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Ricci flow on homogeneous 3-manifolds and the quasi-convergence equivalence relation

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Abstract

Thurston's Geometrization conjecture states that every closed three dimensional manifold can be decomposed in a canonical way into pieces such that each piece admits a locally homogeneous metric. This conjecture was proven by Grisha Perelman with the use of the Ricci flow equation. This equation is a heat type evolution equation for a one parameter family of Riemannian metrics on a manifold where we deform the metrics in the direction of the Ricci curvature tensor. A one parameter family of metrics which satisfies the Ricci flow equation is called a Ricci flow. There are many interesting phenomena regarding Ricci flows. There are Ricci flows that exist for all positive times and converge to a Riemannian metric; there are Ricci flows that exist for all positive times but fail to converge and there exist Ricci flows that exist only up to a finite time. A tool to study Ricci flows that exist for all times is the quasi-convergence. This is an equivalence relation which compares the large time behaviour of two Ricci flows. In this thesis we give an alternative characterization of the quasi-convergence equivalent classes via Lie group actions and compute the quasi-convergence equivalence classes for left invariant Ricci flows on $\mathrm{SL}(2,\mathbb{R})$.

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Θα 'ρθει, ένα απόγευμα ζεστό, θα μπει στον κήπο αυτό όλο το φως που υπάρχει. Θα 'ρθει, μ' ένα ποδήλατο λευκό, θα κοιταχτεί μέσ' στο νερό και θα ρωτάει να μάθει. Πότε γέμισε ο κήπος με πουλιά, πόσο είχε λείψει εκεί μακριά, ποιος τα φροντίζει τ' άνθη;
Ο Κηπουρός - Παύλος Παυλίδης

Contents

1 Introduction

R. Hamilton in 1982, [8], introduced the Ricci flow equation,

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}(g),$$

with the aim to prove Thurston's geometrization conjecture. This conjecture states that every closed three dimensional manifold M can be decomposed in a canonical way into pieces such that each piece in this decomposition admits a locally homogeneous metric. The study of Ricci flow had a profound impact in the field of geometry and topology. G. Perelman [13] and [14] using the Ricci flow was able to solve the geometrization conjecture and with that he proved the famous Poincare conjecture which was unsolved for many years. Another remarkable result that was proven with the use of Ricci flow by S. Brendle and R. Schoen [2] was the Differentiable Sphere conjecture. This conjecture states that if M is complete and simply connected n-dimensional Riemannian manifold with the sectional curvature taking values in the interval $(1, 4]$ then M is diffeomorphic to the n-dimensional sphere S^n. In the study of Riemannian metrics that solve the Ricci flow equation R.Hamilton and J.Isenberg [7] conjectured that the collapsing solutions of the Ricci flow equation will approach in a particular sense the evolution of a locally homogeneous solution of the Ricci flow equation. In order to approach this problem they introduced the notion of quasi-convergence. If $g(t), h(t)$ are two solution of the Ricci flow equation, we say that $g(t)$ quasi-converges to $h(t)$ and we write $h \in [g]$ if and only if for any $\epsilon > 0$ there exist a time t_ϵ such that

$$\sup_{M^n \times [t_\epsilon, \infty)} |h(t) - g(t)|_{g(t)} < \epsilon$$

Thus, quasi-convergence captures the large time behaviour of the solution to the Ricci flow equation. Quasi-convergence is an equivalence relation, therefore we can classify the asymptotic behaviour of solutions to the Ricci flow equation and describe it as equivalence classes of evolving metrics. D. Knopf and K. McLeod ([11]), studied the quasi-convergence classes for homogeneous solutions of the Ricci flow and gave a description of their equivalence classes for almost all homogeneous metrics except for the case of $\mathrm{SL}(2, \mathbb{R})$.

The aim of this master thesis is twofold. First of all, we study the behaviour of the Ricci flow for homogeneous metrics. We describe what kind of curvature singularities, collapsing phenomena or convergences occur in these cases. The second aim of this master thesis is give a good description of the quasi-convergence classes for homogeneous Ricci flows. In particular, using a suitable parametrization of the space of metrics we can characterize the quasi-convergence class of an homogeneous Ricci flow as the orbit space of a particular action of a Lie group. This approach give us two advantages. The first one is to simplify the computations done in [11] and as a result to compute the quasi-convergence class for homogeneous metrics on $\mathrm{SL}(2, \mathbb{R})$ which is a new result that was not done previously. The second one is, via the actions of Lie groups, to give a more rich structure of the quasi-convergence relation for better understanding of this phenomenon.

2 Riemannian Geometry

Before dealing with the Ricci flow equation we first present the basic notions of Riemannian geometry that are going to be used in this thesis. Also we fix the notation that we are going to follow throughout the text. For a detail exposition of Riemannian geometry the reader can consult [3] or [15].

2.1 Basics of Riemannian Geometry

Consider a smooth manifold M of dimension n.

Definition 2.1.1. A Riemannian metric g (also denoted by $\langle .,. \rangle$) on M is a smooth positive-definite section of the bundle of symmetric covariant 2-tensors S^2T^*M. A Riemannian manifold is a smooth manifold M equipped with a Riemannian metric g.

The Riemannian metric enables us to measure lengths of curves and volumes of regions on a manifold. One can show, with the use of partitions of unity, that every smooth manifold M admits a Riemannian metric.

If we want to differentiate vector fields or tensor fileds along vector fields on a manifold we have to use the notion of affine connection in the tangent bundle.

Definition 2.1.2. A connection in TM is a map

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

which satisfies the following properties:

(i) $\nabla_X Y$ is $C^\infty(M)$-linear in X : if $f, h \in C^\infty(M)$ and $X, Y, Z \in \Gamma(TM)$ then

$$\nabla_{(fX+hZ)} Y = f \nabla_X Y + h \nabla_Z Y$$

(ii) $\nabla_X Y$ is $\mathbb{R}$-linear in Y : if $a, b \in \mathbb{R}$ and $X, Y, W \in \Gamma(TM)$ then

$$\nabla_X (aY + bW) = a \nabla_X Y + b \nabla_X W$$

(iii) $\nabla_X Y$ satisfies the Leibniz rule in Y : if $f \in C^\infty$ and $X, Y \in \Gamma(TM)$ then

$$\nabla_X (fY) = X(f)Y + f \nabla_X Y$$

There are many connections defined on a smooth manifold M but there exist only one which makes the metric tensor parallel and the torsion tensor identically zero. This connection is called the Levi-Civita connection and plays a crucial role in the study of geometry of manifolds.

Definition 2.1.3. The Levi-Civita connection in the tangent bundle TM is the unique connection $\nabla : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM)$ which satisfies the following two properties: For all $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} \text{metric compatibility } &: X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ \text{torsion free } &: \nabla_X Y - \nabla_Y X - [X, Y] = 0 \end{aligned}$$

With the use of the Levi-Civita connection we can define the curvature tensor. The curvature tensor is the central object in the study of Riemannian manifolds as it leads to the notion of curvature of a manifold.

Definition 2.1.4. The (3,1)-Riemann curvature tensor Rm is the (3,1)-tensor defined by

$$\mathrm{Rm}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $X, Y, Z \in \Gamma(TM)$.

The (3,1)-Riemann curvature tensor Rm is indeed a tensor. To prove this we must show that Rm is $C^\infty(M)$-linear in all of its arguments. That is we have to show that

$$\mathrm{Rm}(fX,Y)Z = \mathrm{Rm}(X,fY)Z = \mathrm{Rm}(X,Y)fZ = f\,\mathrm{Rm}(X,Y)Z$$

for all smooth functions $f \in C^\infty(M)$. We will only prove $\mathrm{Rm}(fX,Y)Z = f\,\mathrm{Rm}(X,Y)Z$ which more complicated than the other two equalities. The proof of the remaining two follows similarly and can be found in [3]. We have,

$$\begin{aligned}
\mathrm{Rm}(X,Y)fZ =& \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ \\
=& \nabla_X (Y(f)Z + f\nabla_Y Z) - \nabla_Y (X(f)Z + f\nabla_X Z) - [X,Y](f)Z - f\nabla_{[X,Y]} Z \\
=& X(Y(f))Z + Y(f)\nabla_X \nabla_Y Z + X(f)\nabla_Y Z + f\nabla_Y Z \\
&- Y(X(f))Z - X(f)\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - f\nabla_X Z \\
&- X(Y(f))Z + Y(Z(f))Z - f\nabla_{[X,Y]} Z \\
=& f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
=& f\,\mathrm{Rm}(X,Y)Z
\end{aligned}$$

We also consider the (4,0)-Riemann curvature tensor which is the metric contraction of the (3,1)-Riemann curvature tensor

$$R(X,Y,Z,W) = g(R(X,Y)W,Z).$$

The Riemann curvature tensor has many symmetries. Among them the most basic ones are the following:

Proposition 2.1.5. For every X, Y, Z, W smooth vector fields on M we have

$$\begin{aligned}
R(X,Y,Z,W) &= -R(Y,X,Z,W) \\
R(X,Y,Z,W) &= -R(X,Y,W,Z) \\
R(X,Y,Z,W) &= R(Z,W,X,Y) \\
R(X,Y,Z,W) + R(Y,Z,X,W) &+ R(Z,Y,X,W) = 0
\end{aligned}$$

The last equality is called first Bianchi identity.

Proof. The reader can consult [15] for a proof of this result. □

The Riemann curvature tensor is used to define the sectional curvature of a Riemannian manifold. Suppose that x, y are two linear independent tangent vectors in the tangent space T_pM, where $p \in M$. We use the notation $x \wedge y$ to denote the 2-dimensional plane which is spanned by the vectors x and y. The sectional curvature of the plane $x \wedge y$ is defined by

$$K(x \wedge y) = \frac{R(x, y, x, y)}{|x|^2|y|^2 - g(x, y)^2}.$$

The sectional curvature is an invariant that captures, locally, the geometry of a manifold. An important fact that we have to mention here is that the sectional curvature determines the Riemann curvature tensor. This actually comes down to a completely algebraic fact. The proof of this proposition can be found in [3].

Proposition 2.1.6. Suppose that $(V, \langle ., . \rangle)$ is a finite dimensional inner product vector space of dimension n with $n > 2$. Let R and R' be two (3,1)-tensors on V that share the same symmetries with the (3,1)-Riemann curvature tensor. If $x, y \in V$ are two linearly independent vectors we write

$$K(x \wedge y) = \frac{R(x, y, x, y)}{|x|^2|y|^2 - \langle x, y \rangle^2}, \qquad K'(x \wedge y) = \frac{R'(x, y, x, y)}{|x|^2|y|^2 - \langle x, y \rangle^2}$$

for their corresponding sectional curvatures. If $K(x \wedge y) = K'(x \wedge y)$ for every two linearly independent vectors $x, y \in V$ then $R = R'$

The Riemann curvature tensor is a very complicated object. It is usually convenient to study objects which contain part of the information of the Riemann curvature tensor but are simpler. The most notable ones are the Ricci curvature tensor and the scalar curvature.

Definition 2.1.7. The Ricci curvature tensor is (2,0)-tensor given by the trace of the Riemann curvature tensor

$$\mathrm{Ric}(Y, Z) = \mathrm{trace}(X \mapsto \mathrm{Rm}(X, Y)Z),$$

where X, Y, Z are smooth vector fields on M. If $\{e_i\}_i^n$ is an orthonormal frame on M, that is, $g(e_i, e_j) = \delta_{ij}$ then the Ricci curvature tensor is given by

$$\mathrm{Ric}(Y, Z) = \sum_{i,j} R(e_i, Y, e_j, Z).$$

Definition 2.1.8. The scalar curvature is is defined as the pointwise trace of the Ricci tensor,

$$R = \mathrm{tr}(\mathrm{Ric})$$

It is sometimes useful for computational reasons to describe all these geometric quantities that we have defined in a local coordinate system around a point.

We begin with the metric $g \in \Gamma(S^2T^*M)$. Let p be a point in a manifold M. Let U be a coordinate neighborhood around the point $p \in M$ and $\phi = (x^1, x^2, ..., x^n)$ the coordinate map defined on U. We denote by $\frac{\partial}{\partial x^i}$ the coordinate frame and by dx^i its dual frame i.e $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$.

Then g can be written locally in U as

$$g = g_{ij} dx^i \otimes dx^j,$$

for all i, j. The fact that g is a symmetric covariant 2-tensor translates to $g_{ij} = g_{ji}$. If X, Y are smooth vector fields the on U they can be written as

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

then

$$g(X,Y) = g_{ij} dx^i \otimes dx^j (X,Y) = g_{ij} dx^i \otimes dx^j (X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = g_{ij} X^i Y^j$$

Lets now compute the Levi-Civita connection in local coordinates. In order to do that we will use the properties of the connection. We have,

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j} = X^i \nabla_{\frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j} \\ &= X^i \frac{\partial}{\partial x^i}(Y^j) \frac{\partial}{\partial x^j} + X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\end{aligned}$$

We set $\Gamma^k_{ij} \frac{\partial}{\partial x^k} := \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$. So

$$\nabla_X Y = X^i \frac{\partial}{\partial x^i}(Y^k) \frac{\partial}{\partial x^k} + X^i Y^j \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

The functions Γ^k_{ij} are called Christoffel symbols.

With the use of Christoffel symbols we can compute the components of the (3,1)-Riemann curvature tensor in local coordinates. Using the fact that $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ where $\{\frac{\partial}{\partial x^i}\}$ is a local coordinate frame we have,

$$\begin{aligned}\mathrm{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} &= \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \\ &= \nabla_{\frac{\partial}{\partial x^i}} \Gamma^m_{jk} \frac{\partial}{\partial x^m} - \nabla_{\frac{\partial}{\partial x^j}} \Gamma^m_{ik} \frac{\partial}{\partial x^m} \\ &= \frac{\partial}{\partial x^i}(\Gamma^l_{jk}) \frac{\partial}{\partial x^l} - \frac{\partial}{\partial x^j}(\Gamma^l_{ik}) \frac{\partial}{\partial x^l} + \Gamma^m_{jk}\Gamma^l_{im} \frac{\partial}{\partial x^l} - \Gamma^m_{ik}\Gamma^l_{jm} \frac{\partial}{\partial x^l}\end{aligned}$$

Therefore, the components R^l_{ijk} of the (3,1)-Riemann curvature tensor are:

$$R^l_{ijk} = \frac{\partial}{\partial x^i}(\Gamma^l_{jk}) - \frac{\partial}{\partial x^j}(\Gamma^l_{ik}) + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}$$

Taking the metric contraction we can compute the components of the (4,0)-Riemann curvature tensor

$$R_{ijks} = R^l_{ijk} g_s l \tag{2.1}$$

Lastly, the components of the Ricci curvature tensor are

$$R_{js} = R_{ijks} g^{ik} \tag{2.2}$$

and of the scalar curvature tensor are

$$R = g^{jk} R_{jk}. \tag{2.3}$$

2.2 Left invarant metrics on Lie groups

Our attention will mainly be on left invariant metrics on Lie groups. Here we change the notation for a Riemannian metric from g to $\langle ., . \rangle$.

Definition 2.2.1. Let G be a Lie group and $\mathfrak{g}$ the Lie algebra i.e. the space of left invariant vector fields on G. A Riemannian metric on a Lie group G is called left invariant if

$$\langle u, v \rangle_y = \langle d(L_x)_y u, d(L_x)_y v \rangle_{L_x y}$$

for all $x, y \in G$ and $u, v \in T_y G$.

The fact that $\langle u, v \rangle_y = \langle d(L_x)_y u, d(L_x)_y v \rangle_{L_x y}$ for all $x, y \in G$ and $u, v \in T_y G$ is equivalent to the assertion that the left translations $L_x : G \to G$, $L_x(y) = xy$ are isometries for all $x \in G$. Also, The fact that $\langle ., . \rangle$ is left invariant implies that for every X, Y, Z left invariant vector fields on G we have $X\langle Y, Z \rangle = 0$. There is a direct way to obtain a left invariant metric on a Lie group. Suppose that $\langle ., . \rangle_e$ is an inner product on $T_e G$ then $\langle u, v \rangle_x = \langle d(L_{x^{-1}})_x u, d(L_{x^{-1}})_x v \rangle_e$ where $u, v \in T_x G, x \in G$ is a left invariant metric on G.

Now we have to describe the Levi-Civita connection on a Lie group equipped with a left invariant Riemannian metric.

Lemma 2.2.2. Let $\langle ., . \rangle$ be a left invariant metric on a Lie group G. If X, Y, Z be left invariant vector fields on G, then the Levi-Civita connection is given by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle)$$

Proof. Using the Koszul formula

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle + Z\langle Y, X \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle$$

as well as the fact that for every X, Y, Z left invariant vector fields on G, $X\langle Y, Z \rangle = 0$, by direct computation we get

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle).$$

□

We can exploit the simplicity of the Levi-Civita connection to describe the curvature tensor of a left invariant metric on a Lie group G.

Lemma 2.2.3. Suppose that $\langle ., . \rangle$ is a left invariant metric on a Lie group G, ∇ the Levi-Civita connection and X, Y, Z, W left invariant vector fields on G. Then the Riemann curvature tensor is given by

$$\langle \mathrm{Rm}(X,Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle$$

for any left invariant vector fields X, Y, Z on G.

Proof. By the definition of the Riemann curvature tensor

$$\begin{aligned}\langle \mathrm{Rm}(X,Y)Z, W\rangle &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \nabla_Y \nabla_X Z, W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle \\ &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - Y\langle \nabla_X Z, W\rangle \\ &\qquad + \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle \\ &= \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle,\end{aligned}$$

where the last equality follows from the fact that $X\langle Y, Z\rangle = 0$ for any left invariant vector fields X, Y, Z.

□

2.3 Homogeneous metrics and homogeneous models

Suppose that M is a smooth manifold.

Definition 2.3.1. A Riemannian metric g is called locally homogeneous if for every $x, y \in M$ there exist neighborhoods $U_x \subset M$ of x and $U_y \subset M$ of y and an isometry γ from U_x to U_y such that $\gamma(x) = y$. A Riemannian metric is called homogeneous if for every $x, y \in M$ there exist an isometry γ from M to itself such that $\gamma(x) = y$.

Every left invariant metric on a Lie group is locally homogeneous. A nice result that connects the locally homogeneous metrics with the homogeneous ones is the following

Proposition 2.3.2. If the manifold M is simply connected then every locally homogeneous metric is homogeneous.

Proof. We refer to [16] for the proof of this result. □

Definition 2.3.3. A homogeneous model is a pair $(\tilde{M},)$ where $\tilde{M}$ is a simply connected manifold and $\tilde{g}$ a homogeneous metric on $\tilde{M}$. Thus if (M, g) is a manifold with a locally homogeneous metric then its universal cover $\tilde{M}$ with the lifted metric $\tilde{g}$ has the structure of a homogeneous model.

An equivalent description of manifolds equipped with complete locally homogeneous metrics is that of model geometries.

Definition 2.3.4. A model geometry is a triple (M, G, G_*) where M is a simply connected smooth manifold, G_* a Lie group and G is a group of diffeomorphisms which acts transitively on M such that for each $x \in M$, the isotropy group

$$G_x = \{\gamma \in G \mid \gamma(x) = x\}$$

is isomorphic to G_*.

A model geometry is called maximal if the group G is maximal among the subgroups of the diffeomorphism group $\mathrm{Diff}(M)$ that have compact isotropy groups. The concepts of homogeneous model and model geometry are equivalent and it follows from the following proposition.

Proposition 2.3.5. Every model geometry (M, G, G_*) may be regarded as a complete homogeneous space (M, g) and vice versa if (M, g) is a complete homogeneous manifold then $(M, \mathrm{Iso}(M, g), \mathrm{Iso}_x(M, g))$ is a model geometry.

Proof. For a proof we refer to [4], p. 4-8. □

In the three-dimensional case there are nine classes of homogeneous manifolds. They are summarized in the following table ([9]).

Classes of homogeneous manifolds			
Class	Description	Dimensions of isometry group	Thurston's geometry
$\mathbb{R}^3$	Commutative	3 or 4 or 6	E^3
SU(2)	Simple	3 or 4 or 6	S^3
SL(2,$\mathbb{R}$)	Simple	3 or 4	SL(2,$\mathbb{R}$)
Heisenberg	Nilpotent	3 or 4	Nil
E(1,1)	Solvable	3	Solv
E(2)	Solvable	3 or 4 or 6	E^3
$\mathbb{H}^3$	Not a Lie group	6	$\mathbb{H}^3$
$\mathrm{SO}(3) \times \mathbb{R}$	Not a Lie group	4	$S^2 \times \mathbb{R}$
$\mathbb{H}^2 \times \mathbb{R}$	Not a Lie group	4	$\mathbb{H}^2 \times \mathbb{R}$

As we can see most of the homogeneous models are Lie groups but there exist homogeneous models which are not Lie groups.

3 Ricci flow

The Ricci flow equation is a heat-like evolution equation for the metric on a Riemannian manifold where we deform the metric in the direction of the Ricci curvature tensor.

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t)). \tag{3.1}$$

A solution $g(t)$ to the Ricci flow equation is called Ricci flow.

In the view of equation (3.1) we consider the metrics $g(t)$ as a positive sections of the fixed bundle S^2T^*M of symmetric covariant 2-tensors over M. The time derivative makes sense since at each point $p \in M$ we differentiate $g(t)$ in the vector space given by the fibre of this bundle at p (see [1]).

A classical example to illustrate how the Ricci flow equation behaves is the following:

Example 1. Suppose that g_0 is the standard metric on the sphere S^n of radius 1. Consider the 1-parameter family of metrics on the sphere $g(t) = r(t)g_0$. This family is a Ricci flow if and only if $r(t) = (1 - 2(n-1)t)$. Indeed, using that the standard metric on the sphere satisfies the relation $\mathrm{Ric}(g_0) = (n-1)g_0$ as well as the scaling invariance of the Ricci tensor, $\mathrm{Ric}(g) = \mathrm{Ric}(\sigma g)$ for $\sigma > 0$, we have

$$\frac{\partial g(t)}{\partial t} = \frac{\partial(1-2(n-1)t)g_0}{\partial t} = -2(n-1)g_0 = -2\,\mathrm{Ric}(g_0) = -2\,\mathrm{Ric}(g(t))$$

This solution exists on the time interval $(-\infty, \frac{1}{2(n-1)})$ and the sphere will collapse to a point in finite time $T = \frac{1}{2(n-1)}$.

Looking closely at this example we observe that we did not use any specific property of the standard metric on the sphere apart from the fact that it is an Einstein metric. Recall that a metric g is Einstein if there exist a constant $c \in \mathbb{R}$ such that $\mathrm{Ric} = cg$. So we can generalize Example 1 to Einstein manifolds:

Example 2. Consider a manifold with an initial metric g_0 satisfying the property $\mathrm{Ric}(g_0) = \lambda g_0$ where $\lambda \in \mathbb{R}$. Then a solution to the Ricci flow equation with initial data $g(0) = g_0$ is given by $g(t) = (1 - 2\lambda t)g_0$.

Indeed, with direct computation we have

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\lambda g_0 \\ &= -2\,\mathrm{Ric}(g_0) \\ &= -2\,\mathrm{Ric}((1-2\lambda t)g_0) \\ &= -2\,\mathrm{Ric}(g(t))\end{aligned}$$

So, $g(t)$ is a solution to the Ricci flow equation.

An important aspect of the Ricci flow is the diffeomorphism invariance.

Proposition 3.0.1. Suppose that $\bar{g}(t)$ is a solution to the Ricci flow equation and $\phi : M \to \bar{M}$ a time-independent diffeomorphism. Then the pullback metric $g(t) = \phi^*\bar{g}(t)$ is a solution to the Ricci flow equation as well.

Proof. Indeed, we have:

$$\begin{aligned}
\frac{\partial}{\partial t} g(t) &= \frac{\partial}{\partial t} \phi^* \bar{g}(t) \\
&= \phi^* \frac{\partial}{\partial t} g(\bar{t}) \\
&= \phi^*(-2\operatorname{Ric}(\bar{g}(t)) \\
&= -2\phi^* \operatorname{Ric}(\bar{g}(t)) \\
&= -2\operatorname{Ric}(\phi^* \bar{g}(t)) \\
&= -2\operatorname{Ric}(g(t))
\end{aligned}$$

where we used the diffeomorphism invariance of the Ricci curvature tensor $\phi^* \operatorname{Ric}(g) = \operatorname{Ric}(\phi^* g)$ as well as the fact that the time derivative commutes with pullbacks of time-independent diffeomorphisms. □

An interesting consequence of the fact that the Ricci flow equation is invariant under the full diffeomorphism group is that the Ricci flow preserves isometries. Thus, if an initial metric is locally homogeneous then it will remain locally homogeneous as long as the solution exist. Also, since the Ricci flow preserves isometries, if the initial manifold is a quotient of a Riemannian manifold by a group of isometries then it will remain so under the Ricci flow. Therefore, using the fact that the Ricci flow commutes with the covering map $\mu : \tilde{M} \to M$ we can study the Ricci flow of any locally homogeneous metric by studying that of its homogeneous model and vise versa.

In the case of the Ricci flow we do not consider a single manifold equipped with a metric g but rather a manifold with a one parameter family of metrics, parametrized by the flow parameter t, that satisfies the Ricci flow equation. So in order for this consideration to have a meaning we must have a short time existence and uniqueness for this equation. The Ricci flow equation is not strictly parabolic so we cannot assume short time existence and uniqueness. In order to prove short time existence and uniqueness we have to modify the Ricci flow equation in order to be strictly parabolic. Also, we should note that the reason why the Ricci flow equation is not strictly parabolic is exactly the diffeomorphism invariance of the Ricci tensor.

Theorem 3.0.2. If (M, g_0) is a closed Riemannian manifold, there exists an $\epsilon > 0$ and a unique solution $g(t)$, defined for time $t \in [0, \epsilon)$, to the Ricci flow equation such that $g(0) = g_0$.

Proof. We will give a sketch of the proof for short time existence. An interested reader can consult [4], [5] or [1] for a detailed proof.

R. Hamilton, [8], gave a proof for this result using heavy analytic techniques like the Nash–Moser implicit function theorem. Later D. DeTurk, [6], gave a simpler proof for this theorem. The main idea is the following. The Ricci flow equation as noted before is not strictly parabolic since the principal symbol of the non-linear partial differential operator $-2\operatorname{Ric}$ has a non trivial kernel. So we are searching for an equivalent flow that is strictly parabolic, apply short time existence to this new equation and then pullback the solution by a time dependant diffeomorphism. This new equation is called Ricci-DeTurk flow.

Suppose that we have a fixed background metric $\bar{g}$, with a Levi-Civita connection $\bar{\nabla}$. We define the Ricci-DeTurk flow by,

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{R}_{ij} + \nabla_i W_j + \nabla_j W_i,$$

where $g(0) = g_0$ is the initial metric and W_i is an one form given by

$$W_i = g_{ik} g^{pq} (\Gamma^k_{pq} - \bar{\Gamma}^k_{pq}).$$

Note that if $g(s)$ is a 1-parameter family of metrics with variation $\frac{\partial}{\partial s} g_{ij}(s) = v_{ij}$ and $g(0) = g_0$ then

$$\frac{\partial}{\partial s}|_{s=0} W(g(s))_j = X_j + \text{ terms of order zero in } v$$

where $X = \frac{1}{2}\nabla W - \operatorname{div} v$.

Lemma 3.0.3. The following equality holds:

$$\frac{\partial}{\partial s}|_{s=0}(-2\,\mathrm{R}_{ij} + \nabla_i W_j + \nabla_j W_i) = \Delta_L v_{ij} + \text{ terms of order one in } v.$$

Here Δ_L is the Lichnerowicz Laplacian for covariant 2-tensors, which is an elliptic differential operator.

Thus, the Ricci-DeTurk flow is strictly parabolic so given any smooth initial metric g_0 on a closed manifold there exist a unique solution $g(t)$ to this equation such that $g(0) = g_0$.

Now consider the time-dependent diffeomorphisms defined by the ODE:

$$\frac{\partial}{\partial t}\phi_t = -W^*$$
$$\phi_0 = \mathrm{id}$$

where $W^*(t)$ is the dual vector field of W with respect to $g(t)$. Pulling back the solution of the Ricci-DeTurk flow $g(t)$ by the diffeomorphisms ϕ_t we obtain a solution to the Ricci flow $\tilde{g}(t) = \phi_t^* g(t)$. Indeed,

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{g} &= \frac{\partial}{\partial t}\phi_t^* g(t) \\
&= \phi_t^* \frac{\partial}{\partial t} g(t) + \frac{\partial}{\partial s}|_{s=0}(\phi_{t+s}^* g(t)) \\
&= \phi_t^*(-2\,\mathrm{Ric}\, g(t)) + \phi_t^*(\mathcal{L}_{W(t)} g(t)) - \mathcal{L}_{(\phi_t^{-1})_* W(t)}(\phi_t^* g(t)) \\
&= -2\,\mathrm{Ric}(\phi_t^* g(t)) \\
&= -2\,\mathrm{Ric}(\tilde{g}(t)).
\end{aligned}$$

□

We should note that if a solution to the Ricci flow $g(t)$ converges to a metric $g(t) \to g_\infty$ then the limit metric g_∞ must be an Einstein metric. Thus, if a manifold does not admit an Einstein metric there is no hope for any Ricci flow on this manifold to converge.

When we consider a manifold equipped with a one parameter family of metrics which satisfies the Ricci flow equation all the geometric quantities that are defined with the use of the metric change as well. It is usually convenient to modify the Ricci flow equation in order to keep the volume fixed.

Definition 3.0.4. Let M be a compact manifold of dimension n equipped with an one parameter of metrics $g(t)$. Let $\langle r(t)\rangle = \frac{\int_M R(t)d\mu}{\text{Vol}(M)}$ be the average scalar curvature of the metric $g(t)$. The equation

$$\frac{\partial}{\partial t}g(t) = -2\operatorname{Ric}(g(t)) + \frac{2}{n}\langle r(t)\rangle g(t),$$

is called normalized Ricci flow equation since a solution to this equation has fixed volume.

The solution to the Ricci flow and to the normalized Ricci flow are related by a rescaling in space and time. In particular given a solution $g(t)$ to the Ricci flow equation, the metric $\bar{g}(\bar{t})) = c(t)g(t)$ where

$$c(t) = \exp\left(\frac{2}{n}\int_0^t r(\tau)d\tau\right), \qquad \bar{t}(t) = \int_0^t c(\tau)d\tau$$

is a solution to the normalized Ricci flow equation.

A remarkable result due to R. Hamilton ([8]) which illustrates the impact of the study of (normalized) Ricci flow for three dimensional manifolds is the following.

Theorem 3.0.5. Let (M^3, g_0) be a closed Riemannian manifold with positive Ricci curvature. Then there exists a unique solution $g(t)$ to the normalized Ricci flow equation with $g(0) = g_0$ for all times $t \geq 0$. Furthermore, as $t \to \infty$ the metrics $g(t)$ converge exponentially in every C^m-norm to a C^∞ metric g_∞ with constant positive sectional curvature.

This theorem implies that all three-dimension manifolds with positive Ricci curvature are homeomorphic to the three-dimensional sphere. In section 5 we will prove a similar result due to J. Isenberg and M. Jackson ([9]) which states that all homogeneous normalized Ricci flows on the Lie group SU(2) will converge to a homogeneous metric with positive scalar curvature.

The last result we have to mention regarding the Ricci flow is the long-time existence theorem. We start with some definitions.

Definition 3.0.6. Suppose $(M, g(t)), t \in [0, T)$, is a solution to the Ricci flow equation. We say that $[0, T)$ is the maximum interval of existence if either $T = \infty$ or that $T < \infty$ and there does not exist $\epsilon > 0$ and a smooth solution $\tilde{g}(t), t \in [0, T + \epsilon)$ of the Ricci flow such that $\tilde{g}(t) = g(t)$ for $t \in [0, T)$. In the latter case we say that $g(t)$ forms a singularity at time T or simply $g(t)$ is a singular solution.

With this terminology we can talk about the Ricci flow $g(t)$ with initial metric g_0 on a maximal interval $[0, T)$. The next theorem provides the long-time existence for a solution to the Ricci flow equation. Proof of this result can be found in [1].

Theorem 3.0.7. Let M be a compact manifold with a smooth initial metric g_0. The unique solution $g(t)$ of the Ricci flow equation with $g(0) = g_0$ exists on a maximal interval $0 \leq t < T \leq \infty$. Furthermore, $T < \infty$ if and only if

$$\lim_{t \to T} \sup_{x \in M} |\operatorname{Rm}(x,t)| = \infty$$

i.e. as $t \to \infty$ the Riemann curvature tensor is unbounded.

4 Milnor frame

In order to study the behaviour of the Ricci flow equation for left invariant metrics on Lie groups we will first introduce a particular class of frames on the Lie algebra of a Lie group such that the structure constants have a special form. Our ultimate goal is to use this frame to diagonalize a left invariant metric on a Lie group and the Ricci tensor in order for the Ricci flow equation to have a simple form. The diagonalization will of the Ricci flow equation will be proven in the next section.

4.1 Algebraic preliminaries

We begin with some algebraic preliminaries regarding frames on Lie algebras. A Lie algebra structure on the vector space $\mathbb{R}^3$ is given by

$$[. \, , \, .] \in \Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3$$

where $\Lambda^2\mathbb{R}^{3*}$ is the space of alternating 2-covectors in $\mathbb{R}^3$.

Lemma 4.1.1. The space $\mathrm{End}(\mathbb{R}^3)$ is naturally isomorphic with $\Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3$ via the map

$$\phi : \mathrm{End}(\mathbb{R}^3) \to \Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3 \simeq \mathrm{Hom}(\Lambda^2\mathbb{R}^3, \mathbb{R}^3)$$

given by

$$L \mapsto L \circ \rho$$

where $\rho \in \Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3$ is the cross product of $\mathbb{R}^3$,

$$\rho : (u, w) \mapsto u \times w, \; \forall x, y \in \mathbb{R}^3.$$

Proof. The space $\Lambda^2\mathbb{R}^{3*}$ is a 3-dimensional vector space and it is spanned by the basis

$$e_2 \wedge e_3, \; e_3 \wedge e_1, \; e_1 \wedge e_2,$$

where e^1, e^2, e^3 form the dual basis of the standard basis of $\mathbb{R}^3$. These elements are equal to

$$e^1 \circ \rho, e^2 \circ \rho \text{ and } e^3 \circ \rho.$$

Indeed,

$$e^1 \circ \rho(v, w) = (v, w)_1 = v_2 w_3 - v_3 w_2 = e^2 \wedge e^3(v, w),$$
$$e^2 \circ \rho(v, w) = (v, w)_2 = v_3 w_1 - v_1 w_3 = e^3 \wedge e^1(v, w)$$

and

$$e^3 \circ \rho(v, w) = (v, w)_3 = v_1 w_2 - v_2 w_1 = e^1 \wedge e^2(v, w)$$

for any $v, w \in \mathbb{R}^3$. From this we see that the map

$$\gamma : \mathbb{R}^{3*} \to \Lambda^2(\mathbb{R}^{3*})$$

given by

$$\xi \mapsto \xi \circ \rho$$

is an isomorphism. Therefore, we have the isomorphisms

$$\mathrm{End}(\mathbb{R}^3) \simeq \mathbb{R}^{3*} \otimes \mathbb{R}^3 \simeq \Lambda^2(\mathbb{R}^{3*}) \otimes \mathbb{R}^3$$

The first map $\alpha : \mathrm{End}(\mathbb{R}^3) \to \mathbb{R}^{3*} \otimes \mathbb{R}^3$ is defined by $\alpha(L) = \sum_j e^j \circ L \otimes e_j$. So the map

$$\phi : \mathrm{End}(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^{3*}) \otimes \mathbb{R}^3$$

is defined by

$$\phi(L) = \sum_j e^j \circ L \circ \rho \otimes e_j = L \circ \rho$$

□

Now suppose that G is a 3-dimensional Lie group with Lie algebra $\mathfrak{g}$. A frame for $\mathfrak{g}$ is a linear isomorphism

$$\beta : \mathbb{R}^3 \to \mathfrak{g}$$

The group $\mathrm{GL}(3, \mathbb{R})$ acts on the set of frames denoted by, $\mathcal{F}(\mathfrak{g})$ by

$$A_*(\beta) := \beta \circ A^{-1}$$

where $\beta \in \mathcal{F}(\mathfrak{g})$ and $A \in \mathrm{GL}(3, \mathbb{R})$. Given a frame $\beta \in \mathcal{F}(\mathfrak{g})$ we can define the map

$$[.\ ,\ .]_\beta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

by

$$[x, y]_\beta = \beta^{-1}([\beta x.\beta y]).$$

This means that the map $[.\ ,\ .]_\beta$ can be regarded as an element of $\Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3$ hence $[u, v]_\beta = \sigma_\beta(u \times v)$ for a unique matrix $\sigma_\beta \in \mathrm{End}(\mathbb{R}^3) \simeq M_3(\mathbb{R}^3)$. Now, for every $x, y \in \mathbb{R}^3$ we have

$$[\beta(x), \beta(y)] = \beta(\sigma_\beta(x \times y)). \tag{4.1}$$

and so we can determine σ_β uniquely. The next two results describe the transformation of the matrix σ_β under the action of the group $\mathrm{Gl}(3, \mathbb{R})$.

Lemma 4.1.2. Let $A \in \mathrm{GL}(3, \mathbb{R})$. Then

$$Ax \times Ay = \det(A)(A^{-1})^T(x \times y).$$

Proof. Using the formula $\langle a \times b, c\rangle = \det(a, b, c)$ where $a, b, c \in \mathbb{R}^3$ we have that for every $x, y, z \in \mathbb{R}^3$

$$\begin{aligned}
\langle Ax \times Ay, z\rangle &= \det(Ax, Ay, z) \\
&= \det(A)(x, y, A^{-1}z) \\
&= \det(A)\langle x \times y, A^{-1}z\rangle \\
&= \det(A)\langle (A^{-1})^T(x \times y), z\rangle \\
&= \langle \det(A)(A^{-1})^T(x \times y), z\rangle
\end{aligned}$$

Therefore, $Ax \times Ay = \det(A)(A^{-1})^T(x \times y)$ for every $x, y \in \mathbb{R}^3$.

□

Corollary 4.1.3. Let $\beta \in \mathcal{F}(\mathfrak{g})$ and $A \in \mathrm{GL}(3,\mathbb{R})$. Then

$$\sigma_{A_*\beta} = (\det(A))^{-1} A \sigma_\beta A^T$$

Proof. Using the formula $[\beta(x),\beta(y)] = \beta(\sigma_\beta(x \times y))$ we have

$$\begin{aligned}
\beta(A^{-1}\sigma_{A_*\beta}(x \times y)) &= A_*\beta(\sigma_{A_*\beta}(x \times y)) \\
&= [A_*\beta(x), A_*\beta(y)] \\
&= [\beta(A^{-1}x), \beta(A^{-1}y)] \\
&= [\beta(\sigma_\beta(A^{-1}x \times A^{-1}y))] \\
&= \beta(\sigma_\beta(\det(A^{-1})A^T(x \times y))).
\end{aligned}$$

Therefore, for every $x, y \in \mathbb{R}^3$ we have

$$A^{-1}\sigma_{A_*\beta}(x \times y) = \det(A^{-1})\sigma_\beta A^T(x \times y),$$

or equivalently,

$$\sigma_{A_*\beta}(x \times y) = \det(A^{-1}) A \sigma_\beta A^T(x \times y),$$

for every $x, y \in \mathbb{R}^3$. So the transformation of the matrix σ_β is given by,

$$\sigma_{A_*\beta} = \det(A^{-1}) A \sigma_\beta A^T.$$

□

4.2 Relation with left invariant metrics

If $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an inner product on a Lie algebra $\mathfrak{g}$ and $\beta \in \mathcal{F}(\mathfrak{g})$ a frame, we define the inner product g_β on $\mathbb{R}^3$ by

$$g_\beta(x, y) = g(\beta(x), \beta(y)).$$

If we consider the action of the group $\mathrm{Gl}(3,\mathbb{R})$ on the space of frames $\mathcal{F}(\mathfrak{g})$ given by $(A,\beta) = A_*\beta$ then the next Lemma shows the expression of an inner product with respect to the frame $A_*\beta$.

Lemma 4.2.1. Let $A \in GL(3,\mathbb{R})$. Then

$$g_{A_*\beta} = (A^{-1})^* g_\beta,$$

where $g_\beta = \beta^* g$

Proof. For every $x, y \in \mathbb{R}^3$ we have

$$\begin{aligned}
g_{A_*\beta}(x, y) &= g(\beta(A^1 x), \beta(A^1 y)) \\
&= g_\beta(A^{-1}x, A^{-1}y) \\
&= ((A^{-1})^* g)_\beta(x, y)
\end{aligned}$$

Therefore, $g_{A_*\beta} = (A^{-1})^* g_\beta$. □

Remark 4.2.2. In this thesis we will often identify an inner product $\gamma \in IP(\mathbb{R}^3)$ with the associated Hermitian matrix $(\gamma(e_i, e_j))$ where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$. Then for $A \in \mathrm{GL}(3, \mathbb{R})$ we have

$$A^*\gamma = A^T \gamma A. \tag{4.2}$$

In this way the formula of Lemma 4.2.1. becomes:

$$g_{A_*\beta} = (A^{-1})^T g_\beta A^{-1}.$$

Suppose that β is a frame for $\mathfrak{g}$, then

$$[\beta(e_i), \beta(e_j)] = \sum_k c_{ij}^k \beta(e_k)$$

We will express σ_β in terms of the structure constants c_{ij}^k. Recall that σ_β is determined by

$$[\beta(x), \beta(y)] = \beta(\sigma_\beta(x \times y))$$

So by taking $x = e_i$ and $y = e_j$ we get that:

$$\sum_k c_{ij}^k \beta(e_k) = \beta(\sigma_\beta(e_i \times e_j))$$

Hence, $\sigma_\beta(e_i \times e_j) = \sum_k c_{ij}^k e_k$. In a matrix form we can write σ_β as:

$$\sigma_\beta = \begin{pmatrix} c_{23}^1 & c_{31}^1 & c_{12}^1 \\ c_{23}^1 & c_{31}^2 & c_{12}^2 \\ c_{23}^3 & c_{31}^3 & c_{12}^3 \end{pmatrix}$$

Definition 4.2.3. Let g be a left invariant Riemannian metric on a Lie group G. By a Milnor frame for the pair $(\mathfrak{g}, g_e)$ we mean a frame $\beta \in \mathcal{F}(\mathfrak{g})$ such that both σ_β and g_β are diagonal and the entries of σ_β are in the set $\{-2, 0, 2\}$.

A Milnor frame for (G, g) is a left invariant frame β on the Lie group G such that the frame evaluated at the identity element e is a Milnor frame for $(\mathfrak{g}, g_e)$.

We observe that if $MF(\mathfrak{g}, g_e)$ is the collection of all Milnor frames on $(\mathfrak{g}, g_e)$ and $MF(G, g)$ is the collection of all Milnor frames for (G, g) then the evaluation map

$$\begin{aligned} ev : MF(G, g) &\to MF(\mathfrak{g}, g) \\ \beta &\mapsto \beta_e \end{aligned}$$

is a bijection.

A Lie group G is called unimodular if its volume form is bi-invariant. It was proven in [12] that for any left-invariant metric g on a unimodular Lie group G there exists a frame $\beta \in MF(G, g)$ such that this frame is a Milnor frame for the metric g.

Milnor frames will be extremely important not only for the computation of solutions of the Ricci flow equation for left invariant metrics on Lie groups but also for the study of the quasi-convergence classes.

In our definition of Milnor frame we did not specify the order of the entries of the matrix σ. For example if we consider a Milnor frame $\beta = (\beta_1, \beta_2, \beta_3)$ for the pair (G, g) where G is the Heisenberg group and g is a left invariant metric on G then the matrix σ_β of the structure constants can be written in the form

$$\sigma_\beta = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But we can consider also the Milnor frame $\beta' = (\beta_2, \beta_1, \beta_3)$ which has the matrix of structure constants

$$\sigma_{\beta'} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, these two frames are the same up to order and they correspond to isomorphic matrices σ_β and $\sigma_{\beta'}$. To overcome this inconvenience we say that σ has the standard form provided that we have fixed a particular form for the matrix σ. For example in the case of the Heisenberg group we fix the standard for of the matrix σ to be

$$\sigma_\beta = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5 Ricci flow on Homogeneous manifolds

In this section we will describe the Ricci flow for some 3-dimensional homogeneous manifolds and give the general formulas for the Ricci flow for left invariant metrics on Lie groups. We study the Ricci flow equations for the spaces $\mathbb{R}^3, \mathbb{H}^3, \mathrm{SU}(2), S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ ([9]). The Ricci flow of homogeneous metrics on $\mathbb{R}^3$ and $\mathbb{H}^3$ are the simplest among those examples. Their simplicity follows from the fact that these kind of metrics have constant Ricci curvature and are Einstein metrics. In the case of the Ricci flow on SU(2) the situation becomes more complicated. We will prove that the normalized Ricci flow of every left invariant metric on SU(2) converges to a metric of constant positive scalar curvature. The last two examples are unique on their own. There are no Einstein metrics on the $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ so the Ricci flow cannot converge. In the case of $S^2 \times \mathbb{R}$ we will prove that any Ricci flow $g(t)$ develops a curvature singularity in finite time, that is there exist a time $T < \infty$ such that the curvature of $g(t)$ diverges to infinity as $t \to T$. The Ricci flow on $\mathbb{H}^2 \times \mathbb{R}$ has the property that the component of $\mathbb{R}$ shrinks to zero while the components of $\mathbb{H}^2$ expand to infinity. Such a phenomenon is called pancake degeneracy. The computations of the Ricci flow equation for left invariant metrics will be given in later sections where we will also compute the quasi-convergence class for these cases.

5.1 Ricci flow for left invariant metrics on Lie groups

Suppose that G is a 3-dimensional unimodular Lie group with a left invariant metric g. Consider a Milnor frame $\beta = (\beta_1, \beta_2, \beta_3)$ for the pair (G, g) That is g is diagonal

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$$

with respect to the dual frame $\omega = (\omega_1, \omega_2, \omega_3)$ of β and the structure constants satisfy, $[\beta_i, \beta_j] = c_{ij}^k \beta_k$ where $c_{ij}^k \in \{2, 0, -2\}$ and $c_{ij}^k = 0$ unless i, j, k are distinct. Using the Milnor frame we can compute the Riemann curvature tensor and as a result the Ricci curvature tensor. This will lead us to a simplified expression of the Ricci flow equation. First of all we normalize this frame in order to make it orthonormal and compute the Ricci curvature tensor.

Proposition 5.1.1. Let

$$e_1 = A^{-1/2}\beta_1,\ e_2 = B^{-1/2}\beta_2,\ e_3 = C^{-1/2}\beta_3.$$

where $(\beta_1, \beta_2, \beta_3)$ is a Milnor frame. Then this new frame is orthonormal and the structure constants for this frame are

$$[e_1, e_2] = \frac{Cc_{12}^3}{(ABC)^{1/2}} e_3,$$

$$[e_2, e_3] = \frac{Ac_{23}^1}{(ABC)^{1/2}} e_1,$$

$$[e_3, e_1] = \frac{Bc_{31}^2}{(ABC)^{1/2}} e_2.$$

Proof. Indeed by direct computation we have,

$$
\begin{aligned}
[e_1, e_2] &= [A^{-1/2}\beta_1, B^{-1/2}\beta_2] \\
&= A^{-1/2}B^{-1/2}[\beta_1, \beta_2] \\
&= (AB)^{-1/2}c_{12}^3\beta_3 \\
&= (AB)^{-1/2}C^{1/2}c_{12}^3 e_3 \\
&= (ABC)^{-1/2}Cc_{12}^3 e_3 \\
&= \frac{Cc_{12}^3}{(ABC)^{1/2}}e_3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
[e_2, e_3] &= [B^{-1/2}\beta_2, C^{-1/2}\beta_3] \\
&= B^{-1/2}C^{-1/2}[\beta_2, \beta_3] \\
&= (BC)^{-1/2}c_{23}^1\beta_3 \\
&= (BC)^{-1/2}A^{1/2}c_{23}^1 e_1 \\
&= (ABC)^{-1/2}Ac_{23}^1 e_1 \\
&= \frac{Ac_{23}^1}{(ABC)^{1/2}}e_1.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
[e_3, e_1] &= [C^{-1/2}\beta_3, A^{-1/2}\beta_1] \\
&= C^{-1/2}A^{-1/2}[\beta_3, \beta_1] \\
&= (BC)^{-1/2}c_{31}^2\beta_2 \\
&= (BC)^{-1/2}A^{1/2}c_{31}^2 e_2 \\
&= (ABC)^{-1/2}Ac_{31}^2 e_2 \\
&= \frac{Bc_{31}^2}{(ABC)^{1/2}}e_2.
\end{aligned}
$$

□

Putting $l_1 = A, l_2 = B$ and $l_3 = C$ we can compute the components of the Levi-Civita connection as follows:

$$
\begin{aligned}
\langle \nabla_{e_i} e_j, e_k \rangle &= \frac{1}{2}(\langle [e_i, e_j], e_k \rangle + \langle [e_k, e_i], e_j \rangle - \langle [e_j, e_k], e_i \rangle) \\
&= \frac{1}{2(ABC)^{1/2}}(l_k c_{ij}^k + l_j c_{ki}^j - l_i c_{jk}^i).
\end{aligned}
$$

Using the formula for the curvature tensor we have

$$\begin{aligned}K(e_i\wedge e_j)=\langle \mathrm{Rm}\,(e_i,e_j)\,e_j,e_i\rangle =&\left\langle \nabla_{e_i}e_j,\nabla_{e_j}e_i\right\rangle-\left\langle \nabla_{e_j}e_j,\nabla_{e_i}e_i\right\rangle-\left\langle \nabla_{[e_i,e_j]}e_j,e_i\right\rangle\\ =&\frac{1}{4ABC}\left(l_kc_{ij}^k-l_jc_{ik}^j-l_ic_{jk}^i\right)\left(l_kc_{ji}^k-l_ic_{jk}^i-l_jc_{ik}^j\right)\\ &-\frac{1}{2ABC}l_kc_{ij}^k\left(l_ic_{kj}^i-l_jc_{ki}^j-l_kc_{ji}^k\right)\\ =&\frac{1}{4ABC}\left(\left(l_ic_{jk}^i-l_jc_{ki}^j\right)^2-\left(l_kc_{ij}^k\right)^2\right)\\ &+\frac{2}{4ABC}l_kc_{ij}^k\left(l_ic_{jk}^i+l_jc_{ki}^j-l_kc_{ij}^k\right)\end{aligned}$$

Here we used that $\nabla_{e_j}e_j=0$, the skew-symmetry of c_{ij}^k in i and j and the fact that $\langle e_i,e_j\rangle=\delta_i^j$. Thus, the sectional curvatures $K(e_i\wedge e_j)=R(e_i,e_j,e_i,e_j)=\langle \mathrm{Rm}(e_i,e_j)e_j,e_i\rangle$ are given by:

$$\begin{aligned}K\left(e_2\wedge e_3\right)&=\frac{(c_{31}^2B-c_{12}^3C)^2}{4ABC}+c_{23}^1\frac{2c_{31}^2B+2c_{12}^3C-3c_{23}^1A}{4BC},\\ K\left(e_3\wedge e_1\right)&=\frac{(c_{12}^3C-c_{23}^1A)^2}{4ABC}+c_{31}^2\frac{2c_{12}^3C+2c_{23}^1A-3c_{31}^2B}{4AC},\\ K\left(e_1\wedge e_2\right)&=\frac{(c_{23}^1A-c_{31}^2B)^2}{4ABC}+c_{12}^3\frac{2c_{23}^1A+2c_{31}^2B-3c_{12}^3C}{4AB}.\end{aligned}$$

and $K(e_i,e_j)=0$ where $i\neq j$. So, using Definition 2.5.1. we can write the Ricci curvature tensor as follows:

$$\begin{aligned}\mathrm{Ric}\,(e_1,e_1)&=\frac{(c_{23}^1A)^2-(c_{31}^2B-c_{12}^3C)^2}{2ABC},\\ \mathrm{Ric}\,(e_2,e_2)&=\frac{(c_{31}^2B)^2-(c_{12}^3C-c_{23}^1A)^2}{2ABC},\\ \mathrm{Ric}\,(e_3,e_3)&=\frac{(c_{12}^3C)^2-(c_{23}^1A-c_{31}^2B)^2}{2ABC}.\end{aligned}$$

and $\mathrm{Ric}(e_i,e_j)=0$ if $i\neq j$. Thus as we can see the Ricci tensor is diagonal with respect to the frame $\{e_1,e_2,e_3\}$. We want now to prove that the Ricci flow will remain diagonal with respect to a Milnor frame. Let $S^2\mathfrak{g}^*$ denote the space of symmetric bilinear maps $\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$. Then the set of inner products $IP(\mathfrak{g})$ is open in $S^2\mathfrak{g}^*$. Let $g\in\mathfrak{g}$ be an inner product on $\mathfrak{g}$. Then g extends to a left invariant metric $\tilde g$ on G such that $\tilde g_e=g$. We regard the Ricci tensor as a map

$$\begin{aligned}\mathrm{Ric}:\Gamma(S^2T^*G)&\to\Gamma(S^2T^*G)\\ \tilde g&\mapsto\mathrm{Ric}(\tilde g)\end{aligned}$$

Now using the evaluation map ev_e at the identity element e of G we can write $\mathrm{Ric}(\tilde{g})_e$ as an element of $S^2\mathfrak{g}^*$. So we define the map $\overline{\mathrm{Ric}}_e$ by

$$\begin{aligned} \overline{\mathrm{Ric}}_e : IP(\mathfrak{g}) &\to S^2\mathfrak{g}^* \\ g &\mapsto \mathrm{Ric}(\tilde{g})_e \end{aligned}$$

which is evidently a smooth map. Thus it is a smooth vector field on $IP(\mathfrak{g})$. Given a one parameter family $g(t)$ of inner products on a Lie algebra $\mathfrak{g}$ we have the one parameter family $\widetilde{g(t)}$ of left invariant metrics on G given by $\widetilde{g(t)}_e = g(t)$. We consider the Ricci flow equation on G

$$\frac{\partial}{\partial t}\widetilde{g(t)} = -2\,\mathrm{Ric}(\widetilde{g(t)}), \qquad \widetilde{g(0)} = \tilde{g}. \tag{5.1}$$

and the following flow equation on $\mathfrak{g}$,

$$\frac{d}{dt}g(t) = -2\overline{\mathrm{Ric}}_e(g(t)), \qquad g(0) = g. \tag{5.2}$$

We will show that equations (5.1) and (5.2) are equivalent. In particular we will prove the following result.

Proposition 5.1.2. If $\widetilde{g(t)}$ is satisfies the equation (5.1) then $g(t)$ satisfies the equation (5.2). Conversely, if $g(t)$ satisfies the equation (5.2) then $\widetilde{g(t)}$ satisfies the equation (5.1).

Proof. Suppose that $\tilde{g}(t)$ satisfies (5.1) then $g(0) = \widetilde{g(0)}_e = \tilde{g}_e = g$ and

$$\begin{aligned} \frac{d}{dt}g(t) &= \frac{d}{dt}(\widetilde{g(t)}_e) \\ &= \left(\frac{\partial}{\partial t}\widetilde{g(t)}\right)_e \\ &= -2\,\mathrm{Ric}(\widetilde{g(t)})_e \\ &= -2\overline{\mathrm{Ric}}_e(g(t)) \end{aligned}$$

Thus, $g(t)$ satisfies (5.2). Now suppose that $g(t)$ satisfies (5.2). Then $\widetilde{g(0)} = \tilde{g}$ and

$$\begin{aligned} \left(\frac{\partial}{\partial t}\widetilde{g(t)}\right)_e &= \frac{\partial}{\partial t}(\widetilde{g(t)}_e) \\ &= \frac{d}{dt}g(t) \\ &= -2\overline{\mathrm{Ric}}_e(g(t)) \\ &= -2\,\mathrm{Ric}(\widetilde{g(t)})_e \end{aligned}$$

Since left translations by any element y of G is an isometry and Ricci flow preserves isometries (see Proposition 3.0.1) we get that

$$\left(\frac{\partial}{\partial t}\widetilde{g(t)}\right)_y = -2\,\mathrm{Ric}(\widetilde{g(t)})_y$$

for every $y \in G$. Finally,

$$\frac{\partial}{\partial t}\widetilde{g(t)} = -2\,\mathrm{Ric}(\widetilde{g(t)})$$

that is $\widetilde{g(t)}$ satisfies (5.1). □

Since $IP(\mathfrak{g})$ is open in $S^2\mathfrak{g}^*$ the equation (5.2) is the flow equation for the vector field $\overline{\mathrm{Ric}}_e$ on $IP(\mathfrak{g})$. Fix a frame $\beta : \mathbb{R}^3 \to \mathfrak{g}$. Then we will prove the following lemma

Lemma 5.1.3. Let g_t be a solution to the equation (5.2) for $|t| < \delta$. Then if $(g_0)_\beta$ is diagonal we get $(g_t)_\beta$ is diagonal for $|t| < \delta$.

Proof. We define the map S_β by

$$\begin{aligned} S_\beta : S^2\mathfrak{g}^* &\to \mathbb{R}^3 \\ T &\mapsto (T(\beta_1, \beta_2), T(\beta_1, \beta_3), T(\beta_2, \beta_3)) \end{aligned}$$

The set $\mathrm{Ker}\, S_\beta \cap IP(\mathfrak{g})$ is an open subset of $\mathrm{Ker}\, S_\beta$. We claim that if $S_\beta(g) = 0$ then $S_\beta(\overline{\mathrm{Ric}}_e(g)) = 0$. Indeed, we have

$$\begin{aligned} S_\beta(\overline{\mathrm{Ric}}_e(g)) &= (\mathrm{Ric}(\tilde{g})_e(\beta_1, \beta_2), \mathrm{Ric}(\tilde{g})_e(\beta_1, \beta_3), \mathrm{Ric}(\tilde{g})_e(\beta_2, \beta_3)) \\ &= (A^{1/2}B^{1/2}\,\mathrm{Ric}(\tilde{g})_e(e_1, e_2), A^{1/2}B^{1/2}\,\mathrm{Ric}(\tilde{g})_e(e_1, e_3), B^{1/2}C^{1/2}\,\mathrm{Ric}(\tilde{g})_e(e_2, e_3)) \\ &= 0 \end{aligned}$$

where $e_1 = A^{-1/2}\beta_1, e_2 = B^{-1/2}$ and $e_3 = C^{-1/2}$ as in Proposition 5.1.1. Then $\overline{\mathrm{Ric}}_e$ regarded as a vector field on $IP(\mathfrak{g})$ is tangent to $\mathrm{Ker}\, S_\beta \cap IP(\mathfrak{g})$. It follows that

$$\begin{aligned} (g_0)_\beta \text{ is diagonal } &\implies \\ g_0 \in \mathrm{Ker}\, S_\beta \cap IP(\mathfrak{g}) &\implies \\ \forall t \in (-\delta, \delta) : g_t \in \mathrm{Ker}\, S_\beta \cap IP(\mathfrak{g}) &\implies \\ \forall t \in (-\delta, \delta) : (g_t)_\beta \text{ is diagonal.} & \end{aligned}$$

□

Thus we have proven that the Ricci flow equation is diagonal with respect to a Milnor frame. So the Ricci flow equation is equivalent to the system:

$$\frac{dA}{dt} = \frac{(c_{31}^2 B - c_{12}^3 C)^2 - (c_{23}^1 A)^2}{BC}, \tag{5.3}$$

$$\frac{dB}{dt} = \frac{(c_{12}^3 C - c_{23}^1 A)^2 - (c_{31}^2 B)^2}{AC}, \tag{5.4}$$

$$\frac{dC}{dt} = \frac{(c_{23}^1 A - c_{31}^2 B)^2 - (c_{12}^3 C)^2}{AB}. \tag{5.5}$$

The normalized Ricci flow, Definition 3.0.1, for left invariant metrics is of the form:

$$\frac{dA}{dt} = \frac{-4(c_{23}^1 A)^2 + 2(c_{31}^2 B)^2 + 2(c_{12}^3 C)^2 - 4c_{31}^2 B \cdot c_{12}^3 C + 2c_{12}^3 C \cdot c_{23}^1 A + 2c_{23}^1 A \cdot c_{31}^2 B}{3BC}, \tag{5.6}$$

$$\frac{dB}{dt} = \frac{2(c_{23}^1 A)^2 - 4(c_{31}^2 B)^2 + 2(c_{12}^3 C)^2 + 2c_{31}^2 B \cdot c_{12}^3 C - 4c_{12}^3 C \cdot c_{23}^1 A + 2c_{23}^1 A \cdot c_{31}^2 B}{3AC}, \tag{5.7}$$

$$\frac{dC}{dt} = \frac{2(c_{23}^1 A)^2 + 2(c_{31}^2 B)^2 - 4(c_{12}^3 C)^2 + 2c_{31}^2 B \cdot c_{12}^3 C + 2c_{12}^3 C \cdot c_{23}^1 A - 4c_{23}^1 A \cdot c_{31}^2 B}{3AB}. \tag{5.8}$$

5.2 The Ricci flow on $\mathbb{R}^3$

Consider the manifold $\mathbb{R}^3$ equipped with the standard Riemannian metric $g_{\mathbb{R}^3}$ of constant curvature 0. This manifold can be regarded as a homogeneous space for the isometry group $\mathrm{Iso}(\mathbb{R}^3)$ acts transitively on $\mathbb{R}^3$. We want to study the Ricci flow equation for all homogeneous metrics g on $\mathbb{R}^3$. Any homogeneous metric on $\mathbb{R}^3$ can be written as

$$g_0 = A_0 g_{\mathbb{R}^3}$$

where A_0 is a real number. Thus we can write $g(t) = A(t) g_{\mathbb{R}^3}$ for a one parameter family of homogeneous metrics on $\mathbb{R}^3$. Using the scaling invariance of the Ricci tensor, $\mathrm{Ric}(\sigma g) = \mathrm{Ric}(g)$, we have that any family of homogeneous metrics on $\mathbb{R}^3$ is Ricci-flat, that is $\mathrm{Ric}(g(t)) = 0$. Thus the Ricci flow $g(t)$ in this class is trivial:

$$\frac{d}{dt} g(t) = -2\,\mathrm{Ric}(g(t)) = -2\,\mathrm{Ric}(g_0) = 0$$

So $g(t) = g_0$ for every $t > 0$.

5.3 The Ricci flow on $\mathbb{H}^3$

Consider the hyperbolic space $\mathbb{H}^3$ with the standard hyperbolic metric $g_{\mathbb{H}^3}$ of constant curvature -1. We regard $\mathbb{H}^3$ as a homogeneous space for its isometry group acts transitively on $\mathbb{H}^3$. All homogeneous metrics g_0 on $\mathbb{H}^3$ can be written as

$$g_0 = \Lambda_0 g_{\mathbb{H}^3}$$

where Λ_0 is a positive constant. This kind of metrics are Einstein so just like Example 2 of Section 3 we have that

$$g(t) = (\Lambda_0 + 2t) g_0$$

is a solution to the Ricci flow equation. So we can write $g(t) = \Lambda(t) g_0$ where $\Lambda(t) = \Lambda_0 + 2t$. We see that the Ricci flow on $\mathbb{H}^3$ expands linearly in time, in contrast with the case of $\mathbb{R}^3$ where the flow is stationary.

5.4 The Ricci flow of SU(2)

Consider the compact Lie group SU(2). Let g_0 be any left invariant metric and let $\beta \in \mathcal{F}(\mathfrak{g})$ be a Milnor frame for g_0. If ω is the dual frame of β that is $\omega^i(\beta_j) = \delta^i_j$ then g_0 can be written as

$$g_0 = A_0\omega^1 \otimes \omega^1 + B_0\omega^2 \otimes \omega^2 + C_0\omega^3 \otimes \omega^3$$

The matrix σ_β of the structure constants is of the form

$$\sigma_\beta = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In order to study the behaviour of the Ricci flow of SU(2) we will use the normalized Ricci flow in order to fix the volume of SU(2). The result that we want to prove is the following:

Theorem 5.4.1. Every left invariant metric on SU(2) converges exponentially to the round metric of the three-sphere with scalar curvature $R = \frac{3}{2}$

This is a remarkable result since if we choose the initial metric to satisfy $B_0 = C_0 < \frac{A_0}{4}$ then the scalar curvature satisfies $R_0 < 0$ but the flow will converge to a metric of positive scalar curvature.

Proof. Suppose that

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$$

is a one parameter family of left invariant metrics on SU(2) expressed in the Milnor frame β. Note that A, B, C are function of time. The components of the Ricci tensor and the scalar curvature of the metric g on SU(2) are given by:

$$\begin{aligned}
R_{11} &= \frac{1}{2}A\left[A^2 - (B-C)^2\right], \\
R_{22} &= \frac{1}{2}B\left[B^2 - (A-C)^2\right], \\
R_{33} &= \frac{1}{2}C\left[C^2 - (A-B)^2\right], \\
R &= \frac{1}{2}\left\{\left[A^2 - (B-C)^2\right] + \left[B^2 - (A-C)^2\right] + \left[C^2 - (A-B)^2\right]\right\}, \\
\|\,\mathrm{Ric}\,\|^2 &= \frac{1}{4}\left\{\left[A^2 - (B-C)^2\right]^2 + \left[B^2 - (A-C)^2\right]^2 \quad + \left[C^2 - (A-B)^2\right]^2\right\}.
\end{aligned}$$

Thus using the equations (5.4), (5.5) and (5.6) the normalized Ricci flow equation is of the form:

$$
\begin{aligned}
\frac{d}{dt}A &= \frac{2}{3}\left[-A^2(2A-B-C)+A(B-C)^2\right],\\
\frac{d}{dt}B &= \frac{2}{3}\left[-B^2(2B-A-C)+B(A-C)^2\right],\\
\frac{d}{dt}C &= \frac{2}{3}\left[-C^2(2C-A-B)+C(A-B)^2\right].
\end{aligned}
$$

In terms of the metric components if $A = B = C$ then we get a fixed point for the Ricci flow. Indeed, if $A = B = C$ then

$$
\frac{d}{dt}A = \frac{2}{3}\left[-A^2(2A-B-C)+A(B-C)^2\right] = \frac{2}{3}\left[-A^2(2A-A-A)+A(A-A)^2\right] = 0
$$

and similarly for B and C. Those components are the metric components of the round sphere S^3. This lead us to calculate the evolution equations for the differences between A, B, C:

$$
\begin{aligned}
\frac{d}{dt}(A-B) &= \frac{2}{3}\left[-2\left(A^3-B^3\right)+C\left(A^2-B^2\right)+C^2(A-B)\right],\\
\frac{d}{dt}(A-C) &= \frac{2}{3}\left[-2\left(A^3-C^3\right)+B\left(A^2-C^2\right)+B^2(A-C)\right],\\
\frac{d}{dt}(B-C) &= \frac{2}{3}\left[-2\left(B^3-C^3\right)+A\left(B^2-C^2\right)+A^2(B-C)\right].
\end{aligned}
$$

As we can see for the symmetry of the equations that without loss of generality we can assume that $A_0 \geq B_0 \geq C_0$. By using the differential equations for the differences of A, B, C we get that $A \geq B \geq C$ for all t. Now since $2C - A - B \leq 0$ we get that $\frac{d}{dt}C \geq 0$ and so C is non decreasing. We will now estimate the evolution equation for $A - C$.

$$
\begin{aligned}
\frac{d}{dt}(A-C) &= \frac{2}{3}(-2(A^2+AC+C^2)+B(A+C)+B^2)(A-C)\\
&= \frac{2}{3}(-2C^2-AC-(A-B)(A+C)-(A^2-B^2))(A-C)\\
&\leq -2C_0^2(A-C).
\end{aligned}
$$

integrating this inequality yields

$$
A - C \leq (A_0 - C_0)e^{-2C_0^2 t}
$$

Therefore, $A-C$ decays to zero exponentially and since $A \geq B \geq C$ for all t we get that the Ricci flow converges exponentially to the fixed point $A = B = C = 1$, where we have used the normalization $ABC = 1$. From the evolution equation for the scalar curvature

$$
R = \frac{1}{2}\left\{\left[A^2-(B-C)^2\right]+\left[B^2-(A-C)^2\right]+\left[C^2-(A-B)^2\right]\right\}
$$

we conclude that R approaches exponentially the value $\frac{3}{2}$ while the norm of the Ricci tensor $||\,\mathrm{Ric}\,||$ exponentially approach the value $\frac{3}{4}$.

$\square$

5.5 The Ricci flow on $S^2 \times \mathbb{R}$

A nice example of a homogeneous manifold which is not a Lie group is $S^2 \times \mathbb{R}$ equipped with the product metric $g_{\mathbb{R}} + g_{S^2}$, where g_{S^2} is the standard metric on the sphere S^2 of constant curvature 1 and $g_{\mathbb{R}}$ the standard metric on $\mathbb{R}$. We regard $S^2 \times \mathbb{R}$ as a homogeneous space for the isometry group $\mathrm{SO}(2) \times \mathbb{R}$ acts transitively on $S^2 \times \mathbb{R}$. Any homogeneous metric on $S^2 \times \mathbb{R}$ are of the form

$$g_0 = A_0 g_{\mathbb{R}} + B_0 g_{S^2}$$

where A_0, B_0 are positive constants. Thus a one parameter family of homogeneous metrics in There is no Einstein metric on this class so Ricci flows on $S^2 \times \mathbb{R}$ can be written as

$$g = A g_{\mathbb{R}} + B g_{S^2}$$

where A and B are functions of time. $S^2 \times \mathbb{R}$ cannot converge. We will show that actually Ricci flows on $S^2 \times \mathbb{R}$ form curvature singularity in finite time. The components of the Ricci curvature, the scalar curvature and the norm of the Ricci tensor are given by:

$$\begin{aligned} R_{11} &= 0 \\ R_{22} &= 1 \\ R_{33} &= 1 \\ R &= \frac{2}{B} \\ ||\,\mathrm{Ric}\,||^2 &= \frac{2}{B^2} \end{aligned}$$

So the Ricci flow equation takes the form

$$\begin{aligned} \frac{d}{dt} B &= -\frac{2}{3} \\ \frac{d}{dt} A &= \frac{4}{3}\frac{A}{B} \end{aligned}$$

Integrating directly these equations we obtain the solutions

$$\begin{aligned} B &= B_0 - \frac{2}{3} t \\ A &= \frac{A_0 B_0^2}{B_0 - \frac{2}{3} t} \end{aligned}$$

As we can see the round two sphere shrinks linearly in t, while the metric components of $\mathbb{R}$ expand at the indicated rate. The curvature singularity is obtained exactly at time $T = \frac{3}{2} B_0$ where the radius of the sphere reaches zero. In order to avoid this curvature singularity we can normalize the Ricci flow is such a way the the volume of the 2-sphere remain fixed. Let $\langle r(t)\rangle$ be the average of the scalar curvature of S^2. We consider a variant of the normalized Ricci flow:

$$\frac{\partial}{\partial t}g = -2\operatorname{Ric}(g) + \langle r(t)\rangle g$$

Then the system transforms to

$$\begin{aligned}\frac{d}{dt}A &= 0\\ \frac{d}{dt}B &= 2B/A\end{aligned}$$

whose solution is

$$\begin{aligned}A &= A_0,\\ B &= B_0 e^{2/A_0 t}\end{aligned}$$

This solution exist for all times $t \in [0, \infty)$ and it is non-singular for all times $t > 0$.

5.6 The Ricci flow on $\mathbb{H}^2 \times \mathbb{R}$

Consider the manifold $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is the hyperbolic plane equipped with the product metric $g_{\mathbb{R}}+g_{\mathbb{H}^2}$ where $g_{\mathbb{H}^2}$ is the standard metric on the hyperbolic plane of constant curvature -1. We regard $\mathbb{H}^2 \times \mathbb{R}$ as a homogeneous space for its isometry group $\operatorname{Iso}(\mathbb{H}^2) \times \mathbb{R}$ acts transitively on $\mathbb{H}^2 \times \mathbb{R}$. All the homogeneous metrics on this manifold are of the form.

$$g_0 = D_0 g_{\mathbb{R}} + E_0 g_{\mathbb{H}^2}$$

where D_0 E_0 are positive constants. So a one parameter family of homogeneous metrics is of the form

$$g = D g_{\mathbb{R}} + E g_{\mathbb{H}^2}$$

where D and E are functions of time. Just like the case of $S^2 \times \mathbb{R}$ there are no Einstein metrics in the class of homogeneous metrics therefore the Ricci flow equation cannot converge. But, in contrast with the case of $S^2 \times \mathbb{R}$ the Ricci flows do not develop curvature singularities but develop pancake degeneracies.

The components of the Ricci curvature tensor, the scalar curvature and the norm of the Ricci tensor are:

$$\begin{aligned}R_{11} &= 0\\ R_{22} &= -1\\ R_{33} &= -1\\ R &= -2/E\\ ||\operatorname{Ric}||^2 &= 2/E^2\end{aligned}$$

So the Ricci flow equation has the form

$$\frac{d}{dt}E = \frac{2}{3}$$
$$\frac{d}{dt}D = -\frac{4}{3}D/E$$

Therefore we can directly gt the solution to the Ricci flow equation:

$$E = E_0 + \frac{2}{3}t$$
$$D = D_0 E_0^2/(E_0 + \frac{2}{3}t)^2$$

As we can see from the solution the component of the hyperbolic part of the metric scales linearly in time while the other component decreases with rate $1/t^2$. So as $t \to \infty$ the component D goes to zero. Therefore we cannot have convergence. Also, the norm of the Ricci tensor evolves by

$$||\,\mathrm{Ric}\,|| = \sqrt{2}/(E_0 + \frac{2}{3}t)$$

which is a characteristic phenomenon for this kind of degeneracy.

6 Quasi-convergence

We proved that any normalized Ricci flow on SU(2) exists for all times $t > 0$ and converges to a left invariant metric on SU(2) with positive scalar curvature. Unfortunately, there exist Ricci flows, even in the special case of left invariant metrics on Lie groups, that exist for all times but the limit of the solution as $t \to \infty$ is not a Riemannian metric. Such a phenomenon is called collapsing.

Definition 6.0.1. We say that a solution to the Ricci flow equation collapses if the injectivity radius of the corresponding solution to the normalized Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) + \frac{2}{n}\langle r(t)\rangle g(t)$$

goes to zero as $t \to \infty$.

Here $\langle r(t)\rangle = \frac{\int_M R(t) d\mu}{\mathrm{Vol}(M)}$ is the average of the scalar curvature. In order to study the asymptotic behaviour of those collapsing solutions R. Hamilton and J. Isenberg ([7]) introduced the concept of quasi-convergence.

Definition 6.0.2. Suppose that we have two solutions $g(t), h(t)$ of the Ricci flow equation on a manifold M. We say that $g(t)$ quasi-converges to $h(t)$ and we write $h(t) \in [g(t)]$ if and only if for any $\epsilon > 0$ there exist a time t_ϵ such that

$$\sup_{M \times [t_\epsilon, \infty)} |h(t) - g(t)|_{g(t)} < \epsilon.$$

This is the same as saying that $\sup_M |h(t) - g(t)|_{g(t)} \to 0$ as $t \to \infty$.

The quasi-convergence study the large time behaviour of solutions to the Ricci flow equation that exist for all times $t \in (0, \infty)$. We will prove that the quasi-convergence is an equivalence relation on the set $C([0,\infty), \Gamma(S^2_+ T^*M))$, where $S^2_+ T^*M$ is the set of positive definite covariant 2-tensors on M

$$S^2_+ T^*M = \coprod_{m \in M} S^2_+ T^*_m M, \qquad S^2_+ T^*_m M = \{T \in S^2 T^*_m M \ : \ T(x,x) > 0, \forall x \in T_m M \setminus \{0\}\}$$

and $C([0,\infty), \Gamma(S^2_+ T^*M))$ is the space of continuous functions from $[0,\infty)$ to $\Gamma(S^2_+ T^*M)$. Also we denote by $\sim$ the quasi-convergence equivalence relation on $C([0,\infty), \Gamma(S^2_+ T^*M))$. That is if $g(t), h(t) \in C([0,\infty), \Gamma(S^2_+ T^*M))$ then

$$g(t) \sim h(t) \ \Leftrightarrow \ \sup_M |h(t) - g(t)|_{g(t)} \to 0 \text{ as } t \to \infty.$$

Thus, we can divide solutions to Ricci flow equations into equivalence classes. These equivalent classes contain Ricci flows that are close to each other in the sense of quasi-convergence. R. Hamilton conjectured that the large time behaviour of any collapsing solution will be quasi-converge to the evolution of a locally homogeneous metric ([7]).

Before proving that the quasi-convergence is an equivalence relation we first make some remarks about inner products on vector spaces. Let V be a finite dimensional vector space over $\mathbb{R}$ with $\dim(V) = n$. Let S^2V^* be the space of symmetric bilinear forms on V and let

$$IP(V) = \{h \in S^2V^* \mid \forall x \in V \setminus \{0\}, h(x,x) > 0\}$$

be the space of inner product on V. Every inner product $h \in IP(V)$ induces an inner product on the space of covariant symmetric 2-tensors $S^2V^* \subset \otimes^2 V^*$. This inner product induces a norm $|.|_h$ on S^2V^*. In order to prove that quasi-convergence is symmetric we first have to prove the following.

Lemma 6.0.3. Let V be a vector space. There exist a constant C with $0 < C < 1$ such that for any inner product $g \in IP(V)$ and for any covariant symmetric 2-tensor $\beta \in S^2V^*$ the following inequality holds:

$$C|\beta|_g \leq \sup_{x \in V \setminus \{0\}} \frac{|\beta(x,x)|}{g(x,x)} \leq |\beta|_g.$$

Also, the constant C depends only on the dimension of the vector space V.

Proof. Using the Cauchy-Schwartz inequality we get $|\beta(x,x)| \leq |\beta|_g ||x||^2 = |\beta|_g g(x.x)$ for every $x \in V \setminus \{0\}$. Thus by taking the supremum over $x \in V \setminus \{0\}$ we get the right inequality. To get the left inequality, we write $|\beta|_g^2$ as $|\beta|_g^2 = \sum_{i,j} |\beta(e_i, e_j)|^2$. where $\{e_i\}$ is an orthonormal basis for V. So we compute

$$\begin{aligned}
|\beta(e_i, e_j)|_g &= \left| \frac{1}{2}\beta(e_i + e_j, e_i + e_j) - \beta(e_i, e_i) - \beta(e_j.e_j) \right|_g \\
&\leq \left| \frac{\beta(e_i + e_j, e_i + e_j)}{g(e_i + e_j, e_i + e_j)} \right|_g + \frac{1}{2} \left(\left| \frac{\beta(e_i, e_i)}{g(e_i, e_i)} \right|_g + \left| \frac{\beta(e_j.e_j)}{g(e_j.e_j)} \right|_g \right) \\
&\leq 2 \sup \frac{|\beta(x,x)|_g}{g(x,x)}.
\end{aligned}$$

So by taking the sum over i and j we get,

$$|\beta|_g^2 \leq 2\dim(V)^2 \sup \frac{|\beta(x,x)|^2}{g(x,x)},$$

for every $x \in V \setminus \{0\}$. Thus we proved the result for $C = \frac{1}{2\dim(V)^2}$.

□

The fact that the constant C in Lemma 6.0.3 depends only on the dimension on the vector space V enables us to transfer this inequality for every tangent space T_mM of a manifold M for every $m \in M$. Thus we gain the following result

Corollary 6.0.4. Suppose that β_t is a one parameter family of symmetric 2-tensors on M and g_t a one parameter family of Riemannian metrics on M. Then

$$|\beta_t|_{g_t} \to 0 \text{ as } t \to \infty \text{ if and only if } \frac{|\beta_t(x,x)|}{g_t(x,x)} \to 0 \text{ as } t \to \infty \text{ uniformly for } x \in TM \setminus \{0\},$$

where $TM \setminus \{0\}$ is the tangent bundle of M without the zero section.

We are now ready to show that the quasi-convergence is symmetric.

Proposition 6.0.5. Let g_t, h_t be two one parameter families of Riemannian metrics on M with $t \in [0, \infty)$. Then $|g_t - h_t|_{g_t} \to 0$ if and only if $|g_t - h_t|_{h_t} \to 0$.

Proof. Using Corollary 6.0.4. we have (with $x \in TM \setminus \{0\}$),

$$\begin{aligned}
\sup_M |h_t - g_t|_{g_t} &\to 0 \Leftrightarrow \\
\left| \frac{h_t(x,x) - g_t(x,x)}{g_t(x,x)} \right| &\to 0 \text{ uniformly on } TM \setminus \{0\} \Leftrightarrow \\
\frac{h_t(x,x)}{g_t(x,x)} &\to 1 \text{ uniformly on } TM \setminus \{0\} \Leftrightarrow \\
\frac{g_t(x,x)}{h_t(x,x)} &\to 1 \text{ uniformly on } TM \setminus \{0\} \Leftrightarrow \\
\left| \frac{h_t(x,x) - g_t(x,x)}{h_t(x,x)} \right| &\to 0 \text{ uniformly on } TM \setminus \{0\} \Leftrightarrow \\
\sup_M |h_t - g_t|_{h_t} &\to 0
\end{aligned}$$

□

So the quasi-convergence is a reflexive relation. It remains to show that quasi-convergence is a transitive relation.

Proposition 6.0.6. Suppose that g_t, h_t, f_t are one parameter families of Riemannian metrics on a manifold M with $t \in [0.\infty)$. Assume that g_t quasi-converges to h_t and h_t quasi-converges to f_t. Then g_t quasi-converges to f_t.

Proof. Since g_t quasi-converges to h_t we have that

$$|g_t(m) - h_t(m)|_{g_t(m)} \to 0$$

as $t \to \infty$ uniformly in $m \in M$. Also, h_t quasi-converge to f_t, thus,

$$|h_t(m) - f_t(m)|_{h_t(m)} \to 0$$

as $t \to \infty$ uniformly in $m \in M$. Therefore,

$$\begin{aligned}
|g_t(m) - f_t(m)|_{g_t(m)} &= |g_t(m) - h_t(m) + h_t(m) - f_t(m)|_{g_t(m)} \\
&\leq |g_t(m) - h_t(m)|_{g_t(m)} + |h_t(m) - f_t(m)|_{g_t(m)}
\end{aligned}$$

Since g_t quasi-converges to h_t we have $|g_t(m) - h_t(m)|_{g_t(m)} \to 0$ uniformly in $m \in M$. So it suffices to show that $|h_t(m) - f_t(m)|_{g_t(m)} \to 0$ uniformly in $m \in M$. We have by Lemma 6.0.3,

$$\begin{aligned}
|h_t - f_t|_{g_t} &\leq \frac{1}{C} \sup_{m \in M, x \in T_m M \setminus \{0\}} \frac{|h_t(m)(x,x) - f_t(m)(x,x)|}{g_t(m)(x,x)} \\
&= \frac{1}{C} \sup_{m \in M, x \in T_m M \setminus \{0\}} \left(\frac{|h_t(m)(x,x) - f_t(m)(x,x)|}{h_t(m)(x,x)} \left(\frac{h_t(m)(x,x)}{g_t(m)(x,x)} \right) \right) \\
&= \frac{1}{C} \sup_{(m,x) \in TM \setminus \{0\}} \left(\frac{|h_t(m)(x,x) - f_t(m)(x,x)|}{h_t(m)(x,x)} \left(\frac{h_t(m)(x,x)}{g_t(m)(x,x)} \right) \right)
\end{aligned}$$

Since, $|h_t - f_t|_{h_t} \to 0$ uniformly on $TM \setminus \{0\}$ and $|g_t - h_t|_{g_t} \to 0$ uniformly on $TM \setminus \{0\}$ we get that

$$\begin{aligned}
&\sup_{(m,x \in TM \setminus \{0\})} \frac{|h_t(m)(x,x) - f_t(m)(x,x)|}{h_t(m)(x,x)} \frac{h_t(m)(x,x)}{g_t(m)(x,x)} \\
&= \sup_{(m,x) \in S(TM)} \frac{|h_t(m)(x,x) - f_t(m)(x,x)|}{h_t(m)(x,x)} \frac{h_t(m)(x,x)}{g_t(m)(x,x)} \to 0
\end{aligned}$$

where $S(TM)$ is the unit sphere bundle of the tangent bundle TM. This implies that $|g_t - f_t|_{g_t} \to 0$ as $t \to \infty$ uniformly in $S(TM)$ and in $TM \setminus \{0\}$. So g_t quasi-converges to f_t. □

Finally, quasi-convergence is trivially reflexive, therefore we have proven the following theorem.

Theorem 6.0.7. Quasi-convergence is an equivalence relation.

In the following two sections we will describe the quasi-convergence as an equivalent relation and compute the quasi-convergence classes of left invariant Ricci flows on Lie groups. We should note in [11] the authors made some computations for the quasi-convergence classes of left invariant Ricci flows on Lie groups. The method that they used was a direct computation of these quasi-convergence classes. We will follow a different route. We will define an action of the group $\mathrm{GL}(3,\mathbb{R})$ on the space of 3×3 matrices and describe the quasi-convergence class $[g]$ of a left invariant Ricci flow as the orbit of the diagonal part of $[g]$ under a subgroup of $\mathrm{GL}(3,\mathbb{R})$.

7 Strategy to compute the quasi-convergence classes

In this section we are going to present the general strategy to determine the quasi-convergence classes. First of all we will use the so called Iwasawa decomposition for $\mathrm{GL}(3, \mathbb{R})$ to describe the space of metrics on a three dimensional vector space as the space $N \times A \simeq \mathrm{GL}(3, \mathbb{R})/O(3)$ where N is the group of 3 by 3 nilpotent matrices and A the group of 3 by 3 diagonal matrices. Subsets of this space will parametrize the quasi-convergence classes. Next we will define an action of the Lie group $\mathrm{GL}(3, \mathbb{R})$ to the space of 3×3 matrices $M_3(\mathbb{R})$ and determine the stabilizer subgroup of this action. Then the quasi-convergence class can be characterized as the orbit space for a particular subgroup of the stabilizer subgroup.

7.1 Iwasawa decomposition

We begin by recalling the notion of a semisimple Lie algebra and the Cartan decomposition.

Definition 7.1.1. A Lie algebra $\mathfrak{g}$ is called semisimple if the Killing form

$$B(X, Y) = \mathrm{tr}(ad_X \circ ad_Y)$$

is non-degenerate.

Suppose that G is a connected Lie group with semisimple Lie algebra $\mathfrak{g}$. Let B be the Killing form on $\mathfrak{g}$. We denote by $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution of $\mathfrak{g}$. That is θ is a Lie algebra automorphism of $\mathfrak{g}$, with $\theta^2 = -1$ and $(X, Y) \mapsto -B(X, \theta Y)$ is a positive definite bilinear form on $\mathfrak{g}$. The involution θ has two eigenvalues, $+1$ and -1 with corresponding eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$, respectively.

Definition 7.1.2. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ as a direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

That is we decompose the Lie algebra $\mathfrak{g}$ into the eigenspaces of the Cartan involution θ. In our analysis of the quasi-convergence class for homogeneous Ricci flows as well as the the description of the space of inner products on an vector space we will use the Iwasawa decomposition. First we will describe the Iwasawa decomposition on the Lie algebra level and the we will derive a global form of the Iwasawa decomposition for any semisimple Lie group. Proof of these results can be found in [10].

Proposition 7.1.3. (Iwasawa decomposition for semisimple Lie algebras) Suppose that $\mathfrak{g}$ is a real semisimple Lie algebra. We can decompose the Lie algebra $\mathfrak{g}$ as a direct sum of Lie subalgebras

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where $\mathfrak{k}$ is the eigenspace of the Cartan involution θ corresponding to the eigenvalue $+1$, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is an nilpotent subalgebra of $\mathfrak{g}$.

We will now present the global version of the Iwasawa decomposition. We denote by K, A and N the analytic subgroups of $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$.

Proposition 7.1.4. (Iwasawa decomposition) Suppose that G is a real semisimple Lie group with Lie algebra $\mathfrak{g}$. We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$. Then the multiplication map

$$\begin{aligned} K \times A \times N &\to G \\ (k, a, n) &\mapsto kan \end{aligned}$$

is a diffeomorphism.

The subgroups A and N that appear in this decomposition are simply connected. Of particular interest is the Iwasawa decomposition for the Lie group $\mathrm{GL}(n, \mathbb{R})$. The group $\mathrm{GL}(n, \mathbb{R})$ is not semisimple but it admits an Iwasawa decomposition as follows:

Proposition 7.1.5. We can decompose the Lie group $\mathrm{GL}(n, \mathbb{R})$ as

$$\mathrm{GL}(n, \mathbb{R}) = O(n)AN$$

where A is the abelian group which contains all diagonal matrices and N is the nilpotent group which contains all the upper-triangular matrices.

Proof. We refer to [10] for a proof of the Iwasawa decomposition in the more general setting of reductive Lie groups like $\mathrm{GL}(n, \mathbb{R})$. □

A consequence of the Iwasawa decomposition for $\mathrm{GL}(3, \mathbb{R})$ is that the space of all metrics on $\mathbb{R}^3$ can be described as $N \times A$, where N is the 3-dimensional nilpotent group and A the group of diagonal matrices. That is we get an isomorphism

$$N \times A \simeq \mathrm{GL}(3, \mathbb{R})/O(3) \simeq IP(\mathbb{R}^3).$$

7.2 Group actions

Motivated by Corollary 4.1.3. we define an action of the Lie group $\mathrm{GL}(3, \mathbb{R})$ on the space of 3×3 real matrices $M_3(\mathbb{R})$ by,

$$\begin{aligned} \mathrm{GL}(3, \mathbb{R}) \times M_3(\mathbb{R}) &\to M_3(\mathbb{R}) \\ (A, M) &\mapsto A \cdot M = (\det(A))^{-1} A M A^T \end{aligned} \tag{7.1}$$

The next thing we want to describe is the stabilizer subgroup of this action if we have fixed a matrix $\sigma \in M_3(\mathbb{R})$ that corresponds to the matrix of structure constants. That is we want to find all elements A of the group $\mathrm{GL}(3, \mathbb{R})$ that satisfy the equation

$$(\det(A))^{-1} A \sigma A^T = \sigma.$$

We will denote this group by $\mathrm{GL}(3, \mathbb{R})_\sigma$. To simplify the computations we will usually consider the corresponding action on the Lie algebra level,

$$\begin{aligned} \mathfrak{gl}(3, \mathbb{R}) \times M_3(\mathbb{R}) &\to M_3(\mathbb{R}) \\ (X, m) &\mapsto X \cdot m = -\operatorname{tr}(X) m + X m + (X m)^T \end{aligned}$$

and determine the Lie algebra of the stabilizer group for a fixed $\sigma \in M_3(\mathbb{R})$.

$$\mathrm{Lie}(GL(3,\mathbb{R})_\sigma) = \{X \in M_3(\mathbb{R}) \mid -\operatorname{tr} X + X\sigma + (X\sigma)^T = 0\}.$$

Suppose now that g is an inner product on the Lie algebra $\mathfrak{g}$. Consider a Milnor frame β on $(\mathfrak{g}, g)$ such that σ_β has a fixed standard form σ and the inner product g is diagonal with respect to β. Extend g to a left invariant metric $\tilde{g}$ on G such that $\tilde{g}(e) = g$. We denote by $IP(G)^G$ the space of left invariant metrics on G. The map

$$\begin{aligned} IP(\mathfrak{g}) &\to IP(G)^G \\ g &\mapsto \tilde{g} \end{aligned}$$

is a bijection. We define

$$[\tilde{g}]_\beta = \{\tilde{h} \in IP(G)^G \mid \tilde{h}_\beta \text{ diagonal and } |\tilde{g}_t - \tilde{h}_t|_{\tilde{g}_t} \to 0\}.$$

where $\tilde{g}_t$ and $\tilde{h}_t$ are the Ricci flows on G with initial data $\tilde{g}$ and $\tilde{h}$ respectively. We also define

$$[g]_\beta = \{h \in IP(\mathfrak{g}) \mid h_\beta \text{ diagonal and } |g_t - h_t|_{g_t} \to 0\}.$$

The bijection between $\mathfrak{g}$ and $IP(G)^G$ induces a bijection between the sets $[g]$ and $[\tilde{g}]$. Thus, if we denote by $\sim$ the quasi-convergence equivalence relation we have that

$$g \sim h \Leftrightarrow \tilde{g} \sim \tilde{h} \Leftrightarrow |\tilde{g}_t - \tilde{h}_t|_{\tilde{g}_t} \to 0$$

We also observe that $|g_t - h_t|_{g_t} \to 0$ is equivalent to $|g_{t_\beta} - h_{t_\beta}|_{g_{t_\beta}} \to 0$.

A first aim of this section is to describe the space $[g]_\beta$. We start with a frame β on $\mathfrak{g}$ such that the matrix of the structure constants σ_β is diagonal and has the standard form $\sigma_\beta = \sigma$. Let

$$\mathcal{M}_\beta = \{g \in IP(\mathfrak{g}) \mid g_\beta \text{ is diagonal }\}$$

That is we consider the space of inner products on the Lie algebra $\mathfrak{g}$ such that these inner products are diagonal with respect to the frame β. We denote by A the subgroup of $\mathrm{GL}(3,\mathbb{R})$ consisting of the diagonal matrices with non-zero entries. Then we can describe $\mathcal{M}_\beta$ as

$$\mathcal{M}_\beta = \{g \in IP(\mathfrak{g}) \mid \beta^* g \in A.e\}$$

where e is the standard Euclidean metric and $A.e$ is the action of the group A on e given by $a.e = (a^{-1})^* e$. We also consider the map $A \to \mathcal{M}_\beta$ by $a \mapsto (\beta^{-1})^*(a.e)$.

Definition 7.2.1. We define the equivalent relation $\sim_\beta$ on the space of $\mathcal{M}_\beta$ as the restriction of the quasi-convergence relation $\sim$ to the space $\mathcal{M}_\beta$.

This means that $g_t, h_t \in \mathcal{M}_\beta$ are equivalent with respect to the equivalent relation $\sim_\beta$ if and only if the corresponding left invariant Ricci flow $\tilde{g}_t$ quasi-converges to left invariant Ricci flow $\tilde{h}_t$.

Now select a frame $\beta \in \mathcal{F}(\mathfrak{g})$ such that the matrix of the structure constants σ_β is of the standard form σ. Using the frame β we can view any inner product $g \in IP(\mathfrak{g})$ as an inner product on $\mathbb{R}^3$. So we have a bijective map $\beta^* : IP(\mathfrak{g}) \to IP(\mathbb{R}^3)$ given by $g \mapsto \beta^* g := g_\beta$. Now consider the action of the Lie group $\mathrm{GL}(3,\mathbb{R})$ on $IP(\mathbb{R}^3)$,

$$\begin{aligned} \mathrm{GL}(3,\mathbb{R}) \times IP(\mathbb{R}^3) &\to IP(\mathbb{R}^3) \\ (x,\gamma) &\mapsto (x^{-1})^*\gamma \end{aligned}$$

The standard Euclidean inner product e is in $IP(\mathbb{R}^3)$ and so we can consider the stabilizer subgroup of e which is exactly $O(3)$. The group $\mathrm{GL}(3,\mathbb{R})$ acts transitively on $IP(\mathbb{R}^3)$. Thus $IP(\mathbb{R}^3)$ can be described as

$$IP(\mathbb{R}^3) \simeq \mathrm{GL}(3,\mathbb{R}^3)/O(3).$$

Suppose now that we have the group $\mathrm{GL}(3,\mathbb{R})_\sigma$ consisting of all the 3 by 3 matrices that stabilize the matrix of structure constants σ.

Lemma 7.2.2. The map

$$\begin{aligned} \mathrm{GL}(3,\mathbb{R})_\sigma \times A &\to \mathrm{GL}(3,\mathbb{R})/O(3) \\ (x,a) &\mapsto xa \cdot [e] = xa \cdot O(3) \end{aligned}$$

is surjective.

Proof. Consider $h_\beta := (x^{-1})^* e$ a metric on $\mathbb{R}^3$, where $x \in \mathrm{GL}(3,\mathbb{R})$ and e the Euclidean inner product on $\mathbb{R}^3$. Define the inner product h on $\mathfrak{g}$ by $h = (\beta^{-1})^*(h_\beta)$. Now, by [12], h has a Milnor frame α such that $\sigma_\alpha = \sigma$ and $\alpha^* h = h_\alpha$ is diagonal, with respect to the standard basis of $\mathbb{R}^3$ (see Remark 4.2.2.). We have $\sigma_\alpha = \sigma_\beta = \sigma$ and $\beta = \alpha \circ (\alpha^{-1} \circ \beta)$. Combining those two relations we have

$$\sigma = \sigma_\beta = (\alpha^{-1} \circ \beta) \circ \sigma_\alpha = \sigma$$

Thus, $y = \alpha^{-1} \circ \beta \in \mathrm{GL}(3,\mathbb{R})_\sigma$.
Now

$$\begin{aligned} \beta^* \circ (\alpha^{-1})^* \circ a^* e = (x^{-1})^* e &\implies \\ y^* \alpha^* e = (x^{-1})^* e &\implies \\ x^* (ay)^* e = e &\implies \\ ayx \in O(3) &\implies \\ x \in y^{-1} a^{-1} O(3) & \end{aligned}$$

where $y^{-1} \in \mathrm{GL}(3,\mathbb{R})_\sigma$ and $a \in A$. ☐

A corollary of this result is the following.

Corollary 7.2.3. The map

$$\begin{aligned} \mathrm{GL}(3,\mathbb{R})_\sigma \times A &\to IP(\mathbb{R}^3) \\ (y,a) &\mapsto ((ya)^{-1})^* e \end{aligned}$$

is surjective.

Suppose that h is an inner product on the Lie algebra $\mathfrak{g}$. If β is any frame on $\mathfrak{g}$ then h_β is an inner product on $\mathbb{R}^3$. Using Lemma 7.2.1 we have that $h_\beta = ((ya)^{-1})^* e = (y^{-1})^*(a^{-1})^* e$ for some $a \in A$ and $y \in \mathrm{GL}(3, \mathbb{R})_\sigma$. Thus $y^* \beta^* h = (a^{-1})^* e$. Denote by $\alpha = \beta \circ y$, then α is a Milnor frame for $(\mathfrak{g}, h)$. So we have proven the following lemma.

Lemma 7.2.4. If α and β are Milnor frames for h and g, respectively then $\alpha^{-1} \circ \beta \in \mathrm{GL}(3, \mathbb{R})_\sigma$.

7.3 Relation with automorphisms group

Let $\mathfrak{g}$ be a 3-dimensional Lie algebra. Recall (eq. (4.1)) that for a frame $\beta \in \mathcal{F}(\mathfrak{g})$ the matrix of structure constants σ_β is defined by

$$[\beta u, \beta v] = \beta(\sigma_\beta(u \times v)).$$

where u, v are in $\mathbb{R}^3$. We want to prove that the group $\mathrm{GL}(3, \mathbb{R})_\sigma$ is isomorphic to the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. In order to show this we first have to prove some lemmas.

Lemma 7.3.1. Let α, β be two frames on the Lie algebra $\mathfrak{g}$ such that $\sigma_\alpha = \sigma_\beta$. Then the composition

$$\beta \circ \alpha^{-1} : \mathfrak{g} \to \mathfrak{g}$$

is a Lie algebra automorphism.

Proof. For $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}[]
[\beta \circ \alpha^{-1} x, \beta \circ \alpha^{-1} y] &= \beta \sigma_\beta(\alpha^{-1} x \times \alpha^{-1} y) \\
&= \beta \sigma_\alpha(\alpha^{-1} x \times \alpha^{-1}) \\
&= \beta \circ \alpha^{-1}[x, y]
\end{aligned}$$

Thus, $\beta \circ \alpha^{-1}$ is a Lie algebra automorphism. □

Lemma 7.3.2. Suppose that $\phi \in \mathrm{GL}(3, \mathbb{R})$ and β a frame of the Lie algebra $\mathfrak{g}$. Then the following assertions are equivalent.

1. $\phi \in \mathrm{GL}(3, \mathbb{R})_{\sigma_\beta}$
2. $\beta \circ \phi \circ \beta^{-1} \in \mathrm{Aut}(\mathfrak{g})$

Proof. Suppose that $\phi \in \mathrm{GL}(3, \mathbb{R})_{\sigma_\beta}$. Since ϕ is a linear isomorphism the composition $\beta \circ \phi : \mathbb{R}^3 \to \mathfrak{g}$ is a frame for $\mathfrak{g}$ with matrix of structure constants

$$\sigma_{\beta \circ \phi} = \phi^{-1} \sigma_\beta = \sigma_\beta.$$

So by Lemma 7.3.1 we have that $\beta \circ \phi \circ \beta^{-1} \in \mathrm{Aut}(\mathfrak{g})$. For the converse suppose that $\beta \circ \phi \circ \beta^{-1}$ is a Lie algebra automorphism. In order to show that $\phi \in \mathrm{GL}(3, \mathbb{R})_{\sigma_\beta}$ we have to show that

$\sigma_\beta = \sigma_{\beta\circ\phi\circ\beta^{-1}}$. We have, for every $u, v \in \mathbb{R}^3$

$$\begin{aligned}(\beta\circ\phi)\sigma_{\beta\circ\phi}(u\times v) &= [(\beta\circ\phi)u, (\beta\circ\phi)v] \\ &= [(\beta\circ\phi\circ\beta^{-1})\beta u, (\beta\circ\phi\circ\beta^{-1})\beta v] \\ &= (\beta\circ\phi\circ\beta^{-1})[\beta u, \beta v] \\ &= (\beta\circ\phi\circ\beta^{-1})\beta\sigma_\beta(u\times v)\end{aligned}$$

Thus,

$$(\beta\circ\phi)\sigma_{\beta\circ\phi}(u\times v) = (\beta\circ\phi)\sigma_\beta(u\times v)$$

for every $u, v \in \mathbb{R}^3$. Using that $\phi \in \mathrm{GL}(3,\mathbb{R})$ we have that $\sigma_{\beta\circ\phi} = \sigma_\beta$. □

As a corollary of this result we have the following

Corollary 7.3.3. Let $\beta \in \mathcal{F}(\mathfrak{g})$ with $\sigma_\beta = \sigma$. Then the map,

$$\begin{aligned}\beta_* : \mathrm{GL}(3,\mathbb{R}) &\to \mathrm{GL}(\mathfrak{g}) \\ \phi &\mapsto \beta\circ\phi\circ\beta^{-1}\end{aligned}$$

restricts to a group isomorphism $\mathrm{GL}(3,\mathbb{R})_\sigma \to \mathrm{Aut}(\mathfrak{g})$.

We will define an equivalence relation for inner products on a Lie algebra.

Definition 7.3.4. Suppose that g_t and h_t are one parameter families of inner products on the Lie algebra $\mathfrak{g}$ with $t \in [0,\infty)$ that satisfy the flow equation

$$\frac{d}{dt}g_t = -2\overline{\mathrm{Ric}}_e(g_t). \tag{7.2}$$

Then g_t is equivalent to h_t and we write $g_t \sim_{\mathfrak{g}} h_t$ if and only if for every $\epsilon > 0$ there exist a positive number t_ϵ such that for every $t > t_\epsilon$ we have

$$|g_t - h_t|_{g_t} < \epsilon.$$

The next proposition relates the equivalence relation for families of inner products on Lie algebras with the quasi-convergence of their corresponding left invariant metrics on Lie groups.

Proposition 7.3.5. Suppose that g_t and h_t are solutions of the flow equation (7.2). Let $\tilde{g}_t$ and $\tilde{h}_t$ be the corresponding solutions to the Ricci flow equation as in Proposition 5.1.2. Then

$$g_t \sim_{\mathfrak{g}} h_t \Leftrightarrow \tilde{g}_t \sim \tilde{h}_t.$$

Proof. To prove this proposition we will use the left invariance of the Riemannian metrics $\tilde{g}_t$ and $\tilde{h}_t$. We have,

$$\begin{aligned}\tilde{g}_t \sim \tilde{h}_t &\Leftrightarrow |\tilde{g}_t - \tilde{h}_t|_{\tilde{g}_t} \to 0 \\ &\Leftrightarrow \forall x \in G, |\tilde{g}_t(x) - \tilde{h}_t(x)|_{\tilde{g}_t(x)} \to 0 \\ &\Leftrightarrow |\tilde{g}_t(e) - \tilde{h}_t(e)|_{\tilde{g}_t(e)} \to 0 \\ &\Leftrightarrow |g_t - h_t|_{g_t} \to 0 \\ &\Leftrightarrow g_t \sim_{\mathfrak{g}} h_t.\end{aligned}$$

□

Definition 7.3.6. We define the equivalence relation $\sim_{\mathfrak{g}}$ on $IP(\mathfrak{g})$ as follows: if g and h are inner products on $\mathfrak{g}$ then $g \sim_{\mathfrak{g}} h$ if and only if the corresponding solutions g_t and h_t to the equation (7.2), with initial data g and h respectively, satisfy the property $g_t \sim_{\mathfrak{g}} h_t$.

Similarly, we define the equivalence relation $\sim$ on the set of left invariant metrics $IP(G)^G$ on a Lie group G as follows: if $\tilde{g}$ and $\tilde{h}$ are left invariant metrics on G then $\tilde{g} \sim \tilde{h}$ if and only if the corresponding solutions $\tilde{g}_t$ and $\tilde{h}_t$, with initial data $\tilde{g}$ and $\tilde{h}$ respectively, satisfy the property $\tilde{g}_t \sim \tilde{h}_t$ as in Definition 6.0.2.

We will now describe the connection between the Ricci flow equation and the automorphism group of a Lie algebra and of a Lie group. Let $\mathfrak{g}$ be a Lie algebra and let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that g is an inner product on $\mathfrak{g}$ and ϕ an automorphism of the Lie algebra. We denote by ϕ^*g the pullback of the inner product g by the automorphism ϕ. Since ϕ is an automorphism and in particular is a vector space isomorphism ϕ^*g is an inner product on $\mathfrak{g}$. Now consider the left invariant extensions $\widetilde{\phi^*g}$ and $\tilde{g}$ on $\tilde{G}$ of ϕ^*g and g respectively. Also consider the lift $\Phi \in \mathrm{Aut}(\tilde{G})$ of the map ϕ. So we have $T_e\Phi = \phi$. Let $\Phi^*\tilde{g}$ be the pullback of $\tilde{g}$ by the automorphism Φ of $\tilde{G}$. It is easy to check that since Φ is an automorphism of the Lie group $\tilde{G}$, $\Phi^*\tilde{g}$ is a left invariant metric on $\tilde{G}$.

Lemma 7.3.7. With notation as above we have

$$\Phi^*\tilde{g} = \widetilde{\phi^*g}.$$

Proof. We will prove this equality at the identity element e of $\tilde{G}$. Then using that the metrics $\Phi^*\tilde{g}$ and $\widetilde{\phi^*g}$ are left invariant we will have the equality for every element $x \in \tilde{G}$. For every $u, v \in T_e\tilde{G}$ we have

$$\begin{aligned}
(\Phi^*\tilde{g})_e(u,v) &= \tilde{g}_{\Phi(e)}(T_e\Phi(u), T_e\Phi(v)) \\
&= \tilde{g}_e(\phi(u), \phi(v)) \\
&= g(\phi(u), \phi(v)) \\
&= (\phi^*g)(u,v) \\
&= (\widetilde{\phi^*g})_e(u,v)
\end{aligned}$$

Thus

$$(\Phi^*\tilde{g})_e = (\widetilde{\phi^*g})_e.$$

□

Assume that $g(t)$ is a solution to the flow equation

$$\frac{d}{dt}g(t) = -2\overline{\mathrm{Ric}}_e(g(t)). \tag{7.3}$$

The next lemma shows that automorphisms of the Lie algebra commutes with the vector field $\overline{\mathrm{Ric}}_e$.

Lemma 7.3.8. Suppose that ϕ is an automorphism of a Lie algebra $\mathfrak{g}$ and $\tilde{G}$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Let Φ be the corresponding automorphism of the Lie group $\tilde{G}$ such that $T_e\Phi = \phi$. Then we get

$$\phi^*\overline{\mathrm{Ric}}_e(g) = \overline{\mathrm{Ric}}_e(\phi^* g)$$

Proof. By direct computations we have,

$$\begin{aligned}\phi^*\overline{\mathrm{Ric}}_e(g) &= \phi^* \operatorname{Ric}(\tilde{g})_e \\ &= (T_e\Phi)^* \operatorname{Ric}(\tilde{g})_e \\ &= \operatorname{Ric}(\Phi^*\tilde{g})_e \\ &= \operatorname{Ric}(\widetilde{\phi^* g})_e \\ &= \overline{\mathrm{Ric}}_e(\phi^* g)\end{aligned}$$

□

With the use of Lemma 7.3.7 we can show the following theorem.

Theorem 7.3.9. Suppose that $g(t)$ is a solution to the flow equation (7.2). Then $\phi^*(g(t))$ is a solution to the flow equation (7.2). Also we have that $\phi^*(g(t)) = \phi^*(g)(t)$

Proof.

$$\begin{aligned}\frac{d}{dt}\phi^*(g(t)) &= \phi^*(\frac{d}{dt}g(t)) \\ &= \phi^*(-2\overline{\mathrm{Ric}}_e(g(t))) \\ &= -2\overline{\mathrm{Ric}}_e((\phi^* g(t)))\end{aligned}$$

The equality $\phi^*(g(t)) = \phi^*(g)(t)$ follows from the uniqueness of solutions of the equation (7.2) and the fact that $\phi^*(g(0)) = \phi^*(g)(0)$. □

We proved the Theorem 7.3.1 for a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. We want to get this result for every connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that G is a connected lie group with Lie algebra $\mathfrak{g}$. Let $\tilde{G} \to G$ be the universal covering group. That is $\tilde{G}$ is a Lie group and π a lie group homomorphism such that $T_e\pi : T_e\tilde{G} \to T_eG$ is a lie algebra isomorphism. Suppose that $g \in IP(\mathfrak{g})$ is an inner product on $\mathfrak{g}$ and $\tilde{g} \in IP(G)^G$ the corresponding left invariant metric on G with $\tilde{g}_e = g$. Let $\hat{\tilde{g}} = \pi^*\tilde{g}$ the pullback of the metric $\tilde{g}$ on the universal cover $\tilde{G}$. Then $\hat{\tilde{g}}_e = (T_e\pi)^*\tilde{g}_e = (T_e\pi)^*g$. Therefore, $\hat{\tilde{g}} = \widetilde{(T_e\pi)^* g}$. We can now prove the following Lemma.

Lemma 7.3.10. With the same notation as above we have

$$\overline{\mathrm{Ric}}_e((T_e\pi)^*g) = (T_e\pi)^*\overline{\mathrm{Ric}}_e(g)$$

Proof.

$$\begin{aligned}
\overline{\text{Ric}}_e(T\pi)^*_e g) &= \text{Ric}((\widetilde{T_e\pi)^*}g)_e \\
&= \text{Ric}(\hat{\tilde{g}})_e \\
&= \text{Ric}(\pi^*\tilde{g})_e \\
&= \pi^*(\text{Ric}(\tilde{g}))_e \\
&= (T_e\pi)^*(\text{Ric}(\tilde{g})_e) \\
&= (T_e\pi)^*(\overline{\text{Ric}}_e(g)).
\end{aligned}$$

□

Armed with these results we can prove the following theorem.

Theorem 7.3.11. Let $g \in IP(\mathfrak{g})$ an inner product on a Lie algebra $\mathfrak{g}$. For every $\phi \in \text{Aut}(\mathfrak{g})$ we have

$$\phi^*\overline{\text{Ric}}_e(g) = \overline{\text{Ric}}_e(\phi^* g).$$

Proof. In order to prove this theorem we will apply Corollary 7.3.2 to $\psi = (T_e\pi)^* \circ \phi \circ ((T_e\pi)^*)^{-1}$. So for every $g \in IP(\mathfrak{g})$ we have

$$\begin{aligned}
\phi^*\overline{\text{Ric}}_e(g) = \psi &= ((T_e\pi)^*)^{-1} \circ \psi \circ (T_e\pi)^* \\
&= ((T_e\pi)^*)^{-1} \circ \psi\overline{\text{Ric}}_e((T_e\pi)^* g) \\
&= (((T_e\pi)^*)^{-1}\overline{\text{Ric}}_e(\psi^* \circ (T_e\pi)^* g) \\
&= \overline{\text{Ric}}_e((((T_e\pi)^*)^{-1}) \circ \psi \circ (T_e\pi)^* g) \\
&= \overline{\text{Ric}}_e(\phi^* g)
\end{aligned}$$

□

And thus we proved that the operator $\overline{\text{Ric}}_e$ does not depend on the group we chose. This has the following application for the quasi-convergence equivalence relation.

Corollary 7.3.12. Let ϕ be an automorphism of $\mathfrak{g}$. If $g, h \in IP(\mathfrak{g})$ and $g \sim_{\mathfrak{g}} h$, then $\phi^* g \sim_{\mathfrak{g}} \phi^* h$.

Proof. Let g, h be two inner products on the Lie algebra $\mathfrak{g}$. Consider the corresponding Ricci flow equations g_t and h_t with initial data g and h respectively. Assume that g_t quasi-converges to h_t. Then by definition

$$|g_t - h_t|_{g_t} \to 0$$

as $t \to \infty$, which implies that

$$|\phi^*(g_t) - \phi^*(h_t)|_{\phi^*(g_t)} \to 0$$

as $t \to \infty$. Now using Theorems 7.3.8. and 7.3.10. we have

$$|(\phi^* g)_t - (\phi^* h)_t|_{(\phi^* g)_t} \to 0.$$

Therefore, $\phi^* g \sim_{\mathfrak{g}} \phi^* h$. □

We will now describe the equivalent classes for the quasi-convergence in $IP(\mathfrak{g})$. For this we will fix a standard form σ. For any frame $\beta \in \mathcal{F}(\mathfrak{g})$ with $\sigma_\beta = \sigma$ we consider the map

$$\phi_\beta : \mathrm{GL}(3, \mathbb{R})_\sigma \times A \to IP(\mathfrak{g})$$

given by

$$\beta^*(\phi_\beta(x, a)) = xa \cdot e = ((xa)^{-1})^* e$$

where e is the Euclidean inner product on $\mathbb{R}^3$. Now since the map $\beta^* : IP(\mathfrak{g}) \to \mathbb{R}^3$ is a bijection and the map

$$\begin{aligned} \beta^*\phi_\beta : \mathrm{GL}(3, \mathbb{R})_\sigma \times A &\to IP(\mathbb{R}^3) \\ (x, a) &\mapsto ((xa)^{-1})^* e \end{aligned}$$

is surjective by Corollary 7.2.1. we deduce that the map ϕ_β is surjective.

Definition 7.3.13. We define the equivalent relation $\sim_\sigma$ on $\mathrm{GL}(3, \mathbb{R})_\sigma \times A$ as

$$(x, a) \sim_\sigma (y, b) \Leftrightarrow \phi_\beta(x, a) \sim \phi_\beta(x, b)$$

where $\sim$ denotes the quasi-convergence equivalence relation on $IP(\mathfrak{g})$.

Roughly speaking we transfer the quasi-convergence relation on the space of Ricci flows to the space $\mathrm{GL}(3, \mathbb{R})_\sigma \times A$. The next Lemma shows how this equivalent relation on $\mathrm{GL}(3, \mathbb{R})_\sigma \times A$ behaves under the action of $\mathrm{GL}(3, \mathbb{R})_\sigma$.

Lemma 7.3.14. Suppose that $(x, a) \sim_\sigma (y, b)$. Then for all $z \in \mathrm{GL}(3, \mathbb{R})_\sigma$ we have

$$(zx, a) \sim_\sigma (zy, b).$$

Proof. We have

$$\begin{aligned} \beta^*\phi_\beta(zx, a) &= ((zxa)^{-1})^* e \\ &= (z^{-1})^* ((xa)^{-1})^* e \\ &= (z^{-1})^* \beta^* \phi_\beta(x, a). \end{aligned}$$

Therefore

$$\phi_\beta(zx, a) = (\beta z^{-1} \beta^{-1})^* \phi_\beta(x, a).$$

Likewise,

$$\phi_\beta(zy, b) = (\beta z^{-1} \beta^{-1})^* \phi_\beta(y, b).$$

So from the fact that $(x, a) \sim_\sigma (y, b)$ we deduce that $\phi_\beta(x, a) \sim \phi_\beta(y, b)$ that is $\phi_\beta(x, a)$ quasi-converges to $\phi_\beta(y, b)$. Using Corollary 7.3.1. we get that $\beta z^{-1} \beta^{-1}$ is an automorphism of the Lie algebra $\mathfrak{g}$ and thus by Corollary 7.3.2. we have

$$(\beta z^{-1} \beta^{-1})^* \phi_\beta(x, a) \sim (\beta z^{-1} \beta^{-1})^* \phi_\beta(y, b).$$

So we deduce that $\phi_\beta(zx, a) \sim \phi_\beta(zy, b)$ and thus

$$(zx, a) \sim_\sigma (zy, b).$$

□

Now let $\beta \in \mathcal{F}(\mathfrak{g})$ a frame on the Lie algebra $\mathfrak{g}$ with $\sigma_\beta = \sigma$, and let $b \in A$. Let $g = (\beta^{-1})^*(b^{-1})^*(e)$ where e is the Euclidean metric. Then $\beta^*(g) = g_\beta$ is of the form

$$g_\beta = (b^{-1})^* e = b_{11}^{-2} e^1 \otimes e^1 + b_{22}^{-2} e^2 \otimes e^2 + b_{33}^{-2} e^3 \otimes e^3$$

where e^i is the dual frame of β. So β is a σ-Milnor frame for g. Our calculations give us all metrics $h \in IP(\mathfrak{g})$ such that h quasi-converges to g i.e. the equivalence class $[g]$. Indeed, putting $h_\beta = ((xa)^{-1})^* e$ we calculate the set

$$\begin{aligned} S(\beta, b) &= \{(x,a) \in \mathrm{GL}(3,\mathbb{R})_\sigma \times A \mid (\beta^{-1})^*((xa)^{-1})^* e \sim g = (\beta^{-1})^*(b^{-1})^*(e)\} \\ &= \{(x,a) \in \mathrm{GL}(3,\mathbb{R})_\sigma \times A \mid (x,a) \sim_\sigma (e,b)\} \\ &= [(e,b)]_{\sim_\sigma} \end{aligned}$$

From our calculations we can determine the equivalent classes $[(e,b)]_{\sim_\sigma}$ for every $b \in A$. It follows by Lemma 7.3.3

$$[(x,b)]_{\sim_\sigma} = [(xe,b)]_{\sim_\sigma} = L_x[(e,b)]_{\sim_\sigma}.$$

Here $L_x : \mathrm{GL}(3,\mathbb{R})_\sigma \times A \to \mathrm{GL}(3,\mathbb{R})_\sigma \times A, (y,a) \mapsto (xy,a)$

Now take a subgroup $H < \mathrm{GL}(3,\mathbb{R})_\sigma$ such that $HAO(3) = \mathrm{GL}(3,\mathbb{R})_\sigma$ or equivalently

$$\begin{aligned} H \times A &\to IP(\mathbb{R}^3) \simeq \mathrm{GL}(3,\mathbb{R})/O(3) \\ (h,a) &\mapsto ((ha)^{-1})^* e \end{aligned}$$

is surjective. Then we define $\psi_\beta = \phi_\beta|_{H \times A}$. We note that ψ_β is surjective. We pullback the equivalent relation $\sim_\sigma$ to $H \times A$ as follows

$$(h,a) \sim_{\sigma_H} (h',a') \Leftrightarrow \psi_\beta(h,a) \sim \psi_\beta(h',a')$$

We have $[(e,b)]_{\sim_{\sigma_H}} = [(e,b)]_\sim \cap (H \times A)$. Then $[(h,b)]_{\sim_{\sigma_H}} = L_h([(e,b)]_{\sim_{\sigma_H}})$.

Definition 7.3.15. Now consider a left action of a group G on a set S. An equivalent relation $\sim$ on S is called G-invariant if for every $s,t \in S$ and $g \in G$ we have

$$s \sim t \implies g \cdot s \sim g \cdot t.$$

If we have a left action of a group G on a set S with an equivalent relations then there is a unique G-action on the set $S/\sim$ such that the map $p : S \to S/\sim$ is G-equivariant and it is given by

$$g \cdot [s] = [g \cdot s].$$

Definition 7.3.16. A slice for the G-action is a subset Σ of S such that for every $s \in S$ the set $\Sigma \cap G \cdot s$ is non empty.

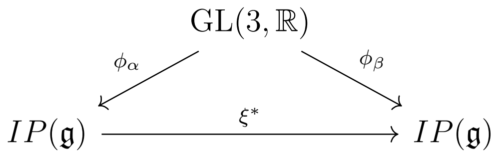
Suppose that Σ is a slice of the G-action such that for every $s_0 \in \Sigma$ the class $[s_0]$ is known. Then each $s \in S$ equals to $g \cdot s_0$ for some $g \in G$ and

$$[s] = [g \cdot s_0] = g \cdot [s_0]$$

In our situation where we have the equivalent relation $\sim_\sigma$ on $\mathrm{GL}(3,\mathbb{R})_\sigma \times A$ a slice is given by $\Sigma = \{(e,a) \mid a \in A\}$ and for the action of $\mathrm{GL}(3,\mathbb{R})_\sigma$ on $\mathrm{GL}(3,\mathbb{R})_\sigma \times A$ the class $[(e,a)]$ can be computed.

A natural question that arises is the following: How does $\phi_\beta : G_\sigma \times A \to IP(\mathfrak{g})$ depend on β? Well suppose that $a \in \mathcal{F}(\mathfrak{g})$ is another frame on $\mathfrak{g}$ such that $\sigma_\alpha = \sigma$.

Lemma 7.3.17. The map $\xi = \alpha \circ \beta^{-1}$ belongs to $\mathrm{Aut}(\mathfrak{g})$ and the following diagram commutes:



Proof. From the definitions of ϕ_α and ϕ_β it follows that

$$\alpha^* \phi_\alpha(x, a) = ((xa)^{-1})^*(e) = \beta^* \phi_\beta(x, a)$$

Therefore,

$$\begin{aligned} \phi_\beta(x, a) &= (\beta^{-1})^* \alpha^* \circ \phi_\alpha(x, a) \\ &= (\alpha \circ \beta^{-1})^* \phi_{\alpha \circ \beta^{-1}}(x, a). \end{aligned}$$

By putting $\xi = \alpha \circ \beta^{-1} = \beta \circ (\beta^{-1} \circ \alpha) \circ \beta^{-1}$ we have that $\beta^{-1} \circ \alpha \in \mathrm{GL}(3, \mathbb{R})_\sigma$, so $\xi \in \mathrm{Aut}(\mathfrak{g})$ by Lemma 7.3.2. □

Corollary 7.3.18. The above diagram shows that

$$\begin{aligned} (x, a) \sim_{\sigma_\alpha} (y, b) &\Leftrightarrow \phi_\alpha(x, a) \sim \phi_\beta(y, b) \\ &\Leftrightarrow \xi^*(\phi_\alpha(x, a)) \sim \xi^*(\phi_\beta(y, b)) \\ &\Leftrightarrow \phi_\beta(x, a) \sim \phi_\alpha(y, b) \\ &\Leftrightarrow (x, a) \sim_{\sigma_\beta} (y, b). \end{aligned}$$

Therefore we have proven the following theorem.

Theorem 7.3.19. If $\alpha, \beta \in \mathfrak{g}$ are Milnor frames on $\mathfrak{g}$ such that $\sigma_\alpha = \sigma_\beta = \sigma$ then the equivalence relation $\sim_{\sigma_\alpha}$ on $\mathrm{Gl}(3, \mathbb{R})$ with respect to α is equal to the equivalence relation $\sim_{\sigma_\beta}$ on $\mathrm{Gl}(3, \mathbb{R})$ with respect to β.

8 Computation of the quasi-convergence classes for Lie groups

In this section we present the computations for the quasi-convergence classes for left invariant Ricci flows on Lie groups.

8.1 The quasi-convergence classes for the Heisenberg group

The first example that we will consider is the Heisenberg group. Let G be the Heisenberg group equipped with a left invariant metric g_0.

Suppose that $\beta = (\beta_1, \beta_2, \beta_3)$ is a Milnor frame for the pair (G, g_0) with dual frame $\omega = (\omega_1, \omega_2, \omega_3)$. The matrix of the structure constants in the case of the Heisenberg group has the standard form

$$\sigma = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The expression of the metric with respect to this frame has the form:

$$g_0 = A_0 \omega^1 \otimes \omega^1 + B_0 \omega^2 \otimes \omega^2 + C_0 \omega^3 \otimes \omega^3,$$

and the Ricci tensor relative to this frame has the expression

$$\mathrm{Ric}(g_0) = 2\frac{A_0^2}{B_0 C_0}\omega^1 \otimes \omega^1 - 2\frac{A_0}{C_0}\omega^2 \otimes \omega^2 - 2\frac{A_0}{B_0}\omega^3 \otimes \omega^3.$$

Suppose now that we have a one parameter family of left invariant metrics on the group G. Then we can write g relative to the Milnor frame β as follows

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3,$$

where A, B, C are functions of time such that $A(0) = A_0, B(0) = B_0$ and $C(0) = C_0$. Similarly, The Ricci tensor for the one parameter family of left invariant metrics g expressed in the Milnor frame β has the form:

$$\mathrm{Ric}(g) = 2\frac{A^2}{BC}\omega^1 \otimes \omega^1 - 2\frac{A}{C}\omega^2 \otimes \omega^2 - 2\frac{A}{B}\omega^3 \otimes \omega^3.$$

So the Ricci flow equation is equivalent to the system.

$$\frac{d}{dt}A = -4\frac{A^2}{BC},\ \frac{d}{dt}B = 4\frac{A}{C},\ \frac{d}{dt}C = 4\frac{A}{B}$$

In order to provide a solution to the Ricci flow equation we first have to determine the quantities that are conserved by the Ricci flow.

Lemma 8.1.1. The quantities AB and (B/C) are conserved by the Ricci flow i.e. $\frac{d}{dt}(AB) = \frac{d}{dt}(B/C) = 0$.

Proof.

$$\frac{d}{dt}(AB) = B\frac{d}{dt}A + A\frac{d}{dt}B = -4\frac{A^2B}{BC} + 4\frac{A^2}{C} = 0$$

and

$$\frac{d}{dt}(B/C) = \frac{C\frac{d}{dt}B - B\frac{d}{dt}C}{C^2} = \frac{\frac{4CA}{C} - \frac{4BA}{B}}{C^2} = 0$$

□

Thus, we introduce positive constants $\Phi = AB = A_0B_0$ and $\Psi = \frac{B}{C} = \frac{B_0}{C_0}$. We have

$$\frac{d}{dt}(A/B^2) = \frac{B^2\frac{d}{dt}A - A\frac{d}{dt}B^2}{B^4} = \frac{-4B^2\frac{A^2}{BC} - 8AB\frac{A}{C}}{B^4} = -12\frac{B}{C}\left(\frac{A}{B^2}\right)^2$$

Integrating this equality we get

$$\frac{A}{B^2} = \frac{C_0/B_0}{12t + B_0C_0/A_0}$$

or equivalently

$$A = \frac{C_0/B_0}{12t + B_0C_0/A_0}B^2$$

Using this we can compute $\frac{d}{dt}A$ as follows:

$$\frac{d}{dt}A = -4\Psi\left(\frac{A}{B^2}\right)A = -4\frac{A}{12t + B_0C_0/A_0}.$$

Therefore,

$$A = A_0^{2/3}B_0^{1/3}C_0^{1/3}\left(12t + B_0C_0/A_0\right)^{-1/3}$$

Lastly we have that $B = \Phi/A$ and $C = B/\Psi$. To sum everything up, the solutions to the Ricci flow equations are given by:

$$A = A_0^{2/3}B_0^{1/3}C_0^{1/3}\left(12t + B_0C_0/A_0\right)^{-1/3}, \tag{8.1}$$

$$B = \frac{\Phi}{A} = A_0^{1/3}B_0^{2/3}C_0^{-1/3}\left(12t + B_0C_0/A_0\right)^{1/3}, \tag{8.2}$$

$$C = \frac{B}{\Psi} = A_0^{1/3}B_0^{-1/3}C_0^{2/3}\left(12t + B_0C_0/A_0\right)^{1/3}. \tag{8.3}$$

We should remark that $A(t) \to 0$ as t goes to infinity while $B(t)$ and $C(t)$ diverge to diverge to infinity as t goes to infinity. Just like the case of $\mathbb{H}^2 \times \mathbb{R}$, the solution of the Ricci flow for any left invariant metric on the Heisenberg group forms a pancake degeneracy.

In order to compute the quasi-convergence equivalence class $[g]$ of a left invariant Ricci flow in the Heisenberg group we first have to compute the quasi-convergence equivalent class $[g]_\beta$ of a left invariant Ricci flow on the Heisenberg group for a fixed Milnor frame β. Suppose that

$$\bar{g} = \bar{A}\omega^1 \otimes \omega^1 + \bar{B}\omega^2 \otimes \omega^2 + \bar{C}\omega^3 \otimes \omega^3$$

is another solution to the Ricci flow equation that has the same Milnor frame β then $\bar{g} \in [g]_\beta$ if $|g-\bar{g}|_g^2 \to 0$ as $t \to \infty$. That is if every term in the following sum converges to zero.

$$|g-\bar{g}|_g^2 = \left(\frac{A-\bar{A}}{A}\right)^2 + \left(\frac{B-\bar{B}}{B}\right)^2 + \left(\frac{C-\bar{C}}{C}\right)^2 \to 0.$$

By using the equations of the solution to the Ricci flow (8.1) - (8.3), $|g-\bar{g}|_g^2 \to 0$ translates to

$$\begin{aligned}
\frac{A-\bar{A}}{A} &= 1 - \frac{\bar{A}_0^{2/3}\bar{B}_0^{1/3}\bar{C}_0^{1/3}}{A_0^{2/3}B_0^{1/3}C_0^{1/3}}\left(\frac{12t + B_0C_0/A_0}{12t + \bar{B}_0\bar{C}_0/\bar{A}_0}\right)^{1/3} \to 1 - \frac{\bar{A}_0^{2/3}\bar{B}_0^{1/3}\bar{C}_0^{1/3}}{A_0^{2/3}B_0^{1/3}C_0^{1/3}}\\
\frac{B-\bar{B}}{B} &= 1 - \frac{\bar{A}_0^{1/3}\bar{B}_0^{2/3}\bar{C}_0^{-1/3}}{A_0^{1/3}B_0^{2/3}C_0^{-1/3}}\left(\frac{12t + \bar{B}_0\bar{C}_0/\bar{A}_0}{12t + B_0C_0/A_0}\right)^{1/3} \to 1 - \frac{\bar{A}_0^{1/3}\bar{B}_0^{2/3}\bar{C}_0^{-1/3}}{A_0^{1/3}B_0^{2/3}C_0^{-1/3}}\\
\frac{C-\bar{C}}{C} &= 1 - \frac{\bar{A}_0^{1/3}\bar{B}_0^{-1/3}\bar{C}_0^{2/3}}{A_0^{1/3}B_0^{-1/3}C_0^{2/3}}\left(\frac{12t + \bar{B}_0\bar{C}_0/\bar{A}_0}{12t + B_0C_0/A_0}\right)^{1/3} \to 1 - \frac{\bar{A}_0^{1/3}\bar{B}_0^{-1/3}\bar{C}_0^{2/3}}{A_0^{1/3}B_0^{-1/3}C_0^{2/3}}
\end{aligned}$$

Lemma 8.1.2. The class $[g]_\beta$ is 1-dimensional. In particular $\bar{g} \in [g]_\beta$ if and only if there exists a positive scaling parameter λ such that

$$\bar{A}_0 = \frac{A}{\lambda},\ \bar{B}_0 = \lambda B_0,\ \bar{C}_0 = \lambda C_0$$

Proof. Let $\bar{A}_0 > 0$ arbitrary and define λ by $\lambda := \frac{A_0}{\bar{A}_0}$. Then $\bar{g} \in [g]_\beta$ only if $\frac{\bar{A}}{A} \to 1 \implies \bar{B}_0\bar{C}_0 = \lambda^2 B_0 C_0$ and $\frac{\bar{C}}{C} \to 1 \implies B_0\bar{C}_0{}^2 = \lambda\bar{B}_0C_0^2$.

Solving these equations for $\frac{\bar{B}_0}{B_0}$ yields

$$\frac{\lambda^2 C_0}{\bar{C}_0} = \frac{\bar{B}_0}{B_0} = \frac{\bar{C}_0^2}{\lambda C_0^2}$$

which implies $\frac{\bar{C}_0}{C_0} = \lambda$ and hence $\frac{\bar{B}_0}{B_0} = \frac{\bar{C}_0^2}{\lambda C_0^2} = \lambda$. □

Having computed the quasi-convergence class $[g]_\beta$ of a left invariant Ricci flow for a fixed Milnor frame we can proceed to the computation of the full quasi-convergence equivalence class. Suppose that g and h are two left invariant Ricci flows on the Heisenberg group. We pick a Milnor frame β for g. Then g_β is diagonal and $\sigma_\beta = \sigma = 2\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. As in the previous section consider the action of the Lie group $\mathrm{GL}(3,\mathbb{R})$ on the space of the 3×3 matrices $M_3(\mathbb{R})$ given by

$$A \cdot \sigma = (\det(A))^{-1} A\sigma A^T. \quad \text{(see (7.1))}$$

In order to determine the stabilizer subgroup $\mathrm{GL}(3,\mathbb{R})_\sigma$ of this action, that is
$\mathrm{GL}(3,\mathbb{R})_\sigma = \{A \in \mathrm{GL}(3,\mathbb{R}) \mid (\det(A))^{-1}A\sigma A^T = \sigma\}$, we will consider the action on the Lie algebra level. By putting $A = \exp(tX)$ for $X \in \mathfrak{g}$ and by taking the derivative of

the relation $(\det(A))^{-1}A\sigma A^T = \sigma$ at zero we find that on the Lie algebra level we have $-\operatorname{tr} X\sigma + X\sigma + (X\sigma)^T = 0$.

We compute,

$$-\operatorname{tr} X\sigma = -(X_{11}+X_{22}+X_{33})\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}+X_{22}+X_{33} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X\sigma = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -X_{11} & 0 & 0 \\ -X_{21} & 0 & 0 \\ -X_{31} & 0 & 0 \end{pmatrix}$$

$$(X\sigma)^T = \left(\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)^T = \begin{pmatrix} -X_{11} & -X_{21} & -X_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $X_{11}+X_{22}+X_{33}-X_{11}-X_{11}=0$ which implies that $X_{11}=X_{22}+X_{33}$. Also, $X_{21}=0$ and $X_{31}=0$.

So the Lie algebra of this stabilizer group for the Heisenberg group is given by:

$$\operatorname{Lie}(\mathrm{GL}(3,\mathbb{R})_\sigma) = \left\{ \begin{pmatrix} X+Y & A & C \\ 0 & X & D \\ 0 & B & Y \end{pmatrix} : X,Y,A,B,C,D \in \mathbb{R} \right\},$$

and finally $(\mathrm{GL}(3,\mathbb{R})_\sigma)_e = G_\sigma$ is equal to,

$$G_\sigma = \left\{ \begin{pmatrix} xy & a & c \\ 0 & x & d \\ 0 & b & y \end{pmatrix} \ : \ x,y,a,b,c,d \in \mathbb{R} \right\}$$

Lemma 8.1.3. The group $N = \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \ : \ n_1,n_2,n_3 \in \mathbb{R} \right\}$ is a subgroup of G_σ.

Proof. Let $n \in N$. Then n is of the form $\begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}$ where n_1,n_2,n_3 are real numbers.

We note that $\det^{-1}(n) = 1$ since $\det(n) = 1$. Thus we have

$$\begin{aligned} n\sigma n^T &= \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n_1 & 1 & 0 \\ n_3 & n_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n_1 & 1 & 0 \\ n_3 & n_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma. \end{aligned}$$

Thus N is a subset of G_σ. Also N is obviously a group and hence N is a subgroup of G_σ. □

In the following we will identify inner products on $\mathbb{R}^3$ with Hermitian matrices an in Remark 4.2.2. So let β be a Milnor frame for g and consider $\alpha = \beta n$ a Milnor frame for h as in Lemma 8.1.3. Therefore, g is diagonal with respect to β and corresponds to the initial data $g_\beta(0) = (A_0.B_0, C_0)$ while h is diagonal with respect to α and corresponds to the initial data $h_\alpha(0) = (\bar{A}_0, \bar{B}_0, \bar{C}_0)$. The expression of the Ricci flow of h in the frame α has the form $h_\beta = n^T h_\alpha n$ where n is the matrix

$$n = \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{aligned} h_\beta &= \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{B} & 0 \\ 0 & 0 & \bar{C} \end{pmatrix} \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \bar{A} & n_1\bar{A} & n_3\bar{A} \\ n_1\bar{A} & n_1^2\bar{A} + \bar{B} & n_1 n_3\bar{A} + n_2\bar{B} \\ n_3\bar{A} & n_1 n_3\bar{A} + n_2\bar{B} & n_3^2\bar{A} + n_2^2\bar{B} + \bar{C} \end{pmatrix} \end{aligned}$$

Theorem 8.1.4. The quasi-convergence class of any left invariant metric g on the Heisenberg group G can be described as

$$[g] = (N_0)_\beta \cdot [g]_\beta$$

where $[g]_\beta$ is the quasi-convergence class of g for a fixed Milnor frame β and $(N_0)_\beta = \{\beta \circ n_0 \circ \beta^{-1} \mid n_0 \in N_0\} < \mathrm{Aut}(\mathfrak{g})$ with N_0 the subgroup of $\mathrm{GL}(3,\mathbb{R})_\sigma$ consisting of the matrices of the form

$$\begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_1, n_2 \in \mathbb{R}.$$

In particular, the quasi-convergence classes for the Heisenberg group are 3-dimensional.

Proof. Let h be a Ricci flow for the Heisenberg group, then $h \in [g]$ if and only if $|h-g|_g^2 \to 0$ as $t \to \infty$, that is if every term in the following sum converge to zero

$$\frac{(\bar{A}-A)^2}{A^2}+\frac{(n_1^2\bar{A}+\bar{B}-B)^2}{B^2}+\frac{(n_3^2\bar{A}+n_2^2\bar{B}+\bar{C})^2}{C^2}+2\frac{(n_1\bar{A})^2}{AB}+2\frac{(n_3\bar{A})^2}{AC}+2\frac{(n_1n_3\bar{A}+n_2\bar{B})^2}{BC} \to 0,$$

as $t \to \infty$. Suppose that every term of the sum converge to zero. Looking at the first term $\frac{(\bar{A}-A)^2}{A^2}$ we get that $\bar{A}-A \to 0$. This in particular implies that $\bar{B}-B \to 0$ and $\bar{C}-C \to 0$ or equivalently the diagonal part of h is in the quasi-convergence class of the $[g]_\beta$ for a fixed Milnor frame β. Now looking at the last term

$$\frac{(n_1n_3\bar{A}+n_2\bar{B})^2}{BC}$$

and using the properties of the solution of the Ricci flow equation i.e. $A(t) \to 0$ while $B(t)$ and $C(t) \to \infty$ we get that $n_2 = 0$. Therefore, a generic Ricci flow in the quasi-convergence class of $[g]$ has the form:

$$h_\beta = \begin{pmatrix} \bar{A} & n_1\bar{A} & n_3\bar{A} \\ n_1\bar{A} & n_1^2\bar{A}+\bar{B} & n_1n_3\bar{A} \\ n_3\bar{A} & n_1n_3\bar{A} & n_3^2\bar{A}+\bar{C} \end{pmatrix}$$

where $(\bar{A},\bar{B},\bar{C})$ is such that $h_\alpha \in [g]_\beta$ for a fixed Milnor frame β and $n_1, n_3 \in \mathbb{R}$. □

8.2 The quasi-convergence classes for $\mathrm{Iso}(\mathbb{E}^2)$

Let G be the Lie group $\mathrm{Iso}(\mathbb{E}^2)$ and g_0 a locally homogeneous metric on G. Let β be a Milnor frame for the pair (G, g_0) with dual frame $\omega = (\omega^1, \omega^2, \omega^3)$. The matrix σ of the structure constants is of the form

$$\sigma_\beta = 2\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider a one-parameter family of left invariant metrics g on G such that $g(0) = g_0$. This family relative to the Milnor frame β can be written in the diagonal form as:

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3.$$

The sectional curvatures are

$$\begin{aligned} K(\beta_2 \wedge \beta_3) &= \frac{(A+B)^2 - 4A^2}{ABC}, \\ K(\beta_3 \wedge \beta_1) &= \frac{(A+B)^2 - 4B^2}{ABC}, \\ K(\beta_1 \wedge \beta_2) &= \frac{(A-B)^2}{ABC}. \end{aligned}$$

and so the Ricci flow is equivalent to the system,

$$\frac{d}{dt}A = 4\frac{B^2 - A^2}{BC},$$
$$\frac{d}{dt}B = 4\frac{A^2 - B^2}{AC},$$
$$\frac{d}{dt}C = 4\frac{(A - B)^2}{AB}.$$

with $A_0 = A(0), B_0 = B(0)$ and $C_0 = C(0)$. To describe the behaviour of a solution to this system we first have to find the conserved of the Ricci flow equation.

Lemma 8.2.1. The conserved quantities for this system of equations are: AB and $C(A+B)$ i.e. $\frac{d}{dt}(AB) = \frac{d}{dt}(C(A+B)) = 0$.

Proof. Indeed, by direct calculation we get

$$\begin{aligned}\frac{d}{dt}(AB) &= A\frac{d}{dt}B + B\frac{d}{dt}A \\ &= A4\frac{A^2 - B^2}{AC} + B4\frac{B^2 - A^2}{BC} \\ &= 4\frac{A^2 - B^2}{C} + 4\frac{B^2 - A^2}{C} = 0\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(C(A+B)) &= (A+B)\frac{d}{dt}C + C\frac{d}{dt}C \\ &= 4(A+B)\frac{(A-B)^2}{AB} + 4C(\frac{B^2 - A^2}{BC} + \frac{A^2 - B^2)}{AC} \\ &= 4(A+B)\frac{(A^2 - 2AB + B^2)}{AB} + 4\frac{AB^2 - A^3 + BA^2 - B^3}{AB} \\ &= 4\frac{A^3 - 2A^2B + AB^2 + A^2B - 2AB^2 + B^3}{AB} + 4\frac{AB^2 - A^3 + BA^2 - B^3}{AB} = 0\end{aligned}$$

□

Therefore we define the the quantities $\Phi = A_0B_0$ and $\Psi = C_0(A_0 + B_0)$.

Lemma 8.2.2. As $t \to \infty$ we have,

- both $A(t)$ and $B(t)$ converge to $\sqrt{A_0B_0}$
- $C(t)$ converge to $\frac{C_0}{2}\left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}}\right)$

Proof. We set $\rho = B/A$. We consider the simplified system

$$\frac{d}{dt}\rho = 8\frac{(1-\rho^2)}{C}, \qquad \frac{d}{dt}C = 4\frac{(1-\rho)^2}{\rho}$$

Since every point on the ray $\rho = A/B = 1$ the component C is a fixed point, we may assume that $\rho_0 \neq 1$. Hence $\rho \neq 1$ for all times from the uniqueness of solutions to the Ricci flow equation. Because $\frac{d\rho}{dt} > 0$ if $0 < \rho < 1$ and $\frac{d\rho}{dt} < 0$ if $\rho > 1$ it follows that the function ρ is bounded and monotone. Therefore, it converges to a limit ρ_∞ which satisfies $\rho_0 < \rho_\infty \leq 1$ or $1 \leq \rho_\infty < \rho_0$. Now, since ρ is monotone we can write C as a function of ρ. So we have

$$\frac{d}{d\rho} \log C = \frac{1}{2\rho} \frac{1-\rho}{1+\rho}$$

But,

$$\log \frac{C(\rho(t))}{C_0} = \int_{\rho_0}^{\rho(t)} \frac{1}{2\rho} \frac{1-\rho}{1+\rho} d\rho \to \log \frac{\sqrt{P_\infty}}{1+\rho_\infty} - \log \frac{\sqrt{P_0}}{1+\rho_0} \text{ as } t \to \infty$$

Therefore we conclude that the limit $\lim_{t\to\infty} C(t)$ exists. As a result $\rho_\infty = 1$. Using this we have

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} B(t) = \sqrt{\Phi} = \sqrt{A_0 B_0}$$

and

$$\lim_{t\to\infty} C(t) = \frac{C_0}{2} \left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}} \right)$$

□

We remark that the limit of the solution $\lim_{t\to\infty} g(t)$ to the Ricci flow is a metric with zero sectional curvature. Using the properties of the solution to the Ricci flow equation we can determine the quasi-convergence class for a fixed Milnor frame.

Lemma 8.2.3. Suppose that g is a left-invariant metric on $\mathrm{Iso}(\mathbb{E}^2)$, then $[g]_\beta$ is 1-parameter family.

Proof. The proof of this lemma is a calculation similar to that of Lemma 8.1.2 and we will not present it here. See [11] for a full proof. □

We will now start to compute the full quasi-convergence class for $\mathrm{Iso}(\mathbb{E}^2)$. Suppose that h, g are two left invariant Ricci flows on the Lie group $\mathrm{Iso}(\mathbb{E}^2)$. Let β be a Milnor frame for g i.e. g is diagonal with respect to this frame and corresponds to the initial data $g_\beta(0) = (A_0, B_0, C_0)$ and let α be a Milnor frame for h i.e. h is diagonal with respect to this frame and corresponds to the initial data $h_\alpha(0) = (\bar{A}_0, \bar{B}_0, \bar{C}_0)$. In the case of $\mathrm{Iso}(\mathbb{E}^2)$ we have that

$$\sigma = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We want to determine the Lie algebra of the stabilizer group $\mathrm{GL}(3,\mathbb{R})_\sigma$ as in the case of the Heisenberg group. That is we have to determine the all the 3×3 matrices X that satisfy the equation $-\operatorname{tr}(X) + X\sigma + (X\sigma)^T = 0$ i.e.

$$-\operatorname{tr}(X) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + X \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \left(X \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)^T = 0$$

So we have:

$$-\operatorname{tr}(X)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}+X_{22}+X_{33} & 0 & 0 \\ 0 & X_{11}+X_{22}+X_{33} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X\sigma = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -X_{11} & -X_{21} & 0 \\ -X_{21} & -X_{22} & 0 \\ -X_{31} & -X_{32} & 0 \end{pmatrix}$$

and

$$(X\sigma)^T = \left(\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)^T = \begin{pmatrix} -X_{11} & -X_{12} & -X_{31} \\ -X_{12} & -X_{22} & -X_{32} \\ 0 & 0 & 0 \end{pmatrix}$$

Thus we get $X_{33} = 0, X_{11} = X_{22}, X_{12} = -X_{21}$ and $X_{32} = X_{31} = 0$. So the Lie algebra of the stabilizer group is

$$\operatorname{Lie}(\mathrm{GL}(3,\mathbb{R})_\sigma) = \left\{\begin{pmatrix} Y & -X & V \\ X & Y & W \\ 0 & 0 & 0 \end{pmatrix} : Y, X, V, W \in \mathbb{R}\right\}$$

So the Lie group $G_\sigma = (\mathrm{GL}(3,\mathbb{R})_\sigma)_e$ is of the form

$$G_\sigma = \left\{\begin{pmatrix} y & x^{-1} & p \\ x & y & q \\ 0 & 0 & 1 \end{pmatrix} \ : \ x, y, p, q \in \mathbb{R}\right\}.$$

By direct calculation, as in Lemma 8.1.3 we have the following,

Lemma 8.2.4. The group $\mathrm{Iso}(\mathbb{E}^2)$ consisting of matrices of the form $\begin{pmatrix} cos\theta & -sin\theta & p \\ sin\theta & cos\theta & q \\ 0 & 0 & 1 \end{pmatrix}$ is a subgroup of G_σ.

Theorem 8.2.5. The quasi-convergence class of any left invariant metric g on the group $G = \mathrm{Iso}(\mathbb{E}^2)$ can be described as

$$[g] = (K_0)_\beta \cdot [g]_\beta$$

where $[g]_\beta$ is the quasi-convergence class of g for a fixed Milnor frame β and $(K_0)_\beta = \{\beta \circ k_0 \circ \beta^{-1} \mid k_0 \in K_0\} < \mathrm{Aut}(\mathfrak{g})$ with K_0 the subgroup of $\mathrm{GL}(3,\mathbb{R})_\sigma$ consisting of the matrices of the form

$$\begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

In particular, every quasi-convergence class is a 2-parameter family.

Proof. Let

$$K = \begin{pmatrix} cos\theta & -sin\theta & p \\ sin\theta & cos\theta & q \\ 0 & 0 & 1 \end{pmatrix}$$

be the matrix which corresponds to the change of frames from α to β i.e. $\beta = \alpha K$. Then $h_\alpha = K^T h_\beta K$ is the Ricci flow h expressed in the Milnor frame β. In particular

$$h_\beta = \begin{pmatrix} cos\theta & -sin\theta & p \\ sin\theta & cos\theta & q \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{B} & 0 \\ 0 & 0 & \bar{C} \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta & p \\ sin\theta & cos\theta & q \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \bar{A}cos^2\theta + \bar{B}sin^2\theta & -\bar{A}cos\theta sin\theta + \bar{B}cos\theta sin\theta & p\bar{A}cos\theta + q\bar{B}sin\theta \\ -\bar{A}cos\theta sin\theta + \bar{B}cos\theta sin\theta & \bar{A}cos^2\theta + \bar{B}sin^2\theta & -\bar{A}psin\theta + \bar{B}qcos\theta \\ p\bar{A}cos\theta + q\bar{B}sin\theta & -\bar{A}psin\theta + \bar{B}qcos\theta & p^2\bar{A} + q^2\bar{B} + \bar{C} \end{pmatrix}$$

Now $h \in [g]$ if and only if every term in the following sum converge to zero

$$\begin{aligned} |h-g|_g^2 =& \frac{(\bar{A}cos^2\theta + \bar{B}sin^2\theta - A)^2}{A^2} + \frac{(\bar{A}cos^2\theta + \bar{B}sin^2\theta - B)^2}{B^2} + \frac{(p^2\bar{A} + q^2\bar{B} + \bar{C} - C)^2}{C^2} \\ &+ 2\frac{(-\bar{A}cos\theta sin\theta + \bar{B}cos\theta sin\theta)^2}{AB} + 2\frac{(p\bar{A}cos\theta + q\bar{B}sin\theta)^2}{AC} + 2\frac{(-\bar{A}psin\theta + \bar{B}qcos\theta)^2}{BC} \end{aligned}$$

Suppose that every term of this sum converge to zero. Looking at the last term

$$2\frac{(-\bar{A}psin\theta + \bar{B}qcos\theta)^2}{BC}$$

and using that both $A(t), B(t) \to \sqrt{A_0B_0}$, (Lemma 8.2.2) we get that

$$2\frac{(-\bar{A}psin\theta + \bar{B}qcos\theta)^2}{BC} \to 2\frac{(-psin\theta\sqrt{A_0B_0} + qcos\theta\sqrt{A_0B_0})^2}{\sqrt{A_0B_0}\frac{C_0}{2}\left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}}\right)}.$$

So we deduce that in order for this term to go to zero we must have $p = q = 0$. Now looking at the third term of the sum

$$\frac{(p^2\bar{A} + q^2\bar{B} + \bar{C} - C)^2}{C^2} = \frac{(\bar{C} - C)^2}{C^2}$$

we get that $\bar{C} - C \to 0$ and as a result $\bar{A} - A \to 0$ and $\bar{B} - B \to 0$. As a result the diagonal part of h with respect to the Milnor frame α is in the class $[g]_\beta$ for a fixed Milnor frame β for g i.e. $h_\alpha \in [g]_\beta$. So $[g]$ is a 2- parameter family. A generic Ricci flow in this equivalence class can be written as

$$h_\beta = \begin{pmatrix} \bar{A}cos^2\theta + \bar{B}sin^2\theta & -\bar{A}cos\theta sin\theta + \bar{B}cos\theta sin\theta & 0 \\ -\bar{A}cos\theta sin\theta + \bar{B}cos\theta sin\theta & \bar{A}cos^2\theta + \bar{B}sin^2\theta & 0 \\ 0 & 0 & \bar{C} \end{pmatrix}$$

where $\theta \in \mathbb{R}$ and $\bar{A} - A \to 0, \bar{B} - B \to 0$ and $\bar{C} - C \to 0$. □

8.3 The quasi-convergence classes for $\mathrm{Iso}(\mathbb{E}^1_1)$

Let G be the Lie group $\mathrm{Iso}(\mathbb{E}^1_1)$ equipped with a left invariant metric g_0 and β a Milnor frame for the pair (G, g_0) with dual frame $\omega = (\omega^1, \omega^2, \omega^3)$. Consider a one parameter family of left invariant metrics g such that $g(0) = g_0$. Relative to the frame β the family g can be written in a diagonal form as

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3.$$

The sectional curvatures are of the form:

$$\begin{aligned} K(\beta_2 \wedge \beta_3) &= \frac{(A-C)^2 - 4A^2}{ABC}, \\ K(\beta_3 \wedge \beta_1) &= \frac{(A+C)^2}{ABC}, \\ K(\beta_1 \wedge \beta_2) &= \frac{(A-C)^2 - 4C^2}{ABC}. \end{aligned}$$

The Ricci flow equation is equivalent to the system

$$\begin{aligned} \frac{d}{dt}A &= 4\frac{C^2 - A^2}{BC}, \\ \frac{d}{dt}B &= 4\frac{(A+C)^2}{AC}, \\ \frac{d}{dt}C &= 4\frac{(A^2 - C^2)}{AB}. \end{aligned}$$

In order to determine the behaviour of the solution to the Ricci flow equation we first have to find the conserved quantities of the system.

Lemma 8.3.1. The conserved quantities of this system are AC and $B(C-A)$ that is $\frac{d}{dt}(AC) = \frac{d}{dt}(B(C-A)) = 0$.

Proof. Indeed,

$$\begin{aligned} \frac{d}{dt}(AC) &= C\frac{d}{dt}A + A\frac{d}{dt}C \\ &= 4C\frac{C^2 - A^2}{BC} + 4A\frac{(A^2 - C^2)}{AB} \\ &= 4\frac{C^2 - A^2 + A^2 - C^2}{B} = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(B(C-A)) &= (C-A)4\frac{(A+C)^2}{AC} + B4\left(\frac{(A^2 - C^2)}{AB} - \frac{C^2 - A^2}{BC}\right) \\ &= 4(C-A)\frac{A^2 + 2AC + C^2}{AC} + 4\left(\frac{A^2C - C^3}{AC} - \frac{AC^2 - A^3}{AC}\right) \\ &= 4\frac{CA^2 + 2AC^2 + C^3 - A^3 - 2A^2C - AC^2}{AC} + 4\frac{A^2C - C^3 + A^3 - AC^2}{AC} = 0 \end{aligned}$$

□

So we introduce the quantities $\Phi = AC = A_0C_0$ and $\Psi = B(C-A) = B_0(C_0 - A_0)$.

Lemma 8.3.2. The solutions to the Ricci flow satisfy the following :

- both $A(t)$ and $C(t)$ converge to $\sqrt{A_0B_0}$,
- while $B(t) \to \infty$.

Proof. Put $\rho = A/C$ and consider the simplified system,

$$\frac{d}{dt}\rho = 8\frac{1-\rho^2}{B}, \qquad \frac{d}{dt}B = \frac{1+\rho}{2\rho(1-\rho)}.$$

If we have the initial condition $A_0 = C_0$ then $\Psi = 0, \rho = 1$ and B grows linearly in time. If this is not the case then by arguing as in the case of $\mathrm{Iso}(\mathbb{E}^2)$, we have that ρ is strictly monotone and approaches a limit ρ_∞ which satisfies $\rho_0 < \rho_\infty \leq 1$ or $1 \leq \rho_\infty < \rho_0$. Since

$$\frac{d}{d\rho}\log B = \frac{1+\rho}{2\rho(1-\rho)}$$

we get that $B \to \infty$ while $A, C \to \sqrt{\Phi}$. □

This kind of behaviour for the Ricci flow where two directions of the metric converge while the other one expands to infinity is called cigar degeneracy. We can now determine the quasi-convergence class $[g]_\beta$ of a left invariant Ricci flow on $\mathrm{Iso}(\mathbb{E}^1_1)$ for a fixed Milnor frame β.

Lemma 8.3.3. The quasi-convergence class $[g]_\beta$ for a fixed Milnor frame β is a 2-parameter family.

Proof. We refer [11] for the full proof of this Lemma. □

Now let h, g be two homogeneous Ricci flows for the Lie group $\mathrm{Iso}(\mathbb{E}^1_1)$ and α, β be Milnor frames for h, g respectively. That is g_β is diagonal and corresponds to the initial data $g_\beta(0) = (A_0, B_0, C_0)$ while h_α corresponds to the initial data $h_\alpha(0) = (\bar{A}_0, \bar{B}_0, \bar{C}_0)$. We want to determine the Lie algebra of the stabilizer group $\mathrm{GL}(3,\mathbb{R})_\sigma$. In order to do this we have to specify the 3×3 matrices that satisfy the equation $-\operatorname{tr}(X)\sigma + X\sigma + (X\sigma)^T = 0$. For the group $\mathrm{Iso}(\mathbb{E}^1_1)$ we have $\sigma = 2\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

So

$$-\operatorname{tr}(X)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + X\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(X\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^T = 0$$

We have

$$-\operatorname{tr}(X)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X_{11}+x_{22}+X_{33} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -X_{11}-X_{22}-X_{33} \end{pmatrix},$$

$$X\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}X_{11}&X_{12}&X_{13}\\X_{21}&X_{22}&X_{23}\\X_{31}&X_{32}&X_{33}\end{pmatrix}\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-X_{11}&0&X_{13}\\-X_{21}&0&X_{23}\\-X_{31}&0&X_{33}\end{pmatrix},$$

and

$$\left(X\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}\right)^T=\begin{pmatrix}-X_{11}&0&X_{13}\\-X_{21}&0&X_{23}\\-X_{31}&0&X_{33}\end{pmatrix}^T=\begin{pmatrix}-X_{11}&-x_{21}&-X_{31}\\0&0&0\\X_{13}&X_{23}&X_{33}\end{pmatrix}.$$

Therefore we get: $X_{11}=X_{33}, X_{22}=X_{21}=X_{23}=0$ and $X_{13}=X_{31}$ So

$$\mathrm{Lie}(\mathrm{GL}(3,\mathbb{R})_\sigma)=\left\{\begin{pmatrix}X&A&Y\\0&0&0\\Y&B&X\end{pmatrix}: X,Y,A,B\in\mathbb{R}\right\}$$

Thus, the group G_σ is equal to

$$G_\sigma=\left\{\begin{pmatrix}x&a&y\\0&1&0\\y&b&x\end{pmatrix}: x,y,a,b\in\mathbb{R}\right\}$$

By direct computations, as in Lemma 8.1.3. we get the following,

Lemma 8.3.4. The group of matrices generated by the matrices of the form

$$\begin{pmatrix}x&a&y\\0&1&0\\y&b&x\end{pmatrix}$$

where $x^2-y^2=1$ is a a subgroup S of the group G_σ.

Theorem 8.3.5. The quasi convergence class of any left invariant metric g on $G=\mathrm{Iso}(\mathbb{E}^1_1)$ can be described as

$$[g]=(S_0)_\beta\cdot[g]_\beta$$

where $[g]_\beta$ is the quasi-convergence class of g for a fixed Milnor frame β and $(S_0)_\beta=\{\beta\circ s_0\circ\beta^{-1}\mid s_0\in S_0\}<\mathrm{Aut}(\mathfrak{g})$ with S_0 the subgroup of $\mathrm{GL}(3,\mathbb{R})$ consisting of the matrices of the form,

$$\begin{pmatrix}1&a&0\\0&1&0\\0&b&1\end{pmatrix},\quad a.b\in\mathbb{R}.$$

In particular, every quasi-convergence class is a 4-parameter family.

Proof. Let

$$R = \begin{pmatrix} x & a & y \\ 0 & 1 & 0 \\ y & b & x \end{pmatrix}$$

where $x^2 - y^2 = 1, x > 0$ be the matrix corresponding to the change of frame from α to β i.e. $\alpha = \beta R$. Then, $h_\beta = R^T h_\alpha R$ is the Ricci flow h expressed in the frame β,

$$\begin{aligned} h_\beta &= \begin{pmatrix} x & a & y \\ 0 & 1 & 0 \\ y & b & x \end{pmatrix}^T \begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{B} & 0 \\ 0 & 0 & \bar{C} \end{pmatrix} \begin{pmatrix} x & a & y \\ 0 & 1 & 0 \\ y & b & x \end{pmatrix} \\ &= \begin{pmatrix} x^2\bar{A} + y^2\bar{C} & ax\bar{A} + by\bar{C} & yx\bar{A} + xy\bar{C} \\ ax\bar{A} + by\bar{C} & a^2\bar{A} + \bar{B} + b^2\bar{C} & ay\bar{A} + bx\bar{C} \\ yx\bar{A} + xy\bar{C} & ay\bar{A} + bx\bar{C} & y^2\bar{A} + x^2\bar{C} \end{pmatrix} \end{aligned}$$

The Ricci flow $h \in [g]$ if and only if every term in the following sum converge to zero:

$$\begin{aligned} |h - g|_g^2 = & \frac{(x^2\bar{A} + y^2\bar{C} - A)^2}{A^2} + \frac{(a^2\bar{A} + b^2\bar{C} + \bar{B} - B)^2}{B^2} + \frac{(y^2\bar{A} + x^2\bar{C} - C)^2}{C^2} \\ & + 2\frac{(ax\bar{A} + by\bar{C})}{AB} + 2\frac{x^2y^2(\bar{A} + \bar{C})^2}{AC} + 2\frac{(ay\bar{A} + bx\bar{C})^2}{BC} \end{aligned}$$

Looking at the first

$$\frac{(x^2\bar{A} + y^2\bar{C} - A)^2}{A^2}$$

and the third term

$$\frac{(y^2\bar{A} + x^2\bar{C} - C)^2}{C^2}$$

we see that

$$\frac{(x^2\bar{A} + y^2\bar{C} - A)^2}{A^2} \to \frac{(x^2\sqrt{\bar{A}_0\bar{B}_0} + y^2\sqrt{\bar{A}_0\bar{B}_0} - \sqrt{A_0B_0})^2}{A_0B_0}$$

and

$$\frac{(y^2\bar{A} + x^2\bar{C} - C)^2}{C^2} \to \frac{(y^2\sqrt{\bar{A}_0\bar{B}_0} + x^2\sqrt{\bar{A}_0\bar{B}_0} - \sqrt{A_0B_0})^2}{A_0B_0}.$$

So in order those terms to converge to zero we must have $x^2 + y^2 = 1$. But $x^2 - y^2 = 1$ and $x > 0$ thus we conclude that $x = 1$ and $y = 0$. In this case we get also that the diagonal part of h with respect to the Milnor frame α is in the quasi-convergence class of g for a fixed Milnor frame β i.e. $h_a \in [g]_\beta$. Therefore, $[g]$ is a 4-parameter family of Ricci flows and a generic Ricci flow in this equivalence class has the expression:

$$h_\beta = \begin{pmatrix} \bar{A}+ & a\bar{A}+ & 0 \\ a\bar{A} & a^2\bar{A} + \bar{B} + b^2\bar{C} & b\bar{C} \\ 0 & b\bar{C} & \bar{C} \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $\bar{A}, \bar{B}, \bar{C}$ are such that $h_\alpha \in [g]_\beta$.

□

8.4 The quasi-convergence classes for $\mathrm{SL}(2,\mathbb{R})$

Consider G the Lie group $\mathrm{SL}(2,\mathbb{R})$ and g_0 a left invariant metric on G. Let β be a Milnor frame for the pair (G, g_0) with dual frame $\omega = (\omega^1, \omega^2, \omega^3)$. The matrix of the structure constants for the case of $\mathrm{SL}(2,\mathbb{R})$ is given by

$$\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let g be a one parameter family of left invariant metrics on $\mathrm{SL}(2,\mathbb{R})$ such that $g(0) = g_0$. Relative to the Milnor frame β the family g can be written in a diagonal form as

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$$

where A, B and C are functions of time. The sectional curvatures are

$$\begin{aligned} K(\beta_2 \wedge \beta_3) &= \frac{(B-C)^2 - A(3A+2B+2C)}{ABC}, \\ K(\beta_3 \wedge \beta_1) &= \frac{[A-(B-C)]^2 - 4B(B-C)}{ABC}, \\ K(\beta_1 \wedge \beta_2) &= \frac{[A+(B-C)]^2 + 4C(B-C)}{ABC}. \end{aligned}$$

So the Ricci flow equation is equivalent to the system

$$\begin{aligned} \frac{d}{dt}A &= 4\frac{(B-C)^2 - A^2}{BC}, \\ \frac{d}{dt}B &= 4\frac{(A+C)^2 - B^2}{AC}, \\ \frac{d}{dt}C &= 4\frac{(A+B)^2 - C^2}{AB}. \end{aligned}$$

Unlike the case of the $\mathrm{Iso}(\mathbb{E}^2)$ or $\mathrm{Iso}(\mathbb{E}^1_1)$ there are not any known conserved quantities for the Ricci flow. The next Lemma describes the properties of the solutions to the Ricci flow equation.

Lemma 8.4.1. The solutions to this system of equations satisfy the following properties:

- A decreases monotonically to $A_\infty > 0$
- Both B and C grow asymptotically with rate $8t$
- B and C approach each other exponetially, $|B - C| \leq Ke^{-kt}$ for some constants $K, k > 0$

Proof. The proof of this result can be found in [11] or in [9] (in the later the authors use a certain normilization to describe the solution to the Ricci flow equation). □

Just like the case of the Ricci flow for the Heisenberg group, $\mathrm{SL}(2,\mathbb{R})$ develops a pancake degeneracy, that is one direction of the metric converges to a real number while the other two directions expand to infinity. In contrast with the Heisenberg group case, the two directions that expand to infinity approach each other exponentially. If we fix a Milnor frame β for g we get the following lemma

Lemma 8.4.2. The class $[g]_\beta$ for an $\mathrm{SL}(2,\mathbb{R})$-metric is a 2-parameter family.

Proof. The proof of this result is lengthy and does not follow from direct computations like in the other cases. We refer [11] for the full proof. □

We are now going to compute the quasi-convergence class for a left invariant Ricci flow on $\mathrm{SL}(2,\mathbb{R})$.

Theorem 8.4.3. The quasi convergence class of any left invariant metric g on $\mathrm{SL}(2,\mathbb{R})$ can be described as

$$[g] = [g]_\beta$$

where $[g]_\beta$ is the quasi-convergence class of g for a fixed Milnor frame β. In particular every quasi-convergence class is a 2-parameter family.

Proof. Suppose that h, g are two left invariant Ricci flows on $\mathrm{SL}(2,\mathbb{R})$. Let α, β be Milnor frames for h and g respectively. That is g_β is diagonal and corresponds to the initial data $g_\beta(0) = (A_0, B_0, C_0)$ and h_α is diagonal and corresponds to the initial data $h_\alpha(0) = (\bar{A}_0, \bar{B}_0, \bar{C}_0)$. Recall that in the case of $\mathrm{SL}(2,\mathbb{R})$ we have that $\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ We will compute the Lie algebra of the stabilizer group G_σ as in the previous cases. That is we have to determine the all the 3×3 matrices X that satisfy the equation $-\operatorname{tr}(X)+X\sigma+(X\sigma)^T = 0$. So we have

$$-\operatorname{tr}(X)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + X\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(X\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^T = 0$$

Therefore we get

$$-\operatorname{tr}(X)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X_{11}+X_{22}+X_{33} & 0 & 0 \\ 0 & X_{11}+X_{22}+X_{33} & 0 \\ 0 & 0 & -X_{11}-X_{22}-X_{33} \end{pmatrix},$$

$$X\sigma = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -X_{11} & -X_{12} & X_{13} \\ -X_{21} & -X_{22} & X_{23} \\ -X_{31} & -X_{32} & X_{33} \end{pmatrix}$$

and

$$(X\sigma)^T = \left(X \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^T = \begin{pmatrix} -X_{11} & -X_{21} & -X_{31} \\ -X_{12} & -X_{22} & -X_{32} \\ X_{13} & X_{23} & X_{33} \end{pmatrix}$$

Thus we get $X_{13} = X_{23}$, $X_{23} = X_{32}$, $X_{21} = -X_{12}$ and $X_{11} = X_{22} = X_{33} = 0$. So the Lie algebra of the stabilizer

$$\text{Lie}(\text{GL}(3,\mathbb{R})_\sigma) = \left\{ \begin{pmatrix} 0 & X & Z \\ -X & 0 & Y \\ Z & Y & 0 \end{pmatrix} : \text{ where } X, Y, Z \in \mathbb{R} \right\}$$

This Lie algebra is isomorphic to the Lie algebra of $\text{SL}(2,\mathbb{R})$ thus the stabilizer group G_σ is isomorphic with the group $\text{SL}(2,\mathbb{R})$ embedded in $\text{GL}(3,\mathbb{R})$. Let $S \in G_\sigma$ be the matrix corresponding to the change of frame from α to β. Then S has the form:

$$S = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 & w \\ 0 & 1 & 0 \\ w & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} = \begin{pmatrix} xu & xwb - ya & xwa - yb \\ yu & xa + ywb & xb + ywa \\ w & ub & ua \end{pmatrix}$$

where $x^2 + y^2 = 1, u^2 - w^2 = 1, u > 0$ and $a^2 - b^2 = 1, a > 0$. So $h_\beta = S^T h_\alpha S$. This 3×3 matrix $h_\beta = S^T h_\alpha S$ has columns e_1, e_2, e_3 where

$$e_1 = \begin{pmatrix} x^2u^2\bar{A} + y^2u^2\bar{B} + w^2\bar{C} \\ (xwb - ya)xu\bar{A} + (xa + ywb)yu\bar{B}uwb\bar{C} \\ (xwa - yb)xu\bar{A} + (xb + ywa)yu\bar{B} + uwa\bar{C} \end{pmatrix}$$

$$e_2 = \begin{pmatrix} xu(xwb - ya)\bar{A} + yu(xa + ywb)\bar{B} + uwb\bar{C} \\ (xwb - ya)^2\bar{A} + (xa + ywb)^2\bar{B} + u^2b^2\bar{C} \\ (wxa - yb)(xwb - ya)\bar{A} + (xa + ywb)(xb + ywa)\bar{B} + u^2ab\bar{C} \end{pmatrix}$$

$$e_3 = \begin{pmatrix} xu(xwa - yb)\bar{A} + yu(xb + ywa)\bar{B} + uwa\bar{C} \\ (wxa - yb)(xwb - ya)\bar{A} + (xa + ywb)(xb + ywa)\bar{B} + u^2ab\bar{C} \\ (xwa - yb)^2\bar{A} + (xb + ywa)^2\bar{B} + u^2a^2\bar{C} \end{pmatrix}$$

now $h \in [g]$ if and only if every term in the following sum converges to zero.

$$\begin{aligned}
|h - g|_g^2 = & \frac{(x^2u^2\bar{A} + y^2u^2\bar{B} + w^2\bar{C} - A)^2}{A^2} \\
& + \frac{((xwb - ya)^2\bar{A} + (xa + ywb)^2\bar{B} + u^2b^2\bar{C} - B)^2}{B^2} \\
& + \frac{((xwb - yb)^2\bar{A} + (xb + ywa)^2\bar{B} + u^2a^2\bar{C} - C)^2}{C^2} \\
& + 2\frac{((xwb - ya)xu\bar{A} + (xa + ywb)yu\bar{B} + uwb\bar{C})^2}{AB} \\
& + 2\frac{((xwa - yb)xu\bar{A} + (xb + ywa)yu\bar{B} + uwa\bar{C})^2}{AC} \\
& + 2\frac{((xwa - yb)(xwb - ya)\bar{A} + (xa + ywb)(xb + ywa)\bar{B} + u^2ab\bar{C})^2}{BC}
\end{aligned}$$

Suppose that every term in the sum converge to zero. Looking at the first term

$$\frac{(x^2u^2\bar{A} + y^2u^2\bar{B} + w^2\bar{C} - A)^2}{A^2} \to 0$$

Since both $\bar{B}$ and $\bar{C}$ diverge to infinity while $\bar{A}$ converge to to a positive constant $\bar{A}_\infty$ (Lemma 8.4.2) we have that the coefficients of $\bar{B}$ and $\bar{C}$ must be zero. So $y^2u^2 = w^2 = 0$ and in particular $w = 0$. Also, using that $u^2 - w^2 = 1, u > 0$ and $w^2 = 0$ we get that $u = 1$ so $y = 0$. Furthermore, using that $x^2 + y^2 = 1$ we get $x^2 = 1$ i.e. $x = \pm 1$. Hence, if this term goes to zero we get $\frac{(\bar{A}-A)^2}{A^2} \to 0$. Now looking at the last term

$$2\frac{((xwa - yb)(xwb - ya)\bar{A} + (xa + ywb)(xb + ywa)\bar{B} + u^2ab\bar{C})^2}{BC} = 2\frac{ab\bar{B} + ab\bar{C}}{BC}$$

we get that $2a^2b^2\frac{(\bar{B}+\bar{C})^2}{BC} \to 0$. But B and C go to infinity with rate $8t$ (Lemma8.4.2) so the ratio $\frac{(\bar{B}+\bar{C})^2}{BC}$ converge to a positive number $\rho > 0$. As a result, in order this term to converge to zero we must have $ab = 0$. Using that $a^2 - b^2 = 1, a > 0$ we get that $b = 0$ and $a = 1$. Finally looking at the second term,

$$\frac{((xwb - ya)^2\bar{A} + (xa + ywb)^2\bar{B} + u^2b^2\bar{C} - B)^2}{B^2} = \frac{(\bar{B} - B)^2}{B^2} \to 0$$

and the third term,

$$\frac{((xwb - yb)^2\bar{A} + (xb + ywa)^2\bar{B} + u^2a^2\bar{C} - C)^2}{C^2} = \frac{(\bar{C} - C)^2}{C^2} \to 0$$

we conclude that $h_\alpha \in [g]_\beta$ that is the diagonal part of h with respect to the Milnor frame α quasi-converge to the diagonal part of g with respect to the Milnor frame β. Thus, $S_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_\alpha \in [g]_\beta$ or in other words $[g] \subset [g]_\beta$. Since the other inclusion always holds we get $[g] = [g]_\beta$. □

9 Further suggestions and remarks

In our work we gave an alternative characterization of the quasi-convergence classes for left invariant Ricci flows which enable us to compute the quasi-convergence classes for Ricci flow on $\mathrm{SL}(2, \mathbb{R})$. Due to lack of time, we could adapt this characterization in the more general setting of homogeneous Ricci flows. We hope that this work will help in the understanding of the large time behaviour for general collapsing solutions. During our research there were number of interesting questions that came out. One of them is that in every example that we have consider, the group G_σ we computed always contains the group that we study. We do not consider this as a mere coincidence but we couldn't derive a proof of this observation. Another question that can be asked is on what extend can we generalize this study to higher dimensional Ricci flows. The behaviour of Ricci flow equation for dimension strictly larger that 3 is a challenging area of research. Few things have been understood about singular or collapsing solutions to the Ricci flow equation in dimension greater that 3. We hope that our work will help the investigation of the long-time behaviour for the Ricci flow in higher dimensions.

References

[1] Ben Andrews and Christopher Hopper. *The Ricci flow in Riemannian geometry: a complete proof of the differentiable 1/4-pinching sphere theorem*. springer, 2010.

[2] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *Journal of the American Mathematical Society*, 22(1):287–307, 2009.

[3] Manfredo Perdigao do Carmo. *Riemannian geometry*. Birkhäuser, 1992.

[4] Bennett Chow and Dan Knopf. *The Ricci Flow: An Introduction: An Introduction*, volume 1. American Mathematical Soc., 2004.

[5] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77. American Mathematical Soc., 2006.

[6] D DeTurck. Deforming metrics in the direction of their ricci tensors, improved version, collected papers on ricci flow, ed. h.-d. cao, b. chow, s;-c. chu and s.-t. yau. *Int. Press, Sommerville, MA*, 2003.

[7] Richard Hamilton and James Isenberg. Quasi-convergence of ricci flow for a class of metrics. *Communications in Analysis and Geometry*, 1(4):543–559, 1993.

[8] Richard S Hamilton et al. Three-manifolds with positive ricci curvature. *Journal of Differential Geometry*, 17(2):255–306, 1982.

[9] James Isenberg, Martin Jackson, et al. Ricci flow of locally homogeneous geometries on closed manifolds. *Journal of Differential Geometry*, 35(3):723–741, 1992.

[10] Anthony W Knapp. *Lie groups beyond an introduction*, volume 140. Springer Science & Business Media, 2013.

[11] Dan Knopf and Kevin McLeod. Quasi-convergence of model geometries under the ricci flow. *Communications in Analysis and Geometry*, 9(4):879–919, 2001.

[12] John Milnor. Curvatures of left invariant metrics on lie groups, 1976.

[13] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

[14] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2003.

[15] Peter Petersen, S Axler, and KA Ribet. *Riemannian geometry*, volume 171. Springer, 2006.

[16] Isadore M Singer. Infinitesimally homogeneous spaces. *Communications on Pure and Applied Mathematics*, 13(4):685–697, 1960.