto (-) \bullet \psi \end{aligned}$$

are isomorphisms of abelian groups.

Proof. — Consider the a natural surjective map $\hbar : H^k(M,\mathbb{Z}) \to \mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{Z})$ given as

$$\hbar(\varphi)(\sigma) = \langle \varphi, \sigma \rangle$$

Therefore, we can rewrite the intersection pairing as

$$\varphi \bullet \psi = \langle \varphi \smile \psi, [M] \rangle = \pm \langle \varphi, \psi \frown [M] \rangle = \pm \langle \varphi, PD(\psi) \rangle = \pm (PD^* \circ \hbar)(\varphi)(\psi),$$

where $PD : H^{n-k}(M,\mathbb{Z}) \to H_k(M,\mathbb{Z})$ is the Poincaré Duality map, $PD(\psi) = \psi \frown [M]$. Therefore, the map $\varphi \mapsto \varphi \bullet (-)$ is precisely $\pm(PD^* \circ \hbar)$.

From the Universal Coefficients Theorem (1.2.4), we find that in general $\hbar$ is not injective and its kernel is precisely the torsion $T^k \subseteq H^k(M,\mathbb{Z})$. Therefore if we quotient out T^k, there is an induced isomorphism

$$\hat{\hbar} : H^k(M,\mathbb{Z})/T^k \to \mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{Z}).$$

Similarly, the Poincaré duality isomorphism descends to an isomorphism

$$\hat{PD} : H_k(M,\mathbb{Z})/T_k \to H^{n-k}(M,\mathbb{Z})/T^{n-k},$$

and thus we find that

$$\hat{PD}^* \circ \hat{\hbar} : H^k(M,\mathbb{Z})/T^k \to \mathrm{Hom}(H^{n-k}(M,\mathbb{Z})/T^{n-k},\mathbb{Z})$$

is an isomorphism. Since $\varphi \mapsto \varphi \bullet (-)$ is precisely $\pm PD^* \circ \hbar$, the result follows once we pass to the quotient. ∎

The name "intersection pairing" seems gratuitous at this point. What does it have to do with intersections? R. Thom [TM07, Corollary II.28] proved that for compact, orientable manifolds, if $\alpha \in H_k(M,\mathbb{Z})$ with $k \leq 8$, then there is an embedded orientable submanifold $i : V_\alpha \hookrightarrow M$ such that $i_*[V_\alpha] = \alpha$. We say that V_α *represents* the class α[11]. We will see this in Theorem 1.3.5 for homology in codegree 2.

With Thom's result, assuming $\dim(M) \leq 8$, the name of the intersection pairing is justified: via Poincaré duality, the intersection pairing passes to a pairing in *homology*, $H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) \to \mathbb{Z}$. For any $\alpha \in H_k(M,\mathbb{Z})$ and $\beta \in H_{n-k}(M,\mathbb{Z})$, let V_α and V_β be embedded submanifolds which represent them. Then $PD(\alpha) \bullet PD(\beta)$ is precisely the oriented intersection number of V_α and V_β. We will see this explicitly for degree 2 (co)in 4-manifolds in Section 4.1.

[11] Lifting the restriction of dimension leaves us with a weaker result: any integral homology class has a *multiple* is representable by a submanifold. The adventurous reader can check [TM07] for the general proof.

1.3 Characteristic classes

Some of the most important tools in algebraic topology are characteristic classes, which are used to classify vector bundles. In this work we will use the first Chern class, which is a classifying tool for complex line bundles, and the first and second Stiefel-Whitney class, which are used to study real vector bundles.

At the outset, We will consider vector bundles over paracompact spaces[12]. The golden standard here is [MS74], but we also take some things from [May99] and [Hat17].

1.3.1 Classifing spaces for line bundles and the first Chern class

First, let's review some definitions. Recall that a topological space E is a $\mathbb{K}$-vector bundle of rank k over a paracompact space X if there is a surjective map $\pi : E \to X$ which is locally trivial: for every point $x \in M$ there is a neighborhood U of x and a homeomorphism $\Psi : \pi^{-1}(U) \to U \times \mathbb{K}^k$, which is linear on the fibers: $\Psi|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{K}^k$ is a linear isomorphism. We call E the *total space* and X the *base space*. We compress all this and say that $\pi : E \to X$ is a vector bundle.

If $\pi : E \to X$ and $\pi' : E' \to X'$ are vector bundles, a **morphism** between them is a pair (f_E, f_X) of continuous maps $f_E : E \to E'$ and $f_X : X \to X'$ for which the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{f_X} & X' \end{array}$$

commutes[13], and for all $x \in X$, f_E is fiberwise a linear map; i.e. $f_E|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \pi'^{-1}(f_X(x))$ is linear. Naturally, we say that (f_E, f_X) is an isomorphism if there exists an inverse morphism (g'_E, g'_X), in which case we say that $E \to X$ and $E' \to X'$ are isomorphic. Note that this implies that the base spaces are homeomorphic. Furthermore, (f_E, f_X) is an isomorphism if and only if f_X is a homeomorphism and $f_E|_{\pi^{-1}(x)}$ is a linear isomorphism on each fiber.

Particularly, we are interested in equivalence classes of vector bundles over a same space. We say that two vector bundles $E \to X$ and $E' \to X$ are **equivalent** if there is an isomorphism (f_E, id) between them which *lifts the identity*. We denote the set of equivalence classes of $\mathbb{K}$-vector bundles of rank k over X by

$$\mathrm{VB}_k(X, \mathbb{K})$$

and we drop the $\mathbb{K}$ if there is no confusion on the field.

Now we introduce a construction that is essential for the discussion of characteristic classes. Let $\pi : E \to X$ be a vector bundle and $f : Y \to X$ a continuous map. We can construct a vector bundle $f^*(E)$ over Y, called the **pullback** of E by f, as the subspace of the product $Y \times E$ given by

$$f^*(E) = \{(x, e) \in Y \times E \mid f(x) = \pi(e)\}.$$

The projection $f^*(E) \to Y$ is the projection on the first component. It is straightforward to check that $f^*(E)$ is indeed a vector bundle over Y which has the same rank as E. It turns out that this construction is only dependent on the homotopy type of f [see May99, section 23.1]:

[12] A topological space X is *paracompact* if any open cover has an open refinement which is *locally finite*. That is, every point has a neighborhood which intersects only finitely many sets in the refinement of the cover.

[13] That is, f_E maps fibers into fibers.

Proposition 1.3.1 (Pullback bundle is homotopy-invariant).
Let $E \to X$ be a vector bundle and $f_0, f_1 : Y \to X$ homotopic maps. Then f_0^*E and f_1^*E are equivalent.

This tells us that the assignment $X \rightsquigarrow \mathrm{VB}_k(X, \mathbb{K})$ is a *contravariant* functor between the category of homotopy classes of manifolds to the category of sets. The surprising fact is that this is a *representable* functor! That is, there is a space $BO_{\mathbb{K}}(K)$ such that there is a natural bijection between $\mathrm{VB}_k(X, \mathbb{K})$ and the set of homotopy classes of maps from X to $BO_{\mathbb{K}}(k)$, which we denote by $[X, BO_{\mathbb{K}}(k)]$. This is to say, there is a natural isomorphism between the functors $\mathrm{VB}_{\mathbb{K}}(-, k)$ and $[-, BO_{\mathbb{K}}(k)]$. We call $BO_{\mathbb{K}}(k)$ a **classifying space** for $\mathbb{K}$-vector bundles of rank k.

The representability of the functor can be proved in purely categorical terms[14]. However we will give an explicit construction of $BO_{\mathbb{K}}(n)$.

Assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $G_k(\mathbb{K}^n)$ be the Grassmannian space of k-subspaces of $\mathbb{K}^n$, whose points are $\mathbb{K}$-vector subspaces of dimension k of $\mathbb{K}^n$. These can be given a natural topology (a manifold structure, even) inherited from $\mathbb{K}^{nk}$. For $k = 1$, the Grassmannians are no more than the projective spaces $G_1(\mathbb{K}^n) = \mathbb{KP}^{n-1}$. We construct a **tautological** or **canonical** bundle of rank k over $G_k(\mathbb{K}^n)$, denoted by γ_k^n, as a subbundle of the product $G_k(\mathbb{K}^n) \times \mathbb{K}^n$ given by

$$\gamma_k^n = \{(V, v) \in G_k(\mathbb{K}^n) \times \mathbb{K}^n \mid v \in V\}.$$

The projection $\gamma_k^n \to G_k(\mathbb{K}^n)$ is the projection on the first component. The name "tautological" comes from the fact that the fiber above a point $V \in G_k(\mathbb{K}^n)$ is precisely the subspace that V represents.

The natural inclusion $\mathbb{K}^n \hookrightarrow \mathbb{K}^n \times \{0\} \subset \mathbb{K}^{n+1}$ induces inclusions $G_k(\mathbb{K}^n) \hookrightarrow G_k(\mathbb{K}^{n+1})$ and $\gamma_k^n \hookrightarrow \gamma_k^{n+1}$ for all n. Then we can take the union over all $n \in \mathbb{N}$ and consider the *infinite Grassmannian* $G_k(\mathbb{K}^\infty)$ and its tautological bundle γ_k^∞. This is precisely the space that represents the functor $\mathrm{VB}_k(-, \mathbb{K})$ [May99, Section 23.2]:

Theorem 1.3.2 (Infinite Grassmanian represents functor $\mathrm{VB}_k(-, \mathbb{K})$).
Given a $\mathbb{K}$-vector bundle of rank k $E \to X$, there is a continuous $f : X \to G_k(\mathbb{K}^\infty)$ which is unique up to homotopy, such that

$$E \cong f^*\gamma_k^\infty.$$

Furthermore, the assignment of a homotopy class of a function $f : X \to G_k(\mathbb{K}^\infty)$ to the equivalence class of $f^*\gamma_k^\infty$ is a natural isomorphism between the functors $[-, G_k(\mathbb{K}^\infty)]$ and $\mathrm{VB}_k(-, \mathbb{K})$. This is to say that $G_k(\mathbb{K}^\infty)$ is a classifying space for $\mathbb{K}$-vector bundles of rank k.

Let R be a ring[15]. A **characteristic class** of degree n for $\mathbb{K}$-vector bundles of rank k is an assignment $c_n : \mathrm{VB}_k(X, \mathbb{K}) \to H^n(X, R)$ which is natural with respect to pullbacks:

$$f^*(c_n(E)) = c_n(f^*E),$$

for vector bundles $E \to X$ and maps $f : Y \to X$. This naturality property implies that it suffices to define characteristic classes on the classifying spaces. In the rest of this section we will focus only on the *first* Chern class, which is defined for complex line bundles. Therefore we need to study the cohomology ring of the classifying space $G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$.

[14] As May[May99] puts it, "on general abstract nonsense grounds, using Brown's representability theorem".
[15] For our purposes either $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{R}$.

Proposition 1.3.3 (Cohomology ring of $\mathbb{CP}^n$).
The cohomology ring of $\mathbb{CP}^n$ is the polynomial ring

$$H^\bullet(\mathbb{CP}^n,\mathbb{Z}) \cong \mathbb{Z}[\mu]\big/\langle \mu^{n+1}\rangle,$$

where $\mu \in H^2(\mathbb{CP}^n,\mathbb{Z})$ is a generator, which can be chosen as the Poincaré dual to the fundamental class $[\mathbb{CP}^{n-1}] \in H_{2n-2}(\mathbb{CP}^n,\mathbb{Z})$.

Proof. — It is a standard result from algebraic topology[16] that the homology of $\mathbb{CP}^n$ is

$$H_k(\mathbb{CP}^n,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \le 2n \text{ is even,} \\ 0 & \text{if } k > 2n \text{ or } k \text{ is odd} \end{cases}.$$

From the Universal Coefficients Theorem (1.2.4), we see that

$$H^k(\mathbb{CP}^n,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \le 2n \text{ is even,} \\ 0 & \text{if } k > 2n \text{ or } k \text{ is odd} \end{cases}.$$

Furthermore, the standard inclusion of $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$ induces isomorphisms[17]

$$H_k(\mathbb{CP}^n,\mathbb{Z}) \xrightarrow{\sim} H_k(\mathbb{CP}^{n+1},\mathbb{Z}),$$
$$H^k(\mathbb{CP}^n,\mathbb{Z}) \xrightarrow{\sim} H^k(\mathbb{CP}^{n+1},\mathbb{Z}),$$

for all $k \le 2n$.

Let's proceed by induction on n. For $n = 1$ the statement is trivially true. Now assume it holds for $\mathbb{CP}^1, \ldots, \mathbb{CP}^n$. Since $H^k(\mathbb{CP}^n, Z) \cong H^k(\mathbb{CP}^{n+1}, Z)$ for all $k \le 2n$, then have that if μ is a generator of $H^2(\mathbb{CP}^{n+1}, Z)$, then μ^k is a generator of $H^k(\mathbb{CP}^{n+1}, Z)$ for all $k \le 2n$. Let's see that μ^{n+1} indeed generates $H^{n+1}(\mathbb{CP}^{n+1}, Z)$.

Since μ^n generates $H^n(\mathbb{CP}^{n+1}, Z)$, then there is a morphism $\varphi : H^n(\mathbb{CP}^n,\mathbb{Z}) \to \mathbb{Z}$ such that $\varphi(\mu^n) = 1$. However, the intersection pairing is non-degenerate (Proposition 1.2.9), which implies that there is a (unique) 2-form $\beta \in H^2(\mathbb{CP}^{n+1},\mathbb{Z})$ such that $\varphi(\alpha) = \alpha \cdot \beta$ for all $\alpha \in H^n(\mathbb{CP}^{n+1},\mathbb{Z})$. However, μ generates $H^2(\mathbb{CP}^{n+1},\mathbb{Z})$, so $\beta = m\mu$ for some m. Therefore,

$$1 = \varphi(\mu^n) = \mu^n \cdot m\mu = m\left\langle \mu^{n+1}, [\mathbb{CP}^{n+1}]\right\rangle.$$

This implies that $m = \pm 1$ and $\left\langle \mu^{n+1}, [\mathbb{CP}^{n+1}]\right\rangle = \pm 1$. Finally, since evaluation on the fundamental class $[\mathbb{CP}^{n+1}]$ is a morphism $H^{n+1}(\mathbb{CP}^{n+1},\mathbb{Z}) \to \mathbb{Z}$, necessarily μ^{n+1} has to be a generator.

Finally, we have that $[\mathbb{CP}^k] \in H_k(\mathbb{CP}^n,\mathbb{Z})$ is a generator because the inclusion $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^n$ induces isomorphisms in homology of degrees up to $2k$. Therefore, the Poincaré dual of $[\mathbb{CP}^{n-1}] \in H_{2n-2}(\mathbb{CP}^n,\mathbb{Z})$ is a generator of $H^2(\mathbb{CP}^k,\mathbb{Z})$, which we can choose for μ. ■

With this definition, we can define the first Chern class of a line bundle.

[16] Easily computed using cellular homology [see Hat02, p. 140].
[17] This can be seen by looking at the standard CW decomposition of $\mathbb{CP}^n$.

Definition 1.3.4 (First Chern class).
Let $L \to M$ be a complex line bundle. There exists a map $f : M \to \mathbb{CP}^\infty$ such that $L \cong f^*(E)$. We define the **first Chern class** *of L, $c_1(L) \in H^2(M, \mathbb{Z})$, as*

$$c_1(L) = f^*(\mu),$$

where μ is the generator of $H^2(\mathbb{CP}^\infty, \mathbb{Z})$ that we found in Proposition 1.3.3.

This definition of the Chern class, and the characterization of the cohomology ring of $\mathbb{CP}^n$, lets us prove the representability of homology classes of codegree 2 with submanifolds.

Theorem 1.3.5 (Thom representability).
Let M be a closed, connected, oriented manifold and $\alpha \in H_{n-2}(M, \mathbb{Z})$ a homology class. Then there exists an embedded submanifold $V_\alpha \subset M$ such that $[V_\alpha] = \alpha$.

Proof (Sketch). — Let $\hat{\alpha} \in H^2(M, \mathbb{Z})$ be the Poincaré dual of α. There exists a complex line bundle $L \to M$ such that $\hat{\alpha} = c_1(L)$. Since $\mathbb{CP}^\infty$ is the classifying bundle for complex line bundles, there is a map $f : M \to \mathbb{CP}^\infty$ such that $L \cong f^*(\gamma_1^\infty)$. By the cellular approximation theorem, we can choose f to be a *cellular map*, so that $f(M) \subseteq \mathbb{CP}^n$ for some large enough n. Furthermore, we can take f to be smooth and transverse to $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. This implies that $V = f^{-1}(\mathbb{CP}^{n-1})$ is an embedded submanifold of M, which is our candidate for the representing α.

Using the Thom isomorphism theorem, it can be shown that indeed $[V] = \alpha$ [see Bre97, Theorem 11.16]. ∎

An additional characterization of the first Chern class comes from considering connections on a U(1)-bundle. Let $L \to M$ be a complex line bundle. Given a Hermitian metric on it, we can construct the unitary frame bundle $\mathrm{U}(1) \hookrightarrow \mathrm{U}(L) \to M$. Choose a connection $\omega \in \Omega^1(\mathrm{U}(L), i\mathbb{R})$, whose curvature is $\Omega = \mathrm{d}\omega$ since U(1) is abelian. Given a cover $\{U_i\}_{i \in I}$ which trivializes L (and therefore $\mathrm{U}(L)$), consider the local field strengths $\mathcal{F}_i \in \Omega^2(U_i, i\mathbb{R})$. According to Proposition 1.1.8, on the overlaps $U_i \cap U_j$, the fields strengths are related by

$$\mathcal{F}_i = \mathrm{Ad}_{g_{ij}^{-1}} \mathcal{F}_j = \mathcal{F}_j.$$

This is because U(1) is abelian, so $\mathrm{Ad}_g = \mathrm{id}$ for all $g \in \mathrm{U}(1)$. This tells us that the collection of 2-forms $\{\mathcal{F}_i\}_{i \in I}$ "glues" together into a unique 2-form $\mathcal{F} \in \Omega^2(M, i\mathbb{R})$. Clearly, $\mathcal{F}$ is closed, so it determines a de Rham cohomology class $[\mathcal{F}] \in H^2_{dR}(M) \otimes i\mathbb{R}$.

Even though we started with a choice of a connection, the class $[\mathcal{F}]$ is independent of it: any other connection ω' is of the form $\omega' = \omega + \pi^*\delta$, where $\delta \in \Omega^1(M, i\mathbb{R})$. Therefore its curvature is $\Omega' = \Omega + \mathrm{d}(\pi^*\delta)$, and so the local field strength satisfies

$$\mathcal{F}' = \mathcal{F} + \mathrm{d}\delta.$$

This implies that $[\mathcal{F}'] = [\mathcal{F}] \in H^2_{dR}(M) \otimes i\mathbb{R}$. In fact, this class is also independent of the choice of Hermitian metric, it is *characteristic*, and more specifically, it satisfies

$$c_1(L) = \frac{-1}{2\pi i}[\mathcal{F}].$$

For proof of these statements, see [MS74, Appendix C, p. 305].

1.3.2 Čech cohomology and the first Stiefel-Whitney class

The other characteristic classes we will use are the Stiefel-Whitney classes. These can be defined similarly to the Chern classes; that is, in terms of the classifying space of *real* vector bundles. However, we will present another face of the first Stiefel-Whitney class, in terms of *Čech* cocycles. Therefore, we will very briefly describe Čech cohomology with coefficients in $\mathbb{Z}_2$. Later, when we discuss Spin structures, we will see the *second* Stiefel-Whitney class show up as an obstruction to their existence over a manifold.

Let X be a topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Given a indices $i_0, \dots, i_k \in I$, write $U_{i_0\cdots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$. For $j = 0, \dots, k$, we have inclusions

$$\iota_j^k : U_{i_0\cdots i_k} \hookrightarrow U_{i_0\cdots \hat{i}_j \cdots i_k},$$

where $\hat{i}_j$ means that the index is omitted. Let G be an abelian group[18]. The set of **Čech k-cochains**, denoted by $\check{C}^k(\mathcal{U}, G)$, is defined as follows: For every tuple of indices $(i_0, \dots, i_k)$, consider locally constant function $f_{i_0,\dots,i_k} : U_{i_0\cdots i_k} \to G$. This defines a map

$$\begin{aligned} f : I^{k+1} &\to \mathrm{LocConst}(U_{i_0\cdots i_k}, G), \\ (i_0, \dots, i_k) &\mapsto f(i_0, \dots, i_k) = f_{i_0,\dots,i_k} \end{aligned}$$

The set of k-cochains is the set of all such maps, that is,

$$\check{C}^k(\mathcal{U}, G) = \prod_{(i_0,\dots,i_k)\in I^{k+1}} \mathrm{LocConst}(U_{i_0\cdots i_k}, G).$$

There is a codifferential $\delta^k : \check{C}^k(\mathcal{U}, G) \to \check{C}^{k+1}(\mathcal{U}, G)$ given by

$$(\delta^k f)(i_0, \dots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j \, f_{i_0,\dots,\hat{i}_j,\dots,i_{k+1}}\Big|_{U_{i_0\cdots \hat{i}_j \cdots i_{k+1}}}.$$

It is a straightforward bookkeeping exercise to show that this codifferential is indeed nilpotent, so that $(\check{C}^\bullet(\mathcal{U}, G), \delta)$ is a cochain complex. The **Čech cohomology** groups are the cohomology groups of the Čech complex:

$$\check{H}^k(\mathcal{U}, G) := {\ker \delta^k}\big/{\mathrm{im}\, \delta^{k-1}}.$$

Of course, this complex, and therefore its cohomology, depend on the choice of cover of X. We can fix this if we restrict ourselves to covers $\mathcal{U}$ for which all intersections $U_{i_0\cdots i_k}$ are either empty or contractible[19]. We call these **good covers**, and it can be shown that if X is a paracompact topological manifold, then Čech cohomology groups are independent of the choice of *good* cover, and in fact they are isomorphic to the singular cohomology groups with coefficients in $G = \mathbb{Z}_2, \mathbb{Z}$ or $\mathbb{R}$[see GQ19, Theorem 10.5].

Now consider a vector bundle $E \to X$ of rank k, and let $\mathcal{U}$ be a good cover. Since each element $U_j \in \mathcal{U}$ is contractible, then necessarily the cover trivializes the bundle. This trivialization has associated transition functions

$$g_{ij} : U_{ij} \to \mathrm{GL}(k, \mathbb{R})$$

[18] In practice, $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{R}$.

[19] Another way to fix it is to see that a refinement $\mathcal{U}'$ of the cover $\mathcal{U}$ induces a morphism $\check{H}^k(\mathcal{U}, G) \to \check{H}^k(\mathcal{U}', G)$. We can then define $\check{H}^k(X, G)$ as the colimit over all covers with respect to the partial order of refinements.

satisfying the cocycle conditions:

$$g_{ji}(x) = g_{ij}(x)^{-1}$$
$$g_{ij}(x)g_{jk}(x)g_{ki}(x) = \mathrm{id}$$

for all x in the appropriate domains. Two sets of cocycles $\{g_{ij}\}$, $\{g'_{ij}\}$ over the trivialization $\mathcal{U}$ determine the *same* bundle if and only if there is a collection of maps $f_i : U_i \to \mathrm{GL}(k, \mathbb{R})$ such that

$$g'_{ij}(x) = f_i^{-1}(x) g_{ij}(x) f_j(x).$$

In particular, for a line bundle, choosing a metric we can obtain transition functions $g_{ij} : U_{ij} \to \mathbb{Z}_2$. Then two sets of transition functions determine the same line bundle if and only if there is a collection of maps $\{f_i\}$ such that

$$g'_{ij} = g_{ij} f_i^{-1} f_j.$$

Note that the collections $\underline{g} = \{g_{ij}\}$ and $\underline{g}' = \{g'_{ij}\}$ determine Čech 1-cocycles $\underline{g}, \underline{g}' \in \check{C}^1(\mathcal{U}, \mathbb{Z}_2)$, whereas the collection $\underline{f} = \{f_j\}$ determines a Čech 0-cocycle $\underline{f} \in \check{C}^0(\mathcal{U}, \mathbb{Z}_2)$. The equivalence condition is precisely

$$\underline{g}' = \underline{g} \cdot \delta \underline{f},$$

written in multiplicative notation. This implies that, in cohomology,

$$[\underline{g}] = [\underline{g}'] \in \check{H}^1(\mathcal{U}, \mathbb{Z}_2).$$

Definition 1.3.6 (First Stiefel-Whitney class).
Let $E \to M$ be a real line bundle of rank k. Choosing a Riemannian metric on E and a good cover $\mathcal{U}$ on M, we can find transition functions $\underline{g} = \{g_{ij} : U_{ij} \to \mathrm{O}(k)\}$ for the bundle. The **first Stiefel-Whitney class** *of the bundle is the Čech cohomology class $w_1(E) \in \check{H}^1(\mathcal{U}, \mathbb{Z}_2)$ of the determinant of the cocycle:*

$$\det(g_{ij}) : U_{ij} \to \mathbb{Z}_2.$$

That is, $w_1(E) = [\det(\underline{g})]$.

From this definition, we see that E is orientable if and only if $w_1(E) = 0$.

1.4 hodge-theory.zip

Let M be a closed, oriented, n-dimensional Riemannian manifold. The metric on M induces a metric on differential forms, which is defined in terms of wedge products of 1-forms as

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det(\langle \alpha_i, \beta_j \rangle),$$

where on the right-hand side we mean the determinant of the matrix whose i, j-th entry is $\langle \alpha_1, \beta_j \rangle$. There exists an isomorphism $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, called the **Hodge star**, such that for all $\alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \,\mathrm{vol}.$$

The $\star$ operator satisfies

$$\star \star \alpha = (-1)^{k(n-k)}\alpha,$$

for all $\alpha \in \Omega^k(M)$.

If M is compact, then we have an L^2-inner product on forms:

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \wedge \star\beta = \int_M \langle \alpha, \beta \rangle \,\mathrm{vol}.$$

Under this inner product, the adjoint of the de Rham differential is $\mathrm{d}^* : \Omega^k(M) \to \Omega^{k-1}(M)$, given by

$$\mathrm{d}^*\alpha = (-1)^{n(k+1)+1} \star \mathrm{d} \star \alpha,$$

for all $\alpha \in \Omega^k(M)$.

Consider the **Hodge Laplacian** $\Delta_H = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. We say that a form ω is **harmonic** if $\Delta_H(\omega) = 0$. Denote by $\mathcal{H}^k(M)$ the set of harmonic k-forms. The key result of Hodge theory is that harmonics forms represent de Rham cohomology.

Theorem 1.4.1 (Hodge).
For all $k \geq 0$, the space of harmonic k-forms $\mathcal{H}^k(M)$ is finite-dimensional, *and there is an orthogonal decomposition*

$$\Omega^k(M) = \mathrm{d}\Omega^{k-1}(M) \oplus \mathrm{d}^*\Omega^{k+1}(M) \oplus \mathcal{H}^k(M).$$

Furthermore, each de Rham cohomology class in $H^k(M)$ has a unique *harmonic representative. That is, $H^k(M) \cong \mathcal{H}^k(M)$.*

For a proof, see [War83, Chapter 6].

1.5 Elliptic regularity

In order to prove that the moduli space of the Seiberg-Witten equations is a smooth, compact manifold, we need to use the implicit function theorem. For this we have to use the *Sobolev* completions of spaces of sections of a bundle.

1.5.1 Sobolev spaces

Let $U \subset \mathbb{R}^n$ be an open set. We say that $f : U \to \mathbb{R}$ is locally p-integrable if every point $p \in U$ has a neighborhood V such that $f|_V \in L^p(V)$. The set of all such functions is denoted by $L^p_{\mathrm{loc}}(U)$. For any given multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we say that a function $g \in L^p_{\mathrm{loc}}(U)$ is a **weak α-th derivative** of f if for all test functions $\varphi \in C^{|\alpha|}_0(U)$,

$$\int_U \varphi g = (-1)^{|\alpha|} \int_U f \partial_\alpha \varphi.$$

We write $g = \partial_\alpha f$. It is a standard result that g is determined uniquely almost everywhere.

For every integer $k \geq 1$ and real $1 \leq p \leq \infty$, define

$$W^{k,p} = \{f \in L^p_{\mathrm{loc}}(U) \mid f \text{ has weak } \alpha\text{-th derivatives in } L^p_{\mathrm{loc}}(U) \text{ for all } |\alpha| \leq k.\}$$

It is straightforward to show that $W^{k,p}$ is a vector space. Furthermore, defining the **Sobolev norm**

$$\|f\|_{k,p} = \left(\sum_{|\alpha| \leq k} \int_U |\partial_\alpha f|^p \right)^{\frac{1}{p}},$$

we have that the space $W^{k,p}$ is a *Banach* space, which we call a **Sobolev space**.

For our purposes, we will need to consider the Sobolev completions of *spaces of sections* of a vector bundle E over a manifold M. In this case, we say that a section $s : M \to E$ is in $W^{k,p}(M, E)$ if it is represented locally by functions in $W^{k,p}$.

The Sobolev spaces satisfy some very useful properties. The adventurous reader can check [Bré11, Chapters 8 and 9], [GT01, Chapters 7 and 8], [DK97, Appendix II], or [see Sal99, Appendix C.1] for more details.

Lemma 1.5.1 (Rellich's lemma).
For all $k \geq 1$*, the inclusion*

$$W^{k+1,p} \hookrightarrow W^{k,p}$$

is compact. In particular, any bounded sequence in $W^{k+1,p}$ *has a subsequence that converges in* $W^{k,p}$.

Proposition 1.5.2 (Sobolev embedding).
Let $n = \dim(M)$*. Then for all finite-dimensional vector bundles* E *over* M*, there is a continuous embedding*

$$W^{k,p}(M, E) \hookrightarrow C^m(M, E),$$

provided that

$$m \leq k - \frac{n}{p}.$$

In particular, if $s \in W^{k,p}(M, E)$ *for all* $k \geq k_0$*, then* s *is smooth.*

1.5.2 Elliptic operators

Let E, F be real or complex vector bundles of ranks p and q, respectively, over a manifold M. A **partial differential operator of order at most** k is a linear map $L : \Gamma(E) \to \Gamma(F)$ that locally, given a choice of trivializations of E and F and coordinates on M, is given by

$$L = \sum_{|\alpha| \leq k} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where α is a multi-index, and each A^α is a $q \times p$ matrix-valued function on the local chart.

For each $m \leq k$, let L_m be the "homogeneous" component of order m of L, that is

$$L_m = \sum_{|\alpha| = m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum_{\alpha_1, \ldots \alpha_m} A^{\alpha_1 \ldots \alpha_m} \frac{\partial}{\partial x^{\alpha_1}} \cdots \frac{\partial}{\partial x^{\alpha_m}}.$$

If we change coordinates to x'^β, but *not the trivialization* of the bundles, then we must have

$$L_m = \sum_{\alpha_1, \ldots, \alpha_m} \sum_{\beta_1, \ldots, \beta_m} A^{\alpha_1 \ldots \alpha_m} \frac{\partial x'^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\beta_m}}{\partial x^{\alpha_m}} \frac{\partial}{\partial x'^{\beta_1}} \cdots \frac{\partial}{\partial x'^{\beta_m}} = \sum_{\beta_1, \ldots, \beta_m} A'^{\beta_1 \ldots \beta_m} \frac{\partial}{\partial x'^{\beta_1}} \cdots \frac{\partial}{\partial x'^{\beta_m}}$$

and therefore, the coefficients of L transform as

$$A'^{\beta_1\ldots\beta_m} = \sum_{\alpha_1,\ldots,\alpha_m} A^{\alpha_1\ldots\alpha_m} \frac{\partial x'^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\beta_m}}{\partial x^{\alpha_m}}.$$

This tells us that the coefficients A^α glue well to form $p \times q$-matrix-valued symmetric tensor field. Even more, following a similar procedure, we can see that these matrices transform well under a change of trivialization, so that L_m has an associated section

$$\tilde{\sigma}_m(L) \in \Gamma(S^m TM \otimes \mathrm{End}(E, F)),$$

whose components are precisely the A^α of above. We interpret it instead as a map

$$\tilde{\sigma}_m(L) : \Gamma(S^m T^*M) \to \Gamma(\mathrm{End}(E, F)).$$

We define the **symbol of order m of** L as the map[20]

$$\begin{aligned}\sigma_m(L) : \Omega^1(M) &\to \Gamma(\mathrm{End}(E, F))\\ \xi &\mapsto \tilde{\sigma}_m(L)(\xi, \xi, \ldots, \xi).\end{aligned}$$

If L is a partial differential operator of order at most k, then we call $\sigma_k(L)$ the **principal symbol** of L. We denote it as $\sigma(L)$ whenever there is no chance of confusion. In local coordinates, for $x \in M$ and $\xi \in T_x^*M$, if we write $\xi = \xi_\mu \, \mathrm{d}x^\mu$, then the symbol $\sigma_m(L)$ becomes[21]

$$\sigma_m(L) = \sum_{\alpha_1,\ldots,\alpha_m} A^{\alpha_1\ldots\alpha_m} \xi_{\alpha_1} \ldots \xi_{\alpha_m}. \tag{1.2}$$

Sometimes, the coordinate expression of a partial differential operator is cumbersome, and it makes it difficult to compute its symbol. Luckily, there is an alternative way to find the symbol in a coordinate-free way. For the sake of simplicity (and since it's our main interest), assume that L is a partial differential operator of order at most 2. Then we can write

$$L = L_2 + L_1 + L_0 = \sum_{\alpha_1,\alpha_2} A^{\alpha_1\alpha_2} \frac{\partial^2}{\partial x^{\alpha_1} \partial x^{\alpha_2}} + \sum_\beta B^\beta \frac{\partial}{\partial x^\beta} + C,$$

where $A^{\alpha_1\alpha_2}$, B^β and C are $q \times p$-matrix-valued functions. For an arbitrary $f \in C^\infty(M)$, define the map $[L, f] : \Gamma(E) \to \Gamma(F)$ as

$$[L, f](s) = L(fs) + fL(s)$$

for all $s \in \Gamma(E)$. A straightforward computation shows that

$$[[L, f], f](s) = 2 \sum_{\alpha_1,\alpha_2} \frac{\partial f}{\partial x^{\alpha_1}} \frac{\partial f}{\partial x^{\alpha_2}} A^{\alpha_1\alpha_2}(s).$$

Note that this expression is precisely Equation (1.2) with a leading factor of 2 and with the components of ξ replaced by $\partial_{\alpha_j} f$. Therefore, a way to compute $\sigma(L)$ is by computing $[[L, f], f]$ for an arbitrary function, substituting $\mathrm{d}f$ with an arbitrary form ξ, and dividing by $k!$.

[20] Note that σ and $\tilde{\sigma}$ are equivalent by polarization.

[21] This follows by writing ξ as $\mathrm{d}f_x$ for some function, and by interpreting the ∂_{α_j} in L_m as the dual basis to the basis $\mathrm{d}x_j^\alpha$.

Example 1.5.3 (Symbol of the Lie derivative).
Let $X \in \mathfrak{X}(M)$ be a vector field, and consider the Lie derivative $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$. In local coordinates x^μ, write $X = X^\mu \partial_\mu$. For any vector field $Y = Y^\nu \partial_\nu$, we have

$$\mathcal{L}_X(Y) = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\frac{\partial}{\partial x^\mu}.$$

Here we identify the first- and zeroth-order terms

$$\mathcal{L}_{X,0}(Y) = -Y^\nu \partial_\nu X^\mu \frac{\partial}{\partial x^\mu} \qquad \mathcal{L}_{X,1}(Y) = X^\nu \partial_\nu Y^\mu \frac{\partial}{\partial x^\mu}.$$

Here, even though the zeroth-order term has a partial derivative, it is *not* acting on the components of Y. From this expression, we see at once that the principal symbol of $\mathcal{L}_X$ is

$$\sigma(\mathcal{L}_X)(\xi) = X^\nu \xi_\nu.$$

Equivalently, fix $f \in C^\infty(M)$. Then

$$[\mathcal{L}_X, f](Y) = \mathcal{L}_X(fY) - f\mathcal{L}_X Y = \mathcal{L}_X(f)Y = \mathrm{d}f(X)\cdot Y,$$

and thus, with the rule "substitute $\mathrm{d}f$ with an arbitrary form ξ", we see that

$$\sigma(\mathcal{L}_X)(\xi) = \xi(X)\cdot .$$

The principal symbols of operators behave well under composition:

Proposition 1.5.4 (Symbol of composition).
Let $L : \Gamma(E) \to \Gamma(F)$ and $D : \Gamma(F) \to \Gamma(G)$ be partial differential operators of order at most k and l, respectively. Then

$$\sigma_{l+k}(D \circ L) = \sigma_l(D) \circ \sigma_k(L).$$

We are interested in a very specific class of operators:

Definition 1.5.5 (Elliptic operator).
A partial differential operator L is **elliptic** *if its principal symbol is a fiberwise isomorphism for all nonzero forms ξ. That is, if for all $x \in M$ and all $\xi \in T^*M$, the map $\sigma(L)(\xi) : E_x \to F_x$ is an isomorphism. Furthermore, if M is a Riemannian manifold and $L : \Gamma(E) \to \Gamma(E)$, we say that L is a* **generalized Laplacian** *if its symbol is*

$$\sigma(L)(\xi) = \pm|\xi|^2 \mathrm{id}.$$

The importance of elliptic operators comes from the fact that the elements of their kernels enjoy a lot of regularity. We will constantly make use of this fact to "recover" lost regularity of spinors when we take derivatives of them, or use Rellich's lemma to find convergent subsequences.

Theorem 1.5.6 (Elliptic regularity).
Let M be a closed, oriented Riemannian manifold, E, F Riemannian vector bundles, and $L : \Gamma(E) \to \Gamma(F)$ an r-th order elliptic operator.

1. *For all* $1 < p < \infty$ *and* $k \geq 1$, *there are constants (depending on* k, p*) such that if* $u \in W^1(E)$ *is a weak* L^p*-solution of*
$$Lu \overset{wk}{=} v,$$
with $v \in W^{k,p}(F)$, *then*
$$u \in W^{k+r,p}(E)$$
and
$$\|u\|_{k+r,p} \leq C(\|u\|_p + \|v\|_{k,p}).$$
2. *If* E, F *are Hermitian, and* $P : L^2(E) \to L^2(E)$ *is the orthogonal projection to* $\ker(L)$, *then for all* $1 < p < \infty$ *and* $k \geq 1$ *there is a constant* C *such that*
$$\|u - Pu\|_{r+k,p} \leq C\|Lu\|_{k,p}$$
for all $u \in W^{r+k,p}(E)$.

CHAPTER 2

Spin Geometry

The Seiberg-Witten equations are, above all, equations about *spinor fields* on manifolds. These spinors have their origin in physics, from the *Dirac* equation, which comes from relativistic quantum mechanics. In a naïve relativistic generalization of quantum mechanics, the evolution of the quantum wavefunction is given by the Klein-Gordon(-Schrödinger) equation (in "natural" units):

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0.$$

Where $\psi : \mathbb{R}^4 \to \mathbb{C}$ is a scalar field, and $m \geq 0$ is the mass of the particle. On physical grounds, Paul Dirac found this equation unsatisfactory, specifically because it was of second order in time[1]. He set out to find a "square root" of the D'Alambertian operator $\partial_t^2 - \nabla^2 + m^2$ which was of first order in time as well. He quickly discovered that such a square root cannot exist as an operator acting on *scalar* functions. However, if he allowed for vector-valued wavefunctions, then by choosing matrices $\gamma^0, \dots, \gamma^3$ satisfying the relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \tag{2.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric[2], he proved that the operator

$$\partial\!\!\!/ \psi = i \sum_\mu \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - m\psi, \tag{2.2}$$

does indeed square to the D'Alambertian.

In general, the γ^μ form a (faithful) representation of a **Clifford algebra** on a vector space S. This Clifford algebra is defined *precisely* as a real algebra with relations given by the anti-commutation relations of Equation (2.1) [3]. Therefore, the wavefunction ψ can no longer be a complex function, but rather a

[1] This leads to a failure in a conservation law that ruins the standard interpretation that the square norm of the wave-function $|\psi|^2$ represents a *probability* distribution in space.

[2] Defined as $\eta^{00} = 1$, $\eta^{ii} = -1$ for $i \geq 1$, and $\eta^{\mu\nu} = 0$ in all other cases.

[3] Clifford himself defined these algebras (which he called *geometric algebras*) as an alternative to the Grassmann algebras, generated by anti-commuting objects but whose squares are 1 or -1. Dirac's rediscovery of them is a nice case of serendipity.

S-valued field on the space-time $\mathbb{R}^3 \times \mathbb{R}$. Such fields we call **spinor**[4] **fields**. In order to generalize this operator to manifolds, we need an additional structure on the manifold, called a *Spin* structure.

Not all 4-manifolds admit a Spin structure (for example, $\mathbb{CP}^2$ does not). However, if we *twist* the structure locally with an additional complex line bundle, we obtain a Spin^c-structure, and these *always* exist for 4-manifolds. Physically speaking, the addition of a complex line bundle is coupling the evolution of the relativistic particle to an electromagnetic field.

Overview of this chapter

This chapter can be roughly divided in two parts. The first is *algebraic*: We define Clifford algebras, review some of their properties, and study their representations. Then we introduce the *Spin groups*, which are subgroups of the Clifford algebras that can be seen as the universal covers of the orthogonal groups. The second part is *geometric*: we promote all these algebraic structures to geometric structures on manifolds, and define the Dirac operator. Finally, we add a U(1) term to everything and study the Spin^c groups, Spin^c-structures, and the coupled Dirac operator. This U(1) term is physically interpreted as coupling an electromagnetic field to the relativistic particle. Topologically, this extends and adds flexibility to the notion of a spin structure.

2.1 Clifford algebras

This section follows [Fig17] and [LM89], with a dash of [Nab05]. Let V be a finite-dimensional vector space over[5] $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $g : V \times V \to \mathbb{K}$ a symmetric bilinear form. We are mostly interested on *non-degenerate* forms, which we call semi-Riemannian metrics, or Riemannian if they are positive definite.

We want to mimic the behaviour of the Dirac γ matrices, which satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu},$$

so we want to make a $\mathbb{K}$-algebra that contains V and satisfies

$$xy + yx = -g(x, y)1.$$

> **Remark.** Here is the first example of slightly different conventions between mathematicians and physicists (which sometimes lead to not-so-slight annoyances). The physics convention is to define the Clifford algebras such that $xy + yx = g(x, y)1$. This follows the behavior of the Dirac γ matrices and the Minkowski metric. This does not change anything in the theory, only the names of things.

2.1.1 Definition and basic properties

We proceed as usual when we want to construct objects that satisfy certain relations: find a larger object of the "same type" and take the quotient over a subset of elements that we need to be identified with zero.

Thus, if V is a vector space with a symmetric, bilinear form g, we construct the Clifford algebra from its tensor algebra $T^\bullet V$ quotienting out the ideal I generated by elements of the form $v \otimes v + g(v, v)$.

[4] Pronounced "spinner", like those fidget toys that were quite popular a few years ago.
[5] Most of the theory, up until the definition of the Spin group, works for *any* field.

Definition 2.1.1 (Clifford algebra).
The **Clifford algebra** *associated to a vector space* (V, g) *with a symmetric, bilinear form is*

$$\mathrm{Cl}(V, g) := {}^{T^\bullet V}\!/_{I},$$

where I is the ideal generated by

$$\{v \otimes v + g(v, v) \mid v \in V\}.$$

Note that the ideal I contains only only elements of even degree of $T^\bullet V$, and hence $V \cap I = \{0\}$. Therefore the map $V \to \mathrm{Cl}(V, g)$ which maps v to its class in $\mathrm{Cl}(V, g)$ is injective, and so we simply think of V as sitting in $\mathrm{Cl}(V, g)$. Furthermore, since I contains only even elements, the quotient $\mathrm{Cl}(V, g)$ inherits a $\mathbb{Z}_2$ grading (which we will view in more detail below).

Therefore, in $\mathrm{Cl}(V, g)$, by definition we have for all $v \in V$, denoting the class of v by the same symbol,

$$v \cdot v = -g(v, v)1.$$

More generally, for all $v, u \in V \subset \mathrm{Cl}(V, g)$:

$$u \cdot v + v \cdot u = -2g(u, v)1.$$

Suppose that we have a linear map $\varphi : V \to A$ into a unital $\mathbb{K}$-algebra A. When can we extend it to a morphism of algebras from $\mathrm{Cl}(V, g)$? Certainly, any extension $\overline{\varphi} : \mathrm{Cl}(V, g) \to A$ must satisfy

$$\varphi(v)^2 = \overline{\varphi}(v)^2 = \overline{\varphi}(v^2) = \overline{\varphi}(-g(v, v)1) = -g(v, v)1$$

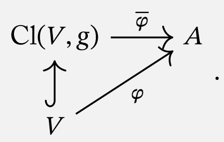
for all $v \in V$. Conversely, since any map can be extended naturally to the tensor product, the condition for it to descend to the quotient $\mathrm{Cl}(V)$ is that it vanishes on the ideal I, which is precisely the condition $\varphi(v)^2 = -g(v, v)1$. This property has its own name name:

Definition 2.1.2 (Clifford map).
Let V *be a* $\mathbb{K}$*-vector space,* g *a symmetric bilinear form on* V*, and* A *an associative, unital* $\mathbb{K}$*-algebra. A* $\mathbb{K}$*-linear map* $\varphi : V \to A$ *is a* **Clifford map** *if for all* $v \in V$*:*

$$\varphi(v)^2 = -g(v, v)1.$$

In the discussion above, we have almost shown that the Clifford algebra $\mathrm{Cl}(V, g)$ satisfies the following universal property:

Proposition 2.1.3 (Universal property of the Clifford algebra).
Let (V, g) *be a vector space with a quadratic form* g*. There is a unique (up to isomorphism) associative unital* $\mathbb{K}$*-algebra* $\mathrm{Cl}(V, g)$*, along with a linear injective morphism* $\iota : V \to \mathrm{Cl}(V, g)$*, such that for every associative, unital* $\mathbb{K}$*-algebra* A*, if* $\varphi : V \to A$ *is a Clifford map then there exists a unique morphism of algebras* $\overline{\varphi} : \mathrm{Cl}(V, g) \to A$ *such that the diagram commutes:*



We haven't shown uniqueness of $\mathrm{Cl}(V,g)$, but it follows from the universal property itself.

This universal property also tells us that linear maps that are orthogonal with respect to the symmetric form can be extended as morphisms of Clifford algebras.

Corollary 2.1.4 (Extension of orthogonal maps).
Let (V,g) and (W,h) be $\mathbb{K}$-vector spaces and g,h symmetric bilinear forms on V, W respectively. If $\varphi : V \to W$ is a $\mathbb{K}$-linear map such that $h(\varphi(u),\varphi(v)) = g(u,v)$ for all $u,v \in V$, then φ extends to a unique morphism $\overline{\varphi} : \mathrm{Cl}(V,g) \to \mathrm{Cl}(W,h)$.

Proof. — This proof is an exercise in using the universal property. First, we have that φ itself can be considered as a map $\varphi : V \to \mathrm{Cl}(W,h)$. It is a Clifford map, since for all $v \in V$,

$$\varphi(v)^2 = -h(\varphi(v),\varphi(v))1 = -g(v,v)1.$$

The first equality comes from the basic relation of $\mathrm{Cl}(W,h)$. Therefore, by the universal property, there is a unique morphism $\overline{\varphi} : \mathrm{Cl}(V,g) \to \mathrm{Cl}(W,h)$ which restricts to φ on V. ■

Universal properties and quotients of tensor algebras are not very pleasant to work with. Fortunately, we can be much more explicit with our description of $\mathrm{Cl}(V,g)$. If we choose a basis $e_1,\dots,e_n$ of V, then the Clifford algebra $\mathrm{Cl}(V,g)$ is the $\mathbb{K}$-algebra generated by $e_1,\dots,e_n$ subject to the relations

$$e_ie_j + e_je_i = -2g(e_i,e_j)1.$$

Notation. We will denote by $\mathbb{R}^{r,s}$ the semi-Riemannian vector space $\mathbb{R}^{n=r+s}$ with the standard semi-Riemannian metric with r positive eigenvalues and s negative eigenvalues. We denote $\mathrm{Cl}(r,s) = \mathrm{Cl}(\mathbb{R}^{r,s})$; and $\mathrm{Cl}(n) = \mathrm{Cl}(\mathbb{R}^n)$ with the standard euclidean metric.

Let's see a few examples:

Example 2.1.5 ($\mathrm{Cl}(1)$, $\mathrm{Cl}(2)$).
For $n = 1$, we have that $\mathrm{Cl}(1)$ is the real associative algebra generated by the elements 1, e with the relation $e^2 = -1$. This is precisely $\mathbb{C}$, seen as an $\mathbb{R}$-algebra. Thus,

$$\mathrm{Cl}(1) \cong \mathbb{C}.$$

Note that the inclusion of $\mathbb{R}$ into $\mathrm{Cl}(1) \cong \mathbb{C}$ is *not* the obvious one: we have

$$\begin{aligned} \mathbb{R} &\hookrightarrow i\mathbb{R} \subseteq \mathrm{Cl}(1) \cong \mathbb{C} \\ 1 &\mapsto i. \end{aligned}$$

Now let's go to the case $\mathrm{Cl}(2)$. This is generated by elements 1, e_1, e_2 subject to the conditions $e_1^2 = e_2^2 = -1$, and $e_1e_j = -e_je_i$. A little bit of playing around with it lets us see that the identification $e_1 = \mathbb{i}$, $e_2 = \mathbb{j}$, $e_1e_2 = \mathbb{k}$ induces an isomorphism between $\mathrm{Cl}(2)$ and the quaternions $\mathbb{H}$:

$$\mathrm{Cl}(2) \cong \mathbb{H}.$$

What is the dimension of the Clifford algebra $\mathrm{Cl}(V, g)$? If $e_1, \dots, e_n$ is an orthonormal basis of V, then the elements

$$\begin{array}{c} 1, \\ e_1, \ \dots, \ e_n, \\ e_1e_2, \ \dots, \ e_1e_n, \ e_2e_3, \ \dots, \ e_{n-1}e_n \\ \vdots \\ e_1 \dots e_n \end{array}$$

are all linearly independent and they span $\mathrm{Cl}(V)$, and so $\dim \mathrm{Cl}(V) = 2^{\dim V}$. We have proved, then:

Proposition 2.1.6 (Dimension of the Clifford algebra).
Let (V, g) be a finite-dimensional vector space with a symmetric bilinear form. Then $\dim \mathrm{Cl}(V, g) = 2^{\dim V}$.

It is time to revisit the $\mathbb{Z}_2$ grading of the Clifford algebra. We have a natural decomposition of $\mathrm{Cl}(V)$ into even and odd components

$$\mathrm{Cl}(V) = \mathrm{Cl}(V)_0 \oplus \mathrm{Cl}(V)_1 \qquad \text{(Only as vector spaces!)}$$

where $\mathrm{Cl}(V)_0$ is the set of even elements, and $\mathrm{Cl}(V)_1$ is the set of odd ones. Note that $\mathrm{Cl}(V)_0$ is a subalgebra of $\mathrm{Cl}(V)$, but $\mathrm{Cl}(V)_1$ is *not*. This decomposition turns $\mathrm{Cl}(V)$ into a $\mathbb{Z}_2$-*graded* algebra, since if $a \in \mathrm{Cl}(V)_i$ and $b \in \mathrm{Cl}(V)_j$, then $ab \in \mathrm{Cl}(V)_{i+j}$ (the sum taken modulo 2).

Consider the orthogonal map $v \mapsto -v$ on (V, g). By Corollary 2.1.4, it induces an automorphism, the **parity** map, which we denote by $p : \mathrm{Cl}(V, g) \to \mathrm{Cl}(V, g)$. Explicitly, for $v_1, \dots, v_k \in V$, this map is

$$p(v_1 \dots v_k) = (-v_1) \dots (-v_k) = (-1)^k v_1 \dots v_k.$$

Therefore, for all $a \in \mathrm{Cl}(V)$, if we write $a = a_0 + a_1$ with $a_0 \in \mathrm{Cl}(V)_0$ and $a_1 \in \mathrm{Cl}(V)_1$, then

$$p(a) = p(a_0 + a_1) = a_0 - a_1.$$

Namely, $\mathrm{Cl}(V)_0$ (resp. $\mathrm{Cl}(V)_1$) is the eigenspace associated to the eigenvalue 1 (resp. -1) of p.

2.1.2 Complex(ified) Clifford algebras

There is a rich theory of real Clifford algebras (see e.g.[LM89]), but we will study their complexified flavours, basically since in the global case we consider *complex* spinors. Given a real vector space V, with a symmetric bilinear form g, we define

$$\mathrm{Cl}_{\mathbb{C}}(V, g) := \mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}.$$

Note that $\mathrm{Cl}_{\mathbb{C}}(V, g)$ is a $\mathbb{C}$-algebra with scalar multiplication

$$z \cdot (a \otimes w) = a \otimes zw,$$

for all $a \in \mathrm{Cl}(V, g)$ and $z, w \in \mathbb{C}$.

The complexified Clifford algebra is precisely the Clifford algebra of the complexified $V_{\mathbb{C}} = V \otimes \mathbb{C}$. This is a straightforward application of the universal property.

Proposition 2.1.7 (Complexification and Cliffordization commute).
Let V be a real vector space and g a semi-Riemannian metric on V. Then there is an isomorphism

$$\mathrm{Cl}_{\mathbb{C}}(V,g) \cong \mathrm{Cl}(V_{\mathbb{C}}, g_{\mathbb{C}}),$$

where $g_{\mathbb{C}}$ is the metric g, extended complex-bilinearly to $V_{\mathbb{C}}$.

Remark. Note that the complex bilinear form $g_{\mathbb{C}}$ is *not* an inner product. Complex inner products (or Hermitian metrics) are sesquilinear, not bilinear.

Furthermore, note that complexifying a bilinear form destroys any definiteness: Suppose that $v \in V$ is such that $g(v,v) > 0$. Then iv is $\mathbb{C}$-linearly dependent with v, but it satisfies

$$g_{\mathbb{C}}(iv, iv) = -g(v,v) < 0.$$

This means that if $e_1, \dots, e_n$ is a g-orthonormal basis of V, we can multiply some of those elements with i to obtain a basis $\varepsilon_1, \dots, \varepsilon_n$ with $g_{\mathbb{C}}(\varepsilon_j, \varepsilon_j) = 1$ for all j. This way, we conclude that for all $r, s \geq 0$,

$$\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{r,s}) = \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{r+s}).$$

Remark. We will be interested in *complex* representations of Clifford algebras. Since these are in bijection with representations of the complexified algebras (see the discussion below Definition 2.3.2), from now on we will only consider *Riemannian* metrics.

Let $\mathrm{Cl}_{\mathbb{C}}(n) = \mathrm{Cl}(n)_{\mathbb{C}} = \mathrm{Cl}(\mathbb{C}^n)$. We consider $\mathrm{Cl}(\mathbb{C}^n)$ as taken with respect to the standard quadratic form

$$q_{\mathrm{st}}(z) = \sum_{i=1}^{n} z_i^2.$$

These complexified Clifford algebras have a *very* simple classification[LM89, Theorem I.4.3]:

Theorem 2.1.8 (Classification of complex Clifford algebras).
For $n \geq 0$, we have

$$\mathrm{Cl}_{\mathbb{C}}(2n) \cong \mathbb{C}(2^n), \text{ and } \qquad \mathrm{Cl}_{\mathbb{C}}(2n+1) \cong \mathbb{C}(2^n) \oplus \mathbb{C}(2^n).$$

This classification will help us with the classification of their irreducible representations.

2.2 The Spin group

The next algebraic ingredients are the *spin groups*. In a pinch, one can define them as the universal covers of the orthogonal groups. However, with this definition, their relation to the Clifford algebras are somewhat obscure. In this section, we will see that the spin groups are the subgroups of the Clifford algebras which are "most like" the orthogonal groups.

2.2.1 Definition and examples

Let V be a Riemannian $\mathbb{K}$-vector space. We denote the inner product as $\langle u, v\rangle$. For every invertible element $a \in \mathrm{Cl}(V)^\times$, we have the **adjoint action** $\mathrm{Ad}_a : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$, given as

$$\mathrm{Ad}_a(b) = aba^{-1},$$

for all $b \in \mathrm{Cl}(V)$. This action is an automorphism of $\mathrm{Cl}(V)$, with inverse $\mathrm{Ad}_a^{-1} = \mathrm{Ad}_{a^{-1}}$.

Note that if $v \in V$ is nonzero, then the Clifford condition

$$v^2 = vv = -||v||^2 1,$$

implies that v is invertible as an element of $\mathrm{Cl}(V)$, with inverse $v^{-1} = -v/||v||^2$. Therefore we can consider the adjoint action of nonzero vectors of V. In particular, if $v \in V$ is nonzero, then for any $u \in V$, we have

$$\mathrm{Ad}_v(u) = vuv^{-1} = -\frac{1}{||v||^2}vuv = -\frac{1}{||v||^2}v(-vu - 2\langle u, v\rangle) = -\left(u - 2\langle u, v\rangle \frac{v}{||v||^2}\right) = -\mathrm{Refl}_v(u),$$

that is, $\mathrm{Ad}_v(u)$ is (minus) the reflection of u across the plane normal to v. This tells us that restricting Ad_v to V gives us an orthogonal map. In fact, for any element of multiplicative subgroup of $\mathrm{Cl}(V)$ generated by V, which we denote by $V^\times$, the adjoint action is still an orthogonal map on V, since

$$\mathrm{Ad}_{v_1 \dots v_k}(u) = (\mathrm{Ad}_{v_1} \circ \dots \circ \mathrm{Ad}_{v_k})(u) = (-1)^k(\mathrm{Refl}_{v_1} \circ \dots \circ \mathrm{Refl}_{v_k})(u).$$

for all $u \in V$. Therefore, we have a *group* homomorphism, which we call the **spinor map** $\mathfrak{z}$, given as

$$\begin{aligned} \mathfrak{z} : V^\times &\to \mathrm{O}(V, g) \\ v &\mapsto \mathfrak{z}(v) = \mathrm{Ad}_v|_V \end{aligned}$$

This morphism is surjective, since every orthogonal map can be decomposed as a set of reflections (this is the **Cartan-Dieudonné theorem** [see Car81, section 10 for an elementary proof]). Any *rotation* can be decomposed as an *even* number of reflections, so if we write $V_0^\times = V^\times \cap \mathrm{Cl}(V)_0$ for the subgroup of *even* elements generated by nonzero vectors, we have a surjective homomorphism

$$\mathfrak{z} : V_0^\times \to \mathrm{SO}(V).$$

In *neither* case $\mathfrak{z}$ is injective, since conjugation is invariant under scalar multiplication: if $a' = \lambda a$ for some nonzero $\lambda \in \mathbb{K}$, then $\mathrm{Ad}_a = \mathrm{Ad}_{a'}$. However, for the even case, this is indeed precisely the kernel [LM89, Proposition I.2.4]:

Lemma 2.2.1 (Kernel of $\mathfrak{z}$).
Let V, g be as above. Then

$$\ker \mathfrak{z}|_{V_0^\times} = \mathbb{K}^\times.$$

This tells us that the map $\mathfrak{z}$ does not identify a copy of $\mathrm{SO}(n)$ inside $\mathrm{Cl}(V)^\times$. However, if we restrict $\mathfrak{z}$ to the subgroup of even elements generated by *unit* vectors, we find an object that is *very similar* to $\mathrm{SO}(n)$. This is what we call the Spin group:

Definition 2.2.2 (Spin(V)).
Let V be a Riemannian vector space. The **spin group** Spin(V) *is the* even *multiplicative subgroup generated by vectors of unit length in V:*

$$\mathrm{Spin}(V) := \left\{ v_1 \ldots v_{2k} \in \mathrm{Cl}(V) \,\middle|\, \langle v_i, v_i \rangle^2 = 1 \text{ for all } k \right\}.$$

We write $\mathrm{Spin}(n) = \mathrm{Spin}(\mathbb{R}^n)$ *with the standard euclidean inner product.*

Since $\mathrm{Spin}(V) \subseteq V_0^\times$ and $\mathscr{s}$ is invariant under rescaling, we have that $\mathscr{s}$ restricted to $\mathrm{Spin}(V)$ is also surjective. This is because a rotation can be decomposed as a series of reflections perpendicular to *unit* vectors[6]. From all our work above with the map $\mathscr{s}$, we obtain:

Theorem 2.2.3 (Short Exact Sequence of Spin).
Let V be a **real**, *finite-dimensional Riemannian vector space. The short sequence*

$$1 \longrightarrow \{-1, 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\ \mathscr{s}\ } \mathrm{SO}(V) \longrightarrow 1$$

is exact. For $V = \mathbb{R}^n$ with $n \geq 2$, the spinor map $\mathscr{s}$ is a non-trivial two-sheeted covering. Furthermore, if $n \geq 3$, then $\mathrm{Spin}(n)$ *is the universal cover of* $\mathrm{SO}(n)$.

Proof. — Suppose that $x \in \mathrm{Spin}(V)$ is in $\ker \mathscr{s}$. By Lemma 2.2.1, since $\mathrm{Spin}(V) \subset V_0^\times$, then $x \in \mathbb{R}^\times$. However, x is a product of elements of unit norm, so $x = \pm 1$.

Finally, we have that $\mathscr{s} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is surjective by the Cartan-Dieudonné theorem (and the fact that we can normalize vectors in $\mathbb{R}$). Therefore the sequence is exact.

Now let's show that $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$. First, note that $\mathscr{s} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is indeed a covering map. Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$ [see Hal15, proposition 13.10], if we can show that $\mathrm{Spin}(n)$ is connected, we will be able to conclude that it is the universal cover[7] of $\mathrm{SO}(n)$.

We can construct a path connecting 1 to -1 in $\mathrm{Spin}(n)$, assuming $n \geq 2$ as follows: let $e_1, e_2 \in V$ be orthonormal. Consider the curve

$$t \mapsto \pm(\cos(t)e_1 + \sin(t)e_2)(\sin(t)e_2 - \cos(t)e_1).$$

Clearly this curve lies entirely in $\mathrm{Spin}(n)$ and connects 1 to -1. ∎

Remark. Many references *define* the Spin groups as the universal covers of the respective special orthogonal groups. Such a definition avoids all the Clifford algebra stuff. However, we *do* need the Clifford algebra stuff to define the Dirac operator that comes in the Seiberg-Witten equations.

---

[6] This is where we need more conditions on the field $\mathbb{K}$. Since the Cartan-Dieudonné theorem states that rotations can be decomposed into an even number of reflections along planes; just could choose a vector v orthogonal to the plane and normalize it. However, this assumes that the equation

$$\langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle = \pm 1$$

can be solved for λ, which is *not* the case for all fields $\mathbb{K}$.

[7] This follows from the following fact: if $\rho : \tilde{X} \to X$ is a covering, then for $x \in X$, the cardinality of the fiber $\rho^{-1}(x)$ is the index of $[\pi_1(X, x) : \rho_* \pi_1(\tilde{X}, \tilde{x})]$[Rot93, Theorem 10.9]. In our case, since $|\mathscr{s}^{-1}(1)| = |\pi_1(\mathrm{SO}(n))|$, then $\mathscr{s}_* \pi_1(\mathrm{Spin}(n), 1) = \{1\}$. However, $\mathscr{s}_* : \pi_1(\mathrm{Spin}(n), 1) \to \pi_1(\mathrm{SO}(n), 1)$ is an injection [Rot93, Theorem 10.7], so necessarily $\pi_1(\mathrm{Spin}(n), 1) = \{1\}$, i.e. $\mathrm{Spin}(n)$ is simply connected, and thus the universal cover of $\mathrm{SO}(n)$.

2.2.2 When $n = 4$

We are building up towards the Seiberg-Witten equations, which are on 4-manifolds. Our main interest is then in the $n = 4$ case. While we can obtain a characterization of $\mathrm{Cl}(4)$ via [LM89, Theorem 4.3], namely that $\mathrm{Cl}(4) \cong \mathbb{H}(2)$, this is not good enough for our purposes and does not help us with the computations that we want. The embedding of $\mathbb{R}^4 \hookrightarrow \mathrm{Cl}(4)$ is needed in order to determine the Spin(4) group, and following the proof of [LM89, Theorem 4.3] (which is constructive) gives us a very complicated embedding. Fortunately, we can express $\mathrm{Cl}(4)$ explicitly.

This example follows [Nab05, pp. 32-37]. Denote by $\mathbb{i}, \mathbb{j}, \mathbb{k}$ the complex units of $\mathbb{H}$. Let e_0, e_1, e_2, e_3 be the standard orthonormal basis of $\mathbb{R}^4$, and consider the map $\mathbb{R}^4 \to \mathbb{H}(2)$ given as

$$e_0 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_1 \mapsto \begin{pmatrix} 0 & \mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$$
$$e_2 \mapsto \begin{pmatrix} 0 & \mathbb{j} \\ \mathbb{j} & 0 \end{pmatrix} \qquad e_3 \mapsto \begin{pmatrix} 0 & \mathbb{k} \\ \mathbb{k} & 0 \end{pmatrix}.$$

It is straightforward to show that this is a Clifford map, which induces an isomorphism of *real* algebras $\mathrm{Cl}(4) \cong \mathbb{H}(2)$. This is called the **Weyl** or **chiral** representation of $\mathrm{Cl}(4)$. In this representation, $\mathbb{R}^4$ is embedded as

$$v \mapsto \begin{pmatrix} 0 & v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k} \\ -v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k} & 0 \end{pmatrix}.$$

For every $v \in \mathbb{R}^4$, let $q(v) = v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k} \in \mathbb{H}$. Then this embedding is simply

$$v \mapsto \begin{pmatrix} 0 & q(v) \\ -\overline{q(v)} & 0 \end{pmatrix}.$$

In particular, products of pairs of vectors go to

$$uv \mapsto \begin{pmatrix} -q(u)\overline{q(v)} & \\ 0 & -\overline{q(u)}q(v) \end{pmatrix}.$$

Thus the even and odd components are the diagonal and antidiagonal matrices:

$$\mathrm{Cl}(4)_0 = \left\{ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} : q_1, q_2 \in \mathbb{H} \right\},$$
$$\mathrm{Cl}(4)_1 = \left\{ \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix} : q_1, q_2 \in \mathbb{H} \right\}.$$

The spin group Spin(4) is the subgroup of $\mathrm{Cl}(4)_0$ which is generated by vectors of unit norm. If $v \in \mathbb{R}^4$ has unit norm, then the corresponding quaternion $q(v)$ also has unit norm, and the product of such elements is again a unit quaternion. Therefore, identifying the unit quaternions[8] with SU(2), we have

$$\mathrm{Spin}(4) \subseteq \left\{ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} : q_1, q_2 \in \mathbb{H} \text{ and } \|q_1\| = \|q_2\| = 1 \right\} \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

[8] The identification is given by $\mathbb{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{i} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\mathbb{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbb{k} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

In fact, this is an equality. This can be seen by explicitly computing the even products of the basis vectors [see Nab05, equation 7.33] and seeing that any pair of unit quaternions can be obtained from an even product of unit vectors[9]. Therefore

$$\mathrm{Spin}(4) = \left\{ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} : q_1, q_2 \in \mathbb{H} \text{ and } \|q_1\| = \|q_2\| = 1 \right\} \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

2.2.3 Lie algebra structures

For any finite-dimensional real or complex vector space V and symmetric bilinear form g on V, the Clifford algebra $\mathrm{Cl}(V, g)$ has a natural smooth structure, arising from its vector space structure. The group of units $\mathrm{Cl}(V)^\times$ is an open subset of $\mathrm{Cl}(V)$, so it is an open smooth submanifold of $\mathrm{Cl}(V)$. Furthermore, the multiplication map $m : \mathrm{Cl}(V) \times \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is bilinear, which automatically grants it smoothness in the finite-dimensional case[10]. The inversion map $\mathrm{inv} : a \mapsto a^{-1}$ is also smooth, and its derivative is $T_a \mathrm{inv}(b) = -a^{-1}ba^{-1}$ but proving that requires a bit more work [see Mur90, Theorem 1.2.3]. With these two remarks we conclude that $\mathrm{Cl}(V)^\times$ is a Lie group.

We can easily find what the Lie algebra of $\mathrm{Cl}(V)^\times$ is. For all $a \in \mathrm{Cl}(V)$, there is a small enough $\epsilon > 0$ such that for all $t \in (-\epsilon, \epsilon)$, the curve $t \mapsto 1 + ta$ lies entirely in $\mathrm{Cl}(V)^\times$ [Mur90, Theorem 1.2.2]. Then the curve $t \mapsto 1 + ta$ is an integral curve of a, if we interpret a as a tangent vector at the identity 1. Therefore, the Lie bracket is[11]

$$[a, b] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (1 + ta)b(1 + ta)^{-1} = ab - ba.$$

We have shown, then:

> **Proposition 2.2.4 (Group of units is a Lie group).**
> *Let V be a real or complex finite-dimensional Riemannian vector space. Then the multiplicative group of units $\mathrm{Cl}(V)^\times$ is a Lie group of dimension $2^{\dim(V)}$, and its Lie algebra is $\mathfrak{cl}(V) = (\mathrm{Cl}(V), [\cdot, \cdot])$, with the commutator as the Lie bracket.*
>
> *Consequently, $\mathrm{Spin}(V)$ is a Lie group as well.*

Let's focus now on $\mathrm{Spin}(n)$ and its Lie algebra. If $n \geq 2$ then Theorem 2.2.3 states that the spinor map $\mathfrak{s} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a non-trivial double cover. This means that $\mathfrak{s}$ is a local diffeomorphism, and so

$$\dim \mathrm{Spin}(n) = \dim \mathrm{SO}(n) = \frac{(n)(n+1)}{2}.$$

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. Consider the curves $\mathbb{R} \to \mathrm{Spin}(n)$ given by

$$t \mapsto (\cos(t)e_i + \sin(t)e_j)(-\cos(t)e_i + \sin(t)e_j) = \cos(2t) + \sin(2t)e_ie_j$$

for $i < j$. These are all curves contained in $\mathrm{Spin}(n)$, passing through 1 at $t = 0$, with tangent vectors $2e_ie_j$. These tangent vectors are elements of the Lie algebra $\mathfrak{spin}(n) = T_1\mathrm{Spin}(n)$. They are linearly independent, and there is in total $(n)(n+1)/2$ of them, so they generate $\mathfrak{spin}(n)$.

Let's collect this in a proposition:

---

[9] This straightforward but not fun.

[10] In general case, for any real or complex algebra A, smoothness of the multiplication map $A \times A \to A$ is guaranteed if A is a **Banach algebra**: That is, A is a Banach space and the multiplication satisfies

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b \in A$.

[11] In fact, this is true of any Banach algebra.

Proposition 2.2.5 (Lie algebra $\mathfrak{spin}(n)$).
For all $n \geq 2$ the Lie group $\mathrm{Spin}(n)$ *has dimension* $(n)(n+1)/2$. *If* $e_1, \dots, e_n$ *is an orthonormal basis of* $\mathbb{R}^n$, *then a basis of* $\mathfrak{spin}(r,s)$ *is given by elements of the form*

$$e_i e_j$$

with $i < j$.

The spinor map $\mathfrak{s} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a Lie group homomorphism, and so its derivative $T_1\mathfrak{s}$: $\mathfrak{spin}(n) \to \mathfrak{so}(n)$ is a Lie algebra morphism. We can see what its explicit action is on the basis that we found in the previous Proposition 2.2.5. We interpret $\mathfrak{so}(n)$ as a subset of $\mathrm{End}(\mathbb{R}^n)$. Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, which induces a basis of $\mathfrak{spin}(n)$ with elements $e_i e_j$ with $i < j$. An integral curve of such elements is given by $\gamma : \mathbb{R} \to \mathrm{Spin}(n)$, with $\gamma(t) = \cos(t) + \sin(t) e_i e_j$. A straightforward computation shows that $\gamma(t)^{-1} = \gamma(-t)$ for all $t \in \mathbb{R}$.

Then for all $v \in \mathbb{R}^n$

$$\begin{aligned} T_1\mathfrak{s}(e_i e_j)(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathfrak{s}(\gamma(t))(v) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) v \gamma(-t) \\ &= e_i e_j v - v e_i e_j \\ &= e_i(-v e_j - 2\langle e_j, v\rangle) - (-e_i v - 2\langle e_i, v\rangle) e_j \\ &= 2(\langle e_i, v\rangle e_j - \langle e_j, v\rangle e_i) \\ &= 2E_{ij}(v). \end{aligned}$$

Here, E_{ij} is represented by the anti-symmetric matrix with a 1 in the i, j entry, -1 in the j, i entry, and zeros everywhere else. These matrices generate $\mathfrak{so}(n)$.

Notation. We denote the induced infinitesimal action of $\mathfrak{spin}(n)$ as

$$\mathfrak{s} := T_1\mathfrak{s} : \mathfrak{spin}(n) \to \mathfrak{so}(n).$$

2.3 Representations

Let's go back for a moment to the Dirac equation (Equation (2.2)). In it, we have a collection of objects γ^μ which satisfy the anti-commutation relations of the Clifford algebra $\mathrm{Cl}(1,3)$, which act on the vector space where the wavefunction ψ lives. That is, they form a representation of $\mathrm{Cl}(1,3)$.

We will only focus on the *complex* representations of the Clifford algebras, and as we will see below, this implies that the signature of the bilinear form on the vector space becomes irrelevant.

2.3.1 Classification and examples

Definition 2.3.1 (Representation of an (associative) algebra).
*Let A be a unital, associative algebra over a field $\mathbb{K}$, and $\overline{\mathbb{K}} \supseteq \mathbb{K}$ an extension. A $\overline{\mathbb{K}}$-**representation** of A is a $\mathbb{K}$-algebra morphism $\rho : A \to \mathrm{End}_{\overline{\mathbb{K}}}(V)$, where V is a $\overline{\mathbb{K}}$-vector space.*

*A subspace $W \subseteq V$ is and **invariant subspace** of the representation ρ if for all $a \in A$,*

$$\rho(a)(W) \subseteq W.$$

> *If ρ admits a nontrivial invariant subspace, we say that it is* **reducible***. Otherwise, we say it is* **irreducible**.

This is the general definition[12] but we will only consider *complex* representations of the real Clifford algebras and their complexifications. That is, for our purposes $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\overline{\mathbb{K}} = \mathbb{C}$. Furthermore, we are interested in the *irreducible* representations of the Clifford algebras.

> **Definition 2.3.2 (Equivalence of representations).**
> *Let $\rho : A \to \mathrm{End}(V)$ and $\rho' : A \to \mathrm{End}(V')$ be two representations of the associative unital algebra A. We say that the two representations are* **equivalent** *if there exists an isomorphism $\varphi : V \to V'$ such that for all $a \in A$:*
> $$\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}.$$
> *We say that φ is an* intertwining operator *and that it* intertwines *ρ and ρ'.*

> **Note.** From now on, all representations are complex, unless otherwise stated. You'll thank me after you read the next paragraph.

Note that since we are considering *complex* representations, it doesn't really matter whether we are representing a real algebra or its complexification. Namely, if A is a unital, associative *real* algebra and $\rho : A \to \mathrm{End}_{\mathbb{C}}(V)$ is a representation, then we can extend ρ to the complexification $A_{\mathbb{C}}$ simply by requiring ρ to be complex-linear:
$$\rho(a + ib) := \rho(a) + i\rho(b).$$
The right-hand side makes sense since $\mathrm{End}_{\mathbb{C}}(V)$ is a complex algebra. Conversely, if we have a representation of the complexification $A_{\mathbb{C}}$, i.e. $\rho : A_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(V)$, then by restricting to the real form $A \subseteq A_{\mathbb{C}}$, we obtain a representation of A. These two operations (extending to the complexification and restricting to the real form) are inverses of one another. Therefore, the representations of a real algebra are in *bijection* with representations of its complexification.

Furthermore, if $\rho : A_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(V)$ is irreducible, then the restriction $\rho|_A : A \to \mathrm{End}_{\mathbb{C}}(V)$ is irreducible as well. We can see this as follows: suppose that $W \subseteq V$ is an invariant subspace of $\rho|_A$. Then $\rho(a)(W) \subseteq W$ for all $a \in A$. However, W is a *complex* vector space so that
$$(\rho(a) + i\rho(b))(W) \subseteq W$$
as well for all $a, b \in A$. Therefore W is an invariant subspace of ρ, which implies that it is either zero or V.

All in all, this means that if we want to find the irreducible representations of an algebra A, it suffices to find the irreducible representations of $A_{\mathbb{C}}$. Fortunately for us, we have a neat classification of the complexified Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(n)$, as we found in Theorem 2.1.8:
$$\mathrm{Cl}_{\mathbb{C}}(2n) \cong \mathbb{C}(2^n)$$

[12] In a few references, there is little or no mention of representations of associative algebras, and instead everything is discussed in terms of modules. This is because every representation $\rho : A \to \mathrm{End}(V)$ turns V into an A-module, with "scalar" multiplication defined by
$$av := \rho(a)(v).$$
An invariant subspace $W \subseteq V$ then corresponds to an A-*submodule* of V. Therefore, an irreducible representation induces an A-module which has no non-trivial submodules. This is known as a **simple** module.

$$\mathrm{Cl}_{\mathbb{C}}(2n+1) \cong \mathbb{C}(2^n) \oplus \mathbb{C}(2^n).$$

These are all (sums) of matrix algebras, and their irreducible representations are determined by the wise mages of algebra:

Theorem 2.3.3 (Irreducible representations of matrix algebras).
Let $\mathbb{K}$ be a field, and let A be a direct sum of matrix algebras over $\mathbb{K}$:

$$A = \bigoplus_{i=1}^{r} \mathbb{K}(n_i).$$

Then the only irreducible $\mathbb{K}$-representations of A are projections onto the components $\mathbb{K}(n_i)$:

$$\rho_i : A \to \mathbb{K}(n_i) \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}^{n_i}),$$

for $i = 1, \dots, r$.

For a reasonably elementary proof, see [Eti+11, Theorem 2.6]. As a corollary of this, we have all the irreducible representations of the complexified (and therefore of the real!) Clifford algebras:

Corollary 2.3.4 (Irreducible representations of Clifford algebras).
For all $n \in \mathbb{N}$, the Clifford algebra $\mathrm{Cl}(2n)$ has a unique irreducible representation, which is 2^n dimensional. In the odd case, $\mathrm{Cl}(2n+1)$ has exactly two inequivalent, irreducible representations, both of dimensions 2^n.

2.3.2 The Spin representation

We want to restrict these representations to obtain representations of the Spin groups. In the case of even dimension, there is only one irreducible representation to begin with. However, in the odd case where there are two inequivalent representations, both restrict to equivalent representations of Spin. Since $\mathrm{Spin}(n) \subseteq \mathrm{Cl}(n)_0$, it suffices to see what happens to irreducible representations when we restrict to the even subalgebras.

Lemma 2.3.5 (Restriction of irreducible representations).
Let $n \geq 0$ be **even** *and let ρ be the unique irreducible representation of $\mathrm{Cl}(n)$. When restricted to the even subalgebra $\mathrm{Cl}(n)_0$, the representation is* reducible. *It can be decomposed into two summands $\rho = \rho_+ \oplus \rho_-$.*

If $n \geq 1$ is **odd***, then both irreducible representations of $\mathrm{Cl}(n)$ become equivalent when restricted to $\mathrm{Cl}(n)_0$, and the resulting representation is still irreducible.*

The proof of this result is a bit laborious (although not too difficult). It can be found in [LM89, section I.5]. Even though we worked with $\mathrm{Cl}_{\mathbb{C}}(n)$, these results hold in general for *complex* representations of all $\mathrm{Cl}(r,s)$. This is because there is a bijection between representations of an algebra and representations of its complexification (see the discussion before Theorem 2.3.3), and because complexifying destroys definiteness, so $\mathrm{Cl}_{\mathbb{C}}(r,s) = \mathrm{Cl}_{\mathbb{C}}(r+s)$ (see the remark after Proposition 2.1.7).

We define the (complex) spin representation as follows:

Definition 2.3.6 (The Spin representation).
Let V be a real, finite-dimensional vector space and g a semi-Riemannian metric on V. The (complex) **spin representation** *is the unique representation $\Delta : \mathrm{Spin}(V) \to \mathrm{End}(S)$ that is obtained by restricting an irreducible representation of $\mathrm{Cl}(V)$ to $\mathrm{Spin}(V)$. If $\dim(V)$ is even, then Δ is reducible, and can be decomposed into two summands $\Delta = \Delta_+ \oplus \Delta_-$. If $\dim(V)$ is odd, Δ is irreducible. We call an element of S a* **spinor**.

Example 2.3.7 (The spin representation: $n = 4$).
In Section 2.2.2, we saw that $\mathrm{Cl}(4)$ is isomorphic to $\mathbb{H}(2)$, and the even subalgebra is

$$\mathrm{Cl}(4)_0 = \left\{ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in \mathbb{H}(2) : q_1, q_2 \in \mathbb{H} \right\}.$$

To complexify $\mathrm{Cl}(4)$, we see each of the four entries of a matrix in $\mathbb{H}(4)$ as a complex 2×2 matrix via the standard representation. All complex-linear combinations of these matrices generate the entire matrix algebra $\mathbb{C}(4)$, so that (as we expect from Theorem 2.1.8), $\mathrm{Cl}_{\mathbb{C}}(4) \cong \mathbb{C}(4)$. Therefore, the only irreducible representation is the identity. When we restrict to $\mathrm{Cl}(4)_0$, the representation is simply interpreting each element of $\mathbb{H}$ as a 2×2 complex matrix:

$$\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in \mathbb{H}(2) \mapsto \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in \mathbb{C}(4).$$

This representation is clearly reducible.

Particularly, the spin representation $\Delta : \mathrm{Spin}(4) \to \mathrm{GL}(4, \mathbb{C})$ splits as $\Delta_\pm : \mathrm{Spin}(4) \to \mathrm{SU}(2)$ given by projections onto the first and second entries of the diagonal:

$$\Delta_+ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = q_1$$
$$\Delta_- \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = q_2.$$

2.3.3 Clifford multiplication and infinitesimal actions

Let V be a Riemannian vector space, and let $\rho : \mathrm{Cl}(V) \to \mathrm{End}(S)$ be a representation of the Clifford algebra. The representation induces a structure of a $\mathrm{Cl}(V)$-module on S, with multiplication given by

$$a \cdot v := \rho(a)(v)$$

for all $a \in A$ and $v \in S$. We call this multiplication **Clifford multiplication**.

If we restrict ρ to the group of units $\mathrm{Cl}(V)^\times$, we obtain a *group* representation

$$\rho : \mathrm{Cl}(V)^\times \to \mathrm{GL}(S).$$

Thus, its differential is a representation the Lie algebra $\mathfrak{cl}(V)$ on S. However, since ρ is linear, then $T_1(\rho) = \rho$, so the induced representation is again Clifford multiplication. Note that we have identified the Lie algebra $\mathfrak{cl}(V)$ with $\mathrm{Cl}(V)$, as in Section 2.2.3.

Similarly, restricting ρ to $\mathrm{Spin}(V)$ gives us a representation whose differential is a representation of the Lie algebra $\mathfrak{spin}(V)$. Once again, this is just Clifford multiplication. We can see this explicitly: Let

$e_1, \dots, e_n$ be an orthonormal basis of V. Then, as we saw in Proposition 2.2.5, the elements $e_i e_j \in \mathrm{Cl}(n)$ with $i < j$ are a basis of $\mathfrak{spin}(n)$, and the curve

$$\gamma(t) = \cos(t) + \sin(t) e_i e_j$$

is an integral curve of $e_i e_j \in \mathfrak{spin}(r, s)$. The action of the differential $T_1 \rho$ is, then

$$T_1\rho(e_i e_j)(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \cos(t)v + \sin(t)\rho(e_i e_j)(v) = \rho(e_i e_j)(v)$$

so indeed $T_1(\rho|_{\mathrm{Spin}(n)}) = \rho|_{\mathfrak{spin}(n)}$.

2.4 Spin structures

Think of what we have done so far as a local model that we want to push up to a global structure on a manifold. The basic ingredient in this global recipe will be a $\mathrm{SO}(n)$-structure, which is to say that our manifold should have a Riemannian metric and be orientable. Then we want to lift the $\mathrm{SO}(n)$-structure to a $\mathrm{Spin}(n)$ structure via a suitable globalization of the adjoint map $\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. Once we have the $\mathrm{Spin}(n)$-structure, then we can construct the spinor bundle, which is the associated vector bundle to the spin representation.

This gives us half of the components of the Dirac equation: the spinor fields, which are sections of the spinor bundle. The second half is the Dirac operator, which is a suitable "globalization" of the operator $i\gamma^\mu \partial_\mu$ that we saw in the introduction.

2.4.1 From local to global

In the previous sections, we constructed the Clifford algebra and Spin group of a Riemannian vector space V. The global analog of V is an orientable vector bundle $E \to M$ of rank n over a smooth manifold M, with a bundle metric on E. Since E has a metric and is orientable, its frame bundle reduces to an orthonormal frame bundle, which is a principal $\mathrm{SO}(n)$ bundle and is denoted by $\mathrm{SO}(E)$ (in the case where $E = TM$ is the tangent bundle, we simply write $\mathrm{SO}(M) := \mathrm{SO}(TM)$).

Let's begin with the Clifford algebra. The fibers E_x of the bundle E are Riemannian vector spaces, so it makes sense to take the Clifford algebra $\mathrm{Cl}(E_x)$ for each $x \in M$. A choice of a frame $e = (e_1, \dots, e_n)$ of E in an open $U \subseteq M$ is equivalent to choosing fiberwise isometries $i_e(x) : E_x \to \mathbb{R}^n$ for all $x \in U$. These isometries, by Corollary 2.1.4, extend to unique isomorphisms $\overline{i_e}(x) : \mathrm{Cl}(E_x) \to \mathrm{Cl}(n)$, which in turn can be interpreted as choices of bases of $\mathrm{Cl}(E_x)$. If we change frames to some other $e' = (e'_1, \dots, e'_n)$, in another open V, then the frames are related by a transition function $g : U \cap V \to \mathrm{SO}(n)$, such that $i_{e'}(x) = g(x) \circ i_e(x)$ for all $x \in U \cap V$. Again, by Corollary 2.1.4, each $g(x)$ extends to an automorphism $c\ell_{g(x)} : \mathrm{Cl}(n) \to \mathrm{Cl}(n)$, such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathbb{R}^n & \hookrightarrow & \mathrm{Cl}(n) \\ & \overset{i_{e(x)}}{\nearrow} & & & \\ E_x & & \Big\downarrow g(x) & & \Big\downarrow c\ell_{g(x)} \\ & \underset{i_{e'(x)}}{\searrow} & & & \\ & & \mathbb{R}^n & \hookrightarrow & \mathrm{Cl}(n) \end{array}.$$

Therefore, a transition function $g : U \cap V \to \mathrm{SO}(n)$ of the orthonormal frame bundle $\mathrm{SO}(E)$ induces a change of frames $c\ell_g : U \cap V \to \mathrm{Aut}(\mathrm{Cl}(n))$ of the collection of Clifford algebras in the fibers. This tells us that we can glue together those fibers with transition maps $c\ell_g$, to obtain a vector bundle. In particular, note that the map $c\ell : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}(n))$, sending $g \mapsto c\ell_g$, is a representation of $\mathrm{SO}(n)$ on $\mathrm{Cl}(n)$. This tells us that the resulting bundle is precisely the associated bundle to $\mathrm{SO}(E)$ via the representation $c\ell$.

Definition 2.4.1 (Clifford bundle).
Let $E \to M$ be an oriented Riemannian vector bundle of rank n and $\mathrm{SO}(E)$ its orthonormal frame bundle. The **Clifford bundle** *is the vector bundle*

$$\mathrm{Cl}(E) = \mathrm{SO}(E) \times_{c\ell} \mathrm{Cl}(n)$$

associated to $\mathrm{SO}(E)$ via the representation $c\ell : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}(n))$ which is obtained by extending each element of $\mathrm{SO}(n)$ to an automorphism of $\mathrm{Cl}(n)$.

In the case where E is the tangent bundle TM, we write $\mathrm{Cl}(M) := \mathrm{Cl}(TM)$.

Note that, by construction, the fibers of $\mathrm{Cl}(E)$ satisfy

$$\mathrm{Cl}(E)_x \cong \mathrm{Cl}(E_x)$$

for all $x \in M$. This means that there is a well-defined inclusion $E \hookrightarrow \mathrm{Cl}(E)$. However, in order to make this inclusion explicit, we need the isomorphism $\mathrm{Cl}(E)_x \cong \mathrm{Cl}(E_x)$, which requires a choice of frame for E_x. Of course, everything is *independent* of this frame, but it is necessary.

Spin structures are required for the existence of spinor fields, such as those that appear in the Dirac equation. They are bundles which are fiberwise lifts of the $\mathrm{SO}(n)$ structure that comes with the Riemannian vector bundle E.

Definition 2.4.2 (Spin structure).
A **spin structure** *on E is a principal $\mathrm{Spin}(n)$ bundle $\mathrm{Spin}(E) \to M$, and a bundle morphism $\Sigma : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ which is fiberwise the spinor map $\mathfrak{s} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. That is, for all $p \in \mathrm{Spin}(E)$ and $g \in \mathrm{Spin}(n)$,*

$$\Sigma(p \cdot g) = \Sigma(p) \cdot \mathfrak{s}(g).$$

Let's consider the question of the *existence* of Spin structures. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of M that trivializes $\mathrm{SO}(E)$, and let $\{g_{ij} : U_i \cap U_j \to \mathrm{SO}(n)\}_{i,j \in I}$ be the transition cocycle. We want to lift the cocyles g_{ij} to cocycles $\{\tilde{g}_{ij} : U_i \cap U_j \to \mathrm{Spin}(n)\}$, following the condition

$$\mathfrak{s} \circ \tilde{g}_{ij} = g_{ij}.$$

On triple overlaps $U_i \cap U_j \cap U_k$, the $\mathrm{SO}(E)$ cocycles satisfy

$$1 = g_{ij} g_{jk} g_{kl} = \mathfrak{s}\left(\tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{kl}\right).$$

This implies that

$$\tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{kl} = \pm 1,$$

which is *a priori* not enough to determine cocyles of a Spin-structure. If we write $w_{ijk} = \tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{kj}$, the collection $\mathscr{w} = \{w_{ijk}\}$ determines a Čech 2-cochain $\mathscr{w} \in \check{C}^2(\mathcal{U}, \mathbb{Z}_2)$. Using the fact that $w_{ijk} \in \mathbb{Z}_2$, so $w_{ijk} = w_{ijk}^{-1} = \omega_{ikj}$ it is a direct (but a bit tedious) computation to show that

$$(\delta \mathscr{w})_{ijkl} = w_{jkl} w_{ikl} w_{ijl} w_{ijk} = 1.$$

Assuming that $\mathcal{U}$ is a good cover, w defines a Čech cohomology class, which we call the *second* Stiefel-Whitney class:

$$w_2(E) := [w] \in \check{H}^2(M, \mathbb{Z}_2).$$

This class is independent of the choice of lifts $\tilde{g}_{ij}$ of g_{ij}. Any two such lifts $\tilde{g}_{ij}$ and $\tilde{g}'_{ij}$ are related by

$$\tilde{g}'_{ij} = \eta_{ij}\tilde{g}_{ij},$$

with $\eta_{ij} \in \mathbb{Z}_2$. The collection of these η_{ij} determines a cochain $\eta \in \check{C}^1(\mathcal{U}, \mathbb{Z}_2)$, and the cocycles $w'_{ijk} = \tilde{g}'_{ij}\tilde{g}'_{jk}\tilde{g}'_{ki}$ satisfy

$$w'_{ijk} = w_{ijk}\eta_{ij}\eta_{jk}\eta_{ki} = w_{ijk}(\delta\eta)_{ijk}.$$

Therefore w and w' differ by a closed Čech 2-cocycle, and thus they determine the same cohomology class. From all these considerations, we find:

Proposition 2.4.3 (Conditions for existence of Spin structure).
Let $E \to M$ be an oriented, Riemannian vector bundle. Then E admits a Spin structure if and only if $w_2(E) = 0$.

2.4.2 Spinor bundles

In the physics literature, as spinor field is often defined as an "object" ψ (whatever that may be) which, under a Lorentz transformation[13] Λ of the underlying spacetime, transforms as

$$\psi \mapsto \psi' = S(\Lambda)^{-1}(\psi \circ \Lambda),$$

where S is a representation of $\mathrm{Spin}(1,3)$ (equivalently, of $\mathrm{Spin}(4)$). In the general case, the Lorentz group acts on the *frame bundle*, so we should think of ψ as a map on the orthonormal frame bundle, which is Lorentz-equivariant, thus a section of the bundle associated to the frame bundle via S.

Definition 2.4.4 (Spinor bundles and the spinor bundle).
Let $E \to M$ be an orientable Riemannian vector bundle with a spin structure $\mathrm{Spin}(E)$, and $\rho : \mathrm{Cl}(n) \to \mathrm{End}(S)$ a representation of the Clifford algebra. The **spinor bundle** *$S(E, \rho)$ is the associated bundle*

$$S(E, \rho) := \mathrm{Spin}(E) \times_\rho S,$$

where we see ρ as a representation of $\mathrm{Spin}(n)$ on S. If we make no explicit reference to the representation ρ, we talk of **the** *spinor bundle $S(E)$, which is associated to $\mathrm{Spin}(E)$ via the spin representation $\Delta : \mathrm{Spin}(n) \to \mathrm{GL}(S)$:*

$$S(E) := S(E, \Delta) = \mathrm{Spin}(E) \times_\Delta S.$$

A section of a spinor bundle is called a **spinor field**.

Since we have a representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}(S)$, we can see S as a $\mathrm{Cl}(n)$-module. The multiplication of elements of $\mathrm{Cl}(n)$ with elements of S is called **Clifford multiplication**. This operation $m : \mathrm{Cl}(n) \otimes S \to S$ exists at the level of linear algebra. Can we promote it to a global operation on $\mathrm{Cl}(E) \otimes S(E, \rho)$? Let's try to brute force it.

[13] The *Lorentz* group is the group $\mathrm{SO}(1,3)$ of linear transformations that preserve the Minkowski metric, i.e. the standard semi-Riemannian metric of signature (1,3).

Consider an open $U \subseteq M$ which trivializes SO(E), with a canonical section $s : U \to \mathrm{SO}(E)_U$, and choose a lift $\tilde{s} : U \to \mathrm{Spin}(E)_U$ (which exists by definition of the existence of a spin structure), such that $\Sigma \circ \tilde{s} = s$. For each $x \in U$, consider the elements in the fibers $p = s(x) \in \mathrm{SO}(E)_x$ and $\tilde{p} = \tilde{s}(s) \in \mathrm{Spin}(E)_x$. Fixing these elements induce isomorphisms

$$\begin{aligned} \mathrm{Cl}(E)_x &\xrightarrow{\sim} \mathrm{Cl}(n) \\ [p, a] &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} S(E, \rho)_x &\xrightarrow{\sim} S \\ [\tilde{p}, v] &\mapsto v, \end{aligned}$$

which trivialize the bundles Cl(E) and $S(E, \rho)$ over U.

Now suppose that we want to define, fiberwise, a Clifford multiplication map $m : \mathrm{Cl}(E)_x \otimes S(E, \rho)_x \to S(E, \rho)_x$. Naïvely, we can fix the trivialization above and let

$$m(a \otimes v) := \rho(a)(v).$$

This is equivalent to naïvely defining

$$[p, a] \cdot [\tilde{p}, v] := [\tilde{p}, \rho(a)(v)].$$

Of course, this definition should be robust under changes of trivialization. If $s' : U \to \mathrm{SO}(E)_U$ is another trivialization, then there is a transition map $g : U \to SO(n)$ such that $s' = s \cdot g$. This map can be lifted to $\tilde{g} : U \to \mathrm{Spin}(n)$, such that $\mathfrak{z}(\tilde{g}) = g$ and we obtain another trivialization $\tilde{s} \cdot \tilde{g}$ of $\mathrm{Spin}(E)_U$. On the fibers above x, we change the point p to $p \cdot g \in \mathrm{SO}(E)_x$ and $\tilde{p}$ to $\tilde{p} \cdot \tilde{g} \in \mathrm{Spin}(E)_x$ so that the frames in Cl(n) and S change as

$$a \mapsto \mathcal{c}\ell_g(a); \quad v \mapsto \rho(\tilde{g})(v);$$

$$a \otimes v \mapsto \mathcal{c}\ell_g(a) \otimes \rho(\tilde{g})(v)$$

for all $a \in \mathrm{Cl}(n)$ and $v \in S$. However, the resulting product $\rho(a)(v)$ is still in S, so it changes as

$$\rho(a)(v) \mapsto \rho(\tilde{g})\rho(a)(v) = \rho(\tilde{g}a)(v).$$

Therefore, if we want m to be robust under changes of frames, it has to satisfy

$$m(\mathcal{c}\ell_g(a) \otimes \rho(\tilde{g})(v)) = \rho(\tilde{g}a)(v),$$

or equivalently,

$$[p, \mathcal{c}\ell_g(a)] \cdot [\tilde{p}, \rho(\tilde{g})(v)] = [\tilde{p}, \rho(\tilde{g}a)(v)]$$

that is,

$$\rho(\mathcal{c}\ell_g(a)\tilde{g}) = \rho(\tilde{g}a)$$

for all $a \in \mathrm{Cl}(n)$. Now we recall that $\mathcal{c}\ell_g$ acts on homogeneous elements of Cl(n) as

$$\mathcal{c}\ell_g(v_1 \cdots v_k) = g(v_1) \cdots g(v_k).$$

This, combined with the fact that $g = \mathfrak{z}(\tilde{g})$, implies that

$$\mathcal{c}\ell_{\mathfrak{z}(\tilde{g})}(a) = \tilde{g}a\tilde{g}^{-1} = \mathfrak{z}_{\tilde{g}}(a)$$

for all $a \in \mathrm{Cl}(n)$ and $\tilde{g} \in \mathrm{Spin}(n)$. This is independent of the choice of preimage of g under $\mathfrak{z} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, since they differ by a sign, which disappears in conjugation. Therefore we have

$$\mathcal{c}\ell_g(a)\tilde{g} = \tilde{g}a\tilde{g}^{-1}\tilde{g} = \tilde{g}a,$$

which means that the fiberwise multiplication map $m : \mathrm{Cl}(E)_x \otimes S(E,\rho)_x \to S(E,\rho)_x$ is independent of the choice of trivialization, so indeed it can be extended to a bundle map

$$m : \mathrm{Cl}(E) \otimes S(E,\rho) \to S(E,\rho),$$

which we call **Clifford multiplication**. The above discussion shows the following proposition:

Proposition 2.4.5 (Clifford multiplication).
Let E be an oriented, Riemannian vector bundle, which admits a spin structure $\mathrm{Spin}(E)$, $\mathrm{Cl}(E)$ *its Clifford bundle, and* $S(E,\rho)$ *the spinor bundle associated to a representation* $\rho : \mathrm{Cl}(n) \to \mathrm{End}(S)$. *Then Clifford multiplication* $m : \mathrm{Cl}(E) \otimes S(E,\rho) \to S(E,\rho)$ *makes* $S(E,\rho)$ *into a bundle of* $\mathrm{Cl}(n)$*-modules.*

In particular, the sections of $S(E,\rho)$ *are a module over the sections of* $\mathrm{Cl}(E)$.

2.5 The Dirac operator

We have spinors and a Clifford module structure on spinor fields. The only thing we need for the Dirac equation is the Dirac operator, for which we need a way to take derivatives of spinor fields. That is, we need a connection on the spinor bundles. Our fundamental structure is the orthonormal frame bundle, so we will begin with a connection on it and work our way to defining a connection on the spinor bundles with it.

2.5.1 Spin connections

Let E be an oriented, Riemannian vector bundle over M which admits a spin structure $\mathrm{Spin}(E)$. Let ∇ be a metric connection on E, which induces an Ehresmann connection $\omega^\nabla \in \Omega^1(\mathrm{SO}(E), \mathfrak{so}(n))$ on $\mathrm{SO}(E)$. If we pull back ω^∇ to $\mathrm{Spin}(E)$ via the morphism $\Sigma : \mathrm{Spin}(E) \to \mathrm{SO}(E)$, we obtain an $\mathfrak{so}(n)$-valued one-form $\Sigma^*\omega^\nabla \in \Omega^1(\mathrm{Spin}(E), \mathfrak{so}(n))$. Finally, composing this with the inverse of the Lie algebra isomorphism $\mathfrak{s} : \mathfrak{spin}(n) \to \mathfrak{so}(n)$, induced by the spinor map $\jmath : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, we obtain an Ehresmann connection $\tilde{\omega}$ on $\mathrm{Spin}(E)$:

$$\tilde{\omega}^\nabla := \mathfrak{s}^{-1} \circ (\Sigma^*\omega^\nabla) \in \Omega^1(\mathrm{Spin}(E), \mathfrak{spin}(n)).$$

Explicitly, for a point $\tilde{p} \in \mathrm{Spin}(E)$ and a vector $\tilde{X} \in T_{\tilde{p}}\mathrm{Spin}(E)$,

$$\tilde{\omega}^\nabla_{\tilde{p}}(\tilde{X}) = (\mathfrak{s}^{-1} \circ \omega^\nabla_{\Sigma(\tilde{p})})(T_{\tilde{p}}\Sigma(\tilde{X})).$$

Finally, with the connection $\tilde{\omega}^\nabla$, we can define a connection on any associated vector bundle of $\mathrm{Spin}(E)$. In particular, for any representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}(S)$, we have the associated spinor bundle $S(E,\rho)$ and thus a connection $\tilde{\nabla}$ on it, which we call the **spin connection**. Schematically:

$$\begin{array}{ccc} \tilde{\omega}^\nabla \text{ on } \mathrm{Spin}(E) & \xleftarrow{\mathfrak{s}^{-1}\circ\Sigma^*} & \omega^\nabla \text{ on } \mathrm{SO}(E) \\ \big\downarrow & & \big\uparrow \\ \tilde{\nabla} \text{ on } S(E,\rho) & & \nabla \text{ on } E \end{array}.$$

Let's see how $\tilde{\nabla}$ acts on a section of $S(E,\rho)$. Note that $\rho : \mathrm{Spin}(n) \to \mathrm{GL}(S)$ induces a representation $\rho_* : \mathfrak{spin}(n) \to \mathfrak{gl}(S)$.

Notation. To avoid cluttering of notation, we write the infinitesimal representation ρ_* as an action of $\mathfrak{spin}(n)$ on S:

$$a \cdot v := \rho_*(a)(v),$$

for all $a \in \mathfrak{spin}(n)$ and $v \in S$.

Let $\Psi \in \Gamma(S(e,\rho))$ be a section. There is a unique function $\tilde{\psi} : P \to V$ such that for all $x \in M$,

$$\Psi(x) = [p, \psi(p)],$$

where $p \in \mathrm{Spin}(E)_x$ is in the fiber above p. For this to be independent of the choice of representative, $\tilde{\psi}$ must be $\mathrm{Spin}(n)$-equivariant, that is

$$\psi(p \cdot \tilde{g}) = \rho(\tilde{g})^{-1}\psi(p)$$

for all $\tilde{g} \in \mathrm{Spin}(n)$. By definition, the connection $\tilde{\nabla}$ acts on Ψ as

$$\tilde{\nabla}_X \Psi(x) = [p, \mathrm{d}^{\tilde{\omega}}\psi_p(\tilde{X})] = [p, (\mathrm{d}\tilde{\psi})_p(\tilde{X}) + \tilde{\omega}^{\nabla}_p(\tilde{X}) \cdot \tilde{\psi}(p)]$$

for all vectors $X \in T_xM$, where $\tilde{X} \in T_p\mathrm{Spin}(E)$ is a vector which projects to X, i.e. $T_p\pi_{\mathrm{Spin}}(\tilde{X}) = X$.

Let's go local now. Let $e = (e_1, \dots, e_n) : U \to \mathrm{SO}(E)$ be a local orthonormal frame of E, which we interpret as a section of $\mathrm{SO}(E)$. This frame has associated connection coefficients $\omega_{j,i} \in \Omega^1(U)$ such that

$$\nabla e_i = \sum_j \omega_{j,i} e_j.$$

These connection coefficients glue together to form the Ehresmann connection $\omega \in \Omega^1(\mathrm{SO}(E), \mathfrak{so}(n))$, precisely in a way such that the pullback $e^*\omega \in \Omega^1(U, \mathfrak{so}(n))$ is a matrix with entries $\omega_{j,i}$. If we write it in terms of the elementary matrices $E_{i,j}$ which form a basis of $\mathfrak{so}(n)$, we have

$$e^*\omega = \sum_{i<j} \omega_{j,i} E_{i,j}.$$

Let $\tilde{e} : U \to \mathrm{Spin}(E)$ be a lift of e, such that $\Sigma \circ \tilde{e} = e$. The section $\tilde{e}$ induces a trivialization of the spinor bundle by "fixing a gauge" everywhere above U. More specifically, for all x in U, there is a preferred element $\tilde{p} = \tilde{e}(x) \in \mathrm{Spin}(E)_x$ in the fiber above x, so we can write $\Psi(x)$ as

$$\Psi(x) = [\tilde{e}(x), \tilde{\psi}(\tilde{e}(x))] := [\tilde{e}(x), \psi(x)],$$

where we have written $\psi = \tilde{\psi} \circ \tilde{e} : U \to S$.

Now we plug this into the expression of $\tilde{\nabla}\Psi$. For a tangent vector $X \in T_xM$, a lift to $T_p\mathrm{Spin}(E)$ is simply $\tilde{X} = \tilde{e}_*X$. Therefore, if $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is the standard orthonormal basis of $\mathbb{R}^n$, we have

$$\begin{aligned}
\tilde{\nabla}_X \Psi(x) &= [\tilde{e}(x), \mathrm{d}\tilde{\psi}_{\tilde{e}(x)}(e_*X) + \tilde{\omega}^{\nabla}_{\tilde{e}(x)}(\tilde{e}_*X) \cdot \psi(x)] \\
&= [\tilde{e}(x), \tilde{e}^*\mathrm{d}\tilde{\psi}_x(X) + \mathfrak{s}^{-1}(\Sigma^*\omega_{\tilde{e}(x)}(\tilde{e}_*X)) \cdot \psi(x)] \\
&= [\tilde{e}(x), \mathrm{d}\psi_x(X) + \mathfrak{s}^{-1}(e^*\omega_x(X)) \cdot \psi(x)] \\
&= [\tilde{e}(x), \mathrm{d}\psi_x(X) + \sum_{i<j}(\omega_{j,i})_x(X)\mathfrak{s}^{-1}(E_{i,j}) \cdot \psi(x)] \\
&= [\tilde{e}(x), \mathrm{d}\psi_x(X) + \sum_{i<j}(\omega_{j,i})_x(X)\mathfrak{s}^{-1}(E_{i,j}) \cdot \psi(x)] \\
&= [\tilde{e}(x), \mathrm{d}\psi_x(X) + \frac{1}{2}\sum_{i<j}(\omega_{j,i})_x(X)\epsilon_i\epsilon_j \cdot \psi(x)].
\end{aligned}$$

In the last line, the dot is the infinitesimal action of $\mathfrak{spin}(n)$ on S which is induced by the representation ρ. However, as we saw in Section 2.3.3, this representation is just Clifford multiplication.

2.5.2 The Dirac operator

Now we specialize on the case where the vector bundle E is the tangent bundle TM. We therefore assume that M is an oriented Riemannian manifold. Furthermore, suppose that M admits a spin structure $\mathrm{Spin}(M)$ that lifts the orthonormal frame bundle $\mathrm{SO}(M)$.

Fix a representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}(S)$ of the Clifford algebra, and let $S(M,\rho)$ its associated spinor bundle. If ∇ is the Levi-Civita connection, we have an induced spin connection $\tilde{\nabla}$ on $S(M,\rho)$.

We define the **Dirac operator** in analogy to Equation (2.2):

Definition 2.5.1 (Dirac operator).
The Dirac operator $\partial\!\!\!/ : \Gamma(S(M,\rho)) \to \Gamma(S(M,\rho))$ *is defined* locally, *for an orthonormal frame* $e = (e_1, \dots, e_m) : U \to \mathrm{SO}(M)$, *as*

$$\partial\!\!\!/ \Psi = \sum_i e_i \cdot \tilde{\nabla}_{e_j} \Psi,$$

where we interpret e_i *as a section of* $\mathrm{Cl}(M)$ *via the natural embedding* $TM \hookrightarrow \mathrm{Cl}(M)$*, and the dot is Clifford multiplication.*

We need to check that this definition is independent of the choice of frame. Let $e' : U' \to \mathrm{SO}(M)$ be a frame of $\mathrm{SO}(M)$ which is related to e via a transition map $\Lambda : U \cap U' \to \mathrm{SO}(n)$:

$$e_i = \sum_j e'_j \Lambda_{j,i}.$$

Then

$$\sum_i e_i \cdot \tilde{\nabla}_{e_j} \Psi = \sum_i \sum_{j,k} \Lambda_{j,i} \Lambda_{k,i} e'_j \cdot \tilde{\nabla}_{e'_k}.$$

However, since $\Lambda \in \mathrm{SO}(n)$ we have that

$$\sum_i \Lambda_{j,i} \Lambda_{k,i} = (\Lambda \Lambda^T)_{j,k} = \delta_{jk},$$

where δ_{jk} is the Krönecker symbol. Therefore, as desired[14]:

$$\sum_i e_i \cdot \tilde{\nabla}_{e_i} \Psi = \sum_i e'_i \cdot \tilde{\nabla}_{e'_i} \Psi.$$

2.6 The world of Spin^c

As we saw above, the condition for the existence of Spin-structures over a vector bundle is rather strict. In this section, we will introduce the Spin^c groups and Spin^c structures. Roughly speaking, a Spin^c structure looks locally like a Spin structure with an additional $\mathrm{U}(1)$ structure. This additional structure gives it more flexibility, to the point that all 4-manifolds admit Spin^c-structures. In physical terms, a Spin^c-structure is necessary for the existence of spinor fields that are *coupled* to electromagnetic fields.

[14] In Einstein notation, for a semi-Riemannian manifold, we might write the Dirac operator as

$$\partial\!\!\!/ \Psi = g^{\nu\mu} e_\nu \tilde{\nabla}_\mu \Psi = e^\mu \tilde{\nabla}_\mu \Psi,$$

with $e^\mu = g^{\mu\nu} e_\nu$ and $g_{\mu,\nu} = \langle e_\mu, e_\nu \rangle$ are the metric coefficients. It is then automatic that the equation is independent of the chosen frame, since the expression has no free indices, only dummy indices.

2.6.1 The Spin^c group

Let V be a *real* vector space with a symmetric bilinear form.

Definition 2.6.1 (Spin^c group).
The group $\mathrm{Spin}^c(V)$ *is the subgroup of* $\mathrm{Cl}_{\mathbb{C}}(V)^\times$ *given by*

$$\mathrm{Spin}^c(V) := \{z\Lambda \in \mathrm{Cl}_{\mathbb{C}}(V)^\times \,:\, z \in \mathrm{U}(1) \textit{ and } \Lambda \in \mathrm{Spin}(V)\}.$$

If $V = \mathbb{R}^n$ *with the standard inner product, we write*

$$\mathrm{Spin}^c(n) = \mathrm{Spin}^c(\mathbb{R}^n).$$

Note that the map $\mathrm{Spin}(V)\times U(1) \to \mathrm{Spin}^c(V)$ given as $(z,\Lambda) \mapsto z\Lambda$ is surjective, with kernel $\{(1,1),(-1,-1)\}$. Therefore

$$\mathrm{Spin}^c(V) \cong \mathrm{Spin}(V) \times U(1)/_{\mathbb{Z}_2}.$$

One of the key properties of the Spin group is that it is a double cover of the corresponding special orthogonal group. Indeed, we have something similar for $\mathrm{Spin}^c(V)$. Define the **determinant map** $\delta : \mathrm{Spin}^c(V) \to \mathrm{U}(1)$ as

$$\delta(z\Lambda) = z^2.$$

This map is well defined, since it is invariant under the change $(z,\Lambda) \mapsto (-z,-\Lambda)$.

Furthermore, the adjoint action $\mathrm{Ad} : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ extends trivially to $\mathrm{Cl}_{\mathbb{C}}(V)$, simply by noting that for all nonzero $z \in \mathbb{C}$ and $a, b \in \mathrm{Cl}(V)$,

$$\mathrm{Ad}_{za}(b) = (za)b(za)^{-1} = aba^{-1} = \mathrm{Ad}_a(b).$$

Therefore, the spinor map $\mathfrak{s} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ also extends to a map $\mathfrak{s}^c : \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$, simply by setting

$$\mathfrak{s}^c(z\Lambda) = \mathfrak{s}(\Lambda).$$

We call this the **complex spinor map**. Very often we will abuse the notation and simply write $\mathfrak{s}$ for $\mathfrak{s}^c$.

With these two maps, we obtain a short exact sequence similar to that of Spin (see Theorem 2.2.3).

Proposition 2.6.2 (Short exact sequence of Spin^c).
Let V *be a real Riemannian vector space. Then the map* $\mathrm{Spin}^c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1)$ *given by*

$$\xi \mapsto (\mathfrak{s}(\xi), \delta(\xi))$$

for all $\xi \in \mathrm{Spin}^c(V)$*, makes the following short sequence exact:*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(V) \longrightarrow \mathrm{SO}(V) \times \mathrm{U}(1) \longrightarrow 1\ .$$

Finally, note that any (complex) representation ρ of $\mathrm{Spin}(V)$ can be extended to $\mathrm{Spin}^c(V)$ by setting

$$\rho(z\Lambda) = z\rho(\Lambda).$$

Remark. The name "determinant" for the map $\delta : \mathrm{Spin}^c(V) \to \mathrm{U}(1)$ comes from the fact that for $n = 4$, if $\Delta_\pm$ are the irreducible components of the Spin representation, then

$$\delta = \det \circ \Delta_\pm.$$

2.6.2 Going global: Spin^c-structures and spinors

Now we take the previous section as a local model that we are going to promote to a global structure over a manifold, in the same way as in Sections 2.4 and 2.5. The first few definitions are essentially the same as those in Section 2.4.

Let $E \to M$ be an oriented, Riemannian vector bundle of rank n with orthonormal frame bundle $\mathrm{SO}(E)$.

Definition 2.6.3 (Complex Clifford bundle).
The representation $c\ell : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}(n))$ *can also be viewed as a representation* $c\ell : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}_\mathbb{C}(n))$. *Thus, we define* **complex Clifford bundle** $\mathrm{Cl}_\mathbb{C}(E)$ *as the associated bundle*

$$\mathrm{Cl}_\mathbb{C}(E) = \mathrm{SO}(E) \times_{c\ell} \mathrm{Cl}_\mathbb{C}(n).$$

Definition 2.6.4 (Spin^c-structure).
A Spin^c*-structure over E is a principal* $\mathrm{Spin}^c(n)$*-bundle denoted by* $\mathrm{Spin}^c(E)$*, along with a bundle morphism* $\Sigma : \mathrm{Spin}^c(E) \to \mathrm{SO}(E)$ *which is fiberwise the spinor map; that is, for all* $p \in \mathrm{Spin}^c(E)$ *and* $\xi \in \mathrm{Spin}^c(n)$,

$$\Sigma(p \cdot \xi) = \Sigma(p) \cdot \mathfrak{s}(\xi).$$

Whenever there is chance of confusion with the spinor morphism of a Spin*-structure, we will denote this complex spinor morphism as* Σ^c.

Since the determinant map $\delta : \mathrm{Spin}^c(n) \to \mathrm{U}(1)$ is, in particular, a representation, then we have a line bundle associated to $\mathrm{Spin}^c(E)$, which we call the **determinant bundle** $L(E)$. Specifically, it is given by

$$L(E) := \mathrm{Spin}^c(E) \times_\delta \mathrm{U}(1).$$

In the same fashion as in Section 2.4.2, given a representation $\rho : \mathrm{Cl}_\mathbb{C}(n) \to \mathrm{End}(S)$, we obtain a representation of $\mathrm{Spin}^c(n)$, with which we can construct the **spinor bundle** $S^c(E,\rho)$. Specifically,

$$S(E,\rho) = \mathrm{Spin}^c(n) \times_\rho S.$$

Once again, if we do not explicitly write the representation, we assume that we are using the extension to Spin^c of the Spin representation $\Delta : \mathrm{Spin}^c(n) \to \mathrm{End}(S)$:

$$S(E) := S(E,\Delta).$$

Similarly to the Spin case, the action of $\mathrm{Cl}_\mathbb{C}(n)$ on S can be extended to a global **Clifford multiplication** morphism

$$m : \mathrm{Cl}_\mathbb{C}(E) \otimes S(E,\rho) \to S(E,\rho).$$

This morphism turns the sections of $S(E,\rho)$ into a bundle over the sections of $\mathrm{Cl}_\mathbb{C}(E)$.

Example 2.6.5 (Spin^c-structure from a Spin-structure).
Let $E \to M$ be an oriented vector bundle with a Riemannian metric, and suppose that E admits a Spin-structure $\mathrm{Spin}(E)$, with spinor map $\Sigma : \mathrm{Spin}(E) \to \mathrm{SO}(E)$. Locally, the fibers of this Spin-structure are copies of $\mathrm{Spin}(n)$. Therefore, if we want to "complete them" to a Spin^c-structure, we should "complete" the fibers with an additional $\mathrm{U}(1)$ group, and then take the fiberwise quotient by $\mathbb{Z}_2$. We can achieve this (effectively) by constructing the Spin^c-bundle from the transition functions of $\mathrm{Spin}(E)$ and the additional $\mathrm{U}(1)$-bundle.

Let $\{U_i\}_{i\in I}$ be a trivializing cover of $\mathrm{SO}(E)$, with transition functions $\Lambda_{ij} : U_i \cap U_j \to \mathrm{SO}(n)$ that lift to the transition functions $\tilde{\Lambda}_{i,j} : U_i \cap U_j \to \mathrm{Spin}(n)$. Then, by definition, for all $i, j \in I$ we have

$$\Sigma \circ \tilde{\Lambda}_{ij} = \Lambda_{ij}.$$

Let L be a complex line bundle over M and let $\mathrm{U}(L)$ be its associated $\mathrm{U}(1)$ frame bundle. Suppose that the cover $\{U_i\}_{i\in I}$ is fine enough so that it also trivializes $\mathrm{U}(L)$, with transition functions $\alpha_{ij} : U_i \cap U_j \to \mathrm{U}(1)$. Then we can construct a Spin^c-bundle over M from the cocycles

$$\beta_{ij} := \alpha_{ij}\tilde{\Lambda}_{ij} : U_i \cap U_j \to \mathrm{Spin}^c(n).$$

We denote this bundle by $\mathrm{Spin}^c(E)$. We can see that this is indeed a Spin^c-structure over $\mathrm{SO}(E)$, since the cocycles satisfy

$$\mathfrak{s} \circ \beta_{ij} = \mathfrak{s} \circ \tilde{\Lambda}_{ij} = \Lambda_{ij},$$

and thus there exists an associated bundle morphism $\Sigma^c : \mathrm{Spin}^c(E) \to \mathrm{SO}(E)$ such that for all $p \in \mathrm{Spin}^c(E)$ and $\xi \in \mathrm{Spin}^c(n)$,

$$\Sigma^c(p \cdot \xi) = \Sigma^c(p) \cdot \mathfrak{s}(\xi).$$

Finally, let $\mathrm{U}(E)$ be the determinant line bundle of the Spin^c-structure. Then the cocycles of $\mathrm{U}(E)$ are

$$\delta \circ \beta_{ij} = \delta \circ (\alpha_{ij}\tilde{\Lambda}_{ij}) = \alpha_{ij}^2,$$

and therefore $\mathrm{U}(E)^2 = L$. Thus, L is the *square root* of the determinant line bundle of the Spin^c-structure.

In general, the process of Example 2.6.5 cannot be done in reverse. That is, given a Spin^c-structure over a vector bundle, we cannot always decompose it into a Spin-structure and a line bundle. This is because the conditions for the existence of a Spin^c-structure are *weaker* than those for the Spin case. However, this decomposition can always be done locally. Morally, we can think of a Spin^c structure as

$$\mathrm{Spin}^c(E) \overset{!}{\sim} \mathrm{Spin}(E) \otimes \mathrm{U}(E)^{\frac{1}{2}},$$

where neither $\mathrm{Spin}(E)$ nor $\mathrm{U}(E)^{\frac{1}{2}}$ can be guaranteed to exist globally[15].

Now we turn to the question of existence of Spin^c-structures.

Proposition 2.6.6 (Condition for existence of Spin^c-structures.).
Let $E \to M$ be an oriented Riemannian vector bundle over M. Then E admits a Spin^c-structure if and

[15] Some references call these **virtual bundles**.

only if there is a class $w \in H^2(M, \mathbb{Z})$ *such that*

$$w_2(E) = w \mod 2.$$

Every such integral lift w *determines a* Spin^c*-structure* Spin$^c(E)$ *satisfying*

$$c_1(L(E)) = w.$$

A proof of this is outlined in [Sco05, p. 423], and with more detail in [Kla13, Lemma 3.2.4].

2.6.3 The complex spin connection and the coupled Dirac operator

We want to repeat the same process of Section 2.5.1: given a metric connection ∇ on a vector bundle E with Spin^c-structure Spin$^c(E)$, can we obtain a connection $\tilde{\nabla}$ on the *complex* spinor bundle $S(E, \rho)$ associated to some representation $\rho : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}(S)$?

In the Spin case, we can do this easily because we have the bundle morphism $\Sigma : \mathrm{Spin}(E) \to \mathrm{SO}(E)$. Fiberwise, this morphism is the spinor map $\mathfrak{s} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, whose differential induces a Lie algebra isomorphism $\mathfrak{S}$ between $\mathfrak{spin}(n)$ and $\mathfrak{so}(n)$. To obtain a connection on Spin(E), we pull back a connection on SO(E) with Σ, and then compose it with $\mathfrak{S}^{-1}$ so that it is a $\mathfrak{spin}(n)$-valued form.

In the complex case, the spinor map $\mathfrak{s} : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ is not a finite cover, so its differential $\mathfrak{S}^c$ is not an isomorphism between $\mathfrak{spin}^c(n)$ and $\mathfrak{so}(n)$. In fact, the entirety of $\mathfrak{u}(1)$ is contained in is kernel! If we pull back a connection on SO(E) with Σ^c, we will be missing the $\mathfrak{u}(1)$ part that is needed to define a connection on Spin$^c(E)$.

How do we get the $\mathfrak{u}(1)$ part? We have a hint: the map $(\mathfrak{s}, \delta) : \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$ *is* a double cover, and it induces an isomorphism $\tilde{\mathfrak{S}} : \mathfrak{spin}^c(n) \to \mathfrak{so}(n) \times \mathfrak{u}(1)$. Therefore, if we can find a SO$(n) \times$U(1)-bundle P and a bundle morphism $\mathcal{S} : \mathrm{Spin}^c(E) \to P$ which is fiberwise the map $(\mathfrak{s}, \delta)$, we can pull back a connection on P, compose it with $\tilde{\mathfrak{S}}^{-1}$, and obtain a connection on Spin$^c(E)$. Of course, P cannot be just any SO$(n) \times$U(1)-bundle. Its SO(n) component has to be SO(E), and since the morphism $\mathcal{S} : \mathrm{Spin}^c(E) \to P$ is $(\mathfrak{s}, \delta)$ fiberwise, then the U(1) component has to be the determinant bundle U(E). Therefore, we are looking for the fiber product[16]

$$P = \mathrm{SO}(E) \times_M \mathrm{U}(E).$$

We have a connection ω on SO(E), but in order to get a connection on the fiber product, we need an additional connection iA on U(E). Since we don't have any *a priori* connection, we will have to introduce it by hand as an independent object.

With both ω and iA, we can define a connection ω_P on the fiber product $P = \mathrm{SO}(E) \times_M \mathrm{U}(E)$ by pulling them back with the projections $\mathrm{pr}_{\mathrm{SO}} : \mathrm{SO}(E) \times_M \mathrm{U}(E) \to \mathrm{SO}(E)$ and $\mathrm{pr}_{\mathrm{U}} : \mathrm{SO}(E) \times_M \mathrm{U}(E) \to \mathrm{U}(E)$:

$$\omega_P := \mathrm{pr}_{\mathrm{SO}}^* \omega + \mathrm{pr}_{\mathrm{U}}^*(iA).$$

[16] The fiber product between two fiber bundles $F \xrightarrow{\pi} M$ and $F' \xrightarrow{\pi'} M$ is defined as the fibers above the diagonal $\Delta_M \subseteq M \times M$ in the bundle $F \times F' \to M \times M$. Specifically:

$$F \times_M F' := (\pi \times \pi')^{-1}(\Delta_M),$$

with projection $\pi : F \times_M F' \to M$ given by the composition of $\pi \times \pi'$ with a projection from the diagonal Δ_M onto M.

The fiber product is precisely the pullback bundle of F' by π (or vice versa):

$$F \times_M F' := \pi^* F' = \{(f, f') \in F \times F' : \pi(f) = \pi'(f')\}.$$

Then, we pull ω_P back with the morphism $\mathcal{S} : \mathrm{Spin}^c(E) \to P$ and compose it with $\tilde{\mathfrak{s}}^{-1} : \mathfrak{so}(n) \times \mathfrak{u}(1) \to \mathfrak{spin}^c(n)$ to obtain a connection $\tilde{\omega}_A$ on $\mathrm{Spin}^c(E)$. Explicitly,

$$\tilde{\omega}_A := \tilde{\mathfrak{s}}^{-1} \circ \mathcal{S}^* \omega_P.$$

Note that we've made explicit the dependence on the U(1)-connection.

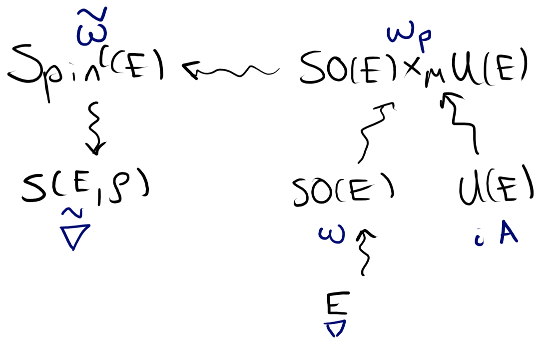


Figure 2.1: The Spin^c connection

With $\tilde{\omega}_A$, we can define a connection $\tilde{\nabla}_A$, called the **complex** or **coupled spin connection**[17] on the spinor bundles $S(E, \rho)$ associated to a representation ρ of $\mathrm{Cl}_{\mathbb{C}}(n)$. How does it act on spinor fields?

Let $\Psi : M \to S(E, \rho)$ be a spinor field, $e = (e_1, \dots, e_n) : U \to \mathrm{SO}(E)$ a local orthonormal frame, and $s : U \to \mathrm{U}(E)$ a section. The map $(e, s) : U \to \mathrm{SO}(E) \times_M \mathrm{U}(E)$ given by

$$x \mapsto (e(x), s(x))$$

is clearly a section of $\mathrm{SO}(E) \times_M \mathrm{U}(E)$, so we can find a section $\tilde{e} : U \to \mathrm{Spin}^c(E)$ such that

$$\mathcal{S} \circ \tilde{e} = (e, s).$$

Since $\mathcal{S}$ is a double cover, there are two such options.

In the local gauge defined by $\tilde{e}$, the spinor field Ψ becomes

$$\Psi(x) = [\tilde{e}(x), \psi(x)],$$

with $\psi : U \to S$. Following the same procedure as in the end of Section 2.5.1, we can show that action of the coupled spin connection is

$$\nabla_A \Psi = [\tilde{e}, \mathrm{d}\psi + (\frac{1}{2} iA + \frac{1}{4} \sum_{i,j} \omega_{j,i} \varepsilon_i \varepsilon_j) \cdot \psi], \tag{2.3}$$

where $\omega_{j,i}$ are the matrix entries of the local gauge potential $e^*\omega$, $\varepsilon_1, \dots, \varepsilon_n$ is the standard orthonormal frame of $\mathbb{R}^n$, and the dot $(\cdot)$ is Clifford multiplication. The factor of $\frac{1}{2}$ comes from the isomorphism $\tilde{\mathfrak{s}} : \mathfrak{spin}^c(n) \to \mathfrak{so}(n) \times \mathfrak{u}(1)$.

With the coupled spin connection, we can define the coupled Dirac operator, in the same was as in Definition 2.5.1. We consider an orientable, Riemannian manifold M that admits a Spin^c structure (that

[17] In physics references, this is called the (minimally coupled) covariant derivative.

is, on its tangent bundle) $\mathrm{Spin}^c(M)$. If $\rho : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}(S)$ is a representation, then we have the spinor bundle $S(M, \rho)$.

Let ∇ be the Levi-Civita connection, and iA a connection on the determinant bundle $\mathrm{U}(M)$. These induce the coupled spin connection $\tilde{\nabla}_A$ acting on spinor fields $\Psi \in \Gamma(S(M, \rho))$.

Definition 2.6.7 (Coupled Dirac operator).
Let $e = (e_1, \dots, e_n)$ be a local orthonormal frame of TM defined in some open set $U \subset M$. The **coupled Dirac operator** $\partial\!\!\!/_A : \Gamma(S(M, \rho)) \to \Gamma(S(M, \rho))$ *is defined as*

$$\partial\!\!\!/_A \Psi = \sum_{i,j} g_{i,j} e_i \cdot \nabla_{A\, e_j} \Psi,$$

where $g_{i,j}$ are the metric coefficients. Here we interpret e_i as a section of $\mathrm{Cl}_{\mathbb{C}}(M)$ via the natural embedding $TM \hookrightarrow \mathrm{Cl}_{\mathbb{C}}(M)$, so that the dot is Clifford multiplication.

In the same way as in Section 2.5.2, we can show that this definition is independent of the choice of frame of TM.

Let's review some properties of the Dirac operator.

Proposition 2.6.8 (Dirac operator is elliptic).
The Dirac operator is an elliptic *partial differential operator of order* 1.

Proof. — From its local expression, it is clear that $\partial\!\!\!/_A$ is a partial differential operator of degree 1. To find its symbol, we fix an orthonormal frame $\{e_i\}$ and a function $f \in C^\infty(M)$, and compute:

$$[\partial\!\!\!/_A, f](\Psi) = \partial\!\!\!/_A(f\Psi) - f\partial\!\!\!/_A\Psi = [s, \sum_i \mathrm{d}f(e_i)e_i \cdot \psi].$$

Note that $\sum_i \mathrm{d}f(e_i)e_i$ is precisely the dual vector of $\mathrm{d}f$ under the isomorphism $T^*M \cong TM$ induced by the metric. Using the rule "replace $\mathrm{d}f$ with an arbitrary form ξ", we find that

$$\sigma(\partial\!\!\!/_A)(\xi)(\Psi) = \xi^\sharp \cdot \Psi,$$

where $(\cdot)^\sharp : T^*M \to TM$ is the isomorphism induced by the metric, and the dot represents Clifford multiplication.

Furthermore, since Clifford multiplication by a non-zero vector is an automorphism, we conclude that $\partial\!\!\!/_A$ is *elliptic*. Note that this also applies to $\partial\!\!\!/_A$ restricted to the space of positive or negative spinors. ∎

Proposition 2.6.9 (Dirac operator is formally self-adjoint).
Let M be a compact, oriented, Riemannian manifold that admits a Spin^c*-structure. Let A be a* U(1)*-connection on the determinant bundle. Then there is an induced Hermitian metric on the spinor bundle $S(M)$, and under it the Dirac operator is formally self-adjoint:*

$$\int_M \langle \partial\!\!\!/_A \psi, \varphi \rangle \,\mathrm{vol} = \int_M \langle \psi, \partial\!\!\!/_A \varphi \rangle \,\mathrm{vol},$$

for all spinor fields $\psi, \varphi \in \Gamma(S(M))$.

Proof. — We work locally around a point $x \in M$. We can choose a *moving* orthonormal frame $\{e_i\}$ such that $\nabla_{e_i} e_i = 0$. Then

$$\begin{aligned}
\langle \partial\!\!\!/_A \psi, \varphi \rangle &= \sum_i \langle e_i \cdot \nabla_{A,e_i} \psi, \varphi \rangle \\
&= -\sum_i \langle \nabla_{A,e_i} \psi, e_i \cdot \varphi \rangle \\
&= -\sum_i \langle \nabla_{A,e_i} \psi, e_i \cdot \varphi \rangle \\
&= -\sum_i (e_i \langle \psi, e_i \cdot \varphi \rangle - \langle \psi, \nabla_{A,e_i} (e_i \cdot \varphi) \rangle) \\
&= -\sum_i (e_i \langle \psi, e_i \cdot \varphi \rangle - \langle \psi, e_i \cdot \nabla_{A,e_i} \varphi \rangle) \\
&= -\sum_i e_i \langle \psi, e_i \cdot \varphi \rangle + \langle \psi, \partial\!\!\!/_A \varphi \rangle .
\end{aligned}$$

Here we used the fact that ∇_A is a unitary connection, and that it satisfies a Leibniz rule $\nabla_A (X \cdot \psi) = \nabla(X) \cdot \psi + X \cdot \nabla_A \psi$. This tells us that

$$\langle \partial\!\!\!/_A \psi, \varphi \rangle - \langle \psi, \partial\!\!\!/_A \varphi \rangle = -\sum_i e_i \langle \psi, e_i \cdot \varphi \rangle .$$

If we define a one-form β by as $\beta(e_i) = \langle \psi, e_i \cdot \varphi \rangle$, we have that the right-hand side is precisely $\mathrm{d}^* \beta$, and so integrating over M, we obtain the result. ■

The Dirac operator is related to the Bochner Laplacian of the spin connection via the *Weitzenböck formula*:

Theorem 2.6.10 (Weitzenböck formula).
The Dirac operator satisfies

$$\partial\!\!\!/_A \partial\!\!\!/_A \psi = \nabla_A^* \nabla_A \psi + \frac{s}{4} \psi - \sum_{i<j} i F_A(e_i, e_j)(e_i e_j \cdot \psi),$$

where s is the scalar curvature of the metric.

This result can be proved by choosing a local moving frame and expanding everything in its local form [see Moo96, p. 56].

2.7 Final ingredient: The squaring map

The final ingredient in our spin soup is a map that relates spinor fields with self-dual two forms on a manifold. This at once tells us that we have to work with four-dimensional manifolds. It also tells us that we need to be careful with the signature of the semi-Riemannian metric, since it will affect the Hodge star operator.

Remark. From now on, we will work exclusively with Riemannian metrics, and set $n = 4$.

2.7.1 The linear squaring map

It is a good time to review and collect all the algebraic results that we have about Cl(4) and Spin(4). Recall that in ?? we showed that the Clifford algebra Cl(4) is precisely the set of 2×2 matrices with entries in the quaternions $\mathbb{H}$. Explicitly, the embedding $\mathbb{R}^4 \hookrightarrow \mathrm{Cl}(4) = \mathbb{H}(2)$ is given by

$$v \mapsto \begin{pmatrix} 0 & v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k} \\ -v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k} & 0 \end{pmatrix} = \begin{pmatrix} 0 & q(v) \\ -\overline{q(v)} & 0 \end{pmatrix},$$

where $q(v) = v_0 + v_1\mathbb{i} + v_2\mathbb{j} + v_3\mathbb{k}$.

Furthermore, in Example 2.3.7 we found that the only irreducible representation $\rho : \mathrm{Cl}(4) \to \mathrm{End}(\mathbb{C}^4)$ is given by

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \mapsto \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \in \mathbb{C}(4),$$

where if $q_j = a_j + b_j\mathbb{i} + c_j\mathbb{j} + d_j\mathbb{k} \in \mathbb{H}$, then $Q_j \in \mathbb{C}(2)$ is the image of q_j under the standard representation of $\mathbb{H}$ in $\mathbb{C}(2)$:

$$Q_j := \begin{pmatrix} a_j + ib_j & c_j + id_j \\ -c_j + id_j & a_j - ib_j \end{pmatrix}.$$

In particular, the image of vectors $v \in \mathbb{R}^4$ under ρ is

$$\rho(v) = \begin{pmatrix} 0 & Q(v) \\ -Q(v)^* & 0 \end{pmatrix},$$

where

$$Q(v) = \begin{pmatrix} v_0 + iv_1 & v_2 + iv_3 \\ -v_2 + iv_3 & v_0 - iv_1 \end{pmatrix}.$$

Note that $\rho(v)$ is anti-Hermitian for all $v \in \mathbb{R}^4$. Therefore, if v is a unit vector, since $-vv = 1$ in Cl(4), it follows that

$$\rho(v)^*\rho(v) = -\rho(v)\rho(v) = \rho(-vv) = \rho(1) = I_4,$$

and so $\rho(v)$ is unitary[18].

In particular, the spin representation $\Delta = \rho|_{\mathrm{Spin}(4)} : \mathrm{Spin}(4) \to \mathrm{SU}(4)$ is

$$\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \mapsto \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \in \mathrm{SU}(4),$$

where q_1 and q_2 are unit quaternions, which implies that $Q_1, Q_2 \in \mathrm{SU}(2)$. Clearly, the spin representation is reducible, with irreducible components $\Delta_\pm : \mathrm{Spin}(4) \to \mathrm{SU}(2)$ being the composition of Δ with the projection on the first and second components.

Let $\varepsilon_0, \dots, \varepsilon_3$ be the standard orthonormal basis of $\mathbb{R}^4$, and $\varepsilon_0^*, \dots, \varepsilon_3^*$ be its dual basis. The map $\Omega^2(\mathbb{R}^4) \to \mathrm{Cl}_0(4)$ given by

$$\varepsilon_i^* \wedge \varepsilon_j^* \mapsto \varepsilon_i \varepsilon_j$$

is a well-defined linear map, so we can define a representation $\mu : \Omega^2(\mathbb{R}^4) \to \mathrm{End}(\mathbb{C}^4)$ using the spin representation:

$$\mu(\eta)(\psi) = \frac{1}{2}\sum_{i,j} \eta_{i,j}\mu(\varepsilon_i^* \wedge \varepsilon_j^*)(\psi) := \frac{1}{2}\sum_{i,j} \eta_{i,j}\Delta(\varepsilon_i\varepsilon_j)\psi.$$

[18] In fact, given a representation $\rho : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}(S)$, there exists an inner product on S such that $\rho(v)$ is unitary for all unit vectors v [see LM89, Theorem 5.17]. In the case of $n = 4$ and the spin representation, this is the standard inner product.

Since Δ is unitary, then μ is anti-Hermitian:

$$\begin{aligned}\left\langle \mu(\varepsilon_i^* \wedge \varepsilon_j^*)(\psi), \phi \right\rangle &= \langle \Delta(\varepsilon_i \varepsilon_j)(\psi), \phi\rangle \\ &= \langle \psi, \Delta(\varepsilon_i\varepsilon_j)^{-1}(\phi)\rangle \\ &= \langle \psi, \Delta(\varepsilon_j\varepsilon_i)(\phi)\rangle \\ &= -\langle \psi, \Delta(\varepsilon_i\varepsilon_j)(\phi)\rangle \\ &= -\left\langle \psi, \mu(\varepsilon_i^* \wedge \varepsilon_j^*)(\phi)\right\rangle.\end{aligned}$$

Since μ acts on even forms, then it preserves the splitting of the spinor space $S = S^+ \oplus S^-$. This can be seen directly by writing $\mu(\eta)$ as a matrix:

$$\begin{aligned}\mu(\eta) &= \rho\begin{pmatrix}(\eta_{01}+\eta_{23})\mathbb{i} + (\eta_{02}+\eta_{31})\mathbb{j} + (\eta_{03}+\eta_{12})\mathbb{k} & 0 \\ 0 & (\eta_{23}-\eta_{01})\mathbb{i} + (\eta_{31}-\eta_{02})\mathbb{j} + (\eta_{12}-\eta_{03})\mathbb{k}\end{pmatrix} \\ &:= \begin{pmatrix}\mu_+(\eta) & 0 \\ 0 & \mu_-(\eta)\end{pmatrix},\end{aligned}$$

Where

$$\begin{aligned}\mu_+(\eta) &= \begin{pmatrix}(\eta_{01}+\eta_{23})i & (\eta_{02}+\eta_{31}) + (\eta_{03}+\eta_{12})i \\ -(\eta_{02}+\eta_{31}) + (\eta_{03}+\eta_{12})i & -(\eta_{01}+\eta_{23})i\end{pmatrix}; \\ \mu_-(\eta) &= \begin{pmatrix}(\eta_{23}-\eta_{01})i & (\eta_{31}-\eta_{02}) + (\eta_{12}-\eta_{03})i \\ -(\eta_{31}-\eta_{02}) + (\eta_{12}-\eta_{03})i & -(\eta_{23}-\eta_{01})i\end{pmatrix}.\end{aligned}$$

Note that the $\mu_+(\eta)$ (resp. $\mu_-(\eta)$) is determined only by the self-dual (resp. anti-self-dual) components of η. Therefore, if η is self-dual, we have that $\eta_{01} = \eta_{23}$, $\eta_{02} = \eta_{31}$ and $\eta_{03} = \eta_{12}$, so that

$$\mu_+(\eta) = 2\begin{pmatrix}\eta_{01} i & \eta_{02} + \eta_{03} i \\ -\eta_{02} + \eta_{03} i & -\eta_{01} i\end{pmatrix}.$$

Futhermore, if we let the coefficients of η to be complex (i.e. if we consider $\Omega^2_+(\mathbb{R}^4, \mathbb{C})$), then μ_+ is an isomorphism onto the space $\mathrm{End}_0(S^\pm)$ of traceless endomorphisms[19].

Proposition 2.7.1 (Traceless endomorphisms are isomorphic to complex self-dual 2-forms).

The morphism $\mu_+ : \Omega^2_+(\mathbb{R}^4, \mathbb{C}) \to \mathrm{End}_0(S^+)$ *is an isomorphism between the space of complex-valued self-dual 2-forms and the traceless endomorphisms.*

Proof. — We can find an explicit inverse. Let $A \in \mathrm{End}_0(S^+)$. In its matrix form, we can write A as

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1\end{pmatrix}.$$

Comparing this expression with the matrix form of $\omega(\eta)$ for a self-dual, complex-valued $\eta \in \Omega^2_+(\mathbb{R}^4, \mathbb{C})$, we have that $\mu_+(\eta) = A$ if and only if

$$\eta_{01} = -\frac{i}{2} z_1$$

[19] Similarly, μ_- is an isomorphism between the space of *anti*-self-dual 2-forms and traceless endomorphisms, but we won't use it.

$$\eta_{02} = -\frac{1}{4}(z_2 - z_3)$$
$$\eta_{03} = -\frac{i}{4}(z_2 + z_3).$$

It is straightforward to check that the map

$$\mu_+^{-1}(A) = \frac{i}{2} z_1(\varepsilon_0^* \wedge \varepsilon_1^* + \varepsilon_2^* \wedge \varepsilon_3^*) - \frac{1}{4}(z_2 - z_3)(\varepsilon_0^* \wedge \varepsilon_2^* + \varepsilon_3^* \wedge \varepsilon_1^*) - \frac{i}{4}(z_2 + z_3)(\varepsilon_0^* \wedge \varepsilon_3^* + \varepsilon_1^* \wedge \varepsilon_2^*)$$

is indeed the inverse of μ_+. ■

Now we are ready to construct the squaring map. Let $\psi \in S^+$ be a spinor. We can construct an endomorphism $\psi \otimes \psi^* \in \mathrm{End}(S^+)$, and make it traceless:

$$(\psi \otimes \psi^*)_0 := \psi \otimes \psi^* - \frac{1}{2}\mathrm{Tr}(\psi \otimes \psi^*)I.$$

In matrix form, if $\psi = (\psi_1 \psi_2)^T$, then $\psi^* = (\overline{\psi}_1 \overline{\psi}_2)$ and

$$(\psi \otimes \psi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\psi_1|^2 - |\psi_2|^2) & \psi_1\overline{\psi}_2 \\ \overline{\psi}_1\psi_2 & \frac{1}{2}(|\psi_2|^2 - |\psi_1|^2) \end{pmatrix}.$$

Definition 2.7.2 ((Linear) Squaring map).
We define the squaring map $\sigma^+ : S^+ \to \Omega_+^2(\mathbb{R}^4, \mathbb{C})$ *as*

$$\sigma^+(\psi) := \mu_+^{-1}((\psi \otimes \psi^*)_0)$$

for all $\psi \in S^+ \cong \mathbb{C}^2$. *We can write* σ^+ *explicitly using Proposition 2.7.1:*

$$\begin{aligned}\sigma^+(\psi) = -\frac{i}{4}\big((|\psi_1|^2 - |\psi_2|^2)&(\varepsilon_0^* \wedge \varepsilon_1^* + \varepsilon_2^* \wedge \varepsilon_3^*) \\ - 2\Im(\psi_1\overline{\psi}_2)&(\varepsilon_0^* \wedge \varepsilon_2^* + \varepsilon_3^* \wedge \varepsilon_1^*) \\ + 2\Re(\psi_1\overline{\psi}_2)&(\varepsilon_0^* \wedge \varepsilon_3^* + \varepsilon_1^* \wedge \varepsilon_2^*)\big).\end{aligned}$$

We can rewrite this more compactly (and also in a basis-independent way). Writing $\varepsilon_i\varepsilon_j \cdot \psi := \Delta_+(\varepsilon_i\varepsilon_j)(\psi)$, we have

$$\langle \psi, \varepsilon_0\varepsilon_1 \cdot \psi \rangle = \langle \psi, \varepsilon_2\varepsilon_3 \cdot \psi \rangle = \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = i(|\psi_1|^2 - |\psi_2|^2)$$

$$\langle \psi, \varepsilon_0\varepsilon_2 \cdot \psi \rangle = \langle \psi, \varepsilon_3\varepsilon_1 \cdot \psi \rangle = \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -2i\Im(\psi_1\overline{\psi}_2)$$

$$\langle \psi, \varepsilon_0\varepsilon_3 \cdot \psi \rangle = \langle \psi, \varepsilon_1\varepsilon_2 \cdot \psi \rangle = \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 2i\Re(\psi_1\overline{\psi}_2).$$

Therefore, we can write

$$\sigma^+(\psi) = -\frac{1}{4}\sum_{i<j} \langle \psi, \varepsilon_i\varepsilon_j \cdot \psi \rangle\, \varepsilon_i^* \wedge \varepsilon_j^*.$$

2.7.2 The global squaring map

Let M be a Riemannian 4-manifold which admits a Spin or Spin^c structure. The spin representation Δ splits, so the spinor bundle $S(M) = S(M, \Delta)$ splits into the sum of the **positive** (or **right-handed**) spinors $S^+(M)$ and the **negative** (or **left-handed**) spinors $S^-(M)$:

$$S(M) = S^+(M) \oplus S^-(M).$$

Analogously with the linear case, the bundles $S(M)$, $S^+(M)$ and $S^-(M)$ have Hermitian metrics such that Clifford multiplication by an unit-length vector field is unitary.

Finally, we promote σ^+ to a definition on the positive spinor bundle.

Definition 2.7.3 (Squaring map).
Let M be an orientable Riemannian 4-manifold that admits a Spin *or* Spin^c *structure. Let $S(M) = S^+(M) \oplus S^-(M)$ be the spinor bundles associated to the (complex) spin representation Δ. We define the* **squaring** *or* **quadratic map** $\sigma^+ : \Gamma(S^+(M)) \to \Omega^2_+(M, \mathbb{C})$ *locally, in terms of an orthonormal frame $e_1, \dots, e_n$ as*

$$\sigma^+(\psi) := -\frac{1}{4} \sum_{i<j} \langle \psi, e_i e_j \cdot \psi \rangle \, e_i^* \wedge e_j^*.$$

Note that $\sigma^+(\psi)$ is a purely imaginary self-dual 2-form.

It is a straightforward exercise to show that σ^+ is well-defined.

CHAPTER 3

The Seiberg-Witten Equations and Moduli Space

We are finally ready to define the Seiberg-Witten equations and their moduli space. Let M be an oriented Riemannian 4-manifold which admits a Spin^c-structure. The **Seiberg-Witten equations** are the equations on positive spinor fields $\psi \in \Gamma(S^+(M))$ and U(1) connections iA, given by

$$\partial\!\!\!/_A \psi = 0 \tag{3.1a}$$
$$F_A^+ = \sigma^+(\psi), \tag{3.1b}$$

where F_A^+ is the self-dual part of the curvature of A, and σ^+ is the squaring map of Section 2.7.2. We call a solution of Equation (3.1) a **Seiberg-Witten monopole**, or just a monopole.

We will see that these equations are equivariant under the natural action of a specific group $\mathcal{G}$ on the space of spinors and connections on U(M). Therefore, the solutions (ψ, A) of the Seiberg-Witten equations will be invariant under the action of the gauge group, and thus, we can consider the **Seiberg-Witten moduli space**

$$\mathcal{M} = \left.\left\{(\psi, A) \in \Gamma(S^+(M)) \times \mathrm{Conn}(\mathrm{U}(M)) \mid \text{Equations 3.1 hold}\right\}\right/\mathcal{G},$$

where Conn(U(M)) denotes the space of U(1)-connections on U(M). The moduli space $\mathcal{M}$ depends both on the Riemannian metric on M and the Spin^c-structure on it.

In most cases, it turns out that $\mathcal{M}$ is a surprisingly *good* object: it is a compact, orientable smooth manifold of finite dimension. Briefly speaking, this regularity comes from two ideas. The first one is that the coupled Dirac operator is *elliptic*, and therefore its kernel is reasonably well-behaved. The second one is that the symmetry group $\mathcal{G}$ is "large" enough to cut down the dimension of the moduli space, while also having a good enough topology so that the quotient is a manifold too.

We will then define the **Seiberg-Witten invariant** of the Spin^c-structure as the integral of a specific form over $\mathcal{M}$. Indeed, this will be an invariant, independent of the Riemannian metric. Furthermore, in most cases, $\mathcal{M}$ will be zero-dimensional, so the invariant will reduce to counting the points of $\mathcal{M}$ with signs depending on the orientation.

Overview of this chapter

The first thing that's in order is proving that the moduli space $\mathcal{M}$ is a smooth manifold. For this, we will use the implicit function theorem, in its infinite-dimensional flavour. First, we need to set some nomenclature: define the **Seiberg-Witten configuration space** $\mathrm{Conf}(M)$ as

$$\mathrm{Conf}(M) = \Gamma(S^+(M)) \times \mathrm{Conn}(\mathrm{U}(M)),$$

and the *target space*

$$\mathcal{Y} = \Gamma(S^-(M)) \times \Omega^2_+(M, i\mathbb{R}).$$

This way, the solutions of the Seiberg-Witten equations are written as the zero set of the **Seiberg-Witten map** $\mathcal{SW} : \mathrm{Conf}(M) \to \mathcal{Y}$, given as

$$\mathcal{SW}(\psi, A) = (\partial\!\!\!/_A \psi, F_A^+ - \sigma^+(\psi)).$$

Ideally, we would show that $(0,0)$ is a regular value of $\mathcal{SW}$, so that by the implicit function theorem, its zero set is a smooth submanifold of $\mathrm{Conf}(M)$. Then, if the quotient by $\mathcal{G}$ is well-behaved, we obtain the moduli space as a smooth manifold too.

However, there's a catch: first we need to endow $\mathrm{Conf}(M)$ and $\mathcal{Y}$ with structures of smooth *Banach* manifolds. Of course, they are the direct product of spaces of sections, so they already are vector spaces. However, upon endowing them with an L^p-norm, smoothness of sections inevitably means that the normed spaces are incomplete. Therefore, we must work with appropriate completions, which in our case will be Sobolev completions of these spaces of sections. We write $\mathrm{Conf}^{k,p}(M)$, $\mathcal{Y}^{k,p}$, $\mathcal{G}^{k,p}(M)$ for the Sobolev versions of the smooth spaces, with k weak derivatives in L^p.

The first important step is showing that even though we've considered a larger configuration space (and thus a larger zero set of $\mathcal{SW}$), the *moduli* space remains the same. That is, any solution with "low" regularity is gauge-equivalent to a smooth one. This is a result that follows from the fact that the Dirac operator is elliptic, and thus the solutions to the Dirac equation enjoy a lot of regularity.

Once we have this, we can try to apply the implicit function theorem. However, we will quickly note that in many cases, $(0,0)$ will not be a regular value of the Seiberg-Witten map. We can fix this by applying a *perturbation*. For any fixed closed self-dual 2-form η, define the **perturbed Seiberg-Witten map** $\mathcal{SW}_\eta : \mathrm{Conf}(M) \to \mathcal{Y}$, as

$$\mathcal{SW}_\eta(\psi, A) = (\partial\!\!\!/_A \psi, F_A^+ - \eta - \sigma^+(\psi)).$$

With the aid of Smale's infinite dimensional version of Sard's theorem, we can show that for a *generic* perturbation η, the value $(0,0)$ will be a regular and thus the zero-set will be a submanifold of $\mathrm{Conf}(M)$, albeit infinite-dimensional.

Now we take the quotient, and calculate the dimension. At every point (ψ, A) in the configuration space, we have an *infinitesimal action* of the "Lie algebra" of the gauge group $\mathcal{G}$, which we denote by $\mathfrak{g}_{(\psi,A)} : T_1\mathcal{G} \to T_{(\psi,A)}\mathrm{Conf}(M)$. If (ψ, A) is furthermore a solution of the perturbed Seiberg-Witten equations, we have a complex, called the **Seiberg-Witten complex**

$$0 \longrightarrow T_1\mathcal{G} \xrightarrow{\ \mathfrak{g}\ } T_{(\psi,A)}\mathrm{Conf}(M) \xrightarrow{T_{(\psi,A)}\mathcal{SW}_\eta} T_{(0,0)}\mathcal{Y} \longrightarrow 0.$$

The homology of this complex tells us about how *good* a point (A, ψ) is. Specifically, we have

$$H^0_{\mathrm{SW}} = 0 \quad \text{if and only if the action of } \mathcal{G} \text{ is free at } (\psi, A),$$

$$H^2_{\mathrm{SW}} = 0 \quad \text{if and only if } (\psi, A) \text{ is a regular point,}$$

and furthermore,

$$H^1_{\mathrm{SW}} = {\ker T_{(\psi,A)}\mathcal{SW}_\eta}/{\operatorname{im}\mathfrak{g}} \cong T_{[A,\psi]}\mathcal{M}.$$

The Fredholm property of the Dirac operator also extends to the entirety of the Seiberg-Witten map, so the kernel $\ker T_{(\psi,A)}\mathcal{SW}_\eta$ is finite dimensional. Therefore, we can compute

$$\dim T_{[\psi,A]}\mathcal{M} = \dim(\ker T_{(\psi,A)}\mathcal{SW}_\eta) - \dim(\operatorname{im}\mathfrak{g}).$$

From the Atiyah-Singer index theorem, we can calculate these dimensions in terms of the indices of the elliptic operators $\partial\!\!\!/_A$ and $\mathrm{d} + \mathrm{d}^*$, and thus obtain

$$\dim\mathcal{M} = \dim T_{[\psi,A]}\mathcal{M} = \frac{1}{4}(c^2 - 2\chi(M) - 3\operatorname{sign}(M)),$$

where $c^2 = \int_M c_1(L(M)) \wedge c_1(L(M))$ with $L(M)$ determinant line bundle of the Spin^c-structure, $\chi(M)$ is the Euler characteristic of M, and $\operatorname{sign}(M) = b_2^+ - b_2^-$ is the signature of M.

Now that we now that $\mathcal{M}$ is a smooth, finite-dimensional manifold, we can define an invariant on it. It turns out that $\mathcal{M}$ is compact, and this follows mainly from an *a priori* bound on ψ and F_A^+ given by the scalar curvature of M and a Weitzenböch formula with the Dirac Laplacian. We can prove that $\mathcal{M}$ is sequentially compact by starting with these *a priori* bounds, and the Sobolev embedding theorems to obtain convergent subsequences of smaller regularity. The elliptic regularity of the solutions to the Dirac equation help "restore" the lost regularity.

Finally, $\mathcal{M}$ is orientable, and an orientation can be inherited from orientations of the cohomology groups $H^1(M,\mathbb{R})$ and $H^2_+(M,\mathbb{R})$. Therefore, we can happily integrate forms over $\mathcal{M}$, and the **Seiberg-Witten invariant** will be defined as the integral of a naturally-chosen form.

3.1 The gauge group and its action

Let M be a smooth manifold that admits a Spin^c-structure Spin$^c(M)$. The **gauge group** or **group of gauge transformations** of Spin$^c(M)$, denoted $\mathcal{G}$, is the group of automorphisms[1] of Spin$^c(M)$ which lift the identity of the orthonormal frame bundle $\mathrm{SO}(M)$. That is, $\Phi \in \mathcal{G}$ if the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\Phi} & \mathrm{Spin}^c(M) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ \mathrm{SO}(M) & \xrightarrow{\mathrm{id}} & \mathrm{SO}(M) \end{array}.$$

Here, $\Sigma : \mathrm{Spin}^c(M) \to \mathrm{SO}(M)$ is the bundle morphism which is fiberwise the spinor map $\mathfrak{s} : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$.

The gauge group seems very unwieldy, but in our case it has a nice presentation.

[1] An automorphism of a principal G-bundle $G \hookrightarrow P \to M$ is a diffeomorphism $\Phi : P \to P$ such that $\pi \circ \Phi = \Phi$ and for all $p \in P$ and $g \in G$, $\Phi(p \cdot g) = \Phi(p) \cdot g$.

Proposition 3.1.1 (Characterization of gauge group).
The gauge group $\mathcal{G}$ is in bijection with the set of smooth maps $\gamma : M \to S^1$:

$$\mathcal{G} \cong C^\infty(M, S^1).$$

Proof. — Let $\gamma : M \to S^1$ be a smooth map. We interpret $S^1 \subset \mathrm{Spin}^c(n)$, so that γ can be seen as a map $\gamma : M \to \mathrm{Spin}^c(n)$. Then we define

$$\begin{aligned} \Phi_\gamma : \mathrm{Spin}^c(M) &\to \mathrm{Spin}^c(M) \\ p &\mapsto p \cdot \gamma(\pi(p)) \end{aligned}$$

for all $p \in \mathrm{Spin}^c(M)$, where $\pi : \mathrm{Spin}^c(M) \to M$. Clearly, Φ_γ is a smooth bundle map, being the composition of smooth maps and mapping fibers into fibers. For all $p \in \mathrm{Spin}^c(M)$:

$$\Sigma(\Phi_\gamma(p)) = \Sigma(p \cdot \gamma(\pi(p))) = \Sigma(p) \cdot \mathfrak{s}(\gamma(\pi(p))) = \Sigma(p),$$

since $\gamma(\pi(p)) \in \mathrm{U}(1)$, and so $\mathfrak{s}(\gamma(\pi(p))) = 1$.

Now let $\Phi \in \mathcal{G}$. We want to see that there is a unique $\gamma : M \to S^1$ such that $\Phi = \Phi_\gamma$. Locally, Φ must act as

$$\Phi(x, \xi) = (x, \varphi(x, \xi)),$$

such that for each x, the map $\varphi(x, -)$ is an automorphism of $\mathrm{Spin}^c(n)$. Since Φ is a gauge transformation, then

$$\Sigma(\Phi(x, \xi)) = (x, \mathfrak{s}(\varphi(x, \xi))) = (x, \mathfrak{s}(\xi)) = \Sigma(x, \xi)$$

and thus $\mathfrak{s}(\varphi(x, \xi)) = \mathfrak{s}(\xi)$, which implies that $\varphi(x, \xi) = e^{i\theta(x,\xi)}\xi$, for some function θ. Therefore, Φ is of the form

$$\Phi(p) = p \cdot \tilde{\gamma}(p),$$

where $\tilde{\gamma} : \mathrm{Spin}^c(M) \to \mathrm{U}(1)$ is a smooth function. Let's see that $\tilde{\gamma}$ is actually constant along the fibers of $\mathrm{Spin}^c(M)$, so that it determines a function $\gamma : M \to \mathrm{U}(1)$. Let $p, p' \in \mathrm{Spin}^c(M)$ be points on the same fiber, so $\pi(p) = \pi(p')$. Then there exists a unique $\xi \in \mathrm{Spin}^c(n)$ such that $p' = p \cdot \xi$, and thus

$$\Phi(p') = \Phi(p \cdot \xi) = \Phi(p) \cdot \xi = p \cdot (\tilde{\gamma}(p)\xi).$$

On the other hand,

$$\Phi(p') = p' \cdot (\tilde{\gamma}(p')) = p' \cdot (\xi\tilde{\gamma}(p')).$$

Since $\tilde{\gamma}(p)$ and $\tilde{\gamma}(p')$ are in $\mathrm{U}(1)$, then they commute with ξ and thus

$$\tilde{\gamma}(p) = \tilde{\gamma}(p').$$

From this, indeed we see that $\Phi_\gamma = \Phi$. ■

Now let's see that $\mathcal{G}$ acts on the space of spinors and U(1)-connections, and that the Seiberg-Witten equations are equivariant under this action.

The **action of $\mathcal{G}$ on spinor fields** is given as follows. Let $\gamma \in \mathcal{G}$, seen as a smooth map $\gamma : M \to S^1 \subseteq \mathbb{C}$. For any left, right, or total spinor field Ψ, we define

$$(\Psi \cdot \gamma)(x) = \gamma(x)^{-1}\Phi(x).$$

The **action of $\mathcal{G}$ on connections on the determinant bundle** are a bit more subtle. Given a map $\gamma : M \to \mathrm{U}(1)$, we have an induced automorphism of the U(1)-bundle $\Phi_\gamma : \mathrm{U}(L) \to \mathrm{U}(L)$, given as

$$\Phi_\gamma(p) = p \cdot \delta(\gamma(\pi(p))).$$

Given a U(1)-connection $\omega \in \Omega^1(\mathrm{U}(M), i\mathbb{R})$, we define

$$\omega \cdot \gamma = \Phi_\gamma^* \omega.$$

How does this action look like locally? Let $s : U \to \mathrm{U}(M)$ be a local trivializing section, and write $iA = s^*\omega$. Then

$$s^*(\omega \cdot \gamma) = (\Phi_\gamma \circ s)^* \omega.$$

However,

$$(\Phi_\gamma \circ s)(x) = s(x) \cdot \delta(\gamma(x)) = s(x) \cdot \gamma(x)^2$$

which is another section s' related to s by a transition function $g = \gamma^2$. Therefore, writing $iA' = s^*(\omega \cdot \gamma)$,

$$iA' = iA + g^{-1}\,\mathrm{d}g = iA + 2\gamma^{-1}\,\mathrm{d}\gamma\,.$$

Since the action of γ looks like a change of local gauge, then the curvature is unchanged:

$$F_A = F_{A'}.$$

With this we prove:

Proposition 3.1.2 (Invariance of solutions of SW equations).
If $\Psi \in \Gamma(S^+(M))$ and $A \in \mathrm{Conn}(\mathrm{U}(M))$ form a solution to the Seiberg-Witten equations, i.e.,

$$\begin{aligned} \partial\!\!\!/_A \psi &= 0 \\ F_A^+ &= \sigma^+(\psi), \end{aligned}$$

then for all $\gamma \in \mathcal{G}$, the gauge-transformed monopole $(\psi \cdot \gamma, A \cdot \gamma)$ is also a solution.

Proof. — We need to see how the Dirac operator $\partial\!\!\!/_A$ and the squaring map $\sigma^+(\psi)$ change under a gauge transformation.

Recall that locally, the action of $\mathcal{G}$ on A is

$$iA' = iA \cdot \gamma = iA + 2\gamma^{-1}\,\mathrm{d}\gamma\,.$$

Therefore, in a local frame $s = (e_1, \dots, e_n)$ of $\mathrm{SO}(M)$ that lifts to a frame $\bar{s}$ of $\mathrm{Spin^c}(M)$, if Ψ is represented locally as $\Psi = [\bar{s}, \psi]$, then the spin connection acts as

$$\begin{aligned} \nabla_{A'}\Psi' &= [\bar{s}, \mathrm{d}(\gamma^{-1}\psi) + \left(\frac{i}{2}A' + \frac{1}{4}\sum_{j,k}\omega_{k,j}e_je_k\right)\cdot(\gamma^{-1}\psi)] \\ &= [\bar{s}, -\gamma^{-2}\,\mathrm{d}\gamma\,\psi + \gamma^{-1}\,\mathrm{d}\psi + \left(\frac{i}{2}A + \gamma^{-1}\,\mathrm{d}\gamma + \frac{1}{4}\sum_{j,k}\omega_{k,j}e_je_k\right)\cdot(\gamma^{-1}\psi)] \\ &= \gamma^{-1}[\bar{s}, \mathrm{d}\psi + \left(\frac{i}{2}A + \frac{1}{4}\sum_{j,k}\omega_{k,j}e_je_k\right)\cdot\psi] \end{aligned}$$

$$= \gamma^{-1}\nabla_A\Psi.$$

This implies that $\partial\!\!\!/_{A'}\Psi' = \gamma^{-1}\partial\!\!\!/_A\Psi$.

Finally, let's see how the squaring map changes. Again, locally,

$$\begin{aligned}\sigma^+(\psi') &= -\frac{1}{4}\sum_{i<j}\langle\gamma^{-1}\psi, e_ie_j\cdot\gamma^{-1}\psi\rangle e_i^*\wedge e_j^* \\ &= -\frac{1}{4}\sum_{i<j}\langle\psi, e_ie_j\psi\rangle e_i^*\wedge e_j^* \\ &= \sigma^+(\psi).\end{aligned}$$

In summary, under a gauge transformation γ, the Seiberg-Witten equations transform as

$$\begin{matrix}\partial\!\!\!/_A\Psi \\ F_A^+ - \sigma^+(\Psi)\end{matrix} \xrightarrow{\cdot\gamma} \begin{matrix}\gamma^{-1}\partial\!\!\!/_A\Psi \\ F_A^+ - \sigma^+(\Psi)\end{matrix},$$

and thus, (Ψ', A') is a solution of the Seiberg-Witten equations if and only if (Ψ, A) is. ■

The fact that the action of $\gamma \in \mathcal{G}$ on a spinor Ψ is $\gamma^{-1}\Psi$ means that the stabilizer of any configuration (Ψ, A) is trivial if and only if $\Psi \equiv 0$. In that case, from the local expression of the action on connections, we have that $\gamma \in \mathrm{Stab}(0, A)$ if and only if $\mathrm{d}\gamma = 0$. We have proved:

Proposition 3.1.3.
The stabilizer of an element (Ψ, A) *under the action of* $\mathcal{G}$ *is trivial if and only if* Ψ *is not uniquely zero. If* $\Psi \equiv 0$, *then*

$$\mathrm{Stab}(0, A) = S^1.$$

We say that a configuration (Ψ, A) with $\Psi \equiv 0$ is a *reducible* configuration. If $\Psi \neq 0$, then we say that it is *irreducible*. As we saw above, reducible configurations are an obstruction to obtaining a nice manifold structure the moduli space.

3.2 Topology of the Moduli Space

This section follows [Nic07, Section 2.2.1] with a bit of cosmetic changes. The proper setup of the Seiberg-Witten equations and their moduli space is, then, in the Sobolev completions. Fix, once and for all, a smooth reference connection $A_0 \in \mathrm{Conn}(\mathrm{U}(M))$. We define the configuration space as

$$\mathrm{Conf}^{2,2} = W^{2,2}(S^+(M)) \times \mathrm{Conn}^{2,2}(\mathrm{U}(M)),$$

where

$$\mathrm{Conn}^{2,2}(\mathrm{U}(M)) = \{A_0 + \pi^*\alpha \mid \alpha \in W^{2,2}(iT^*M)\}$$

is the space of connections[2] in $W^{2,2}$. The target space is

$$\mathcal{Y}^{1,2} = W^{1,2}(S^-(M)) \times W^{1,2}(i\Lambda^2_+T^*M),$$

[2] Recall that the space of $\mathrm{U}(1)$ connections is an affine space modelled on $\Omega^1(M)$.

so that Seiberg-Witten map $\mathcal{SW}$ is

$$\mathcal{SW} : \mathrm{Conf}^{2,2} \to \mathcal{Y}^{1,2}.$$

The regularity decreases since $\mathcal{SW}$ is a first-order differential operator. Note both $\mathrm{Conf}^{2,2}$ and $\mathcal{Y}^{1,2}$ are Hilbert spaces, so if $\mathcal{SW}$ is smooth, we can use the implicit function theorem. We will often drop the superscripts and write Conf, $\mathcal{Y}$, etc., but we remark that we are always using these Sobolev completions.

Just as we want it the Seiberg-Witten map is a *smooth* map between Hilbert manifolds [Proposition 2.1.7 Nicoo] Similarly, we enlarge the gauge group $\mathcal{G}$ so that it becomes a Hilbert-Lie group [Nicoo, Proposition 2.1.8]. Write

$$\mathcal{G}^{3,2} = W^{3,2}(M, S^1).$$

Proposition 3.2.1 ($\mathcal{G}^{3,2}$ is a Hilbert-Lie group).
$\mathcal{G}^{3,2}$ is a Hilbert-Lie group, and its "Lie algebra" is

$$T_1\mathcal{G}^{3,2} = W^{3,2}(M, i\mathbb{R}).$$

Furthermore, the action of $\mathcal{G}^{3,2}$ on $\mathrm{Conf}^{3,2}$ is smooth.

The next thing that's in order is proving that, even though we have enlarged the configuration space and the gauge group, the moduli space *has not changed*. This is a prime example of the usefulness of elliptic regularity.

Proposition 3.2.2 (Moduli space is the same).
Let $\eta \in W^{1,2}(\Lambda^2 T^*M)$ be a perturbation and $(\Psi, A) \in \mathrm{Conf}^{3,2}$ be a solution to

$$\mathcal{SW}_\eta(\Psi, A) = 0.$$

Then there exists a gauge transformation $\gamma \in \mathcal{G}^{3,2}(M)$ such that $(\Psi \cdot \gamma, A \cdot \gamma)$ is smooth. *Consequently, there is a bijection between the moduli space of $W^{2,2}$ solutions*

$$\mathcal{M}^{2,2}(\eta) = \left\{(\Psi, A) \in \mathrm{Conf}^{2,2} \mid \mathcal{SW}_\eta(\Psi, A) = 0\right\} / \mathcal{G}^{3,2}(M)$$

and the moduli space $\mathcal{M}(\eta)$ of smooth solutions.

Proof (Sketch). — This proof is an excellent example of the use of elliptic regularity. Let $(\Psi, A) \in \mathrm{Conf}^{2,2}$ be a solution, with $A = A_0 + i\alpha$. By Hodge decomposition, we can write

$$\alpha = [\alpha] + \mathrm{d}f + \mathrm{d}^*\beta,$$

where $[\alpha]$ is the harmonic part, $f \in W^{3,2}(M)$ and $\beta \in W^{3,2}(\Lambda^2 T^*M)$. Consider the gauge transformation $\gamma = e^{-\frac{i}{2}f}$. Then

$$(\Psi, A) \cdot \gamma = (e^{-\frac{i}{2}f}\Psi, A_0 + i[\alpha] + i\mathrm{d}^*\beta).$$

Writing $a = [\alpha] + \mathrm{d}^*\beta$, we have that $\mathrm{d}^*a = 0$.

Since (Ψ, A) are a solution to the Seiberg-Witten equations, then so is $(\Psi, A) \cdot \gamma$. That is,

$$\not{\partial}_{A_0}\Psi - \frac{1}{2}(ia) \cdot \Psi = 0$$

$$F_{A_0}^{+} + i\mathrm{d}^{+}a + \eta = \sigma^{+}(\Psi).$$

We see then that Ψ is a solution to a homogeneous Dirac equation with respect to a *smooth* connection. Here is where the elliptic regularity comes into play. Since $a, \psi \in W^{2,2}$, then by the Sobolev embeddings we have that $a, \psi \in L^p$ for all $1 < p < \infty$. Therefore, the multiplication $ia \cdot \Psi$ is in L^p as well for all $1 < p < \infty$, and therefore $\partial\!\!\!/_{A_0}\Psi$ too. By elliptic regularity of the Dirac operator, since $\Psi \in W^{2,2}$ and $\Psi \in L^p$ for all $p > 1$ is the solution of an elliptic equation, then $\Psi \in W^{1,p}$ for all $p < \infty$.

From the second equation since $\mathrm{d}^* a = 0$, we have that $(d^+ + d^*)a + \eta \in W^{1,p}$ for all $p < \infty$. By Sobolev embedding, we find that $\eta \in W^{1,p}$ for all $p < \infty$, and thus $(d^+ + d^*)a$ as well. From elliptic regularity of $(d^+ + d^*)$, we obtain that $a \in W^{2,4}$, and so $ia \cdot \psi \in W^{1,p}$ for all $p < \infty$. We repeat this process to obtain that $\Psi \in W^{2,p}$, and so we obtain that Ψ and a are smooth. ■

Let $\mathcal{B}^{2,2}$ be the quotient space

$$\mathcal{B}^{2,2} = \mathrm{Conf}^{2,2}/\mathcal{G}^{3,2}(M),$$

We can show that it is Hausdorff with the quotient topology [Mor96, Section 4.5]. Furthermore, we can show that the action of the gauge group admits *local slices*; that is, for every point $(\Psi, A) \in \mathrm{Conf}$, there is a neighborhood U of (Ψ, A) and an embedded submanifold $S \subseteq U$ which "parameterizes" the orbits of $\mathcal{G}$ close to (Ψ, A)[Mor96, Section 4.5][Nic00, Section 2.2.2]. Intuitively, this shows that the quotient space $\mathcal{B}$ looks like S locally. For irreducible solutions, these local structures stitch together well to form a global smooth structure.

Theorem 3.2.3 (Manifold structure of quotient space of irreducible configurations).
Let $\mathcal{B}^* \subseteq \mathcal{B}$ be the open subset of gauge classes of irreducible *configurations. Then $\mathcal{B}^*$ is a Hilbert manifold.*

Now we want to show that $\mathcal{M}$ is also a smooth, finite-dimensional manifold. The strategy is showing that the zero set of $\mathcal{SW}_\eta$ is a smooth submanifold of Conf. If the action of $\mathcal{G}$ behaves well enough, $\mathcal{M}$ will be a smooth manifold and we can compute its dimension with the strategy outlined at the beginning of this chapter.

3.2.1 Dimension of $\mathcal{M}$

Fix a perturbation $\eta \in W^{1,2}(\Lambda^2 T^* M)$, and denote the zero set of $\mathcal{SW}_\eta$ as

$$\mathcal{Z}_\eta^{2,2} = \left\{(\Psi, A) \in \mathrm{Conf}^{2,2} \mid \mathcal{SW}_\eta(\Psi, A) = 0\right\}.$$

Then, if $\mathfrak{G}_{(\Psi,A)} : \mathcal{G} \to \mathrm{Conf}$ is the action $\mathfrak{G}_{(\Psi,A)}(\gamma) = (\Psi\cdot\gamma, A\cdot\gamma)$, necessarily we have that for all $(\Psi, A) \in \mathcal{Z}_\eta^{2,2}$, the composition $\mathcal{SW}_\eta \circ \mathfrak{G}_{(\Psi,A)}$ is exactly zero. Taking differentials, and denoting by $\mathfrak{g}_{(\Psi,A)} = T_1\mathfrak{G}_{(\Psi,A)}$, we have the Seiberg-Witten complex

$$0 \longrightarrow T_1\mathcal{G}(M) \xrightarrow{\mathfrak{g}_{(\Psi,A)}} T_{(\psi,A)}\mathrm{Conf}(M) \xrightarrow{T_{(\Psi,A)}\mathcal{SW}_\eta} T_{(0,0)}\mathcal{Y} \longrightarrow 0.$$

The homology of this complex is

$$\begin{aligned} \mathcal{H}_0 &= \ker \mathfrak{g}_{(\Psi,A)} \\ \mathcal{H}_1 &= \ker T_{(\Psi,A)}\mathcal{SW}_\eta / \operatorname{im} \mathfrak{g}_{(\Psi,A)} \cong T_{[\Psi,A]}\mathcal{M}(\eta) \\ \mathcal{H}_2 &= \operatorname{coker} T_{(\Psi,A)}\mathcal{SW}_\eta. \end{aligned}$$

We have that $\mathcal{H}_0$ is trivial if and only if the action of $\mathcal{G}$ is free on (Ψ, A), and therefore it its trivial if and only if (Ψ, A) is irreducible.

On the other side of the complex, $\mathcal{H}_2$ is trivial if and only if (Ψ, A) is a regular point of $\mathcal{SW}_\eta$. Therefore, if 0 is a regular value of $\mathcal{SW}_\eta$, then by the implicit function theorem, the set of *irreducible* solutions will form a smooth submanifold of Conf. Since the action of $\mathcal{G}$ is free on irreducible solutions, the quotient $\mathcal{Z}/\mathcal{G}$ will be a smooth submanifold of $\mathcal{B}^*$.

Therefore, we have two obstructions to having a good structure of the moduli space: First, the reducibility of some solutions, and second, the failure of 0 to be a regular value of $\mathcal{SW}_\eta$. We will see below that under some conditions on M, we can choose a *generic* perturbation η such that there are no reducible solutions and such that 0 is a regular value of $\mathcal{SW}_\eta$, and therefore the moduli space $\mathcal{M}(\eta)$ is *smooth*.

But for the rest of this section, let's assume that all is good. Let's assume that there exists a perturbation η for which all solutions are irreducible and 0 is a regular value. Therefore $\mathcal{H}_0 = 0$ and $\mathcal{H}_2 = 0$. Our main result is that $\mathcal{M}(\eta)$ is, surprisingly, finite-dimensional. Then we need to find explicit expressions for all the elements of the Seiberg-Witten complex. From Proposition 3.2.1, we know that $T_1\mathcal{G} \cong W^{3,2}(M, i\mathbb{R})$. We know that Conf is the product of a Hilbert space with an affine space modeled on $W^{2,2}(T^*M)$, and so its tangent space at any configuration (Ψ, A) is

$$T_{(\Psi,A)}\text{Conf} \cong W^{2,2}(S^+(M)) \oplus W^{2,2}(T^*M).$$

The target space $\mathcal{Y}$ is the product of two Hilbert spaces, so

$$T_{(0,0)}\mathcal{Y} \cong \mathcal{Y} = W^{1,2}(S^-(M)) \oplus W^{1,2}(i\Lambda^2_+ T^*M).$$

Let's find the differential of the Seiberg-Witten map. First, we vary the spinor. Consider, for some $t > 0$, a slight perturbation from a spinor Ψ. We have

$$\partial\!\!\!/_A(\Psi + t\varphi) = \partial\!\!\!/_A(\Psi) + t\partial\!\!\!/_A\varphi.$$

On the other hand, we have the endomorphism

$$(\Psi + t\varphi)^* \otimes (\Psi + t\varphi) = \Psi^* \otimes \Psi + t(\varphi^* \otimes \Psi + \Psi \otimes \varphi^*) + \mathcal{O}(t^2),$$

and thus,

$$\sigma^+(\Psi + t\varphi) = \sigma^+(\Psi) + t\mu_+^{-1}((\varphi^* \otimes \Psi + \Psi \otimes \varphi^*)_0) + \mathcal{O}(t^2).$$

Putting these results together, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{SW}_\eta(\Psi + t\Phi, A) = T_{(\Psi,A)}\mathcal{SW}_\eta(\Phi, 0) = (\partial\!\!\!/_A\Phi, \mu_+^{-1}((\varphi^* \otimes \Psi + \Psi \otimes \varphi^*)_0)).$$

Let's see what happens when we vary the connection. For a one-form $i\alpha \in \Omega^1(M, \mathbb{R})$, write $A' = A + it\pi^*\alpha$. Then

$$F_{A'} = F_A + it\mathrm{d}\alpha.$$

The Dirac operator changes as as

$$\partial\!\!\!/_{A'}\Psi = \sum_i e_i \cdot \nabla_{A,e_i} \cdot \Psi = \sum_i e_i \cdot (\nabla_{A,e_i}\Psi + \frac{i}{2}\alpha(e_i)\Psi) := \partial\!\!\!/_A\Psi + \frac{1}{2}i\alpha \cdot \Psi.$$

Putting these two results together, we have

Lemma 3.2.4 (Differential of Seiberg-Witten map).
The differential of the Seiberg-Witten map $\mathcal{SW}_\eta$ is

$$T_{(\Psi,A)}\mathcal{SW}_\eta(\varphi, i\alpha) = (\not{\partial}_A\varphi + \frac{1}{2}i\alpha\cdot\Psi, \mu_+^{-1}((\varphi^*\otimes\Psi + \Psi\otimes\varphi^*)_0) + \mathrm{d}^+(i\alpha)).$$

In a little bit more legible matrix form:

$$T_{(\Psi,A)}\mathcal{SW}_\eta\begin{pmatrix}\varphi\\ i\alpha\end{pmatrix} = \begin{pmatrix}\not{\partial}_A & \frac{1}{2}\bullet\cdot\Psi\\ \mu_+^{-1}((\bullet^*\otimes\Psi+\Psi\otimes\bullet^*)_0) & \mathrm{d}^+\end{pmatrix}\begin{pmatrix}\varphi\\ i\alpha\end{pmatrix}.$$

Finally, let's see what the differential of the action of $\mathcal{G}$ on Conf is. Let $\gamma_t = e^{itf} \in \mathcal{G}$. Then, for a configuration (Ψ, A), we have

$$\begin{aligned} T_1\mathfrak{G}(if) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\Psi\cdot\gamma_t, A\cdot\gamma_t) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(e^{-itf}\Psi, A + 2e^{-itf}\,\mathrm{d}e^{itf}) \\ &= (-if\Psi, 2i\,\mathrm{d}f) \end{aligned}$$

We have proved:

Lemma 3.2.5 (Infinitesimal action of gauge group).
For every configuration (Ψ, A), the infinitesimal action $\mathfrak{g}_{(\Psi,A)} = T_1\mathfrak{G}_{(\Psi,A)}$ of the gauge group is given by

$$\mathfrak{g}_{(\Psi,A)}(if) = (-if\Psi, 2i\,\mathrm{d}f).$$

The strategy to computing the dimension of the tangent space is to "fold" the Seiberg-Witten complex in half and present the quotient $\ker T_{(\Psi,A)}\mathcal{SW}_\eta/\operatorname{im}(\mathfrak{g}_{(\Psi,A)})$ as the kernel of a *surjective Fredholm*[3] map whose index is known. For it, we will use the L^2-adjoint of the infinitesimal action.

Lemma 3.2.6 (Adjoint of infinitesimal action).
Given a configuration $(\Psi, A) \in$ Conf, the L^2-adjoint of the infinitesimal action is the map $\mathfrak{g}^*_{(\Psi,A)}$: $T_{(\Psi,A)}$Conf $\to T_1\mathcal{G}$ given by

$$\mathfrak{g}^*_{(\Psi,A)}(\varphi, i\alpha) = -i\Im\langle\Psi,\varphi\rangle + 2\mathrm{d}^*(i\alpha).$$

Proof. — We will drop the subindex (Ψ, A). Note that $T_1\mathcal{G} \cong W^{3,2}(M, i\mathbb{R})$ is a *real* vector space, and so we must take the adjoint with respect to the *real* part of the inner product on $T_{(\Psi,A)}\mathrm{Conf} = W^{2,2}(S^+) \oplus W^{2,2}(iT^*M)$:

$$\begin{aligned} \langle\mathfrak{g}(if), (\varphi, i\alpha)\rangle &= \int_M \Re\langle -if\Psi, \varphi\rangle + \langle 2i\,\mathrm{d}f, i\alpha\rangle\,\mathrm{vol} \\ &= \int_M f\Im\langle\Psi,\varphi\rangle + \langle 2if, \mathrm{d}^*(i\alpha)\rangle\,\mathrm{vol} \end{aligned}$$

[3] Recall that a bounded operator between Banach spaces is *Fredholm* if its kernel and algebraic cokernel are finite dimensional [see Con07, Section XI.2]. Its index is the difference between the dimension of its kernel and the dimension of its cokernel.

$$= \int_M if(-i\Im\langle\Psi,\varphi\rangle + 2\mathrm{d}^*(i\alpha))\mathrm{vol}$$

Comparing with

$$\langle if, \mathfrak{g}(\varphi, i\alpha)\rangle = \int_M if\mathfrak{g}^*(\varphi, i\alpha)\mathrm{vol},$$

we obtain the result. ■

Now we proceed with "folding" the Seiberg-Witten complex. Define a map $\mathcal{D}_{(\Psi,A)} : T_{(\Psi,A)}\mathrm{Conf} \to T_{(0,0)}\mathcal{Y} \oplus T_1\mathcal{G}$ as

$$\begin{aligned}\mathcal{D}_{(\Psi,A)}(\varphi, i\alpha) &= (T_{(\Psi,A)}\mathcal{SW}_\eta, \mathfrak{g}^*_{(\Psi,A)})(\varphi, i\alpha)\\ &= (\partial\!\!\!/_A\varphi + \frac{i}{2}\alpha\cdot\Psi, \mathrm{d}^+(i\alpha) - \frac{1}{2}\mu^+(\Psi^*\otimes\varphi), 2\mathrm{d}^*(i\alpha) - i\Im\langle\Psi,\varphi\rangle).\end{aligned}$$

Then we have that

$$\ker\mathcal{D}_{(\Psi,A)} = \ker T_{(\Psi,A)}\mathcal{SW}_\eta \cap \ker\mathfrak{g}^*_{(\Psi,A)} = \ker T_{(\Psi,A)}\mathcal{SW}_\eta \cap (\mathrm{im}\,\mathfrak{g}_{(\Psi,A)})^\perp \cong {}^{\ker T_{(\Psi,A)}\mathcal{SW}_\eta}\!/_{\mathrm{im}\,\mathfrak{g}_{(\Psi,A)}}.$$

This is the operator we are looking for. Let's see that it is Fredholm and find its index.

Proposition 3.2.7 (Fredholm index of $\mathcal{D}_{(\Psi,A)}$).
The map $\mathcal{D}_{(\Psi,A)}$ is Fredholm, and its real *index is*

$$\mathrm{ind}(\mathcal{D}_{(\Psi,A)}) = \dim_{\mathbb{R}}\ker\mathcal{D}_{(\Psi,A)} - \dim_{\mathbb{R}}\mathrm{coker}\,\mathcal{D}_{(\Psi,A)} = \frac{1}{4}(c^2 - 2\chi(M) - 3\,\mathrm{sign}(M)),$$

where $c^2 = \int_M c_1(L(M)) \wedge c_1(L(M))$.

Proof. — Let A_0 be the smooth reference connection, so that $A = A_0 + \pi^*(ia)$. Then

$$\mathcal{D}_{(0,A_0)}(\varphi, i\alpha) = (\partial\!\!\!/_{A_0}\varphi, \mathrm{d}^+(i\alpha), 2\mathrm{d}^*(i\alpha)),$$

and so $\mathcal{D}_{(0,A_0)} = \partial\!\!\!/_{A_0} \oplus (\mathrm{d}^+, 2\mathrm{d}^*)$ is the sum of elliptic operators, and thus it is Fredholm. Furthermore, we have that

$$\mathcal{D}_{(\Psi,A)} = \mathcal{D}_{(0,A_0)} + K,$$

where

$$K(\varphi, i\alpha) = (\frac{1}{2}ia\cdot\varphi - \frac{1}{2}i\alpha\cdot\Psi, -\frac{1}{2}\mu_+^{-1}(\Psi^*\otimes\varphi + \varphi^*\otimes\Psi)_0, -i\Im\langle\Psi,\varphi\rangle)$$

is a $C^\infty(M)$-linear operator (i.e. a zeroth-order differential operator), which is compact. Thus, $\mathcal{D}_{(\Psi,A)}$ is also Fredholm and it has the same index as $\mathcal{D}_{(0,A_0)}$.

Let's compute the index of $\mathcal{D}_{(0,A_0)}$ then. Write $\mathfrak{d} : W^{2,2}(iT^*M) \to W^{1,2}(i\Lambda^2_+T^*M) \oplus W^{3,2}(M, i\mathbb{R})$ as $\mathfrak{d}(i\alpha) = (\mathrm{d}^+(i\alpha), 2\mathrm{d}^*(i\alpha))$. Suppose that $\alpha \in \ker(\mathfrak{d})$, i.e., $\mathrm{d}^+\alpha = 0$ and $\mathrm{d}^*\alpha = 0$. Since $\mathrm{d}^*\mathrm{d}^+ = \frac{1}{2}\mathrm{d}^*\mathrm{d}$, then $\mathrm{d}^*\mathrm{d}\alpha = 0$, and thus

$$\langle\mathrm{d}\alpha, \mathrm{d}\alpha\rangle = \langle\alpha, \mathrm{d}^*\mathrm{d}\alpha\rangle = 0,$$

which implies that $\mathrm{d}\alpha = 0$. Conversely, if $\mathrm{d}\alpha = 0$ then $\mathrm{d}^+\alpha = 0$, and so $\ker\mathrm{d}^+ = \ker\mathrm{d}$. With this we conclude that

$$\ker\mathfrak{d} = \ker\mathrm{d} \cap \ker\mathrm{d}^* \cong H^1(M, \mathbb{R})$$

is precisely the space of harmonic 1-forms.

Now let's find coker $\mathfrak{d} \cong \ker \mathfrak{d}^*$. We have that

$$\mathfrak{d}^*(\beta, if) = \mathrm{d}^*\beta + i\,\mathrm{d}f\,.$$

These two terms are orthogonal, since $\langle \mathrm{d}^*\beta, \mathrm{d}f\rangle = \langle \beta, \mathrm{d}^2 f\rangle = 0$. Therefore $(\beta, if) \in \ker \mathfrak{d}^*$ if and only if $\beta \in \ker \mathrm{d}^*$ and $f \in \ker \mathrm{d}f$. Thus

$$\ker \mathfrak{d}^* = H^0(M,\mathbb{R}) \oplus \ker(\mathrm{d}^*).$$

However, since on 2-forms, $\mathrm{d}^* = -\star\mathrm{d}\star$, if β is self-dual then $\mathrm{d}^*\beta = -\star\mathrm{d}\beta$, and thus $\mathrm{d}^*\beta = 0$ if and only if $\mathrm{d}\beta = 0$. Thus, *on self-dual forms*,

$$\ker \mathrm{d}^* = \ker \mathrm{d} = \ker \mathrm{d}^* \cap \ker \mathrm{d} \cong H^2_+(M,\mathbb{R}).$$

With this we see that

$$\ker \mathfrak{d}^* \cong H^0(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R}).$$

Therefore, since we've assumed that M is connected, $\dim H^0(M,\mathbb{R}) = 1$ and so

$$\operatorname{ind} \mathfrak{d} = b^1 - b^0 - b^2_+ = b^1 - 1 - b^2_+.$$

Finally, from the Atiyah-Singer index theorem[Mor96, p. 47], we can deduce that for any connection A,

$$\operatorname{ind}_{\mathbb{R}} \partial\!\!\!/_A = \frac{1}{4}\left(\int_M c_1(L)\wedge c_1(L) - \operatorname{sign}(M)\right) := \frac{1}{4}(c^2 - \operatorname{sign}(M)),$$

where L is the determinant bundle of the Spin^c-structure and we've written $c^2 = \int_M c_1(L)\wedge c_1(L)$. This, together with the index of $\mathfrak{d}$, tells us that

$$\operatorname{ind}_{\mathbb{R}}(\mathcal{D}_{(\Psi,A)}) = \operatorname{ind}_{\mathbb{R}}(\mathcal{D}_{(0,A_0)}) = \frac{1}{4}(c^2 - 2\chi(M) - 3\operatorname{sign}(M)). \qquad \blacksquare$$

Now we have all the tools to find the dimension of the moduli space, assuming that there are no reducible points and that 0 is a regular value of $\mathcal{SW}_\eta$.

Theorem 3.2.8 (Dimension of the moduli space).
Let $\eta \in W^{1,2}(i\Lambda^2_+ T^*M)$ be a perturbation such that the Seiberg-Witten map $\mathcal{SW}_\eta$ has no reducible solutions and 0 *is a regular value. Then the moduli space $\mathcal{M}(\eta)$ is a smooth manifold of dimension*

$$\dim \mathcal{M}(\eta) = \frac{1}{4}(c^2 - 2\chi(M) - 3\operatorname{sign}(M)),$$

where $c^2 = \int_M c_1(L)\wedge c_1(L)$ is the integral of the squared Chern class of the determinant line bundle L of the Spin^c*-structure.*

Note that this depends only on M and the Spin^c*-structure (which possibly depends on the metric on M).*

Proof. — As we have shown in Proposition 3.2.7, for all configurations (Ψ, A), the map $\mathcal{D}_{(\Psi,A)} = (\partial\!\!\!/_A, \mathfrak{g}^*_{(\Psi,A)})$ is Fredholm with index $\frac{1}{4}(c^2 - 2\chi(M) - 3\operatorname{sign}(M))$. Furthermore, we have that

$$\ker \mathcal{D}_{(\Psi,A)} \cong T_{[\Psi,A]}\mathcal{M}(\eta).$$

Since all solutions (Ψ, A) are irreducible, then the infitesimal action $\mathfrak{g}_{(\Psi,A)}$ is injective, and thus its adjoint $\mathfrak{g}^*_{(\Psi,A)}$ is surjective. Furthermore, since 0 is a regular value of $\mathcal{SW}_\eta$, then its differential $T_{(\Psi,A)}\mathcal{SW}_\eta$ is surjective. Therefore $\mathcal{D}_{(\Psi,A)}$ is surjective and so coker $\mathcal{D}_{(\Psi,A)} = 0$. With this, we have

$$\dim T_{[\Psi,A]}\mathcal{M}(\eta) = \dim\ker \mathcal{D}_{(\Psi,A)} = \operatorname{ind} \mathcal{D}_{(\Psi,A)} = \frac{1}{4}(c^2 - 2\chi(M) - 3\operatorname{sign}(M)).$$

■

3.2.2 Generic smoothness

Now the question remains: How likely are we to find a perturbation η such that there are no reducible solutions and all points are smooth? Spoiler: very.

First, let's see what conditions are necessary to be able to find a perturbation for which all solutions are irreducible. Suppose that $\eta \in W^{1,2}(i\Lambda^2_+ T^*M)$ is a perturbation that admits a *reducible* solution $(0, A)$. Then

$$F_A^+ + \eta = 0$$

For any differential form ω, we write $[\omega]$ for its harmonic part in its Hodge decomposition. Then necessarily,

$$[F_A^+] = -[\eta].$$

Since F_A is a representative of ($-2\pi i$ times)the Chern class of the determinant bundle $L(M)$, we have

$$2\pi i[c_1(L)] = [\eta].$$

The converse statement is also true.

Proposition 3.2.9 (Perturbations that admit reducible solutions).
A perturbation η admits reducible solutions if and only if its harmonic part satisfies

$$2\pi i[c_1(L)] = [\eta].$$

Proof. — We've already proved above that if η admits reducible solutions, then $2\pi i c_1(L)_h = \eta_h$. Conversely, assume that $2\pi i c_1(L)_h = \eta_h$. Given any connection A, we have that[4] $[F_A] = -2\pi i[c_1(L)]$, and so

$$[F_A^+] = -[\eta].$$

If we can extend this equality of harmonic parts to an equality of self-dual parts, we're done. Write $\eta = [\eta] + \mathrm{d}\alpha$. For any given connection A, we have that $F_A = [F_A] + \mathrm{d}\beta$ for some β. If we let A be such that $F_A = [F_A] - \mathrm{d}\alpha$, then necessarily

$$F_A^+ = -\eta,$$

and thus $(0, A)$ is a reducible solution. ■

Corollary 3.2.10 (Condition for perturbations without reducible solutions).
If $b^2_+ = 0$, then there all *closed perturbations have reducible solutions. If $b^2_+ > 0$, then there exist closed perturbations which do not have reducible solutions.*

[4] this is from Hodge theory and the isomorphism of cohomology with harmonic forms. Two forms have the same harmonic part if and only if they are in the same cohomology class.

Alright, so the question of the existence of reducible solutions is settled. What about smoothness? The main tool in our arsenal is Sard's theorem, or rather, Smale's infinite-dimensional version of it[Nicoo, Section 1.5.2]:

Theorem 3.2.11 (Sard-Smale theorem (with parameters)).
Let X, Λ, Y be Banach manifolds and $F : X \times \Lambda \to Y$ a smooth map. For each $\lambda \in \Lambda$, write $F_\lambda : X \to Y$ as $F_\lambda(x) = F(x, \lambda)$. If $y_0 \in Y$ is a regular value of F, then the set of parameters $\lambda \in \Lambda$ for which y_0 is a regular value of F_λ is dense in Λ.

This is to say, for "generic" λ, the map F_λ has y_0 as a regular value and thus $F_\lambda^{-1}(y_0)$ is a smooth submanifold of X.

The part of the theorem that says that $F_\lambda^{-1}(y_0)$ is a smooth submanifold can be copied *almost verbatim* from the implicit function theorem for finite-dimensional manifolds.

In lieu of this theorem, we should consider the map

$$\begin{aligned} SW : \mathrm{Conf}^{2,2} \times W^{1,2}(i\Lambda^2_+ T^*M) &\to \mathcal{Y}^{1,2} \\ (\Psi, A, \eta) &\mapsto \mathcal{SW}_\eta(\Psi, A). \end{aligned}$$

If 0 is a regular value of SW, then we're done. Let's see that this is the case.

Note that SW is linear in the perturbation. Therefore, we can easily find its differential in terms of the differential of $\mathcal{SW}_\eta$:

$$T_{(\Psi,A,\eta)} SW(\varphi, i\alpha, \xi) = (\partial\!\!\!/_A \varphi + \frac{1}{2} i\alpha \cdot \Psi, \mu_+^{-1}((\varphi^* \otimes \Psi + \Psi^* \otimes \varphi)_0) + \mathrm{d}^+(i\alpha) + \xi).$$

From this, we see that the differential is surjective on the second factor, because of that clean ξ term. Let's see that it is surjective in the first one: define $G : W^{2,2}(S^+(M)) \oplus W^{2,2}(T^*M) \to W^{1,2}(S^-(M))$ as

$$G(\varphi, i\alpha) = \partial\!\!\!/_A \varphi + \frac{1}{2} i\alpha \cdot \Psi.$$

If we prove that G is surjective as well, then we are done. We argue by contradiction: assume that $\psi \in W^{1,2}(S^-(M))$ is in the orthogonal complement of the image of G. Then in particular, for all φ, we have

$$\langle \partial\!\!\!/_A \varphi, \psi \rangle = \langle \varphi, \partial\!\!\!/_A \psi \rangle = 0.$$

This is to say that ψ satisfies the Dirac equation $\partial\!\!\!/_A \psi \overset{wk}{=} 0$ in the weak sense. By elliptic regularity, it satisfies it in the strong sense as well, so $\partial\!\!\!/_A \psi = 0$. Choose a point $x_0 \in M$ and a neighborhood U of x_0 such that neither Ψ nor ψ have zeros in U. Make U small enough so that there is a coordinate chart defined on U *and* so that the spinor bundles $S^\pm(M)$ are trivialized.

Note that, given the expression of the spin representation, we have that the composition

$$\mathbb{R}^4 \otimes \mathbb{C} \hookrightarrow \mathrm{Cl}_\mathbb{C}(4) \to \mathrm{Hom}_\mathbb{C}(S^+, S^-)$$

is a linear injective map, and by dimension counting, it is an isomorphism (since $\dim_\mathbb{C} S^\pm = 2$.) This implies that given two spinors $\psi_+ \in S^+$ and $\psi_- \in S^-$, we can find an element $a \in \mathbb{R}^4 \otimes \mathbb{C}$ such that

$$ia \cdot \psi_+ = \psi_-.$$

Locally, in terms of the trivialization, this means that for all $x \in U$, we can find a tangent vector $a_x \in T_x U$ such that

$$ia_x \cdot \Psi(x) = \psi(x).$$

Then the assignment $x \mapsto a_x$ is a vector field a on U. If we multiply it with a bump, we can construct a global vector field vanishing outside of U such that

$$ia_x \cdot \Psi(x) = \psi(x).$$

Let $a_\flat$ be the 1-form dual to a under the isomorphism induced by the metric. Then $a_\flat$ is a one-form supported in U and

$$\int_M \langle ia_\flat \cdot \Psi, \varphi \rangle \,\mathrm{vol} = \int_U \langle ia_\flat \cdot \Psi, \varphi \rangle \,\mathrm{vol} = \int_U |\varphi|^2 \mathrm{vol} = 0.$$

This implies that $\varphi = 0$ on U, and thus, by unique continuation of Dirac spinors [see Sal99, Theorem E.8], we conclude that $\varphi = 0$. With this we conclude that $T_{(\Psi,A,\eta)}SW$ is indeed surjective. By the Sard-Smale theorem, we conclude the following:

Theorem 3.2.12 (Generic smoothness).
For a generic *perturbation* $\eta \in W^{1,2}(i\Lambda^2_+ T^*M)$*, the moduli space of irreducible solutions* $\mathcal{M}^*(\eta)$ *is a smooth manifold. If* $b^2_+ > 0$*, then we can furthermore choose* η *such that* $\mathcal{M}(\eta) = \mathcal{M}^*(\eta)$*.*

3.2.3 Orientation

Recall that we described the tangent space to the moduli space at a point $[\Psi, A]$ in terms of the map $\mathcal{D}_{\Psi,A} : T_{(\Psi,A)}\mathrm{Conf} \to T_{(0,0)}\mathcal{Y} \oplus T_1\mathcal{G}$, given by

$$\mathcal{D}_{\Psi,A}(\varphi, i\alpha) = (\partial\!\!\!/_A \varphi + \frac{i}{2}\alpha \cdot \Psi, \mathrm{d}^+(i\alpha) - \frac{1}{2}\mu^+(\Psi^* \otimes \varphi), -2\mathrm{d}^*(i\alpha) - i\Im\langle \Psi, \varphi \rangle).$$

Specifically, we proved that

$$\ker \mathcal{D}_{(\Psi,A)} \cong T_{[\Psi,A]}\mathcal{M}(\eta).$$

The collection of all operators $\mathcal{D}_{(\Psi,A)}$ is a smooth family of Fredholm operators, parameterized by the configuration space Conf. Then there is an associated line bundle over Conf, called the *determinant* of the family $\mathcal{D}$, whose fiber over (Ψ, A) is

$$\det(\mathcal{D})_{(\Psi,A)} = \Lambda^{\mathrm{top}}(\ker \mathcal{D}_{(\Psi,A)}) \otimes \Lambda^{\mathrm{top}}(\mathrm{coker}\, \mathcal{D}_{(\Psi,A)})^*,$$

see [Nic07, Section 1.5.1] and [Section 5.2.1 DK97]. Assuming that we've chosen a perturbation so that all solution to the perturbed equations are smooth, then the restriction of the bundle $\det(\mathcal{D})$ to the set of solutions is

$$\det(\mathcal{D})_{(\Psi,A)} = \Lambda^{\mathrm{top}}(\ker \mathcal{D}_{(\Psi,A)}) = \det\big(T_{[\Psi,A]}\mathcal{M}(\eta)\big) \quad \text{assuming } (\Psi, A) \text{ is a solution.}$$

Clearly, we see that the resulting bundle $\det(\mathcal{D})|_{\mathcal{Z}}$ is constant along the orbits of the action of the gauge group, so it descends to precisely the determinant bundle of $\mathcal{M}(\eta)$. This is to say that it suffices to find an orientation of the determinant bundle $\det(\mathcal{D})$ to find an orientation of $\mathcal{M}(\eta)$.

To find such and orientation, we note that if there are two homotopic families of Fredholm operators parameterized by a smooth manifold have *isomorphic* determinant bundles Therefore, consider the curve $t \mapsto \mathcal{D}^t_{(\Psi,A)}$, given as

$$\mathcal{D}^t_{(\Psi,A)}(\varphi, i\alpha) = (\partial\!\!\!/_A \varphi + t\frac{i}{2}\alpha \cdot \Psi, \mathrm{d}^+(i\alpha) + t\mu_+^{-1}(\Psi^* \otimes \varphi + \varphi^* \otimes \Psi)_0, -2\mathrm{d}^*(i\alpha) - ti\Im\langle \Psi, \varphi \rangle).$$

Then $\mathcal{D}^1_{(\Psi,A)} = \mathcal{D}_{(\Psi,A)}$, and

$$\mathcal{D}^0_{(\Psi,A)}(\varphi, i\alpha) = (\partial\!\!\!/_A\varphi, \mathrm{d}^+(i\alpha), -2\mathrm{d}^*(i\alpha)) = \mathcal{D}_{(0,A)}(\varphi, i\alpha).$$

This is precisely the simplified map that we used in Proposition 3.2.7. For it, we proved that

$$\begin{aligned} \ker \mathcal{D}_{(0,A)} &= \ker \partial\!\!\!/_A \oplus H^1(M,\mathbb{R}) \\ \operatorname{coker} \mathcal{D}_{(0,A)} &= \operatorname{coker} \partial\!\!\!/_A \oplus H^1(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R}). \end{aligned}$$

Since $\ker \partial\!\!\!/_A$ and $\operatorname{coker} \partial\!\!\!/_A$ are complex vector spaces, they have canonical orientations. Similarly, since M is connected, then $H^0(M,\mathbb{R}) \cong \mathbb{R}$ has a natural orientation. Therefore, it suffices to choose an orientation of $H^1(M,\mathbb{R})$ and $H^2_+(M,\mathbb{R})$ to find an orientation of $\det(\mathcal{D}_{(0,A)}) = \det\big(\mathcal{D}^0_{(\Psi,A)}\big)$. Note that such an orientation is made globally. If the action of $\mathcal{G}$ is orientation-preserving, then this orientation descends to an orientation of $\mathcal{M}(\eta)$.

Then fix $\gamma \in \mathcal{G}$, and consider the action $\mathrm{Conf} \to \mathrm{Conf}$, $(\Psi, A) \mapsto (\Psi\cdot\gamma, A\cdot\gamma)$. Recall that $\Psi\cdot\gamma = \gamma^{-1}\Psi$ and $A\cdot\gamma = \Phi^*_\gamma A$, where $\Phi_\gamma : \mathrm{Spin}^c(M) \to \mathrm{Spin}^c(M)$ is the bundle morphism $\Phi_\gamma(p) = p\cdot\delta(\pi(p))$. This action is $\mathbb{C}$-linear on spinors. Furthermore since the space of connections is an affine space modelled on $\Omega^1(M, i\mathbb{R})$ and Φ_γ is a bundle morphism that lifts the identity, then for any $i\alpha \in \Omega^1(M, i\mathbb{R})$ we have that $\Phi^*_\gamma(A + \pi^*(i\alpha)) = \Phi^*_\gamma + \pi^*(i\alpha)$. This is to say that the differential of the action $(\Psi, A) \mapsto (\Psi\cdot\gamma, A\cdot\gamma)$ is

$$(\varphi, i\alpha) \mapsto (\gamma^{-1}\varphi, i\alpha).$$

The induced isomorphisms $\ker \mathcal{D}_{(\Psi,A)} \to \ker \mathcal{D}_{(\Psi\cdot\gamma,A\cdot\gamma)}$ and $\operatorname{coker} \mathcal{D}_{(\Psi,A)} \to \operatorname{coker} \mathcal{D}_{(\Psi\cdot\gamma,A\cdot\gamma)}$ are orientation-preserving, so indeed the orientation of the determinant bundle descends to the quotient.

We have proved, then:

Proposition 3.2.13 (Orientation of the moduli space).
The moduli space $\mathcal{M}(\eta)$ is orientable, and an orientation is obtained by the choice of an orientation on $H^1(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R})$.

3.2.4 Compactness

Arguably, the most remarkable feature of the Seiberg-Witten moduli space is that it is *compact*. This is the greatest contrast with Donaldson theory, where the moduli space of self-dual forms is not compact, and finding an appropriate compactification requires a lot of work.

Our main idea is showing that $\mathcal{M}(\eta)$ is sequentially compact. Since $\mathcal{M}(\eta)$ is a submanifold of a Hilbert manifold, then sequential compactness implies compactness. There is a key *a priori* estimate on the norm of a solution that is given by the Weitzenböck formula.

Proposition 3.2.14 (Curvature bound on Dirac spinors).
Let $(\Psi, A) \in \mathrm{Conf}$ and $\eta \in W^{1,2}(i\Lambda^2_+T^*M)$ be a perturbation. Then for all $x \in M$:

$$|\Psi(x)|^2 \le \frac{1}{2}\max(0, \sup(-s) + 4\|\eta\|_\infty).$$

Proof. — Without loss of generality, assume that the solution is smooth. Since $\partial\!\!\!/_A\Psi = 0$, then naturally $\partial\!\!\!/_A\partial\!\!\!/_A\Psi = 0$, and by the Weitzenböck formula, we have

$$\nabla^*_A\nabla_A\psi + \frac{s}{4}\Psi + \frac{1}{2}|\Psi|^2\Psi - \mu_+(\eta)(\Psi) = 0.$$

Here we used the fact that $\mu_+(F_A^+) = (\Psi \otimes \Psi^*)_0 - \mu_+(\eta)$. Taking the pointwise inner product with Ψ, we obtain

$$\left\langle \nabla_A^* \nabla_A \psi, \Psi \right\rangle + \frac{s}{4}|\Psi|^2 + \frac{1}{2}|\Psi|^4 - \langle \mu_+(\eta)(\Psi), \Psi \rangle = 0.$$

Since the connection is compatible with the metric on the spinor fields, we have

$$\sum_i L_{e_i} L_{e_i} \langle \Psi(x), \Psi(x) \rangle = \sum_i \left\langle \nabla_{A,e_i} \nabla_{A,e_i} \Psi(x), \Psi(x) \right\rangle + 2 \sum_i \left\langle \nabla_{A,e_i} \Psi(x), \nabla_{A,e_i} \Psi(x) \right\rangle + \sum_i \left\langle \Psi(x), \nabla_{A,e_i} \nabla_{A,e_i} \Psi(x) \right\rangle,$$

which implies that

$$\Delta(|\Psi|^2) + 2 \sum_i |\nabla_{A,e_i} \Psi|^2 = 2\Re \sum_i \left\langle \nabla_{A,e_i} \nabla_{A,e_i} \Psi, \Psi \right\rangle.$$

Now we use the fact that $\nabla_A^* = - \star \nabla_A \star$ on 1-forms[5] to see that

$$\begin{aligned} \nabla_A^* \nabla_A \Psi &= - \star \nabla_A \star \left(\sum_i \nabla_{A,e_i} \Psi \, \mathrm{d}e_i \right) \\ &= - \star \sum_i \nabla_{A,e_i} \nabla_{A,e_i} \Psi \mathrm{vol} \\ &= - \sum_i \nabla_{A,e_i} \nabla_{A,e_i} \Psi. \end{aligned}$$

Therefore

$$\Delta(|\Psi|^2) + 2 \sum_i |\nabla_{A,e_i} \Psi|^2 = 2\Re \left\langle \nabla_A^* \nabla_A \Psi, \Psi \right\rangle.$$

From the Weitzenböck formula, we see that if $\partial\!\!\!/_A \Psi = 0$, then $\left\langle \nabla_A^* \nabla_A \Psi, \Psi \right\rangle$ is real, and thus necessarily $\langle \mu_+(\eta)(\Psi), \Psi \rangle$ is real as well. Then

$$\begin{aligned} \Delta(|\Psi|^2) &\leq 2 \left\langle \nabla_A^* \nabla_A \Psi, \Psi \right\rangle \\ &= -\frac{s}{2}|\Psi|^2 - |\Psi|^4 + 2 \langle \mu_+(\eta)(\Psi), \Psi \rangle \\ &\leq -\frac{s}{2}|\Psi|^2 - |\Psi|^4 + 4\|\eta\|_\infty |\Psi|^2. \end{aligned}$$

If x is a point where $|\Psi(x)|^2$ achieves its maximum, then $\Delta(|\Psi(x)|^2) = 0$ and so, assuming that Ψ is not identically zero,

$$|\Psi|^2 \leq -\frac{s}{2} + 2\|\eta\|_\infty |\Psi|^2.$$

This proves the proposition. ■

As an immediate consequence, we have that for $\eta = 0$ (or small enough), if M has a metric with *positive* scalar curvature, then the only possible solutions of the Seiberg-Witten equations have $\Psi \equiv 0$.

The next step is refining the gauge-fixing that we used in the proof of the regularity of solutions.

[5] This can be checked directly, or proved by using an explicit expression of d^*.

Lemma 3.2.15 (Gauge-fixing lemma).
Let $A_0 \in \mathrm{Conn}(U(L))$ be a smooth reference connection. There is a constant C, which depends only possibly on the metric, such that for $A = A_0 + ia \in \mathrm{Conn}^{k,2}(\mathrm{U}(L))$, there is a gauge transformation $\gamma \in \mathcal{G}^{k+1,2}$ such that $A \cdot gamma = A_0 + ia'$ satisfies

$$\mathrm{d}^* a' = 0$$
$$\|[a']\|_2 \leq C,$$

where $[a']$ is the harmonic part of a'.

Theorem 3.2.16 (Moduli space is compact).
Let $(\Psi_n, A_n) \in Conf^{2,2}$ be a sequence of solutions of the perturbed Seiberg-Witten equations, with perturbation $\eta \in W^{4,2}(i\Lambda^2_+ T^*M)$. Then there is a sequence of gauge transformations $\gamma_n \in \mathcal{G}^{3,2}$ such that $(\Psi_n \cdot \gamma_n, A_n \cdot \gamma_n)$ has a convergent subsequence.

Proof (Sketch). — The proof is quite laborious (and that of the previous lemma), so we will only sketch it. It can be found in all its splendor in [Nic00], [Sal99], and [Mor96].

The key idea is to use the *a priori* bounds to "start" the convergence. Since we have a priori bounds on Ψ, from the monopole equation we find a bound on A. By Sobolev embedding, this implies that there are convergent subsequences but with lower regularity. We then "restore" the regularity using gauge transformations and elliptic regularity.

Without loss of generality, from the previous lemma we can assume that $A_n = A_0 + ia_n$, with $\mathrm{d}^* a_n = 0$. Then the curvature bound, together with the monopole equation $F^+_{A_n} + \eta = \sigma^+(\Psi_n)$, gives us a bound

$$\|(\mathrm{d}^+ + \mathrm{d}^*) a_n\|_\infty \leq C.$$

But $(\mathrm{d}^+ + \mathrm{d}^*)$ is an elliptic operator, and therefore, we have a bound on $\|a_n - [a_n]\|_{1,p}$ for all $p < \infty$. However, since $\|[a'_n]\|_2$ is bounded, we obtain that $\|a_n\|_{1,p}$ is bounded for all $p < \infty$. Then by Sobolev multiplication, we can find a sup-bound on $\|ia_n \cdot \Psi_n\|_\infty$. And now, since

$$\partial\!\!\!/_{A_n} \Psi = \partial\!\!\!/_{A_0} \Psi + \frac{1}{2} ia_n \cdot \Psi_n = 0$$

we can use the elliptic regularity of $\partial\!\!\!/_{A_0}$ to obtain that $\|\Psi_n\|_{1,p}$ is bounded for all $p < \infty$. Since we had already an $W^{1,p}$ bound on a_n, we have that $\|ia_n \cdot \Psi_n\|_{1,p}$ is bounded too. Therefore, again by elliptic regularity of $\partial\!\!\!/_{A_0}$, we have that $\|Psi_n\|_{2,p}$ is bounded for all $p < \infty$. Here we found a better bound with boosted regularity of Ψ_n. Repeating this same argument again, we also find that $\|a_n\|_{2,p}$ is bounded for all $p < \infty$. Then by the Sobolev embedding theorem, there is a subsequence that converges in weakly $W^{1,p}$. If we repeat this process, we can find that there is a subsequence that converges strongly in $W^{2,p}$. ■

3.3 The Seiberg-Witten invariant

Even though we will not use it in the proof of Donalson's theorem, we will briefly discuss the Seiberg-Witten invariants, since no discussion of Seiberg-Witten is complete with at least a mention of them. See [Nic00, Section 2.3] and [Section 7.4 Sal99].

We have proved that if $b^2_+ > 0$, then for a generic perturbation η, the moduli space is a *smooth, compact, oriented, finite-dimensional* manifold. In this section we define and invariant depending only on the smooth structure and the Spin^c-structure on it. We distinguish the case where $\dim \mathcal{M}(\eta) = 0$ and $\dim \mathcal{M}(\eta) > 0$.

If $\dim \mathcal{M}(\eta) = 0$, then $\mathcal{M}$ is a finite set of points. We will see that these points have a canonical orientation that might differ from the orientation which comes from a choice of an orientation of $H^1(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R})$ (Proposition 3.2.13). The invariant is defined as a signed sum over these points.

First, let's see that there is a canonical orientation. Recall that we wrote the tangent space $T_{[\Psi,A]}\mathcal{M}(\eta)$ as the kernel of the map $\mathcal{D}_{(\Psi,A)}$. This family of Fredholm map induces a line bundle $\det(\mathcal{D})$, which on each fiber is

$$\det\big(\mathcal{D}_{(\Psi,A)}\big) = \Lambda^{\mathrm{top}}(\ker \mathcal{D}_{(\Psi,A)}) \otimes \Lambda^{\mathrm{top}}(\operatorname{coker} \mathcal{D}_{(\Psi,A)})^*.$$

Since $\mathcal{M}(\eta)$ is zero-dimensional, and assuming that we've chosen η to be a perturbation for which all solutions are smooth, then

$$\ker \mathcal{D}_{(\Psi,A)} \cong 0 \cong \operatorname{coker} \mathcal{D}_{(\Psi,A)},$$

and thus,

$$\det\big(\mathcal{D}_{(\Psi,A)}\big) = \Lambda^{\mathrm{top}}(0) \otimes \Lambda^{\mathrm{top}}(0)^* \cong \mathbb{R},$$

where the isomorphism is *canonical*[6,7]. This canonical fiberwise isomorphism induces an orientation of $\mathcal{M}(\eta)$, and this one we compare with the one obtained from the orientation chosen from $H^1(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R})$.

Definition 3.3.1 (Zero-dimensional invariant).
Let M be a smooth, oriented, Riemannian 4-manifold with a Spin^c-structure, for which the moduli space $\mathcal{M}(\eta)$ is zero-dimensional. Choose an orientation of $H^1(M,\mathbb{R}) \oplus H^2_+(M,\mathbb{R})$.

We define the **Seiberg-Witten invariant***, which depends possibly on the Spin^c-structure, as*

$$\mathrm{SW} = \sum_{[\Psi,A]\in\mathcal{M}(\eta)} \pm 1,$$

where the sign is $+1$ if the canonical orientation at $[\Psi,A]$ agrees with the one induced from the orientation of the cohomology groups, and -1 if it doesn't.

When the dimension is positive, the definition of the invariant is a bit more subtle. Consider the space of irreducible configurations

$$\mathrm{Conf}_{\mathrm{irr}} = \{(\Psi,A) \in \mathrm{Conf} \mid \Psi \neq 0\}.$$

Fix a point $x_0 \in M$, and consider **based gauge group** $\mathcal{G}_0$ as

$$\mathcal{G}_0 = \{\gamma \in \mathcal{G} \mid \gamma(x_0) = 1\}.$$

Clearly there is a bijection $\mathcal{G} \cong \mathcal{G}_0 \times S^1$. The action of $\mathcal{G}_0$ is *always free*: for $\gamma \in \mathcal{G}_0$, if $A \cdot \gamma = A$, this implies that $d\gamma = 0$, and therefore since $\gamma(x_0) = 1$, then $\gamma \equiv 1$.

Let $\mathcal{B}_0 = \mathrm{Conf}/\mathcal{G}_0$. Then there is a natural S^1-action on $\mathcal{B}_0$, and the orbits of this action coincide precisely with the points of $\mathcal{B} = \mathrm{Conf}/\mathcal{G}$. Thus, we have an S^1-bundle

$$\mathcal{B}_0 \to \mathcal{B}.$$

[6] For any finite-dimensional (real) vector space V, there is an isomorphism $\Lambda^{\mathrm{top}}V \otimes \Lambda^{\mathrm{top}}V^* \to \mathbb{R}$ given on homogeneous terms as

$$v_1 \wedge \cdots \wedge v_n \otimes u_1^* \wedge \cdots \wedge u_n^* \to \det\big([u_j^*(v_i)]\big).$$

[7] Note that if V is zero-dimensional, we define $\Lambda^{\mathrm{top}}(V) = \mathbb{R}$. This is for consistency with $\Lambda^{\mathrm{top}}(V \oplus W) = \Lambda^{\mathrm{top}}V \otimes \Lambda^{\mathrm{top}}W$.

Definition 3.3.2 (Seiberg-Witten invariant).
Let μ *be a representative of the Chern class of the* S^1*-bundle* $\mathcal{B}_0 \to \mathcal{B}$. *If* $d = \dim \mathcal{M}(\eta)$ *is even, we define*

$$\mathrm{SW} = \int_{\mathcal{M}(\eta)} \mu^{\frac{d}{2}}.$$

If d *is odd, we define* $\mathrm{SW} = 0$.

The idea for proving *invariance* of the Seiberg-Witten invariant, very briefly, goes as follows:

Assume that $b_2^+ > 1$, and let g_1, g_2, η_1, η_2 be Riemannian metrics and parameters such that $\mathcal{M}_{g_i}(\eta_i)$ are smooth. Consider a smooth path of metrics g_t connecting g_0 and g_1, and for each t choose equivalent Spin^c-structures. When $b_2^+ > 1$, the space of *good* parameters is (path)-connected, and therefore we can consider a smooth path of good parameters η_t connecting η_0 and η_1. The collection of moduli spaces $\mathcal{M}(\eta_t)$ can be organized as a manifold $\tilde{\mathcal{M}}$, exhibiting a cobordism between $\mathcal{M}(\eta_0)$ and $\mathcal{M}(\eta_1)$, in such a way that the form μ chosen above can be extended. Then from Stokes theorem, since μ is closed, this implies that its integrals over $\mathcal{M}(\eta_0)$ and $\mathcal{M}(\eta_1)$ are equal.

If $b_2^+ = 1$, then the space of *good* parameters is disconnected: the obstruction is $2\pi i[c_1(L)] = [\eta]$, and since $b_2^+ = 1$, the space of good perturbations has two components. If the perturbations η_0 and η_1 are on same component, then we can repeat the procedure above, and thus the invariants are the same. However, if the perturbations are on different components, then there is a *wall crossing* formula:

$$\mathrm{SW}(\eta_1) - \mathrm{SW}(\eta_0) = (-1)^{\frac{d}{2}},$$

where $d = \dim \mathcal{M}(\eta_1) = \dim \mathcal{M}(\eta_2)$.

CHAPTER 4

The Intersection Form and Donaldson's Theorem

Roughly speaking, the intersection form of a four-manifold is a bilinear, symmetric, nondegenerate integral form that codifies information of the intersection numbers of its embedded surfaces. This simple algebraic object is an excellent classifying tool for *topological* four-manifolds: every such form is the intersection form of exactly one or two of them. Therefore, a study of the algebraic characteristics of these forms gives us a wealth of information about the topology of four-manifolds.

However, once we go to *smooth* realm, the situation changes. In this chapter, we will prove *Donaldson's theorem*, which imposes severe restrictions on the intersection forms of smooth four-manifolds; namely, the only positive- or negative-definite intersection forms that are allowed are *diagonal*. The proof we present is attributed to Kronheimer and Elkies, and it is based on looking closely at the restrictions that the intersection form imposes on the Seiberg-Witten moduli space.

Overview of this chapter

The main idea to have in mind is Thom's representability theorem for homology in degree 2, which we briefly discussed in Theorem 1.3.5. It tells us that classes of degree 2 are precisely the fundamental classes of embedded surfaces. This allows us to freely move back and forth between algebraic and geometric concepts. With this, we will define the intersection form, its main properties and state some results which exhibit it as a powerful classification tool for four-manifolds. We will look at the algebraic aspects of non-degenerate, symmetric bilinear integral forms (which we call *unimodular* forms), and show a partial classification of them.

The main course of the meal is Donaldson's theorem. Briefly speaking, its proof is goes as follows: first, we show that there is a close relation between the Spin^c-structures on a manifold and *characteristic* vectors of the intersection form. Afterwards, we see that the Seiberg-Witten moduli space of a manifold with positive- or negative-definite intersection form must be empty or be zero-dimensional. Finally, we see that intersection forms that are *not* diagonal allow for characteristic vectors whose corresponding Spin^c-

structures have moduli spaces with *positive* dimension. This implies that if a smooth four-manifold has a definite intersection form, it is necessarily diagonal.

4.1 The intersection form of 4-manifolds

Let M be a closed, connected, oriented 4-manifold. The intersection pairing is a map on cohomology classes of degree 2. By Poincaré duality, the intersection pairing can also be seen as a pairing between *homology* classes of degree 2. We call this the *intersection form*.

Definition 4.1.1 (Intersection form).
Let M be a closed, connected, oriented 4-manifold. The intersection form is the bilinear, symmetric map $q_M : H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$ given as

$$q_M(\sigma,\eta) := PD(\sigma) \bullet PD(\eta).$$

Explicitly,

$$q_M(\sigma,\eta) = \langle PD(\sigma), \beta\rangle = \langle PD(\beta), \sigma\rangle\,.$$

Indeed, q_M is symmetric since the cup product is symmetric between even-degree classes. We had shown that the intersection pairing is non-degenerate (outside of torsion), and this implies at once that q_M is non-degenerate. This means that once we choose generators of $H_2(M,\mathbb{Z})$, q_M can be represented by an integer-valued matrix which is invertible *over the integers*. This property is called *unimodularity*[1].

Proposition 4.1.2 ("Intersection form" is not a gratuitous name).
Let M be a closed, connected, oriented 4-manifold with intersection form q_M. Let $\alpha,\beta \in H_2(M,\mathbb{Z})$ be homology classes represented by surfaces $S_\alpha, S_\beta \subset M$, respectively. Then

$$q_M(\alpha,\beta) = S_\alpha \cdot S_\beta,$$

where the $\cdot$ on the right-hand side is the oriented intersection number of the surfaces.

Proof. — The idea of the proof is as follows. If α,β are not torsion, then their Poincaré duals can be represented as de Rham cohomology classes in $\hat\alpha,\hat\beta \in H^2(M,\mathbb{R})$. Furthermore, since α, β are represented by S_α and S_β, then we can choose $\hat\alpha$ and $\hat\beta$ vanishing everywhere except for a neighborhood of S_α and S_β, respectively.

We may assume that S_α and S_β intersect transversely. Since they have complementary dimensions, they intersect at a finite set of points. For each point $p \in S_\alpha \cap S_\beta$, consider a neighborhood U and coordinates (x_1,x_2,y_1,y_2) such that locally, S_α is given by $y_1 = y_2 = 0$ and S_β is given by $x_1 = x_2 = 0$. Then we can write

$$\hat\alpha = f(x_1,x_2)\,\mathrm{d}y_1 \wedge \mathrm{d}y_2\,,$$

with f a function with compact support such that $\int f = 1$. Since under the de Rham isomorphism, cup products pass to wedge products, we have

$$q_M(\alpha,\beta) = \big\langle \hat\alpha \smile \hat\beta, [M]\big\rangle = \int_M \hat\alpha \wedge \hat\beta = \int_{S_\beta} \hat\alpha = \pm 1,$$

[1] Equivalently, its determinant is ± 1.

and the sign depends on whether $dy_1 \wedge dy_2$ coincides with the orientation of S_β or not.

Repeating for all points in the intersection, we obtain

$$\alpha \bullet \beta = \sum_{p \in S_\alpha \cup S_\beta} \pm 1,$$

where the sign is chosen depending on whether the orientation of $T_pS_\alpha \oplus T_pS_\beta$ is the same as T_pM. This is precisely the intersection number.

Now suppose α is torsion, i.e., for some k, $k\alpha = 0$. Therefore $k\alpha$ is represented by a surface $S_{k\alpha}$ bounding a 3-submanifold, which implies that $S_{k\alpha} \cdot S_\beta = 0$. ■

Example 4.1.3 (Intersection form of $S^2 \times S^2$).
We have that $H_2(S^2 \times S^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Take two points $p, q \in S^2$, so that $\alpha = [S^2 \times \{q\}]$ and $\beta = [\{p\} \times S^2]$ are generators. Then

$$q_{S^2 \times S^2}(\alpha, \beta) = (S^2 \times \{q\}) \cdot (\{q\} \times S^2) = 1,$$

whereas

$$q_{S^2 \times S^2}(\alpha, \alpha) = (S^2 \times \{q\}) \cdot (S^2 \times \{q'\}) = 0.$$

Here we perturbed $S^2 \times \{q\}$ a little so that there is no self-intersection. Therefore, with this basis,

$$q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example 4.1.4 (Intersection form of $\mathbb{CP}^2$).
We have that $H_2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, with generator $[\mathbb{CP}^1]$. Let μ be the Poincaré dual of $[\mathbb{CP}^1]$. Then

$$\left\langle \mu \smile \mu, [\mathbb{CP}^2] \right\rangle = \left\langle \mu, \mu \frown [\mathbb{CP}^2] \right\rangle = \left\langle \mu, [\mathbb{CP}^1] \right\rangle = 1$$

if the orientation of $\mathbb{CP}^2$ is the canonical one inherited from its complex structure. Then

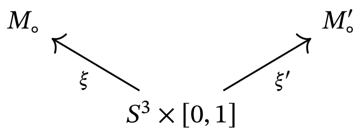
$$q_{\mathbb{CP}^2} = (1).$$

Note that changing the orientation of M means changing the sign of the fundamental class $[M]$. From the definition of the intersection pairing, we see that changing the orientation of M corresponds to changing the sign of q_M.

Proposition 4.1.5 (Intersection form of connected sum).
Let M, M' be closed, connected, oriented four-manifolds. Then

$$q_{M \# M'} = q_M \oplus q_{M'}$$

Proof. — Recall that the connected sum $M \# M'$ is obtained as follows: Take small open 4-balls $B^4 \subset M$ and $B^4 \subset M'$, and consider the manifolds $M_\circ = M - B^4$ and $M'_\circ = M' - B^4$. Both of them have S^3 for boundaries. Consider the cylinder $S^3 \times [0,1]$, and embeddings



which map $\xi(S^3 \times \{1\}) = \partial M_\circ$, $\xi(S^3 \times \{0\}) = \partial M'_\circ$, and whose images are precisely collar neighborhoods of the boundaries. The connected sum $M \# M'$ is obtained by identifying the collar neighborhoods via the images of the embeddings ξ, ξ'. The choice of this embedding guarantees that the resulting manifold is smooth, and has an orientation compatible with the orientations of M and M'.

The connected sum $M \# M'$ only alters 4-cells, so the homology in degree two in unaltered. As far as the 1, 2, and 3-cells are concerned, $M \# M'$ is a disjoint union of M and M'. More explicitly, let M_c and M'_c be the images of $M_\circ \sqcup S^3[0,1)$ and $S^3 \times (0,1] \sqcup M'_\circ$ in the quotient $M \# M'$. Intuitively, M_c and M'_c are just $M_\circ$ and $M'_\circ$ plus a little cylinder that extends into the "other side". Then $M_c \cap M'_c \cong S^3 \times (0,1)$, and from the Mayer-Vietoris sequence (in reduced homology) we obtain

$$H_k(M_c) \oplus H_k(M_c) \cong H_k(M \# M')$$

for $k = 2$ and $k = 3$. However, we have that M_c is just M minus a disk; i.e. $M_c \cong M - D^4$, so again, considering a Mayer-Vietoris sequence with a slightly larger disk $D \subset M$ such that $M - D^4 \cap D \cong S^3 \times (0,1)$, we find that

$$H_k(M_c) \cong H_k(M)$$

for $k = 2$ and $k = 3$. In particular, this tells us that $H_2(M \# M') \cong H_2(M) \oplus H_2(M')$, and so the result follows. ■

4.1.1 Unimodular symmetric forms

In order to make use of the intersection form, we need to study its algebraic properties. W e saw that it is an integral, nondegenerate, symmetric bilinear form defined on a finitely-generated free abelian group. We call all such forms **unimodular symmetric forms**. In this section, we will state some basic results about them. The standard reference is [MH73].

Definition 4.1.6 (Invariants of unimodular symmetric forms).
Let Z be a finitely-generated free abelian group, and let $q : Z \times Z \to \mathbb{Z}$ be a unimodular form. To it we can assign three **invariants***:*

- **Rank***: Since q is unimodular, it has full rank. We write it* $\operatorname{rank}(q) = \operatorname{rank}(Z)$.
- **Type** *or parity: We say that q is* even *if for all $x \in Z$, $q(x,x)$ is even. Otherwise, we say that q is* odd. *This means that that q is odd if $q(x,x)$ is odd for* at least one *element $x \in Z$.*
- **Signature***: Since q is represented by a symmetric matrix, it can be diagonalized over the rationals. The signature of q, denoted* $\operatorname{sign}(q)$*, is*

$$\operatorname{sign}(q) = \#\{\textit{positive eigenvalues}\} - \#\{\textit{negative eigenvalues}\}.$$

If we choose generators of Z, by Sylvester's Law of Inertia, this is indeed an invariant of q.

In relation to the signature, we can say that q is **positive-definite** *(resp.* **negative-definite***) if for all $x \in Z$, $q(x,x) \geq 0$ (resp. ≤ 0). Equivalently, q is positive (resp. negative) definite if* $\text{sign}(q) = \text{rank}(q)$ *(resp.* $\text{sign}(q) = -\text{rank}(q)$*). If q is neither positive-definite nor negative-definite, we say that it is* **indefinite**.

Notation. In [MH73], unimodular symmetric forms on Z are called *inner products* on Z. In their notation, a forms of *type I* are odd and forms of *type II* are even.

Given two symmetric bilinear forms, $q, q' : Z \times Z \to \mathbb{Z}$, we say that they are **equivalent** or **isomorphic** if there is an automorphism $\varphi : Z \to Z$ such that

$$q'(x,y) = q(\varphi(x), \varphi(y))$$

for all $x, y \in Z$.

We can easily classify all indefinite *odd* forms in terms of their rank and signature.

Proposition 4.1.7 (Classification of indefinite odd forms).
Let q be a symmetric unimodular form over Z. Then there is a set of generators of Z for which q is a diagonal matrix (and therefore with only ± 1 on the diagonal). Consequently, indefinite odd forms are uniquely determined by their rank and signature.

Proof. — This proof follows [MH73, Theorem 4.3]. We proceed by induction on the rank of Z. If $\text{rank}(Z) = 1$, the result follows immediately. Assume $\text{rank}(Z) = n > 0$, and take an element $x_1 \neq 0 \in Z$ such that $q(x_1, x_1) = 0$. Such elements always exist [MH73, Lemma II.4.1]. Furthermore, assume that x_1 is a generator of an cyclic component of Z, and complete this to a basis $x_1, \dots, x_n$. By non-degeneracy of q, there exist unique elements $y_1, \dots, y_n$ such that $q(x_i, y_i) = 1$, which generate Z as well. Now, by hypothesis, q is odd so there is a vector $v \in Z$ such that $q(v,v)$ is odd. Writing v in terms of $y_1, \dots, y_n$, we see that necessarily there is a y_j for which $q(y_j, y_j)$ is odd as well.

Define a subgroup $Z_0 = \langle x_1, y \rangle \subset Z$ as follows: if $q(y_1, y_1)$ is odd, let $y = y_1$. Otherwise, let $y = y_1 + y_k$. Note that $q(y,y)$ is odd. When restricted to Z_0, q has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 2m+1 \end{pmatrix}.$$

We can find a set of generators of Z_0 which diagonalizes $q|_{Z_0}$ as follows: Write $x' = y - mx_1$ and $y' = y - (m+1)x_1$. Then

$$\begin{aligned} q(x', x') &= q(y,y) - 2mq(y, x_1) + m^2 q(x_1, x_1) = 2m + 1 - 2m = 1, \\ q(y', y') &= q(y,y) - 2(m+1)q(y, x_1) + (m+1)^2 q(x_1, x_1) = 2m + 1 - 2(m+1) = -1, \\ q(x', y') &= q(y,y) - (2m+1)q(y, x_1) + m(m+1)q(x_1, x_1) = 0. \end{aligned}$$

Therefore in the basis x', y', the form $q|_{Z_0}$ is $(1) \oplus (-1)$. Now we split Z as $Z_0 \oplus Z_0^\perp$, where $Z_0^\perp$ is the q-orthogonal complement of Z_0. Then we have that

$$q \simeq (1) \oplus (-1) \oplus q|_{Z_0^\perp}.$$

If $q|_{Z_0}^\perp$ is indefinite, by induction we are done. If not, then either $(1) \oplus q|_{Z_0^\perp}$ or $(-1) \oplus q|_{Z_0^\perp}$ are indefinite, so we can proceed by induction and obtain the result. ∎

There is an equivalent result for the *even* case, but its proof requires some lattice theory [see MH73, Theorem II.5.3]. With this, we conclude:

Theorem 4.1.8 (Indefinite forms are determined by rank, signature and type).
Let q, q' be indefinite *unimodular, symmetric forms. Then q and q' are equivalent if and only if they have the same rank, signature, and type.*

The notion of a *characteristic vector* is crucial in our proof of Donaldson's theorem. We will see that characteristic vectors of the intersection form of a manifold are precisely lifts of the second Stiefel-Whitney class of a manifold.

Definition 4.1.9 (characteristic vector).
Let q be a unimodular, symmetric form over Z. A vector $w \in Z$ is characteristic *if for all $x \in Z$,*

$$q(w, x) = q(x, x) \mod 2.$$

Lemma 4.1.10 (Existence of characteristic vectors).
Every unimodular, symmetric form admits a characteristic vector.

Proof. — Consider the $\mathbb{Z}_2$-vector space $X = {}^{Z}\!/_{2Z}$. For every $x \in Z$, denote by $\overline{x} \in X$ its class in the quotient. The form q descends to a bilinear form $\overline{q} : X \times X \to \mathbb{Z}_2$ defined as

$$\overline{q}(\overline{x}, \overline{y}) := q(x, y) \mod 2,$$

where x, y are representatives of $\overline{x}, \overline{y}$. This form is non-degenerate: suppose that there is $\overline{x}$ such that $\overline{q}(\overline{x}, \overline{y}) = 0$ for all $\overline{y} \in X$. This is to say that $q(x, y)$ is even for all $y \in Z$. Choose generators $e_1, \dots, e_n$ of Z, such that $x = k_1 e_1 + k_n e_n$. If x is not of the form $2x'$ for some $x' \in Z$, then necessarily there is some k_j which is odd, which implies that for all $y \in Z$, $q(e_j, y)$ is even. However, this violates non-degeneracy of q, and therefore x must be of the form $2x'$, i.e., $\overline{x} = 0$.

Consider the map $s : X \to Z_2$ given by $s(\overline{x}) = \overline{q}(\overline{x}, \overline{x})$. A priori, the maps looks quadratic. However, it is linear:

$$\begin{aligned} s(\overline{x} + \overline{y}) = q(x + y, x + y) &= q(x, x) + 2q(x, y) + q(y, y) \mod 2 \\ &= q(x, x) + q(y, y) \mod 2 \\ &= s(\overline{x}) + s(\overline{y}). \end{aligned}$$

By non-degeneracy of $\overline{q}$, there exists a *unique* class $\overline{w} \in X$ such that

$$s(\overline{x}) = \overline{q}(\overline{w}, \overline{x})$$

for all $\overline{x} \in X$. Choosing a representative $w \in Z$ of $\overline{w}$ gives us a characteristic vector. ■

Lemma 4.1.11 (Signature in terms of characteristic vector).
Let q be a unimodular symmetric form over Z and w a characteristic vector. Then

$$\operatorname{sign}(q) = q(w, w) \mod 8.$$

Proof. — First, suppose that q is odd and indefinite. Then there is a q-orthonormal basis $x_1, \dots, x_p, y_1, \dots, y_p$ of Z which exhibit q as a direct sum $q \simeq \oplus_p(1) \oplus_q(-1)$. Here, where $p+q = \text{rank}(q)$ and $\text{sign}(q) = p-q$. Then

$$w = x_1 + \cdots + x_p + y_1 + \cdots + y_q$$

is a characteristic vector and clearly $q(w, w) = p - q = \text{sign}(q)$. Note that if w' is any other characteristic vector, then as we saw in the proof of Lemma 4.1.10, $w' = w + 2x$ for some $x \in Z$, and thus

$$q(w', w') = q(w, w) + 4q(w, x) + 4q(x, x) = q(w, w) + 8q(x, x) + 8a.$$

Here we used the fact that $q(w, x) = q(x, x) + 2a$ for some $a \in \mathbb{Z}$. Then indeed, if q is odd and indefinite, for all characteristic vectors we have the result.

Suppose that q is arbitrary, and consider the form $q' = q \oplus (1) \oplus (-1)$ on $Z \oplus \mathbb{Z}^2$, which is odd and indefinite. If w is a characteristic vector of q, and x, y are generators of $\mathbb{Z}^2$, we have that $w' = w + x + y$ is a characteristic vector of $q \oplus (1) \oplus (-1)$ and by the previous result for odd indefinite forms,

$$\begin{aligned} \text{sign}(q) = \text{sign}(q') &= q'(w' + x + y, w' + x + y) \mod 8 \\ &= q(w', w') \mod 8. \end{aligned}$$

∎

With this result we can explicitly exhibit all the indefinite forms, if we study two examples:

Example 4.1.12 (Hyperbolic form).
Consider the form H on $\mathbb{Z}^2$ given by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is straightforward to show that H is even, indefinite with $\text{sign}(H) = 0$. As we saw in Example 4.1.3, H is the intersection form of $S^2 \times S^2$.

Example 4.1.13 (E_8).
Define the form E_8 on $\mathbb{Z}^8$ as

$$E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 1 \end{pmatrix}.$$

This form is unimodular, positive-definite and even. Therefore $\text{rank}(E_8) = \text{sign}(E_8) = 8$. This can be proved by row-reducing the matrix to see that its reduced form (over the rationals) is diagonal, with all positive entries, and that its determinant is 1 [see Sco05, p. 126].

Surprisingly, the form E_8 is the intersection form of a four-manifold, which we also call E_8. The construction of this manifold requires some techniques that fall out of the scope of this work [see Sco05, pg. 86].

Theorem 4.1.14 (All indefinite forms).
Let q be an indefinite unimodular symmetric form.

1. *If q is odd, then q is of the form*
$$\oplus_r(1) \oplus_s (-1),$$
with $r + s = \operatorname{rank}(q)$ *and* $r - s = \operatorname{sign}(q)$.

2. *If q is even, then q is of the form*
$$\pm \oplus_r E_8 \oplus_s H,$$
where $\operatorname{rank}(q) = 8r + 2s$ *and* $\operatorname{sign}(q) = \pm 8r$.

Proof. — We already know this result for *odd* forms. If q is even, then 0 is a characteristic vector, and therefore $\operatorname{sign}(q) = 0 \mod 8$. If we write $\operatorname{sign}(q) = a - b$, then necessarily $\operatorname{rank}(q) = a + b$ is even as well. We can see that all combinations of the form $\pm \oplus_r E_8 \oplus_s H$ exhaust all such ranks and signatures, and since indefinite forms are uniquely determined in this way, the result follows. ■

4.1.2 The topology of four-manifolds

In this section, we will see the immense power of the intersection form as a classification tool for topological four-manifolds. Unfortunately, the proof of most of these results fall out of the scope of this work.

The first result says that the intersection forms classify four-manifolds up to homotopy equivalence:

Theorem 4.1.15 (Whitehead).
Let M and M' be closed, oriented, simply connected topological *four-manifolds. Then M and M' are homotopy-equivalent if and only if their intersection forms are isomorphic.*

A proof can be found in [MH73, Section V.1].

The question of which unimodular symmetric forms show up as intersection forms was answered by Freedman: *all of them*, and in an almost unique way:

Theorem 4.1.16 (Freedman).
Let q be a unimodular, symmetric form.

1. *If q is even, there is exactly one closed, oriented, simply-connected* topological *four-manifold whose intersection form is q.*

2. *If q is odd, there are exactly two closed, oriented, simply-connected* topological *four-manifolds whose intersection forms are q, at least one of which does not admit any smooth structures.*

A proof of this theorem can be found in [FQ90, Chapter 10]. As an immediate consequence of this, we have that if two closed, oriented, simply-connected *smooth* four-manifolds have isomorphic intersection forms, then they must be *homeomorphic*.

4.2 Finale: Donaldson's theorem

We're finally here. The final stretch. Let M be a *smooth*, oriented, closed 4-manifold with negative-definite intersection form q_M. We will first show that, given a choice of a Spin^c-structure on M, the Seiberg-Witten moduli space is either empty or zero-dimensional (i.e., the expected dimension is non-positive). Then, we will show that if the intersection form is not diagonal, then there is a Spin^c-structure on M for which the moduli space has a strictly *positive* dimension. Therefore, q_M is necessarily diagonal.

The first step is an application *Wu's formula* which exhibits the second Stiefel-Whitney class of a manifold as the mod-2 reduction of (the Poincaré dual of) a characteristic vector of its intersection form.

Proposition 4.2.1 ($w_2(M)$ is mod-2 reduction of characteristic vector).
Let M be a closed, connected, oriented four-manifold with intersection form q_M. Then for any characteristic vector $w \in H_2(M, \mathbb{Z})$,

$$w_2(M) = PD(w) \mod 2.$$

Consequently, for all $\alpha \in H_2(M, \mathbb{Z})$,

$$\langle w_2(M), \alpha \rangle = q_M(\alpha, \alpha) \mod 2.$$

The proof of this theorem comes from studying the definition of the Stiefel-Whitney classes in terms of the Steenrod operations and the Thom class of the tautological bundle of $\mathbb{RP}^\infty$, see [May99, section 23.6] and [MS74, p. 130].

Corollary 4.2.2 (Closed four-manifolds admit Spin^c-structures).
Every closed, connected, oriented four-manifold admits Spin^c-structures.

Proof. — This follows immediately from the previous proposition: The intersection form of a manifold always admits characteristic vectors, and these are integral lifts of the second Stiefel-Whitney class. This is precisely the condition for the existence of Spin^c-structures that we saw in Proposition 2.6.6. ∎

Lemma 4.2.3.
Let M be a smooth, closed, oriented 4-manifold with negative-definite intersection form q_M. If $H^1(M, \mathbb{R}) = 0$, then for all characteristic vectors u of q_M,

$$\frac{1}{4}(q_M(u, u) + b_2(M)) \leq 0.$$

Proof. — Since the intersection form is negative definite, then $b_2^+ = 0$, so that $b_2 = b_2^-$ and $\mathrm{sign}(M) = -b_2^-$. Let $w \in H^2(M, \mathbb{Z})$ be the Poincaré dual of a characteristic vector of q_M. By Wu's formula, w is an integral lift of $w_2(M)$, and so we can consider a Spin^c-structure over M whose determinant bundle L satisfies $c_1(L) = w$. Therefore, the expected dimension of the Seiberg-Witten moduli space $\mathcal{M}$ is

$$d = \dim \mathcal{M} = \frac{1}{4}(w \bullet w - 3\,\mathrm{sign}(M) - 2\chi(M)) = \frac{1}{4}(w \bullet w + b_2) - 1.$$

Since w is (dual to) a characteristic vector, then $w \bullet w = \mathrm{sign}(M) = -b_2 \mod 8$, and thus $w \bullet w + b_2 = 0 \mod 8$. This means that $\dim \mathcal{M}$ is *odd*.

Now, assuming that $H^1(M, \mathbb{R}) = 0$, there is a unique gauge class of reducible solutions to the Seiberg-Witten equations. To see this, take any closed perturbation η, and consider the smooth reference connection A_0. Any other connection is of the form $A = A_0 - i\alpha + i\beta$, where β is a closed form. Note that $F_A = F_{A_0} - i\, d\alpha$, and it trivially satisfies

$$F_A^+ + \eta = 0,$$

since $F_A^+ + \eta$ is a harmonic self-dual form and $b_+^2 = 0$. Furthermore, since $b^1 = 0$, then $\beta = \mathrm{d}f$ for some $f \in C^\infty(M)$, and thus

$$A = A_0 - i\alpha + i\,\mathrm{d}f\,,$$

which is precisely the gauge transformation by $\gamma = e^{if}$ of the connection $A = A_0 - i\alpha$. Therefore indeed all reducible solutions are on the same gauge class.

Consider the space of solutions modulo the based gauge group $\mathcal{M}_0(\eta) = \mathcal{Z}_\eta / \mathcal{G}_0$. The tangent space at 1 of the based gauge group is simply

$$T_1\mathcal{G}_0 = \{if \in \mathbb{C}^\infty(M, i\mathbb{R}) \mid f(x_0) = 0\}\,.$$

And the infinitesimal action, which we denote with the same symbol $\mathfrak{g}_{(0,A)}$, is given by

$$\mathfrak{g}_{(0,A)}(if) = (0, -2i\,\mathrm{d}f).$$

We still have a "based" Seiberg-Witten complex

$$T_1\mathcal{G}^0 \xrightarrow{\mathfrak{g}_{(0,A)}} T_{(0,A)}\mathrm{Conf} \xrightarrow{T_{(0,A)}\mathcal{SW}_\eta} T_{(0,0)}\mathcal{Y}\ ,$$

and here the differential of the Seiberg-Witten map reduces to

$$T_{(0,A)}\mathcal{SW}_\eta = \partial\!\!\!/_A \oplus \mathrm{d}^+.$$

Therefore the tangent space of the based moduli space $\mathcal{M}_0(\eta)$ at the reducible point $[0,A]$ is

$$T_{[0,A]}\mathcal{M}_0(\eta) \cong {\ker T_{(0,A)}\mathcal{SW}_\eta}\big/{\mathrm{im}\,\mathfrak{g}_{(0,A)}} = \ker \partial\!\!\!/_A \oplus H^1(M,\mathbb{R}) = \ker \partial\!\!\!/_A.$$

This is because we've assumed that $H^1(M,\mathbb{R}) = 0$.

Recall that we still have a free action of U(1) on the based moduli space. Since this action is free, the induced linear action of U(1) on the tangent space $T_{[0,A]}\mathcal{M}$ is also free (except at the origin). Furthermore, since $\partial\!\!\!/_A$ is elliptic and $\mathbb{C}$-linear, then $\ker \partial\!\!\!/_A$ is a finite-dimensional complex vector space, whose (real) dimension is necessarily $\dim \mathcal{M}(\eta) + 1$. Therefore, if we quotient out by the U(1) action, we get that $T_{[0,A]}\mathcal{M}_0(\eta)\big/\mathrm{U}(1)$ is a cone over

$$S^{\frac{d+1}{2}}\big/\mathrm{U}(1) \cong \mathbb{CP}^{\frac{d-1}{2}}$$

Of course, all these considerations on the tangent space at $[0,A]$ extend to a small neighborhood U of it.

That is to say that the (unbased) moduli space $\mathcal{M}(\eta)$ is a smooth manifold except the single reducible class $[0,A]$, and there is a neighborhood U of it which is isomorphic to a cone over $\mathbb{CP}^{\frac{d-1}{2}}$. Its complement $\mathcal{M}(\eta) - U$ is, then a smooth compact manifold with boundary $\mathbb{CP}^{\frac{d-1}{2}}$.

If $d = 1$, then $\mathcal{M}(\eta) - U$ is a compact, one-dimensional manifold whose boundary is a single point, which is impossible. Suppose now that $d > 1$. Then the restriction of the U(1)-bundle $\mathcal{M}_0(\eta) \to \mathcal{M}(\eta)$ to the boundary $\partial(\mathcal{M}(\eta) - U) \cong \mathbb{CP}^{\frac{d-1}{2}}$ as a U(1) bundle. The fiber above a point $[\Psi, A']$ is the orbit of the U(1) action on $\mathcal{M}_0(\eta)$, that is, it is precisely the circle that it comes from in the quotient. This is to say that the restriction $\mathcal{M}_0(\eta)|_{\partial(\mathcal{M}(\eta)-U)}$ is precisely the universal bundle over $\mathbb{CP}^{\frac{d-1}{2}}$.

Let μ be the first Chern class of this bundle. Then, by definition, μ is the Poincaré dual to the fundamental class of $[\mathbb{CP}^1]_{\mathbb{CP}^{(d-1)/2}}$, and since the cohomology ring of $\mathbb{CP}^{(d-1)/2}$ is generated by this fundamental class, we have

$$\left\langle \mu^{(d-1)/2}, [\mathbb{CP}^{(d-1)/2}] \right\rangle = \int_{\partial(\mathcal{M}(\eta)-U)} \mu^{(d-1)/2} = \pm 1.$$

On the other hand, since the bundle $\mathcal{M}_0 \to \mathcal{M}$ is defined everywhere else, then μ extends to $\mathcal{M}(\eta) - U$, and by the Stokes theorem,

$$\int_{\partial(\mathcal{M}(\eta)-U)} \mu^{(d-1)/2} = \int_{\mathcal{M}(\eta)-U} \mathrm{d}\mu^{(d-1)/2} = 0.$$

We have then arrived at a contradiction in both cases when $d = 1$ and $d > 0$. This implies that $d \leq 0$, and the result follows. ■

The following result, which is purely algebraic, tells us that unimodular symmetric forms that are not diagonal have "short" characteristic vectors. Combining this with the previous lemma, we will obtain Donaldson's theorem, at least in the case where $H^1(M, \mathbb{R}) = 0$.

Lemma 4.2.4 (Elkies).
Let $q : Z \times Z \to \mathbb{Z}$ be a symmetric, unimodular bilinear form. If q is not $\oplus(-1)$ *nor* $\oplus(+1)$, *then there exists a characteristic vector $w \in Z$ such that*

$$|q(w, w)| < \operatorname{rank}(q).$$

We call w a short *characteristic vector.*

The proof of this lemma requires the theory of theta series and modular forms, and can be found in [Elk95].

Lemma 4.2.5 (Killing cohomology with surgery).
Let M be a smooth, closed, oriented 4-manifold. Then we can perform surgery on M and obtain a manifold M' with $H^1(M', \mathbb{R}) = 0$, but with $q_{M'} = q_M$.

Proof. — In this proof we will take all groups with integral coefficients. Consider a non-trivial element $c \in H_1(M)$ which is not torsion. It can be represented by an embedded $S^1 \hookrightarrow M$. Since it's nontrivial, then S^1 (nor any multiple of it) bounds a surface in M.

Let's do the intuition first. We take a tubular neighborhood of S^1, which is of the form $S^1 \times D^3$. If we remove this neighborhood we obtain a manifold with boundary $\partial S^1 \times D^3 = S^1 \times S^2$. Each circle $S^1 \times \{p\}$ in the boundary is "equivalent" in homology to the initial circle. We want to make all these circles in the boundary trivial, so we want to make them bound a surface, and thus we replace each $S^1 \times \{p\}$ by a disc $D^2 \times p$ with matching boundary. The resulting manifold is the same as attaching the handle $D^2 \times S^2$ along $S^1 \times S^2$, and since each of the circles $S^1 \times \{p\}$ is now the boundary of the corresponding disk, then their homology class is zero.

How do we know that this procedure doesn't alter $H_2(M)$? Nontrivial homology classes in $H_2(M)$ can be thought of as closed surfaces that do not bound any 3-submanifolds. When we attach the "filling" disk D^2 to each circle $S^1 \times \{p\}$ on the boundary, we are not "closing off" any surface that might introduce new homology. This is because such "half sphere" that we're capping off with D^2 would have the circle $S^1 \times \{p\}$ as its boundary.

So that's the intution.

Let $T \subset M$ be a tubular neighborhood of S^1, which is diffeomorphic to $S^1 \times D^3$. From the long, exact sequence in (reduced) homology for the pair (T, M), we have

$$\dots \longrightarrow H_2(M,T) \longrightarrow H_1(T) \longrightarrow H_1(M) \longrightarrow H_1(M,T) \longrightarrow 0.$$

However, the map $H_1(T) \to H_1(M)$ induced by inclusion is injective, since S^1 is nontrivial in $H_1(M)$. Then by exactness of the sequence, the map $H_2(M,T) \to H_1(T)$ is zero. We are left with the short exact sequence

$$0 \longrightarrow H_1(T) \longrightarrow H_1(M) \longrightarrow H_1(M,T) \longrightarrow 0,$$

which implies that $H_1(M,T) \cong H_1(M)/H_1(T)$.

Let M' be the manifold obtained as follows: after removing the tubular neighborhood T from M, we obtain a manifold with boundary $\partial(M - T) \cong S^1 \times S^2$. The manifold M' is the result of attaching a 2-handle $S^2 \times D^2$ along this boundary:

$$M' = (M - T) \cup_{\partial(M-T)} (S^2 \times D^2).$$

Denote the handle $S^2 \times D^2$ as $T' \subset M'$. Repeating the same process as above, we have that $H_1(M', T') \cong H_1(M')/H_1(T')$. However, $H_1(T') = 0$, so $H_1(M', T') \cong H_1(M')$. On the other hand, since $M - T = M' - T'$, then by excision, $H_1(M,T) \cong H_1(M', T')$. Putting all these together we find

$$H_1(M') \cong H_1(M) \big/ H_1(T) = H_1(M) \big/ \langle c \rangle .$$

Repeating this process for all generators of the free part of $H_1(M)$, we successfully kill $H_1(M)$.

An argument using the Mayer-Vietoris sequence of T and $M - T$ shows that $H_2(M)$ remains invariant. This surgery can be performed away from surfaces representing generators of $H_2(M)$, and therefore the intersection form is preserved [see Kat95, Section 6]. ■

We are finally ready to prove Donaldson's theorem.

Theorem 4.2.6 (Donaldson).
Let M be a smooth, closed, oriented 4-manifold, with definite intersection form q_M. Then q_M is diagonal.

Proof. — Without loss of generality, suppose that q_M is negative-definite (if not, consider M with the opposite orientation). Suppose that $H^1(M, \mathbb{R}) = 0$. If q_M is not diagonal, the by Elkies's lemma, there is a short characteristic vector w for which

$$|q_M(w,w)| = -q_M(w,w) < \operatorname{rank}(q_M) = b^2,$$

and so

$$q_M(w,w) + b^2 > 0.$$

However, this contradicts Lemma 4.2.3. Therefore q_M must be diagonal.

If $H^1(M, \mathbb{R})$ is not trivial, then from Lemma 4.2.5, we can perform surgery on M to obtain a manifold M' with $H^1(M', \mathbb{R}) = 0$ but $q_{M'} = q_M$. After applying the results above to M', we obtain the result. ■

At once, Donaldson's theorem exhibits a wealth of *non-smoothable* four-manifolds. Given any definite unimodular form which is not diagonal, by Freedman's theorem there exists a topological manifold that represents it, but it cannot have any smooth structure. For example the manifold E_8 is non-smoothable.

We also immediately obtain the result that any smooth manifold with definite intersection form is *homeomorphic* to a connected sum of several copies of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$.

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