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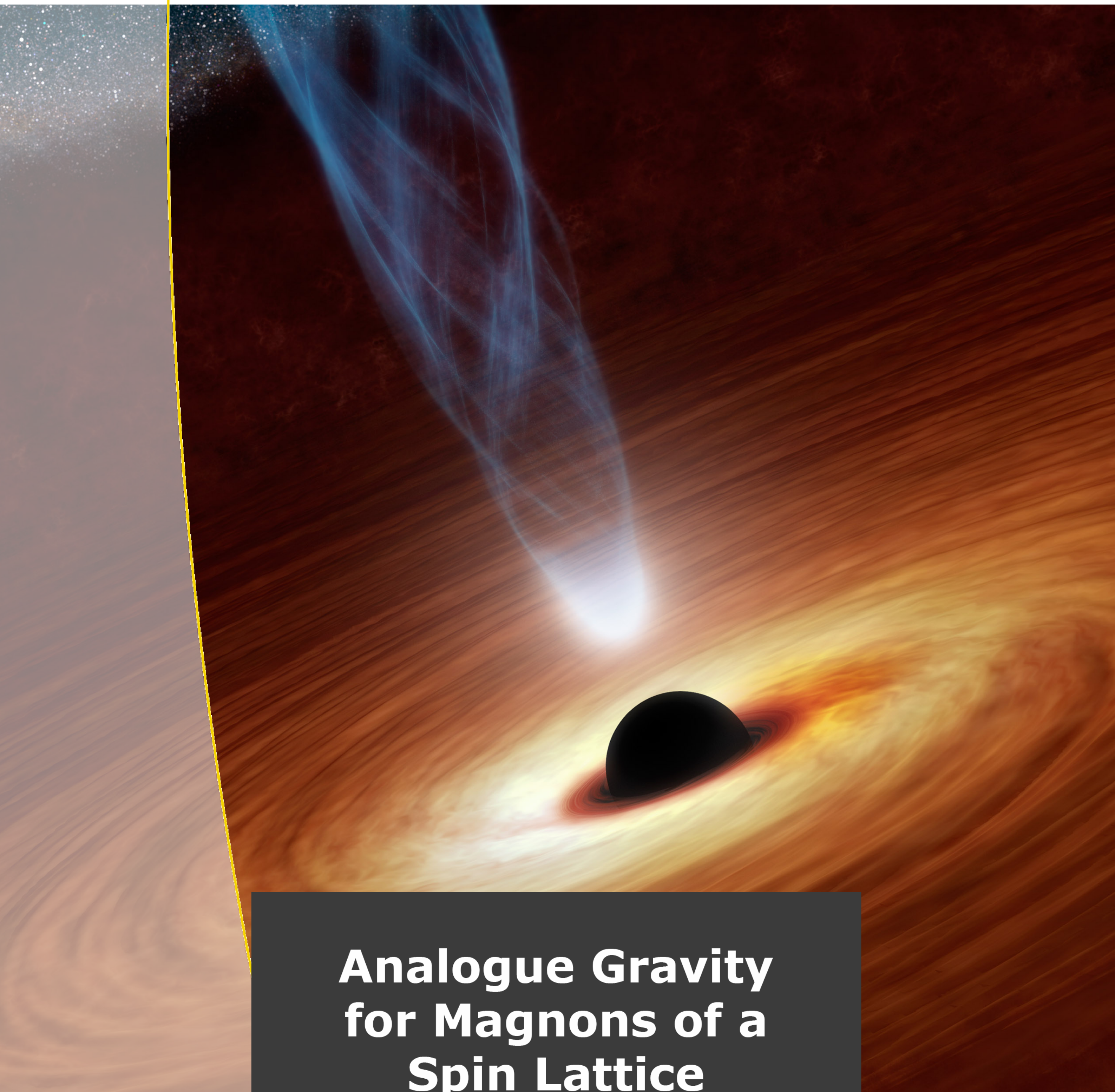

Analogue Gravity for Magnons of a Spin Lattice

Bachelor Thesis
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Abstract

Theoretical research into analogue gravity systems demonstrate promising results for observing gravitational phenomena. In this thesis we give an introduction of the theory behind analogue gravity and investigate the possibility of Cherenkov radiation occurring at a one dimensional magnonic black or white hole. We do this by determining the critical background spin velocity for an energetic instability to be possible and examening the behavour of magnons as they cross the event horizon where the background spin velocity exceeds this critical velocity. This way, we find expressions for the reflection and transmission coefficients of spin waves crossing the event horizon, which can be used to further explore this phenomenon.

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Chapter 1

Introduction

It is perhaps a feat of devine irony that one of the biggest challenges in physics today concerns the same topic that helped bring about physics as we know it; understanding the nature of gravity. Most will be familiar with the anecdote of Galileo Galilei dropping two canonballs with different masses from the top of the *Torre pendente di Pisa* [1], which may or may not be entirely fictional, to demonstrate them being subject to the same acceleration. It was Newton however, who in his 1687 publication *PhilosophiæNaturalis Principia Mathematica* used the three laws of nature he proposed to accurately explain and predict the motions of the heavenly bodies. He formalized discription of the force of gravity as

$$F_g = G\frac{Mm}{r^2}, \tag{1.1}$$

a version of his second law, where M and m are masses, G is the gravitational constant, and r is de distance between the masses. Still, there were some discrepancies between theory and practice. Nearly two and a half centuries later Albert Einstein revolutionized the way we think about gravity, space, and even time as in 1915 he published his theory of General Relativity [3], which has come to be one of the pillars of the modern physics era.

One of the problems in studying gravity is that it is incredibly weak compared to the other three fundamental forces. It is 10^{29} times weaker than the weak interaction, making it quite difficult to conduct any kind of experimental research into gravity. In 1981 Unruh published a paper proposing a system gouverned by fluid dynamics which can be described using some of the mathematics analogous to a black hole [4]. Here the velocity of the (bulk) fluid models the curvature of space, whilst perturbations of the fluid behave like matter particles. This theoretical framework of analogue gravity has been further developed and appears to be a promising direction for experimental research into general relativity. Recently, Duine et al published a paper suggesting that features of analogue gravity may also be found in magnon systems [5]. In this paper we will explore this idea further.

Thesis Structure

Structurally, this thesis is devided into two parts. In the first part of this thesis we will start by introducing a few concepts from General Relativity which we will need for the calculations in chapter 3. There we will demonstrate how Unruh proved that a particular fluid system can be described in mathematical terms analogous to a black hole. The second part of this thesis is concerned with magnons and analogue gravity for magnon systems, as in chapter 4 we will introduce the concept of magnons and a few of their properties and thereafter that calculate the dispersion relation for magnons in various situations. This will then be used in chapter 5 to study Cherenkov radiation at the event horizon of a magnonic white or black hole. To conclude we will summarize the primary results in this thesis and give an idea of the outlook of further research in this area.

Part I:

Introduction to Analogue Gravity

Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things.

Sir Isaac Newton

Chapter 2
Preliminary Concepts from General Relativity

Before we can start our discussion on analogue gravity, we first need some concepts from general relativity. In this chapter, we will give a brief introduction on tensors, the Schwarzschild metric, and the Klein-Gordon Equation, all of which we will utilize in the following chapter. We will label the dimensions of space using $0-3$, x^0 being time and (x^1, x^2, x^3) the spatial coordinates. In some instances the Einstein summation convention might be used, where an indice which appears twice is being summed over (i.e. $a_i\mathbf{e}_i = \sum_i a_i\mathbf{e}_i$).

2.1 | Tensors

A tensor is a mathematical object used in linear algebra and differential geometry, and represents a further abstraction of vectors and matrices. The rank of a tensor refers to the number of indices needed to identify an element of the tensor, i.e. a tensor of rank 0 is a scalar (which consists of only one element), a tensor of rank 1 is a vector, and rank 2 represents a matrix. A 2^{nd}-rank tensor could for example be $T^{m,n} = V^n W^m$

Tensors play a large role in how a physical quantity reacts to a change of basis. If the quantity decreases when the scale of the basis increases this is called contravariant, and when the quantity increases with the increased scale of the basis it is called covariant. This relation is denoted by raised or lowered indices on the tensor. The tensors $T^{n,m}(x)$ and $T^{n,m}(y)$ are the same tensor in respectively the x and y-basis

$$T^{m,n}(y) = \sum_{r,s} \frac{\partial y^m}{\partial x^r}\frac{\partial y^n}{\partial x^s} T^{r,s}(x) \qquad \text{(Contravariant transformation)} \tag{2.1}$$

$$T_{m,n}(y) = \sum_{r,s} \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n} T_{r,s}(x) \qquad \text{(Covariant transformation)} \tag{2.2}$$

To illustrate this we will briefly look at the Metric Tensor. Consider Pythagoras' theorem

$$ds^2 = \sum_m dx^m dx^m, \tag{2.3}$$

$$= \sum_{m,n} \delta_{m,n} dx^m dx^n. \tag{2.4}$$

We can transform our coordinates from the x-basis to the y-coordinate system using

$$dx^m = \sum_r \frac{\partial x^m}{\partial y^r} dy^r, \tag{2.5}$$

from which we obtain

$$ds^2 = \sum_{m,n} \delta_{m,n} \sum_r \frac{\partial x^m}{\partial y^r} dy^r \sum_s \frac{\partial x^n}{\partial y^s} dy^s, \tag{2.6}$$

$$= \sum_{m,n}\sum_{r,s} \delta_{m,n} \frac{\partial x^m}{\partial y^r}\frac{\partial x^n}{\partial y^s} dy^r dy^s. \tag{2.7}$$

We identify the Metric Tensor as

$$g_{m,n} = \sum_{m,n} \delta_{m,n} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s}, \tag{2.8}$$

so that we can rewrite Pythagoras'theorem as

$$ds^2 = \sum_{r,s} g_{m,n} dy^r dy^s, \tag{2.9}$$

using the Metric Tensor. In case we are looking at flat space $g_{m,n}$ simply reduces to $\delta_{m,n}$.

2.2 | The Schwarzschild Metric

Einstein's theory of General Relativity is formalized mathematically in the Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c} T_{\mu\nu}, \tag{2.10}$$

which amongst others features the Metric Tensor we discussed above. It also features the Ricci Curvature Tensor $R_{\mu\nu}$, the Curvature Scalar R, the Cosmological constant Λ, the Gravitational Constant G, speed of light c, and the Stress-Energy Tensor. In general it can be stated that the terms on the lefthand side of the Field Equations have to do with the curvature of spacetime and everything on the lefthand side has to do with mass. Although a complete discussion of the Einstein Field Equations and how to solve them is beyond the scope of this thesis, it suffices to say that solutions take the form of a metric. The simplest solution (with the exception of the trivial flat space solution) is the Schwarzschild metric, which describes space around a non-rotating, zero charge black hole in a vacuum. Near the horizon of the black hole the Schwarzschild metric is given by

$$ds^2 \approx \left[\frac{\hat{r} - 2M}{2M}\right] d\hat{t}^2 - \left[\frac{2M}{\hat{r} - 2M}\right] d\hat{r}^2. \tag{2.11}$$

2.3 | Klein-Gordon Equation and the D'Alembertian

From special relativity Einstein derived his famous equation

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4, \tag{2.12}$$

relating mass and energy. For a plane wave the energy is given by $\hbar\omega$ and the momentum by $\hbar\mathbf{k}$, thus in case we are working with waves quantum operators need to be substituted

$$E \to i\hbar\partial_t, \tag{2.13}$$

$$\mathbf{p} \to -i\hbar\nabla. \tag{2.14}$$

Therefore we find the mass-energy relation

$$(\hbar^2\partial_t^2 - c^2\hbar^2\nabla^2 + m^2c^4)\psi = 0, \tag{2.15}$$

$$(\Box + \chi^2)\psi = 0, \tag{2.16}$$

where $\Box = c^{-2}\partial_t - \nabla^2$ and for now we defined $\chi = mc/\hbar$. This version of the mass-energy relation is known as the Klein-Gordon Equation. Here $\Box$ denotes the d'Alembertian operator, which is the Laplacian for Minkowski space and is generally defined as

$$\Box = g^{\mu\nu}\partial_\nu\partial_\mu, \tag{2.17}$$

with $g^{\mu\nu}$ the inverse Minkowski metric. For curved space the scalar d'Alembertian becomes [6, p. 188]

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi). \tag{2.18}$$

Chapter 3
Analogue Gravity from Fluid Dynamics

In this chapter we will expand upon the derivation of a wave equation governing the propagation of phonons in a non-homogeneous or moving fluid given by Visser et al [7, p. 8]. This derivation provides a clear illustration of how analogue black holes, and more specifically their event horizon, can emerge in classical system in way which is very similar to Unruh's original 1981 paper [4]. Under certain constraints the behaviour of the phonons in the fluid can be described using the same techniques used for describing a Lorentzian manifold. Thereafter we shall show how in this Lorentzian geometry we can recover a metric which is in essence the Schwarzschild metric with different constants.

3.1 | Fluid Dynamics

Originally, Feynman derived a version of the wave equation that describes the propagation of sound waves in homogeneous matter with three spatial dimensions [8]. For a velocity potential, his equation can be written as

$$\partial_t^2 \phi = c^2 \nabla^2 \phi, \tag{3.1}$$

where ϕ is the potential, and c denotes the speed of sound within the medium. The product of our derivation will be a wave equation which does have the ability to describe propagations in a medium of a non-homogeneous nature, e.g. a fluid where the bulk is not at rest. During our calculations we will rely mainly on two fundamental equations of fluid dynamics:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \text{(Equation of continuity)} \tag{3.2}$$

$$\rho \frac{d\mathbf{v}}{dt} \equiv \rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = \mathbf{f}. \qquad \text{(Euler's equation)} \tag{3.3}$$

Here ρ is the density of the fluid, $\mathbf{v}$ is its velocity, and $\mathbf{f}$ are the internal forces. We will also utilize the specific enthalpy of the fluid, which is the enthalpy per unit of mass.

3.2 | Generalizing the Wave Equation

Before we can commence our derivation, we have to make a few assumptions about the fluid through which the sound waves propagate. The fluid has to be:

1. Inviscid. If a fluid has zero viscosity, the only forces present are those due to pressure.
2. Vorticity free. The fact that the fluid is locally irrotational implies $\mathbf{v} = -\nabla\phi$.
3. Barotropic, i.e. ρ is a function of p only.

Additionaly, we wil make use of the following result. A full proof of this result can be found in appendix A.

Result 1. *In three spatial dimensions, with $v = \|\mathbf{v}\|$, we find the identity*

$$-(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla \left(\frac{1}{2} v^2 \right). \tag{3.4}$$

We shall start our analysis with the Euler equation. Since we have assumed the fluid to be inviscid we can write

$$\mathbf{f} = -\nabla p, \tag{3.5}$$

which combined with the Euler equation in (3.3) gives us

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p \tag{3.6}$$

$$\partial_t \mathbf{v} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{\rho}\nabla p. \tag{3.7}$$

Using result 1 this equation can be rewritten as

$$\partial_t \mathbf{v} = \mathbf{v}\times(\nabla\times\mathbf{v}) - \frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}v^2\right). \tag{3.8}$$

Because of the barotropic condition, we are now able to define the specific enthalpy as

$$h(p) = \int_0^p \frac{dp'}{\rho(p')}, \tag{3.9}$$

such that

$$\nabla h = \partial_i \int_0^p \frac{dp'}{\rho(p')} = \frac{1}{\rho(p)}\partial_i p_i = \frac{1}{\rho}\nabla p. \tag{3.10}$$

From the second assumption we made concerning the fluid we see that $\partial_t \mathbf{v} = -\nabla\partial_t\phi$. Additionally we utilize the relation $\nabla h = \rho^{-1}\nabla p$ to conclude that equation (3.8) is equivalent to

$$\nabla[-\partial_t\phi + h + \frac{1}{2}(\nabla\phi)^2] = 0, \qquad \text{or} \qquad -\partial_t\phi + h + \frac{1}{2}(\nabla\phi)^2 = 0. \tag{3.11}$$

We now want to separate the motion of the sound waves proagating through the medium from the bulk motion of the fluid. This can be done by linearising around the equations of motion around some assumed background (ρ_0, p_0, ϕ_0), so that

$$\rho = \rho_0 + \epsilon\rho_1 + \mathcal{O}(\epsilon^2), \tag{3.12}$$

$$p = p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2), \tag{3.13}$$

$$\phi = \phi_0 + \epsilon\phi_1 + \mathcal{O}(\epsilon^2). \tag{3.14}$$

The linearisations for ρ and p can thereafter be entered into the equation of continuity, so that

$$\partial_t(\rho_0 + \epsilon\rho_1) + \nabla[(\rho_0 + \epsilon\rho_1)(\mathbf{v}_0 + \epsilon\mathbf{v}_1)] = 0, \tag{3.15}$$

where $\mathbf{v}_0 + \epsilon\mathbf{v}_1 = -\nabla(\phi_0 + \epsilon\phi_1) = \mathbf{v}$. Separating the ϵ from the non-ϵ terms produces the two equations

$$\partial_t\rho_0 + \nabla\cdot(\rho_0\mathbf{v}_0) = 0, \tag{3.16}$$

$$\partial_t\rho_1 + \nabla\cdot(\rho_1\mathbf{v}_0 + \rho_0\mathbf{v}_1) = 0. \tag{3.17}$$

Furthermore, from our definition of h in equation (3.9) we gather

$$h(p) = h(p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2)) = h_0 + \epsilon\frac{p_1}{\rho_0} + \mathcal{O}(\epsilon^2)). \tag{3.18}$$

Subsequent substitution of this linearisation of h in equation (3.11) gives us

$$-\partial_t(\phi_0 + \epsilon\phi_1) + \left(h_0 + \epsilon\frac{p_1}{\rho_0}\right) + \frac{1}{2}[\nabla(\phi_0 + \epsilon\phi_1)]^2 = 0 \tag{3.19}$$

$$-\partial_t(\phi_0 + \epsilon\phi_1) + \left(h_0 + \epsilon\frac{p_1}{\rho_0}\right) + \frac{1}{2}\nabla^2(\phi_0^2 + 2\epsilon\phi_0\phi_1 + \epsilon^2\phi_1^2) = 0, \tag{3.20}$$

so that separation of terms gives one two distinct equations

$$-\partial_t\phi_0 + h_0 + \frac{1}{2}(\nabla\phi_0)^2 = 0 \tag{3.21}$$

$$-\partial_t\phi_1 + \frac{p_1}{\rho_0} - \mathbf{v}_0\cdot\nabla\phi_1 = 0. \tag{3.22}$$

One can rearrange this last equation to find

$$p_1 = \rho_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) \tag{3.23}$$

Once again utilizing the barotropic condition we can relate

$$\rho_1 = \frac{\partial\rho}{\partial p}p_1 = \frac{\partial\rho}{\partial p}\rho_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) \tag{3.24}$$

Utimately the result of this linearised Euler equation can be incorporated in the linearised equation of continuity, resulting up to an overall sign in the wave equation

$$-\partial_t\left(\frac{\partial\rho}{\partial p}\rho_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1)\right) + \nabla\cdot\left(\rho_0\nabla\phi_1 - \frac{\partial\rho}{\partial p}\rho_0\mathbf{v}_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1)\right) = 0. \tag{3.25}$$

Using the fact that the speed of sound within the fluid is defined as

$$\frac{1}{c^2} \equiv \frac{\partial\rho}{\partial p}, \tag{3.26}$$

we can algebraically simplify this equation somewhat to obtain

$$\frac{\rho_0}{c^2}\left[-\partial_t(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) + \nabla\cdot\left(c^2\nabla\phi_1 - \mathbf{v}_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1)\right)\right] = 0. \tag{3.27}$$

Before we proceed, we shall define the following 4×4 matrix (the matrix can be found in it's entirety on page 26).

$$f^{\mu\nu}(t,\mathbf{x}) \equiv \frac{\rho_0}{c^2}\begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & \cdot & \cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j) \end{bmatrix} \tag{3.28}$$

Henceforth for the sake of clearity the notations 0−3 and t, x, y, z will at times be used interchangeably, i.e. $(v_0^x, v_0^y, v_0^z) = (v_0^1, v_0^2, v_0^3)$, time being the 0^{th} dimension. Our definition of $f(t,\mathbf{x})$ as given in (3.28) enables us to write

$$\begin{aligned}\partial_\mu(f^{\mu\nu}\partial_\nu\phi_1) = \frac{\rho_0}{c^2}[&-\partial_t(\partial_t\phi_1 + \sum_{i=1}^{3} v_0^i\partial_i\phi_1) \\ &+ \partial_x(c^2\partial_x\phi_1 - v_0^x\partial_t\phi_1 - v_0^x\sum_{j=1}^{3} v_0^j\partial_j\phi_1) \\ &+ \partial_y(c^2\partial_y\phi_1 - v_0^y\partial_t\phi_1 - v_0^y\sum_{k=1}^{3} v_0^k\partial_k\phi_1) \\ &+ \partial_z(c^2\partial_z\phi_1 - v_0^z\partial_t\phi_1 - v_0^z\sum_{l=1}^{3} v_0^l\partial_l\phi_1)]\end{aligned} \tag{3.29}$$

$$\begin{aligned}= \frac{\rho_0}{c^2}[&-\partial_t(\partial_t\phi_1 + \sum_{i=1}^{3} v_0^i\partial_i\phi_1) \\ &+ \sum_{i=1}^{3}\partial_i c^2\partial_i\phi_1 - \sum_{j=1}^{3}\partial_j v_0^j\left(\partial_t\phi_1 + \sum_{k=1}^{3} v_0^k\partial_k\phi_1\right)]\end{aligned} \tag{3.30}$$

$$= \frac{\rho_0}{c^2}[-\partial_t(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) + \nabla\cdot(c^2\nabla\phi_1 - \mathbf{v}_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1))] \tag{3.31}$$

This result is identical to equation (3.27). Consequently the wave equation we found in equation (3.25) and the equation

$$\partial_\mu (f^{\mu\nu} \partial_\nu \phi_1) = 0 \tag{3.32}$$

are identical statements describing the propagation of sound waves in the fluid.

3.3 | Applying Techniques of Lorentzian Geometry

In order to obtain a metric from this newly found wave equation, we have to use some Lorentzian geometry techniques. In a curved space (3+1)-dimensional Lorentzian manifold the curved space scalar d'Alembertian is given in terms of the metric $g_{\mu\nu}(t, \mathbf{X})$ by

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi), \tag{3.33}$$

where $\det(g_{\mu\nu}) \equiv g$. Hence we can rewrite equation (3.32) in terms of the d'Alembertian once we identify

$$\sqrt{-g} g^{\mu\nu} = f^{\mu\nu}. \tag{3.34}$$

Now using standard properties of the determinant we can calculate

$$\det(f^{\mu\nu}) = \det(\sqrt{-g} g^{\mu\nu}) = (\sqrt{-g})^4 (\det(g_{\mu\nu}))^{-1} = (\sqrt{-g})^4 g^{-1} = g. \tag{3.35}$$

Additionally we can find the determinant of $f^{\mu\nu}$ by expanding the matrix in minors.

$$\det(f^{\mu\nu}) = \left(\frac{\rho_0}{c^2}\right)^4 \cdot [(-1) \cdot (c^2 - v_0^2) - (-v_0)^2] \cdot [c^2] \cdot [c^2] = -\frac{\rho_0^4}{c^2} \tag{3.36}$$

Therefore we can deduce that

$$g = -\frac{\rho_0^4}{c^2}, \qquad \text{meaning} \qquad \sqrt{-g} = \frac{\rho_0^2}{c}. \tag{3.37}$$

$$g^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} f^{\mu\nu} = \frac{1}{\rho_0 c} \left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots & \cdot & \cdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{array} \right] \tag{3.38}$$

We can now find the metric $g_{\mu\nu}$ itself by inverting $g^{\mu\nu}$.

$$g_{\mu\nu} \equiv \frac{\rho_0}{c} \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots & \cdot & \cdots \\ -v_0^i & \vdots & \delta^{ij} \end{array} \right] \tag{3.39}$$

This metric can also be expressed as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = \frac{\rho_0}{c} [-c^2 dt^2 + (dx^i - v_0^i dt) \delta_{ij} (dx^j - v_0^j dt)]. \tag{3.40}$$

3.4 | Analogue to the Schwarzschild Metric

Unruh demonstrates that from this metric found we can recover a metric for the fluid that is analogous to the Schwarzschild metric for a gravitational black hole [4]. Before we proceed to construe his derivation however, we shall have to add three constraints to the fluid.

1. The fluid has to converge to a certain point.
2. Around this point of convergence the fluid has to be spherically symmetric.
3. The background flow of the fluid has to be stationary.

We see that up to an overall sign we can rewrite the metric in equation (3.40) as

$$ds^2 = \frac{\rho_0}{c}[(c^2 - \mathbf{v}_0 \cdot \mathbf{v}_0)dt^2 + 2dt\mathbf{v}_0 \cdot d\mathbf{x} - d\mathbf{x} \cdot d\mathbf{x}]. \tag{3.41}$$

Furthermore, because of the spherical symmetry $\mathbf{v}_0 = v_0^r(r)\hat{\mathbf{r}}$, and switching to spherical coordinates $d\mathbf{x} \to d\mathbf{r}$. We find that

$$d\mathbf{r} \cdot d\mathbf{r} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.42}$$

Result 2. *If we define a new time τ so that*

$$\tau = t + \int \frac{v_0^r}{c^2 - (v_0^r)^2} dr, \tag{3.43}$$

then

$$dt^2 = d\tau^2 - \frac{2v_0^r d\tau dr}{(c^2 - (v_0^r)^2)} + \frac{(v_0^r)^2 dr^2}{(c^2 - (v_0^r)^2)^2}. \tag{3.44}$$

Combining the metric with result (2) and the above and dropping the angular part of the metric we see that

$$\frac{c}{\rho_0} ds^2 = (c^2 - (v_0^r)^2)dt^2 + 2v_0^r dt dr - [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.45}$$

$$= (c^2 - (v_0^r)^2)\left(d\tau^2 - \frac{2v_0^r d\tau dr}{(c^2 - (v_0^r)^2)} + \frac{(v_0^r)^2 dr^2}{(c^2 - (v_0^r)^2)^2}\right) + 2v_0^r dr\left(d\tau - \frac{v_0^r dr}{c^2 - (v_0^2)^2}\right) - dr^2 \tag{3.46}$$

$$= (c^2 - (v_0^r)^2)d\tau^2 - 2v_0^r d\tau dr + \frac{(v_0^r)^2 dr^2}{(c^2 - (v_0^r)^2)} + 2v_0^r d\tau dr - \frac{2(v_0^r)^2 dr^2}{c^2 - (v_0^r)^2} - \frac{(c^2 - (v_0^r)^2)dr^2}{(c^2 - (v_0^r)^2)} \tag{3.47}$$

$$= (c^2 - (v_0^r)^2)d\tau^2 - \frac{c^2}{c^2 - (v_0^r)^2} dr^2. \tag{3.48}$$

Subsequently, if one assumes the fluid to smoothly transition from subsonic to sonic speed, one can linearise around the point of transiton,

$$v_0^r = -c + \alpha(r - R) + \mathcal{O}((r - R)^2). \tag{3.49}$$

After we enter this linearisation into the metric we obtain

$$ds^2 \approx \frac{\rho_0}{c}\left(2c\alpha(r - R)d\tau^2 - \frac{cdr^2}{2\alpha(r - R)}\right). \tag{3.50}$$

If we demand the system to obey the constraints $r = (R/2M)\hat{r}$, $\tau = (2M/R)\hat{t}$, and $\alpha = R/8M^2$, then up to a factor ρ_0 we recover

$$ds^2 \approx \left[\frac{\hat{r} - 2M}{2M}\right] d\hat{t}^2 - \left[\frac{2M}{\hat{r} - 2M}\right] d\hat{r}^2, \tag{3.51}$$

which is equal to the Schwarzschild metric near the horizon of the black hole. This remarkable result demonstrates that it is possible to create an artificial analogue black hole using fluid dynamics. One can also ommit those last constraints on r, τ, and α. In that case, even thouh one does not recover such an analogue black hole necisarrily, it still becomes possible to study a large number of analogue gravity phenomena.

3.5 | The Draining Bathtub Metric

In addition to the Schwarzschild metric, utilizing our findings above we will demonstrate a method for constructing a metric which like the Kerr metric allows for an ergoregion around the event horizon [9]. Once again the system we will consider is that of a non-homogeneous or moving fluid, where pertubations propagate against some bulk background motion converging to a central point. This time, however, the background motion has not only a radial but also a azimuthal component, similar to when one pulls the plug out of the bathtub (hence the name of the metric). Initially we make the same assumptions on the fluid as in chapter 3, i.e. the fluid is *inviscid*, *vorticity free*, and *barotropic*. We will be working in polar coordinates (r, θ), thus our result will be in $(2+1)$-dimensions. It can be expanded to be $(3+1)$-dimensional by adding a z dimension.

Like before, for the equation of motion of the perturbations we assume $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, where $\mathbf{v}_0$ represents the bulk background motion and $\mathbf{v}_1$ that of the perturbations. In order to be able to construct this metric, we first need to find $\mathbf{v}_0$. Since we are using polar coordinates, the equation of continuity is written as

$$\partial_t \rho + \frac{1}{r}\partial_r(r\rho v^r) + \frac{1}{r}\partial_\theta(\rho v^\theta), \tag{3.52}$$

thus giving

$$\partial_r(r\rho v^r) = 0 \tag{3.53}$$

for the radial component, meaning that $\rho v^r \sim 1/r$. Because of the irrotationality we know that $v^\theta \sim 1/r$ and due to conservation of momentum $\rho v^\theta \sim 1/r$ meaning that ρ is constant. In combination with the barotropic condition this tells us that the pressure p and speed of sound c are constant as well. Hence we find

$$\mathbf{v}_0 = \frac{-A\hat{e}_r + B\hat{e}_\phi}{r} \tag{3.54}$$

for the motion of the bulk fluid. From chapter 3 we can now use equation (3.42) and substitute $\mathbf{v}_0$, resulting in

$$ds^2 = \frac{\rho}{c}\left[\left(c^2 - \frac{A^2+B^2}{r^2}\right)dt^2 - \frac{2Adrdt}{r} + 2Bd\theta dt - dr^2 - r^2 d\theta^2\right]. \tag{3.55}$$

By taking the non-rotating limit $B = 0$ we can identify the horizon as being located at

$$r_H = \frac{A}{c}. \tag{3.56}$$

The boundary of the ergosphere can be found at

$$r_{es} = \frac{\sqrt{A^2+B^2}}{c}. \tag{3.57}$$

Trivially, the angular velocity of the horizon can be calculated using

$$\omega = \frac{\mathbf{r}\times\mathbf{v}_0}{|\mathbf{r}|^2} = \frac{B}{r_H^2}. \tag{3.58}$$

Part II:

Analogue Gravity and Magnons

Look deep into nature, and then you will understand everything better.

Albert Einstein

Chapter 4

Magnonic Dispersion Relation

Recently, it has been suggested that a magnon lattice might be a viable candidate for a system which allows us to observe certain features of analogue gravity. In this chapter, after giving a brief introduction of the concept of magnons, we will calculate the dispersion relation for a magnon. To do this, we will first calculate two ground states by minimizing the energy of our spin lattice. We will then consider small perturbations which give rise to spin waves and calculate the dispersion relation for these waves. We will do this for two diffrent situations, first with the background spin velocity and spin damping turned off, and than with both turned on.

4.1 | Spin Waves

Consider a regular Bravais lattice of particles, each of which has an intrinsic magnetic spin. The Heisenberg ferromagnetic Hamiltonian of this lattice is given by

$$\mathcal{H} = -\frac{1}{2}J_s \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i, \tag{4.1}$$

where J_s is the exchange energy ($J_s > 0$ for a ferromagnet), g is the Landé g-factor (characterizing the magnetic moment and gyromagnetic ratio for an electron with both spin and angular momenta), and μ_B is the Bohr-magneton (an expression for the magnetic moment of electron brought about by its orbital or spin angluar momentum). Furthermore, $\mathbf{H}$ is the magnetic field, and $\mathbf{S}$ are spin operators.

One of the ground states, as we will see in the next chapter, is when all of the spins are lined up in the same direction. For an magnet such as modeled by the Ising model, where a spin can be in one of two dicrete positions, the first excited state would logically be when one spin is flipped in the opposite direction (i.e. the other discrete position). However, if we look at a lattice with continuous symmetry so that spins can also at a tilted position, we find that this is not the case. Rather we find a collective disturbance in the form of a superposition of states with one reduced spin. This way, the exchange energy penalty paid for misaligned spins in lower, because the difference in orientation between two neighbouring spins is minimized. From a quantum mechanical point of view such a collective excitation or spin wave can be viewed as a quasiparticle called a *magnon*. These magnons have a spin of -1, meaning they exhibit boson-like behaviour. A typical energy for a magnon would be in the order of μeV, which is about the same as the Curie temperature at which a ferromagnet loses its permanent magnetic properties. The general dynamics of magnons is gouverned by the Landau-Lifshitz-Gilbert equation, which will be introduced later on in this chapter.

4.2 | Calculating the Ground States

In order to find the ground states of our spin lattice, we first have to minimize the energy. This can be done by taking the functional derivative of the energy E to $\mathbf{n}$. In order to do this we choose $\mathbf{n} = \mathbf{n}_g + \delta\mathbf{n}(\mathbf{x}, t)$, i.e. a ground state plus a minor perturbation. We define the the functional derivative $\delta E[\mathbf{n}]/\delta \mathbf{n}$ using

$$E[\mathbf{n} + \delta\mathbf{n}] = E[\mathbf{n}] + \int d\mathbf{x} \frac{\delta E[\mathbf{n}]}{\delta \mathbf{n}} \cdot \delta\mathbf{n}. \tag{4.2}$$

For an easy-plane configuration spin lattice the energy is given by

$$E[\mathbf{n}] = \int \frac{d\mathbf{x}}{a^3} \left[-\frac{J_s}{2} \mathbf{n} \cdot \nabla^2 \mathbf{n} + \frac{K}{2} n_z^2 + B n_z \right], \tag{4.3}$$

with a^3 the volume of a unit cell, J_s the spin stiffness, and B the strength of a magnetic field in the z-direction. The anisotropy is given by the constant K [**?**]. We now enter our choice for $\mathbf{n}$ giving

$$\begin{aligned} E[\mathbf{n} + \delta\mathbf{n}] &= \int \frac{d\mathbf{x}}{a^3} \left[-\frac{J_s}{2} (\mathbf{n} + \delta\mathbf{n}) \cdot \nabla^2 (\mathbf{n} + \delta\mathbf{n}) + \frac{K}{2} (n_z + \delta n_z)^2 + B(n_z + \delta n_z) \right] && (4.4) \\ &= E[\mathbf{n}] + \int \frac{d\mathbf{x}}{a^3} \left[-\frac{J_s}{2} (\mathbf{n} \cdot \nabla^2 \delta\mathbf{n} + \delta\mathbf{n} \cdot \nabla^2 \mathbf{n}) + K n_z \delta n_z + B \delta n_z + \mathcal{O}(\delta\mathbf{n}^2) \right]. && (4.5) \end{aligned}$$

To continue we consider that double partial integration yields

$$\begin{aligned} \int dx_i [n_i \partial_{x_i}^2 \delta n_i] &= \int dx_i [n_i \partial_{x_i} \delta n_i] - \int dx_i [\partial_{x_i} n_i \partial_{x_i} \delta n_i] && (4.6) \\ &= \int dx_i [n_i \partial_{x_i} \delta n_i - \delta n_i \partial_{x_i} n_i] + \int dx_i [\delta n_i \partial_{x_i}^2 n_i], && (4.7) \end{aligned}$$

thus after dropping the boundary terms we can repalce $\mathbf{n} \cdot \nabla^2 \delta\mathbf{n}$ by $\delta\mathbf{n} \cdot \nabla^2 \mathbf{n}$. Also dropping any higher order $\delta\mathbf{n}$-terms gives us

$$E[\mathbf{n} + \delta\mathbf{n}] = E[\mathbf{n}] + \int \frac{d\mathbf{x}}{a^3} \left[-J_s \nabla^2 \mathbf{n} + K n_z \hat{\mathbf{z}} + B \hat{\mathbf{z}} \right] \cdot \delta\mathbf{n}, \tag{4.8}$$

so [if we take the volume of a unit cell $a^3 = 1$] we obtain

$$\frac{\delta E[\mathbf{n}]}{\delta \mathbf{n}} = -J_s \nabla^2 \mathbf{n} + K n_z \hat{\mathbf{z}} + B \hat{\mathbf{z}}. \tag{4.9}$$

Additionally, since $\mathbf{n}$ is a unit vector we have to incorporate the constraint $\|\mathbf{n}\| = 1$. To this end we have to add a Lagerange multiplier to equation (4.3) in the form of $-\lambda(\mathbf{n}^2 - 1)$. After taking the functional derivative this amounts to

$$\frac{\delta E[\mathbf{n}]}{\delta \mathbf{n}} = -J_s \nabla^2 \mathbf{n} + K n_z \hat{\mathbf{z}} + B \hat{\mathbf{z}} - 2\lambda \mathbf{n}. \tag{4.10}$$

Taking the derivative with respect to λ and letting the result equal zero gives

$$\frac{\partial}{\partial \lambda} \int \frac{d\mathbf{x}}{a^3} \left[-\frac{J_s}{2} \mathbf{n} \cdot \nabla^2 \mathbf{n} + \frac{K}{2} n_z^2 + B n_z - \lambda(\mathbf{n}^2 - 1) \right] = \mathbf{n}^2 - 1 = 0. \tag{4.11}$$

To proceed our minimalization let $\delta E[\mathbf{n}]/\delta \mathbf{n} = 0$, i.e.

$$-J_s \nabla^2 \mathbf{n} + K n_z \hat{\mathbf{z}} + B \hat{\mathbf{z}} - 2\lambda \mathbf{n} = 0. \tag{4.12}$$

This leaves us with four equations determining the ground states,

$$\begin{aligned} -J_s \nabla^2 n_{x,y} &= 2\lambda n_{x,y} && (4.13) \\ -J_s \nabla^2 n_z + K n_z + B &= 2\lambda n_z && (4.14) \\ n_x^2 + n_y^2 + n_z^2 &= 1 && (4.15) \end{aligned}$$

Since a uniform latice gives the lowest energy, we assume $\nabla^2 \mathbf{n}$ to be zero. This means that either $n_{x,y} = 0$ or $\lambda = 0$. In the case of $n_{x,y} = 0$ the solution kan only be $n_z = \pm 1$, with $\lambda = (B \pm K)/B$, whereby we are using the $n_z = -1$ solution due to the direction of the magnetic field. Our solution for this ground state then becomes

$$\mathbf{n}_{g1} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \tag{4.16}$$

whereby the spins are aligned in the $-\hat{\mathbf{z}}$ direction. Now let us consider the second case, where λ is equal to zero. From equation (4.14) we gather that $n_z = -B/K$, so for $B < K$ a state satisfying the equations above is given by the unit vector

$$\mathbf{n}_{g2} = \begin{pmatrix} \sqrt{1 - B^2/K^2}\cos\theta \\ \sqrt{1 - B^2/K^2}\sin\theta \\ -B/K \end{pmatrix}, \tag{4.17}$$

which describes a spin vector with a fixed inclination $-B(K^2 - B^2)^{-1/2}$ away from the $-\hat{\mathbf{z}}$ vector .

4.3 | The Landau-Lifshitz-Gilbert Equation

A general description of the dynamics of spin density is given by the Landau-Lifschitz-Gilbert equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla\right)\mathbf{n} - \mathbf{n} \times \mathbf{H}_{eff} = -\alpha\mathbf{n} \times \left(\frac{\partial}{\partial t} + \frac{\beta}{\alpha}\mathbf{v}_s \cdot \nabla\right)\mathbf{n}, \tag{4.18}$$

where $\mathbf{v}_s$ parameterizes the reactive spin transfer torque on the left and the dissipative torque on the right-hand side of the equation, and α is the dimensionless Gilbert damping parameter accounting for spin relaxation. Both of these values will be set to zero for now leaving us with

$$\partial_t\mathbf{n} - \mathbf{n} \times \mathbf{H}_{eff} = 0. \tag{4.19}$$

Furthermore, the effective field $\mathbf{H}_{eff}$ is defined as

$$\mathbf{H}_{eff} = -\frac{\delta E[\mathbf{n}]}{\hbar\delta\mathbf{n}} = \frac{1}{\hbar}\left[J_s\nabla^2\mathbf{n} - Kn_z\hat{\mathbf{z}} - B\hat{\mathbf{z}}\right], \tag{4.20}$$

hence we have

$$\hbar\partial_t\mathbf{n} = [J_s\mathbf{n} \times \nabla^2\mathbf{n} - Kn_z\mathbf{n} \times \hat{\mathbf{z}} - B\mathbf{n} \times \hat{\mathbf{z}}]. \tag{4.21}$$

First we will linearize around the $\mathbf{n}_{g1}$ ground state. To do this, we will assume that

$$\mathbf{n}_1 = \mathbf{n}_{g1} + \delta\mathbf{n}(\mathbf{x}, t) \approx \begin{pmatrix} \delta n_x(\mathbf{x}, t) \\ \delta n_y(\mathbf{x}, t) \\ -1 \end{pmatrix}, \tag{4.22}$$

allowing for only very minor fluctuations $\delta n_{x,y}$ in the x and y direction. Correction for these fluctuations in n_z are quadratic and therefore left out. Upon entering $\mathbf{n}_1$ we find the following two equations.

$$\hbar\partial_t\delta n_x = +J_s\nabla^2\delta n_y + (K - B)\delta n_y \tag{4.23}$$
$$\hbar\partial_t\delta n_y = -J_s\nabla^2\delta n_x - (K - B)\delta n_x \tag{4.24}$$

It is convenient to model these minor fluctuations of spin torque as plane waves using the dispersion relation $\delta n_{x,y} = C_{1,2}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, from which we obtain the linear equation

$$\begin{pmatrix} -\hbar i\omega & J_s\mathbf{k}^2 - (K - B) \\ -J_s\mathbf{k}^2 + (K - B) & -\hbar i\omega \end{pmatrix}\begin{pmatrix} C_1e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ C_2e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \end{pmatrix} = \mathbf{0}. \tag{4.25}$$

For this equation to be solvable, the determinant of the matrix needs to equal zero, i.e.

$$(-\hbar i\omega)^2 + (J_s\mathbf{k}^2 - (K-B))^2 = 0. \tag{4.26}$$

Rearranging gives

$$\hbar\omega = J_s\mathbf{k}^2 + \hbar\Omega \tag{4.27}$$

for the angular frequency if we define $\hbar\Omega = B - K$. These plane wave perturbatios of spin torque are called magnons, and $\hbar\Omega$ is the magnon gap. Secondly, we will linearize around the $\mathbf{n}_{g2}$ ground state. We will make the approximation

$$\mathbf{n}_2 = \mathbf{n}_{g2} + \delta\mathbf{n}(\mathbf{x},t) \approx \begin{pmatrix} \cos\theta \\ \sin\theta \\ -1+\delta n \end{pmatrix}, \tag{4.28}$$

where $\theta = \theta(\mathbf{x},t)$ and $\delta n = \delta n(\mathbf{x},t)$. If we enter $\mathbf{n}_2$ in Eq. (4.21) we get

$$\hbar\partial_t\cos\theta = J_s[\sin\theta\nabla^2\delta n + (1-\delta n)\nabla^2\sin\theta] - \sin\theta(B-K+K\delta n) \tag{4.29}$$

$$\hbar\partial_t\sin\theta = J_s[-\cos\theta\nabla^2\delta n - (1-\delta n)\nabla^2\cos\theta] + \cos\theta(B-K+K\delta n) \tag{4.30}$$

$$\hbar\partial_t\delta n = J_s[\cos\theta\nabla^2\sin\theta - \sin\theta\nabla^2\cos\theta]. \tag{4.31}$$

Using a few indentities from vector calculus we find that $\nabla^2\sin\theta = \nabla\cdot(\cos\theta\nabla\theta) = \cos\theta\nabla^2\theta - \sin\theta(\nabla\theta)^2$, and similarly $\nabla^2\cos\theta = \nabla\cdot(-\sin\theta\nabla\theta) = -\sin\theta\nabla^2\theta - \cos\theta(\nabla\theta)^2$. In both cases we drop the $(\nabla\theta)^2$-term, since these are quadratic and therefore negligible for small $\nabla\theta$. Our equations now become

$$-\sin\theta\hbar\partial_t\theta = -\sin\theta[B-K+K\delta n - J_s\nabla^2\delta n] + J_s(1-\delta n)\cos\theta\nabla^2\theta \tag{4.32}$$

$$\cos\theta\hbar\partial_t\theta = \cos\theta[B-K+K\delta n - J_s\nabla^2\delta n] + J_s(1-\delta n)\sin\theta\nabla^2\theta \tag{4.33}$$

$$\hbar\partial_t\delta n = J_s\nabla\cdot(\cos^2\theta\nabla\theta + \sin^2\theta\nabla\theta). \tag{4.34}$$

Subtracting $\sin\theta$ times Eq. (4.32) from $\cos\theta$ times Eq. (4.33) gives us

$$\hbar\partial_t\theta = [B-K+K\delta n - J_s\theta\nabla^2\delta n], \tag{4.35}$$

so as a result we find

$$\hbar\frac{d\delta n}{dt} = J_s\nabla^2\theta \tag{4.36}$$

$$\hbar\frac{d\theta}{dt} = B-K+K\delta n - J_s\nabla^2\delta n, \tag{4.37}$$

as the equations gouverning the dynamics of the fluctuations around the tilted ground state. Now we can take the second order derivative of Eq. (4.37)

$$\hbar\frac{d^2\theta}{dt^2} = K\frac{d\delta n}{dt} - J_s\nabla^2\delta n, \tag{4.38}$$

and substitue Eq. (4.36) resulting in

$$\hbar^2\frac{d^2\theta}{dt^2} = J_sK\nabla^2\theta - J_s\nabla^4\theta. \tag{4.39}$$

To continue, we take the perturbations in θ to be $\theta = \theta_0 + C_3e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, so that

$$(-i\hbar\omega)^2(\theta-\theta_0) = J_sK(i\mathbf{k})^2(\theta-\theta_0) - J_s^2(i\mathbf{k})^4(\theta-\theta_0). \tag{4.40}$$

We can therefore identify the relation

$$\hbar\omega = \sqrt{J_sK\mathbf{k}^2 + J_s^2\mathbf{k}^4} \approx \hbar c|\mathbf{k}|, \tag{4.41}$$

with $c = \sqrt{J_sK}/\hbar$, whereby we dropped the $\mathbf{k}^4$-term due to its negligibly small size.

4.4 | Linearization with Damping and Background Velocity

Although our previous approximation will proof sufficient for some calculations, many analogue gravity phenomenae only arise when looking at a more complex linearization which takes into account factors such as the background velocity of the spin waves. We will therefore repeat our linearization of the Landau-Lifshitz-Gilbert equation, this time assuming α and $\mathbf{v}_s$ to be non-zero. First of all, we will use the $\mathbf{n}_{g1}$ (axial) ground state. This gives us the two equations

$$\hbar\partial_t\delta n_x + \hbar(\mathbf{v}_s\cdot\nabla)\delta n_x = +J_s\nabla^2\delta n_y + (K-B)\delta n_y - \alpha\hbar\partial_t\delta n_y - \beta\hbar(\mathbf{v}_s\cdot\nabla)\delta n_y \tag{4.42}$$

$$\hbar\partial_t\delta n_y + \hbar(\mathbf{v}_s\cdot\nabla)\delta n_y = -J_s\nabla^2\delta n_x - (K-B)\delta n_x + \alpha\hbar\partial_t\delta n_x + \beta\hbar(\mathbf{v}_s\cdot\nabla)\delta n_x. \tag{4.43}$$

Again, we will use the plane wave solutions $\delta n_{x,y} = C_{1,2}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ giving us the

$$\begin{pmatrix} i\hbar(\mathbf{v}_s\cdot\mathbf{k}-\omega) & -[-J_s\mathbf{k}^2 + (K-B) + i\alpha\hbar\omega - i\beta\hbar\mathbf{v}_s\cdot\mathbf{k}] \\ -J_s\mathbf{k}^2 + (K-B) + i\alpha\hbar\omega - i\beta\hbar\mathbf{v}_s\cdot\mathbf{k} & i\hbar(\mathbf{v}_s\cdot\mathbf{k}-\omega) \end{pmatrix}\begin{pmatrix}\delta n_x \\ \delta n_y\end{pmatrix} = \mathbf{0}. \tag{4.44}$$

linear equation. In order for there to be a singular solution the determinant of the 2×2-matrix has to equal zero, thus

$$i^2\hbar^2(\mathbf{v}_s\cdot\mathbf{k}-\omega)^2 + (-J_s\mathbf{k}^2 + (K-B) + i\alpha\hbar\omega - i\beta\hbar\mathbf{v}_s\cdot\mathbf{k})^2 = 0, \tag{4.45}$$

which can be rearranged to read

$$\hbar(\mathbf{v}_s\cdot\mathbf{k}-\omega) = -J_s\mathbf{k}^2 + (K-B) + i\alpha\hbar\omega - i\beta\hbar\mathbf{v}_s\cdot\mathbf{k}. \tag{4.46}$$

Rewriting gives

$$(1+i\alpha)\hbar\omega = J_s\mathbf{k}^2 + (B-K) + (1+i\beta)\hbar\mathbf{v}_s\cdot\mathbf{k}, \tag{4.47}$$

subsequently we find

$$\hbar\omega = \frac{1}{(1+i\alpha)}[J_s\mathbf{k}^2 + \hbar\Omega + (1+i\beta)\hbar\mathbf{v}_s\cdot\mathbf{k}], \tag{4.48}$$

where $\hbar\Omega = B-K$. Taking the Taylor series for small values for α and β gives

$$\hbar\omega = (1-i\alpha)[J_s\mathbf{k}^2 + \hbar\Omega + \hbar\mathbf{v}_s\cdot\mathbf{k}] + i\hbar\beta\mathbf{v}_s\cdot\mathbf{k}. \tag{4.49}$$

Finally, we will linearize around a modified version of the $\mathbf{n}_{g2}$ tilted ground state. Using

$$\mathbf{n}_{g2} = \begin{pmatrix}\sqrt{2n_c}\cos\theta \\ \sqrt{2n_c}\sin\theta \\ -1+n_c\end{pmatrix}, \tag{4.50}$$

where $n_c = 1 - B/K$, will enable us to calculate fluctuations not only in the x and y but also in the z direction. By entering this $\mathbf{n}_{g2}$ into the Landau-Lifshitz-Gilbert equation we obtain the two equations

$$2n_c\left[\hbar(\partial_t + \mathbf{v}_s\cdot\nabla)\theta + (K-B) + (J_s\nabla^2 - K - \alpha\hbar\partial_t - \beta\hbar\mathbf{v}_s\cdot\nabla)n_c\right] = -\hbar\beta(n_c-1)\mathbf{v}_s\cdot\nabla n_c \tag{4.51}$$

$$2n_c[\alpha\partial_t + \beta\mathbf{v}_s\cdot\nabla - J_s\nabla^2]\theta + (\partial_t + \mathbf{v}_s\cdot\nabla)n_c = 0. \tag{4.52}$$

We now enter $n_c \to n_c + \delta n$ and $\theta \to \theta_0 + \delta\theta$, and drop any terms containing more than one δn or $\delta\theta$, resulting in

$$2(K-B)\left[\hbar(\partial_t + \mathbf{v}_s\cdot\nabla)\delta\theta + (J_s\nabla^2 - K - \alpha\hbar\partial_t - \beta\hbar\mathbf{v}_s\cdot\nabla)\delta n\right] = \hbar\beta B\mathbf{v}_s\cdot\nabla\delta n \tag{4.53}$$

$$2(K-B)[-J_s\nabla^2 + \alpha\partial_t + \beta\mathbf{v}_s\cdot\nabla]\delta\theta + K(\partial_t + \mathbf{v}_s\cdot\nabla)\delta n = 0. \tag{4.54}$$

This can be rewritten as

$$i\hbar(\mathbf{v}_s \cdot \mathbf{k} - \omega)\delta\theta + [-J_s\mathbf{k}^2 - K + i\hbar(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k}) + i\hbar\beta\frac{B}{2(B-K)}\mathbf{v}_s \cdot \mathbf{k}]\delta n = 0 \tag{4.55}$$

$$[J_s\mathbf{k}^2 - i\hbar(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k})]\delta\theta - i\hbar\frac{K}{2(B-K)}(\mathbf{v}_s \cdot \mathbf{k} - \omega)\delta n = 0, \tag{4.56}$$

or as the matrix equation

$$\begin{pmatrix} i\hbar(\mathbf{v}_s \cdot \mathbf{k} - \omega) & -J_s\mathbf{k}^2 - K + i\hbar(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k}) + i\hbar\beta\frac{B}{2(B-K)}\mathbf{v}_s \cdot \mathbf{k} \\ J_s\mathbf{k}^2 - i\hbar(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k}) & -i\hbar\frac{K}{2(B-K)}(\mathbf{v}_s \cdot \mathbf{k} - \omega) \end{pmatrix} \begin{pmatrix} \delta\theta \\ \delta n \end{pmatrix} = \mathbf{0}. \tag{4.57}$$

Since this equation has a solution, we know that

$$K\hbar^2(\mathbf{v}_s \cdot \mathbf{k} - \omega)^2 - 2K(B-K)[J_s\mathbf{k}^2 - i\hbar(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k})] = 0, \tag{4.58}$$

thus

$$(\mathbf{v}_s \cdot \mathbf{k} - \omega) = c|k| + i\frac{(B-K)}{c\hbar|k|}(\alpha\omega - \beta\mathbf{v}_s \cdot \mathbf{k}) \tag{4.59}$$

where $c = \sqrt{2(B-K)J_s}/\hbar$, giving

$$(\mathbf{v}_s \cdot \mathbf{k} - \omega) = c|k| + i\frac{(B-K)}{c\hbar|k|}\alpha(\mathbf{v}_s \cdot \mathbf{k} - \omega) + i\frac{(B-K)}{c\hbar|k|}(\alpha - \beta)\mathbf{v}_s \cdot \mathbf{k} \tag{4.60}$$

4.5 | Results

We have now found the dispersion relations of the magnons arising from perturbations of the spin lattice. We see that for the fully alligned ground state $\mathbf{n}_{g1}$ the dispersion relation is quadratic both with the background velocity and damping turned on and off. For the tilted $\mathbf{n}_{g2}$ ground state both dispersion relations are linear. The dispersion relations can be seen in Table 4.5.

Ground State	$\mathbf{v}_s, \alpha, \beta = 0$	$\mathbf{v}_s, \alpha, \beta \neq 0$
$\mathbf{n}_{g1}$	$\hbar\omega = J_s\mathbf{k}^2 + \hbar\Omega$	$\hbar\omega = J_s\mathbf{k}^2 + \hbar\Omega + \hbar\mathbf{v}_s \cdot \mathbf{k}$
$\mathbf{n}_{g2}$	$\hbar\omega = \hbar c|\mathbf{k}|$	$\hbar\omega = c^*|\mathbf{k}|$

Table 4.1: An overview of the real parts of different dispersion relations for magnons, where $\hbar\Omega = B - K$, $c = \sqrt{J_s K}/\hbar$, and $c^* = \sqrt{J_s(B-K)}/\hbar$.

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Chapter 5
Cherenkov Radiation

Before we can further study black or white holes for magnons, we first have to know more about possible effects that can play a role in our experiments. One of such effects which might be observable is the Cherenkov effect [6]. An example of the Cherenkov effect that most will be familiar with is a Mach cone on a jet breaking through the sound barrier, or the radiation emmitted by a charged particle moving faster than the phase velocity of the medium it is in.

When working with a magnonic black or white hole there will by definition be a section of the setup where the bulk spin velocity is higher than the velocity of the magnons. It has already been demonstrated that magnonic Cherenkov radiation occurs when a current is driven through a permalloy strip with a magnetic field inducing cobalt Cylinder on top of it, acting as an impurity of the spin lattice [10]. It is therefore possible that presens of an event horizon might also induce Cherenkov radiation.

In this chapter we will calculate the critical speed which has to be surpassed to form a black or white hole. We will then find the boundary conditions for the spin waves on the event horizon, so that finally we can calculate reflection and transmission coefficients for these spin waves. For this, we will use the axial approximation we found in the preceding chapter.

5.1 | Finding an instability

An energetic instability occurs when the real part of the dispersion relation changes sign. For the axial situation (with background velocity and damping taken into account) we have found the dispersion relation

$$\hbar\omega = (1 - i\alpha)[J_s\mathbf{k}^2 + \hbar\Omega + \hbar\mathbf{v}_s \cdot \mathbf{k}] + i\hbar\beta\mathbf{v}_s \cdot \mathbf{k}. \tag{5.1}$$

We are looking at a one dimensional situation where for some $x = x_0$ the bulk velocity on the lefthand side is below and on the righthand side above the critical velocity. Because our setup is one dimensional $\mathbf{k} = k$ and $\mathbf{v}_s = v_s$. We will start off by asuming $\alpha, \beta = 0$ (i.e. no damping). In that case

$$\hbar\omega = J_s k^2 + \hbar\Omega + \hbar v_s k. \tag{5.2}$$

We first locate the minimum using

$$\partial_k(\hbar\omega)n = 2J_s k + \hbar v_s = 0, \tag{5.3}$$

from which we find the minimum to be placed at $k = -\hbar v_s/2J_s$. If we enter this value for k in equation (5.2) we obtain

$$\hbar\omega = \frac{\hbar^2 v_s^2}{4J_s} + \hbar\Omega - \frac{\hbar^2 v_s^2}{2J_s}, \tag{5.4}$$

$$= \hbar\Omega - \frac{\hbar^2 v_s^2}{4J_s}. \tag{5.5}$$

This means a change of sign when $|v_s| > 2\sqrt{J_s(B-K)}/\hbar$, remembering that $\hbar\Omega = B - K$. We shall now determine at what v_s the change of sign happens for the imaginary component of $\hbar\omega$, which reads

$$\hbar\omega_{im} = -\alpha J_s k^2 - \alpha\hbar\Omega - (\alpha - \beta)\hbar v_s k. \tag{5.6}$$

Taking the derivative with respect to k gives

$$\partial_k \hbar\omega_{im} = -2\alpha J_s k - (\alpha - \beta)\hbar v_s = 0, \tag{5.7}$$

from which we find the minimum to be located at $k = -(\alpha-\beta)\hbar v_s/2\alpha J_s$. Pluging this into Eqn. (5.6) leads to

$$\hbar\omega_{im} = \frac{(\alpha-\beta)^2}{4\alpha J_s}\hbar^2 v_s^2 - \alpha\hbar\Omega, \tag{5.8}$$

from which we find that a change in sign occurs when

$$|v_s| > \left|\frac{\alpha}{\alpha-\beta}\right| \frac{2\sqrt{J_s(B-K)}}{\hbar}. \tag{5.9}$$

We see that in the case that $\beta = 0, \alpha \ll 1$ this v_s becomes the same as for $\hbar\omega_{re}$. This makes sense since $\hbar\omega_{im} = -i\alpha\hbar\omega_{re}$, so that when $\hbar\omega_{re}$ changes sign $\hbar\omega_{im}$ will as well.

5.2 | Schrödinger-like equation

Now that we have found the critical velocity, we have to decide on what boundary conditions to use for when the magnons cross the event horizon. Combining Eqn. (4.42) gives

$$\hbar(\partial_t + \mathbf{v}_s\cdot\nabla)(i\delta n_x + \delta n_y) = [J_s\nabla^2 + (K-B)](i\delta n_y - \delta n_x) + \hbar(\alpha\partial_t + \beta\mathbf{v}_s\cdot\nabla)(\delta n_x - i\delta n_y). \tag{5.10}$$

We know define $\bar{\Psi} = \delta n_x - i\delta n_y$, so that

$$i\hbar(\partial_t + \mathbf{v}_s\cdot\nabla)\bar{\Psi} = (-J_s\nabla^2 + \hbar\Omega)\bar{\Psi} + \hbar(\alpha\partial_t + \beta\mathbf{v}_s\cdot\nabla)\bar{\Psi}. \tag{5.11}$$

Our first boundary condition is $\bar{\Psi}_L = \bar{\Psi}_R$, in other words; the spin wave has to be continous. Integrating from $x_0 - \epsilon$ to $x_0 + \epsilon$ and then letting $\epsilon \to 0$ leaves only the non-continuous terms

$$i\hbar v_s(\bar{\Psi}(x_0+\epsilon) - \bar{\Psi}(x_0-\epsilon)) = -J_s\partial_x(\bar{\Psi}(x_0+\epsilon) - \bar{\Psi}(x_0-\epsilon)), \tag{5.12}$$

and thus our second boundary condition becomes

$$J_s\partial_x\bar{\Psi}_L + i\hbar v_L\bar{\Psi}_L = J_s\partial_x\bar{\Psi}_R + i\hbar v_R\bar{\Psi}_R. \tag{5.13}$$

Let us define

$$\bar{\Psi}_L = e^{ik_1x} + re^{-ik_2x}, \tag{5.14}$$
$$\bar{\Psi}_R = te^{ik_3x}, \tag{5.15}$$

then taking our basis so that $x_0 = 0$ gives

$$J_s i(k_1 - k_2 r) + i\hbar v_L(1+r) = J_s ik_3 t + i\hbar v_R t, \tag{5.16}$$
$$= i(J_s k_3 + \hbar v_R)(1+r), \tag{5.17}$$

where we used $t = 1 + r$ which we found from our first boundary condition. Rearranging gives

$$r = \frac{\hbar(v_R - v_L) + J_s(k_3 - k_1)}{\hbar(v_L - v_R) - J_s(k_2 + k_3)}, \tag{5.18}$$

and since $t = 1 + r$ we also find

$$t = -\frac{J_s(k_1 + k_2)}{\hbar(v_L - v_R) - J_s(k_2 + k_3)}. \tag{5.19}$$

We have now found an expresion for our reflection and transmission coefficients.

5.3 | Rewriting r and t

Although we have found an elegant expression for both the reflection and the transmission coefficient, for future research it might be usefull to have them expressed in terms of the background velocity, rather than the wave vectors. We will therefore rewrite the result above in a different form. From Eqn. (5.1) and the *abc*-formula we find

$$2J_s k = -\hbar v_s \pm \sqrt{\hbar^2 v_s^2 - 4J_s(\hbar\Omega - \hbar\omega)}. \tag{5.20}$$

Entering this in the equation for r gives

$$r = \frac{\hbar(v_R - v_L) \pm (\sqrt{\hbar^2 v_R^2 - 4J_s(\hbar\Omega - \hbar\omega)} - \sqrt{\hbar^2 v_L^2 - 4J_s(\hbar\Omega - \hbar\omega)})}{\hbar(3v_L - v_R) \pm (\sqrt{\hbar^2 v_L^2 - 4J_s(\hbar\Omega - \hbar\omega)} + \sqrt{\hbar^2 v_R^2 - 4J_s(\hbar\Omega - \hbar\omega)})}. \tag{5.21}$$

Now let

$$v_R = v_c + \xi_R, \tag{5.22}$$
$$v_L = v_c - \xi_L, \tag{5.23}$$

where $v_c = 2\sqrt{J_s(B-K)}/\hbar$. Assuming ξ_R, ξ_L to be small we find $\hbar^2 v_R^2 = 4J_s(B-K) + 2\hbar^2 v_c \xi_R$ and the same for $\hbar^2 v_L$ if we substitute $\xi_R \to -\xi_L$.

$$r = \frac{\hbar(\xi_R - \xi_L) \pm (\sqrt{4J_s\hbar\omega + 2\hbar^2 v_c \xi_R} - \sqrt{4J_s\hbar\omega - 2\hbar^2 v_c \xi_L})}{\hbar(2v_c - 3\xi_L - \xi_R) \pm (\sqrt{4J_s\hbar\omega + 2\hbar^2 v_c \xi_R} + \sqrt{4J_s\hbar\omega - 2\hbar^2 v_c \xi_L})} \tag{5.24}$$

Taking the Taylor series around small ξ_R gives $(4J_s\hbar\omega + 2\hbar^2 v_c \xi_R)^{1/2} = \sqrt{4J_s\hbar\omega} + \hbar^2 v_c \xi_R / \sqrt{4J_s\hbar\omega}$. We do the same thing for ξ_L, which we then enter into the equation for r to obtain

$$r = \frac{2\sqrt{J_s\hbar\omega}(\xi_R - \xi_L) \pm \hbar v_c(\xi_R - \xi_L)}{2\sqrt{J_s\hbar\omega}(2v_c - 3\xi_L - \xi_R) \pm \hbar v_c(\xi_R + \xi_L)}. \tag{5.25}$$

Seeing as $t = 1 + r$ we also find

$$t = \frac{2\sqrt{J_s\hbar\omega}(2v_c - 4\xi_L) \pm 2\hbar v_c \xi_R}{2\sqrt{J_s\hbar\omega}(2v_c - 3\xi_L - \xi_R) \pm \hbar v_c(\xi_R + \xi_L)}. \tag{5.26}$$

We now have expressions for the reflection and transmission coefficients that are only dependent on the bulk speed at both sides of the event horizon.

Chapter 6

Conclusion and Outlook

We will conclude this thesis by recapitulating the main points and results of the preceding discussion. After that we will discuss the outlook of further research into magnonic analogue gravity.

In the first part of this thesis, we focussed on the theory behind analogue gravity. We started by discussing a few relevant concepts from General Relativity such as tensors, the Schwarzschild metric and the d'Alembertian. These concepts were needed in order to understand the calculations done by Unruh and later Visser et al. In these calculations it is demonstrated how an inviscid, barotropic, vorticity free fluid that is spherically symmetric coverging to a point can be described using the mathematics of a black hole. More specifically, we found two metrics; first the Schwarzschild metric (for a non-rotating, zero charge balck hole), which we then generalized to accomodate for rotation of the analogue black hole.

The second part of this thesis proceded to focus on analogue gravity for magnonic systems. Firstly we introduced the concept of magnons and some of their properties. We then calculated the two ground states of a spin lattice by minimizing its energy. We found two ground states (one axial and one tilted), which allowed us to model magnons as minor perturbations around these states. We did this for both ground states, first withouth and then with taking into account background velocity and damping. This way, we were able to calculate the dispersion relations for each of these four situations. For the axial groud state we found the dispersion relations to be quadratic, whilst for the tilted state they turned out to be linear.

Using the dispersion relation of the axial ground state we were able to calculate a critical velocity at which an energetic instability occurs. To create a magnonic black or white hole, on one side of the event horizon the background velocity needs to be below this critical velocity, and on the other side this velocity needs to be surpassed. We then found the boundary conditions for the magnons passing through the event horizon, from which we were ultimately able to find expressions for the reflection and transmission coefficients of these spin waves.

6.1 | Outlook

From this point on, a further study into what exactly happens at the event horizon would be of interest. It could give us more insight into the phenomenae that occur around a magnonic black or white hole horizon. Once the setup we used for or research becomes experimentally viable, these results could be used to make observable predictions. This way we can verify part of the theoretical framework.

Another particular setup worthy of further study would be a magnonic black hole-white hole pair. Specifically one could investigate the occurence of super resonance, which if it were to emerge would could possibly be utilized for designing a magnon laser.

Part III:

Appendices

All physics is either impossible or trivial. It is impossible until you understand it, and then it becomes trivial.

Ernest Rutherford

Appendix A
Supplementary Proofs

A.1 | Proof of Result 1 (See Page 6)

Result 1. *In three spatial dimensions, with $v = \|\mathbf{v}\|$, we find the identity*

$$-(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{v}\times(\nabla\times\mathbf{v}) - \nabla\left(\frac{1}{2}v^2\right). \tag{3.4}$$

We will provide two prove this result using two distinct methods. The first method will be by writing the equation out into its components.

Proof 1. We start by writing out the left side of the equation.

$$-(\mathbf{v}\cdot\nabla)\mathbf{v} = (v_1\partial_x + v_2\partial_y + v_3\partial_z)\mathbf{v} \tag{A.1}$$

$$= -\begin{pmatrix} v_1\partial_x v_1 + v_2\partial_y v_1 + v_3\partial_z v_1 \\ v_1\partial_x v_2 + v_2\partial_y v_2 + v_3\partial_z v_2 \\ v_1\partial_x v_3 + v_2\partial_y v_3 + v_3\partial_z v_3 \end{pmatrix} \tag{A.2}$$

We then do the same for the first term on the right-hand side of the equation.

$$\mathbf{v}\times(\nabla\times\mathbf{v}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} \tag{A.3}$$

$$= \begin{pmatrix} v_2(\partial_x v_2 - \partial_y v_1) - v_3(\partial_z v_1 - \partial_x v_3) \\ v_3(\partial_y v_3 - \partial_z v_2) - v_1(\partial_x v_2 - \partial_y v_1) \\ v_1(\partial_z v_1 - \partial_x v_3) - v_2(\partial_y v_3 - \partial_z v_2) \end{pmatrix} \tag{A.4}$$

Lastly we write out the second term on the right-hand side.

$$-\nabla(\frac{1}{2}v^2) = -\nabla[\frac{1}{2}(v_1^2 + v_2^2 + v_3^2)] \tag{A.5}$$

$$= -\begin{pmatrix} v_1\partial_x v_1 + v_2\partial_x v_2 + v_3\partial_x v_3 \\ v_1\partial_y v_1 + v_2\partial_y v_2 + v_3\partial_y v_3 \\ v_1\partial_z v_1 + v_2\partial_z v_2 + v_3\partial_z v_3 \end{pmatrix} \tag{A.6}$$

Finally we combine these written out terms to see that

$$\mathbf{v}\times(\nabla\times\mathbf{v}) - \nabla(\frac{1}{2}v^2) = -\begin{pmatrix} v_1\partial_x v_1 + v_2\partial_y v_1 + v_3\partial_z v_1 \\ v_1\partial_x v_2 + v_2\partial_y v_2 + v_3\partial_z v_2 \\ v_1\partial_x v_3 + v_2\partial_y v_3 + v_3\partial_z v_3 \end{pmatrix} \tag{A.7}$$

$$= -(\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{A.8}$$

which completes our proof. ■

Additionally, we provide a proof utilizing the Levi-Civita tensor below.

Proof 2.

$$
\begin{aligned}
\mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla\left(\frac{1}{2}v^2\right) &= v_j \mathbf{e}_j \times \partial_\beta v_\alpha \epsilon_{i\beta\alpha} \mathbf{e}_i - \partial_\gamma \frac{1}{2} v_\alpha^2 \mathbf{e}_\gamma && \text{(A.9)}\\
&= v_j \partial_\beta v_\alpha \epsilon_{i\beta\alpha} \epsilon_{ij\gamma} \mathbf{e}_\gamma - v_\alpha \partial_\gamma v_\alpha \mathbf{e}_\gamma && \text{(A.10)}\\
&= [v_j \partial_\beta v_\alpha (\delta_{\alpha j}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{j\beta}) - v_j \partial_\gamma v_\alpha \delta_{\alpha j}\delta_{\beta\gamma}]\mathbf{e}_\gamma && \text{(A.11)}\\
&= -v_j \partial_\beta v_\alpha \delta_{\alpha\gamma}\delta_{j\beta} \mathbf{e}_\gamma && \text{(A.12)}\\
&= -v_\beta \partial_\beta v_\gamma \mathbf{e}_\gamma && \text{(A.13)}\\
&= -(\mathbf{v}\cdot\nabla)\mathbf{v}. && \text{(A.14)}
\end{aligned}
$$

■

A.2 | Proof of Result 2 (See Page 10)

Result 2. *If we define a new time τ so that*

$$\tau = t + \int \frac{v_0^r}{c^2 - (v_0^r)^2} dr, \tag{3.43}$$

then

$$dt^2 = d\tau^2 - \frac{2v_0^r d\tau dr}{(c^2 - (v_0^r)^2)} + \frac{(v_0^r)^2 dr^2}{(c^2 - (v_0^r)^2)^2}. \tag{3.44}$$

Proof. First of all, we define

$$\tau = t + \int \frac{v_0^r}{c^2 - (v_0^r)^2} dr \qquad \Longrightarrow \qquad t = \tau - \int \frac{v_0^r}{c^2 - (v_0^r)^2} dr. \tag{A.15}$$

We consider t to be sufficiently smooth to take an exact differential.

$$
\begin{aligned}
dt &= \left(\frac{\partial t}{\partial \tau}\right) d\tau + \left(\frac{\partial t}{\partial r}\right) dr && \text{(A.16)}\\
&= d\tau - \frac{v_0^r}{c^2 - (v_0^r)^2} dr. && \text{(A.17)}
\end{aligned}
$$

Squaring dt gives

$$dt^2 = d\tau^2 - \frac{2v_0^r d\tau dr}{(c^2 - (v_0^r)^2)} + \frac{(v_0^r)^2 dr^2}{(c^2 - (v_0^r)^2)^2}. \tag{A.18}$$

■

Appendix B

Expanded Calculations

B.1 | Calculations on Page 8

$$f^{\mu\nu}(t,\mathbf{x}) \equiv \frac{\rho_0}{c^2}\begin{bmatrix} -1 & -v_0^1 & -v_0^2 & -v_0^3 \\ -v_0^1 & c^2-(v_0^1)^2 & -v_0^1v_0^2 & -v_0^1v_0^3 \\ -v_0^2 & -v_0^2v_0^1 & c^2-(v_0^2)^2 & -v_0^2v_0^3 \\ -v_0^3 & -v_0^3v_0^1 & -v_0^3v_0^2 & c^2-(v_0^3)^2 \end{bmatrix} \tag{B.1}$$

$$\begin{aligned} \partial_\mu(f^{\mu\nu}\partial_\nu\phi_1) = \frac{\rho_0}{c^2}[&-\partial_t(\partial_t\phi_1 + v_0^x\partial_x\phi_1 + v_0^y\partial_y\phi_1 + v_0^z\partial_z\phi_1) \\ &-\partial_x(v_0^x\partial_t\phi_1 - (c^2-(v_0^x)^2)\partial_x\phi_1 + v_0^xv_0^y\partial_y\phi_1 + v_0^xv_0^z\partial_z\phi_1) \\ &-\partial_y(v_0^y\partial_t\phi_1 + v_0^yv_0^x\partial_x\phi_1 - (c^2-(v_0^y)^2)\partial_y\phi_1 + v_0^yv_0^z\partial_z\phi_1) \\ &-\partial_z(v_0^z\partial_t\phi_1 + v_0^zv_0^x\partial_x\phi_1 + v_0^zv_0^y\partial_y\phi_1 - (c^2-(v_0^z)^2)\partial_z\phi_1)] \end{aligned} \tag{B.2}$$

$$\begin{aligned} \partial_\mu(f^{\mu\nu}\partial_\nu\phi_1) = \frac{\rho_0}{c^2}[&\partial_t(-\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) \\ &+\nabla\cdot\begin{pmatrix} c^2\partial_x\phi_1 - v_0^x\partial_t\phi_1 - (v_0^x)^2\partial_x\phi_1 - v_0^xv_0^y\partial_y\phi_1 - v_0^xv_0^z\partial_z\phi_1 \\ c^2\partial_y\phi_1 - v_0^y\partial_t\phi_1 - v_0^yv_0^x\partial_x\phi_1 - (v_0^y)^2\partial_y\phi_1 - v_0^yv_0^z\partial_z\phi_1 \\ c^2\partial_z\phi_1 - v_0^z\partial_t\phi_1 - v_0^zv_0^x\partial_x\phi_1 - v_0^zv_0^y\partial_y\phi_1 - (v_0^z)^2\partial_z\phi_1 \end{pmatrix}] \end{aligned} \tag{B.3}$$

$$\begin{aligned} = \frac{\rho_0}{c^2}[&-\partial_t(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) \\ &+\nabla\cdot(c^2\nabla\phi_1 - \mathbf{v}_0(\partial_t\phi_1 + v_0^x\partial_x\phi_1 + v_0^y\partial_y\phi_1 + v_0^z\partial_z\phi_1))] \end{aligned} \tag{B.4}$$

$$= \frac{\rho_0}{c^2}[-\partial_t(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1) + \nabla\cdot(c^2\nabla\phi_1 - \mathbf{v}_0(\partial_t\phi_1 + \mathbf{v}_0\cdot\nabla\phi_1))] \tag{B.5}$$

Appendix B
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