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Duality between a (3+1) Einstein-Maxwell-Dilaton gravity theory and a (2+1) strange metal quantum theory

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Abstract

In this thesis we will discuss the holographic description of cuprate superconductors in the strange metal phase. The strange metal phase occurs as the high-temperature behaviour above the superconducting phase transition. From experimental results it is observed that around the Fermi surface the imaginary part of the (nodal) self energy obeys a power law in terms of both temperature and energy. We introduce a holographic description of this strange metal by means of a dual Einstein-Maxwell-Dilaton (EMD) gravity theory. With this model some features of the experimental behaviour of the strange metal can be reproduced. We also present some additional results that could lead to new experimental insights.

Contents

1 Introduction

In this thesis we will discuss how condensed matter systems can be described by a dual classical gravity theory. This idea is known as the Anti-de Sitter versus Condensed Matter Theory (or AdS/CMT correspondence). To avoid chaos, it is important to realize that all the following words (in the context of this thesis) mean exactly the same

- Anti-de Sitter versus Conformal Field Theory AdS/CFT correspondence/duality
- Weak gravity versus strong quantum theory correspondence/duality
- The Holographic principle/duality

For clarity the words duality, correspondence and principle mean exactly the same in the context of this thesis.

The rest of this thesis is divided into chapters as follows:

Chapter 2: Condensed matter/strange metals.

In this chapter we will discuss the condensed matter system of interest. We will discuss why this certain metal can not be described with standard methods from condensed matter theory.

Chapter 3: The holographic principle.

In this section we will explicitly investigate all the known theory about the holographic principle and it's assumptions. We will focus mostly on what is important to describe the condensed matter system of interest.

Chapter 4: Holographic description of strange metals.

In this chapter we will use the concepts from chapter 3, to describe the proposed dual weak gravity theory to the quantum mechanical theory from chapter 2.

Chapter 5: Results.

In this chapter we will explore what the consequences are of what has been assumed in all previous chapters. We will do this numerically and analytically.

Chapter 6: Discussion and outlook.

In the last chapter we will interpret the results and propose possible future direction for the research on strange metals.

2 Condensed matter/strange metals

2.1 When to use holography

This section is based on [12, 19] for the quasiparticle description and on [8] relating this problem to holography.

Condensed matter is interested in solving the many-body problem in quantum mechanics. Standard ways of doing this are starting with a free theory of particles and adding interactions. We can start with a free particle description and add interactions in terms of a new potential. This implies that the system behaves with the original particles including their properties (like mass and charge), but their movement is now altered by the external potential. Another way of describing interactions can be done by the so called quasiparticle description.

The quasiparticle description starts with strongly interacting particles that are difficult to describe as no perturbation theory can be applied. A new particle can be proposed, the quasiparticle, which is weakly interacting with other quasiparticles and thus we come back to the much simpler problem of describing weakly interacting particles. One of the most famous examples is Landau's Fermi liquid theory describing electrons in metals.

It is however almost impossible to ensure that no quasiparticle description exist. One could try numerous types of different quasiparticles and exclude them one by one. More on this can be read in [8].

For strongly correlated systems, the quasiparticle description does not work. Therefore physicists have to resort to other theoretical methods.

Luckily there are some methods for studying such strongly correlated systems. One promising method is known as the AdS/CFT (or holographic) correspondence.

2.2 Strange metals

The strongly correlated system that will be studied in this thesis is the strange metal. Strange metallic behaviour is observed as the high-temperature behaviour of superconductors [17] and also recently in the magical-angle bilayer graphene for example [2]. From a technological standpoint understanding high-temperature superconductors is of huge importance. The critical temperature, T_c, determines the maximum temperature for which the material is a superconductor. As mentioned in [8], studying strange metals is a prerequisite for the understanding of the value of T_c. In figure 3, the phase diagram for cuprates (copper oxide metals) can be seen. Here it can be seen that the strange metal regime is indeed the high-temperature behaviour of certain superconductors.

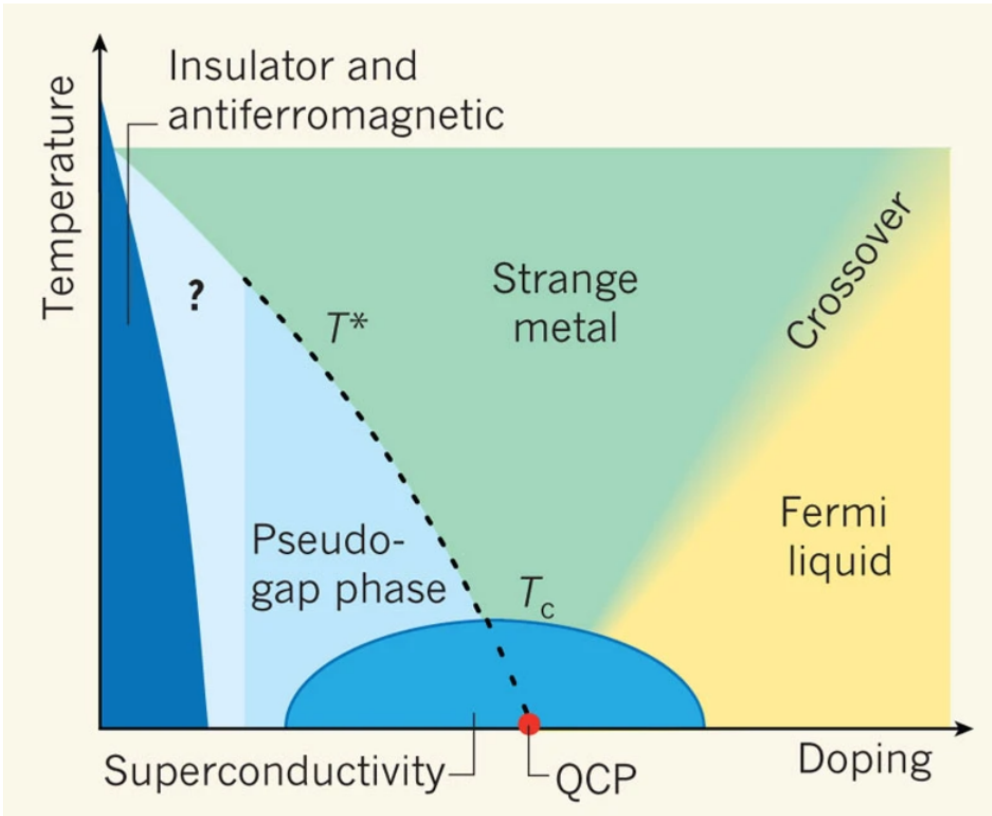

Figure 1: Phase diagram for the cuprates, the strange metal regime of interest is clearly indicated, source:[20].

One peculiar property of the strange metal regime is the linear in T resistivity, which is very unusual in typical metals, therefore the name strange metal. Usually the resistivity of metals can be characterized by means of the Mott-Ioffe-Regel condition, we will not go into detail, but more on this can be read in the master thesis by Joren Harms[1].

It is believed that the strange metals can be understood in terms of the dynamical critical exponent z and the anomalous scaling dimension θ. Explicitly $z \to \infty$ and $\theta \to -\infty$ with z/θ remaining a constant. With the use of these two parameters a dual theory can be introduced that has the property of the linear scaling in resistivity with temperature. When introducing a dual gravity theory to this quantum theory the meaning and implementation of these two parameters will be explained (section 4.1.1).

With the use of Angle-Resolved Photoemission Spectroscopy (ARPES), the so called spectral function can be observed experimentally. This function obtaining its name from spectroscopy itself, is directly related to one of the most favorite functions of any theoretical physicist; the Green's function. The spectral function can be seen as the one particle density of states. And provides the probability of a particle to be in a certain energy and momentum. Also the dispersion relation can be seen in the spectral function as the somewhat blurry line in the energy momentum plane. In figure 2, the spectral function for a cuprate as a function of energy (ω) and momentum (k) is given.

[1]Link is mentioned on page 10.

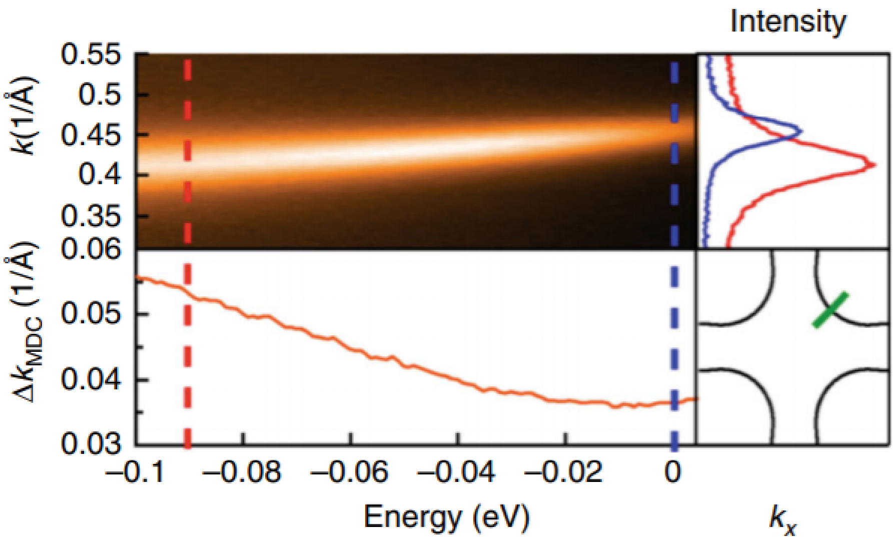

Figure 2: Spectral function of a cuprate as a function of energy and momentum in top left, momentum distribution curves (MDC) in the top right, imaginary part of the self energy (full width at half maximum of the peaks in the MDC) as a function of energy in the bottom left and the node in the bottom right. Source: [17].

The goal of this thesis is to describe the behaviour (i.e. the calculation of the Green's function) of these cuprate metals by introducing a dual gravity theory. In the bottom right of the figure above, the nodal direction in which the spectral function is observed is given. This puts two more constraints on the fermionic theory, together with the earlier mentioned constraints we need to describe the following[2]:

- A 2-dimensional massless fermionic theory containing 2 space dimensions (and one time dimension).
- The critical dynamcial exponent $z \to \infty$.
- The anomalous scaling dimension $\theta \to -\infty$.
- z/θ remaining constant.

The remaining of this thesis will focus on how these four constraints are implemented with the use of dual gravity theory and how this might reveal new insights on the condensed matter side. The proposed phenomenological law for the imaginary part of the self energy (Σ'') is given by

$$\Sigma'' \propto (\omega^2 + \beta T^2)^\alpha \tag{2.1}$$

Where β and α are constants. These constants are found experimentally for several different doping levels and the results of this can be seen in the paper. In the case of the strange metal, $\alpha = 0.5$ and for all doping levels β is determined to be roughly equal to π.

[2]The first two constraints come from the nodal direction, more can be read in [17].

3 The holographic principle

This chapter is mainly based on the book "Holographic duality in condensed matter physics"[21]. Other references will be mentioned.

Calculating n-point functions in a QFT by performing calculations in general relativity (GR), this sounds like magic but it might work. The AdS/CFT correspondence is a duality between gauge and gravity theories and was discovered by Maldacena in 1997 [14]. The idea originates from string theory, but applications are found in studying strongly coupled quantum field theories, such as QCD. From around 2007 this duality is also used in condensed matter physics to study strongly correlated condensed matter systems. In condensed matter physics the duality is also referred to as AdS/CMT, where CMT stands for condensed matter theory.

In this section we will look into how holography works. This chapter will start with the basics and the AdS/CFT correspondence. We will work towards explaining how the holographic principle can be used regarding condensed matter physics.

3.1 AdS/CFT

A physical theory can typically be well described by its symmetries. Thus different theories possessing the same symmetries can describe the same physics, that idea is known as a duality. In this section we will introduce the AdS/CFT correspondence and show explicitly how symmetries of AdS space times can correspond to the symmetries of CFT's.

3.1.1 Anti-de Sitter spacetimes

Here we will mainly follow section 4.4 of [21] to provide a short description of AdS spacetimes. For a more elaborate description of spacetimes and general relativity there are plenty of books explaining symmetries of spacetimes in much more detail such as sections 3.8 and 3.9 of [3]. Since the AdS/CFT correspondence can be viewed as a symmetry correspondence, it is necessary to explore the symmetries of AdS spacetimes. The symmetries of spacetimes are known as isometries, which are coordinate transformations that leave the metric invariant, explicitly

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x). \tag{3.1}$$

Coordinate transformation of the form $x_\nu \to x_\nu + \xi_\nu$, result into $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x)$[3]. For the equation above to be true we need

$$\nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) = 0, \tag{3.2}$$

which is known as the killing equation. The amount of independent killing vectors, $\xi_\mu(x)$ determines the degree of symmetry of the spacetime. There can only be $D(D+1)/2$ independent killing vectors. Spacetimes that posses this amount of killing vectors are known as maximally symmetric spaces. Maximally symmetric spaces can also be understood quite literary as they look the same at every point. This implies that each point should also posses the same constant curvature. These maximally symmetric spaces are solutions to Einsteins equation in vacuum with a cosmological constant, i.e. solutions to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 0. \tag{3.3}$$

By simply contracting this equation with $g^{\mu\nu}$, we obtain $R = 2D\Lambda/(D-2)$. Now there are 3 possibilities for maximally symmetric spacetimes

$R = 0,\ \Lambda = 0$	Minkowski space
$R > 0,\ \Lambda > 0$	de Sitter space
$R < 0,\ \Lambda < 0$	Anti-de Sitter space

[3] $\nabla_\mu V_\nu$ is defined as $\partial_\mu V_\nu - \Gamma^{\delta}_{\mu\nu} V_\delta$, when working on a vector.

In this thesis we will be interested in this last case (as the name AdS/CFT suggests). As always there are different choices for a coordinate system that would solve equation 3.3 with $\Lambda < 0$, a convenient choice in the context of AdS/CFT is the Poincaré patch:

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2, \tag{3.4}$$

or with $z = L^2/r$

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right). \tag{3.5}$$

Note that these coordinates do not cover the whole spacetime. A special property of this spacetime is that it has a boundary (at $r \to \infty$ or $z \to 0$). In the next section we will discuss conformal field theory and its relation to the boundary of the AdS spacetime.

3.1.2 Conformal field theory

Whole books have been written on the subject of conformal field theories. In this section only the essential parts will be mentioned. We will follow chapter 4 of the book "Conformal Field Theory"[6].

The conformal transformation is defined as a transformation between coordinates x and x', that leaves the metric tensor $g_{\mu\nu}(x)$ invariant up to a scale

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x). \tag{3.6}$$

$\Lambda(x)$ is known as the scale/conformal factor. The subgroup $\Lambda(x) = 1$ represents the Poincaré group (the symmetry group of special relativity). From expression 3.6, the possible forms of the conformal transformations can be derived (again see [6]). The results are the following possible transformations: [4]

Translations:	$x'^\mu = x^\mu + a^\mu$	(3.7)
Dilatations:	$x'^\mu = \alpha x^\mu$	(3.8)
Rigid rotations:	$x'^\mu = M^\mu_\nu x^\nu$	(3.9)
Special conformal transformations:	$x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}.$	(3.10)

Note here that α, a^μ, b^μ and M^μ_ν are all coordinate independent.

Now going back to the AdS spacetime, we can show that the boundary can indeed be viewed as conformal. Since the space is maximally symmetric translations and rotations are already a symmetry at every point. Dilatations are also a symmetry since changing $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, leaves the metric (equation 3.5) invariant. Introducing the concept of inversion, $I : x^\mu \to \frac{x^\mu}{x^2}$, we can show that the special conformal transformation is indeed a symmetry of the AdS boundary. First perfoming an inversion, then a translation and then another inversion:

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu - b^\mu}{x^2 - 2b\cdot x + b^2} \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}. \tag{3.11}$$

Now one can also show that the metric is invariant under the following transformation

$$x^\mu \to \frac{x^\mu}{z^2 + \eta_{\mu\nu}x^\mu x^\nu},\ z \to \frac{z}{z^2 + \eta_{\mu\nu}x^\mu x^\nu}, \tag{3.12}$$

which as $z \to 0$ becomes an inversion. Thus we can conclude that the boundary of AdS is invariant under all the possible conformal transformations as mentioned above.

[4] For $d \geq 3$, see [6] for more details.

3.2 GKPW rule

Stating that an AdS boundary corresponds to a CFT is already interesting, but not that useful. Luckily there is more! The duality between the gravity and the quantum theory goes a bit further. The Gubser–Klebanov–Polyakov–Witten (GKPW) rule is the mathematical equation that directly shows the correspondence between the quantum and the gravity theories. This rule can be applied if the spacetime is assymptotically AdS, meaning that as $r \to \infty$, the spacetime reduces to an AdS spacetime. The rule is still conjectural and its support comes from string theory. Therefore the rule will be stated and only slightly motivated as it will be used as a mathematical working tool in this thesis. The GKPW rule relating the QFT to the Gravity can be written as

$$\left\langle e^{\int d^{d+1}x J(x)\mathcal{O}(x)} \right\rangle_{\text{QFT}} = \int \mathcal{D}\phi \; e^{iS_{\text{bulk}}(\phi(x,r))|_{\phi(x,r=\infty)=J(x)}}. \tag{3.13}$$

In QFT physical observables can be calculated starting with calculating correlators, n-point functions or Green's functions. Normally this is done by altering the Lagrangian and taking functional derivatives

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle = \prod_{i=1}^{n} \frac{\delta}{\delta J_i(x_i)} \ln Z_{\text{QFT}}\Bigg|_{J=0} = \prod_{i=1}^{n} \frac{\delta}{\delta J_i(x_i)} \ln Z_{\text{Grav}}\Bigg|_{J=0}. \tag{3.14}$$

Since there is a straightforward connection between the gravity and quantum side we can just replace Z_{QFT} with Z_{Grav}, where we defined

$$Z_{QFT} = \left\langle e^{\int d^{d+1}x J(x)\mathcal{O}(x)} \right\rangle_{\text{QFT}}, \tag{3.15}$$

$$Z_{Grav} = \int \mathcal{D}\phi \; e^{iS_{\text{bulk}}(\phi(x,r))|_{\phi(x,r=\infty)=J(x)}}. \tag{3.16}$$

Now we can directly observe how n-point functions of a QFT can be calculated by performing calculations on the gravity side. But there is a catch, as can be see in equation 3.16, one has to do a path integral on the gravity side. In other words performing quantum gravity. This is notably hard to say the least. In a certain limit, the large N-limit, the path integral disappears and only the most dominant term is left. This limit can also be seen as the classical limit and fortunately the classical limit of quantum gravity is known, this is general relativity. The large N-limit and its relation to condensed matter physics will be explained in section 3.5. How to encode the sources on the gravity side will be explained by means of an example in section 3.2.2.

3.2.1 CFT correlators

When a QFT is conformal, a certain form of the correlators in that QFT can be derived. Following section 4.3 of [6], we will shortly show how this works. First performing a general coordinate transformation (for a 2-point function, also note that x and x' can be vectors)

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \mathcal{O}_1(x_1')\mathcal{O}_2(x_2')\rangle, \tag{3.17}$$

and then performing a scale transformation on $x \to \lambda x$ we find

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\rangle. \tag{3.18}$$

Due to rotational and translation invariance we also find that $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ is of the form $f(|x_1 - x_2|)$, where $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$. Thus a 2-point function of a correlator in a CFT is obliged to be of the form of

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{3.19}$$

where C_{12} is a constant coefficient. By performing the special conformal transformation we also find that the 2-point function is only non-zero when the so called scaling or conformal dimensions Δ_1 and Δ_2 are equal[5]. General expressions for higher order (CFT) n-point functions are also known, but we will only be interested in the 2-point function or the Green's function in this thesis.

Knowing the conformal dimension of a field theory can be used to fix certain parameters of the AdS theory. An example of this can be found in [16], where the mass of a scalar field in the bulk can be fixed in order to obtain the correct conformal dimension of the correlator on the boundary.

3.2.2 Scalar field example

One of the most simple examples of an application of this rule can be found in [8, 21]. Or in other master thesis that have earlier been written on this topic like (see: https://web.science.uu.nl/ITF/Teaching/Master'sTheses.htm)

- Daan Peerlings, Holographic nodal surfaces
- Joren Harms, Thermoelectric transport in strange metals
- Enea Mauri, Fluctuations in the holographic superconductor

3.3 Fermions from holography

This part is based on [10]. In this thesis we are interested in calculating the Green's function for a fermionic system. In order to this we will also need a holographic encoding for fermions. First we will discuss the GKPW rule and then proceed with how to treat fermions in curved spacetime.

3.3.1 GKPW rule for fermions

The GKPW rule for fermions can be stated as

$$\left\langle \exp\left[\int d^d x \left(\bar{\chi}_0 \mathcal{O} + \bar{O}\chi_0\right)\right]\right\rangle_{\text{QFT}} = e^{-S_{\text{grav}}[\chi_0,\bar{\chi}_0]}, \tag{3.20}$$

note that here $iS_{grav} \to -S_{grav}$, this is the difference in Euclidean description where $t \to -i\tau$. This is the Euclidean convention and some properties are easier to calculate in this manner. Here χ_0 and $\mathcal{O}$ are spinors. The main difference in the standard GKPW rule comes from the gamma matrices that are relevant for fermions. In the expression above, only the zero or time gamma matrix is visible by $\bar{\chi} = \chi^\dagger\gamma^t$. The importance of the other gamma matrices will become clear the next few sections. In the context of condensed matter we are interested in the fermionic Green's function which is proportional to $G_R \sim \langle \mathcal{O}\mathcal{O}^\dagger \rangle$. Later we will see that at the boundary the action can be written as

$$S_{grav} \sim \int d^d x \, \bar{\chi}_0 S \chi_0, \tag{3.21}$$

here S is a matrix that relates the source and the operator, by $\langle \mathcal{O} \rangle = S\chi_0$. Taking functional derivatives of equation 3.21, we can find $\langle \mathcal{O}\bar{\mathcal{O}} \rangle \sim S$, and thus $\langle \mathcal{O}\mathcal{O}^\dagger \rangle \sim S\gamma^t$.

Thus to calculate the Green's function, an expression for the matrix S has to be derived. A set of differential equations for a more general matrix, called ξ in this thesis, can be derived which is valid in the whole bulk. The relation between these matrices is proportional to $S \sim \lim_{r\to\infty} \xi$, a normalization is also needed which is related to the conformal dimension of the operator. The details on this will become clear in section 4.1.3.

To calculate the matrix ξ, a description for fermions/spinors in curved spacetime is necessary.

[5] See [6] for the full derivation.

3.3.2 Dirac equation in curved spacetime

In this section we will first give a short description of how fermions are treated in flat spacetime, more on this can be read in chapter 10 of [18]. We will proceed with the treatment in curved spacetime, more on this can be read in [8, 21]. The equation for describing free spin-1/2 particles, fermions, is known as the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \quad \text{or} \quad (\partial\!\!\!/ - m)\psi = 0. \tag{3.22}$$

Here $A\!\!\!/ = \gamma^\mu A_\mu$, the defining property of the Dirac equation is that the gamma/Dirac matrices, γ^μ, anti commute as

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \pm 2\eta^{\mu\nu}. \tag{3.23}$$

The sign of the equation above depends on which version of equation 3.22 is used and on the convention of $\eta^{\mu\nu}$. In this thesis we will work with the metric sign convention of $\eta^{\mu\nu} = diag(-,+,+,...)$ and with the Dirac equation without the i in front of the covariant derivative. This will lead to a plus sign in equation 3.23. Fermions may also interact, the interaction of fermions with charge q, with the electromagnetic gauge field can be added yielding

$$(\partial\!\!\!/ - iqA\!\!\!/ - m)\psi = 0. \tag{3.24}$$

The Dirac equation in curved spacetime is given by

$$(\nabla\!\!\!\!/ - iqA\!\!\!/ - M)\psi = 0, \tag{3.25}$$

here, $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\underline{ab}\mu}\Gamma^{\underline{ab}}$ and $X\!\!\!/ = \Gamma^\mu X_\mu$, where Γ^μ are now the gamma matrices in curved spacetime. For the gamma matrices in curved spacetime it holds that $\Gamma^\mu = e^\mu_{\underline{a}}\Gamma^{\underline{a}}$, here the underlined indices represent the gamma matrices in flat space time. The definition of the vielbeins is given by

$$e^\mu_{\underline{a}} e^\nu_{\underline{b}} \eta^{\underline{ab}} = g^{\mu\nu}. \tag{3.26}$$

Now we can easily calculate how the Dirac matrices in curved spacetime anti commute

$$\{\Gamma^\mu, \Gamma^\nu\} = e^\mu_{\underline{a}} e^\nu_{\underline{b}} \{\Gamma^{\underline{a}}, \Gamma^{\underline{b}}\} = 2 e^\mu_{\underline{a}} e^\nu_{\underline{b}} \eta^{\underline{ab}} = 2g^{\mu\nu}. \tag{3.27}$$

The Dirac equation in curved spacetime is invariant under both local Lorentz transformation and under the rules of general relativity. The spin connection is the property that make sure that the Dirac equation in curved spacetime is indeed invariant under global coordinate transformation. We will not go into detail in this thesis, since we will find that we do not even need the spin connection. We do however need the vielbeins to describe the fermions in curved spacetime. More on the topic of fermions in curved spacetime can be found in for example [8]. And for specific examples in the context of AdS/CMT: [16, 13] and many more papers.

3.4 Holographic thermodynamics

The goal is to describe a fermionic many-body quantum theory. Therefore we need to explore how to describe the fermions and the thermodynamics. In this section we will describe the relevant thermodynamics for this system.

3.4.1 Temperature

A remarkable result of holography is how the temperature of the QFT is directly related to the temperature of a black hole in the bulk. In this section we will follow chapter 6 of [21]. In QFT the temperature is related to the periodicity of the field in Euclidean time ($\tau = it$). Where explicitly the field is periodic in $\hbar/k_BT$, in natural units ($\hbar = k_B = 1$), the periodicity is simply $1/T$. k_B is Boltzmann's constant. Transforming to Euclidean time can be problematic, but in the case of stationary space-times this transformation can be done without problems. Stationary space-times are dual to equilibrium configurations on the boundary in which we are interested. To get a temperature from the geometric bulk theory, we will use that at the horizon of a black hole the metric should be smooth. We will start with a general metric of a theory with a horizon

$$ds^2 = -g_{\tau\tau}(r)dt^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)d\vec{x}^2. \tag{3.28}$$

Transforming to Euclidean time, $t = -i\tau$, yields

$$ds_E^2 = g_{\tau\tau}(r)d\tau^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)d\vec{x}^2. \tag{3.29}$$

At the horizon, $r = r_0$, $g_{\tau\tau}(r_0) = g^{rr}(r_0) = 0$, thus expanding the metric around the horizon gives up to first order

$$ds_E^2 = g'_{\tau\tau}(r_0)(r - r_0)d\tau^2 + \frac{dr^2}{g^{rr\prime}(r_0)(r - r_0)} + g_{xx}(r_0)d\vec{x}^2. \tag{3.30}$$

By defining $R_0 = 2\sqrt{r - r_0}/\sqrt{g^{rr\prime}(r_0)}$ and $\theta = \sqrt{g'_{\tau\tau}g^{rr\prime}(r_0)}\tau/2$ we get a simpler metric

$$ds_E^2 = R_0^2 d\theta^2 + dR_0^2 + g_{xx}(r_0)d\vec{x}^2. \tag{3.31}$$

Smoothness at $R_0 = 0$, the horizon, implies that this point can be viewed as the center of a polar coordinate system with coordinates θ and R_0. In polar coordinates θ is periodic with 2π, knowing that τ is periodic in $1/T$. we must thus solve $\theta(\tau) + 2\pi = \theta(\tau + 1/T)$ to get an expression for the temperature, this yields

$$T = \frac{\sqrt{g^{rr\prime}(r_0)g'_{\tau\tau}(r_0)}}{4\pi}. \tag{3.32}$$

The result that black holes have a temperature was theorized by Stephen Hawking in 1974 [9].

3.4.2 Entropy

Hawking and Bekenstein also showed that black holes have entropy. And they explicitly showed that the entropy of a black hole is proportional to its area. This is in stride with naive thermodynamics. Because in naive thermodynamics temperature should scale with the volume. This result was one of the first hints towards theorizing the holographic principle. Entropy is the logarithm of the amount of micro-states of the black hole. We can therefore see that it seems to be that all degrees of freedom of a black hole. i.e. the micro-states are thought to be represented on the boundary of the black hole.

3.4.3 Chemical potential

The one thermodynamic quantity that is left to describe is the chemical potential. We will denote the chemical potential of the boundary theory by μ. The chemical potential on the boundary can be described by a temporal gauge field (A_t) in the gravity theory. This can be understood by the idea that local symmetries in the gravity theory correspond to global symmetries on the boundary quantum theory.

3.4.4 Resistivity and Viscosity

From condensed matter physics it is known that resistivity and viscosity are linear with respect to each other. From earlier holographic work it is known that viscosity is linear with respect to entropy. In the next section we will show that for a certain metric, also temperature and entropy are linear with respect to each other. This is the condition that is needed to describe strange metals, since we need that resistivity is proportional to temperature.

3.5 Large N-limit and condensed matter

As mentioned earlier, the GKPW rule only works in the large N-limit. In the original work, this N was the N in a SU(N) theory. Condensed matter is clearly not an SU(N) theory, but the limit can still be used. In condensed matter the N roughly corresponds to the amount of degrees of freedom of a system. A derivation of the original work and how this works in condensed matter can be found in [21].

4 Holographic description of strange metals

In this section we will analytically calculate the differential equations for the ξ matrix. By means of an Einstein-Maxwell-Dilaton (EMD) background we enrich the fermionic theory with the z, θ parameters.

4.1 EMD Background

In this section we will calculate the background equation for the EMD action. This action is given (in a 4 dimensional bulk theory) by [21]

$$S = \int \mathrm{d}^4x\sqrt{-g}\big(R - \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{Z(\phi)}{4}F^2 - V(\phi)\big) = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}, \tag{4.1}$$

here $Z(\phi) = e^{\phi/\sqrt{3}}$ and $V(\phi) = -6\, Cosh(\phi/\sqrt{3})$. We can recognize the gravity and electromagnetic parts of the action by R and $F^2 = F^{\mu\nu}F_{\mu\nu}$. With these parts we can already describe some thermodynamics by means of temperature and chemical potential. The scalar dilaton (ϕ) field provides us with the possibility to alter the $\mathbf{z}$ and θ parameters.

4.1.1 Hyperscaling violating metrics

Obtaining the correct z and θ on the boundary can be done by introducing so called hyperscaling violating metrics. In this section we will look into how this works for both the boundary and the bulk theory.

Setting $\phi = 0$, the action reduces to the action of just gravity and electromagnetism. The solution to that action is well known as the Reissner–Nordström (RN) black hole. The RN black hole already possessed the correct $\mathbf{z} \to \infty$ parameter. Calculating the Green's function for both spinors and scalars in that background has already extensively been done, a review can be found in [5]. There is even a claim that this can already produce the correct strange metallic behaviour [4].

There is however a problem with the RN black hole. At zero temperature the black hole still has a horizon. The horizon/area of the black hole is proportional to the entropy of the black hole. The problem is that this implies that at zero temperature there is still some entropy left. Holographically this thus also occurs at the fermionic boundary theory. A finite entropy at zero temperature in condensed matter implies that the ground state of the system is degenerate[6]. A degenerate ground state is only possible in a few extraordinary cases, but generally goes against the laws of thermodynamics [15].

The EMD black hole solution can solve this problem. The parameters $\mathbf{z}$ and θ will provide this solution, these parameters can be introduced (for d=2) as

$$\theta = \frac{4\beta}{\alpha + \beta}, \quad \mathbf{z} = 1 + \frac{\theta}{2} + \frac{(2(2-\theta)+\theta)^2}{2(2-\theta)\alpha^2}, \tag{4.2}$$

where α and β are characterized by the infrared behaviour[7] of $V(\phi) = -V_0 e^{-\beta\phi}$ and $Z(\phi) = Z_0 e^{\alpha\phi}$. By analyzing the initial forms of $Z(\phi)$ and $V(\phi)$, we can easily calculate that $\alpha = -\beta = 1/\sqrt{3}$. Since $\phi \to \infty$

[6] Since $S \propto ln$(microstates), thus S is only 0 when there is only one (non-degenerate) ground state.
[7] There is a minus mistake in the book: "HOLOGRAPHIC DUALITY IN CONDENSED MATTER PHYSICS" [21].

when $r \to r_0$, where r_0 is the position of the black hole horizon. Pluggin α and β in equation 4.2 we see that for this specific background we have $\theta \to -\infty$ and $\mathbf{z} \to \infty$ with $\eta = -\theta/\mathbf{z} = 1$.

The meaning of these two parameters will now be explained. $\mathbf{z}$, the dynamical critical exponent, corresponds to how much stronger the correlation functions depend on ω versus k. In terms of the so called local IR Green's function

$$\mathcal{G}(\omega, k) \propto \left(-\omega^{2\mathbf{z}} + k^2\right)^{\Delta}. \tag{4.3}$$

A conformal theory in which time and space scale the same way, $\mathbf{z} = 1$. When $z \to \infty$, this IR Green's function only scales with ω as

$$\mathcal{G}(\omega, k) \propto \omega^{\Delta'}, \tag{4.4}$$

this IR Green's function is directly proportional to the imaginary part of the self energy. And the imaginary part of the self energy is again directly proportional the width of the peaks of the spectral function. Thus knowing the IR scaling function enables us to understand the experiment. Since the experimental results from section 2 suggest that the imaginary part of the self energy only scales with energy (ω), we indeed need a $z \to \infty$ to describe this.

Another requirement from the experiment is that specifically for the strange metal, the resistivity should scale linearly with the temperature. As explained in section 3.4 resisitivity scales with the entropy. Thus we need that in the IR region of the bulk $T \propto S$. Hyperscaling can be understood as "The scaling dimension of an operator predicted by its "physical" dimension is called the naive scaling dimension, the canonical scaling dimension, or, for obscure reasons, the engineering dimension."[1]. When thermodynamic quantities scale differently this is called hyperscaling violating. And the dual gravity theories are therefore known as hyperscaling violating metrics. The hyperscaling/naive scaling dimensions of the thermodynamic quantities S and T is d, the amount of space dimensions by $S \sim T^d$. The anomalous scaling dimension θ can be seen as a correction to the naive dimension by $d \to d - \theta$, due to $\mathbf{z}$ there is another correction to this and we obtain

$$S \sim T^{(d-\theta)/\mathbf{z}}. \tag{4.5}$$

For the EMD background used in this thesis we can now see that $S \sim T^{\eta} = T$. A result from holography is that the viscosity of a system holographically corresponds to the entropy [11]. From a condensed matter point of view it is known that viscosity scales linearly with the resisitivity. This can be roughly understood as that is more difficult for electrons to move around if the viscosity is higher and therefore viscosity (δ) is linearly dependent on the resisitivity (ρ). Now we can draw the conclusion that $\rho \sim \delta \sim S \sim T$. In other words $\rho \sim T$, which is the desired result, since this is observed experimentally. Note that this also solves the degenarate groundstate issue of RN black hole, since the EMD background has $S = 0$ at $T = 0$. The next sections will focus on the explicit calculations of the Green's function for the fermions.

4.1.2 Background dynamics

The equations of motion can be found by varying the action with respect to ϕ, A_t and $g^{\mu\nu}$.

$$\delta S = \int \mathrm{d}^4 x \delta(\sqrt{-g})\mathcal{L} + \int \mathrm{d}^4 x \sqrt{-g}\delta\mathcal{L}, \tag{4.6}$$

using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, the equations of motion after varying with respect to the metric are

$$-\frac{1}{2}g_{\mu\nu}\mathcal{L} + R_{\mu\nu} - \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{Z(\phi)}{2}F_{\mu\tau}F_\nu^{\ \tau} = 0. \tag{4.7}$$

The other two equations of motion (coming from varying w.r.t. ϕ and A_t) are

$$\partial_r(\sqrt{-g}Z(\phi)F^{rt}) = 0, \tag{4.8}$$

$$\partial_\nu(\sqrt{-g}\partial^\nu\phi) + \sqrt{-g}\frac{dZ(\phi)}{d\phi}\frac{F^2}{4} + \sqrt{-g}\frac{dV(\phi)}{d\phi} = 0. \tag{4.9}$$

The equations of motion for this action are

$$f\phi'' + 4f\alpha'\phi' + f'\phi' + 2\sqrt{3}e^{-2\alpha}Sinh(\frac{\phi}{\sqrt{3}}) + \frac{4f'\alpha' + f''}{2\sqrt{3}} = 0, \tag{4.10}$$

$$A_t'' + \frac{A_t'\phi'}{\sqrt{3}} + 2A_t'\alpha' = 0, \tag{4.11}$$

$$f'' + 4f'\alpha' - e^{-2\alpha + \frac{\phi}{\sqrt{3}}}(A_t')^2 = 0, \tag{4.12}$$

$$4\alpha'' + (\phi')^2 + 4(\alpha')^2 = 0, \tag{4.13}$$

here the first and second equation come from varying w.r.t. ϕ and A_t respectively. The other two equation come from linear combinations of the rr, tt and xx components of equation 4.7. Solutions to these equations are

$$A_t = \frac{\mu}{q}(1 - \frac{r_0}{r}), \tag{4.14}$$

$$\phi = \frac{\sqrt{3}}{2} ln(\frac{r}{r - \frac{(\mu/q)^2}{3r_0}}), \tag{4.15}$$

$$\alpha = ln(r - \frac{(\mu/q)^2}{3r_0}) + \frac{3}{4} ln(\frac{r}{r - \frac{(\mu/q)^2}{3r_0}}), \tag{4.16}$$

$$f = 1 - (\frac{r_0}{r})^3. \tag{4.17}$$

The ansatz for the metric is given by

$$ds^2 = e^{2\alpha}(-fdt^2 + dx^2 + dy^2) + \frac{e^{-2\alpha}}{f}dr^2. \tag{4.18}$$

Note that we can also calculate the temperature for this metric using equation 3.32, which yields

$$T_{EMD} = \frac{(fe^{2\alpha})'}{4\pi}\Big|_{r=r_0} = \frac{\sqrt{9r_0^2 - 3(\frac{\mu}{q})^2}}{4\pi}. \tag{4.19}$$

There is a symmetry in the action, equation 4.1 and thus also in the equations of motion. This symmetry is given by

$$r \to ar, \quad (t, \mathbf{x}) \to (t, \mathbf{x})/a, \quad e^\alpha \to ae^\alpha, \quad A_t \to aA_t, \tag{4.20}$$

this symmetry can be used to fix either r_0 or $\mu' = \frac{\mu}{q}$. Note that the metric only depends on μ', different conventions are known and most of the times μ is used instead of μ'. The reason why we choose to use μ/q in the background is to make sure that the S.I. units are correct as can be seen in appendix 7.1 and the next section.

4.1.3 EMD fermions

Now it is time to put fermions on top of the background. This means that the fermions do not influence the background, but merely propagate in this background. Physically this means that the fermions themselves do not influence the thermodynamics in the way of curving the background. The fermions however still do have 2 degrees of freedom, the mass and the charge. Later these 2 degrees will be used numerically to explore the different kind of behaviour The vielbeins for this metric are

$$e^t_{\underline{t}} = \sqrt{-g^{tt}} = \frac{e^{-\alpha}}{\sqrt{f}}, \quad e^i_{\underline{i}} = e^{-\alpha}, \quad e^r_{\underline{r}} = \sqrt{f}e^\alpha. \tag{4.21}$$

If the vielbeins/metric only depend on the r-coordinate, the covariant derivative can be replaced by the simpler partial derivative, yielding

$$(\partial\!\!\!/ - iqA\!\!\!/ - m)\phi = 0. \tag{4.22}$$

Here ϕ and ψ are related by $\phi = p(r)\psi$, where p(r) is some function only depending on r, this rescaling will not affect the matrices S and thus also not G_R that we would like to calculate. The exact form of $p(r) = (-gg^{rr})^{-1/4}$. After Fourier transforming $\phi(r,x)$ by

$$\phi(r,x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\phi(r,k)e^{ikx}, \tag{4.23}$$

we can write equation 4.22 as

$$(\Gamma^r\partial_r + ik\!\!\!/ - iq\Gamma^0 A_0 - m)\phi = 0, \tag{4.24}$$

where we used that A_μ only has a zero component. A convenient basis to work in is given by

$$\Gamma^{\underline{r}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^{\underline{0}} = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \Gamma^{\underline{1}} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \Gamma^{\underline{2}} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{4.25}$$

It can be checked that these matrices indeed obey the Dirac algebra given by equations 3.23 and 3.27[8]. The fermionic boundary theory is determined by 2-dimensional spinors. That boundary can be described with the 2-dimensional matrices γ^μ, which are exactly the components of matrices in equation 4.25. Thus $\gamma^0 = i\sigma_2$, $\gamma^1 = \sigma_1$ and $\gamma^2 = \sigma_3$. Now we choose $k^\mu = (\omega, k_i)$ and multiply with $\Gamma^{\underline{r}}$ and we find that equation 4.24 reduces to

$$(\sqrt{g^{rr}}\partial_r + i\Gamma^{\underline{r}}(\sqrt{g^{ii}}\Gamma^{\underline{i}}k_{\underline{i}} - \sqrt{-g^{tt}}(\omega + qA_t)\Gamma^{\underline{r}}\Gamma^{\underline{0}}) - m\Gamma^{\underline{r}})\phi = 0. \tag{4.26}$$

This can now be written using $\phi = \begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix}$ as

$$\sqrt{\frac{g_{ii}}{g_{rr}}}\left(\partial_r \mp m\sqrt{g_{rr}}\right)\phi_\pm = \mp i(-i\sigma_2 u + k\sigma_1)\phi_\mp, \tag{4.27}$$

where $u = \sqrt{\frac{g_{ii}}{-g_{tt}}}(\omega + qA_t)$ and we have set $k_1 = k$ and $k_2 = 0$, this can be done due to the rotational symmetry. At the boundary, when r goes to infinity, we can see that equation 4.27 behaves as

$$r^2(\partial_r \mp \frac{m}{r})\phi_\pm = \mp ik_\mu\gamma^\mu\phi_\mp. \tag{4.28}$$

Here $k_\mu = (-(\omega+\mu), k_i)$. Assuming that ϕ_+ is the dominant term (setting $\phi_- = 0$ at the boundary) and plugging in a simple ansatz of $\phi_+ \sim r^\lambda$, we can find that up to first order $\phi_+ \sim r^m$ at the boundary. The same can be done for ϕ_- yielding that it goes like $\phi_- \sim r^{-m}$ at the boundary. The second order term can be found by again an ansatz like $\phi_+ \sim r^\lambda$, but now taking $\phi_- \sim r^{-m}$. Which gives

$$r^2(\partial_r - \frac{m}{r})r^\lambda = (\lambda - m)r^{\lambda+1} \sim r^{-m}, \tag{4.29}$$

the non-trivial solution can be found by matching the powers, thus $\lambda = -m-1$. The same can be done to find the behaviour of ϕ_-. This leads to the following boundary behaviour for both $\phi_\pm$ of

$$\phi_+ = Ar^m + Br^{-m-1},\ \phi_- = Cr^{m-1} + Dr^{-m}. \tag{4.30}$$

Plugging this back into equation 4.28, we find

[8]The Pauli matrices are given by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$C = \frac{i\gamma^\mu k_\mu}{2m-1} A,\ B = \frac{i\gamma^\mu k_\mu}{2m+1} D. \tag{4.31}$$

Since ϕ_+ is in this case the dominant term we can identify the coefficient A as the source and D can be seen as the expectation value. We can assume that these two are related by $S = \frac{D}{A}$[9]. We can also switch the role of the source and the expectation value by switching from m to $-m$, these two quantization methods are allowed for $0 \leq m < \frac{1}{2}$.[10] Fully written out equation 4.27 looks like

$$\sqrt{\frac{g_{ii}}{g_{rr}}} \left(\partial_r \mp m\sqrt{g_{rr}}\right) \begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i \begin{pmatrix} \mp(k-u)z_\mp \\ \pm(k+u)y_\mp \end{pmatrix}, \tag{4.32}$$

here $\phi_\pm = \begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix}$. Introducing $\xi_+ = \frac{iy_-}{z_+}, \quad \xi_- = -\frac{iz_-}{y_+}$, the retarded Green's function can be written (using equation....) as

$$G_R = S\sigma_2 = \lim_{\epsilon \to 0} \epsilon^{-2m} \left(\begin{array}{cc} \xi_+ & 0 \\ 0 & \xi_- \end{array} \right) \Bigg|_{r=\frac{1}{\epsilon}}. \tag{4.33}$$

Here we should note that in the final expression for the Green's function only the finite terms are extracted. Here the equations for $\xi_\pm$ can be derived from equation 4.32 yielding

$$\sqrt{\frac{g_{ii}}{g_{rr}}} \partial_r \xi_\pm = -2m\sqrt{g_{ii}}\xi_\pm \mp (k \mp u) \pm (k \pm u)\, \xi_\pm^2. \tag{4.34}$$

4.1.4 In-falling boundary condition.

In this section we will calculate the boundary condition for $T \neq 0$ and $w \neq 0$. Now we are left with solving two first-order differential equations. Since the differential is of first order, one boundary condition is needed in order to fully specify the solution. The boundary condition is typically given by expanding equation 4.34 around the horizon of the black hole (at $r = r_0$). Following the PHD thesis of Vivian Jacobs[11]. With the form of our metric, equation 4.32 can be written as

$$\sqrt{e^{4\alpha} f(r)} \left(\partial_r \mp m\sqrt{\frac{e^{-2\alpha}}{f(r)}}\right) \begin{pmatrix} y_\pm \\ z_\mp \end{pmatrix} = i \begin{pmatrix} \mp(k - \sqrt{\frac{1}{f(r)}}(\omega + qA_t))z_\mp \\ \pm(k + \sqrt{\frac{1}{f(r)}}(\omega + qA_t))y_\pm \end{pmatrix}, \tag{4.35}$$

Since the boundary condition is at the horizon we can write $\phi_\pm$ as a power expansion in terms of $(r - r_0)$. This power expansion can be done in terms of $V = \sqrt{f(r)}$

$$y_\pm = v_\pm V^{\alpha_\pm}(r), \quad z_\pm = w_\pm V^{\beta_\pm}(r). \tag{4.36}$$

Thus we can write equation 4.35 as

$$\frac{e^{2\alpha}}{2\sqrt{f(r)}} f'(r) \begin{pmatrix} \alpha_\pm y_\pm \\ \beta_\mp z_\mp \end{pmatrix} \mp m\sqrt{e^{2\alpha}} \begin{pmatrix} y_\pm \\ z_\mp \end{pmatrix} = i \begin{pmatrix} \mp(k - \sqrt{\frac{1}{f(r)}}(\omega + qA_t))z_\mp \\ \pm(k + \sqrt{\frac{1}{f(r)}}(\omega + qA_t))y_\pm \end{pmatrix}. \tag{4.37}$$

Multiplying this equation by $\sqrt{f(r)}$ and then letting $r \to r_0$ we are left with

$$e^{2\alpha} f'(r_0) \begin{pmatrix} \alpha_\pm y_\pm \\ \beta_\mp z_\mp \end{pmatrix} = \pm 2i \begin{pmatrix} \omega z_\mp \\ \omega y_\pm \end{pmatrix}, \tag{4.38}$$

[9]Here A, D are still spinors, since $\phi_\pm$ are also spinors.
[10]The reason for this is that only for this range of m, A and D indeed correspond to the dominant terms in equation 4.30.
[11]See the UU archive for PHD thesis: http://dspace.library.uu.nl/handle/1874/317823.

here we used that $f(r_0) = 0$ and $A_t(r_0) = 0$. The non-trivial solution is the one where the powers match, such that $\alpha_\pm = \beta_\mp$. Putting this back in leads to the following two equations

$$\begin{pmatrix} \alpha_\pm \\ \beta_\mp \end{pmatrix} = \frac{\pm 2i\omega}{e^{2\alpha} f'(r_0)} \begin{pmatrix} w_\mp / v_\pm \\ v_\pm / w_\mp \end{pmatrix}. \tag{4.39}$$

Using that $\alpha_\pm = \beta_\mp$, we can write $\alpha_\pm^2 = \alpha_\pm \beta_\mp$, finding

$$\left(\frac{w_-}{v_+} \right)^2 = 1, \quad \left(\frac{w_+}{v_-} \right)^2 = 1. \tag{4.40}$$

Thus $w_-/v_+ = \gamma_1$ and $w_+/v_- = \gamma_2$, where $\gamma_{1,2} = \pm 1$. Putting this back in equation 4.39, we obtain

$$\begin{pmatrix} \alpha_+ \\ \alpha_- \\ \beta_+ \\ \beta_- \end{pmatrix} = \frac{2i\omega}{e^{2\alpha} f'(r_0)} \begin{pmatrix} \gamma_1 \\ -\gamma_2 \\ \gamma_1 \\ -\gamma_2 \end{pmatrix}, \tag{4.41}$$

now we can set $\gamma_1 = -1$ and $\gamma_2 = 1$. Plugging the powers α and β back into equation 4.36 we can find that $\phi_\pm \sim (r - r_0)^{-2i\omega/e^{2\alpha} f'(r_0)}$.[12] Plugging this back into equation 4.23 we find

$$\phi(r, x) \sim \int \frac{\mathrm{d}^3 k}{(2\pi)^3} (r - r_0)^{-2i\omega/e^{2\alpha} f'(r_0)} e^{-i\omega t + i k_i x^i} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i(\omega + \tilde{r})t + i k_i x^i}, \tag{4.42}$$

here $\tilde{r} = 2ln(r - r_0)/e^{2\alpha} f'(r_0)$, which is a growing function for positive r and this implies the equation above can be seen as an in-falling wave at the horizon. The in-falling boundary condition for $\xi_\pm$ can then easily be calculated using its definition and is given by

$$\xi_\pm|_{r=r_0} = i. \tag{4.43}$$

4.2 Spectral function and self energy

We will mostly be interested in the spectral function, which is basically the imaginary part of the Green's function [16]

$$\rho(\omega, k) = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} G_R(\omega, k). \tag{4.44}$$

At zero temperature and zero energy there is a predicted delta peak in the spectral function. The $k = k_F$ position where this occurs is known as the the Fermi surface. Around k_F, the Green's function can be written as [4]

$$G_R(\omega, k) = \frac{Z}{-\omega + v_F (k - k_F) - \Sigma(\omega, k)}, \tag{4.45}$$

where Z and v_F are constants and $\Sigma(\omega, k)$ is the renormalized self energy. Separating the real (Σ') and imaginary (Σ'') part by defining: $\Sigma \equiv \Sigma' + i\Sigma''$, we can write[13]

$$\rho(\omega, k) = \frac{Z}{\pi} \frac{\Sigma''(\omega, k)}{(-\omega + v_F(k - k_F) - \Sigma'(\omega, k))^2 + \Sigma''(\omega, k)^2}. \tag{4.46}$$

[12] In this derivation we have used that $w_\pm$ and $v_\pm$ are real and that $e^{2\alpha} f'(r_0)$ is positive.

[13] Here we use $\frac{1}{a+bi} = \frac{a-bi}{a^2-b^2}$, and thus we find $\operatorname{Im} \frac{1}{a+bi} = \frac{-b}{a^2-b^2}$.

4.3 The dictionary

The known rules about the duality between classical gravity and strong quantum field theory are written in the dictionary. Thus combining everything that is mentioned in this thesis. We can write down exactly what this boils down to in the dictionary.

weak gravity theory	strong quantum theory
Black hole temperature (T)	Temperature (T)
Black hole entropy (S)	Entropy (S)
Time component of the electromagnetic field (A_t)	Chemical potential (μ)
Number of space dimensions ($d = 3$)	Number of space dimensions ($d - 1 = 2$)
Number of time dimensions (1)	Number of time dimensions (1)
$\theta \to -\infty$, $\mathbf{z} =\to \infty$ with $\eta = -\theta/\mathbf{z} = 1$	Entropy scales linearly with temperature ($S \propto T$, also same θ and $\mathbf{z}$)
Fermions	Fermionic operators (In this thesis: Green's function, thus also the spectral function)

4.4 Numerical calculation

Now we finally have all the ingredients to calculate the fermionic Green's function. For clarity, we are left with solving

$$\sqrt{\frac{g_{ii}}{g_{rr}}}\partial_r \xi_\pm = -2m\sqrt{g_{ii}}\xi_\pm \mp (k \mp u) \pm (k \pm u)\,\xi_\pm^2,$$
$$\xi_\pm|_{r=r_0} = i.$$

These two differential equations depend on the parameters q, m, and μ which determine the physical behaviour. These equations also depend on k and ω, which can be varied to study the physical behaviour at different energy and momenta.

In this thesis we are interested in the phenomenological behaviour of fermions close to the Fermi surface at fixed chemical potential. This behaviour is characterized by the scaling of the imaginary self energy with temperature and energy. Recent experimental work can be found in [17]. The energy can be varied by studying the behaviour at different ω. The temperature behaviour is characterized by μ' and r_0, since $T_{EMD} = \frac{\sqrt{9r_0^2 - 3\mu'^2}}{4\pi}$.

Now we can also use the scale symmetry in the equations of motion, equation 4.20, to fix either r_0 or μ'. Physically fixing r_0 corresponds to a fixed horizon radius and thus a fixed temperature. To fix μ' implies fixing the chemical potential. It is however numerically easier to fix r_0. If $r_0 = 1$ is fixed, the differential equations for ξ can be solved in the whole bulk for a fixed temperature. Because of the symmetry in the equations of motion the result of fixed T can be scaled to a solution with fixed μ' by introducing the scale invariant quantities: T/μ', ω/μ', k/μ'. We are interested in a fixed chemical potential, since this corresponds to a fixed doping of a certain metal.

We should solve the equations in the whole bulk, from $r = r_0 = 1$ up till $r = \infty$. It is numerically more convenient to solve the equation with the coordinate $z = 1/r$, since we can then solve the equation from $z = 1$ to $z = 0$. Solving the equations exactly in this way will lead to numerical problems since we divide by 0 then. So we actually solve the equations from $z = 1 - \delta$ until $z = \epsilon$. Here δ and ϵ are chosen small. When doing the calculations, convergence tests are always done. For small ω/μ' and T/μ', there are poles in the differential equations. Therefore at those cases ϵ and δ have to be chosen much smaller than when performing the calculations at high ω/μ' and T/μ'. Note that we will also always work at nonzero but small $T/\mu', \omega/\mu'$.

5 Results

5.1 Default case: $q = 1$, $m = -1/4$ and $\omega \approx 0$.

By default we will work with $q = 1$ and $m = -1/4$, since it was found that the desired predicted Fermi surfaces appeared in alternative quantization, i.e. when the mass is negative[14]. The reason for this is not researched, but this effect also occurs in other cases like [7, 16].

Now we present some results for both the normalized $\tilde{\xi}_{\pm} = \lim_{r\to\infty} r^{-2m}\xi_{\pm}$ and $\rho = \mathrm{Im}(\tilde{\xi}_{+} + \tilde{\xi}_{-})$. We will omit the tildes in this section, thus we write ξ again ($\tilde{\xi} \to \xi$).

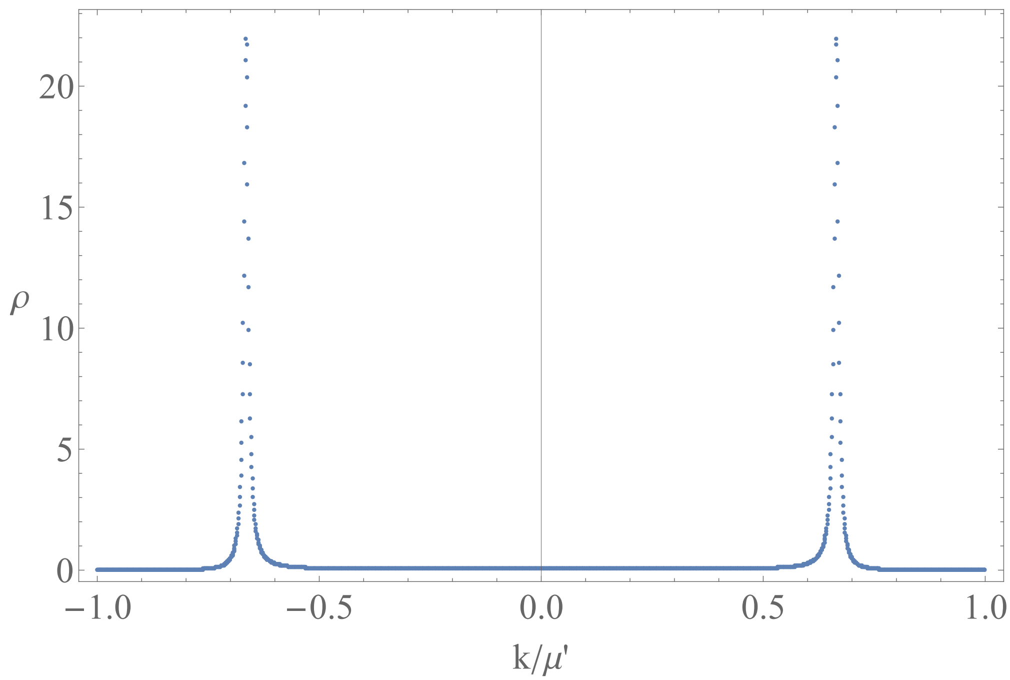

Figure 3: The spectral function at $T/\mu' = 0.05$. The positions of the peaks determine the Fermi surface. Since the system is still rotational invariant in the k_x, k_y-plane.

[14] $| m |< 1/2$

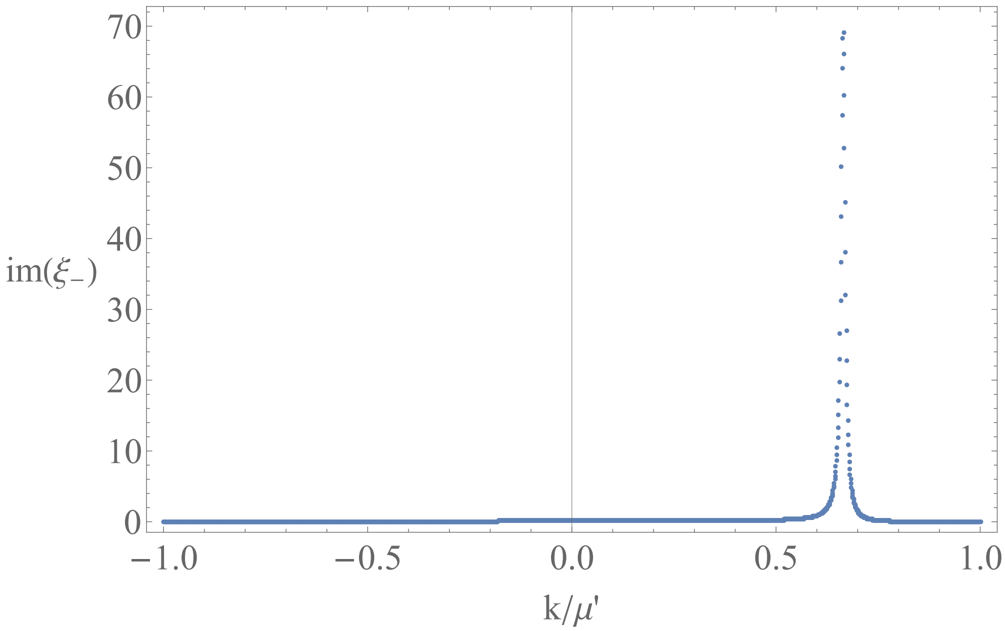

Figure 4: Imaginary part of ξ_- at $T/\mu' = 0.05$.

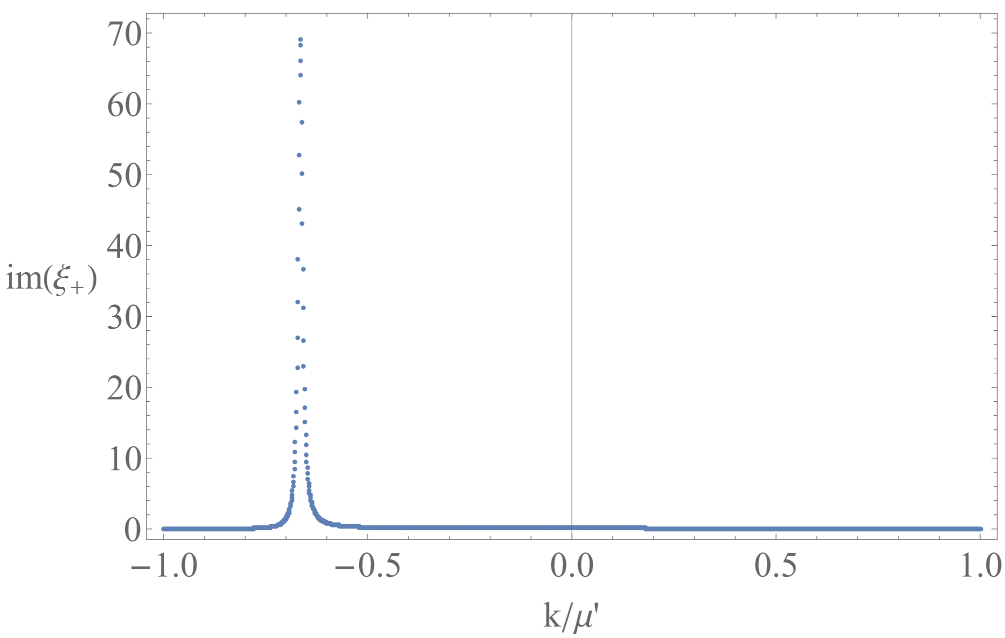

Figure 5: Imaginary part of ξ_+ at $T/\mu' = 0.05$.

As can be seen in the figures above, there is clearly a Fermi surface appearing around $k_F/\mu' \approx 0.67$. It also seems to be that $\xi_-(k) = \xi_+(-k)$. From the differential equations for $\xi_\pm$ (4.34), this symmetry can also directly be observed. Therefore we can just focus on either one of $\xi_\pm$. We choose to look at the positive momentum (k) side, thus at ξ_-. The one thing we are most interested in is the scaling of the width of these peaks as function of increasing T and increasing (negative) ω as is observed in experiments. Since the scaling is independent of the prefactor, we can just focus on the imaginary part of ξ_- (instead of ρ). Now we will look closer at ξ_- and fit the peaks with the proposed form of equation 4.46.

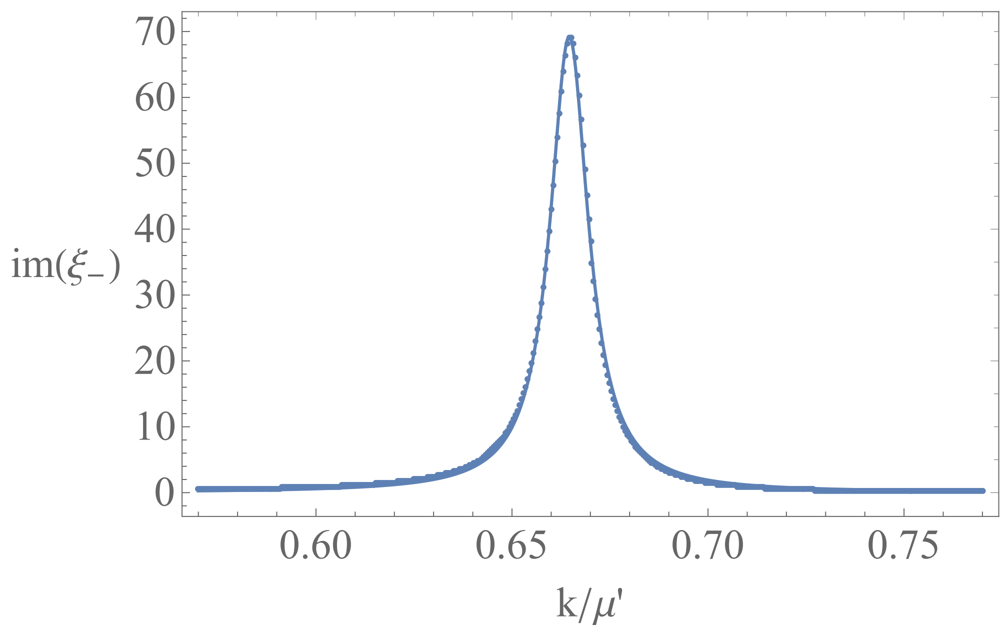

Figure 6: Imaginary part of ξ_- with a lorentzian fit according to equation 4.46 at $T/\mu' = 0.05$.

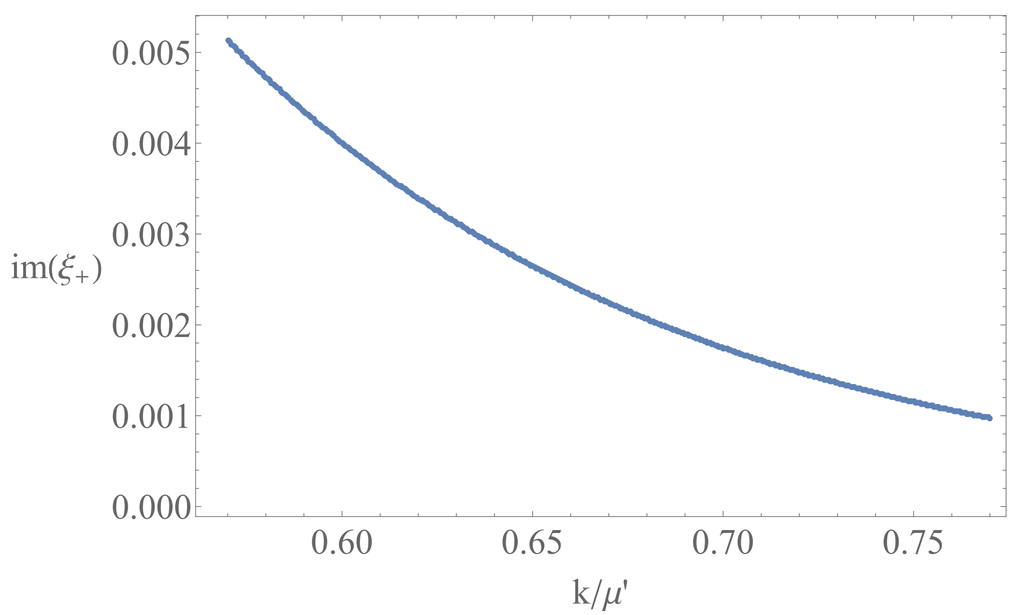

Figure 7: Imaginary part of ξ_+ where $T/\mu' = 0.05$.

Since we actually have $\rho \propto \mathrm{Im}(\xi_- + \xi_+)$, we could be a bit ignorant in just looking at ξ_-. But as can be seen in the figure above, close to the Fermi surface it is a good assumption to ignore the contribution of ξ_+. Now we can look at how the width of these peaks depends on the temperature.

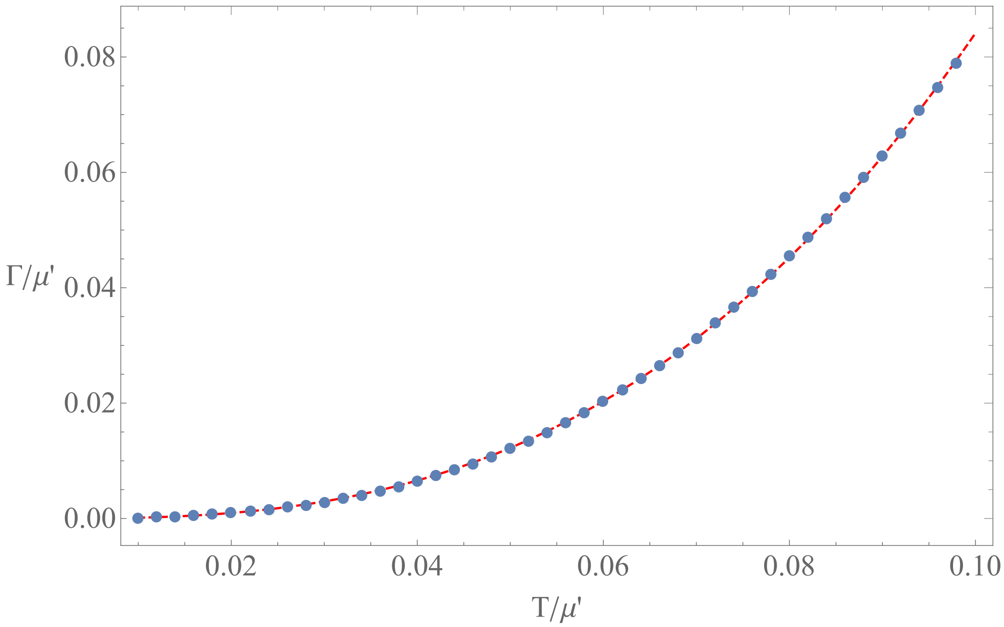

Figure 8: A plot and fit of the FWHM of the spectral function versus the temperature. We approximately find $\Gamma/\mu'' \propto \Sigma'' \propto T^{2.76}$.

As explained in section 2, $\Gamma \propto \Sigma''$. For this case we find approximately $\Sigma'' \propto T^{2.76}$. To confirm that this is indeed the correct power, we remove several data points from either side to make sure we still obtain approximately the same power. This is indeed found to be the case for this low T behaviour. What is also found is that the position of the peak shifts (now called k_0 in the non-zero T case) with altering T/μ' as can be seen in figure 9.

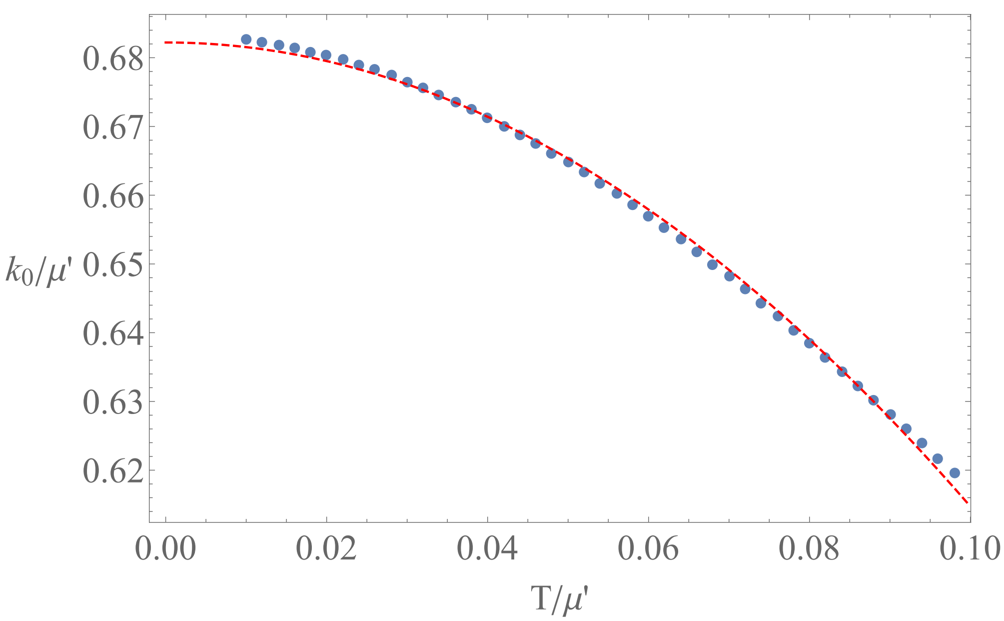

Figure 9: A plot and a rought quadritic fit of the temperature dependence versus the peak position.

Note that the assumptions that Σ'' is only T dependent only holds in the case where $T \gg \omega$. Another way to show that the T dependence is truly a power law, we can plot the relation between Σ'' and T on a double log scale, which was done for consistency. What is also interesting is that at higher temperature the conformal dimension of G_R is also found. It was found that indeed $\Sigma'' \propto (-\omega^2 + k^2)^{2m} \approx k$. Now we will look at the energy (ω) dependence.

5.2 Default case: $q = 1$, $m = -1/4$ and $T \approx 0$.

We found that in the case of fixed small temperature (T) and varying energy (ω) that we could not find a single power law. The problem seems to lie in that the peak is in this case more shifting than in the other case.

5.3 Analytic calculation for the power law

According to the experiment, the power law is fixed for a certain metal. However this seems not to be the case for the ω dependence. We earlier left the definition of Δ' in equation 4.4 a bit vague. But from the literature it is known that for the RN black hole, $\Delta' \equiv \Delta'(k, q, m)$. So it is naive to assume that a precise power law would be found for the ω dependence. The IR Green's function from 4.4, can be found analytically in some cases, as is done for the RN black hole in [5]. This is done by looking at the zero-T metric. That metric should then be of the form where there is a strong asymmetry between space and time (the Lifshitz or $z \to \infty$ backgrounds), to get an only ω dependent term, as is explained in section 4.1.1.

In the literature, they normally define: $\Sigma'' \equiv h(k)\mathcal{G}(\omega, k)$, where $\mathcal{G}(\omega, k) = c(k)\omega^{2\nu_k}$. Where for a fixed m and q we obtain $\Delta'(k, q, m) \equiv \nu_k$. In this section we will use arguments from [7], to eventually show that $\nu_k = 2k/\mu'$.

5.3.1 Near horizon expansion Zaanen conventions.

From the literature it is known that the exponent $\Delta'(k, q, m)$ is determined by the near horizon (IR region) of the metric at small energy and temperature. In the book "HOLOGRAPHIC DUALITY IN CONDENSED MATTER PHYSICS"[21] there is already given what the metric for zero temperature looks like. They use slightly different conventions, but we will clearly mention what this calculation means in our conventions.

$$A_t = \frac{\sqrt{3Q\mu}}{r+Q} - \frac{\sqrt{3Q\mu^{1/3}}}{L^{2/3}}, \tag{5.1}$$

$$\phi = \frac{\sqrt{3}}{2} \ln\left(1 + \frac{Q}{r}\right), \tag{5.2}$$

$$\alpha = \ln\left(\frac{r}{L}\right) + \frac{3}{4}\ln\left(1 + \frac{Q}{r}\right), \tag{5.3}$$

$$f = 1 - \frac{\mu L^2}{(r+Q)^3}. \tag{5.4}$$

We will look around the horizon at zero T, here $r = 0$ and $\mu = \frac{Q^3}{L^2}$.

$$e^{2\alpha} = \frac{r^{1/2}(Q+r)^{3/2}}{L^2} \approx \frac{r^{1/2}Q^{3/2}}{L^2} \text{ around } r = 0 \tag{5.5}$$

$$f = 1 - \frac{Q^3}{(r+Q)^3} \approx \frac{3r}{Q} \tag{5.6}$$

Here the metric becomes

$$ds^2 = -3\frac{r^{3/2}Q^{1/2}}{L^2}dt^2 + \frac{r^{1/2}Q^{3/2}}{L^2}(dx^2+dy^2) + \frac{L^2}{3r^{3/2}Q^{1/2}}dr^2. \tag{5.7}$$

Which can be written as

$$ds^2 = \left(\frac{L_2^2}{2Q\zeta}\right)(\frac{L_2^2}{\zeta^2}\left(-dt^2+d\zeta^2\right) + \frac{Q^2}{L^2}(dx^2+dy^2)). \tag{5.8}$$

With $\zeta = \frac{L_2^2}{2\sqrt{Qr}}$ and $L_2 = \frac{2L}{\sqrt{3}}$. This also yields $A_t = -\frac{L_2^3}{2Q\zeta^2}$.

5.3.2 Nonzero T Zaanen.

Now we will extend this to nonzero temperature by expanding around a nonzero horizon r_h, which is close to 0 in the limit of low temperature. We have that $(r_h+Q)^3 = \mu$. (Here we set $L=1$ which can be done by re scaling as is shown in the appendix.) We now have

$$e^{2\alpha} = r^{1/2}(Q+r)^{3/2}, \tag{5.9}$$

$$f = 1 - \frac{\mu}{(r+Q)^3} = 1 - \frac{(r_h+Q)^3}{(r+Q)^3} \tag{5.10}$$

and

$$A_t = \frac{\sqrt{3Q\mu}}{r+Q} - \sqrt{3Q\mu^{1/3}} = \frac{\sqrt{3Q(r_h+Q)^3}}{r+Q} - \sqrt{3Q(r_h+Q)}. \tag{5.11}$$

We are interested in what the metric looks like for $T \ll \mu'$ and $\omega \ll \mu'$[15]. However we still have two different cases that are $\omega \gg T$ and $\omega \ll T$. Starting with the expansion of f we will have in both regimes

$$f = \frac{3(r-r_h)}{r_h(Q/r_h+1)} - \frac{6(r-r_h)^2}{r_h^2(Q/r_h+1)^2} + h.o.t. = \frac{3(r-r_h)}{Q} + h.o.t. \tag{5.12}$$

Here we made use of $\frac{r-r_h}{Q} \ll 1 (\omega \ll \mu')$ in the first line and we used $Q/r_h \gg 1$ $(T \ll \mu')$ in the second line. Doing a straightforward Taylor expansion of $e^{2\alpha}$ around the horizon, will lead to terms that look like $(r-r_h)/r_h$ this leads to a problem in the regime where $\omega \gg T$ since here $(r-r_h)/r_h \gg 1$. Thus that expansion only works in the the case where $\omega \ll T$. In both cases we do have

$$A_t = -\frac{\sqrt{3Q}}{\sqrt{Q+r_h}}(r-r_h) + h.o.t. = -\sqrt{3}(r-r_h) + h.o.t. \tag{5.13}$$

5.3.3 Dirac equation for $\omega \gg T$.

For the Dirac equation to be invariant under this coordinate transformation from $r \to \zeta$ we need to make sure that $\Gamma^r\partial_r = \Gamma^\zeta\partial_\zeta$. By definition $g_{\zeta\zeta}d\zeta^2 = g_{rr}dr^2$ and in this case specifically $\sqrt{g_{\zeta\zeta}}d\zeta = -\sqrt{g_{rr}}dr$ as in [5]. Thus in this case we need to be more careful in choosing the basis of the gamma matrices, specifically $\Gamma^{\underline{\zeta}} = -\Gamma^{\underline{r}}$. We can still write (as in equation 4.26)

$$(\sqrt{g^{\zeta\zeta}}\partial_\zeta + i\Gamma^{\underline{\zeta}}(\sqrt{g^{ii}}\Gamma^{\underline{i}}k_{\underline{i}} - \sqrt{-g^{tt}}(\omega+qA_t)\Gamma^{\underline{\zeta}}\Gamma^{\underline{0}}) - m\Gamma^{\underline{\zeta}})\phi = 0. \tag{5.14}$$

this will now lead to

$$\sqrt{\frac{g_{ii}}{g_{\zeta\zeta}}}\left(\partial_\zeta \pm m\sqrt{g_{\zeta\zeta}}\right)\phi_\pm = \pm i(-i\sigma_2 u + k\sigma_1)\phi_\mp \tag{5.15}$$

[15]In these conventions, $\mu' \sim Q$, $r_h \sim T$ and $r - r_h \sim \omega$.

and also

$$\sqrt{\frac{g_{ii}}{g_{\zeta\zeta}}}\left(\partial_\zeta \pm m\sqrt{g_{\zeta\zeta}}\right)\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i\begin{pmatrix} \pm(k-u)z_\mp \\ \mp(k+u)y_\mp \end{pmatrix}. \tag{5.16}$$

Putting in the metric components we obtain

$$\sqrt{\frac{3\zeta^2Q^2}{4}}\left(\partial_\zeta \pm m\sqrt{\frac{8}{9\zeta^3 Q}}\right)\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i\begin{pmatrix} \pm(k-\sqrt{\frac{3\zeta^2Q^2}{4}}(\omega-\frac{4q}{3\sqrt{3}Q\zeta^2}))z_\mp \\ \mp(k+\sqrt{\frac{3\zeta^2Q^2}{4}}(\omega-\frac{4q}{3\sqrt{3}Q\zeta^2}))y_\mp \end{pmatrix}. \tag{5.17}$$

Which we can write as

$$\partial_\zeta\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = \mp m\frac{2\sqrt{2}}{3\zeta^{3/2}Q^{1/2}}\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} + i\begin{pmatrix} \pm(\frac{2k}{\sqrt{3}\zeta Q}-(\omega-\frac{4q}{3\sqrt{3}Q\zeta^2}))z_\mp \\ \mp(\frac{2k}{\sqrt{3}\zeta Q}+(\omega-\frac{4q}{3\sqrt{3}Q\zeta^2}))y_\mp \end{pmatrix}. \tag{5.18}$$

Doing something similar to Gubser [7], we can argue that that close to the horizon the electric field becomes negligible. This term goes as ζ^{-2}. We can also argue based on the same idea. That the mass, which goes as $\zeta^{-3/2}$ can also be neglected in the IR region. Thus we should look at the most dominant terms near $\omega\zeta \to \infty$[16]. The only terms left are the k and ω terms.

$$\partial_\zeta\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i\begin{pmatrix} \pm(\frac{2k}{\sqrt{3}\zeta Q}-\omega)z_\mp \\ \mp(\frac{2k}{\sqrt{3}\zeta Q}+\omega)y_\mp \end{pmatrix}. \tag{5.19}$$

Now we can solve this at the boundary of the IR region where $\omega\zeta \to 0$, which yields for the Green's function up to zeroth order:

$$G_R = \frac{B_1 + B_2\mathcal{G}(\omega,k)}{A_1 + A_2\mathcal{G}(\omega,k)} \tag{5.20}$$

Where as earlier mentioned $\mathcal{G}(\omega,k) \propto \omega^{2\nu_k}$ for $T \ll \omega$. The spectral function is thus given by

$$\rho(k,\omega) = \frac{1}{\pi}\operatorname{Im} G_R(k,\omega) = \frac{B_2\mathcal{G}(\omega,k)}{(A_1 + A_2\operatorname{Re}\mathcal{G}(\omega,k) + \operatorname{Im}(\mathcal{G}(\omega,k)))^2} \equiv \frac{Z}{\pi}\frac{\Sigma''(\omega,k)}{(-\omega+v_F(k-k_F)-\Sigma'(\omega,k))^2+\Sigma''(\omega,k)^2}. \tag{5.21}$$

Now we should realize we obtain $\Sigma''(\omega,k)$. Whereas the experiment claims to observe $\Sigma''(\omega)$.

5.4 Back to the numerics

Knowing ν_k, it should be easier to understand the results. Knowing $\mathrm{d}\nu_k$ makes it possible to adjust the power law to find the true powers determined by k_F/k_0. It was found that indeed fitting the equations with the adjusted power for the ω dependence results in finding a convergent power law. So we now fit with $\omega^{2\nu} \to \omega^{2\nu_{k_F}-4v_F\omega}$, this resulted into indeed finding around $\nu_{k_F} = 2k_F/\mu'$. Note that for the T dependence we already found this result, since $4 * 0.67 \approx 2.76$. This result is confirmed by looking at several cases of $\{q,m\}$ where k_F/μ' is different.

We can calculate the position of k_F/μ' and thus ν_{k_F} as a function of m and q, see figure 10.

[16]Note that ζ involves the definition of ω by $\zeta \propto \omega$. Doing this specific limit still keeps ω a finite quantity. Therefore we have to take the limit $\omega \to 0$ later.

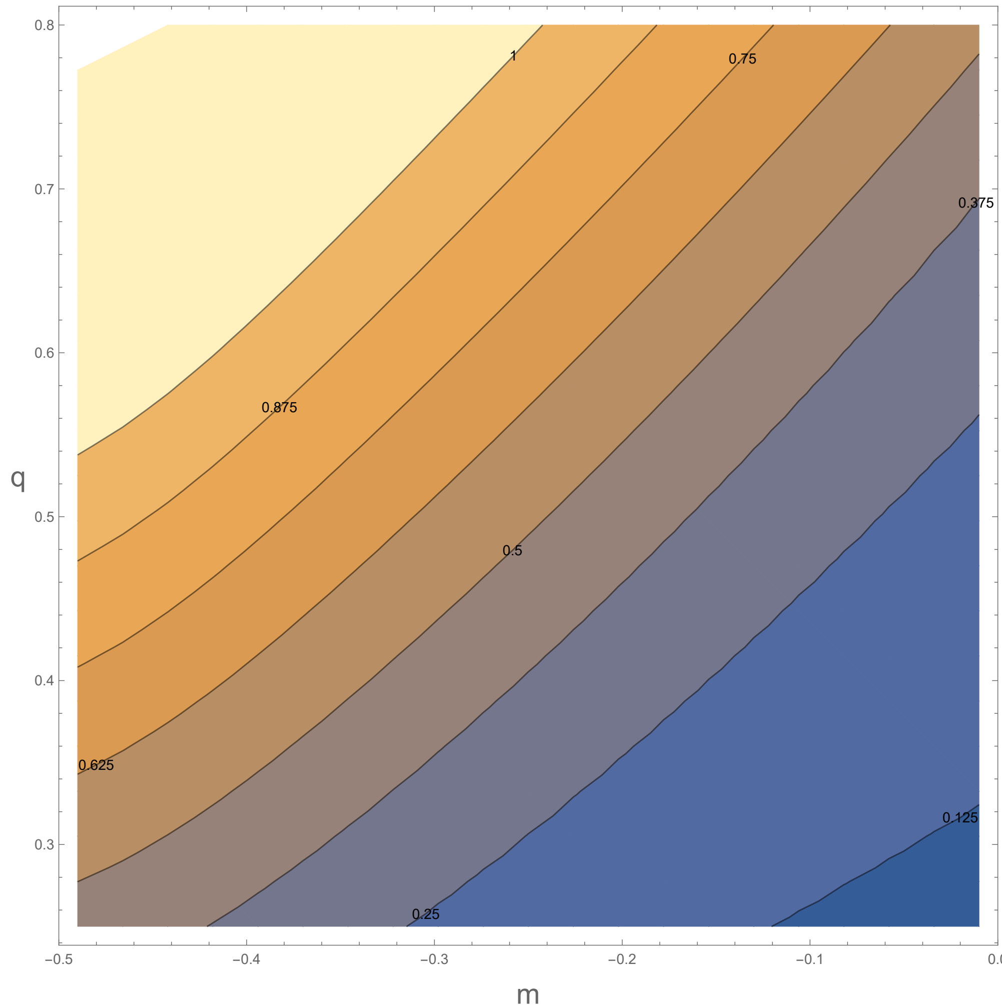

Figure 10: A plot of $\nu_k = 2k_F/\mu'$, in this figure we defined k_F/μ' as the peak position at $T/\mu' = 0.001$. The strange metal line is the line where $\nu_k = 1/2$.

The particular line of interest is what we denote the strange metal line given roughly by a quadratic fit [17]

$$q = 0.68(m + 1.04)^2 + 0.06. \tag{5.22}$$

This equation is also useful numerically, since we can simply plug in any value of $0 < m < -1/2$. This will then always lead roughly to strange metallic behaviour in terms of the scaling behaviour.

5.5 Self Energy

In ARPES experiments there can only be looked at the spectral function. Thus analyzing the spectral function seems to be a good way to start theoretically. There is however theoretically also a way to directly calculate the imaginary (and real) part of the self energy

$$\Sigma''(\omega, k) = -Z \,\mathrm{Im}\left(\frac{1}{G_R(\omega, k)}\right) \approx -Z \,\mathrm{Im}\left(\frac{1}{\lim_{r\to\infty} r^{2m}\xi_-(\omega, k)}\right). \tag{5.23}$$

[17]Note that we calculated all the peak positions (k_F/μ') at $T/\mu' = 0.001$

In this way we can fix k, which cannot be done for the spectral function. Since the peak position k_0 changes when either T or ω is varied. We now present the results for the 2 distinct cases where we have chosen $\{m, q\} = \{-0.3, 0.436\}$ according to the strange metal line.

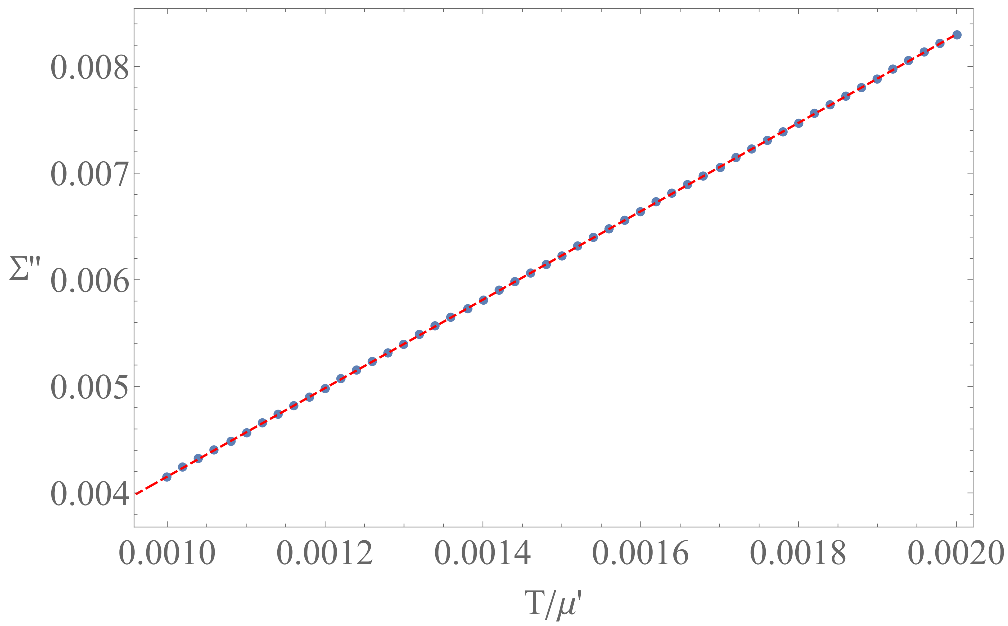

Figure 11: The imaginary part of the self energy as a function of temperature at fixed $k/\mu' = k_F/\mu' = 0.25$.

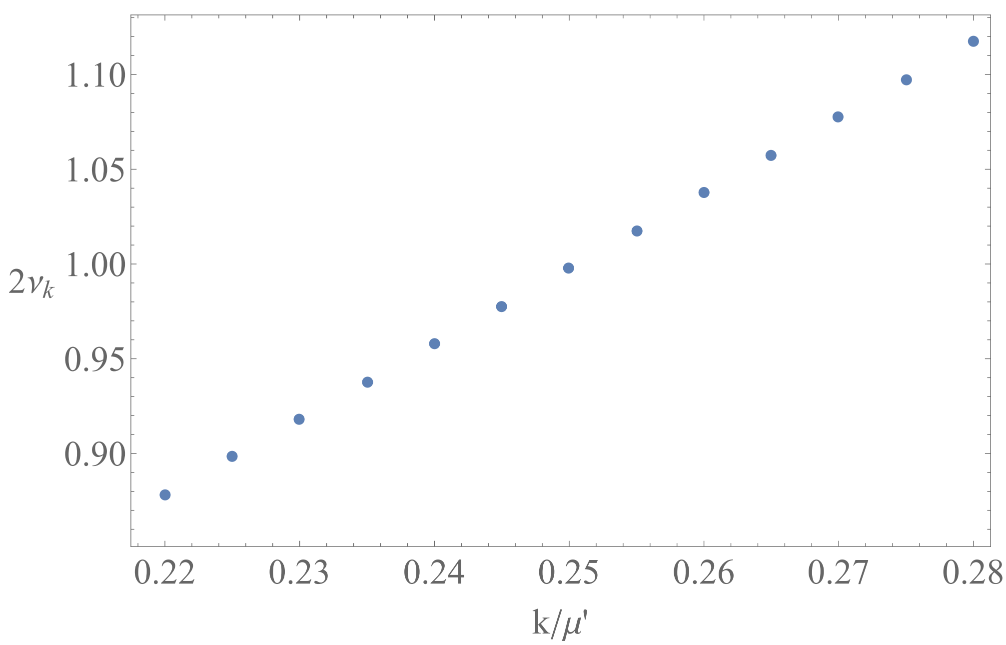

Figure 12: The power ν_k as a function of k/μ' for small $T \gg \omega$.

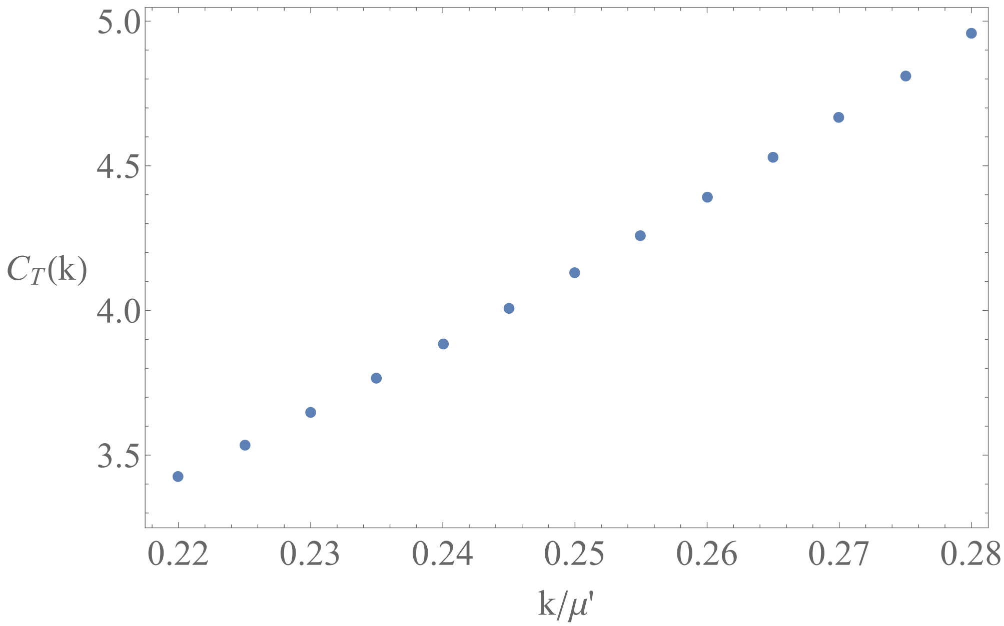

Figure 13: The k-dependent prefactor on the T dependence of the self energy.

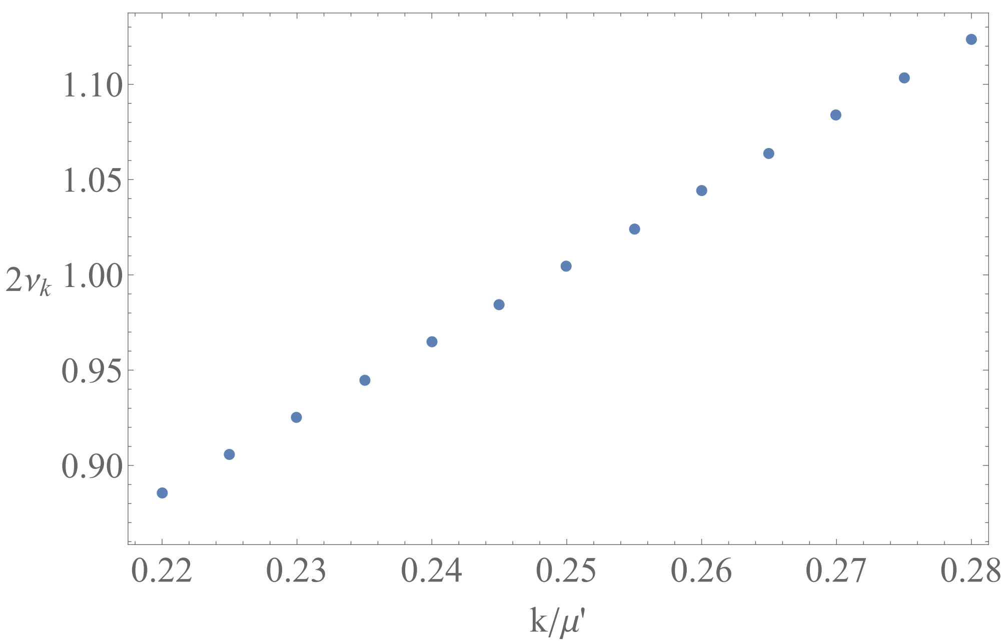

Figure 14: The power ν_k as a function of k/μ' for small $T \ll \omega$.

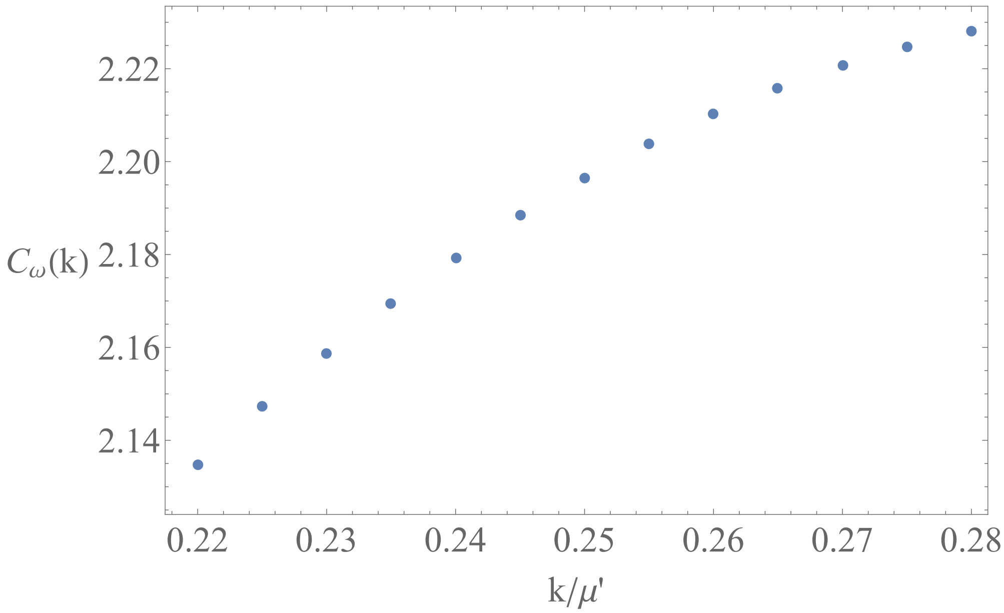

Figure 15: The k-dependent prefactor on the ω dependence of the self energy.

Here we defined the following $\Sigma'' = C_T(k)T^{2\nu_k}$ for $T \gg \omega$ and $\Sigma'' = C_\omega(k)\omega^{2\nu_k}$ for $T \ll \omega$. The function for the self energy is fitted with the proposed form as above and we find $C_T(0.25) \approx 4.12$ and as predicted $\nu_k = 1/2$ see figure 11. We also present the results for the both cases and find as predicted approximately $\nu_k = 2k_F/\mu'$. The parameter $C_T(k)/C_\omega(k)$ determines the coupling strength difference for the fermions between energy and temperature. According to experiment (and Fermi liquid theory) this parameter should be around (exactly) π[17]. The result for both cases can be seen in figures 13 and 15. The last result that we present is the k dependence on this comparative strength in figure 5.5.

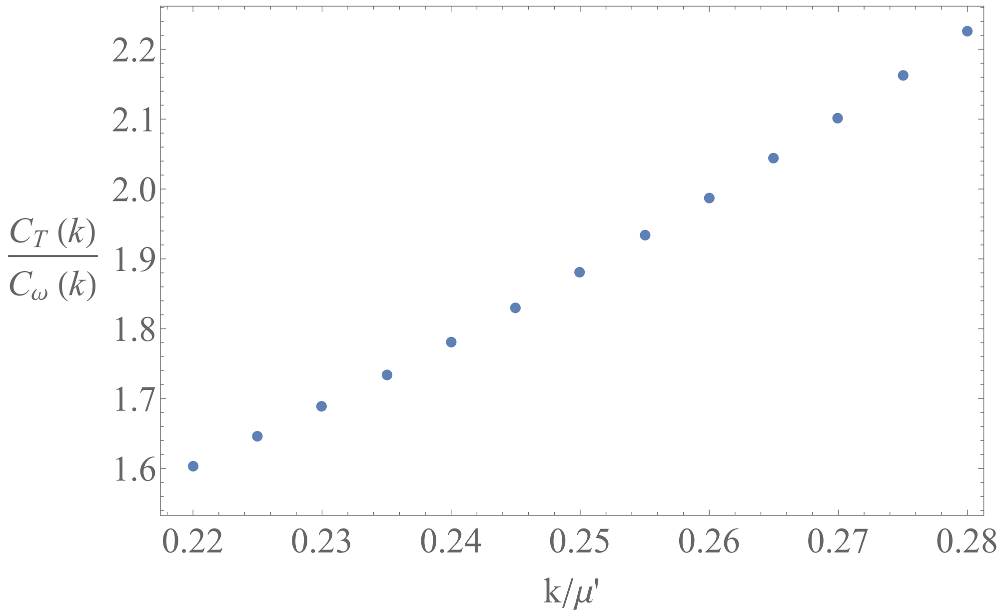

Figure 16: The division of the two calculated prefactors which corresponds to the relative strength between energy and temperature.

6 Discussion and outlook

So what did we add to the scientific tree of knowledge?

It boils down to that we have added a rule to the dictionary of the AdS/CMT correspondence. Specifically, we added a rule that can describe a strange metal quantum theory by a weak EMD gravity theory. In the previous chapter we have shown this rule $\nu_k = 2k/\mu'$, representing α on the experimental side at $k = k_F$. We have motivated the rule both analytically and numerically. We have also shown numerically how the position of the peak depends on the energy and temperature. We do not have a strict rule for how the peak position depends on all the parameters yet. First we present the full dictionary thus with the rule added:

weak gravity theory	strong quantum theory
Black hole temperature (T)	Temperature (T)
Black hole entropy (S)	Entropy (S)
Time component of the electromagnetic field (A_t)	Chemical potential (μ)
Number of space dimensions ($d = 3$)	Number of space dimensions ($d - 1 = 2$)
Number of time dimensions (1)	Number of time dimensions (1)
$\theta \to -\infty$, $\mathbf{z} =\to \infty$ with $\eta = -\theta/\mathbf{z} = 1$	Entropy scales linearly with temperature ($S \propto T$, also same θ and $\mathbf{z}$)
Fermions	Fermionic operators (In this thesis: Green's function, thus also the spectral function)
IR region of black hole	$\nu_k = 2k/\mu'$ (In $\Sigma'' = (\omega^2 + T^2)^{\nu_k/\mu'}$)

These results mean that we can indeed roughly describe a strange metal at the boundary. We can choose the only free parameters q, m according to the earlier roughly defined strange metal line:

$$q = 0.68(m + 1.04)^2 + 0.06.$$

6.1 Outlook

To fully describe the result, only one parameter is still left to be found, called β in the experiments. We have to further explore the self energy on the strange metal line. To investigate if indeed we can fix $\beta = \pi$.

There should also be looked at if the experiments might also observe a changing scaling exponent. Thus there can be experimentally tested if indeed $\alpha \to \alpha(k)$ and if this is even observed.

7 Appendix

7.1 Units

The action in equation 4.1 can be written in dimension full quantities as

$$S_{EMD} = \int \mathrm{d}^4x\sqrt{-g}\Big(\frac{c^3}{16\pi G}(R - V(\phi) - \frac{1}{2}\left(\partial_\mu\phi\right)^2) - \frac{Z(\phi)}{4\mu_0 c}F^2\Big). \tag{7.1}$$

The Dirac action in SI units is given by [16]

$$S = ig_f\int \mathrm{d}^4x\sqrt{-g}\bar{\psi}\left(\not{D} - \frac{Mc}{\hbar}\right)\psi + ig_f\int \mathrm{d}^3x\sqrt{-h}\bar{\psi}_R\psi_L. \tag{7.2}$$

By using the following conventions (also $A_t = cA_0$),

$$\tilde{R} = RL^2 \tag{7.3}$$

$$\tilde{V}(\tilde{\phi}) = V(\phi)L^2 \tag{7.4}$$

$$\tilde{A}_{\tilde{\mu}} = \sqrt{\frac{16\pi G}{\mu_0 c^4}}A_\mu \tag{7.5}$$

$$\tilde{A}_{\tilde{t}} = \sqrt{\frac{16\pi G}{\mu_0 c^6}}A_t \tag{7.6}$$

$$\tilde{Z}(\tilde{\phi}) = Z(\phi) \tag{7.7}$$

$$(\tilde{t}, \tilde{\mathbf{x}}, \tilde{r}) = (ct, \mathbf{x}, r)/L \tag{7.8}$$

$$\tilde{\phi} = \phi \tag{7.9}$$

$$\tilde{M} = \frac{McL}{\hbar} \tag{7.10}$$

$$\tilde{q} = \sqrt{\frac{\mu_0 c^6}{16\pi G}}\frac{L}{\hbar c}q \tag{7.11}$$

Plugging this back in leads to the Lagrangian's given earlier in chapter 4.

7.2 The loose end of the ω case

7.2.1 Dirac equation for $\omega \ll T$.

For the dirac equation to be invariant under this coordinate transformation from $r \to \zeta$ we need to make sure that $\Gamma^r\partial_r = \Gamma^\zeta\partial_\zeta$. By definition $g_{\zeta\zeta}d\zeta^2 = g_{rr}dr^2$ and in this case specifically $\sqrt{g_{\zeta\zeta}}d\zeta = \sqrt{g_{rr}}dr$. Thus in this case we can simply choose $\Gamma^{\underline{\zeta}} = \Gamma^{\underline{r}}$. The derivation of the dirac equation is therefore equivalent to the case in which the r coordinate is used. And we obtain

$$\sqrt{\frac{g_{ii}}{g_{\zeta\zeta}}}\left(\partial_\zeta \mp m\sqrt{g_{\zeta\zeta}}\right)\phi_\pm = \mp i(-i\sigma_2 u + k\sigma_1)\phi_\mp, \tag{7.12}$$

and also

$$\sqrt{\frac{g_{ii}}{g_{\zeta\zeta}}}\left(\partial_\zeta \mp m\sqrt{g_{\zeta\zeta}}\right)\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i\begin{pmatrix} \mp(k-u)z_\mp \\ \pm(k+u)y_\mp \end{pmatrix}. \tag{7.13}$$

This can now be written as

$$\sqrt{3r_h}Qe^{-\zeta/2}\left(\partial_\zeta \mp m\sqrt{\frac{e^\zeta}{3Q^{1/2}r_h^{1/2}}}\right)\begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i\begin{pmatrix} \mp(k - \sqrt{\frac{Q}{3e^\zeta}}(\omega - \sqrt{3}qe^\zeta))z_\mp \\ \pm(k + \sqrt{\frac{Q}{3e^\zeta}}(\omega - \sqrt{3}qe^\zeta))y_\mp \end{pmatrix}. \tag{7.14}$$

Now we can look at the dominant term at the boundary, thus $\zeta \to \infty$ and we obtain

$$\sqrt{3r_h}Qe^{-\zeta}\partial_\zeta \begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix} = i \begin{pmatrix} \mp\sqrt{\frac{Q}{3}}(\sqrt{3}q)z_\mp \\ \mp\sqrt{\frac{Q}{3}}(\sqrt{3}q)y_\mp \end{pmatrix}. \tag{7.15}$$

This equation above we can solve, but this will not give a power law. I believe the solution to this will look something like $\phi \sim e^{\nu(r-r_h)}$, where this ν are dependent on q, Q, r_h.

Near horizon expansion in the case of $\omega \gg T$.

In the regime where $(r - r_h)/r_h \gg 1$ we can not use the expansion of equation **??**, since this will result in the domination of higher order terms. Thus we rewrite $e^{2\alpha}$ as

$$e^{2\alpha} = r^{1/2}(Q+r)^{3/2} = r^{1/2}Q^{3/2} + h.o.t. = Q^{3/2}(r-r_h)^{1/2}(1+\frac{r_h}{r-r_h})^{1/2} + h.o.t. = Q^{3/2}\sqrt{r-r_h} + h.o.t. \tag{7.16}$$

Here we first used $Q/r \gg 1$ and then $r_h/(r-r_h) \ll 1$. In these limits the metric becomes

$$ds^2 = \sqrt{r-r_h}Q^{3/2}(-\frac{3(r-r_h)}{Q}dt^2 + dx^2 + dy^2) + \frac{1}{3(r-r_h)\sqrt{r-r_h}Q^{1/2}}dr^2. \tag{7.17}$$

Now we define $\zeta = \frac{2}{3\sqrt{Q(r-r_h)}}$ such that $d\zeta = -\frac{1}{3Q^{1/2}(r-r_h)^{3/2}}dr$ and we obtain

$$ds^2 = \frac{2Q}{3\zeta}(\frac{4}{3\zeta^2Q^2}(-dt^2 + d\zeta^2) + dx^2 + dy^2) \tag{7.18}$$

We also obtain

$$A_t = -\frac{4}{3\sqrt{3}Q\zeta^2}. \tag{7.19}$$

Note that in this case if $r_h \to 0$ we obtain the zero temperature solution as derived in section 5.3.1. In these coordinates the horizon is at $\zeta = \infty$ and the boundary is at $\zeta = 0$.

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