M-theory backgrounds with torus actions and brane–anti-brane systems

Author:
Paraskevas Tzitzimpasis

Supervisors:
Dr Thomas Grimm
Dr Gil Cavalcanti

June 2020

A thesis submitted to the Departments of Physics and Mathematics
at Utrecht University for the degrees of

Master in Theoretical Physics and Master in Mathematical Sciences

Utrecht University

Abstract

Given an M-theory background, a reduction along the orbits of an isometric circle action must be performed in order to retrieve Type IIA String Theory. We study M-theory backgrounds with isometric torus actions. In this case, the choice of a reduction circle is highly non-unique. However, different reduction circles give rise to dual Type IIA backgrounds. We consider actions on eleven dimensional backgrounds, allowing for possible fixed points of codimension four. The latter are interpreted as D6-branes and D6-anti-branes. We then use the torus action to conclude the existence of special reduction circles that lead to Type IIA on a manifold with boundary. We demonstrate this phenomenon explicitly for two special backgrounds and we examine some key aspects of their structure.

Contents

Chapter 1

Introduction

1.1 Why D-branes?

This is a thesis about D-branes. Therefore, it should be appropriate to initiate our discussion with a quick review regarding their origins and their importance in the development of String Theory. Branes are objects that appear in the most fundamental form of String Theory which goes under the name *bosonic String Theory*. This theory necessarily lives in 26 dimensions [1], often called the *critical dimension*. At first sight, this might appear as an unreasonably large number of dimensions. However, since it could in principle be any natural number, it is fairly close to the number of dimensions in our universe. The necessity for so many dimensions is dictated by a consistency requirement, the cancellation of a *conformal anomaly* [1]. It was quickly realized that the introduction of open strings allowed for Dirichlet boundary conditions so that an open end of the string is confined to lie on a hypersurface which was later called a *D-brane* with the "D" standing for Dirichlet.

Even from this vague definition, it is clear that D-branes are objects of arbitrary dimension as long they do not exceed the dimension of spacetime itself. The dimension of a D-brane is the most important piece of information that we can have about it. This is why, when we refer to a D-brane of known dimension equal to $p+1$, we call it a Dp-brane. Like particles, branes extend in the time direction of spacetime. In this sense, a Dp-brane is an object which extends in p spacelike dimensions and the one time direction. It follows that a D0-brane is an ordinary particle and a D1-brane is a one-dimensional object, namely a string. In the same way, a D(-1)-brane is a point in spacetime and corresponds to an instant of time. For this reason, we call it a *D-instanton*. A a pair of D-branes is shown in Figure 1.1.

From this premature perspective, a D-brane is nothing but an artifact of the human brain. It simply helps us visualize the allowed motion of an open string. This was nothing new for physics. Every physics student has encountered such surfaces arising from constraints, for example in the study of a space vehicle moving on the surface of a planet. Such constraint surfaces are ubiquitous and they can be handled quite efficiently in the context of Lagrangian and Hamiltonian mechanics. However, there is something quite remarkable about D-branes which makes them worth spending our valuable time on:

D-branes are dynamical objects in String Theory!

If we think about it, such a statement makes sense since in a theory of gravity like String Theory, what we mean by a hypersurface is ambiguous as long as spacetime itself is dynamical and can

[1] There are also formulations of String Theory in dimensions different from the critical dimension. Those are rightfully called non-critical String Theories.

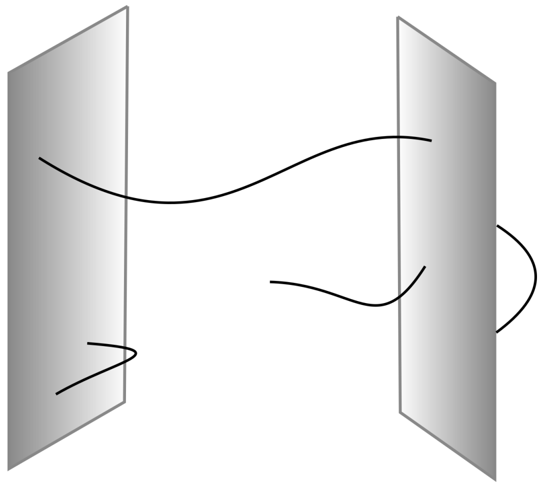


Figure 1.1: Two D-branes and some open strings that can end on them.

fluctuate. Under the light of this revolutionary discovery, String Theory should probably be called Brane Theory. Indeed, from our current understanding of String Theory, strings are as fundamental as D-branes. Just as strings have an action, the Polyakov action, which dictates how they evolve in space and time, D-branes also have their action and can be treated like multi-dimensional particles.

If D-branes are to be promoted to fundamental objects, then we should expect them to have various traits in common with ordinary particles in field theories. One such feature is the existence of *anti-particles*. Particles and anti-particles are identical in their mass and spin but they have opposite electric charge. The most famous example is the electron whose anti-partner is called anti-electron or, more commonly, *positron*. Collectively, we refer to anti-particles as anti-matter. Anti-matter behaves much the same way as ordinary matter and the dominance of matter over anti-matter in our universe had been a long standing conundrum for physicists after the discovery of the positron. Great progress has been made in answering this fundamental question after the discovery of *CP violation* in the weak interactions which was first observed in 1964 and later explained theoretically.

In any case, irrespective of whether they are around us or not, anti-particles exists and if String Theory treats them on equal footing with D-branes, then there should also exist *anti-branes*. Indeed, anti-branes exist and they appear naturally as we will see in various instances of this thesis. In addition, there is a nice way to interpret them as generalized anti-particles. Indeed, recall from standard field theory that in terms of Feynman diagrams, anti-particles can be thought of as particles going back in time. In slightly more abstract language, we can say that the world-lines of particles and anti-particles have opposite orientations. In this way, we can think of anti-branes as being obtained by a change of orientation of the world-volume of a standard D-brane. For the anti-particles, the world-volume is one-dimensional so there is only one direction whose orientation can be switched. However, for general D-branes of higher dimension there are many directions whose orientations we can change. This is why it would be misleading to say that anti-branes are branes going backwards in time.

1.2 Superstring theory

Bosonic String Theory was successful in planting the seed for a theory of quantum gravity but was hampered by the existence of a tachyon in the spectrum of closed strings. This closed string tachyon still poses a subject of controversy and uncertainty as to what it might imply for bosonic

String Theory itself. Nevertheless, the tachyon is not the only herald of distress since, as its name accurately indicates, bosonic String Theory is a theory of bosons and only accounts for a world without fermions. Around the dawn of the seventies, theoretical physicists hit two (or even more) birds with one stone when they discovered a consistent way to introduce fermions [2] that also works as a miracle that eliminates the tachyonic mode of closed strings. The theory that emerged was called *Superstring Theory*.

The "super" in Superstring is due to the use of supersymmetry, often abbreviated SUSY, which is the magical ingredient that was missing from bosonic String Theory. Physical theories are invariant (at least locally) under the Poincaré group which consists of rotations and translations. Put a little differently, the Poincaré group is the group of isometries of Minkowski spacetime. The corresponding Lie algebra, called the *Poincaré algebra* is given by $\mathfrak{t} \rtimes \mathfrak{so}(1, n-1)$ with n being the dimension of spacetime and $\mathfrak{t} \simeq \mathbb{R}^n$ corresponding to translations. The use of supersymmetry extends the Poincaré algebra by adding anti-commuting generators. The resulting algebra is called *super-Poincaré algebra*. The new anti-commuting generators act on particle states by changing their spin by $\frac{1}{2}$. This is how fermions are introduced in a theory of bosons like bosonic String Theory. Every bosonic particle has its *superpartners* which are the set of states that can be reached by applying supersymmetry generators on the given state.

We can choose the number of supersymmetry generators that we want in our theory. The most elementary supersymmetric theories have one supersymmetry generator Q^α and are called $\mathcal{N} = 1$ theories. We can add more supersymmetry generators Q_i^α with $i = 1, ..., \mathcal{N}$. Theories with $\mathcal{N} \geq 2$ are said to have *extended supersymmetry*. For example in an $\mathcal{N} = 1$ theory, a bosonic scalar field of spin zero can be acted upon by the the generator Q^α to give a spin-$\frac{1}{2}$ field. Applying the same generator twice will simply return zero, since the anti-commutation relations of the supersymmetry generators imply $(Q^\alpha)^2 = 0$. In the presence of extended supersymmetry we could act with different generators to obtain higher spin states all of which will be related with each other by supersymmetry.

With the addition of this invaluable ingredient, Superstring Theory was born. However, due to the existence of some choices involved in the incorporation of supersymmetry in String Theory, there is not one but five theories. What they have in common is that contrary to the bosonic theory, those five Superstring Theories are defined in ten dimensions [3] which already brings us considerably closer to our phenomenological goal of describing a world of four dimensions. The five superstring theories are

- ***Type IIA and Type IIB***: Those are $\mathcal{N} = 2$ theories. They will be our main focus throughout this thesis.
- ***Heterotic String Theories***: Those are $\mathcal{N} = 1$ theories. There are two types of Heterotic strings called *Heterotic* $E_8 \times E_8$ and *Heterotic SO(32)*. As their names suggest, their difference lies in the gauge groups that appear in those theories.
- ***Type I***: A theory similar to Type II theories but the strings (both open and closed) are unoriented. This is the only theory with unoriented strings.

Those five theories are in principle different by construction. However, it has been discovered that they are different sides of the same coin. More precisely, those theories are interrelated by a web of dualities. We will not try to give a complete account of those dualities since as already proclaimed, our main tools will be Type II theories. From now on, we will practically forget about the existence of the Heterotic and Type I strings but it is useful to keep in mind that due to the aforementioned dualities, the physical results one gets in one superstring theory are universal.

1.3 Type II supergravities

One of the achievements of bosonic String Theory is that at low energies it reproduces the Einstein equations. We would expect that something similar should hold for Superstring theories. The low energy limits of the Type IIA and Type IIB Superstring Theories are called Type IIA and Type IIB supergravities (or SUGRA's for short). Those are $\mathcal{N} = 2$ supersymmetric theories of gravity (thus the name supergravity). They comprise the standard Einstein gravity together with additional bosonic and (due to supersymmetry) fermionic fields.

Instead of presenting the ten-dimensional $\mathcal{N} = 2$ supergravities, we start with a theory which is believed to be more fundamental, in a sense that we will shortly explain. This is eleven dimensional supergravity. It is an $\mathcal{N} = 1$ theory and is in fact maximal in the sense that we cannot construct higher dimensional supergravities with Lorentzian signature [2] and in that it is the only supergravity in eleven dimensions. Finally, the reason why it deserves the status of a more fundamental theory compared to Type II supergravities is that in a sense 11d supergravity contains them. What this means is that we can derive Type II supergravities by starting with this maximal 11d supergravity. Let us demonstrate in broad strokes how this is achieved. The bosonic part of the action of the 11d theory reads

$$\mathcal{S}_{d=11} = \frac{1}{2\kappa_{11}^2} \int \left[R * 1 - \frac{1}{2} * F_4 \wedge F_4 - \frac{1}{6} F_4 \wedge F_4 \wedge A_3 \right] \tag{1.1}$$

Here the first term is the standard Einstein-Hilbert term giving rise to pure gravity in eleven dimensions, the 3-form A_3 is an additional bosonic field of the theory and $F_4 = dA_3$ is its field strength. There is a U(1) gauge symmetry for which A_3 is the gauge field and it transforms as $A_3 \to A_3 + d\Lambda_2$ for an arbitrary 2-form Λ_2. The last term in 1.1 is the Chern-Simons term which under the aforementioned gauge transformation picks up a total derivative. Everything else is manifestly invariant and the invariance of the action follows. The second term of the action is a standard kinetic term. The fermionic degrees of freedom, which are absent in the action we wrote down, comprise a gravitino. Finally, κ_{11} denotes Newton's constant in eleven dimensions and * is the Hodge star operator in eleven dimensions.

Having examined some key points of 11d supergravity, we move on to Type IIA. This was known to be a consistent supergravity theory in ten dimensions but it was later realized that it is in fact deeply related to 11d SUGRA. The Type IIA SUGRA contains a variety of fields unlike 11d SUGRA which contains only three. Those are listed below.

- A graviton and a gravitino.
- A dilaton (scalar field) ϕ and a dilatino which is its superpartner.
- A 2-form gauge potential B_2 with field strength $H_3 = dB_2$.
- A 1-form and a 3-form C_1, C_3 which are again gauge fields. We can dualize those fields to get their "magnetic duals" in a process similar to how one obtains a magnetic field strength in classical Electromagnetism. We first consider the field strengths $F_2 = dC_1$ and $F_4 = dC_3$ and dualize them to get $F_6 = *F_4$ and $F_8 = *F_2$. Those "magnetic" field strengths should correspond to some potentials such that $F_8 = dC_7$ and $F_6 = dC_5$. In this way, one concludes that the existence of the gauge fields C_1, C_3 implies the existence of C_5, C_7. The argument can of course be reversed.

[2]The argument for this is pretty simple. A supersymmetric theory in 12 dimensions could be toroidaly compactified to 4d and would give rise to a theory with at least $\mathcal{N} = 16$. This amount of supersymmetry is excluded because it implies the existence of fields with spin higher than two.

Owing to the way they are derived, the bosonic fields are divided into two categories. The (bosonic) fields contained in the first three bullets belong to the *NS-NS sector* and the fields of the last bullet to the *RR-sector*. This is why they are often referred to as NS-NS or RR fields respectively. Before we write down the action of Type IIA SUGRA, it is convenient to define the invariant field strength

$$\tilde{F}_4 := F_4 - C_1 \wedge dB_2$$

The bosonic action of Type IIA SUGRA is the following:

$$\begin{aligned} S_{IIA} =& \frac{1}{2\kappa_{10}^2} \int \Big[e^{-2\phi} R * 1 + 4e^{-2\phi} d\phi \wedge * d\phi + \frac{1}{2} e^{-2\phi} H_3 \wedge * H_3 - \frac{1}{2} F_2 \wedge * F_2 - \frac{1}{2} F_4 \wedge * F_4 \\ & - \frac{1}{2} B_2 \wedge dC_3 \wedge dC_3 \Big] \end{aligned} \tag{1.2}$$

This action can be obtained as a low energy effective action of Type IIA String Theory and the length scale set by κ_{10} can be given in terms of the string length α' which is the only length scale in String Theory. Their relation is

$$\kappa_{10} = \frac{1}{4\pi} (4\pi^2 \ell_s^2)^4$$

We now set to demonstrate how this Type IIA SUGRA can be obtained as a certain limit of 11d supergravity [4, 5]. Their relation lies in the heart of this thesis and a complete account of the underling mathematical ideas will be given later in Section 2.5. However, we will attempt here a quick take on the subject with particular emphasis on the physics of the story. Intuitively, the idea is to start with the theory in eleven dimensions and assume that one dimension is circular [3] so that we can perform a *dimensional reduction* which pictorially amounts to shrinking the circle so that we get an effective theory in ten dimensions. This is a special case of the more general *compactification* which is a fundamental concept in String Theory.

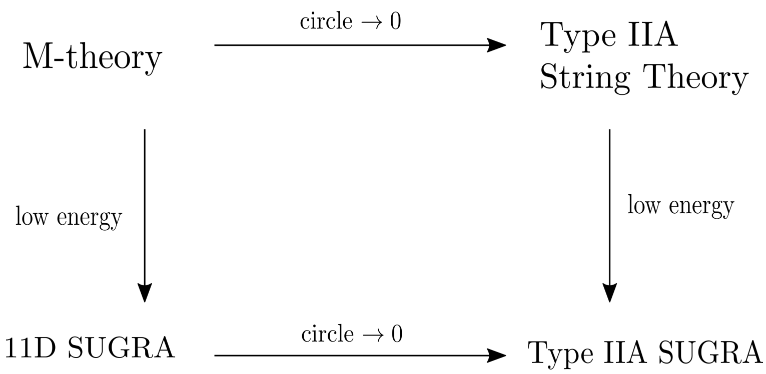
Under the reduction, different modes of the higher dimensional theory, regroup to form the lower dimensional degrees of freedom. Denoting the circular dimension with x^{11}, the eleven dimensional metric decomposes into the component $g_{11\,11}$ which is a scalar in ten dimensions, the 1-form $g_{m\,11}$ and the symmetric 2-tensor g_{mn} with ten dimensional indices. Those comprise the dilaton, RR 1-form C_1 and metric of the Type IIA supergravity. The 3-form A_3 decomposes into a 2-form A_{mn11} and a 3-form A_{mnk} in ten dimensions which correspond to the NS-NS 2-form B and the RR 3-form C_3. An expansion of the action 1.1 in terms of the lowest dimensional modes indeed gives the Type IIA action 1.2. Note that an important assumption in order to transform the eleven dimensional integral into a ten dimensional one is that the components of the fields do not explicitly depend on the circular dimension. This will be later explained more rigorously to translate to the requirement that the circle dimension is an isometric direction. Note that the size of the eleventh direction (the circle) becomes the dilaton in ten dimensions which is known from String Theory arguments (see e.g. [6]) to be related to the string coupling constant g_s via

$$g_s = e^{\phi}$$

Therefore, a small circle implies a small string coupling and consequently an accurate perturbative ten dimensional Type IIA String Theory. It is natural to ask what happens when the size of the circle grows large. It is conjectured, and indeed a lot of evidence corroborates this idea, that for

[3] The precise statement here would be the existence of a circle action which we will define and explore in the following chapter.

arbitrary circle size the theory is a non-perturbative formulation of String Theory. We do not know what this theory is but we know that at low energies it must reduce to eleven dimensional supergravity. This putative non-perturbative formulation of String Theory goes under the name *M-theory*. More pictorially we have the following commutative diagram.



The next important building block for our understanding of the low energy dynamics of String Theory is Type IIB supergravity, which is the low energy limit of Type IIB String Theory. We will present it in the same way we did for Type IIA by starting with an overview of its field content which is

- A graviton and a gravitino.
- A dilaton (scalar) and the dilatino.
- A 2-form gauge field B_2 with $H_3 = dB_2$
- A 0-form C_0, a 2-form C_2 and a 4-form C_4 along with their magnetic duals C_8 and C_6 (note that the dual of C_4 is itself).

It is worth noting here that the NS-NS sectors of Type IIA and Type IIB supergravities are the same. We introduce again, as we did for Type IIA, some field redefinitions on the basis of gauge invariance:

$$\tilde{F}_3 := F_3 - C_0 \wedge H_3$$
$$\tilde{F}_5 := F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$$

Here we stick to the notation $F_n := dC_{n-1}$. There is an additional restriction which cannot be derived from an action but has to be imposed by hand. This is the self-duality of $\tilde{F}_5$ which reads

$$\tilde{F}_5 = *\tilde{F}_5$$

Having defined the necessary fields, we can now express the action of Type IIB supergravity in a concise form.

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int \Big[e^{-2\phi}R * 1 + 4e^{-2\phi}d\phi \wedge *d\phi - \frac{1}{2}e^{-2\phi}H_3 \wedge *H_3 - \frac{1}{2}F_1 \wedge *F_1 - \frac{1}{2}\tilde{F}_3 \wedge *\tilde{F}_3 - \frac{1}{4}\tilde{F}_5 \wedge *\tilde{F}_5 - \frac{1}{2}C_4 \wedge H_3 \wedge F_3\Big] \tag{1.3}$$

Note that it is indeed impossible to impose the self-duality constraint on $\tilde{F}_5$ at the level of the action since this would imply $\tilde{F}_5 \wedge *\tilde{F}_5 = \tilde{F}_5 \wedge \tilde{F}_5 = 0$. This is why this condition has to be imposed on top of the equations of motion. Note also the additional factor of 1/2 in front of the $\tilde{F}_5 \wedge *\tilde{F}_5$ term which accounts for the fact $\tilde{F}_5$ has twice as many degrees of freedom in the action before the self-duality condition is imposed.

Now that we have outlined some key features of Type II supergravities, we would like to discuss how branes fit into this framework. Recall that in bosonic String Theory we had branes of arbitrary dimension. In principle, this is also true for Type II theories. We can define submanifolds of arbitrary dimension less or equal than ten. Nevertheless, the dynamical nature of branes raises an additional question pertaining to their stability. This turns out to be more subtle than in bosonic strings. Consider a classical particle. It couples to an electromagnetic potential (1-form) A_μ via the action

$$S \sim \int_W A = \int_W A_\mu \frac{dx^\mu}{d\tau} d\tau$$

where W is the world-line of the particle parametrized by the proper time τ as $x^\mu = x^\mu(\tau)$. As we have already remarked, from a more abstract point of view, a particle is a D0-brane so we would expect that analogous properties hold not only for particles but also for higher dimensional objects. It was known for example since the times of bosonic String Theory that the fundamental string [4], which we denote F1, couples to the Kalb-Ramond 2-form B_2 via

$$S \sim \int_W B_2$$

with W the world-volume (or world-sheet) of the string. However, before the advent of superstring theory there seemed to be insufficient fields for the D-branes to couple to. It turns out that a Dp-brane couples to an RR (p+1)-form field C_{p+1} via the action

$$S \sim \int_{Dp} C_{p+1}$$

with the integral carried on the volume of the Dp-brane. In this sense, Dp-branes are sources of the RR field C_{p+1}. This remarkable property of D-branes was first discovered in [7] and although simple in its core, it has important consequences for the stability of the branes. This is because a Dp-brane is stable as long as it couples to some field in the theory. Just as it happens with electrically charged particles, the conservation of charge prohibits the spontaneous annihilation of the particle itself. On the other hand, Dp-branes whose dimension is not suitable to couple to an RR potential of the theory are unstable. This leads to the following conclusion regarding the brane content of the two Type II theories

- ***Type IIA theory***: Since we have the RR-forms C_1, C_3, C_5, C_7 we expect that there are D0, D2, D4 and D6-branes.
- ***Type IIB theory***: Here we have all the even RR-fields C_0, C_2, C_4, C_6, C_8. The corresponding branes are the D(-1) (the instanton), D1, D3, D5, D7 and D9-branes.

The branes that couple to dual potentials are often called dual branes. For example, in Type IIA the D0 and D6 couple to the C_1 and C_7 which are the magnetic duals of each other. This is why

[4] The fundamental string is the object that we start with when we study String Theory. It is not the same as a D1-brane although they have the same dimension. This makes sense since if this was the case, then only Type IIB theory would have strings. This would apparently be inconsistent since the starting point of all String Theories (including Superstrings) is the quantization of a one-dimensional object.

Type IIA	F1	NS5	D0	D2	D4	D6	D8	
Type IIB	F1	NS5	D(-1)	D1	D3	D5	D7	D9

Table 1.1: Branes in Type II theories

branes come in pairs in each theory just like RR fields come in pairs. We can play the same game for the NS-NS gauge field B_2. This is the field to which the fundamental string couples and has a 3-form strength H_3 whose dual is a 7-form coming from a 6-form potential. This should couple to a six-dimensional brane which should exist in both Type II theories (since the NS-NS 2-form does). The brane which couples to this potential is called *NS5-brane*. We could say that Type IIA contains all odd dimensional branes and Type IIB contains all the even dimensional ones. However something is still missing since there is no D8-brane in Type IIA. It turns out that such a brane exists in a version of Type IIA supergravity called *massive supergravity* or *Romans supergravity* which includes a scalar field analogous to a cosmological constant. Associated to this cosmological constant there is a 9-form to which the D8-brane couples (see e.g. [8]). To recap, we present the brane content of Type II supergravities in Table 1.1.

1.4 Dualities

Dualities in String Theory are equivalences between seemingly different theories. In some sense, they serve as dictionaries that allow us to translate between different formulations of the same underlying physics. In principle one could propose that we only use one language, namely one String Theory so that dualities would no longer be useful and our lives would be easier. However, due to the intricate nature of Sting Theory many aspects of which remain elusive, it is often imperative to use a certain picture which is well suited for the problem at hand and then use dualities to compare it with what we know from a different formulation. We have already noted that the five different Superstring Theories are in fact related by dualities. This holds true for the Type II theories which are related among themselves and with M-theory via the mechanism described in the previous section. Here we will attempt a quick overview of the two main dualities.

1.4.1 T-duality

T-duality [9, 10] is a symmetry with a long history since it existed even in the context of bosonic String Theory where it relates the theory to itself. In Superstring theory, T-duality relates the two Type II supergravities. In order to perform this duality, we need to pick out a special circular direction in our ten dimensional spacetime. This is completely analogous to the step required to get from M-theory to Type IIA String Theory and whose details and implications will be thoroughly explored in the next chapter. Let us start in Type IIA, call the coordinate in the circular direction X^{10} and suppose that its length scale is R. Then applying T-duality lands us to Type IIB with a circular direction of length scale inversely proportional to R. More specifically, if ℓ_s denotes the string length then under T-duality we have

$$R \longleftrightarrow \frac{\ell_s^2}{R} \qquad g_s \longleftrightarrow g_s \frac{\ell_s}{R}$$

In this sense T-duality relates Type IIA/IIB compactified on a circle of radius R to Type IIB/IIA compactified on a circle of radius ℓ_s^2/R. T-duality also changes the field content and this is of course to be expected since Type II theories have different RR fields. Under this duality, the "10" index (with x^{10} the circular dimension) is added to the anti-symmetric indices of the field C_p if it was not there (so that it becomes a $(p+1)$-form) or the "10" index is deleted if it was already there (so that we get a $(p-1)$-form). By passing to branes and using the correspondence between RR fields and Dp-branes we find that a Dp-brane becomes a D(p+1)-brane if it did not extend along the x^{10} circle direction or it becomes a D(p -1)-brane if it wrapped the circular direction. This holds for the branes that are charged under the RR fields. The fundamental string F1 and NS5-brane are simply mapped to themselves.

1.4.2 S-duality

As we saw, T-duality relates the Type II theories. There exists another duality, called *S-duality* that was first discovered in the context of heterotic theory [11]. It was subsequently discovered that S-duality relates Type IIB theory to itself [12]. In contrast to T-duality, it is a non-perturbatve duality in the sense that it relates theories at strong coupling to theories at weak coupling. In it's simplest form S-duality acts on the coupling constant as follows

$$g_s \longleftrightarrow \frac{1}{g_s}$$

Such an exchange begets an exchange in the degrees of freedom of the theory that are excited. For example, branes are inherently non-perturbative objects. What we mean by this can be understood if we examine the tension (mass density) of a Dp-brane which can be shown to be given by [7]

$$T_{Dp} = \frac{2\pi}{g_s} \frac{1}{(2\pi \ell_s)^{p+1}}$$

This justifies why one first discovers D-branes as immovable, non-dynamical objects. It is because the approach that we have to String Theory is purely perturbative meaning that the coupling is very small so that the tension of a Dp-brane is indeed so large that it practically behaves like a solid wall. This is frustrating because it reminds us of the limitations that a perturbative approach entails but at the same time fascinating since every bit of information that we get about D-branes is a peek to the world of non-perturbative String Theory. Contrary to Dp-branes, the fundamental string has tension

$$T_{F1} = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi\ell_s^2}$$

This is why the fundamental string is a perturbative object as it should be since it serves as the starting point in the definition of perturbative String Theory. This also clarifies the sense in which the D1-brane (sometimes called D-string) is different from the fundamental string. Indeed, although they have the same dimension, they interact with gravity in a different way since they have different tension. The tension of the D-string is

$$T_{D1} = \frac{1}{2\pi g_s \ell_s^2}$$

This implies that if the coupling becomes very large, it can be the case that the D-string becomes lighter than the fundamental string and in this sense it will rightfully claim the title of the more "fundamental" object. Indeed, this is what happens under S-duality. It is often the case (especially in the context of supergravity) that non-perturbative configurations are referred to as *solitons*, a term which will be more thoroughly explained later. This is why it is frequently said that S-duality exchanges the elementary (perturbative) excitations of a theory with the solitonic (non-perturbative) ones and vice versa.

Now that we have gotten a feeling of what S-duality is and its significance for uncovering non-perturbative effects that would otherwise be completely out of reach with our humble perturbative methods, we will present it in more detail. In fact, we will discuss a much larger symmetry of Type IIB supergravity which contains S-duality. This global symmetry corresponds to the non-compact group SL(2,$\mathbb{R}$). As usual, when we want to discover symmetries, the right place to look is the action of a theory. We will therefore go back to the Type IIB action 1.3 and in order to make the sought after symmetry manifest, we will perform some redefinitions. We first introduce a vector (of 2-forms) given by

$$\vec{B} = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

This vector contains the NS-NS and RR form to which the fundamental and D-string couple. Consider an element $\Lambda \in$ SL(2,$\mathbb{R}$) given by

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

Under the action of Λ the 2-forms transform as a doublet and the action is given concisely in terms of the vector $\vec{B}$ as follows:

$$\vec{B} \longrightarrow \Lambda \vec{B}$$

with the action on the right-hand side being the standard matrix multiplication. Since the matrix elements of Λ do not have any spacetime dependence, by taking the differential of the two sides, we find that the same transformation law applies to the 3-form field strengths. The next redefined field we introduce is a complex scalar field called the *axio-dilaton* field τ which plays an important role in String Theory compactifications. As its name suggests, it is formed by combining the field C_0 (which is traditionally called the axion field) with the dilaton ϕ. In particular we have

$$\tau = C_0 + ie^{-\phi} \tag{1.4}$$

Recall the relation between the dilaton and the string coupling which implies that the complex part of τ is the inverse of the coupling constant. Under the action of the element Λ as above, the transformation of the axio-dilaton reads

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

In order to make our notation even more compact, we can introduce the symmetric SL(2,$\mathbb{R}$) matrix

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\tau|^2 & -C_0 \\ -C_0 & 1 \end{pmatrix}$$

This essentially helps to simplify the admittedly complicated transformation law for τ. The transformation of the new object $\mathcal{M}$ follows the simpler linear rule

$$\mathcal{M} \longrightarrow (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1}$$

Finally, the last field that transforms under the action of Λ is the metric. This is evident if we look at the action 1.3 where the Einstein-Hilbert term is multiplied by a function of the dilaton. Since the dilaton transforms non-trivially under the SL(2,$\mathbb{R}$) action, if we want the action to be invariant, the metric has to transform as well. In order to overcome this problem, it is useful to make a field redefinition such that the Einstein-Hilbert term is not multiplied by anything. This redefinition only needs to involve the metric and the dilaton. We define a new metric given by

$$g^E_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$$

The new metric $g^E_{\mu\nu}$ is often called *Einstein metric* and the old one $g_{\mu\nu}$ is referred to as the *string metric*. This redefinition amounts to going from the *string frame* to the *Einstein frame*. This change of frame has the following effect on the first term of the Type IIB action 1.3

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}R = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g^E}\Big(R - \frac{9}{2}\partial^\mu\phi\partial_\mu\phi\Big)$$

In the right hand side, the Ricci scalar is calculated with the Einstein metric. We chose to write the integral in coordinates and explicitly write the integration measure to stress out the use of a different metric. Having introduced this new notation we can now proceed to re-express the Type IIB action as

$$\begin{aligned} S_{IIB} =& \frac{1}{\kappa_{10}^2}\int d^{10}x\sqrt{-g}\Big(R - \frac{1}{12}\vec{H}^T\mu\nu\rho\mathcal{M}\vec{H}^{\mu\nu\rho} + \frac{1}{4}tr(\partial^\mu\mathcal{M}\partial_\mu\mathcal{M}^{-1})\Big) \\ & - \frac{1}{8\kappa_{10}^2}\Big(\int \tilde{F}_5 \wedge *\tilde{F}_5 + \epsilon_{ij}C_4 \wedge H^{(i)} \wedge H^{(j)}\Big) \end{aligned}$$

Here we used the Einstein metric. In the last term we chose to express the matrix multiplication in components so that $H^{(i)}$ denotes the i'th component of the vector $\vec{H}$. We can also express $\tilde{F}_5$ in a manifestly invariant form as

$$\tilde{F}_5 = F_5 + \frac{1}{2}\epsilon_{ij}B_2^{(i)} \wedge H_3^{(j)}$$

Additionally, the Hodge star operator is invariant under a rescaling of the metric like the one that we used to go from the string to the Einstein frame so that $*\tilde{F}_5$ is also invariant. This shows that not only the action but also the self-duality condition of $\tilde{F}_5$ is invariant under the SL(2,$\mathbb{R}$) action. The invariance of all the equations of motion shows that this is a global symmetry. We claimed in the beginning of this discussion that S-duality is contained in this SL(2,$\mathbb{R}$) symmetry. To see this take a background with vanishing C_0 and pick the SL(2,$\mathbb{R}$) element

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This acts on the axio-dilaton by $\tau \longrightarrow -1/\tau$ so that when $C_0 = 0$ we get the S-duality transformation $g_s \longrightarrow 1/g_s$.

The discovery of the global SL(2,$\mathbb{R}$) is great news. However, recall that Type IIB supergravity, for which we proved that the symmetry holds, is only the low-energy limit of Type IIB String Theory. The natural question then arises whether this is an honest symmetry or just a low-energy artifact. It turns out that this symmetry is actually broken by quantum effects and it is believed that the subgroup SL(2,$\mathbb{Z}$) survives as a true symmetry of the full theory. One argument for why it is imperative to only consider integer coefficients in the transformation can be given in terms of the violation of the Dirac quantization condition (see section 18.3 of [13]).

1.5 A geometric SL(2,$\mathbb{Z}$) transformation

Having established this powerful symmetry for the Type IIB superstring theory, we would like to address the question of what it begets for the Type IIA theory. To understand that, it is useful to consider an M-theory with an *isometric torus action* or in the common physics language M-theory on a torus. This means that we have two isometric directions on which we can reduce M-theory to get two Type IIA backgrounds. Those backgrounds are in principle different. Our purpose is to argue that they are dual. It can be shown [14, 15] that the SL(2,$\mathbb{Z}$) symmetry of Type IIB supergravity is the same as the SL(2,$\mathbb{Z}$) symmetry of the torus on which eleven dimensional supergravity is compactified on.

To demonstrate explicitly what this means, consider the independent cycles c_1 and c_2 in the isometric T^2 of M-theory and suppose that they are related by an element $\Lambda \in \mathrm{SL}(2,\mathbb{Z})$ so that $\Lambda c_1 = c_2$ where c_1, c_2 are represented as column vectors. The more rigorous way to do that is to consider c_1, c_2 as elements of the homology group $H_2(T^2)$ which is indeed a two dimensional vector space on which the action of $\mathrm{SL}(2,\mathbb{Z})$ is standard matrix multiplication. The statement that this action is the same as the $\mathrm{SL}(2,\mathbb{Z})$ action on Type IIB amounts to the commutativity of the diagram in Figure 1.2. The dashed arrows denote the duality at the level of Type IIA which geometrically, when lifted to M-theory, translates to the torus symmetry. This dashed arrow is a type of relates different Type IIA theories. However, it is crucial that the existence of the diagram heavily relies on the existence of an isometric circle in the ten dimensional theory [16]. This is why it is important that we start with a torus isometry in eleven dimensions.

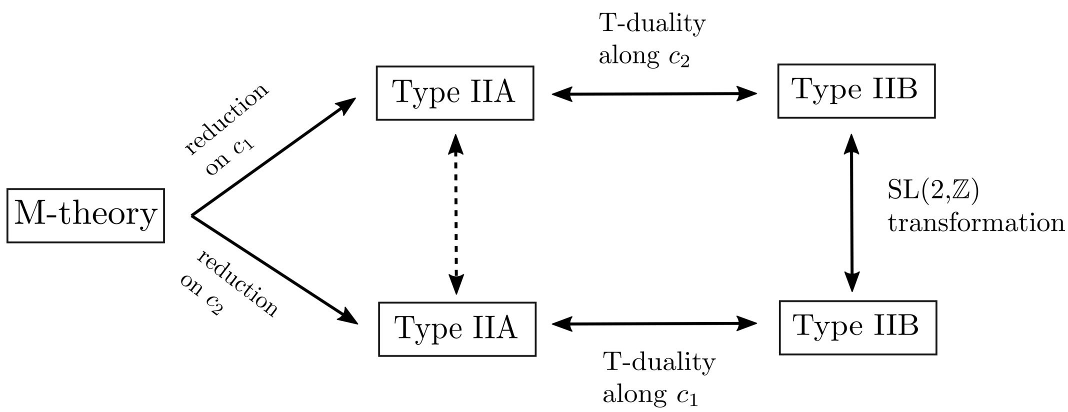


Figure 1.2: The correspondence between an SL(2,$\mathbb{Z}$) transformation on the torus in M-theory and an SL(2,$\mathbb{Z}$) transformation of the Type IIB theory.

The commutativity of the diagram can be proved by an explicit application of the various dualities and transformations (see e.g. section 8.4 of [17]). It turns out that the axio-dilaton τ on the Type IIB side corresponds to the complex structure parameter τ of the torus on the M-theory side. By this we mean that we think of T^2 as the quotient of $\mathbb{C}$ by the lattice generated by 1 and $\tau \in \mathbb{C}$. Every different choice of τ results in a different induced complex structure in the quotient. However any two complex structure parameters related by an SL(2,$\mathbb{Z}$) transformation are equivalent (they define the same complex structure). The group SL(2,$\mathbb{Z}$) in this context arises very naturally as the *modular group* of the torus.

We have therefore discovered that the non-perturbative SL(2,$\mathbb{Z}$) symmetry of Type IIB superstring theory admits a purely geometric interpretation in terms of the modular group of the torus that acts isometrically on an M-theory background. Additionally, we now understand that in the presence of an isometric T^2 acting on an M-theory background, Type IIA backgrounds related by reductions along different cycles of the torus should be dual. We thus expect them to exhibit the same kind of physics. In this thesis, we want to exploit this duality to understand the dynamics between D6-branes and D6-anti-branes in Type IIA superstring theory. Before we do that we will need to understand their geometric origin. Indeed, we will soon set out understand the theory of group actions and why they are important for studying string backgrounds. But before we do that, we will give a short motivation for why studying brane–anti-brane systems is a cornerstone for the development of a better understanding of string theory.

1.6 Brane–anti-brane systems and tachyons

We have already mentioned the existence of tachyonic degrees of freedom that plagued bosonic string theory since the first days of its development. This tachyon was believed to indicate an instability and its elimination is still considered one of the most important landmarks of superstring theory. The removal of the closed string tachyon is possible due to the *GSO projection* [18], which ensures that the closed string tachyon can be removed from the theory spectrum without violating some fundamental consistency requirements related to modular invariance. This miracle projection however cannot alleviate the open string tachyonic modes that arise in brane–anti-brane systems. Indeed, there are two tachyonic modes arising from open strings stretching from the brane to the anti-brane and vice-versa. We therefore have the following:

In superstring theory, there are tachyonic modes associated to open strings stretching between branes and anti-branes.

The cryptic nature of tachyons as carriers of imaginary mass is demystified once a field theoretic approach is adopted. This can be easily illustrated by considering a scalar field ϕ with potential $V(\phi)$. A perturbation expansion around $\phi = 0$ reveals that $V''(0)$ is the quadratic coefficient and therefore has the interpretation of a mass squared term (ignoring cubic and higher order terms). From this perspective, a negative mass squared simply means that we are expanding around a maximum of the theory and the tachyon is the harbinger of the breakdown of perturbation theory. Put differently, since our theory is unstable under small perturbations around the expanding point $\phi = 0$, using perturbation techniques makes no sense. At this point, the sensible thing to do is perform our perturbation expansion around a minimum of the potential.

The inherent instability of the spaces containing brane–anti-brane pairs will be a guiding principle for us in the following. The main question that we seek to answer is how the brane–anti-brane system evolves and how to understand its unstable nature. The methods that we will employ are inherently geometrical and the field theoretic nature of the tachyonic instability is obscured but we should not forget that it is our main argument for expecting the system to be unstable in the first place.

Chapter 2

Group actions on manifolds

2.1 General properties of group actions

In this chapter we attempt to give a fairly general account of group actions on four-manifolds. In particular we will focus on circle and torus actions.

Definition 2.1.1. A **left action** of a group G on a set M is a map

$$\begin{aligned}\Phi : G \times M &\longrightarrow M \\ (g, p) &\mapsto \Phi(g, p) := g \cdot p = gp\end{aligned}$$

satisfying the following properties

- $(g \cdot h) \cdot p = g \cdot (h \cdot p)$ for all $p \in M$ and $g, h \in G$.
- $e \cdot p = p$ for all $p \in M$ where e is the identity element of G.

The case of interest for us is when M is a smooth manifold and G a Lie group in which case we can require the map Φ to be a smooth map. Such actions are called *smooth*. A space M acted upon by a group G is often called a ***G-space***. The natural morphisms between G-spaces are G-equivariant maps (or simply equivariant maps)

Definition 2.1.2. An ***equivariant map*** is a map $f : M \to N$ between G-spaces that commutes with the group action namely

$$f(g \cdot p) = g \cdot f(p) \qquad \forall p \in M$$

An equivariant map which is also a diffeomorphism is called an equivalence of G-spaces.

We clarify some common terminology that we will use.

Definition 2.1.3. Let G be a group acting on a manifold M via an action $\Phi : G \times M \to M$

1. Given some point $p \in M$ the **orbit map** is $\Phi_p := \Phi(-, p) : G \to M$.
2. The **orbit** $\mathcal{O}_p$ of G through a point $p \in M$ is the image of the orbit map Φ_p.
3. The **fixed point set** of a group element $g \in G$ is the set

$$M^g = \{p \in M | g \cdot p = p\}$$

4. The **isotropy group** or **stabilizer** of a point $p \in M$ is

$$G_p = \{g \in G | g \cdot p = p\}$$

5. The group action is called **free** if all of the stabilizers are trivial.
6. The group action is called **effective** if $M^g = M$ only for $g = e$. We introduce the ***kernel*** of an action $\ker\Phi := \{g \in G | g \cdot x = x \ \ \forall x \in M\}$. Then effectiveness is equivalent to $\ker\Phi = \emptyset$.
7. The group action is called **semi-free** if the stabilizers are either trivial or the entire G.
8. The group action is called **locally free** if all the stabilizers are discrete.

In many cases we want to assume our actions to be effective. This is not very restrictive because any ineffective action naturally gives rise to an effective one.

Proposition 2.1.1. Let $\Phi : G \times M \to M$ be a group action and let $N = \ker\Phi$. Then there is a canonically induced effective action $\tilde{\Phi}$ of G/N on M.

Proof. Let $\tilde{\Phi} : G/N \times M \to M$ defined by $[g] \cdot x = g \cdot x$ for $[g] \in G/N$. This is clearly well-defined and has trivial kernel so we only need to check continuity. We have the commutative diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\Phi} & M \\ {\scriptstyle \pi\times\mathrm{id}}\big\downarrow & \nearrow_{\tilde{\Phi}} & \\ G/N \times M & & \end{array}$$

Then for any open $U \subseteq M$ we have $\tilde{\Phi}^{-1}(U) = \pi(\times\mathrm{id})(\Phi^{-1}(U))$ and since the projection $\pi : G \to G/N$ is open the claim follows. □

Example 2.1.1. As a first example of a Lie group action on a manifold we discuss the action of S^1 on S^3. Consider S^3 as a subset of $\mathbb{C}^2$ via:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ \mid \ |z_1|^2 + |z_2|^2 = 1\}$$

and $S^1 \subset \mathbb{C}$ as the unit complex numbers $p, q \in \mathbb{Z}$ and define the group action

$$\begin{aligned} \Phi : S^1 \times S^3 &\longrightarrow S^3 \\ (\lambda, z_1, z_2) &\mapsto (\lambda^p z_1, \lambda^q z_2) \end{aligned}$$

This is in fact an action on $\mathbb{C}^2$ that restricts to an action on S^3. In general it is not effective with $\ker\Phi = \{\lambda \in S^1 \mid \lambda^p = \lambda^q = 1\} \simeq \mathbb{Z}_{\mathrm{gcd(p,q)}}$ which is empty if and only if p, q are coprime. In any other case we can divide by the kernel as in Proposition 2.1.1 to obtain an effective action. In the case $p = q = 1$ the action is called the ***Hopf action***. When $p = 1, q = -1$ (or vice versa) the action is called the ***anti-Hopf action***. These are the only two cases in which Φ is free.

An interesting case arises when M is a Riemannian manifold endowed with a G-invariant metric. Initially this might seem to restrict the allowed types of actions but in fact when the group G is compact (as will be in our considerations) we can always assume this as the following proposition shows:

Proposition 2.1.2. If G is a compact Lie group acting smoothly on a smooth manifold M, then there exists a G-invariant metric on M.

Proof. First note that every smooth manifold M admits a Riemannian metric. Choose a partition of unity $\{\psi_\alpha\}$ subordinate to an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n\}$. On V_α we have the standard metric induced from $\mathbb{R}^n$ which we call $g^{(\alpha)}$. Then there is a natural pullback metric $\phi_\alpha^*(g^{(\alpha)})$ on U_α and by trivially extending the functions ψ_α to be zero outside U_α we can define a metric g on TM such that for every $X, Y \in \mathbf{Vect}(M)$ we have

$$g(X, Y) = \sum_\alpha \psi_\alpha \phi_\alpha^*(g^{(\alpha)})(X, Y) = \sum_\alpha \psi_\alpha g^{(\alpha)}(D\phi_\alpha X, D\phi_\alpha Y)$$

If the cover $\{U_\alpha\}$ is locally finite, which we can always assume, then this sum is finite at each point and it is easy to check that the properties of a metric are satisfied. Now let $\Phi : G \times M \to M$ be the given group action with G a compact Lie group. By compactness, G admits a Haar measure $d\mu$ which is bi-invariant and is such that G has unit volume with respect to it. Then we can define a new metric $\tilde{g}$ on M given by

$$\tilde{g}(X,Y) = \int_G g(D\Phi_h X, D\Phi_h Y) d\mu_h$$

where h denotes the integration variable. This metric is G-invariant. Indeed if $h' \in G$ then we have

$$\begin{aligned}(\Phi_{h'}^* \tilde{g})(X,Y) &= \tilde{g}((D\Phi_{h'})X, (D\Phi_{h'})Y) = \int_G g((D\Phi_{h'\cdot h})X, (D\Phi_{h'\cdot h})Y) d\mu_h \\ &= \int_G g((D\Phi_{h'\cdot h})X, (D\Phi_{h'\cdot h})Y) d\mu_{h'\cdot h} = \int_G g(D\Phi_h X, D\Phi_h Y) d\mu_h = \tilde{g}(X,Y)\end{aligned}$$

where in the second equality we used the naturality of the pushforward and in the third the invariance of the Haar measure while in the last equality we changed the integration variable. This completes the proof. □

We can therefore talk about isometric actions without loss of generality as long as we do not have any reason to focus on a special metric on a manifold. This will change later, when we consider physical applications in which the metric is specified as a solution to the Einstein equations but for the moment the assumption of isometric actions is harmless.

Given a group action $\Phi : G \times M \to M$ we often find it useful to consider the *quotient space* or *space of orbits* M/G which comes with the canonical projection $\pi : M \to M/G$. As a topological space, M/G is endowed with the quotient topology. This means that a subset $U \subseteq M/G$ is open if and only if $\pi^{-1}(U) \subset M$ is open. The smooth structure of the quotient space M/G is not evident and in fact it can easily fail to be a manifold. However the situation becomes simpler when the action is free and G is compact.

Theorem 2.1.1. Let G be a compact group acting freely on a manifold M. Then there exists a smooth structure on M/G such that $\pi : M \to M/G$ is a principal G-bundle and in particular a submersion.

Sketch of the proof. To show the manifold structure on the quotient space we work as follows. By Proposition 2.1.2 there exists a G-invariant metric on M. Therefore we consider the Riemannian manifold (M, g) such that G acts with isometries. Then for $p \in M$ consider the map $\Phi_p : G \to M$. This map is an injective immersion as we will prove in Theorem 2.1.3 and since G is compact it is also an embedding. Therefore there exists a tubular neighbourhood

$$N_\epsilon(\mathcal{O}_p) = \exp(\nu^{<\epsilon}(\mathcal{O}_p))$$

where $\nu^{<\epsilon}(\mathcal{O}_p)$ denotes an open neighbourhood of the zero section in the normal bundle of the orbit $\mathcal{O}_p$. This is a G-invariant tubular neighbourhood since the Riemannian exponential map commutes with isometries. If we let $S_p := \exp(\nu_p^{<\epsilon}(\mathcal{O}_p)$ be the image of the an open neighbourhood of 0 in the normal space at p then $\pi(S_p)$ is open in M/G since

$$\pi^{-1}(\pi(S_p)) = \bigcup_{p \in \mathcal{O}_p} \exp(\nu_p^{<\epsilon}(\mathcal{O}_p)) = \exp(\nu^{<\epsilon}(\mathcal{O}_p)) = N_\epsilon(\mathcal{O}_p)$$

The situation can be visualized in Figure 2.1. Then a chart around a point $[p] \in M/G$ can be given by taking some representative $p \in \pi^{-1}([p])$ and identifying S_p with $\mathbb{R}^k$ where $k = \dim M - \dim G$ so that we have

$$\mathbb{R}^k \to \nu^{<\epsilon}(\mathcal{O}_p) \xrightarrow{\exp} S_p \to \pi(S_p)$$

The advantage of those coordinates is that it is easy to see that π is a submersion (by the submersion theorem). Then one can show that these charts give M/G a smooth structure. To show that it has a principal bundle structure we use the same neighbourhoods $\pi(S_p)$ and the local trivializations

$$t : \pi(S_p) \times G \to \pi^{-1}(\pi(S_p)) \qquad (q, g) \mapsto g \cdot q$$

Given two trivializiations $(\pi(S_{p_1}), h_1)$ and $(\pi(S_{p_2}, h_2)$ we choose some $h \in G$ such that $h \cdot p_1 = p_2$ and we calculate the transition functions. For this it is useful to write the arbitrary element $q \in S_p$ as $q = \exp(v)$ and then we calculate

$$\begin{aligned}(h_2^{-1} \circ h_1)(\pi(q), g) &= h_2^{-1}(g \cdot q) = h_2^{-1}(g \cdot \exp_{p_1}(v)) = h_2^{-1}(gh^{-1} \exp_{p_2}(h_* v)) \\ &= (\pi(\exp_{p_2}(h_* v)), gh^{-1}) = (\pi(q), gh^{-1})\end{aligned}$$

and so the transition function is $g \mapsto gh^{-1}$. □

The following is an immediate consequence.

Corollary 2.1.1. If G is compact and $H \subset G$ is a closed subgroup then $G \to G/H$ is a principal H-bundle.

Proof. Consider the right action of H on G. This is clearly free and therefore by the above theorem the result follows. □

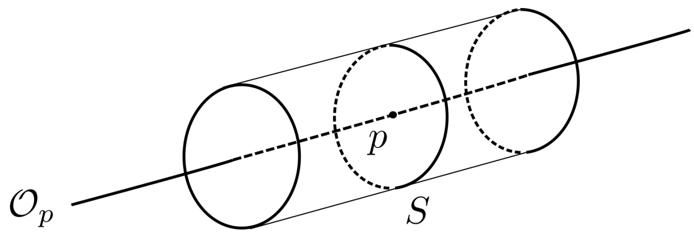


Figure 2.1: An equivariant tubular neighbourhood of an orbit $\mathcal{O}_p$

Example 2.1.2. As an example we can again consider the Hopf action which is free. It is not very hard to show that its quotient space is isomorphic to $\mathbb{CP}^1 \simeq S^2$. Indeed, the obvious map $\phi : S^3/S^1 \to \mathbb{CP}^1$ such that $[(z_1, z_2)] \mapsto [z_1 : z_2]$ is an isomorphism, with the right hand side expressed in homogeneous coordinates. We then have a principal bundle structure:

$$\pi : S^3 \to \mathbb{C}P^1 \quad , \quad (z_1, z_2) \mapsto [z_1 : z_2]$$

We call this bundle the Hopf fibration. The non-triviality of this fibration is evident, for example by noticing that $\pi(S^3) = 0$ while $\pi(S^2 \times S^1) = \mathbb{Z}$.

Example 2.1.3. As an example of an action which is not free consider the S^1 action on the sphere S^2 given by rotations around some axis. The orbits are clearly circles and there are two fixed points, the two poles P, Q. The quotient map geometrically collapses the orbits to points as in Figure 2.2. Note how the fixed points act as boundaries for the disk S^2/S^1. This is not a coincidence but a general occurrence as we will see later.

In Example 2.1.2 the action is free and the isotropy groups are all trivial. Additionally, all the orbits are diffeomorphic to S^1 and this allows for the bundle structure. In general however, this is not the case as we saw in Example 2.1.3. First note that for any $g \in G$, $p \in M$ we have $G_{g \cdot p} = gG_pg^{-1}$ so that isotropy groups along the same orbit are not just isomorphic but conjugate.

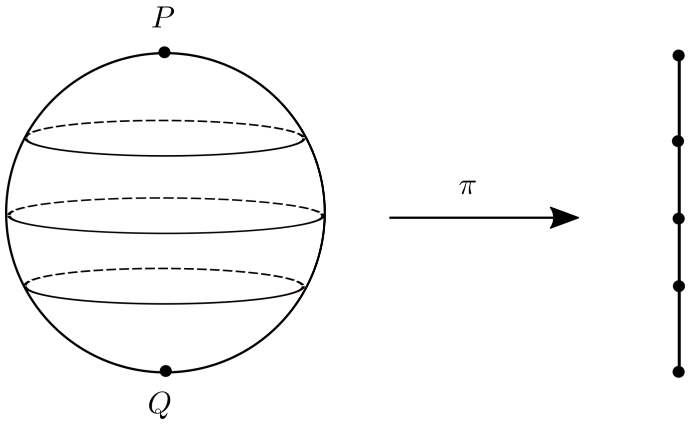


Figure 2.2: The quotient map $S^2 \to S^2/S^1$

In order to understand the structure of the set of isotropy groups $\{G_p \mid p \in M\}$ we impose an equivalence relation declaring two isotropy groups $G_p, G_q \subseteq G$ equivalent if they are conjugate as subgroups of G. The resulting equivalence classes are called *isotropy types*. In a similar fashion, we declare two orbits $\mathcal{O}_p, \mathcal{O}_q$ to be equivalent if the isotropy subgroups at p and q are conjugate. Such an equivalence class is called an *orbit type*. This definition makes sense because the orbit $\mathcal{O}_p$ is determined by the isotropy subgroup G_p. Indeed, it easily follows from the definitions that the orbit $\mathcal{O}_p$ is in bijection with G/G_p which makes them complementary to one another. We can additionally define a partial ordering in the set of isotropy types. If G_p, G_q are two isotropy groups with isotropy types $[G_p], [G_q]$ we say that $[G_p] \leq [G_q]$ if and only if $[G_p]$ is conjugate to a subgroup of $[G_q]$. This clearly induces a partial ordering in the set of orbit types by reversing the inequality.

Lemma 2.1.1 (Kleiner's lemma)**.** Let G act isometrically on M via $\Phi : G \times M \to M$. If $c : [0,1] \to M$ is a minimal length curve from $\mathcal{O}_{c(0)}$ to $\mathcal{O}_{c(1)}$ then there exists a subgroup $H \subset G$ such that $G_{c(t)} = H$ for $t \in (0,1)$ and H is a subgroup of $G_{c(0)}$ and $G_{c(1)}$.

Proof. Let $H := \{g \in G \mid g \cdot c(t) = c(t) \quad \forall\ t \in [0,1]\}$. Suppose there exists $t_0 \in (0,1)$ and $g \in G_{c(t_0)}$ such that $g \notin H$. Consider $\Phi_g := \Phi(g, -) : M \to M$. Then $D_{c(t_0)}\Phi_g(\dot{c}(t_0)) \neq \dot{c}(t_0)$. Define a piecewise smoth path $\tilde{c} : [0,1] \to M$ by setting $\tilde{c}|_{[0,t_0]} := c|_{[0,t_0]}$ and $\tilde{c}|_{[t_0,1]} := \Phi_g \circ c$. Then $\tilde{c}$ joins $\mathcal{O}_{c(0)}$ and $\mathcal{O}_{c(1)}$ and has the same length as c. This is a contradiction since length minimizing geodesics are smooth. □

Theorem 2.1.2. Let G be a compact Lie group acting on a Riemannian manifold M. Then there exists a unique maximal orbit type (equivalently a unique minimal isotropy type).

Proof. Applying Zorn's lemma we need to show that any decreasing chain

$$G \geq K_1 \geq K_2 \geq ...$$

of isotropy groups stabilizes. Notice that K_{i+1} is conjugate and therefore isomorphic to a subgroup of K_i. Additionally, K_i are closed subgroups of G (see Theorem 2.1.3). It follows that $\dim(K_{i+1}) \leq \dim(K_i)$ and this clearly stabilizes after some sufficiently large i. This implies the Lie algebras $\mathrm{Lie}(K_{i+1})$ and $\mathrm{Lie}(K_i)$ are isomorphic. This means that the identity components of K_i and K_{i+1} are isomorphic as Lie groups. It could be the case that some K_i has infinitely many components so that the chain does not have to end. However, this is excluded by the compactness of G and the compactness of K_i which are closed in G and therefore compact.

To prove uniqueness suppose there exist two different isotropy groups G_p, G_q that are minimal but not conjugate. By Kleiner's lemma we can consider a length minimizing geodesic $c : [0,1] \to M$

and the subgroup H in Kleiner's lemma 2.1.1 will be conjugate to a subgroup of both G_p and G_q. But since G_p, G_q are minimal, H is conjugate to both of them and therefore they are conjugate to each other proving that $[G_q] = [G_p]$. □

We often call the isotropy groups of minimal isotropy type the *principal isotropy groups* and the orbits of maximal orbit type *principal orbits*. Next we wish to understand the local behavior of M around the orbits. We already have Theorem 2.1.2 but we want to be more general.

Definition 2.1.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let G act smoothly on M via the action $\Phi : G \times M \to M$ and let $\Phi_p : G \to M$ be the orbit map at p. Then for any $X \in \mathfrak{g}$ the ***fundamental vector field*** $X^{\#} \in \mathrm{Vect}(M)$ corresponding to X is

$$X^{\#}(p) := (D_e\Phi_p)(X) = \frac{d}{dt}\bigg|_{t=0} \Phi(\exp(tX), p)$$

where $\exp : \mathfrak{g} \to G$ is the exponential map of the Lie algebra.

Intuitively, fundamental vector fields indicate the direction towards which points on M move by an "infinitesimal" action of a one-dimensional subspace of G which is the image under the exponential map of a one-dimensional linear subspace of $\mathfrak{g}$ spanned by the element $X \in \mathfrak{g}$. In Example 2.1.3 the fundamental vector field would be tangent to circles of constant latitude and would vanish at the poles which agrees with the interpretation we just gave, since the poles do not move in any direction under the group action.

Theorem 2.1.3. [19] Let G be a Lie group acting on a manifold M and let $p \in M$.

- The stabilizer G_p is a closed subgroup in G, with Lie algebra $\mathfrak{h} = \{X \in \mathfrak{g} | \ X^{\#}(p) = 0\}$ where $X^{\#}$ is the vector field on M corresponding to X.
- The orbit map $G/G_p \to M$ is an immersion whose image coincides with $\mathcal{O}_p$.

Proof. For the first claim it suffices to show that in some neighbourhood U of $1 \in G$ the intersection $U \cap G_m$ is a submanifold with tangent space $T_eG_m = \mathfrak{h}$. Then the same will hold around any point $g \in G$ and neighbourhood gU. It can be shown by some elementary differential geometry that $[X,Y]^{\#} = [X^{\#}, Y^{\#}]$ where the bracket on the left hand side is the Lie algebra bracket and the bracket on the right hand side the Lie bracket of vector fields on M. Using this we deduce that $\mathfrak{h}$ is closed under commutator and therefore a subalgebra of $\mathfrak{g}$. Additionally since $tX \in \mathfrak{h}$, we have that the path $\Phi(\exp(tX), p)$ in M must be constant since the derivative vanishes everywhere so that $\Phi(\exp(tX), p) = p$ and therefore $\exp(tX) \in G_p$.

Next we choose a vector subspace $u \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus u$. As before we have $D_e\Phi_p : \mathfrak{g} \to T_pM$ whose kernel is (by definition) $\ker(D\Phi_p) = \mathfrak{h}$. Since $u \cap \mathfrak{h} = \emptyset$ the kernel of $D_e\Phi_p|_u : u \to T_pM$ which is now restricted to u must be empty and therefore $D_e\Phi_p|_u$ is an injection. Using the implicit function theorem this means that the map $u \to M$ given by $Y \mapsto \Phi(\exp(Y), p)$ (whose differential is $D_e\Phi_p$) is injective for Y in a sufficiently small neighbourhood of $0 \in u$ so

$$\exp(Y) \in G_p \Leftrightarrow \Phi(\exp(Y), p) = p \Leftrightarrow \Phi(\exp(Y), p) = \Phi(\exp(0), p) \Leftrightarrow Y = 0 \qquad (2.1)$$

It follows from the definition of the exponential map (see any textbook in Lie theory) that the pushforward $D_0 \exp : \mathfrak{g} \to \mathfrak{g}$ is the identity. Therefore, using the inverse function theorem we a neighbourhood V of $0 \in \mathfrak{g}$ and a neighbourhood U of $e \in G$ such that $\exp : V \to U$ is a diffeomorphism. Therefore, we can express any $g \in U$ as $g = \exp(X+Y) = \exp(Y)\exp(X)$ for unique $X \in \mathfrak{h}$ and $Y \in u$ which are sufficiently close to the zero vector. We then get

$$g \in G_p \Leftrightarrow \exp(Y)\exp(X) \in G_p \Leftrightarrow \exp(Y) \in G_p \Leftrightarrow Y = 0$$

where in the last step we used 2.1. We therefore conclude that $g \in G_p \Leftrightarrow g \in \exp(\mathfrak{h})$. Since $\exp(\mathfrak{h})$ is a submanifold in a neighbourhood of $0 \in \mathfrak{g}$ and the exponential map is a local diffeomorphim, it follows that G_p is a submanifold in the neighbourhood U of the identity $e \in G$.
The same arguments show that $T_e(G/G_p) \simeq \mathfrak{g}/\mathfrak{h} \simeq u$ and injectivity of the map $D_e\Phi_p|_u : u \to T_pM$ shows that the map $G/G_p \to M$ is an immersion. □

This shows that the orbits are images of injective immersions. However, they are not always submanifolds. This is true however for the case of compact manifolds. The local structure of a quotient M/G is also well behaved as long as the action is smooth. In order to explain the previous sentence we would like to investigate the manifold M in the neighbourhood of an orbit $\mathcal{O}_p$ of some point $p \in M$. To do this we introduce some terminology. Let $p \in M$ and let $A \subset X$ be a space on which G_p acts. We call a ***tube about*** $p \in M$ a G-equivariant embedding

$$\varphi : G \times_{G_p} A \to M$$

onto an open neighbourhood of $\mathcal{O}_p$ where the twisted product $G \times_{G_p} V := (G \times V)/G_p$ is defined using the free action of G_p on the product $G \times V_p$ by

$$h \cdot (g, v) = (gh^{-1}, h \cdot v) \qquad \forall g \in G\ ,\ v \in V\ ,\ h \in G_p$$

Additionally, if $p \in S$ and $G_p(S) = S$ then we call S a ***slice*** at p if the map $G \times_{G_p} S \to M$ is a tube about $\mathcal{O}_p$. We are interested in the case when the space A (which can now be though of as a general "slice") is not arbitrary but has a vector space structure. Let V be a Euclidean space on which G_p acts orthogonally, namely by orthogonal transformations. Then, a ***linear tube*** about $\mathcal{O}_p$ is a G-equivariant embedding

$$\varphi : G \times_{G_p} V \to M$$

onto an open neighbourhood of $\mathcal{O}_p$. In this case the space looks like part of a vector bundle over the orbit as we show in Proposition 2.1.3. Additionally, if S is a slice at $p \in M$ then we call it a ***linear slice*** if the canonically associated tube $G \times_{G_p} S \to M$ is equivalent to a linear tube, that is if the G_p-space S is equivalent to an orthogonal G_p-space. The intuitive picture behind this definition is illustrated in Figure 2.1. If there exists a linear tube around each orbit then we say that the action is ***locally smooth***. In this case we also call the G-space M locally smooth.

Definition 2.1.5. The action of a Lie group G on a manifold N is called ***locally smooth*** if for every $x \in N$ there exists a slice which is a disk on which the action of the stabilizer G_x is equivalent to an orthogonal action.

Recall that in the proof of Theorem 2.1.1 we gave an explicit construction of such equivariant neighbourhoods for the case of free actions. The orthogonality follows from the action being isometric. Therefore we have the following

Lemma 2.1.2. A free, smooth action of a compact Lie group G is locally smooth.

Proof. The proof is given in the first part of the proof of Theorem 2.1.1. □

We want to extend this result to general smooth actions that are not necessarily free. For this reason we formalize the construction of Theorem 2.1.1 and put it in a more general setting where the isotropy group is non-trivial. Suppose G_p is the isotropy group of p and consider the diffeomorphism $\phi_g := \Phi(g, -) : M \to M$ for $g \in G_p$. Then p is a fixed point of ϕ_g so the pushforward $D_p\phi_g : T_pM \to T_pM$ is an automorphism of T_pM. Additionally, the restriction of ϕ_g to the orbit $\mathcal{O}_p$ is a diffeomorphism $\phi_g|_{\mathcal{O}_p} : \mathcal{O}_p \to \mathcal{O}_p$ with an induced pushforward $D_p(\phi_g|_{\mathcal{O}_p}) : T_p\mathcal{O}_p \to T_p\mathcal{O}_p$. Therefore, the pushforward induces a well defined automorphism of the quotient vector space $V_p := T_pM/T_p\mathcal{O}_p$ which can be thought of as the tangent space at p that is normal to the orbit where "normal" is to be understood with respect to a G-invariant metric. We call this induced automorphism of V_p the *isotropy representation*.

Proposition 2.1.3. Let $\rho : G_p \to \mathrm{GL}(V_p)$ be a linear representation. The quotient space $G \times_{G_p} V_p$ has the structure of a vector bundle over G/G_p with projection:

$$\begin{aligned} \pi : \ G \times_{G_p} V_p &\longrightarrow G/G_p \\ [g, v] &\longmapsto [g] \end{aligned}$$

and fibers isomorphic to V_p.

Proof. Fix a representative $[g] \in G/G_p$ then the map $\pi^{-1}([g]) \to V_p$ given by $[g, v] \mapsto v$ is well defined since in general $[g, v_1] = [g, v_2]$ implies $v_1 = v_2$. This map gives the desired bijection $\pi^{-1}([g]) \simeq V_p$. Next we construct a trivialization. Let (U, ϕ_U) be a local trivialization for the principal bundle $G \to G/G_p$ with $U \subseteq G/G_p$ open such that

$$\begin{aligned} \phi_U : G|_U &\longrightarrow U \times G_p \\ g &\longmapsto ([g], \beta_U(g)) \end{aligned}$$

for some map $\beta_U : G \to G_p$. Then we define the trivialization (U, ψ_U) given by:

$$\begin{aligned} \psi_U : (G \times_{G_p} V_p)|_U &\longrightarrow U \times V_p \\ [g, v] &\longmapsto ([g], \rho(\beta_U(g))v) \end{aligned}$$

The map ψ_U is a diffeomorphism with inverse

$$\begin{aligned} \psi_U^{-1} : U \times V_p &\longrightarrow (G \times_{G_p} V_p)|_U \\ ([g], v) &\longmapsto [\phi_U^{-1}([g], e), v] \end{aligned}$$

where e denotes as always the identity element of G. Finally, the restriction of ψ_U to the fiber is a linear isomorphism. $\square$

What we have described so far is essentially $G \times_{G_p} V_p$ as an associated vector bundle to the principal bundle $G \to G/G_p$ and the isotropy representation $\rho : G_p \to \mathrm{GL}(V_p)$. Then we can view G/G_p as a submanifold of $G \times_{G_p} V_p$ via the zero section $s : G/G_p \to G \times_{G_p} V_p$ whose image is the set $\{[g, 0] \mid g \in G\}$. We can then ask if there exists a map $f : G \times_{G_p} V_p \to M$ such that the following diagram commutes

$$\begin{array}{ccc} G/G_p & \xrightarrow{\ s\ } & G \times_{G_p} V_p \\ {\scriptstyle \Phi(-,p)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{O}_p & \hookrightarrow & M \end{array}$$

If such a map exists, then it will essentially be an extension of the orbit map $\Phi(-, p) : G/G_p \to M$. The following theorem, often referred to as the slice theorem, asserts that such a map f can be found in a neighbourhood of the zero section.

Theorem 2.1.4. (Slice theorem) Let G be a compact Lie group which acts smoothly on M. Then there exists an equivariant diffeomorphism from an equivariant open neighbourhood of the zero section in $G \times_{G_p} V_p$ to an open neighbourhood of $\mathcal{O}_p$ in M which sends the zero section G/G_p onto the orbit $\mathcal{O}_p$ by the natural map f. Here the action of G on $G \times_{G_p} V_p$ is defined by $h \cdot [g, q] = [hg, q]$.

Proof. The map $[g, v] \mapsto g \cdot \exp(v)$ is the desired map f from a neighbourhood of the zero section in $G \times_{G_p} V_p$ (this follows from exp being a local diffeomorphism) onto an equivariant neighbourhood of $\mathcal{O}_p$. The G-invariance again follows from the fact that exp commutes with isometries just like in Theorem 2.1.1. $\square$

Corollary 2.1.2. If G acts on M with a unique isotropy type then M/G is a smooth manifold.

Proof. The construction of charts is the same as in the proof of Theorem 2.1.1. □

The essence of the above theorem is that smooth actions of compact groups are locally smooth and therefore locally behave in a well understood way around an orbit.

2.2 Metrics on orbit spaces

Let $\pi : E \to B$ be a submersion. It is often the case that the space B is a Riemannian manifold $(B, \tilde{g})$. Our current goal is to develop the necessary tools in order to be able to endow the orbit space with a metric which is in some sense natural. Firstly, for any $x \in B$ the submersion theorem implies that $\pi^{-1}(x)$ is a submanifold that we call the *fiber* at x. Then for some $u \in \pi^{-1}(x)$ the tangent space T_uE splits in a horizontal and vertical subspace. To do this we define the *vertical subspace* of T_uE to be $\mathcal{V}_u := \ker D_u\pi$. To see that this definition makes sense consider a path which lies entirely in the fiber $c : [0,1] \to \pi^{-1}(x)$ such that $c(0) = u$ so that we have

$$D\pi\left(\frac{dc(t)}{dt}\right) = D\pi \circ Dc\left(\frac{d}{dt}\right) = D(\pi \circ c)\left(\frac{d}{dt}\right) = 0$$

since $\pi \circ c$ is constant. Therefore every vector that lies in T_uF is also in $\mathcal{V}_u$ so that $T_uF \subseteq V_u$ is a linear subspace and because they have the same dimension (by the submersion theorem) we conclude that $\mathcal{V}_u = T_uF$. Now suppose that E is endowed with some Riemannian metric g. Then we define the *horizontal subspace* $\mathcal{H}_u$ at $u \in E$ to be the orthogonal complement of $\mathcal{V}_u$ in T_uE (with respect to the metric g) such that $T_uE = \mathcal{V}_u \oplus \mathcal{H}_u$. An example is illustrated in Figure 2.3. It is clear that $H_u \simeq T_xB$ as vector spaces. In this sense we have constructed a splitting of the space E into a "fiber component" and a "base component". Note that without a metric, there would be no canonical way to perform this splitting. Performing this splitting at every point $u \in E$ we obtain the *vertical and horizontal subbundles* $\mathcal{V}$ *and* $\mathcal{H}$ of E. Sections of $\mathcal{H}$ are called *horizontal vector fields.* We are now ready to define what we mean by a "natural" metric on the total space E. The intuition is that we want a metric g such that when we restrict to horizontal vectors in T_uE , it agrees with the metric on the base space B.

Definition 2.2.1. A submersion $(E, g) \to (B, \tilde{g})$ is called a ***Riemannian submersion*** if at each point $u \in E$, $D_u\pi$ preserves the length of horizontal vectors

$$g_u(X, Y) = \tilde{g}_{\pi(u)}((D_u\pi)X, (D_u\pi)Y) \qquad \forall\, X, Y \in \mathcal{H}_u \tag{2.2}$$

An equivalent way of phrasing this is that $D_u\pi : T_uE \to T_{\pi(u)}B$ is a linear isometry when restricted to $\mathcal{H}_u$.

In the context of the previous section, the submersion we are interested in is the quotient map of a group action $\pi : M \to M/G$. There is of course no a priori reason for this map to be a submersion. However, when the group action is smooth and free Theorem 2.1.1 ensures that we have a principal bundle structure. Of course this is in general not true and in fact the actions that will be of central interest for us are those that have fixed points. We can choose to ignore those and focus on the regular part of the manifold. Then everything works out fine as shown by the following theorem.

Theorem 2.2.1. Let G be a compact Lie group acting isometrically on a Riemannian manifold M.

1. The union M_0 of maximal orbits is open and dense in M.

2. The G-action on M restricts to M_0 and $M_0 \to M_0/G$ is a Riemannian submersion. It is also a fiber bundle with fiber G/H where H is a minimal isotropy group.

3. The quotient M_0/G is open dense and connected in M/G.

Proof. For (1) let $p \in M_0$ and $\varphi : G \times_{G_p} V \to M$ an invariant tubular neighbourhood as in Theorem 2.1.4 so that $p = \varphi([e, 0])$. Then if $q = \varphi([g, v])$ and $h \in G_q$ so that $h \cdot q = q$ we have

$$\varphi([g, v]) = \varphi([hg, v]) \Rightarrow [g, v] = [hg, v] \Rightarrow [e, v] = [g^{-1}hg, v]$$

from which we conclude that $g^{-1}hg \in G_p$ and G_q is conjugate to a subgroup of G_p so that $[G_q] \leq [G_p]$. However by the assumption of maximality we conclude $[G_q] = [G_p]$. Therefore the orbit $\mathcal{O}_q$ is also maximal and the set M_0 is open. In order to show that M_0 is dense we pick an arbitrary point $p \in M$ and a length minimizing geodesic $c : [0, 1] \to M$ connecting the orbit $\mathcal{O}_p$ with a principal orbit $\mathcal{O}_q$. By Kleiner's lemma 2.1.1 $G_{c(t)}$ for $t \in (1, 0)$ is a subgroup of $G_{c(1)}$ and the latter is of minimal type by assumption so that $G_{c(t)}$ must be of minimal type for all $t \in (0, 1)$. This shows that all points in $c((0, 1])$ have minimal isotropy groups and those can be chosen arbitrarily close to p.

For (2) we first note that $G \cdot M_0 = M_0$ since M_0 consists of entire orbits. From corollary 2.1.2 it follows that M_0/G is a smooth manifold. Since G acts with isometries on the Riemannian manifold (M, g) where g is some G-invariant metric, we can endow M/G with the quotient metric $\tilde{g}$ defined by

$$\tilde{g}_{\pi(u)}((D_u\pi)X, (D_u\pi)Y) = g_u(X, Y)$$

This is well defined because using $h \cdot u$ instead of u clearly leaves the left hand side invariant and the invariance of the right hand side follows from the isometric action of G. The The quotient map $\pi : M \to M/G$ is then a Riemannian submersion.

For (3) note that M_0/G is open and dense by $\pi : M \to M/G$ being open and continuous. To show connectedness we again use Kleiner's lemma to connect two principal orbits $\mathcal{O}_p$, $\mathcal{O}_q$ by a length minimizing geodesic $c : [0, 1] \to M$. Since the isotropy groups G_p, G_q are minimal, the isotropy groups of all the points in the path c are of minimal type and therefore this path lies in M_0 so its image under the quotient map is a path in M_0/G. □

When E is a principal G-bundle and π a bundle map then the fiber is $F \simeq G$ and $\mathcal{V}_u = T_uG \simeq \mathfrak{g}$. The explicit isomorphism between V_u and $\mathfrak{g}$ is given by assigning to each $X \in \mathfrak{g}$ its fundamental vector field $X^\#(u)$ evaluated at u which is clearly vertical from Definition 2.1.4. The map $X \mapsto X^\#$ is an isomorphism when G acts freely.

As usual, we assume that the action of G is effective so that the maximal orbits are isomorphic to G. What we have proven so far is that given such an action $G \times M \to M$ we can consider the union of principal orbits M_0 which is a G-bundle over M_0/G. Then if M_0 is endowed with a metric such that G acts with isometries, we can always construct a quotient metric on M_0/G as in the proof of Theorem 2.2.1 which makes the quotient map a Riemannian submersion (which is unique by construction).

The reader familiar with the theory of principal bundles might have noticed a striking similarity between the decomposition in terms of horizontal and vertical subbundles and the theory of connections on principal bundles. This is no coincidence as we explain in Appendix A.

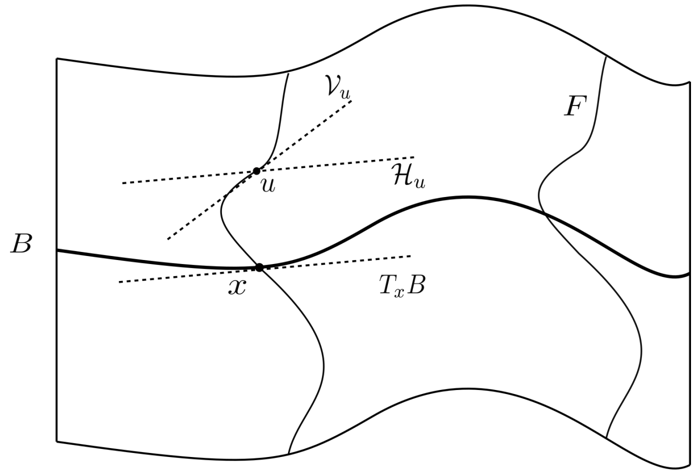


Figure 2.3: A submersion $E \to B$ and the horizontal and vertical subspaces.

2.3 Circle actions

The restriction to the class of abelian groups simplifies the above setup. We are mainly concerned with circle and torus actions and therefore we assume G to be abelian from now on.

Lemma 2.3.1. Effective actions of abelian groups have trivial principal isotropy type.

Proof. Recall that principal isotropy type means minimal isotropy type. Let $[G_p]$ be of such minimal type. Then for every $q \in M$ we have $[G_p] \leq [G_q]$ implying that $G_p = gHg^{-1}$ for a subgroup H of G_q. But since G is abelian $gHg^{-1} = H$ and G_p is a subgroup of G_q for every $q \in M$. This implies that $G_p \subseteq \bigcap_{q \in M} G_q$ but the assumption of effective action is equivalent to $\bigcap_{q \in M} G_q = \emptyset$ and the claim follows. □

The above lemma implies that maximal orbits are diffeomorphic to G and their union is a G-bundle over M_0/G which follows from Theorem 2.2.1. From now on in this section we will be only interested in circle actions $G = S^1$. Since the stabilizers are trivial for a dense subset M_0 of M, let us explore what can happen in $M \setminus M_0$. We first consider fixed points where the stabilizer is the whole of S^1. Let S^1 act on an oriented $2n$-dimensional manifold M via an action $\Phi : S^1 \times M \to M$ with a discrete set of fixed points and let $p \in M$ be a fixed point. Then $\Phi_p : M \to M$ induces a pushforward $D\Phi_p : T_pM \to T_pM$ which makes T_pM an S^1-module. This is nothing more than the isotropy representation that we have already encountered but now the isotropy group is the entire group G and the normal space is identified with the whole tangent space. We can decompose T_pM into irreducible representations as

$$T_pM = \bigoplus_{i=1}^{n} L_i$$

where the L_i are vector spaces isomorphic to $\mathbb{C}$ on which the S^1 action is given as multiplication by $g^{w_p^i}$ where $g \in S^1$ and w_p^i are non-zero integers called the ***weights*** at $p \in M$. In terms of real representations and under the identification $\mathbb{C} \simeq \mathbb{R}^2$ the S^1-action is just a direct sum of 2×2 rotation matrices. If the dimension of M is odd then

$$T_pM = \bigoplus_{i=1}^{n} L_i \oplus \mathbb{R}$$

where the action on the $\mathbb{R}$ component is just ± 1. In particular, we focus on the even dimensional case and everything carries on trivially to the odd case. The sign of each weight depends on whether the induced action agrees with the chosen orientation of each plane $\mathbb{C}$ or not. However, we can always change the orientation of individual components L_i as long as we preserve the orientation of T_pM induced by the orientation of M. For this reason, it is the sign of the product of the weights that really matters. We denote this product sign by $\epsilon(p) = \pm 1$ and denote the *fixed point data at* p by $\Sigma_p = \{\epsilon(p), w_p^1, ..., w_p^n\}$ where we take all the weights w_p^i to be positive.

2.3.1 Circle actions from Killing vector fields and topological invariants

We now turn to a different description of circle actions. We mainly follow [20, 21]. We first recall the notion of a fundamental vector field 2.1.4. When we use fundamental vector fields we may lose a lot of information about the action of the entire group G. In particular we have to pick a one-dimensional vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ spanned by an element $X \in \mathfrak{g}$ and then the vector field $X^\#$ encodes the infinitesimal action of elements arbitrarily close to the identity $e \in G$ that lie "in the direction" of X. The integral curves of $X^\#$ are then the one-dimensional orbits of the one-parameter-subgroup $\exp(\mathfrak{h}) \subseteq G$.

Lemma 2.3.2. Let G be a compact Lie group and $X \in \mathfrak{g}$. Consider the one-parameter-subgroup $\varphi : \mathbb{R} \to G$ where $t \mapsto \exp(tX)$. Then if $H := \mathrm{Im}(\varphi)$ is closed, it is a circle subgroup of G.

Proof. H is an abelian group since $\exp(t_1X)\exp(t_2X) = \exp((t_1 + t_2)X) = \exp(t_2X)\exp(t_1X)$. It is also closed and G is compact, so H is also compact. Connectedness of H follows by the continuity of the exponential map. Therefore H is a torus and since it is one-dimenensional it is S^1. □

Conversely, the Lie algebra of every circle subgroup of G is a one-dimensional subspace of $\mathfrak{g}$. By taking a basis vector of this subspace and considering the associated fundamental vector field, we see that every circle action gives rise to a vector field. If the circle action is isometric, then the vector field generates an isometry and is therefore a Killing vector field. This is why the term *infinitesimal isometry* is sometimes used instead. The above discussion, including the proof of Lemma 2.3.2 can be generalized for a torus subgroup T^k. Then $\mathrm{Lie}(T^k)$ is an abelian Lie subalgebra of $\mathfrak{g}$. By passing to fundamental vector fields, any such subalgebra corresponds to a set of commuting vector fields which, if the torus action is isometric, are commuting Killing vector fields.

Since the following sections are devoted to torus actions, we will often use this language of Killing vector fields instead of that of a torus action. Of course the correspondence between torus actions and commuting Killing fields should be used with caution since it is not a bijection. There can be commuting sets of Killing fields that do not arise as fundamental vector fields associated to a torus subgroup of the isometry group. This can happen when the condition for closedness in Lemma 2.3.2 is not satisfied. In the physics language we say that a family of Killing fields *integrate* to an isometric torus action. To account for the cases when this fails to be the case, the more general term *local isometry* is used to refer to the image of the map φ in Lemma 2.3.2. In this sense Killing fields are in one-to-one correspondence with local isometries. The language of vector fields is very convenient for applications in physics and also useful in mathematical applications as we will see below. First, we need to recast some of the notions we have been discussing into this new language.

We begin with fixed points of circle actions. Those are points $p \in M$ whose orbit is a point and their isotropy group is $G_p = S^1$. If $X \in \mathrm{Vect}(M)$ is the associated vector field then the orbits of the circle action are the integral curves of X namely paths $\gamma : (0, 1) \to M$ such that $(D_{t_0}\gamma)(\partial/\partial t) = X|_{\gamma(t_0)}$ for all $t_o \in (0, 1)$. A fixed point $p \in M$ is then a point for which the integral curve is the constant path at p so that $X(p) = 0$. Such points are called the *fixed points of a vector field.*

The second important concept is that of the isotropy representation. As we already saw in the previous section, this is a powerful tool to differentiate between different types of fixed points. If K is a Killing vector field and ∇ the covariant derivative, then the endomorphism $\nabla X : T_pM \to T_pM$ is skew-symmetric with respect to an orthonormal basis in view of the Killing equation. It therefore admits the canonical form

$$(\nabla X) = \begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & a_r & & & \\ & & & -a_r & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

Under the exponential map we get the induced action of the local isometry

$$\begin{pmatrix} \cos(a_1 t) & \sin(a_1 t) & & & & \\ -\sin(a_1 t) & \cos(a_1 t) & & & & \\ & & \ddots & & & \\ & & & \cos(a_r t) & \sin(a_r t) & \\ & & & -\sin(a_r t) & \cos(a_r t) & \\ & & & & & I_{n-2r} \end{pmatrix} \tag{2.3}$$

This is precisely the decomposition of T_pM in terms of irreducible representations of S^1 that we saw in the previous section. The numbers a_i are the weights. As before, those can be taken to be positive up an overall sign. Now that we have clarified how we can use Killing fields, we will use them some important results. The first one indicates that the fixed point set of a circle action has a nice structure from the point of view of differential geometry.

Theorem 2.3.1. Let M be a Riemannian manifold with $\dim M = n$ and X be a Killing vector field. Let F be the fixed point set of X and consider the decomposition $F = \cup N_i$ into connected components. Then each N_i is a totally geodesic closed submanifold of M of even codimension.

Sketch of the proof. Let $p \in F$ and consider the isotropy representation of the local isometry $\exp(\nabla X) : T_pM \to T_pM$ as in 2.3. If $n = 2r$ then p is an isolated point and both of the claims of the theorem hold trivially. In any other case there is a $(n-2r)$-dimensional subspace of T_pM that is left invariant by the action of the local isometry. We call this space W_p. Then if U is an open neighbourhood of the origin in T_pM and since the exponential map commutes with isometries, $\exp(W_p \cap U)$ is a $(n-2r)$-dimensional submanifold of $\exp(U)$ and it is invariant under the local isometry. It is also totally geodesic as the image of the exponential map. $\square$

Theorem 2.3.2. Let M be a compact Riemannian manifold and $F = \bigcup N_i$ be the fixed point set of a Killing vector field decomposed in connected components. Then

$$\chi(M) = \sum_i \chi(N_i)$$

Proof. Since N_i are fixed point loci, for any $p \in M$ the isotropy group is G and therefore the equivariant neighbourhoods of the slice theorem 2.1.4 will be diffeomorphic to $G \times_G V_p \simeq V_p$ where V_p is the normal space at p. These are open m-disks of dimension $m = \dim V_p$. Consider an ϵ-neighbourhood of N_i and A_i its closure, where we choose ϵ small enough so that every point of A_i can be joined to the nearest point of N_i by a unique geodesic of length $\leq \epsilon$ and so that $A_i \cap A_j = \emptyset$ for $i \neq j$. Then each A_i is a disk bundle over N_i. Let $A := \cup_i A_i$ and $B := \overline{M \setminus A}$. Then $A \cap B = \partial A$. Now whenever we have a long exact sequence of vector spaces

$$... \longrightarrow U_k \longrightarrow V_k \longrightarrow W_k \longrightarrow U_{k-1} \longrightarrow W_{k-1} \longrightarrow ...$$

it follows that

$$\sum_k (-1)^k \dim U_k - \sum_k (-1)^k \dim V_k + \sum_k (-1)^k W_k = 0$$

This can be easily shown by applying the rank-nullity theorem at every stage of the long exact sequence, then taking the appropriate alternating sum and using exactness. We want to use this formula for the long exact sequence of the pair (M, B) which is given by

$$... \longrightarrow H_n(B; \mathbb{K}) \longrightarrow H_n(M; \mathbb{K}) \longrightarrow H_n(M, B; \mathbb{K}) \longrightarrow H_{n-1}(B; \mathbb{K}) \longrightarrow ...$$

where the coefficients are taken in some field $\mathbb{K}$. Using the definition of the Euler characteristic as $\chi(X) := \sum_k H_k(X; \mathbb{K})$ we find that

$$\chi(M) = \chi(B) + \chi(M, B) \tag{2.4}$$

Applying the same formula for the long exact sequence of the pair $(A, A \cap B)$ we find

$$\chi(A) = \chi(A \cap B) + \chi(A, A \cap B) \tag{2.5}$$

Next we observe that $\text{int}(B) = M \setminus A$ and therefore $M \setminus \text{int}(B) = M \setminus (M \setminus A) = A$ and $B \setminus \text{int}(B) = \partial B = \partial A = A \cap B$. We also have that $\overline{\text{int}(B)} = B$ since B is closed so that we can apply the Excision theorem to deduce that $H_n(M, B; \mathbb{K}) \simeq H_n(A, A \cap B; \mathbb{K})$ for all n and therefore $\chi(M, B) = \chi(A \cap B)$ which combined with 2.4 and 2.5 gives

$$\chi(M) + \chi(A \cap B) = \chi(A) + \chi(B) \tag{2.6}$$

Now both $A \cap B$ and B are free of fixed points and by the Lefschetz fixed point theorem (see for example [22]) we conclude that their Euler characteristic vanishes so that 2.6 gives $\chi(M) = \chi(A)$. As we have already mentioned, A_i is a disk a bundle and the disk is contractible so that $\chi(\mathbb{D}^m) = 1$ so $\chi(A_i) = \chi(N_i)\chi(\mathbb{D}^m) = \chi(N_i)$. From this it follows that

$$\chi(M) = \sum_i \chi(N_i)$$

□

It is interesting that fixed points always appear as submanifolds of even codimension. The intuition behind Theorem 2.3.1 is quite simple and pertains to the fact that each circle action will rotate a number of planes.

2.3.2 Local structure around fixed points

Of particular interest, especially for applications in physics, are circle actions on four-manifolds which are going to be the backbone of the rest of our discussions. In this situation, fixed point loci come in dimension zero and two. Those will appear frequently in our discussions and deserve their own definition.

Definition 2.3.1. Let S^1 act on a four-manifold M. If P is an isolated fixed point with weights $(\epsilon(P), w_1, w_2)$ such that $w_1/w_2 = p/q$ with p,q co-prime integers, then we call P a ***(p,q)-nut*** if $\epsilon(P) = +1$ or ***(p,q)-anti-nut*** if $\epsilon(P) = -1$. More specifically, if $p = q = 1$ we call the fixed point a ***nut*** or ***anti-nut*** when $\epsilon(P) = +1$ and $\epsilon(P) = -1$ respectively. A two-dimensional submanifold of fixed points is called a ***bolt***.

Let $p \in M$ be a fixed point (either isolated or a point on a bolt). Then the normal space is just $V_p = T_pM$ and from the slice theorem it follows that there exists an equivariant neighbourhood of p diffeomorphic to $S^1 \times_{S^1} V_p \simeq T_pM$ on which the group action is given by the isotropy representation. The isotropy representation in this case is the rotation of the two copies of $\mathbb{C}$ with a pair of associated weights $(\epsilon(p), w_1, w_2)$ and in fact it exactly coincides with the action on $\mathbb{C}^2$ given in Example 2.1.1. The integers p, q that were introduced there are the weights of the circle action. In particular, when the weights are ± 1 the action is the Hopf or anti-Hopf action. This simple observation will prove crucial in our subsequent discussions and we state it in the form of the following lemma.

Lemma 2.3.3. Let M be a four-manifold with an S^1 locally smooth action and let $p \in M$ be a fixed point with unit weights. Then there exists an equivariant neighbourhood of p diffeomorphic to $\mathbb{R}^4$ on which the action looks like a radial extension of the Hopf or anti-Hopf action.

The closure of this neighbourhood is a four-disk $\mathbb{D}^4$ whose boundary S^3 is acted upon by a Hopf or anti-Hopf action giving the standard quotient $S^3 \to S^2$. For the more general case, we first note that S^3 can be identified with the set of lines passing through the origin in $\mathbb{R}^4$. Then if the action is characterized by weights $(\epsilon(p), w_1, w_2)$ the quotient of this S^3 will give a *weighted projective space* $\mathbb{C}P^{[w_1,w_2]}$. Of course, in the case of unit weights (Hopf or anti-Hopf action) we recover $\mathbb{C}P^1 \simeq S^2$ but this is the only case for which the quotient space can be a manifold. For every fixed point with a non-unit weight this fails to be the case. This is why the Hopf and anti-Hopf actions are so important.

However, in this work we want to treat general circle actions and not only those that locally look like the Hopf or anti-Hopf ones. Let us consider the action of S^1 on $\mathbb{C}^2$ with weights $k, \ell \in \mathbb{N}$ and k, ℓ coprime so that the action is effective (see Example 2.1.1). We have

$$e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\ell\theta} z_2) \tag{2.7}$$

In this case the points $(z_1, 0)$ have non-zero isotropy given by

$$\{\theta \in (0, 2\pi] \mid e^{ik\theta} z_1 = z_1\} = \{\theta \in (0, 2\pi] \mid k\theta = 2\pi n \ , \ n \in \mathbb{Z}\} \simeq \mathbb{Z}_k$$

Similarly, the points $(0, z_2)$ have isotropy $\mathbb{Z}_\ell$. The points (z_1, z_2) with $z_1, z_2 \neq 0$ have trivial isotropy due to ℓ and k being coprime. We have therefore discovered that our circle action has *exceptional orbits*, namely orbits with discrete stabilizer. Therefore it cannot be semi-free. In other words, if we wanted a semi-free action we should only consider actions with unit weights on the fixed points and in this case the quotient would automatically be smooth.

As already stated, when the circle action is restricted to $S^3 \subset \mathbb{C}^2$ the quotient is in general a weighted projective space. This space is an *orbifold* namely a manifold with singularities that are locally isomorphic to quotient sngularities of the form $\mathbb{R}^n/\Gamma$ where $n = \dim M$ and Γ is a finite subgroup of GL(n,$\mathbb{R}$). The weighted projective space $S^3/S^1 = \mathbb{C}P^{[k,\ell]}$ is indeed an orbifold which is often called a *spindle* when $k, \ell \neq 1$ and *teardrop orbifold* when only one of k, ℓ is different from one. Those names are justified by their shape which is illustrated in Figure 2.4. The singular points of those orbifolds are locally of the form $\mathbb{R}^2/\Gamma$ where $\Gamma = \mathbb{Z}_k$ for some $k \in \mathbb{N}$ and $\mathbb{Z}_k$ acts diagonally on $\mathbb{R}^2$.

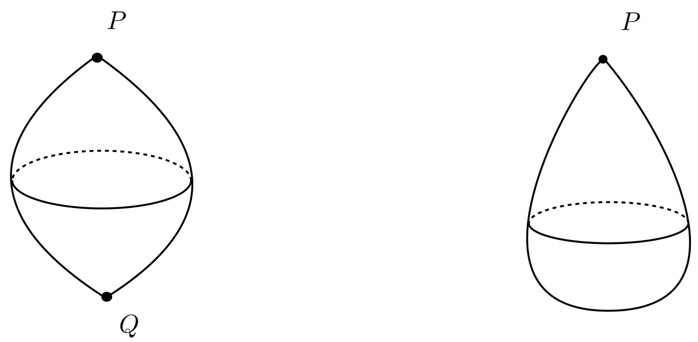


Figure 2.4: The spindle orbifold $\mathbb{C}P^{[k,\ell]}$ (left) has two singular poles P, Q with neighbourhoods diffeomorphic to $\mathbb{R}^2/\mathbb{Z}_k$ and $\mathbb{R}^2/\mathbb{Z}_\ell$ respectively. The teardrop orbifold $\mathbb{C}P^{[p,1]}$ (right) has a singular pole P, a neighbourhood of which is diffeomorphic to $\mathbb{R}^2/Z_p$ for some $p \in \mathbb{N}$.

In the quotient space $\mathbb{C}^2/S^1$, these singularities become one dimensional and can end on the images of fixed points. Therefore we can have one singular line (radial extension of the teardrop) or two singular lines (radial extension of the spindle) emanating from the image of a fixed point. Those singular points are known as *cones* and we will see later on, that they deserve some attention from a physical point of view.

2.4 T^2 actions on four-manifolds

In this section we focus on torus actions so that all the theorems of the previous sections apply. We will use M^* to denote the quotient space of a torus action. This notation is often used in the literature. The first thing we want to address is the topology of the orbit space which turns out to be quite simple. We start with two lemmata whose proofs can be found in [23].

Lemma 2.4.1. Let G be a compact Lie Group acting locally smoothly on a manifold M with M^* connected. If $\dim M^* \leq 2$ then M^* is a manifold (with boundary).

Lemma 2.4.2. If M is an arc-wise connected G-space, with G compact Lie group and if there is an orbit which is connected or there are fixed points of the G-action then the fundamental group of M maps onto that of M^*. Thus if M is simply connected, then so is M^*.

Therefore, for simply connected four-manifolds with T^2 actions, we have a concrete description since the orbit space is a two dimensional simply-connected manifold with possible boundary as long as the conditions of Lemma 2.4.2 are satisfied. This is in fact always the case because simply connected manifolds turn out to necessarily have fixed points.

Lemma 2.4.3. Let T^2 act on a simply connected manifold M. Then the action contains fixed points.

Proof. Suppose there are no fixed points. Then by Theorem 2.3.2 we get $\chi(M) = 0$. Since M is simply connected we have $H_1(M, \mathbb{Z}) = 0$. By Poincaré duality and Universal coefficients we get

$H_3(M,\mathbb{Z}) = 0$ so that

$$\chi(M) = 2 + \text{rank}H_2(M,\mathbb{Z}) \geq 2$$

which is a contradiction. □

We have established that for a simply connected four-manifold M, there are non-principal orbits and the orbit space is a simply connected manifold. If M is also compact then the orbit space is also compact (since the quotient map is continuous) and therefore it has to be a topological disk $\mathbb{D}^2$. This will not be the case later, when we consider non-compact manifolds in the context of String Theory where we will encounter a quotient space that looks like the upper half plane. Since we understand the very simple topology of the orbit space, we now turn to the question of what types of orbits there are and how they are allocated. A thorough study of how the torus can act on closed, orientable four-manifolds was carried out in [24]. Most of the qualitative results carry on to the case of non-compact manifolds. The possible orbits and their isotropy groups and representations are depicted in Table 2.1 which we now turn to explain.

Consider T^2 parametrized by two angles $\theta, \phi \in [0, 2\pi)$. Consider the following subgroups:

$$G(m,n) := \{(\theta,\phi) \in T^2 \mid m\phi + n\theta = 0\} \tag{2.8}$$

When $m = n = 0$ we identify $G(0,0) = T^2$. Otherwise, we let $n \neq 0$ and we have the following cases:

- When m/n is rational then $G(m,n)$ is a closed subgroup of T^2 which can be identified with the image of an embedding $S^1 \to T^2$ by the closed subgroup theorem for Lie groups. We can then take m, n to be coprime and all inequivalent such subgroups are labeled by distinct pairs of coprime integers (m, n).
- When m/n is irrational then $G(m,n)$ never closes back to itself. It is the image of an injective immersion $\mathbb{R} \to T^2$ which is dense in T^2 and often referred to as the *irrational winding of the torus.* Note that this map cannot be an embedding since then its image would have to be closed and dense and therefore the entire T^2. On the other hand, we have seen in Theorem 2.1.3 that isotropy groups are closed Lie subgroups and we conclude that those $G(m,n)$ with m/n irrational cannot occur as isotropy groups of torus actions.

From now on we use $G(m,n)$ to denote a circle subgroup with m/n rational and we also take m, n to be coprime. The orbits of the point with isotropy group a circle $G(m,n)$ are circles and this is why those are often called $C-orbits$. The other types of orbits appearing in Table 2.1 are principal orbits and fixed points which we have already seen but in addition to them, we see two types of exceptional orbits where the stabilizer is discrete. The following theorem [24] which is fundamental in understanding the structure of the orbit space, asserts that the various types of orbits do not appear randomly but instead have a very specific pattern

Theorem 2.4.1. Let M be a closed, simply-connected, oriented four-manifold with a T^2 action. The orbit space M^* is a 2-manifold with boundary. All principal orbits and exceptional orbits are in the interior and the boundary consists of C-orbits and fixed points.

In practice, we will not deal with exceptional orbits of torus actions from now on. They rarely occur in practical applications and in fact under mild assumptions it can be shown that they are completely absent [1] (see [23] section IV3). Therefore, we are left with the last three isotropy types

[1]Note that exceptional orbits will often appear in the circle actions since they are associated with the conical points of the previous section. However, when we let the entire T^2 act on them, their isotropy type becomes S^1.

Isotropy group	**Orbit**	**Slice**	**Isotropy action on slice**	**Image of orbit in M^***
$\mathbb{Z}_n \times e$	T^2	$\mathbb{D}^2$	rotation	isolated interior point
$\mathbb{Z}_n \times \mathbb{Z}_m$	T^2	$\mathbb{D}^2$	rotation	isolated interior point when (n,m)=1 (otherwise not possible)
G(m,n)	S^1	$\mathbb{D}^3$	rotation	boundary point
T^2	fixed point	$\mathbb{D}^4$	rotation in two planes by $G(m_1, n_1)$ and $G(m_2, n_2)$	isolated boundary point (only possible when $m_1n_2 - n_1m_2 = \pm 1$)
e	T^2 (principal orbit)	$\mathbb{D}^2$	rotation	interior point

Table 2.1: The possible isotropy groups, orbit and slice types (adapted from [24])

of Table 2.1. Since the fixed points count the Euler characteristic, there can only be a finite number of them if M is compact (the same conclusion also follows since fixed points are isolated and using the compactness of M). Therefore we can think of the orbit space as a disk whose boundary ∂M^* contains a finite number of points $\{P_1, ..., P_k\}$ with k the number of fixed points. The arcs on ∂M^* joining two consecutive points P_i, P_{i+1} are orbits of isotropy group $G(m, n)$ and since the stabilizer of a point cannot change discontinuously (see the slice theorem 2.1.4) we conclude that the isotropy group must be constant along the arc. A crucial conclusion made in [24] is that if $G(m, n)$ and $G(m', n')$ are the stabilizers on two adjacent arcs joined at P_i then they must satisfy

$$\det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix} = mn' - m'n = \pm 1$$

Geometrically, this is the condition that $G(m, n)$ and $G(m', n')$ generate the homology of T^2 and intersect transversally at one point. An illustration of how the orbit space looks like is show in Figure 2.5.

Now consider the quotient map of the T^2 action $\pi : M \to M^*$ and let $M_0^* = M^* \setminus \partial M^*$ be the interior of the orbit space consisting of principal points. Then $\pi^{-1}(M_0^*)$ is the union of principal orbits in M which we denote with M_0 as in before. This is a principal T^2-bundle from Theorem 2.2.1 and since now the base space is an open disk which is contractible, we conclude that the bundle is trivial, namely $M_0 = M_0^* \times T^2$. By the first assertion of the same Theorem, M_0 is open and dense and now we see that $M \setminus M_0 = \pi^{-1}(\partial M^*)$ is the union of C-orbits and fixed points. Since its complement is a trivial bundle, $\pi^{-1}(\partial M^*)$ can be thought of as an obstruction to the triviality of the principal bundle. Additionally, there is a nice intuitive picture of what the fixed point loci look like. Indeed, if γ_i is an arc on ∂M^* of isotropy $G(m, n)$ joining the points P_i, P_{i+1} then $\pi^{-1}(\gamma_i)$ is a sphere on which the circle action of $G(m, n)$ looks like that of Example 2.1.3 while the action of a $G(m', n')$ with $mn' - m'n = \pm 1$ leaves the sphere invariant. In other words, $\pi^{-1}(\gamma_i)$ is a bolt for the circle action of $G(m', n')$.

Finally, we close this section with another celebrated result of [24] that concerns the topological

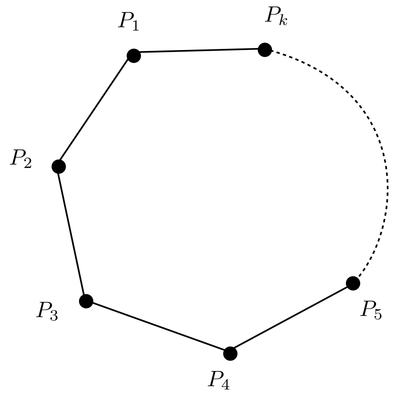


Figure 2.5: The orbit space of a T^2 action on a closed, connected four-manifold. The dots correspond the fixed points P_i and the arcs joining them correspond to C-orbits. The interior of the disk comprises principal orbits (or exceptional orbits in the more general case).

classification of simply connected four-manifolds with T^2 actions [2], demonstrating that the existence of an effective T^2-action is highly non-generic and has important topological consequences.

Theorem 2.4.2. Let T^2 act effectively and smoothly on a closed, oriented, simply connected four-manifold M. Then M is equivariantly diffeomorphic to one of the following: S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $S^2 \times S^2$ or an equivariant connected sum of those.

There is in fact more we can say about how the orbit space data determines the total four-manifold for the case of the four building blocks of Theorem 2.4.2. Let $P_1, ..., P_k$ and γ_i be as before and let $G(m_i, n_i)$ be the isotropy group on γ_i. Then we denote

$$e_i := \begin{vmatrix} m_{i-1} & n_{i-1} \\ m_i & n_i \end{vmatrix} = \pm 1 \qquad i = 2, 3, ...k$$

$$e_1 := \begin{vmatrix} m_k & n_k \\ m_1 & n_1 \end{vmatrix} = \pm 1$$

When $k > 2$ we also define the determinants for non-consecutive arcs, namely $r_2 = m_1 n_3 - m_3 n_1$ and $r_3 = m_2 n_4 - m_4 n_2$. Then the possible manfiolds for $k = 1, 2, 3, 4$ are given in Table 2.2. Although legitimate from a mathematical standpoint, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ are of lesser significance in physics since they are not spin, that is their Stiefel-Whitney class is non-zero. However, those spaces and their fixed point structure have also been considered in physics [27].

2.5 From group actions to String Theory

We now want to combine the results that we have so far obtained through this arduous mathematical journey of the previous sections and get some contact with String Theory. As we have already mentioned, String Theory and in particular Type IIA theory lives in ten dimensions. Additionally, this theory is conjectured to arise from the eleven dimensional M-theory. The low energy picture of this correspondence is well understood since it is a correspondence between the eleven dimensional and Type IIA supergravities. But first, let us properly define what an M-theory background actually is.

[2]A similar classification exists for circle actions on four-manifolds [26, 25].

Number of fixed points k	Four-manifold M	Additional condition
2	S^4	
3	$\mathbb{C}P^2$	$e_1e_2e_3 = -1$
	$\overline{\mathbb{C}P^2}$	$e_1e_2e_3 = 1$
4	$S^2 \times S^2$	$e_1e_4 = e_2e_3$, both r_2, r_3 are even and of them zero

Table 2.2: The orbit space data of the simplest compact simply-connected four-manifolds (table adjusted from [28])

Definition 2.5.1. An ***M-theory background*** consists of a triplet (M, g, F_4) where M is an eleven dimensional spin manifold, g is a metric on M, and F_4 is a closed four-form.

Ten dimensional Type IIA String Theory arises as a reduction of M-theory in eleven dimensions. In the language of Section 2.2, we first view the M-theory background as a principal S^1-bundle and subsequently identify the Type IIA background with the base space, or in different terms with the horizontal distribution with respect to the metric g. Apparently, in order to do that we need a free circle action. We can phrase this requirement in terms of a Killing field $X \in \mathrm{Vect}(M)$ which integrates to a circle action and has non-vanishing norm such that there are no fixed points. However, since the four-form is also part of our geometric data, additionally to $\mathcal{L}_X g = 0$ (the Killing equation) we must also require $\mathcal{L}_X F_4 = 0$.

As long as those requirements are satisfied, we can start exploring the Type IIA picture. As we saw in Section 2.2 there is a natural metric that the base space is endowed with and this is the unique metric g_{10} that makes the quotient map a Riemaniann isometry. This is indeed the metric that is relevant for us. In a sense, this can be taken to be the definition of a Type IIA background since it has been shown that such a reduction from eleven dimensions encodes all the information that is needed to define the most general Type IIA supergravity background [29]. The converse is not true though and this is why eleven dimensional supergravity (M-theory) should be thought of as the more fundamental theory.

Let us now focus on a coordinate-based algorithm that will allow us to go back and forth between M-theory and Type IIA backgrounds. We focus on the metric ignoring for the moment F_4. Given the Killing vector X, the splitting in terms of a horizontal and vertical subspace can be made explicit by the induced connection one-form ω (see Appendix A). In this case since the Lie algebra of S^1 is $\mathbb{R}$, ω can be regarded as an element of $\Omega^1(M)$. Locally, if x is a coordinate along the orbit of the Killing field X (therefore a coordinate along the fiber) we can write $\omega = dx + A$ for some locally defined one-form A on the base manifold $N = M/S^1$. Then, by the local splitting of M in terms of horizontal and vertical distributions we can express the eleven dimensional metric as

$$g = h + |X|^2(dx + A)^2$$

This form is often called a *connection metric.* The norm of the Killing vector $|X|^2$ is by assumption non-zero everywhere and here we denote the ten-dimensional metric on N (and its pullback on M via the quotient map) with h. It is evident that in this setting, the norm $|X|$ has the geometric interpretation of the length of the fiber. In String Theory we choose to parametrize $|K| = e^{2\varphi/3}$ where $\varphi : M \to \mathbb{R}$ is called the ***dilaton.*** For reasons of convenience and convention, in String

Theory it is useful to conformally rescale the metric h on N by the factor $e^{-2\varphi/3}$ such that the metric becomes a *warped connection metric* which is explicitly given by

$$g = e^{-2\varphi/3}h + e^{4\varphi/3}(dx + A)^2$$

We now turn to the reduction of the four-form F_4. First note that the decomposition $\omega = dx + A$ implies that the curvature form $d\omega = dA$ is independent of the fiber direction $\iota_X d\omega = d\omega(X) = 0$ so that $d\omega$ is a horizontal 2-form. Additionally, using Cartan's magic formula we find

$$\mathcal{L}_X d\omega = d\iota_X d\omega + \iota_X d^2\omega = 0$$

This shows that $d\omega$ is both horizontal and invariant. Such forms are called *basic*. They are very important in this context because they are pullbacks of forms on the base manifold N. In this sense the curvature form $d\omega$ is the pullback of a two-form $H_2 = dA$ defined on N which we call the RR two-form (we stick to the physics convention of omitting the pullback when it is obvious). It turns out after some elementary manipulations (see [30] for an excellent review) that the four-form can be expressed as

$$F_4 = H_4 - dx \wedge H_3 \tag{2.9}$$

where H_4 and H_3 are basic forms that correspond to the RR two-form and NS-NS three-form of the Type IIA background. We therefore have the bijection between M-theory with free circle actions and Type IIA backgrounds given by

$$\Big\{\text{11d with } (M, g, F_4) \text{ and free } S^1\text{-action}\Big\} \longleftrightarrow \Big\{\text{10d with } (N, h, \varphi, H_2, H_3, H_4)\Big\}$$

We have already seen that are topological obstructions (like the Euler characteristic) that force any circle action to have fixed points. In String Theory the fixed points have a deep physical meaning. Suppose we start with an eleven dimensional M-theory background of the form $C_7 \times Z_4$ with a circle action being trivial on C_7 and being effective on Z_4. Then the theory of circle actions on four manifolds that we have developed applies. If there are isolated fixed points of the circle action (nuts) then we can remove them and consider their complement which is a principal bundle according to Theorem 2.2.1. The action is then free and we can perform the reduction to Type IIA hoping that we can be cavalier about forgetting those fixed points. Notice that after the dimensional reduction, the fixed points have codimension three so they appear as 7-dimensional objects in the ten-dimensional String Theory. These have the right dimension to be D6-branes and we will see a physical proof later that this is indeed the case. For now we take it as the definition.

Definition 2.5.2. Let (M, g) be an oriented, Riemannian manifold such that $\dim M = 11$ with an isometric S^1 action and let $F := M^{S^1}$ denote the fixed point locus. Then a connected component N_i of F of codimension four is a D6-brane if its fixed point data has $\epsilon(p) = 1$ and an anti-D6-brane if $\epsilon(p) = -1$ and all the weights are units.

Since in order to perform the reduction, we excluded the fixed points, we must be careful regarding the behavior of the reduced space. In general, this process of removing fixed points and reducing is pretty common in physics and the images of the fixed points in the quotient space are often called *topological defects*, a well-deserved name since they encode topological information about the total space. In general, we can be confident that for most spaces, the metric on the base space will extend to a smooth metric when we try to include the topological defects. However, even if the base metric smoothly extends to the fixed points, does the same apply to the RR-forms? Usually those questions are addressed for each case separately. For the case of the H_2 form however which as we saw has a very geometric origin, it has been shown [31] that there always exists an extension on the base manifold N.

The topological defect can be a general geodesic submanifold of even codimension in the total space and odd codimension after the reduction. In our case, it can be either isolated points or a spherical bolt. In physics, we like to work with scalar quantities instead of cohomology classes and it is useful to define the *topological charge* of a fixed point set (in the reduced space) given by

$$N := \frac{1}{4\pi} \int_{\Sigma} H_2$$

If the topological defect is a point (nut) then Σ is a surface surrounding it. If the defect is a bolt, then the integration is carried on the bolt. It can be shown [32, 33] that this integration around a (p, q) nut gives $N = (pq)^{-1}\beta/4\pi$ where β is the period of the Killing vector generating the circle action. Similarly, in the case of the bolt, the result is $N = Y\beta/4\pi$ where Y is the self intersection of the bolt.

An interesting point can be made when N_4 admits more than one isometric circle actions. In this case it is not a priori clear which one is the right one. There can be therefore a variety of Type IIA backgrounds that lift to the same eleven dimensional M-theory background and therefore they are all dual in the sense that their physical properties must be the same.

2.5.1 Backgrounds with isometric torus actions

We want to take the discussion one step further and consider M-theory backgrounds with isometric torus actions. We take M to be a four-manifold with the action $\Phi : T^2 \times M \to M$. As explained in the previous sections, in order to get a Type IIA background we need to choose an isometric circle action. In this new setting, we have a whole $SL(2, \mathbb{R})$ of choices and in particular we can choose any $G(m, n)$ subgroup of T^2. Additionally, the obtained ten dimensional backgrounds for different $G(m, n)$ should be dual to each other. We want to understand the qualitative differences between those backgrounds which will allow us to explore this kind of peculiar duality.

Our main goal is to study backgrounds with both D6 branes and anti-branes. Therefore, the natural setting is a four-manifold with a circle action of two fixed points, a nut and an anti-nut. This is why we focus on the simplest toy model, the S^4. As we will see later in Chapter 4, there exist backgrounds of eleven dimensional supergravity of the form $C_7 \times S^4$ with the metric on S^4 being T^2-invariant. Therefore, the results that we discuss now can indeed fit in String Theory models. The structure of the orbit space of S^4 is depicted in Figure 2.6. We take the C-orbits to have isotropy $G(0, 1)$ and $G(1, 0)$. This can always be achieved by an $SL(2, \mathbb{R})$ transformation of T^2.

As we have discussed, when we choose a circle action such that the weights of the fixed points are unit, then the fixed points are interpreted as a D6-$\overline{\text{D6}}$ pair in the Type IIA background. In this case the slice theorem ensures the existence of an equivariant neighbourhood of the fixed point diffeomorphic to $\mathbb{R}^4$ where the action is a radial extension of the Hopf action. The quotient space is $\mathbb{R}^3$ and by performing a one-point compactification (adding the fixed point at infinity and the corresponding point in the quotient space) we find that for such an action the quotient space is $S^4/S^1 \simeq S^3$.

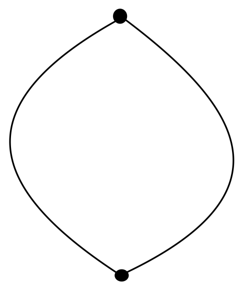


Figure 2.6: The obit space of S^4. We take the collapsing cycles at the boundary to be (0,1) and (1,0).

However, we notice that there are two peculiar choices of reduction circle that we can make, the two circles that collapse at the boundary of the orbit space. For instance, if we choose the cycle $G(0,1)$ then the sphere $\pi^{-1}(\gamma_1) \subset S^4$ is left invariant under the induced circle action since all the points are of isotropy $G(0,1)$. Therefore, $\pi^{-1}(\gamma_1)$ is a spherical codimension two bolt of this circle action. The same happens of course for the circle action of the cycle $G(1,0)$ and the bolt $\pi^{-1}(\gamma_2)$. On the other hand, if we choose any other cycle $G(p,q)$, none of the spheres $\pi^{-1}(\gamma_1), \pi^{-1}(\gamma_2)$ is fixed and the only fixed points are the fixed points of the T^2 action P, Q.

The question that we now want to answer is what the base space looks like if we reduce along say $G(0,1)$. We know in advance that the topological defect will not consist of points but will instead be a sphere, the image of $\pi^{-1}(\gamma_1)$ under the quotient map of the circle action $\pi_{(0,1)} : S^4 \to S^4/S^1$. The discussion of Section 2.3 applies here identically and therefore around each fixed point of the bolt, say for instance the "pole" P, we can find using the slice theorem 2.1.4 an equivariant neighbourhood diffeomorphic to $T_P M \simeq \mathbb{R}^4$ on which the S^1-action is characterised by the fixed point data $\{\epsilon(P), w_1, w_2\}$. Due to the fixed point locus being a codimension two bolt, one of the weights w_1, w_2 must be zero. This is easily seen either from the discussion in the beginning of Section 2.3 or (for those who prefer Killing vectors over actions) from the expression 2.3. Assume that $w_2 = 0$ and $w_1 \neq 0$. We can think of the invariant copy of $\mathbb{C}$ as the tangent space to the bolt and the copy on which the action is non-trivial as the normal space. To see what the base space locally looks like we take the quotient of this neighbourhood $\mathbb{C} \oplus \mathbb{C}/S^1$ where the circle acts on $\mathbb{C}$ by $(e^{i\theta}, z) \mapsto e^{iw_1\theta} z$. By rescaling the generating Killing vector, we can set $w_1 = 1$. It is not hard to see that quotienting the plane by rotations gives $\mathbb{C}/S^1 \simeq [0, \infty)$ so that $\mathbb{C} \oplus \mathbb{C}/S^1 \simeq \mathbb{C} \times [0, +\infty)$ which is the upper half of $\mathbb{R}^3$. By adding the point at infinity we conclude that $S^4/S^1 \simeq \mathbb{D}^3$ is a three dimensional ball with an S^2 boundary as expected. The boundary is identified with the locus of the topological defect. This locus is spherical since the quotient map is a homeomorphism on the fixed point locus. We have therefore discovered a type of duality that arises due to the different choices of reduction cycles and can be summarized as follows

- Reduction of M-theory along a cycle of the torus whose action has two isolated fixed points of opposite charge. The images of the fixed points are isolated topological defects. In the case of unit weights, the defects represent D6-branes and anti-branes.

- Reduction of M-theory along a cycle whose action has a codimension two fixed point set (bolt). The topological defect is a sphere in the reduced space which appears as a spherical boundary.

In the following chapter we will focus on how those mathematical ideas are applied in String

Theory and we will delve into the world of brane–anti-brane dynamics. Our main focus in this thesis is to describe systems consisting of a single D6-$\overline{\text{D6}}$ pair and therefore S^4 is the manifold of special importance for us since it is the natural choice of compact, simply-connected four-manifold providing a natural candidate for an M-theory background with a brane–anti-brane pair. This background, as we will see more explicitly in Chapter 4 is realized when the M-theory four-form is non-zero. In fact, there is a simpler M-theory background which is purely geometric in the sense that $F_4 = 0$. This manifold has the topology $\mathbb{R}^2 \times S^2$ and admits a torus action with two fixed points. At first sight, one might wonder how such a space is compatible with the classification of Theorem 2.4.2. The caveat is that Theorem 2.4.2 only concerns compact manifolds. The minimality of this background will allow us to get some insight in the wild world of brane–anti-brane dynamics. It will also provide the stage for testing the duality that we just introduced. This space will be the main subject of the following chapter.

Chapter 3

Kaluza-Klein branes in String Theory

3.1 The geometry of solitons

The term *soliton* in physics and specifically in the context of gravity is broad enough to include general topologically stable solutions of the field equations that describe localized matter. We will see that D6-branes are among the large variety of objects that fall in this category. In particular, what we will discover throughout the course of this Chapter is that D6-branes are intimately related to *magnetic monopoles*, a certain type of soliton which is mangetically charged.

In the previous chapter, we exhibited the ideas behind dimensional reduction, the process of constructing connection metrics and how we can go back and forth between a principal bundle and its base space. This was all demonstrated for the particular case of eleven dimensional M-theory and its reduction to the ten-dimensional Type IIA String Theory. Historically, the idea of reduction goes way before the introduction of String Theory and was known in physics as Kaluza-Klein theory. This remarkably beautiful geometric idea aspired to explain gravity and electromagnetism in our four-dimensional world as the result of reduction of a five-dimensional geometry with pure gravity. To star with, recall that by *gravity in d dimensions* we mean a metric on a d-dimensional manifold extremizing the Einstein-Hilbert action given by

$$S = -\frac{1}{16\pi G_k}\int dx^d \sqrt{-g}\; R^{(d)}$$

In the above, $R^{(d)}$ is the Ricci scalar in d dimensions. We then make the usual ansatz that the total space splits in the product $M_4 \times M_c$ of a four dimensional space M_4 and an internal space M_c. More generally, we can consider M_c-bundles over M_4. The resulting physics at low energy scales compared to the compactification scale is gravity coupled to a Yang-Mills theory with the group of isometries of the internal space, $\mathrm{ISO}(M_c)$, being the gauge group. In the following we restrict to $d = 5$ which is the standard Kaluza-Klein approach. In this case the spectrum of options for a compact one-dimensional internal manifold is restricted to S^1. As a consequence, the four-dimensional theory will be ordinary gravity with a U(1) gauge theory, namely electromagnetism. Let us first see how this gauge theory in four dimensions comes about. Consider pure gravity in five dimensions

$$S = -\frac{1}{16\pi G_K}\int dx^5 \sqrt{-\mathrm{det} g}\; R^{(5)} \tag{3.1}$$

Then we consider the background $\mathbb{R}^{1,3} \times S^1$ with $\mathbb{R}^{1,3}$ being Minkowski spacetime. The spirit of what follows is very similar to the horizontal and vertical splitting of a connection metric in "fiber directions" and "base directions" as in Section 2.2. The traditional (for the physics literature) approach that we consider here is useful because it can be applied in a wide range of internal manifolds, not confined to circles and indeed we will have the chance to see its power in Chapter 4. Next, we expand the only field in our theory, the five dimensional metric, in terms of the Fourier modes

$$g_{MN}(x^\mu, y) = \sum_{n=0}^{+\infty} g^n_{MN}(x^\mu) e^{iny/R}$$

where x^μ denotes the coordinates on $\mathbb{R}^{1,3}$ and y the circle coordinate. We assume that the five-dimensional metric does not depend on the circle coordinate so that translation along the circle generates an isometry. Since nothing in the action 3.1 depends on y, the circle can be integrated out to obtain the four dimensional effective theory. One then makes without loss of generality the following parametrization of the five dimensional metric

$$g_{MN} = \varphi^{-1/3} \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu \varphi & A_\mu \varphi \\ A_\nu \varphi & \varphi \end{pmatrix}$$

whose form serves the purpose of isolating the Einstein-Hilbert term in four dimensions so that the reduced action becomes

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-\det g(x^\mu)} \Big[R^{(4)} + \frac{1}{6\varphi^2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} \varphi F^{\mu\nu} F_{\mu\nu} \Big] \tag{3.2}$$

Here, we set $G = G_K/2\pi R$ and $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the U(1) field strength, completely analogous to the the 2-form H_2 in Section 2.5. Recall that such a two-form is well-defined in the base space which is here four-dimensional. The gauge symmetry associated to this field is generated by the translation in the circle direction namely $y \to y + \lambda(x^\mu)$ in which case the transformation of the gauge field A_μ is $A_\mu \to A_\mu + \partial_\mu \lambda$ and $F_{\mu\nu}$ is manifestly invariant. The gauge field A_μ is sometimes called the *graviphoton* to emphasize its gravitational origin. The original idea of the Kaluza-Klein theory was to obtain a pure theory of gravity and electromagnetism by setting the scalar φ to be constant. The field equation for φ is of the form $\Box \varphi \sim F^{\mu\nu} F_{\mu\nu}$ which means that φ can be constant only provided that $F_{\mu\nu} = 0$ in which case there is no gauge theory but only gravity in four dimensions. This was the incurable flaw of the Kaluza-Klein idea that led people to initially abandon it. However, the fact that it naturally incorporates electromagnetism and gravity renders it a natural setup to construct gravitational backgrounds of magnetic monopoles. This is what we do next.

We can start by exploring the simplest type of metrics, namely those that are static. This makes sense intuitively, since a magnetic monopole system should be time independent. We can express this as $\partial g_{AB}/\partial t = 0$. Another sensible assumption is to take the time direction to be totally flat namely $g_{0A} = \delta_{0A}$, with x^0 denoting the time coordinate. The complete five dimensional equations of motion $R_{AB} = 0$ will be trivially satisfied in the time direction and the only non-trivial ones will be:

$$R_{ij} = R_{4i} = R_{44} = 0 \tag{3.3}$$

Where $i, j \in \{1, 2, 3\}$ and the index 4 is reserved for the circular direction whose coordinate we still denote by y. These equations imply that the constant time, four dimensional slices will be Ricci flat. Manifolds satisfying this constraint played a key role in the development of Euclidean quantum gravity and due to their frequent use, they have been granted the special name *gravitational instantons*. This general term describes four-manifolds with Riemannian, complete, Ricci flat metrics.

It is worth making a small digression here to stress out the necessity of starting with a five dimensional theory. If we had started with four dimensional gravity then the resulting equation for the three dimensional slice $R_{ij} = 0$ would imply that the manifold is flat, since in three dimensions the Riemann tensor is completely characterized by the Ricci tensor (they have the same number of independent components). This is why the simplest solitons are obtained from the $M_4 \times S^1$ ansatz instead of, for instance, $M_3 \times S^1$.

Back to our five dimensional setting, we notice that solutions of $R_{ij} = 0$ (with the constraint that one direction is a circle) can be obtained from four-dimensional solutions of (Euclidean) pure gravity. More precisely, if M_4 is a four manifold with a Ricci flat Riemannian metric $ds^2 = g_{\mu\nu} dx^\mu dx^{nu}$ then the manifold $\mathbb{R} \times M_4$ with metric $ds^2 = -dt^2 + g_{\mu\nu} dx^\mu dx^\nu$ is a solution to the five dimensional Einstein equations 3.3.

The first interesting example of such a Riemannian four-manifold that can arise in this way is the Taub-NUT instanton. The four-dimensional metric is given by:

$$ds^2 = V^{-1}(dy + A_\phi d\phi)^2 + V(dr^2 + r^2 d\Omega^2) \tag{3.4}$$

where

$$A_\phi = 2m(1 - \cos\theta) \qquad V = 1 + \frac{4m}{r} \tag{3.5}$$

The space is topologically $\mathbb{R}^4$ and the metric is Ricci flat but is not the flat metric on $\mathbb{R}^4$. Here r, θ, ϕ are the standard spherical coordinates on $\mathbb{R}^3$ and y is a periodic coordinate which parametrizes a circular direction in our space. A simple computation of the volume of a geodesic ball reveals that it does not scale with r^4 as would be the case in flat space but with r^3. The period of the coordinate y is $2\pi R$ with $R = 8m$ and m being a free parameter in the metric which is related to the ADM mass of the solution. The periodicity of y is fixed by the requirement that the metric is complete. The metric seems to have a singularity at the origin $r = 0$ so in principle it is only defined on $\mathbb{R}^4 \setminus 0$ but it can be shown that it smoothly extends to a metric on $\mathbb{R}^4$. Let us explore the structure of this metric by considering constant $r \neq 0$ slices. The induced metric on the three spheres is

$$ds^2 = V^{-1}(dy + A_\phi d\phi)^2 + V r^2 d\Omega^2 \tag{3.6}$$

where now V is a constant. This nothing else than the Hopf metric on the three sphere, namely the metric that makes the Hopf map $S^3 \to S^2$ a Riemannian subemrsion. In other words the (isometric) circle action corresponding to translations $y \to y + a$ is the Hopf action on each S^3-shell. Therefore, the Taub-NUT is a radial extension of the Hopf metric from S^3 to the entire $\mathbb{R}^4 \setminus 0$. We have already remarked that the Hopf action is free on the entire $\mathbb{R}^4 \setminus 0$ and therefore it can be seen as a principal S^1-bundle. The Taub-NUT metric is then a connection metric on $\mathbb{R}^4 \setminus 0$ (with connection form A_ϕ) which also makes the quotient map $\pi : \mathbb{R}^4 \setminus 0 \to \mathbb{R}^3 \setminus 0$ a Riemannian submersion. It is not hard to prove that the metric 3.7 is Ricci-flat. The Taub-NUT metric is also special because it is hyper-Kahler and has SU(2) holonomy also implying Ricci flatness.

Recall from our discussion in 2.3.2 that the radial extension of the Hopf (or anti-Hopf) action in $\mathbb{R}^4$ is the natural action in an equivariant neighbourhood of a fixed point with unit weights. The requirement of unit weights was then natural for the quotient to be a manifold. What we have just presented is a canonical Ricci-flat metric in the neighbourhood of a nut. Had we chosen $-A_\phi$ for the connection, the circle action on S^3 would be an anti-Hopf action and the metric would describe the equivarant neighbourhood of an anti-nut. Historically, the Taub-NUT metric was discovered long before the importance of fixed points was realized. This is why the isolated fixed points with unit weights are called nuts and anti-nuts.

There exists a natural generalization of the Taub-NUT space called the *multi Taub-NUT metric*. This turns out to describe an array of fixed points. The metric is given by:

$$ds^2_{TN_k} = \frac{1}{U(\vec{r})}(dy + \chi)^2 + U(\vec{r})d\vec{r}^2 \tag{3.7}$$

where $d\vec{r}^2 = dr^2 + r^2 d\Omega^2$ is the flat metric on $\mathbb{R}^3$. The space TN_k is again a hyperkäler four-manifold with the topology of an S^1-fibration over $\mathbb{R}^3$. The coordinate $\vec{r}$ is a vector in $\mathbb{R}^3$. The function $U(\vec{r})$ and the connection form χ satisfy:

$$U(\vec{r}) = 1 + \frac{R}{2}\sum_{a=1}^{k}\frac{1}{|\vec{r} - \vec{r_a}|} \qquad , \qquad d\chi = *_3 dU$$

The latter condition (which is a consequence of the field equations) was satisfied in the case of the Taub-NUT metric for $\chi = A_\phi$ and $U(\vec{r}) = V(r)$ but in the more general case at hand, it can be quite non-trivial and therefore we must be careful about those choices. In the above expressions we interpret $\vec{r_a}$ as the positions of the (localized) solitons which are points in the space TN_k. They correspond to isolated fixed points of the circle action generated by translation along the y direction and therefore they are the locations of the solitons.

As we have already mentioned, the other topological invariant that is relevant for our discussion is the Euler characteristic. If we consider an open ball $B \subset \mathbb{R}^3$ containing all points $\vec{r_a}$ then its boundary S^2 is the base of a regular S^1 bundle over S^2 so that we can use the same arguments as in the case of the Dirac monopole. It can be shown that χ is a well-defined connection form and the resulting total charge is:

$$\int_{S^2} d\chi = 2\pi k$$

which, just like in the case of the Dirac monopole, can be reinterpreted as the Chern number of the bundle being $k \in \mathbb{Z}$. This result matches with the additive nature of the magnetic charge.
If in the metric 3.7 instead of $d\chi = *_3 dU$ we had taken $d\chi = - *_3 dU$ then we would be describing objects with opposite magnetic charge which are called anti-monopoles. The corresponding singularities would then be anti-nuts. When considered separately, monopoles and anti-monopoles are no different in terms of their intrinsic topology (they differ however in terms of their embeddings in the higher dimensional space). An interesting scenario arises when one wants to consider systems comprising both of them. It is then natural to ask if there are known solutions for a system of a monopole and an anti-monopole.

The above question has been answered to the positive in [34]. First note that one way of generating four dimensional Euclidean solutions is by taking Lorentzian solutions of the standard vacuum field equations and making the time coordinate periodic and Euclidean, corresponding to the circular of the Kaluza-Klein anstatz. This makes it relatively simple to generate solitons by employing already known solutions of general relativity. This is precisely the strategy we will follow. The way to proceed is to start with the Kerr solution in 1+3 dimensions and use the aforementioned procedure to get a five dimensional solitonic solution. Doing this, we obtain the following metric which is often referred to as the *Kerr instanton*:

$$\begin{aligned} ds^2 = &- dt^2 + \frac{1}{r^2 - a^2\cos^2\theta}\left[\Delta(d\tau + a\sin^2\theta d\phi)^2 + \sin^2\theta((r^2 - a^2)d\phi - ad\tau)^2\right] \\ &+ (r^2 - a^2\cos^2\theta)\left[\frac{dr^2}{\Delta} + d\theta^2\right] \end{aligned} \tag{3.8}$$

where $\Delta = r^2 - 2mr - a^2$. As a reality check, one can verify that the constant t slices correspond to Euclidean Kerr metrics with periodic, Euclidean time τ. The angular momentum of this space is imaginary and equal to iaM. It is also worth noting that the metric 3.8 is asymptotically flat since the rotation parameter a enters in the metric to second order and does not appear in the asymptotic expansion. Similar to what happened with the Taub-NUT space, in order to ensure smoothness of the metric at the horizon $r_+ = m + (m^2 + a^2)^{1/2}$ where $\Delta = 0$, we have to impose the following periodicity condition:

$$(r, \tau, \theta, \phi) \sim (r, \tau + 2\pi\gamma n_1, \theta, \phi + 2\pi\gamma\Omega n_1 + 2\pi n_2) \qquad n_1, n_2 \in \mathbb{Z}$$
$$\text{where } \gamma = \frac{2mr_+}{(m^2 + a^2)^{\frac{1}{2}}} \;\; , \;\; \Omega = \frac{a}{r_+^2 - a^2} = \frac{a}{2mr_+} \tag{3.9}$$

Regularity of the metric (and of course constancy of the signature) also requires $r > r_+$ so that $\Delta > 0$. This also ensures that $r^2 - a^2 \cos^2\theta > 0$ so we do not have to worry about singularities coming from this term. A special feature of the Kerr instanton is that it has two Killing vectors associated to U(1) isometries, the standard $\partial/\partial\tau$ and the co-rotating Killing vector $\partial_\tau + \Omega\partial_\phi$. In the other words, this space can be viewed as a principal bundle in two different ways. The corresponding perspectives are different although apparently they describe the same space.

- If we choose the U(1) isometry associated to ∂_τ then the fixed points of the isometry are given by the vanishing of $K^\mu K_\mu = g_{\tau\tau} = 0$ which is the surface $r = m + \sqrt{m^2 + a^2\cos^2\theta}$. Since we imposed $r \geq r_+$ the only fixed points correspond to $r = r_+$ and $\theta = 0, \pi$. These can be shown to correspond to a nut and an anti-nut. Note that we assumed that $a \neq 0$ since in the case $a = 0$ we recover the Schwarzschild limit of the Kerr solution which has a different behavior.
- If we choose the U(1) isometry of the co-rotating Killing vector then a straightforward calculation shows that there is a fixed surface $r = r_+$. This constitutes a regular spherical bolt of self intersection $Y = 0$ and the resulting manifold has the topology of a complex line bundle over S^2.

As a consistency check, note that if we apply the formulas for the Euler characteristic and signature, both descriptions give the same Euler characteristic and signature. In those formulas, the Chern number k is zero because the Kerr instanton is asymptotically flat so that it looks like a trivial $S^2 \times S^1$ at infinity. This gives $\chi = 2$ and $\tau = 0$ which means that the Kerr instanton solution is not in the same topological sector as the vacuum. Using further arguments, it was shown in [34] that the Kerr instanton describes a monopole– anti-monopole system. Note also that there is a physical interpretation for the parameter a in this solution which can be thought of as the distance between the monopole and the anti-monopole.

In passing, we mention that the Kerr instanton is in many ways similar to the Bonnor solution [35] in General Relativity which also describes a magnetic dipole. Their relation can be understood in the context of Einstein-Maxwell theory coupled to a dilaton with the coupling between the Maxwell term and the dilaton being arbitrary which leads to the following action

$$S = -\frac{1}{16\pi G}\int dx^4 \sqrt{-\det g}\Big(R^{(4)} - 2\partial^\mu\varphi\partial_\mu\varphi - e^{-2b\varphi}F^{\mu\nu}F_{\mu\nu}\Big)$$

Then, the Bonnor solution is a solution of this theory for $b = 1$. On the other hand, the Kaluza-Klein theory fixed this coupling by the requirement that the action comes from pure gravity in higher dimensions imposing $b = \sqrt{3}$. Indeed it can be checked by a straightforward computation that the Kaluza-Klein action 3.2 reduces to the Einstein-Maxwell-dilaton action after a redefinition of the dilaton $\varphi \mapsto -4e^{-2\sqrt{3}\varphi}$. The Kerr solution was found to be a solution for this value of the coupling. There is also a generalized Bonnor solution for arbitrary value of b of which the Kerr instanton and the Bonnor solution are just special cases.

3.2 From Kaluza-Klein to String Theory backgrounds

In this section we utilize the Kaluza-Klein soliton backrounds that we previously discussed, in order to make contact with the ten dimensions of String Theory. The route we will take is the one already announced in Section 2.5 which amounts to considering products of seven and four dimensional manifolds to construct M-theory backgrounds and then reduce along isometric cycles to obtain Type IIA backgrounds. We will first exhibit how the Taub-NUT space gives rise to a single D6-brane (or anti-brane). In the next section, we will apply the same procedure for the Kerr instanton and we will discover our first background with a D6$-\overline{\text{D6}}$ pair.

3.2.1 The D6-brane

We would like to interpret the above solutions as objects living in the ten dimensions of String Theory. The mechanism that relates solitonic solutions like the ones we obtained to string theory configurations is relatively well understood. Let us see how it works.

We start with a qualitative description. We have already discussed the relationship between M-theory and Type IIA String Theory from a mathematical standpoint. Now we want to motivate this connection in physical terms. The mass of the supergraviton in eleven dimensions is $M_{11}^2 = -p^M p_M = 0$. We can dimensionally reduce this on the circle whose radius length scale is R_c and use that the momenta in the circular dimension are quantized $p_{11} = n/R$, $n \in \mathbb{Z}$ to conclude that in 10 dimensions we get a tower of massive states with masses $M_n^2 = (n/R_c)^2$. These massive states are all BPS [1] and carry n units of U(1) charge. For $n = 1$ we can identify this with the D0-brane obtained in the context of type IIA string theory which also carries a unit charge and has a mass given by $(\ell_s g_s)^{-1}$. By matching the masses we get that in order for this correspondence to hold we need:

$$R_c = \ell_s g_s \tag{3.10}$$

which gives an important relation between the string coupling and the radius of the eleventh dimension. This relation is also the reason why we can interpret perturbative type IIA string theory as a weak coupling limit of M-theory, since the limit $R_c \to 0$ in which we obtain a legitimate 10-dimensional description is the same as $g_s \to 0$. From the above discussion it also becomes clear that in this limit the tension of the D0-brane diverges, which renders it a non-perturbative object. The relevant gauge field corresponding to the $U(1)$ charge is the one coming from the reduction of the eleven dimensional metric g_{MN}, namely $C_m = g_{m11}$. This charge is precisely what we defined as topological charge in Section 2.5 where we identified the D6-branes as the natural objects that carry it. From the new perspective that we just developed, D0-branes also deserve this title. Indeed, D0 and D6 are the duals of each other and they both share the same origin. This is consistent with the fact that they are the magnetic duals of each other.

Armed with the previous results, we can now try to give a quantitative interpretation of the gravitational instanton metrics in the context of String Theory. One particularly intriguing aspect of String Theory is that any Type IIA background can be lifted to M-theory which is the content

[1] Meaning that they saturate the BPS bound. Although we will not be heavily concerned with this notion, it is very useful and will be mentioned throughout this thesis. We refer to [36] for more details.

of Section 2.5. Let a solution of type IIA be given by the metric $g_{\mu\nu}$, the dilaton φ and the Ramond-Ramond 1-form C_μ. Then the eleven dimensional metric:

$$ds^2 = e^{-\frac{2\varphi}{3}} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4\varphi}{3}} (dx_{11} + C_\mu dx^\mu)^2 \tag{3.11}$$

is a solution of the eleven dimensional supergravity. In order to account for the additional massless bosonic fields in type IIA, we must turn on fluxes. In other words, we have to consider non-trivial configurations for the four-form of eleven dimensional supergravity. If this form vanishes, we call the background purely gravitational. Conversely, given a purely gravitational M-theory setup, it is possible to obtain an associated type IIA solution by dimensional reduction. This was shown in its generality. In the following we will demonstrate how this mapping works for the specific metrics of gravitational instantons that we found in the previous section.

Let us first consider the Taub-NUT solution which is obtained as a special case of the multi Taub-NUT for k=1 with the unique defect sitting at $\vec{r} = 0$. Then we can construct an eleven dimensional metric given by:

$$ds^2 = -dt^2 + \sum_{m=5}^{10} (dy^m)^2 + ds^2_{TN} \tag{3.12}$$

Here, ds^2_{TN} is shorthand for the Taub-NUT metric 3.7. The metric 3.12 is a solution to the eleven-dimensional Einstein equations. It thus follows that it is a solution to the field equations of eleven dimensional supergravity with all other fields set to zero. We now want to dimensionally reduce along the M-theory circle in order to study the underlying String Theory picture. This was first done in [5]. Comparing 3.12 with 3.11 we immediately get

$$U(r)^{-1} = e^{\frac{4}{3}\phi} \Rightarrow e^\phi = \left(1 + \frac{R}{2r}\right)^{-\frac{3}{4}}$$

This gives the 10-dimensional dilaton. It follows that our dilaton is not constant but depends on the radial coordinate r. One could raise some objections here. In the derivation of 3.10 we assumed a constant radius for the S^1-fibration which translates to a constant dilaton. Now that the dilaton and consequently the fiber radius is varying with respect to the base manifold, 3.10 has to be modified. It turns out that 3.10 is still valid but with R_c substituted by the asymptotic radius R of the Taub-NUT fibration. Having clarified this issue, we can go on and identify the ten dimensional metric:

$$\begin{aligned} ds^2_{10} &= e^{\frac{2}{3}\phi}\Big(- dt^2 + \sum_{m=5}^{10} (dy^m)^2 + \left(1 + \frac{R}{2r}\right) d\vec{r}^2 \Big) \\ &= \left(1 + \frac{R}{2r}\right)^{-\frac{1}{2}} \Big(- dt^2 + \sum_{m=5}^{10} (dy^m)^2 + \left(1 + \frac{R}{2r}\right) d\vec{r}^2 \Big) \end{aligned} \tag{3.13}$$

A simple change of coordinates given by $\tilde{r} = R/2 + r$ transforms the ten dimensional metric to a more easily recognizable form:

$$ds^2_{10} = \left(1 - \frac{R}{2\tilde{r}}\right)^{\frac{1}{2}} \Big(- dt^2 + \sum_{m=5}^{10} (dy^m)^2 \Big) + \left(1 - \frac{R}{2\tilde{r}}\right)^{-\frac{1}{2}} d\tilde{r}^2 + \tilde{r}^2 \left(1 - \frac{R}{2\tilde{r}}\right)^{\frac{3}{2}} d\Omega_2^2 \tag{3.14}$$

This metric was found in [37] to give the 6-brane solution in the context of 10 dimensional 2A supergravity which is the low energy limit of Type IIA String Theory. We thus conclude that the Taub-NUT soliton in M-theory corresponds to a D6-brane in Type IIA. It can also be shown using the same argumentation, that the multi Taub-NUT background in M-theory reduces to a system of k D6-branes in type IIA.

To make this argument even more solid we can prove that the energy content of the above eleven dimensional spacetime agrees with this interpretation as a system of D6-branes. This is done by considering the energy density of the spacetime but instead of integrating over the entire ten dimensional spacetime (which would give a mass), we integrate over the the four dimensional space transverse to the brane (the Taub-NUT space) which gives a quantity with the right dimensions for a tension:

$$T = \frac{1}{16\pi G_{11}} 2\pi R \int d^3x \nabla^2 V(r) = \frac{2\pi}{(2\pi\ell_p)^9}(2\pi R)^2 = \frac{1}{g_s}\frac{2\pi}{(2\pi\ell_s)^7} = T_{D6}$$

This agrees with the value of the tension of a D6-brane in string theory. In this manner we succeeded in identifying the eleven dimensional background $\mathbb{R}^7 \times TN$ with the M-theory lift of a system of a D6-brane of Type IIA String Theory. We are led to the following:

Proposition 3.2.1. The metric 3.12 is the M-theory lift of a D6 brane in type IIA string theory.

This proposition and the preceding discussion constitutes the physical proof that we promised in Section 2.5 when we identified isolated fixed points in four-manifolds with unit weights (nuts and anti-nuts) with D6-branes. Since we showed that the Taub-NUT metric is nothing more than the metric realizing the Hopf (or anti-Hopf) action in an equivariant neighbourhood of a fixed point, it becomes more than plausible that Definition 2.5.2 is well-motivated from a physical perspective. Note also that the Taub-NUT space admits two Killing spinors, being hyper-Kahler. This means that the solution 3.14 breaks half of the supersymmetry of the M-theory vacuum. This is also consistent with the status of D6-branes as half-BPS objects.

An almost immediate generalization is that the description of an array of D6-branes in Type IIA lifts to M-theory on a multi-Taub-NUT space $\mathbb{R}^7 \times TN_k$. The k-1 spheres connecting those k monopoles (fixed points) look like the sphere in Figure 2.2 and their image under the quotient map is a one-dimensional line stretching from one monopole to the other. Therefore, an M2-brane wrapped around this sphere in the eleven dimensional theory, has the interpretation of an open string with its ends on two different branes in ten dimensions. If we call S_{ij} the sphere connecting the i^{th} and j^{th} fixed points then the area of this surface is

$$\int_{S_{ij}} U^{-1/2}(\vec{r})U^{1/2}(\vec{r})dy|d\vec{r}| = 2\pi R\ell = 16\pi m\ell \qquad \ell := \int |d\vec{r}|$$

Here ℓ is the distance between r_i and r_j in $\mathbb{R}^3$ along some arbitrary path that we have chosen. Since the restriction of the Taub-NUT metric on $\mathbb{R}^3$ is the flat metric, the geodesics are just straight lines and the minimal distance is $\ell = |r_i - r_j|$. Therefore, the minimal sphere S_{ij} has surface area $16\pi m|r_i - r_j|$. This looks a lot like the area of a cylinder of length $|r_i - r_j|$ and radius $R = 8m$ and this would indeed be the case if the metric was the flat metric on $\mathbb{R}^4$. This is therefore another manifestation of the differences between the Taub-NUT and the flat metrics. If we now wrap an M2-brane around a sphere S_{ij} then the mass of this membrane will be

$$m_{ij} = 16\pi m T_{M2}|r_i - r_j|$$

From a ten dimensional perspective, this is exactly the product of the string tension $T_s = 16\pi m T_{M2}$ times the distance between the branes $|r_i - r_j|$, in agreement with the standard formula for the mass of the open string between D-branes. In particular, the mass of the open string goes to zero as the monopoles approach and this gives rise to a gauge enhancement from $U(1)\times U(1) \longrightarrow SU(2)$.

Another way to see this enhancement is through the intersection form of the four-manifold. In particular, the $k-1$ submanifolds $S_{i,i+}$ with $1 \leq i \leq k-1$ generate H_2 and it is straightforward to calculate the intersection form which will be a $(k-1)\times(k-1)$ matrix. The diagonal elements are the self-intersections of the spheres, a quantity which in Section 2.5 we identified with the topological charge carried by it or equivalently carried by the fixed points at the two poles. In particular, for the case at hand, the fixed points are both nuts so the self intersection of $S_{i,i+1}$ is 2. To compute the off-diagonal elements note that only adjacent spheres intersect and therefore the only non-zero entries are -1 for the $(i, i+1)$ element since each $S_{i,i+1}$ intersects its neighbor in exactly one fixed point

$$I = \begin{pmatrix} 2 & -1 & 0 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & 0 & ... & 0 & 0 \\ 0 & -1 & 2 & -1 & ... & 0 & 0 \\ . & . & . & . & ... & . & . \\ . & . & . & . & ... & . & . \\ 0 & 0 & 0 & 0 & ... & 2 & -1 \\ 0 & 0 & 0 & 0 & ... & -1 & 2 \end{pmatrix}$$

This is the Cartan matrix of the A_{k-1} algebra. When the fixed points of the circle action coincide, this will give rise to an A_{k-1} singularity and a corresponding gauge enhancement $U(1)^k \longrightarrow SU(k)$. The signature of this matrix is k-1. This concludes our discussion regarding systems of many D6-branes.

3.3 D6-$\overline{\text{D6}}$ pairs in String Theory

Having identified the right geometry for D6-branes (Taub-NUT) and for an array of D6-branes (multi-Taub-NUT) and their anti-brane analogs, it would now be desirable to know what is the M-theory description of a $\text{D6} - \overline{\text{D6}}$ pair. For this we need metrics with actions of fixed points of both nut and anti-nut type. It is natural to consider the monopole–anti-monopole solution 3.8 as an obvious candidate for this. It turns out that this is indeed the correct interpretation. The main arguments for this were made in [38]. We will review the main points made there.

We start by embedding the Kerr instanton in the eleven dimensional space of M-theory just as we did for the Taub-NUT space. We recall that the nut and anti-nut associated to the Killing vector ∂_τ are located at the poles of the horizon $r = r_+$, namely at $\theta = 0, \pi$. We expect those two points to signify the location of the D6-$\overline{\text{D6}}$ in the transverse space. In particular the metric becomes:

$$\begin{aligned} ds^2 = &- dt^2 + \sum_{m=1}^{6}(dy^m)^2 + \frac{1}{r^2 - a^2\cos^2\theta}\left[\Delta(d\tau + a\sin^2\theta d\phi)^2 + \sin^2\theta((r^2-a^2)d\phi - a d\tau)^2\right] \\ &+ (r^2 - a^2\cos^2\theta)\left[\frac{dr^2}{\Delta} + d\theta^2\right] \end{aligned} \tag{3.15}$$

This geometry was discussed in [38]. The main idea is to prove that the *local geometry* around the nut and the anti-nut is the monopole geometry 3.12, proving that the metric 3.15 has a monopole and an anti-monopole embedded in it. What we mean by local geometry in this context can be intuitively understood as follows. First of all, consider the geodesic distance between the two fixed points located at $r = r_+, \theta = 0, \pi$. It is not hard to show that moving along θ is a geodesic connecting the two points and therefore the geodesic distance is

$$\ell := \int_0^\pi \sqrt{r_+^2 - a^2 \cos^2\theta}\, d\theta$$

For $a \gg M$ we have $\ell \simeq 2a$ and we can think of a as being a measure of the monopole–anti-monopole distance. In any case, we have a concrete interpretation for the parameter a. Next, consider the dimensionless quantity r/a where r is some arbitrary coordinate that measures the distance from one of the poles. Then a local geometry arises as the limit when $r/a \to 0$ which gives a quantitative criterion for being close enough to one of the monopoles. There are two ways to do that, either by keeping r arbitrary and taking $a \to \infty$ or by keeping a arbitrary and taking $r \to 0$. Those two limits turn out to give the expected geometry which is of course the Taub-NUT geometry. The mass parameter of those Taub-NUT geometries is equal to $\gamma/2$.

The above arguments serve to convey the content of the local geometry around the monopoles but fail to give a complete account of the dynamics in this spacetime. In order to achieve that, let us explore what the ten-dimensional theory looks like. First, we observe that due to the identifications 3.9 which were necessary to make the metric regular (and the space complete) the periods of the τ, ϕ coordinates are intertwined. We introduce a new coordinate

$$\tilde{\phi} := \phi - B\tau \qquad B := \Omega - \frac{1}{\gamma}$$

For this coordinate, using 3.9 we have:

$$\tilde{\phi} \sim \phi + 2\pi n_2 + 2\pi\gamma\Omega n_1 - \left(\Omega - \frac{1}{\gamma}\right)(\tau + 2\pi\gamma n_1) = \tilde{\phi} + 2\pi(n_1 + n_2) \tag{3.16}$$

so that $\tilde{\phi}$ has period 2π. The metric in those coordinates reads:

$$\begin{aligned} ds_{11}^2 &= -dt^2 + \sum_{i=1}^{6}(dy^i)^2 + \Sigma\left[\frac{dr^2}{\Delta} + d\theta^2\right] + \frac{\Delta + a^2\sin^2\theta}{\Sigma}d\tau^2 \\ &+ \frac{2[\Delta - (r^2 - a^2)]a\sin^2\theta}{\Sigma}d\tau(d\tilde{\phi} + Bd\tau) + \frac{\sin^2\theta}{\Sigma}[(r^2 - a^2)^2 + \Delta a^2 \sin^2\theta](d\tilde{\phi} + Bd\tau)^2 \end{aligned} \tag{3.17}$$

Now we perform a Kaluza-Klein reduction of this metric along the Killing field ∂_τ which in the old coordinates is the field $\partial_\tau + B\partial_\phi$. For now, we forget the specific value of B and we think of it as an arbitrary parameter. This will help us uncover the many different ten dimensional backgrounds for reductions along different cycles. The reduction gives [39]

$$\begin{aligned} ds_{10}^2 &= \Lambda^{1/2}\left\{ - dt^2 + \sum_{i=1}^{6}(dy^i)^2 + \Sigma\left[\frac{dr^2}{\Delta} + d\theta^2\right]\right\} + \Lambda^{-1/2}\Delta\sin^2\theta d\tilde{\phi}^2 \\ e^{\frac{4}{3}\varphi} &= \Lambda \\ A_{\tilde{\phi}} &= \Lambda^{-1}\frac{\sin^2\theta}{\Sigma}\left\{B[(r^2 - a^2)^2 + \Delta a^2\sin^2\theta] - a[(r^2 - a^2) - \Delta]\right\} \\ \Lambda &:= \frac{1}{\Sigma}\left\{[\Delta + a^2\sin^2\theta] - 2Ba\sin^2\theta[(r^2 - a^2) - \Delta] + B^2\sin^2\theta[(r^2 - a^2)^2 + \Delta a^2\sin^2\theta]\right\} \end{aligned} \tag{3.18}$$

where φ is the ten dimensional dilaton as usual. Let us now try to understand the dynamics of the theory if we ignore the brane–anti-brane pair. If we want to have a notion of space in which the brane–anti-brane lives, we must look at the asymptotic space. The quick way to do this is to set $a = m = 0$ and we expect that this is the "background" in which the brane–anti-brane pair is embedded. Note that we take $a, m \to 0$ only in the metric but we leave the periodicities 3.9 intact since they remain unchanged in the asymptotic region we are exploring. In this case we get:

$$\begin{aligned}
ds_{10}^2 &= \Lambda^{1/2}\Big[-dt^2 + \sum_{i=1}^{6}(dy^i)^2 + dr^2 + r^2 d\theta^2\Big] + \Lambda^{-1/2} r^2 \sin^2\theta d\tilde{\phi}^2 \\
e^{\frac{4}{3}\varphi} &= \Lambda \\
A_{\tilde{\phi}} &= \frac{Br^2\sin^2\theta}{1 + B^2 r^2 \sin^2\theta} \\
\Lambda &= 1 + B^2 r^2 \sin^2\theta
\end{aligned} \tag{3.19}$$

This however is not flat space which is to say that even if we ignore the presence of the branes, there is still a non-trivial one-form field $A_{\tilde{\phi}}$. Let us explore this geometry. The space is known as a ***fluxbrane*** and there is a solid way to understand its structure [41, 40, 42] and its subtle relation to flat space. Consider $\mathbb{R}^d$ with the flat metric and isometry group $\mathrm{ISO}(\mathbb{R}^d) = \mathrm{SO}(d) \ltimes \mathbb{R}^d$. We want to consider the possible isometric circle actions which are given by a group homomorphism $S^1 \hookrightarrow \mathrm{SO}(d) \ltimes \mathbb{R}^d$. More generally we can consider linear embeddings of the Lie algebra $\mathrm{Lie}(S^1) = \mathbb{R}$ in $\mathfrak{so}(d) \ltimes \mathbb{R}^d$. To specify such a map we need only specify an element of the Lie algebra (the image of $1 \in \mathbb{R}$) which consists of a pair (ω, λ) with $\omega \in \mathfrak{so}(d)$ an anti-symmetric matrix and $\lambda \in \mathbb{R}^d$ a translation. The action on elements $x \in \mathbb{R}^d$ is given by

$$(t, x) \mapsto \exp(t\omega) \cdot x + t\lambda \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

When $\omega = 0$ this action is just a translation and the quotient map $\mathbb{R}^d \to \mathbb{R}^{d-1}$ is the map that collapses the λ-direction. In this case the action is clearly free. When $\lambda = 0$ then we have pure rotations and there will necessarily be fixed points. In general if we want the induced action to be free we must require that

$$\omega \cdot x + \lambda = 0 \tag{3.20}$$

has no solutions (those would be fixed points). A sufficient condition for this is that the rotations generated by ω occur in a hyperplane that is orthogonal to λ. This can always be arranged by redefining the origin by $O \mapsto O + a$ for some vector $a \in \mathbb{R}^d$. In this case the new action is given by the pair $(\omega, \lambda + \omega \cdot a)$. The rotation part then takes place in the orthogonal $\mathbb{R}^{d-1}$ plane and is characterized by the [(d-1)/2] eigenvalues of ω which we denote B_i. In order for this action to descend into a circle action we must impose further identifications. This is common when performing Kaluza-Klein reductions along non-compact directions. Let $P \in \mathbb{R}^d$ and P' be the point obtained by moving P by an amount $2\pi R$ along the orbit where R is arbitrary but sets the scale of the circular dimension. In more precise terms, we started with a manifold M admitting an $\mathbb{R}$-action and by quotienting with a cocompact subgroup $\Gamma \subset \mathbb{R}$ we consider the circle action of $\mathbb{R}/\Gamma$ on M/Γ. An illustration for $\mathbb{R}^3$ can be seen in Figure 3.1 where the two cases of zero and non-zero rotation are depicted.

The flat metric can be written in such a way to make the decomposition of $\mathbb{R}^d$ into $[d/2]$ two-planes manifest

$$ds^2 = \sum_{i=1}^{m}(d\rho_i^2 + \rho_i^2 d\phi_i^2) + dy^2 + dx^2$$

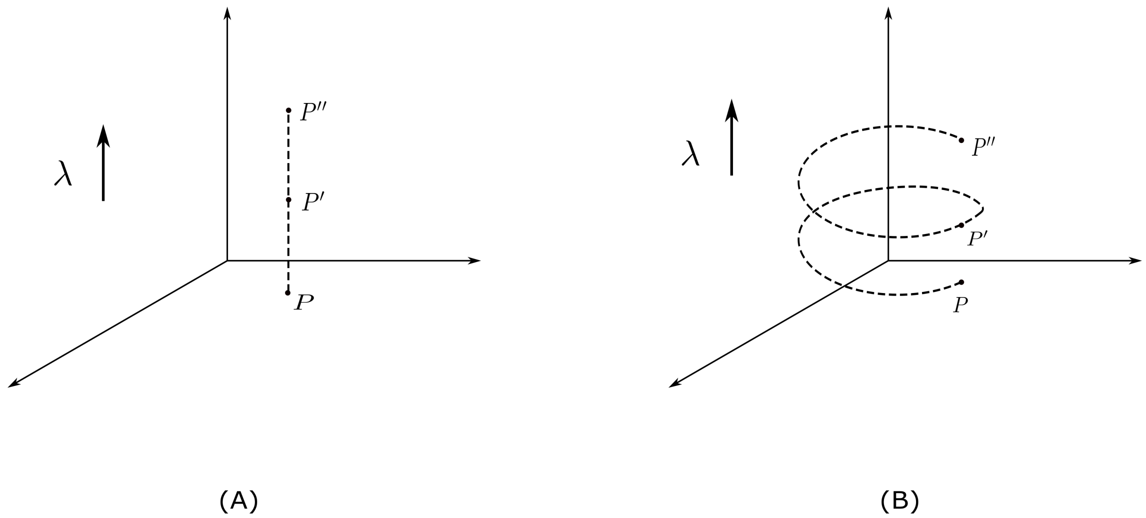


Figure 3.1: (A) The orbit of P when $\omega = 0$ (B) The orbit of P when $\omega \neq 0$. In both cases the points P, P', P'' are identified in order to obtain a circle action.

Here y is the coordinate in the direction of λ and (ρ_i, ϕ_i) are polar coordinates on each two-plane on which the ω action acts as a rotation with parameter B_i. The sign of the parameter B_i is related to the orientation of the rotation. The dx^2 term is absent when the dimension $d = 2m + 1$ is odd. When $d = 2m + 2$ is even, x, y span the two-plane on which the ω action is trivial. In those coordinates the identifications we have introduced so far can be summarized by the equivalence relation:

$$(\phi_i, y) \sim (\phi_i + 2\pi n_1 B_i R + 2\pi n_2, y + 2\pi n_1 R) \qquad n_1, n_2 \in \mathbb{Z}$$

The identifications $\phi_i \to 2\pi n_2$ and $y \to 2\pi n_1 R$ are of course the standard periodicities that would be present in a regular or "untwisted" reduction. The new ingredient here is the periodicity $\phi_i \to \phi_i + 2\pi n_1 B_i R$ when $y \to y + 2\pi n_1 R$. Note that this is exactly the type of identifications that we encountered when we constructed the Kerr instanton 3.9. Since the angular coordinates ϕ_i are identified under two operations, we can freely change B_i by a multiple of $1/R$ and the identifications remain unchanged (the effect is similar to what we saw in 3.16). In other words, inequivalent spacetimes are obtained only for [2]

$$-\frac{1}{2R} < B \leq \frac{1}{2R} \tag{3.21}$$

Let us now proceed with the Kaluza-Klein reduction along the orbits of the Killing field. The Killing field is

$$q = \partial_y + \sum_{i=1}^{m} B_i \partial_{\phi_i} \tag{3.22}$$

The idea now is to introduce coordinates along the orbits of the action as we always do when we perform a reduction. The canonical coordinates are $\tilde{\phi}_i := \phi_i - B_i y$. In the new coordinates the flat

[2] In fact, this holds if we consider only bosons. If we want to classify backgrounds which differ only in the spin structure as inequivalent, this argument needs a little more care. For more information on this issue, in the context of fluxbranes, see [43].

metric takes the form

$$ds^2 = \Lambda\Big[dy + \frac{1}{\Lambda}\sum_{i=1}^{m} B_i\rho_i^2 d\tilde{\phi}_i\Big]^2 + \sum_{i=1}^{m}(d\rho_i^2 + \rho_i^2 d\tilde{\phi}_i^2) - \frac{1}{\Lambda}\Big(\sum_{i=1}^{m} B_i\rho_i^2 d\tilde{\phi}_i\Big)^2 + dx^2$$

$$\Lambda := 1 + \sum_{i=1}^{m} B_i^2\rho_i^2 \tag{3.23}$$

It is then straightforward to reduce this using a reduction ansatz. If we consider $d = 11$ then we can use 3.11 to obtain:

$$ds_{10}^2 = \Lambda^{1/2}\Big[\sum_{i=1}^{5}(d\rho_i^2 + \rho_i^2 d\tilde{\phi}_i^2) - \frac{1}{\Lambda}\Big(\sum_{i=1}^{m} B_i\rho_i^2 d\tilde{\phi}_i\Big)^2\Big]$$

$$e^{\frac{4}{3}\varphi} = \Lambda \qquad A = \frac{1}{\Lambda}\sum_{i=1}^{m} B_i\rho_i^2 d\tilde{\phi}_i$$

Where A denotes the gauge field and φ the dilaton. This shows that a gauge field is produced whenever ω acts non-trivially on some two-plane so that $B_i \neq 0$. Note also that the parameter B_i is related to the field strength by

$$B_i^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}|_{\rho_i=0}$$

When only one of the B_i is non-zero, then we call the solution a *fluxbrane* [3]. In this case there is field strength in the $(\rho_i, \tilde{\phi}_i)$ plane which is thought of as being the plane transverse to the fluxbrane. Therefore, the fluxbrane is an object that extends along the remaining eight dimensions. In order to emphasize its dimensionality and its eight dimensional Poincaré invariance, we often refer to this object as an $F7$-brane. When more than one B_i are non-zero, then we interpret this configuration as various intersecting fluxbranes each transverse to some $(\rho_i, \tilde{\phi}_i)$ plane. The fluxbbranes are regarded as a generalization of the Melvin universe [45] in String Theory.

It is important that due to the form of the dilaton in the fluxbrane background (see either 3.19 or 3.23) the ten-dimensional spacetime asymptotically decompactifies. This indicates a breakdown of the basic assumption that the compactified dimension can be made "small enough". Therefore, although we started with a valid M-theory background, the type IIA picture is only adequate at certain distances ρ_i. This reveals the non-perturbative nature of the spacetime we are considering here. We can view those backgrounds as an approximation to a constant magnetic field which is valid when $\rho \ll 1/|B|$ so that the geometry looks locally as a circle bundle over $\mathbb{R}^3$ where the circles have approximately constant length R and the string coupling is $g_s = R/\sqrt{\alpha'}$. What is more, the standard compactification assumption dictates a regime of validity only for length scales larger than the compactification scale so that $\rho \gg R$. Combining those relations together gives the following consistency condition:

$$R\,|B| \ll 1 \tag{3.24}$$

We can always arrange the magnetic field to satisfy this condition by slightly shifting the Killing vector field on which we reduce from 3.22 to

$$q' = \partial_y + \Big(\frac{n}{R} + B\Big)\partial_\phi$$

[3] We could have chosen the rotation to take place in a plane containing the time direction. This would give rise to a different object called the *nullbrane*. Apparently, taking the vector λ to be in the time direction would not be sensible since it would result in compactifying time. For more details on nullbranes see [44]

This has the effect of shifting the magnetic field by n/R while preserving the periodicity of the canonical coordinates $\tilde{\phi} = \phi - (n/R + B)y$. It is easy to see then that there is a unique value of n such that 3.24 is satisfied. This procedure can be performed for the case when more magnetic fields B_i are present to ensure that 3.24 holds for all of them separately. This result is of great importance since it picks out a "correct" cycle on which the reduction is to be performed such that the ten dimensional theory is indeed perturbatively accurate.

Next we observe that 3.19 is precisely this fluxbrane spacetime containing only one fluxbrane originating from a non-trivial action on the plane with polar coordinates $\rho = r\sin\theta$ and ϕ. In a more pictorial way, we have shown the commutativity of the diagram of Figure 3.2

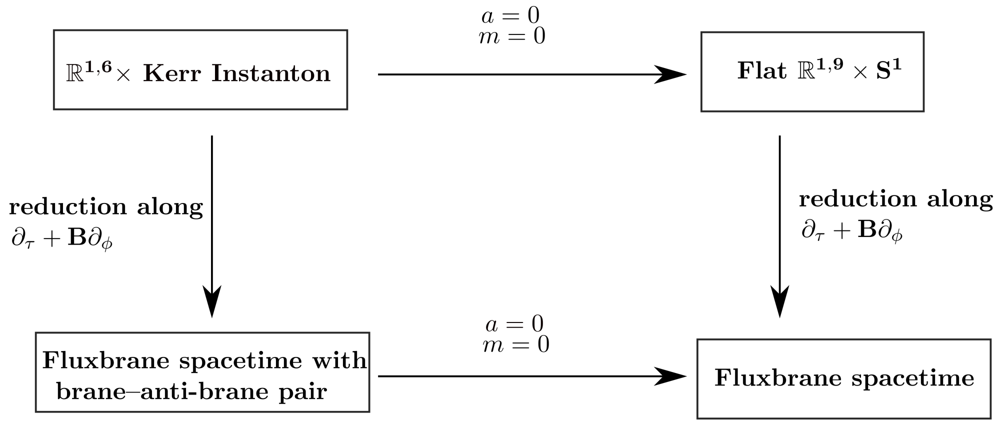


Figure 3.2: An illustration of the relation between the Kerr instanton and the fluxbrane spacetime.

We have argued for the existence of the fluxbrane in the Kerr instanton geometry by reducing and setting $m = a = 0$. Since the Killing vector along which we reduced was $\partial_\tau + (\Omega - 1/\gamma)\partial_\phi$ we see that at asymptotic infinity the Kerr spacetime will have a magnetic field equal to $\Omega - 1/\gamma$. The compactification radius of our reduction is $R = \gamma$ (the "radius" of τ) and therefore we have

$$BR = \left(\Omega - \frac{1}{\gamma}\right)\gamma = \Omega\gamma - 1 = \frac{a}{\sqrt{m^2 + a^2}} - 1$$

which for $a \gg m$ 3.24 is clearly satisfied. Therefore the reduction along this Killing vector field is valid if we are interested in the regime $a \gg m$ which reflects the physical scenario of the brane–anti-brane being a large distance apart. In this case the magnetic field for $a \gg m$ is approximated by

$$B = \Omega - \frac{1}{\gamma} \simeq -\frac{m}{4a^2}$$

What is remarkable about this value is that it is precisely the magnetic field required to keep the brane–anti-brane pair apart [38]. To see this first note that as we have already mentioned, the mass of each monopole can been identified with the quantity $\gamma/2$ (a requirement that is derived by demanding that close to each monopole the metric becomes a Taub-NUT metric). Then in the regime $a \gg m$ that we are considering the mass of each brane is given by

$$M_{D6} = \frac{\gamma}{2} = \frac{m(m + \sqrt{a^2 + m^2})}{\sqrt{a^2 + m^2}} \longrightarrow m \qquad \text{as } \frac{m}{a} \to 0$$

Next we calculate the magnetic repulsion due to this magnetic field that we found above

$$F_{\text{mag}} \simeq 2qB \simeq 2M_{D6}\frac{m}{4a^2} = \frac{m^2}{2a^2}$$

Where we used that the D6 are maximal p-branes so that their charge is equal to their mass. For the same reason their gravitational and magnetic attraction (due to their own fields) will be equal and given by

$$F_{\text{grav}} = \frac{M_{D6}^2}{r^2} \simeq \frac{m^2}{4a^2}$$

Where the distance was taken to be the geodesic distance of the two monopoles for large a which was found to be equal to $2a$. We see that $2F_{\text{grav}} = F_{\text{mag}}$ so that the system perfectly balances under the influence of the magnetic field caused by the fluxbrane.

To recap, we have identified the content of M-theory on a Kerr instanton as being a D6-$\overline{\text{D6}}$ pair immersed in a magnetic fluxbrane background where the brane–anti-brane pair balances. This has been possible due to the twisted identifications 3.9 that give rise to a magnetic field. It is important that the twisting is "measured" in the coordinates in which the (asymptotic) metric assumes its canonical flat form. Indeed, we can always (and we did) introduce untwisted coordinates as in 3.16 but then the asymptotic metric will not have the standard flat form since now a magnetic field exists everywhere.

3.4 A closer look at the brane–anti-brane system

In this section we explore the structure of the Kerr instanton under the torus action. Our aim is to understand the disparities between different circle actions and the geometric properties of the bolts and nuts both in the eleven dimensional geometry and after the reduction. Additionally, we will investigate the issue of the brane–anti-brane stability which so far has been attributed to a fortuitous conspiracy of the various fields and parameters.

3.4.1 Fixed points of general Kiling fields

In the previous section we discussed various aspects of M-theory on a Kerr instanton and by performing a Kaluza-Klein reduction on a circle we examined the String Theory dynamics in ten dimensions. However, many crucial points remain nebulous. To begin with, the choice of a Killing vector along which the reduction takes place seemed ad hoc since there is no preferred element in the isometry algebra $\mathfrak{iso}(M)$ or at least we did not argue about it. Another problem along the same lines is the stability of the brane–anti-brane system which relied on the magnetic field that exactly canceled the brane–anti-brane attraction. This was a seemingly miraculous intervention, heavily dependent upon the choice of the Killing vector's finely tuned magnitude. In this section, we seek answers to those questions.

To begin with, the isometry group of the Kerr instanton is $U(1) \times U(1)$, namely a 2-torus, spanned by the ϕ, τ coordinates on which the metric 3.8 does not depend. Therefore, we have an isometric torus action on the manifold $\mathbb{R}^2 \times S^2$. Reducing to ten dimensions requires choosing an embedded circle $G(m,n) \hookrightarrow T^2$ for which we unquestionably chose the one whose Lie algebra is spanned by $K = \partial_\tau + (\Omega - 1/\gamma)\partial_\phi$. Let us generalize this by considering an arbitrary linear combination:

$$K = \kappa\partial_\tau + \lambda\partial_\phi \quad , \quad \kappa, \lambda \in \mathbb{R} \tag{3.25}$$

It is straightforward to compute the magnitude of the Killing vector

$$K^\mu K_\mu = \kappa^2 g_{\tau\tau}^2 + 2\kappa\lambda g_{\tau\phi} + \lambda^2 g_{\phi\phi}^2 \tag{3.26}$$

$$= \frac{1}{r^2 - a^2\cos^2\theta}\Big\{\Delta(\kappa + \lambda a \sin^2\theta)^2 + \sin^2\theta(\kappa a - \lambda(r^2 - a^2))^2\Big\} \tag{3.27}$$

The fixed points of the corresponding isometry will be given by the vanishing locus of $K^\mu K_\mu$. Since $\Delta \geq 0$ we have

$$\Delta(\kappa + \lambda a \sin^2\theta)^2 = 0 \tag{3.28}$$
$$\sin^2\theta(\kappa a - \lambda(r^2 - a^2))^2 = 0 \tag{3.29}$$

In order to find all possible solutions, we distinguish the following cases:

- If $\Delta = 0 \Rightarrow r = r_+ = m + \sqrt{m^2 + a^2}$ then 3.28 is satisfied and 3.29 can be satisfied in two ways:
  - If $\sin\theta = 0 \Rightarrow \theta = 0, \pi$ then the fixed points are two isolated points. This is therefore, a nut–anti-nut pair. The reduction that we performed in the previous section falls in this category. In this case there is no constraint on the κ, λ so these two points are fixed by any circle subgroup. Those are therefore the fixed points of the torus action.
  - If $\kappa a - \lambda(r_+^2 - a^2) = 0 \Rightarrow \lambda/\kappa = \Omega$ where $\Omega = a/(r_+^2 - a^2)$ as before. In this case the fixed point locus is the surface $r = r_+$.
- If $\Delta \neq 0$ then we must demand that $\kappa + \lambda a \sin^2\theta = 0$ so that in order to satisfy 3.29 we again have two possibilities
  - If $\sin\theta = 0 \Rightarrow \theta = 0, \pi$ then $\kappa = 0$ and the Killing vector is in the direction of ∂_ϕ. In this case the bolt is the disjoint union of the surfaces $\theta = 0$ and $\theta = \pi$. We will have more to say about those shortly.
  - If $\sin\theta \neq 0$ (and $\Delta \neq 0$) then the equations that we get have no solutions.

Summarizing, there are three types of fixed point loci, two bolts and a nut–anti-nut pair depending on which circle subgroup of the torus we choose as our eleventh dimension. The nuts are the fixed points of the torus while the bolts constitute points with isotropy group S^1 where geometrically one of two independent torus cycles "collapses". This is in perfect agreement with our conclusion regarding fixed points sets in Section 2.5. However, there is a small difference owing to the non-compactness of the manifold we are studying here. In our treatment of compact manifolds, the codimension two fixed point loci (bolts) had to be compact but this is not longer the case. Let us explore what they look like in more detail:

- The surface $r = r_+$ has induced metric

  $$ds^2 = (r_+^2 - a^2\cos^2\theta)d\theta^2 + \frac{\sin^2\theta}{r_+^2 - a^2\cos^2\theta}\Big[(r_+^2 - a^2)d\phi - a d\tau\Big]^2$$

  A more appropriate coordinate $\tilde{\phi} := \phi - \frac{a}{(r_+^2 - a^2)}\tau = \phi - \Omega\tau$ makes the geometry of this surface more transparent

  $$ds^2 = (r_+^2 - a^2\cos^2\theta)d\theta^2 + \frac{\sin^2\theta(r_+^2 - a^2)^2}{r_+^2 - a^2\cos^2\theta}d\tilde{\phi}^2$$

  This is a metric on a sphere and therefore $r = r_+$ is a spherical bolt. As a reality check, we can calculate the Euler characteristic $\chi(\mathbb{R} \times S^2) = \chi(S^2) = 2$ which is indeed correct and agrees with the existence of two fixed points of the entire torus action which would again give $\chi(\mathbb{R}^2 \times S^2) = 2$ according to Theorem 2.3.2.

- The surfaces $\theta = 0, \pi$ both have the same induced metric given by

$$ds^2 = \frac{\Delta}{r^2} d\tau^2 + (r^2 - a^2)\frac{dr^2}{\Delta}$$

This is a warped product of the circle (parametrized by τ) and the line (parametrized by r) with the radius of the circle vanishing at $r = r_+$. This therefore has the topology of a disk $\mathbb{D}^2$. The asymptotic radius of the circle is equal to γ (the period of τ). Note that we can use this bolt to calculate the Euler characteristic $\chi(\mathbb{R}^2 \times S^2) = \chi(\mathbb{D}^2 \sqcup \mathbb{D}^2) = 1 + 1 = 2$.

The geometry of those surfaces is illustrated in Figure 3.3. Note the structure of the orbit space under the entire torus action. Consider the right picture in Figure 3.3 and quotient out the azimuthial coordinate $\tilde{\phi}$. The resulting space is isomorphic to the upper half Euclidean plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ with the boundary being the line $\theta = 0$ which is indeed a line after the quotient is taken. This is an interesting example of the quotient structure of a torus action on a non-compact manifold, in this case $\mathbb{R}^2 \times S^2$. The orbit space is indeed simply connected and has a boundary in agreement with our treatment in 2.4. What is noteworthy is the existence of this new type of bolt which is not connected.

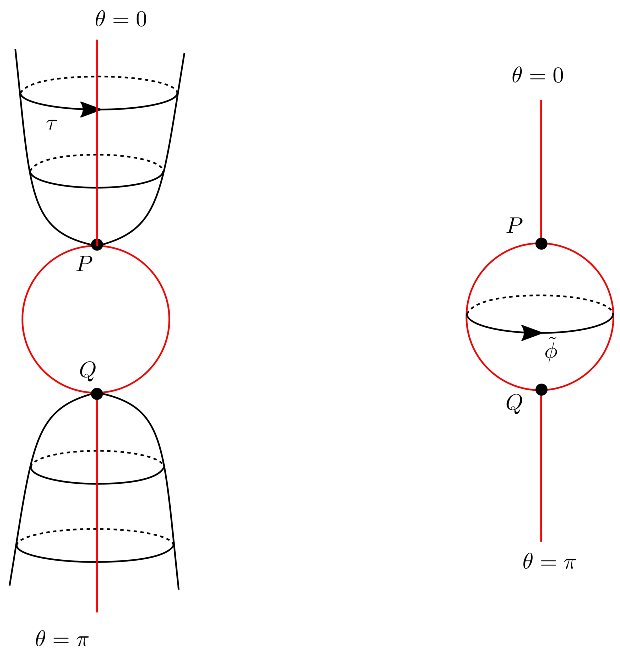


Figure 3.3: The bolt B_1 (left) which is the disjoint union of two disks and the bolt B_2 (right) which is a two-sphere.The nut P and the anti-nut Q are located at the intersection of the two bolts. Note that the two bolts should not be depicted in the same image since the azimuthial coordinate is different in the two cases.

3.4.2 Isotropy representations of different Killing fields

Now that we have uncovered the geometry of the fixed points, we move on to investigate the fixed point data. This is done using the isotropy representation which was introduced in Section 2.3. As we mentioned there, the isotropy representation of a circle action that is generated by a Killing field $K \in \text{Vect}(M)$ is given by ∇K whose eigenvalues constitute the fixed point data. One problem that often arises is that the metrics we are using are adapted to the isometries under consideration and therefore the coordinate system is singular on the fixed point locus. Those quantities can be calculated in different coordinates and the result is that the weight associated to $\partial_\tau + \Omega \partial_\phi$ is

$w_1 = \gamma^{-1}$ and the weight associated to ∂_ϕ is $w_2 = -1$. Those are calculated on the bolts B_1 and B_2 respectively.

Since we have the weights for those two fields, we can consider an arbitrary linear combination $K = \kappa\partial_\tau + \lambda\partial_\phi$ which will in general have two fixed points, the points P and Q of Figure 3.3. The isotropy representation at P is then found to be:

$$\nabla K|_P = \kappa \begin{pmatrix} 0 & \gamma^{-1} & 0 & 0 \\ -\gamma^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & -\Omega & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \kappa/\gamma & 0 & 0 \\ -\kappa/\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa\Omega - \lambda \\ 0 & 0 & -\kappa\Omega + \lambda & 0 \end{pmatrix}$$

From this we deduce that the point P is a (p, q)-nut if

$$\frac{\kappa/\gamma}{\kappa\Omega - \lambda} = \frac{p}{q} \tag{3.30}$$

Here, p, q are coprime integers. We can ask for example what are the values of κ, λ such that the point P corresponds to a fixed point with $p = q = 1$ which is the requirement for the action to locally be the Hopf action on a three-sphere surrounding the fixed point as we saw in 2.3.2. Solving 3.30 with $p/q = 1$ gives

$$\frac{\lambda}{\kappa} = \Omega - \frac{1}{\gamma} \tag{3.31}$$

This corresponds precisely to the Killing field $K = \partial_\tau + (\Omega - \gamma^{-1})\partial_\phi$ that we used in Section 3.3 and which was used for the reduction in [38]. The argument that was given in [38] to identify the system with a D6-$\overline{\text{D6}}$ relied on a comparison between the Taub-NUT metric and the local geometry around the fixed point. The proof that we presented here has a topological flavor and depends on the existence of a Hopf action around the fixed point under consideration. A similar calculation for the point Q shows that when 3.31 holds, the point Q is indeed an anti-nut with weights $\{+1, -1\}$. We have therefore shown the following

Reduction along the Killing field $K = \partial_\tau + (\Omega - \gamma^{-1})\partial_\phi$ corresponds to a D6-$\overline{D6}$ pair.

It is also interesting to calculate the nut charge of the branes. The weights of the isotropy representation of $K = \partial_\tau + \Omega - \gamma^{-1})\partial_\phi$ are $\{\gamma^{-1}, \gamma^{-1}\}$ at P and $\{\gamma^{-1}, -\gamma^{-1}\}$ at Q. Therefore following Section 2.5, the nut charge at P is

$$N = \frac{1}{4\pi}\int_{S^2} F = \frac{2\pi\gamma}{4\pi} = \frac{\gamma}{2}$$

This is the mass parameter of the D6-brane that we used in the previous section and which was derived in [38] by a direct comparison of a local model of the Kerr metric with the Taub-NUT metric. The charge is reversed at the point Q which is an anti-nut.

According to our discussion in 2.3.2, after the reduction along K, the quotient space is a smooth manifold. The quotient space is this case is $\mathbb{R}^3$ and as we will again verify in the next Section, that the metric is indeed smooth. Now, consider a general reduction along the Killing field $K' = \partial_\tau + B\partial_\phi$ with B left unspecified. The isotropy representation at P is given by:

$$\nabla K'|_P = \begin{pmatrix} 0 & \gamma^{-1} & 0 & 0 \\ -\gamma^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega - B \\ 0 & 0 & B - \Omega & 0 \end{pmatrix} \tag{3.32}$$

Due to the non-unit weights, the rotation in the two planes (which are the tangents to the two bolts B_1 and B_2) will introduce non-zero isotropy and the space will develop orbifold singularities as we saw in 2.3.2. Let us use $\pi(B_1), \pi(B_2)$ for the images of the two bolts under the quotient map. Generally, those are one dimensional and we think of them as such. If we rescale the Killing field, we can arrange for the orbifold group on either $\pi(B_1)$ or $\pi(B_2)$ to be arbitrary. Indeed, by rescaling the Killing field by $k\gamma$ we find

$$\nabla(k\gamma K')|_P = \begin{pmatrix} 0 & k & 0 & 0 \\ -k & 0 & 0 & 0 \\ 0 & 0 & 0 & k\gamma(\Omega - B) \\ 0 & 0 & k\gamma(B - \Omega) & 0 \end{pmatrix} \tag{3.33}$$

In this case, B_1 is locally acted upon by the upper left block of 3.33 and $\pi(B_1)$ has a conical angle of $2\pi/k$. On the other hand, the compact one-dimensional line $\pi(B_2)$ will have a cone singularity with angle

$$\frac{2\pi}{k\gamma(\Omega - B)} = \left(1 + \frac{m^2}{a^2}\right)^{1/2} \left(1 - \frac{2Bmr_+}{a}\right)^{-1} \frac{2\pi}{k}$$

We note that when the bolt B_1 is fixed by the action of the Killing field, then no singularity appears on $\pi(B_1)$ and similarly for B_2. We should also stress out that under a general reduction, the (p, q) type of the fixed points will change and therefore the charge of the topological defect in the quotient space will be different. More specifically, the charge will in general be a fraction of the D6-brane charge $\gamma/2$. Those branes with fractional charges always sit on orbifold singularities as we just saw.

3.4.3 Conical singularities and cosmic strings

We now investigate the singular structure from the point of view of the metric which corresponds to a more traditional approach in the physics language. The concept of *conical singularities* is a rather common theme encountered in various physical systems. We say that a manifold M has a *conical singularity* if there is a region in which the metric admits the following form:

$$ds^2 = dr^2 + r^2 d\Omega^2_{n-1}$$

where $d\Omega^2_{n-1}$ is the metric of an $(n-1)$-dimensinal manifold N. In this case, the metric is smooth only if N is the $(n-1)$-sphere S^{n-1} and $d\Omega^2_{n-1}$ is the round metric. For four dimensional manifolds, if N is S^3/Γ with $\Gamma \subset SU(2)$ a finite subgroup then the singularity is of ADE type. When the dimension of N is two, then we can measure how much it deviates from the flat plane geometry by defining the *deficit angle* δ. Given an origin O in the plane, let C denote the proper length of the circumference of the circles centered at O and r denote the radial coordinate. Then the deficit angle δ is defined by the relation

$$\left.\frac{dC}{dr}\right|_{r=0} = 2\pi - \delta$$

In some sense, the deficit angle is a measure of how "conical" a singularity is. The appearance of such deficit angles is a herald of instabilities. In the solution that we have been discussing, the Kerr instanton, the eleven dimensional metric is of course smooth, since by construction the periodicities were chosen so. However, the reduced metric can still possess conical suingularities. The locations of such conical singularities have been investigated in [46, 47] and they coincide with our findings in 2.3.2 which indicate that they can only appear on the images of the bolts $\pi(B_1), \pi(B_2)$. If $\Delta\tilde{\phi}$ denotes the periodicity of some angular coordinate $\tilde{\phi}$ in the reduced metric then the deficit angles are found to be given by

$$\delta_{\pi(B_1)} = 2\pi - \frac{\Delta\tilde{\phi}}{\sqrt{g_{\theta\theta}}} \frac{d\sqrt{g_{\tilde{\phi}\tilde{\phi}}}}{d\theta}\Big|_{\theta=0,\pi} = 2\pi - \Delta\tilde{\phi} \tag{3.34}$$

$$\delta_{\pi(B_2)} = 2\pi - \frac{\Delta\tilde{\phi}}{\sqrt{g_{rr}}} \frac{d\sqrt{g_{\tilde{\phi}\tilde{\phi}}}}{dr}\Big|_{\theta=0,\pi} = 2\pi - \left(1 + \frac{m^2}{a^2}\right)^{1/2} \left(1 - \frac{2Bmr_+}{a}\right)^{-1} \Delta\tilde{\phi} \tag{3.35}$$

Those are precisely the same values that we calculated in the previous section using isotropy representations. The only difference is that in the previous section we calculated periodicities instead of deficit angles. Again, we can easily verify that the only way to get rid of those deficit angles is to require

$$\Delta\phi = 2\pi$$
$$B = \frac{a - \sqrt{m^2 + a^2}}{2mr_+} = \Omega - \frac{1}{\gamma}$$

After this observation, one might wonder what is the physical interpretation of deficit angles and why their existence is so tightly related to the stability of the brane-anti-brane system. The answer is that such deficit angles correspond to cosmic strings [39] with tension T, related to the deficit angle by $8\pi T = \delta$ (see e.g. [48]). When there is non-vanishing deficit angle on $\pi(B_1)$ (two infinite lines stretching to infinity) then the ten dimensional interpretation is that of two cosmic strings pulling the brane–anti-brane pair apart against the attractive forces of their gravity and RR fields. On the other hand, a non-vanishing deficit angle on $\pi(B_2)$ (a compact line starting on the nut and ending on the anti-nut) is interpreted as a strut pushing the nut–anti-nut pair away (see Figure 3.3).

The existence of a magnetic field in which the brane–anti-brane system balances was seen as a random artifact but under this new perspective, it can be attributed to the delicate cancellation of the deficit angles, leaving all the burden of the suspension of the brane–anti-brane pair to the magnetic field (since the cosmic strings are absent).

3.5 Reduction along special cycles

In this section want to consider reductions along different cycles in the isometrically acting torus parametrized by the (τ, ϕ) coordinates in the Kerr instanton. Since there is nothing particularly special about the Killing vector $K = \partial_\tau + (\Omega - 1/\gamma)\partial_\phi$ (other than eliminating the conical singularities) we can perform an $SL(2, \mathbb{R})$ transformation to generate different ten dimensional backgrounds. In particular, as we have already argued in Chapter 2 choosing to reduce along circle actions with fixed point bolts will result in ten dimensional backgrounds with boundaries. We explore the geometry of those backgrounds and make some comments on their stability.

3.5.1 Reduction on the spherical bolt B_2

Having understood the structure of reductions along various different cycles, it is now time to further investigate some of them. First, as we have noted before, requiring that the ten dimensional string coupling does not diverge due to the existence of the fluxbrane background led us to the condition 3.24. We know that equivalent backgrounds arise when $B = \Omega + n/\gamma$ for some integer $n \in \mathbb{Z}$ and for those the angular coordinate has period 2π so that no conical singularity exists on the bolt B_1. In this case the only possible deficit angle can occur on the bolt B_2. We have seen that B_2 is non-singular when we pick $B = \Omega - 1/\gamma$ but we have already pointed out that this leads to a valid perturbative picture only when the D6-$\overline{\text{D6}}$ pair is far apart. For this more general B the condition 3.24 becomes:

$$R\,|B| = \gamma\Big(\Omega + \frac{n}{\gamma}\Big) = \frac{a}{(m^2+a^2)^{1/2}} + n \ll 1$$

When $a \sim m$ we expect that the brane–anti-brane pair has come close to the critical distance in which the open string stretching between them becomes tachyonic. In this case if $a/m \sim 1$ we get $R|B| \simeq 0.7 + n$ so that there is no integer n which ensures $R|B| \ll 1$. This regime is therefore manifestly non-perturbative. On the other hand, when the brane–anti-brane pair coincide we have $a/m \sim 0$ and $n = 0$ gives the required perturbative background. In this case the Killing vector is $K = \partial_\tau + \Omega\partial_\phi$ and the fixed point locus is the bolt B_2. Reduction along the orbits of this vector gives the ten dimensional solution

$$\begin{aligned}
ds^2_{10} &= \Lambda^{1/2}\Big\{-dt^2 + \sum_{i=1}^{6}(dy^i)^2 + \Sigma\Big[\frac{dr^2}{\Delta} + d\theta^2\Big]\Big\} + \Lambda^{-1/2}\Delta\sin^2\theta d\tilde{\phi}^2\\
e^{\frac{4}{3}\varphi} &= \Lambda\\
A_{\tilde{\phi}} &= \Lambda^{-1}\frac{\sin^2\theta}{\Sigma}\Big\{\Omega[(r^2-a^2)^2 + \Delta a^2\sin^2\theta] + a[(r^2-a^2) - \Delta]\Big\}\\
\Lambda &:= \frac{1}{\Sigma}\Big\{\Delta(1+\Omega a\sin^2\theta)^2 + a^2\sin^2\theta\Big(1 - \frac{(r^2-a^2)^2}{(r_+^2-a^2)^2}\Big)\Big\}
\end{aligned}$$

We have obtained a Type IIA geometry with no D6-$\overline{\text{D6}}$ but with a spherical bolt where the string coupling vanishes. This bolt is a spherical boundary so that $\partial N_{10} = \mathbb{R}^7 \times S^2$. The boundary does not carry any topological charge. In other words, the self-intersection of the bolt vanishes. This follows from charge conservation. As we just argued, this background corresponds to the more accurate perturbative picture in the limit when the brane–anti-brane coincide. The reduced space is free of conical singularities.

3.5.2 Reduction on the bolt B_1

The last qualitatively different option we have is to reduce along the Killing vector field $K = \partial_\phi$. The two connected components of the bolt B_1 are boundaries of the ten dimensional space and the metric on the reduced space becomes

$$\begin{aligned}
ds^2_{10} &= \Lambda^{1/2}\Big\{-dt^2 + \sum_{i=1}^{6}(dy^i)^2 + \Sigma\Big[\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{(r^2-a^2)^2 + a^2\Delta\sin^2\theta}d\tau^2\Big]\Big\}\\
e^{\frac{4}{3}\varphi} &= \Lambda\\
A_\tau &= \frac{\Delta a\sin^2\theta - \sin^2\theta a(r^2-a^2)}{\Delta a\sin^4\theta + \sin^2\theta(r^2-a^2)^2} = \frac{\Delta a - a(r^2-a^2)}{\Delta a\sin^2\theta + (r^2-a^2)^2}\\
\Lambda &:= \frac{1}{\Sigma}\Big(\Delta a^2\sin^4\theta + \sin^2\theta(r^2-a^2)^2\Big)
\end{aligned}$$

Indeed, now $\theta = 0, \pi$ is a weak coupling locus. The self intersection of those bolts in the four dimensional space $\mathbb{R}^2 \times S^2$ is zero which follows from the vanishing signature of the space $\mathbb{R}^2 \times S^2$

[32]. Therefore, the boundaries do not carry any topological charge in the sense of Section 2.5 and do not attract each other.

Concluding this section we have demonstrated the existence of different ten dimensional backgrounds corresponding to the same eleven dimensional geometry. We have been concerned with the reduction on the Kerr instanton $\mathbb{R}^2 \times S^2$ which has the same fixed point structure as the S^4. This could have been expected since we can write the decomposition $S^4 = \mathbb{D}^2 \times S^2 \sqcup \mathbb{D}^3 \times S^1$ where the two manifolds are glued along their common boundary $S^2 \times S^1$. This shows that the Kerr instanton differs from S^4 by a manifold which admits a free circle action.

The non-compactness of $\mathbb{R}^2 \times S^2$ allows for a greater freedom in the types of bolts and indeed we discovered both a compact and a non-compact bolt. In total we can have three different topologies in the reduced space. Reducing along a generic circle with two isolated fixed points gives $\mathbb{R}^3$. On the other hand, reducing along the special circles gives, as expected, quotient spaces with boundaries. For the case of the non-compact bolt B_1 the quotient space is $\{(x, y, z) \in \mathbb{R}^3 | z \in [-1, 1]\}$ which has two infinite walls at the two ends. For the bolt B_2 the quotient space was $\mathbb{R}^3 \setminus \mathbb{D}^3$ namely $\mathbb{R}^3$ with a hole. In both cases the boundaries are not charged due to their vanishing self-intersections (in four dimensions).

Finally, we mention that a similar approach to brane–anti-brane systems has been considered in [49]. A class of supergravity solutions with $\mathrm{ISO}(1,p) \times SO(9-p)$ and RR fields were found in [50] and later interpreted in [51] as coincident brane–anti-brane pairs in type II theories. Subsequently, [49] considered the purely geometric M-theory lift of those backgrounds (for $p = 6$) and investigated the reduction along different circles. In this work, they considered orbifold singularities in the eleven dimensional space (before the reduction). These orbifolds have localized closed string tachyons and were discussed in [52].

Chapter 4

Flux vacua and S^4

4.1 Fluxes

In constructing string vacua it is useful to consider *fluxes*. Those correspond to non trivial p-forms F that can be integrated over closed p-cycles Σ to give terms of the form $\int_\Sigma F$. In eleven dimensions the only possibility is the four-form flux. We have already seen in Section 2.5 that the inclusion of a non-trivial four-form F_4 in an eleven dimensional supergravity background induces different non-zero forms in the reduced Type IIA background. When we further compactify on a manifold X of dimension $\dim X = d$, the inclusion of such terms can be shown [53] to generate a potential for the moduli fields of the form

$$V_F = \frac{m_p^2}{V_X^2} \int_X \sqrt{g} g^{mr} ... g^{ns} F_{m...n} F_{r...s}$$

Although those are great news since we seem to have overcome the moduli stabilization problem, there is still a small implication. Since our metric is not fixed (it has its own moduli) we can consider a one parameter family of metrics given by $g_{mn} = r^2 g^0_{mn}$ where g^0_{mn} is some fiducial metric which is chosen such that the volume is unit $\mathrm{Vol}(g^0_{mn}) = 1$. Then the corresponding potential from all the possible p-forms and the additional contribution from the Einstein-Hilbert term would give

$$\frac{V_F}{m_p^4} = \sum_p r^{-2p-d} \int_X \sqrt{g^0_{mn}} F_p^2|_0 - r^{-2-d} \int_X \sqrt{g^0} R^0 \tag{4.1}$$

If the Ricci scalar is negative then the potential acquires its minimum as $r \to \infty$ which signifies an instability. Therefore, we necessarily assume from now on that the internal manifold X has positive Ricci curvature. In the case when the fluxes are absent, only the second term in 4.1 contributes so $r \to 0$ which indicates that the internal manifold collapses to zero size. This is why we need non-zero fluxes to support the manifold and acquire a minimum of V_F for an intermediate value of r. It is important to note that the behavior of the potential near $r \to \infty$ is dictated by the "least negative" power so that when $F_0 = F_1 = 0$ the curvature term dominates. On the other hand when only $F_1 \neq 0$, both terms scale in the same way so that the potential has no extrema. Finally when $F_0 \neq 0$ the p-form term dominates at $r \to \infty$.

4.2 $\mathrm{AdS}_7 \times S^4$

Among the various solutions of eleven dimensional supergravity $\mathrm{AdS}_7 \times S^4$ and its dual $\mathrm{AdS}_4 \times S^7$ are of special interest. The first reason is their crucial role in the AdS /CFT correspondence. The second reason is that spontaneous compactification induced by bosonic fields of eleven dimensional supergravity can only happen when the internal space is of dimension seven or four. It is then clear why in this context the maximally symmetric solutions $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$ are significant. The latter one has received more attention owing to the parallelizability of S^7 which facilitates many calculations and of course because of its relevance to four dimensional physics. However, in the context of our work, we are interested in four dimensional internal manifolds and therefore $\mathrm{AdS}_7 \times S^4$ is the one we will investigate.

Our starting point is the eleven dimensional supergravity whose bosonic sector is described by the Lagrangian

$$\mathcal{L}_{11} = R * 1 - \frac{1}{2} * G_4 \wedge G_4 - \frac{1}{6} G_4 \wedge G_4 \wedge A_3 \tag{4.2}$$

where $G_4 = dA_3$ is the four form field strength and $*$ denotes the eleven dimensional Hodge star. The corresponding equations of motion read

$$R_{MN} = \frac{1}{12}\Big(G_{MRST} G_N^{\ RST} - \frac{1}{12} G_{RSTL} G^{RSTL} g_{MN} \Big) \tag{4.3}$$

$$\nabla_M G^{MNRS} + \frac{3 \cdot 3!}{(144)^2} \epsilon^{NRSM_1N_1R_1S_1M_2N_2R_2S_2} G_{M_1N_1R_1S_1} G_{M_2N_2R_2S_2} = 0 \tag{4.4}$$

It is worth explaining how the aforementioned solution comes about. Consider the eleven dimensional background $M_7 \times N_4$ in M-theory. If no fluxes are present, supersymmetry forces N_4 to be a Calabi-Yau and our only possibilities are K3 or $\mathbb{T}^4$. We can also turn on the fluxes. In M-theory the only option is the G_4 flux. As before, we usually want the isometries of M_7 to be preserved which translates to a G_4 living entirely in N_4. Those flux compactifications fall into the class of Freund-Rubin compactifications where only the top dimensional fluxes of the internal space are non-zero (or their lowest dimensional duals). Those compactifications are very constrained because the above requirements imply the existence of a Killing spinor in N_4. This has the non-trivial consequence that the cone $C(N_4)$ over N_4 admits a covariantly constant spinor and can only be of the type $\mathbb{R}^4/\Gamma \times \mathbb{R}$ where Γ is a subgroup of SU(2) with its natural action on $\mathbb{R}^4 \simeq \mathbb{C}^2$. In the simplest case where Γ is trivial, we have $C(N_4) = \mathbb{R}^5$ which gives that $N_4 = S^4$.

Since our internal manifold has positive curvature, we expect that M_7 will be negatively curved. For a maximally symmetric space this has to be AdS_7. This is indeed the case and the total background solution is $\mathrm{AdS}_7 \times S^4$. Let us see why this is a valid supergravity solution [54]. We endow S^4 with the round metric which we call $R^2 g^s_{mn}$ where R denotes the radius of the S^4 and g^s_{mn} is the round metric on the unit four-sphere. We use Latin indices for the internal space S^4, Greek indices for AdS_7 and uppercase Latin indices for the total eleven dimensional space. For AdS_7 we use the metric:

$$ds^2_{AdS_7} = \frac{L^2}{u^2}\Big(du^2 + \eta_{\mu\nu} d\gamma^\mu d\gamma^{nu} \Big)$$

so that our total eleven dimensional metric becomes

$$ds^2 = \frac{L^2}{u^2}\Big(du^2 + \eta_{\mu\nu} d\gamma^\mu d\gamma^{nu} \Big) + R^2 g^s_{mn} dx^m dx^n \tag{4.5}$$

with $\eta_{\mu\nu}$ the Minkowski metric. The flux G_4 being a top-dimensional form in S^4 can be expressed as $G_4 = h \mathrm{vol}_{S^4}$ with $(\mathrm{vol}_{S^4})_{\mu\nu\rho\sigma} = \sqrt{g^s} \epsilon_{\mu\nu\rho\sigma}$ the volume form on the unit S^4 with $g^s := \det(g^s_{mn})$ and $h \in C^\infty(S^4)$. However this is not the complete picture and in fact h has to be a constant. To see this we need to look at the corresponding equation of motion 4.4. Since the flux G_4 is required to be non-zero only along the four components of the internal space, the second term vanishes. What is left, just like in ordinary Maxwell theory coupled to gravity, is the conservation equation:

$$\frac{1}{\sqrt{g^s}}\partial_m\left(\sqrt{g^s}G^{mnrs}\right) = \nabla_m G^{mnrs} = 0$$

So that the solution is of the form $G^{mnrs} = h\epsilon^{mnrs}/\sqrt{g^s}$ but now with $h \in \mathbb{R}$. Lowering the indices and remembering the tensorial nature of the object $\epsilon^{mnrs}/\sqrt{g^s}$ (but not of the Levi-Civita symbol alone) we also find $G_{mnrs} = h\sqrt{g^s}\epsilon_{mnrs} = h\text{vol}_{S^4}$. Then we use the quantization of flux to get:

$$T_{M_2}\int_{S^4} G_4 = 2\pi N \Rightarrow T_{M_2} h R^4 \frac{8\pi^2}{3} = 2\pi N \Rightarrow h = \frac{6\pi N}{T_{M_2} R^4 8\pi^2} = \frac{3\pi(\ell_{11})^3}{R^4} N \tag{4.6}$$

with $T_{M_2} = (2\pi)^{-2}\ell_{11}^{-3}$ the M2-brane tension which plays the role of the elementary charge, ℓ_{11} the eleven dimensional planck length and $N \in \mathbb{Z}$. In the above we also used the volume of S^4 which is given by $\text{vol}(S^4) = 8\pi^2 R^4/3$. The remaining eleven dimensional supergravity equations of motion are satisfied by the ansatz 4.5 provided that:

$$R^2 = \frac{L^2}{4} = \frac{216}{G^2} \quad \text{where} \quad G^2 := G_{mnrs}G^{mnrs} = h^2 \cdot 4! \tag{4.7}$$

Therefore, the fluxes fix the length scales R, L. This is consistent with our general discussion about fluxes. In particular, using 4.6 we find that the length scales are related to the quantization number N by

$$R = \frac{L}{2} = (\pi N)^{\frac{1}{3}}\ell_{11}$$

In particular we notice that when $N = 0$, the radius R of the internal manifold goes to zero corresponding to the expected collapsing behavior in the absence of flux. Another important remark is the relation between the scale of the sphere R and the AdS scale L which are of the same order. This is a typical hierarchy problem in flux compactifications.

At this point we have a valid M-theory background. According to our previous discussion the natural thing to do is to dimensionally reduce it to obtain the corresponding type IIA picture. The four-sphere S^4 with its round metric admits a circle fibration with base space S^3 and one singularity sitting at each pole. The flat metric on S^4 can be conveniently expressed as the spherical suspension of S^3 and is given by

$$ds^2_{S^4} = R^2 g^s_{mn}dx^m dx^n = R^2\left(d\alpha^2 + \sin^2(\alpha)ds^2_{S^3}\right)$$

In the above, $\alpha \in [0, \pi]$ is the suspension parameter. The metric on S^3 is the Hopf metric, namely the metric that makes the Hopf fibration $S^3 \to S^2$ a submersion of Riemannian manifolds when the metric on the base S^2 is the round metric. In local coordinates this can be expressed as

$$ds^2_{S^3} = \frac{1}{4}ds^2_{S^2} + (dy + C_1)^2 \quad \text{with} \;\; dC_1 = \frac{1}{2}\text{vol}_{S^2} \tag{4.8}$$

where $ds^2_{S^2}$ is the standard round metric on S^2. The total metric then becomes:

$$ds^2 = \frac{L^2}{u^2}(du^2 + \eta_{\mu\nu}d\gamma^\mu d\gamma^\nu) + R^2 d\alpha^2 + \frac{R^2\sin^2(\alpha)}{4}(d\theta^2 + \sin^2(\theta)d\phi^2) + R^2\sin^2(\alpha)(dy + C_1)^2 \tag{4.9}$$

The connection form C_1 is often taken to be of the Wu-Yang form

$$C_1^{(n)} = -\frac{1}{2}(1-\cos\theta)d\phi \qquad C_1^{(s)} = \frac{1}{2}(1+\cos\theta)d\phi$$

where $C_1^{(n)}$ and $C_1^{(s)}$ are the connection forms around the north ($\theta = 0$) and south ($\theta = \pi$) pole respectively. In this setting, y is the coordinate along the fiber S^1 and also the direction of the M-theory circle on which we perform the dimensional reduction. In the language of section 1, we view S^3 as a circle fibration using the U(1) isometry corresponding to the Killing vector field ∂_y.

If we want the U(1) reduction to preserve some supersymmetry then we must require that the action of the U(1) isometry on the spin bundle leaves at least one spinor invariant. Since ISO(S^4)= SO(5) every isometry is characterized by two angles that appear in the canonical form of the associated element in $\mathfrak{so}(5)$ which are antisymmetric matrices. The existence of invariant spinors then forces those two angles that we call a_1, a_2 to either be equal or opposite namely $a_1 = \pm a_2$ [55]. In other words the supersymmetry preserving isometries have a canonical form

$$\exp\begin{pmatrix} 0 & a\tau & 0 & 0 & 0 \\ -a\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm a\tau & 0 \\ 0 & 0 & \mp a\tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(a\tau) & \sin(a\tau) & 0 & 0 & 0 \\ -\sin(a\tau) & \cos(a\tau) & 0 & 0 & 0 \\ 0 & 0 & \cos(a\tau) & \pm\sin(a\tau) & 0 \\ 0 & 0 & \mp\sin(a\tau) & \cos(a\tau) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.10}$$

where τ is the parameter of the U(1) isometry. Does ∂_y generate such an isometry? To answer this we must identify the isometry and then carefully investigate how it acts on the S^4 in appropriate coordinates. The isometry generated by ∂_y is clearly the translation in the y-direction by some constant τ. To find its canonical form and compare it to 4.10 we embedd S^4 in $\mathbb{R}^5$. We take x^5 to be the coordinate along the suspension axis, namely the line connecting the points $\alpha = 0$ and $\alpha = \pi$. Due to the suspension construction, the x^5 will be unaffected by a shift in the y-coordinate. The rest of the coordinates x^i can be thought of as the coordinates in $\mathbb{R}^4$ in which an S^3 is embedded. It is very convenient to work in the coordinates y, θ, ϕ as in 4.8 where θ, ϕ parametrize the S^2 as usual. These coordinates are often called Hopf coordinates because they make the Hopf structure manifest. They are related to the ambient space coordinates by:

$$x^1 = \sin(\alpha)\cos\left(\frac{\theta}{2}\right)\sin(y) \qquad x^2 = \sin(\alpha)\cos\left(\frac{\theta}{2}\right)\cos(y) \qquad x^3 = \sin(\alpha)\sin\left(\frac{\theta}{2}\right)\sin(\phi+y)$$
$$x^4 = \sin(\alpha)\sin\left(\frac{\theta}{2}\right)\cos(\phi+y) \qquad x^5 = \cos(\alpha)$$

It is now a matter of straightforward algebra to verify that under the isometry transformation $y \to y+\tau$ the coordinates x^i of $\mathbb{R}^5$ transform as

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} \to \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) & 0 \\ 0 & 0 & -\sin(\tau) & \cos(\tau) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}$$

which shows that the isometry generated by ∂_y does indeed have an invariant spinor. We go on to perform the reduction as in [55] by using the string-frame ansatz $ds_{11}^2 = e^{-\frac{2}{3}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi}(dy + C_1)^2$. Comparing this antsatz to our eleven dimensional background 4.5 and using 4.7 we get:

$$e^{\frac{4}{3}\phi} = R^2\sin^2(\alpha) \Rightarrow e^{\frac{2}{3}\phi} = R\sin(\alpha) \tag{4.11}$$

$$ds_{10}^2 = R^3\sin(\alpha)\left[\frac{4}{u^2}(du^2 + \eta_{\mu\nu}d\gamma^\mu d\gamma^\nu) + d\alpha^2 + \frac{1}{4}\sin^2(\alpha)\, ds_{s^2}^2\right] \tag{4.12}$$

We therefore see that the ten dimensional space is a warped product of AdS_7 and S^3. Note that we had established from purely topological considerations in Section 2.5 that the quotient space must be an S^3 in the absence of a boundary. This agrees with the three dimensional part of 4.12 which is a spherical suspension of S^2 showning that this is indeed a metric on S^3. The next thing to do is compute the fluxes associated to this ten dimensional solution. The result is [55] that there are non-trivial F_2 and NS H-fluxes given by:

$$F_2 = -\frac{1}{2}\mathrm{vol}_{S^2} \qquad H = -\frac{3}{4}R^3 \sin^3(\alpha) d\alpha \wedge \mathrm{vol}_{S^2} \tag{4.13}$$

Note that there is a difference in the definition of our R which differs by a factor of 2 from the radius R in [55]. The four-from flux H_4 vanishes in this background as is evident from 2.9 since now the eleven dimensional four-form has no component that is independent of the reduction coordinate. We can easily integrate the H-flux to get the B-field

$$B_2 = \frac{3}{4}R^3\Big(\cos(\alpha) - \frac{\cos^3(\alpha)}{3}\Big)\mathrm{vol}_{S^2} + b \tag{4.14}$$

where b is a closed two-form. In the above integration process we implicitly used that the volume form of a closed manifold like S^2 cannot be exact. If it was, a straightforward application of Stoke's theorem would imply a zero volume. Therefore, 4.14 is the most general form of a primitive for H.

4.2.1 Characterization of the fixed points

In Chapter 2 we have given a prescription for characterizing the type of a fixed points in terms of the isotropy representation. Here we show that the two fixed points we found in S^4 are indeed of the expected type. In particular, we will show that the point at $\alpha = 0$ is a nut and the point at $\alpha = \pi$ is an anti-nut.

Let us call the stationary points of the circle action P (for $\alpha = 0$) and Q (for $\alpha = \pi$). First, we focus on the point P. Remember from Section 2.3 that what we are interested in is the matrix form of the endomorphism $\phi_P : T_P M \to T_P M$ from which we can read off the (p,q) nut type. The point P is an extremum of the coordinate α and therefore the infinitesimal displacements around P do not change α and consequently x^5. This means that $T_P M$ is spanned by $\{\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{x^4}\}$ which we take to be an oriented basis of $T_P M$ agreeing with the orientation induced on the plane $x^5 = 1$ by the ambient space $\mathbb{R}^5$. For this reason we can identify $T_P M$ with the plane $x^5 = R$. The action of the chosen U(1) isometry on $T_P M$ will then be given by 4.11 restricted on the first four coordinates so that the matrix form of the endomorphism ϕ_P becomes

$$(\phi_P)_{ab} = \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{pmatrix}$$

which verifies that P is a (1,1) nut singularity. Now we turn to the point Q. Here we cannot apply the same logic since by choosing an orientation for $T_P M$ we have picked an orientation for S^4. The question now is whether $\{\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{x^4}\}$ is an oriented basis of $T_Q M$ in which case $T_Q M$ is identified with the plane $x^5 = -1$. If this is true, then the point Q becomes a (1,1) nut as well. Otherwise, if the orientation is reversed, then we can take $\{\partial_{x^1}, \partial_{x^2}, \partial_{x^4}, \partial_{x^3}\}$ as an oriented basis in which case the second block of the matrix in 4.11 is inverted and the induced action of the U(1) isometry on $T_Q M$ given by the matrix form of the endomorphism $\phi_Q : T_Q M \to T_Q M$ becomes

$$(\phi_Q)_{ab} = \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{pmatrix}$$

which means that the point Q is a (1,-1) anti-nut. This is indeed what happens and in total we have a nut and an anti-nut. Before giving a general proof, it is instructive to understand geometrically why this holds by turning to some lower dimensional examples. Consider the more familiar case of the unit circle S^1 embedded in $\mathbb{R}^2$. There are two orientations for S^1, the "clockwise" and "counter-clockwise" corresponding to the two generators of $H_2(S^1) = \mathbb{Z}$. As shown in Figure 4.1 the point P has a tangent space which can be identified with the plane $x_2 = 1$. Their orientations agree if we choose the "clockwise" orientation since ∂_{x_1} is an oriented basis for both. However, at the antipodal point Q the element ∂_{x_1} which is an oriented basis for the plane $z = -1$ has the wrong orientation on S^1 since it corresponds to the "counter-clockwise" orientation.

The same holds true for S^2 as shown in Figure 4.1. It is clear that if we choose the "outward" orientation of the sphere, then $\{\partial_{x_1}, \partial_{x_2}\}$ is an oriented basis for $T_P M$ but not for $T_Q M$ where the orientation has to be reversed. Therefore, we must take $\{\partial_{x_2}, \partial_{x_1}\}$ as an oriented basis there.

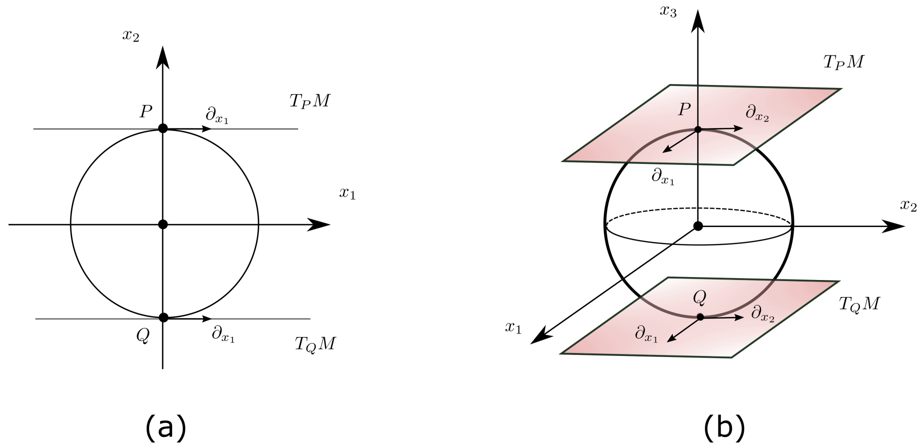


Figure 4.1: (a) The tangent spaces of two antipodal points of S^1. (b) The antipodal tangent spaces for S^2.

Now that we have made it clear that this holds for S^1, we want to prove the same for any sphere S^m including the S^4 that we are mainly interested in. In order to formalize our intuitive picture and turn it into a rigorous statement, some definitions are in order. First, we introduce the reflection map

$$f_m : S^m \to S^m \quad , \qquad (x^1, x^2, ..., x^{m+1}) \mapsto (x^1, x^2, ..., -x^{m+1})$$

where x^i are the coordinates of the $\mathbb{R}^m$ in which S^m is embedded. We take the two poles P,Q to be the intersection points of the x_{m+1} axis with S^m. Let the orientation of S^m be such that $\{\partial_{x^1}, \partial_{x^2}, ..., \partial_{x^m}\}$ is an oriented basis at P. Since f_m only affects the last coordinate x^{m+1}, the induced pushforward $(f_m)_* : T_P S^m \to T_Q S^m$ sends the oriented basis $\{\partial_{x^1}, \partial_{x^2}, ..., \partial_{x^m}\}$ to itself. The map f_m is smooth and bijective and its image is S^m. If f_m preserves the orientation of S^m then $\{\partial_{x^1}, \partial_{x^2}, ..., \partial_{x^m}\}$ is an oriented basis at Q. Of course this certainly cannot be true for m=1 since it would contradict our geometric picture for S^1 so we already know that f_1 is orientation-reversing. We prove here that this holds for every m.

Proposition 4.2.1. The map $f_m : S^m \to S^m$ is orientation-reversing for every $m \geq 1$.

Proof. First we note that f_m induces a group homomorphism on homology groups $(f_m)_* : H_m(S^m) \to H_m(S^m)$. Since $H_m(S^m) = \mathbb{Z}$ for $m \geq 1$, any such group homomorphism is of the form $(f_m)_*(a) = k \cdot a$ for some $k \in \mathbb{Z}$ and for every $a \in \mathbb{Z}$. The integer k is called the *degree* of the map f_m and is denoted by $\deg(f_m) = k$. Since our map f_m is bijective, the induced map $(f_m)_*$ is an isomorphism of abelian groups and the only possiblities are $\deg(f_m) = \pm 1$. Clearly, if $\deg(f_m) = 1$, the map f_m is orientation preserving, otherwise it is orientation-reversing. This is immediate if one takes the fundamental class as the definition of orientation. In this language we want to prove that $\deg(f_m) = -1$.

We proceed by induction on m. For m=1 we have $f_1 : S^1 \to S^1$ acting as $\{x_1, x_2\} \mapsto \{x_1, -x_2\}$ and as we have already seen geometrically, this map reverses orientation. To be more precise we can introduce a coordinate $\theta = \arctan(x_1/x_2)$ in terms of which the map f_1 acts as $\theta \mapsto -\theta$. This has the effect of reversing the orientation of the S^1 proving that f_1 has $\deg(f_1) = -1$.

We now let $D^m = \{(x^1, ..., x^{m+1}) \in S^m | x^1 \leq 0\}$ be the southern hemisphere in S^m and view S^{m-1} as the boundary of D^m or, in other words, the equator of S^m. Then if we restrict f_m on the equator we get $f_m|_{S^{m-1}} = f_{m-1}$. Using the naturality of the long exact sequence of the pair (D^m, S^{m-1}) we get a commutative square

$$\begin{array}{ccccc} H_m(S^m) & \xrightarrow{\simeq} & H_m(D^m, S^{m-1}) & \xrightarrow{\delta} & H_{m-1}(S^{m-1}) \\ \Big\downarrow{\scriptstyle (f_m)_*} & & & & \Big\downarrow{\scriptstyle (f_{m-1})_*} \\ H_m(S^m) & \xrightarrow{\simeq} & H_m(D^m, S^{m-1}) & \xrightarrow{\delta} & H_{m-1}(S^{m-1}) \end{array}$$

where δ denotes the corresponding connecting homomorphisms of the long exact sequence which are in this case isomorphisms. From the commutativity of the diagram we get $\deg(f_m) = \deg(f_{m-1})$. □

This concludes the proof that S^4 has a nut–anti-nut pair of singularities with respect to the isometry generated by the Killing vector field ∂_y. This is in accordance with the requirement that on a compact manifold like S^4 the total nut charge must be zero. It is also worth noting how supersymmetry restricts the allowed types of singularities. Indeed, as already mentioned the existence of invariant spinors implies the action 4.10 on the ambient space and we have shown how this is related to the action of the isometry on the tangent spaces of two antipodal points. In particular, it follows that one always gets precisely one (1,1) and one (1,-1) singularity for any isometry that preserves supersymmetry, not just for the one we picked. We therefore have the following statement

Every U(1) reduction of M-theory on S^4 which preserves some supersymmetry must contain a D6-$\overline{\text{D6}}$ pair.

The appearance of a brane–anti-brane pair in a supersymmetric background might at first glance be surprising since we know that D-branes introduced in a supersymmetric background break half of the supersymmetries and anti-branes break exactly the other half (see section 18.5 of [13]). However, this reasoning only applies to flat space. In our case the supersymmetry parameters change from the south to the north pole so that both the D6 and the $\overline{\text{D6}}$ can be BPS [55].

4.3 The S^4 ansatz

We will now consider the compactification of eleven dimensional supergravity on S^4. We have already discussed the existence of a background of the form $\text{AdS}_7 \times S^4$ which is a solution to the eleven dimensional supergravity equations of motion. The next thing we want to do is perform a Kaluza-Klein reduction on this background. This means that we want to go back to the eleven dimensional supergravity and expand all the fields around this background $\mathrm{AdS}_7 \times S^4$. We illustrate how this works with the simplest example, a free scalar field in five dimensions with the action

$$S_0 = -\frac{1}{2} \int d^5x \, \partial_M \varphi \partial^M \varphi \tag{4.15}$$

Recall from Section 3.1 that the idea behind Kaluza-Klein theory is to assume that the five dimensional background can be split as a metric product $M_5 = M_4 \times S^1$ with M_4 being Minkowski space. If we use the coordinates x^μ on M_4 and the coordinate y on the circle then the equation of motion can be written as

$$\Box \varphi = 0 \Rightarrow \partial^\mu \partial_\mu \varphi + \partial_y^2 \varphi = 0$$

The field φ can then be expanded in Fourier modes as

$$\varphi(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{n=\infty} \varphi_n(x^\mu) e^{iny/R}$$

The choice of Fourier expansion is because it corresponds to an expansion in terms of eigenfunctions of the Laplacian on S^1 which are the familiar exponentials $e^{imy/R}$ (with a normalization factor) where R is the radius of the circle. Substituting the Fourier expansion in the equation of motion and using that the eigenfunctions of the Laplacian are independent we obatain:

$$\partial^\mu \partial_\mu \varphi_n - \frac{n^2}{R^2} \varphi_n = 0$$

This shows that the fields φ_n are massive modes from the point of view of the lower dimensional theory and their masses are inversely proportional to the radius of the internal circle. An obvious problem is that now we have a theory in four dimensions but the price we had to pay is the introduction of an infinitude of fields φ_n. The compactification ansatz corresponds to taking the limit $R \to 0$ in which case we can neglect all the massive fields φ_n and keep only zero mode φ_0. This is called a *truncation.*

In general we can apply the above procedure for an arbitrary theory specified by an action and an arbitrary compactification on a background $M \times N$ with metric $g = g^M \oplus g^N$ where N is the compact manifold on which we compactify. Since this is a metric product, the Laplacian splits as $\Delta_{M \times N} = \Delta_M + \Delta_N$ and we expand all the fields of the given theory in terms of the eigenvalues of Δ_N which we denote by Y_q and we have

$$\Delta_N Y_q(y) = -m_q^2 Y_q(y)$$

where y collectively denote the coordinates on N. Therefore, a scalar field in $M \times N$ will be expanded as $\varphi(x, y) = \sum_q \varphi_p(x) Y_q(y)$. We could repeat the above procedure step by step. It is true that the eqigenvalues of the Laplacian m_q will be inversely proportional to the volume of N so taking the limit $\mathrm{vol}(N) \to 0$ should be enough to justify why we need to keep only the zero mode $\varphi_0(x^\mu)$. However, this is not quite right. To see why, consider a φ^3 theory on $M \times N$. Then the expanded action will contain terms of the form

$$\int d^m \sqrt{\det g^M} \varphi_q(x) \varphi_0^2(x) \times \int d^n y \sqrt{\det g^N} Y_q(y) Y_0^2(y)$$

which will contribute to the equation of motion for the field ϕ_q with terms of the form

$$(\Box - m_q^2)\varphi_q(x) = (...)\varphi_0^2 + ...$$

What we observe now is that keeping only the zero mode but putting $\varphi_q = 0$ contradicts the equations of motion. When this does not happen and our truncated fields are in accordance with the equations of the higher dimensional theory, we say that we have a *consistent truncation*. In the above example, the truncation to the zeroth mode could be consistent if it happens that $\int d^n y \sqrt{\det g^N} Y_q(y) Y_0(y)^2 = 0$. This term only depends on the properties of our compactification manifold N. It turns out that compactifications on tori of arbitrary dimension are always consistent with a truncation to the zeroth mode. However, for general spaces, this is far from common. If we realize that the zero mode truncation is inconsistent then one thing we could try to circumvent the problem is to make a *non-linear ansatz* in which case the fields $\varphi(x, y)$ are expanded non-linearly in terms of the $Y_q(y)$.

In the context of string theory, consistency is an important requirement when performing a compactification since only then can the solutions of the lower dimensional theory also be solutions of the initial theory (M-theory in our case). It is important to realize however, that there can be many inequivalent consistent truncations. Recall for example our discussion in Section 3.1 regarding the Kaluza-Klein reduction from five dimensional gravity to four dimensional gravity with a U(1) gauge field A_μ coupled to a scalar φ. Although we did not explicitly mention it there, we saw two different consistent truncations. The one retained the full isometry group $\text{ISO}(S^1) = U(1)$ which appeared as a gauge field in the reduced action

$$S = -\frac{1}{16\pi G}\int d^4x \sqrt{-\det g(x^\mu)}\Big[R^{(4)} + \frac{1}{6\varphi^2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{4}\varphi F^{\mu\nu}F_{\mu\nu}\Big]$$

However, we also mentioned a further truncation in which the scalar φ is constant and $F_{\mu\nu} = 0$ in which case we obtain the action of pure gravity in four dimensions. This is also consistent with the equations of motion. The important lesson to get from this story is that when we compactify on a background $M \times N$ with N compact, consistency does not necessarily imply that we have to keep all the fields of the $\text{ISO}(N)$ gauge group but we can have gauge groups G that are subgroups of $\text{ISO}(N)$. In the Kaluza-Klein example above, the second truncation with $\varphi =$ constant retains the trivial subgroup of U(1) and this physically means that all the information about the higher dimensional theory is lost. Therefore, although the solutions of the lower dimensional theory satisfy the higher dimensional equations of motion, not all of the higher dimensional solutions can be realized in this way.

Our goal now is to reduce the eleven dimensional supergravity on the $\text{AdS}_7 \times S^4$ background which as we already discussed is a valid solution. Note that the S^4 has four Killing spinors η^I satisfying

$$D_\mu \eta^I = \frac{1}{2} i \gamma_\mu \eta^I$$

This means that after compactification from eleven to seven dimensions, we get a maximally supersymmetric $\mathcal{N} = 4$ theory. We want this reduction to be consistent. However, for general n-dimensional spheres S^n, consistent truncations containing all the gauge fields of the gauge group $\text{ISO}(S^n) = \text{SO}(n+1)$ cannot be obtained. Nevertheless, it has been shown that it is possible for reduction of eleven dimensional supergravity on S^7 and S^4. The result that concerns the S^4 was obtained in [57, 56] where it was proved that such a reduction is indeed possible and leads to maximal $\mathcal{N} = 4$ *gauged supergravity* in seven dimensions, a result which had been expected [58] but hard to prove. The term *gauged* is used to distinguish this compactification from the one of the same theory on a torus in which case the the lower dimensional theory is called *ungauged.* The complete ansatz for the bosonic part of the metric and the four-form is [59]

$$ds_{11}^2 = \Delta^{1/3} ds_7^2 + \frac{1}{g^2}\Delta^{-2/3} T_{ij}^{-1} DY^i DY^j \tag{4.16}$$

$$\begin{aligned} F_{(4)} =& \frac{1}{4!}\epsilon_{i_1\dots i_5}\Big[-\frac{1}{g^3} U\Delta^{-2} Y^{i_1} DY^{i_2}\wedge \dots \wedge DY^{i_5} + \frac{4}{g^3}\Delta^{-2} T^{i_1 m} DT^{i_2 n} Y^m Y^n DY^{i_3}\wedge\dots\wedge DY^{i_5} \\ &+ \frac{6}{g^2}\Delta^{-1} F_{(2)}^{i_1 i_2}\wedge DY^{i_3}\wedge DY^{i_4} T^{i_5 j} Y^j\Big] - T_{ij} * S_{(3)}^i Y^j + \frac{1}{g} S_{(3)}^i \wedge DY^i \end{aligned} \tag{4.17}$$

with

$$\begin{aligned} &U := 2T_{ij}T_{jk}Y^iY^k - \Delta T_{ii} && \Delta := T_{ij}Y^iY^j \\ &F_{(2)}^{ij} := dB_{(1)}^{ij} + gB_{(1)}^{ik}\wedge B_{(1)}^{kj} && DY^i := dY^i + gB_{(1)}^{ij}Y^j \\ &DT_{ij} := dT_{ij} + gB_{(1)}^{ik}T_{kj} + gB^{jk}T_{ik} && Y^iY^i = 1 \end{aligned} \tag{4.18}$$

In the above expressions, Y^i are the embedding coordinates of S^4. The symmetric matrix T_{ij} has unit determinant and parametrizes the scalar coset $\mathrm{SL}(6,\mathbb{R})/\mathrm{SO}(6)$. It can be expressed as $T_{ij} = \Pi_i^{-1A}\Pi_j^{-1B}\eta_{AB}$ with Π_i^{-1A} being a type of vielbein for the scalar coset manifold $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)$, A, B being gauge indices and the metric η_{AB} being the SO(5) metric which is simply δ_{AB}. For later convenience it also useful to introduce the object $T^{AB} := \Pi_i^{-1A}\Pi_i^{-1B}$. The $B_{(1)}^{ij}$ denote SO(5) gauge fields, the $S_{(3)}^i$ are three-forms, g is the gauge coupling and Δ plays the role of a warp factor.

The above ansatz is admittedly complicated. For this reason it is useful that we make ourselves familiar with a simpler version of it. For the sake of simplicity we want to work with a reduction ansatz in which the SO(5) fields are further truncated. It is known that there exists a consistent truncation of the $\mathcal{N} = 4$ theory to $\mathcal{N} = 2$ seven dimensional supergravity which consists of the metric, a 2-form potential, three vectors and a dilaton, coupled to a vector multiplet which consists of a vector and three scalars. This considerably simplifies the setup but we can do even better. A further truncation was introduced in [60, 61] in which only two U(1) gauge fields and two scalars survive. The reduction anstatz for the eleven dimensional metric and the four-form field strength of this model is given by

$$ds_{11}^2 = \tilde\Delta^{1/3} ds_7^2 + g^{-2}\tilde\Delta^{-2/3}\Big(X_0^{-1} d\mu_0^2 + \sum_{i=1}^{2} X_i^{-1}(d\mu_i^2 + \mu_i^2(d\psi_i + gA_{(1)}^i)^2)\Big) \tag{4.19}$$

$$\begin{aligned} *F_{(4)} =& 2g\sum_{\alpha=0}^{2}\Big(X_\alpha^2\mu_\alpha^2 - \tilde\Delta X_\alpha\Big)\epsilon_{(7)} + g\tilde\Delta X_0\ \epsilon_{(7)} + \frac{1}{2g}\sum_{\alpha=0}^{2} X_\alpha^{-1}\bar{*} dX_\alpha\wedge d(\mu_a^2) \\ &+ \frac{1}{2g^2}\sum_{i=1}^{2} X_i^{-2} d(\mu_i^2)\wedge(d\psi_i + gA_{(1)}^i)\wedge \bar{*} F_{(2)}^i \end{aligned} \tag{4.20}$$

In this ansatz, μ_i are coordinates satisfying $\mu_0^2+\mu_1^2+\mu_2^2 = 1$ and together with the ψ_1, ψ_2 they constitute the set of coordinates on S^4. The parameters X_1, X_2 are the scalar fields that parametrize the metric with $X_0 = (X_1X_2)^{-2}$. Together with the two gauge fields $A_{(1)}^i$ they constitute the allowed deformations of the background (round) metric on S^4. The gauge fields $A_{(1)}^i$ form a basis of the Cartan subalgebra $\mathfrak{u}(1)\times\mathfrak{u}(1)$ of $\mathfrak{so}(5)$. There is also a warping factor $\tilde\Delta = \sum_{\alpha=0}^{2} X_\alpha\mu_\alpha^2$. We can then notice that if we set $X_i = 1$ and $A^i_{(1)} = 0$ then $\tilde{\Delta} = 1$ and the metric becomes

$$ds^2_{11} = ds^2_7 + g^{-2}\Big[d\mu_0^2 + \sum_{i=1}^{2}(d\mu_i^2 + \mu_i^2 d\psi_i^2)\Big]$$

The expression in squared brackets is a round metric on the unit S^4. Therefore, this is our background metric around which we expand. In this case, we can also identify the gauge coupling parameter g with the inverse of the radius of the S^4. Concluding the explanation of the various terms in 4.20 we note that $*$ is the Hodge star in eleven dimensions, $\bar{*}$ the Hodge star in S^4, $\epsilon_{(7)}$ is the volume form in seven dimensions and $F^i_{(2)} = dA^i_{(1)}$ is the field strength associated to the gauge fields $A^i_{(1)}$. This ansatz can be obtained from 4.16 4.17 by the following relations between the old and new variables:

$$\begin{aligned} &Y^5 = \mu_0 \\ &Y^1 = \mu_1 \sin\psi_1 \qquad Y^2 = \mu_1 \cos\psi_1 \\ &Y^3 = \mu_2 \sin\psi_2 \qquad Y^4 = \mu_2 \cos\psi_2 \end{aligned} \tag{4.21}$$

The condition $Y^iY^i = 1$ translates to $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$ as required and the remaining relations between the scalars and gauge fields are

$$T_{AB} = \mathrm{diag}(X_1, X_1, X_2, X_2, X_0) \tag{4.22}$$

$$A^1_{(1)} = -B^{12}_{(1)} - B^{34}_{(1)} \qquad A^1_{(2)} = B^{12}_{(1)} - B^{34}_{(1)} \tag{4.23}$$

It is convenient to parametrize the scalars X_i by two dilaton fields that can be written as a vector $\vec{\varphi} = (\varphi_1, \varphi_2)$ in the following way

$$X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}} \quad , \quad \vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - \frac{8}{5} \tag{4.24}$$

where the normalization of the $\vec{a_i}$ can be fixed by choosing $\vec{a}_1 = (\sqrt{2}, \sqrt{\frac{2}{5}})$ and $\vec{a}_2 = (-\sqrt{2}, \sqrt{\frac{2}{5}})$. In order to proceed, one should now substitute 4.19 and 4.20 in the eleven dimensional supergravity equations of motion and then try to construct the effective Lagrangian in seven dimensions which reproduces those equations. This is explicitly done in [60] where the Lagrangian is found to be

$$\frac{\mathcal{L}}{\sqrt{-\det g}} = R - \frac{1}{2}(\partial\vec{\varphi})^2 - V - \frac{1}{4}\sum_{i=1}^{2} e^{\vec{a}_i\cdot\vec{\varphi}}(F^i_{(2)})^2 \tag{4.25}$$

with the scalar potential V given by

$$V = g^2\Big(- 4X_1X_2 - 2X_1^{-1}X_2^{-2} - 2X_2^{-1}X_1^{-2} + \frac{1}{2}(X_1X_2)^{-4}\Big) \tag{4.26}$$

It is straightforward to check that the potential V has two stationary points one of which is a maximum at $X_1 = X_2 = 1$ and the other one being a saddle point at $X_1 = X_2 = 2^{-1/5}$.

4.4 The torus action

Having explained the physical origin of the S^4 ansatz in the supergravity context, we now set to put it in under the microscope of torus actions. A quick inspection of the truncated background 4.19 4.20 reveals that there is an obvious isometric torus action corresponding to translation of the ignorable variables ψ_1, ψ_2. The corresponding Killing fields are $K_i = \partial_{\psi_i}$. The four-form is also independent of the two coordinates so that

$$\mathcal{L}_{K_1} F_4 = \mathcal{L}_{K_2} F_4 = 0$$

Since the tori are parametrized by ψ_1, ψ_2 the coordinates μ_0, μ_1, μ_2 are the coordinates on the base space. As we saw, those are coordinates in $\mathbb{R}^3$ satisfying $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. From their definition 4.21 it turns out that since $Y_i \in [-1, 1]$ we must have $\mu_0 \in [-1, 1]$ and $\mu_1, \mu_2 \in [0, 1]$ so that in total they do not parametrize a sphere but rather a quarter of a sphere. This is consistent with the conclusions of Section 2.4. To further explore the orbit space structure, we consider the fixed points of the action. The norm of the Killing fields is given by

$$|K_i|^2 = g^{-2} \tilde{\Delta}^{-2/3} X_i^{-1} \mu_i^2 \quad , i = 1, 2$$

which vanishes precisely when $\mu_i = 0$. In the orbit space, these are indeed the two edges forming the boundary. Their intersection comprises the fixed points of the entire torus action and they are given by $\mu_1 = \mu_2 = 0$ or $\mu_0 = \pm 1$. The Killing fields K_1, K_2 are therefore the infinitesimal generators of the only two embedded cycles in T^2 whose restricted action on S^4 has a bolt as fixed point locus. To verify this, we consider an arbitrary linear combination $K' = \kappa K_1 + \lambda K_2$ with norm

$$|K'|^2 = \kappa^2 g_{\psi_1 \psi_1} + 2\kappa\lambda g_{\psi_1 \psi_2} + \lambda^2 g_{\psi_2 \psi_2} = g^{-2} \tilde{\Delta}^{-2/3} (\kappa^2 X_1^{-1} \mu_1^2 + \lambda^2 X_2^{-1} \mu_2^2)$$

Indeed, the vanishing locus is given by $\mu_1 = \mu_2 = 0$ for $\kappa\lambda \neq 0$ showing that a generic circle in T^2 has two fixed points P, Q. The circle actions induced by the Killing fields K_1 and K_2 have weights equal to ± 1. This is easily deduced from the definitions 4.21. The translation $\psi_1 \mapsto \psi + \beta$ only rotates the plane $Y^1 - Y^2$ with the rotation matrix being

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \tag{4.27}$$

Similarly, the rotation $\psi_2 \mapsto \psi_2 + \beta$ rotates the $Y^3 - Y^4$ plane in the same way. The structure of the orbit space is shown in Figure 4.2.

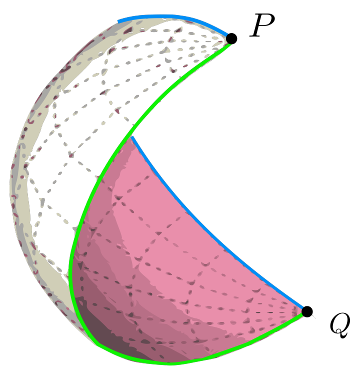


Figure 4.2: The quarter sphere as the orbit space of the torus action on S^4. The blue edge denotes the locus $\mu_1 = 0$ and the green edge denotes the locus $\mu_2 = 0$. They are loci of isotropy $(0, 1)$ and $(1, 0)$ respectively. The points P, Q are the fixed points of the torus given by $\mu_0 = \pm 1$.

Performing the reduction along ψ_1 of the metric 4.19 we obtain the ten dimensional background

$$
\begin{aligned}
&e^{4\varphi/3} = g^{-2}\tilde{\Delta}^{-2/3}X_1^{-1}\mu_1^2 \\
&ds_{10}^2 = e^{2\varphi/3}\Big[\tilde{\Delta}^{1/3}ds_7^2 + g^{-2}\tilde{\Delta}^{-2/3}\Big(X_0^{-1}d\mu_0^2 + X_1^{-1}d\mu_1^2 + X_2^{-1}d\mu_2^2 + X_2^{-1}\mu_2^2(d\psi_2 + gA^{(2)})^2\Big)\Big] \\
&A_i = gA_i^{(1)} \text{ (RR 1-form)}
\end{aligned}
\tag{4.28}
$$

In order to see what the three dimensional space is, we can set the moduli fields to zero. The three dimensional part of the metric then becomes (up to a conformal factor):

$$
ds_3^2 = d\mu_0^2 + d\mu_1^2 + d\mu_2^2 + \mu_2^2 d\psi_2^2
$$

The first three terms are simply the round metric on the quarter sphere as in Figure 4.2. The last term is a warped metric on a circle with the size of the circle vanishing at $\mu_2 = 0$ (green line in Figure 4.2). This is indeed a metric on $\mathbb{D}^3$ which according to our discussion in Section 2.5 is the space that we should obtain when the fixed point locus is a bolt.

Note that the RR 1-form in 4.28 does not depend on the coordinates of the three dimensional reduced space. Therefore, the pullback of the field strength on the three dimensional space $S^4/S^1 \simeq \mathbb{D}^3$ is zero. This is consistent with the fact that our spherical bolt (which is now the boundary of $\mathbb{D}^3$) does not carry any topological charge as it was defined in Section 2.5. Note that this agrees with the interpretation that there are no D6-branes or anti-branes in this reduced spacetime to act as sources for the RR-form.

Chapter 5

Conclusion

We have developed a range of techniques to study eleven dimensional M-theory backgrounds with isometric torus actions. The interest in this direction of research was ignited, as we saw in Chapter 1, by the existence of the SL(2,$\mathbb{Z}$) duality in Type IIA theory which ensures that reductions on different circles $S^1 \hookrightarrow T^2$ give rise to equivalent backgrounds. This immediately brought up the question of whether this duality has some non-trivial implications on the dynamics of the various theories that it relates with each other.

The curious duality led us to investigate how groups can act on manifolds with special focus on circles and tori acting on four-manifolds. The prominent position of four-manifolds in this discussion was dictated by the interpretation of codimension four fixed point surfaces as D6-branes and D6-anti-branes in the reduced ten dimensional theory (where they become codimension three objects). Therefore, it was natural from this point of view to adopt the assumption that our eleven dimensional background is a product of a seven dimensional manifold which we think as the brane worldvolume and a four-manifold which we interpret as the space transverse to the branes. In this intuitive picture, isolated fixed points of the four manifold correspond to branes and anti-branes.

The structure of torus actions on four-manifolds revealed that the quotient space can look very different depending on the circle subgroup of the torus that we choose to quotient out. In physics terms, this translates to the existence of various Type IIA backgrounds which are dual despite their apparent disparities. In particular, we were interested in torus actions with two fixed points since our initial endeavor revolved around the investigation of brane–anti-brane systems. We found that in this situation a generic circle subgroup of the torus has two isolated fixed points corresponding to the brane and the anti-brane. However, we also discovered that there are two special circles whose fixed point locus consists of a codimension two surface, a bolt. The slice theorem also allowed us to conclude that the quotient space under those special circle actions will have a boundary.

Armed with a solid understanding of the geometry behind torus actions, we moved on to investigate certain backgrounds in String Theory. After a brief detour to the world of Kauza-Klein theory, we demonstrated a physical argument that identifies fixed points with D6-branes from the perspective of supergravity. We then proceeded to the Kerr instanton, a space which combined the simplicity of a purely geometric background with the intricacy of the brane–anti-brane dynamics. We found out that the D6-$\overline{\text{D6}}$ pair can balance in the presence of a finely tuned magnetic field. By inspecting different reduction circles, we discovered conical singularities and of course the sought after special boundary inducing circle actions. The bolts of those two actions were found to be of two different types. One of them is a sphere and the other is the disjoint union of two planes. We discussed some of the implications for the lower dimensional theory.

While discussing the dynamics of the D6-$\overline{\text{D6}}$ pair, we introduced the notion of a fluxbrane which provided the necessary force for the brane–anti-brane to balance. The existence of the fluxbrane provided further evidence for the non-perturbative nature of the Type IIA background. This is due to the uncontrollably large values of the string coupling away from the fluxbrane axis. This, in turn, led to a criterion 3.24 ensuring that there exists a region where the perturbative picture is reliable. Applying this criterion picks out a preferred reduction circle. We found that in the limit of coincident brane–anti-brane the reliable background is indeed the one with a spherical boundary and no branes. This seems to suggest that when the branes get close to each other, there is no notion of branes and anti-branes.

Finally, we turned our attention to backgrounds with fluxes and in particular, the $\text{AdS}_7 \times S^4$ solution of eleven dimensional supergravity. Our interest in this example is also supported by the existence of a full non-linear ansatz for the S^4 compactification of eleven dimensional supergravity. We demonstrated that by performing a certain consistent truncation, we obtain an infinite family of eleven dimensional solutions which admit an isometric torus action. The four-sphere had also been a prime example of a manifold with a torus action from a purely mathematical perspective. Using the slice theorem and the isotropy representation in Chapter 2 we had already deduced that the quotient of S^4 should give rise to either S^3 or $\mathbb{D}^3$ which possesses the S^2 boundary. We confirmed that these findings are reproduced by our supergravity solutions and we discussed the geometry of the reduced space.

Our overall exposition has revealed an intriguing prospect regarding the end state of a D6-$\overline{\text{D6}}$ pair. However, the ultimate fate of those spacetimes with boundaries remains elusive. What is more, although we have established that spacetimes with boundaries do exist and have to be taken into account, the mechanism that induces the topological transition is unclear. We have argued in Section 3.5 that when the D6-brane and anti-brane come together, the perturbative picture is highly non-reliable and non-perturbative effects are believed to take control. It would be highly desirable to obtain a better understanding of the processes that take place in this non-perturbative regime and to possibly shed light on the issue of the topology change.

Appendices

Appendix A

Principal bundles and connections

Definition A.0.1. Let G be a Lie group and let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \Big\downarrow \pi \\ & & M \end{array}$$

be a fiber bundle with fiber G and a smooth action $G \times P \to P$. Then P is a ***principal G-bundle*** if

1. The action of G preserves the fibers of π and is simply transitive on them, i.e. the action restricts to

   $$G \times P_x \to P_x$$

   and the orbit map

   $$\begin{aligned} G &\to P_x \\ g &\mapsto g \cdot p \end{aligned}$$

   is a bijection for all $x \in M$, $p \in P_x$.

2. There exists a bundle atlas of G-equivariant bundle charts $\phi_i : P_{U_i} \to G \times U_i$ satisfying

   $$\phi_i(g \cdot p) = g \cdot \phi_i(p) \quad \forall p \in P_{U_i},\ g \in G$$

   where on the right hand side G acts on $(a, x) \in G \times U_i$ via

   $$g \cdot (a, x) \mapsto (g \cdot a, x)$$

The group G is called the ***structure group*** of the principal bundle P.

We first recall that if $\Phi : G \times E \to E$ is the free group action then $\phi_g := \Phi(g, -) : E \to E$ is a fiber preserving diffeomorphism and $D_u\phi_g : T_uE \to T_{g \cdot u}E$ is an isomorphism that preserves the vertical subspace $\mathcal{V}_u$. If we then choose a horizontal space $\mathcal{H}_u$ we can ask whether $(D_u\phi_g)\mathcal{H}_u$ is also a horizontal space at $g \cdot u$. There is no canonical choice of horizontal subspaces unless we specify some additional structure (like a metric). The concept of a connection is precisely the additional structure required to make this non-canonical choice.

Definition A.0.2. Let $E \to B$ be a principal G-bundle. A ***connection*** (or Ehresmann connection) on E is a distribution $\mathcal{H}$ of horizontal subspaces satisfying the following property

$$(D_u\phi_g)\mathcal{H}_u = H_{g \cdot u} \tag{A.1}$$

The definining condition A.1 is called left-invariance (or right-invariance if we choose a right action). Sometimes it is useful to consider connections that are not invariant. In this context, invariant connections are also called *principal connections*. This definition, albeit intuitive, is not tailored for practical applications. For this reason, we introduce the following related concept.

Definition A.0.3. Let $E \to B$ be a principal G-bundle. A ***connection one-form*** $\omega \in \Omega^1(P, \mathfrak{g}) := \mathfrak{g} \otimes T^*P$ is a projection of T_uP onto the vertical component $\mathcal{V}_uP \simeq \mathfrak{g}$. The projection property is summarized by the following requirements

- $\omega(X^\#) = X$ for all $X \in \mathfrak{g}$
- $\phi_g^*\omega = \mathrm{Ad}_g\omega$ for all $g \in G$.

A connection one-form is also called a *gauge field*. The choice of a connection form is equivalent to the choice of an Ehresmann connection as shown in the following theorem.

Theorem A.0.1. There is a bijective correspondence between Ehresmann connections on a principal G-bundle $E \to B$ and connection 1-forms given as follows:

- Let $\mathcal{H}$ be an Ehresmann connection on P. Then every element $Y \in T_uE$ splits as $Y = X^\# + Y^H$ with $Y^H \in \mathcal{H}_u$ a horizontal vector and $X^\#$ a vertical vector which can be uniquely written as the fundamental vector of some $X \in \mathfrak{g}$. We then define a connection one-form ω such that $\omega_u(Y) = X \in \mathfrak{g}$.
- Conversely, let $\omega \in \Omega^1(E, \mathfrak{g})$ be a conenction one-form. Then $\mathcal{H}_u := \ker \omega$ defines an Ehresmann connection.

A proof can be found in any introductory textbook on gauge theory like [62] to which we refer for further reading. Locally the connection one-form can be given as follows. Let $\{U_i\}$ be an open cover of the base B and $s_i : U_i \to E$ be local sections that trivialize the bundle. Then $\mathcal{A}_i := s_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$ are called *local gauge potentials*. On an overlap $U_i \cap U_j$ if we let $g_{ij} : U_{ij} \to G$ be the transition function then the local gauge fields transform as

$$\mathcal{A}_j = \mathrm{Ad}_{g_{ij}}\mathcal{A}_i + g_{ij}^{-1}\, dg_{ij}$$

Conversely, a collection of such local gauge fields transforming in this way define a unique global connection one-form. This local way of expressing connections is used in physics and we will explicitly use it in all of our applications.

Bibliography

[1] Claud Lovelace. “Pomeron form factors and dual Regge cuts”. In: *Physics Letters B* 34.6 (1971), pp. 500–506.

[2] J-L Gervais and Bunji Sakita. “Field theory interpretation of supergauges in dual models”. In: *Nuclear Physics B* 34.2 (1971), pp. 632–639.

[3] John H Schwarz. “Physical states and pomeron poles in the dual pion model”. In: *Nuclear Physics B* 46.1 (1972), pp. 61–74.

[4] Edward Witten. “String theory dynamics in various dimensions”. In: (1995).

[5] Paul K Townsend. “The eleven-dimensional supermembrane revisited”. In: *Physics Letters B* 350.2 (1995), pp. 184–188.

[6] David Tong. *String theory*. Tech. rep. 2009.

[7] Joseph Polchinski. “Dirichlet branes and Ramond-Ramond charges”. In: *Physical Review Letters* 75.26 (1995), p. 4724.

[8] E Bergshoeff et al. “The IIA super-eightbrane”. In: *arXiv preprint hep-th/9511079* (1995).

[9] Thomas Henry Buscher. “A symmetry of the string background field equations”. In: *Physics Letters B* 194.1 (1987), pp. 59–62.

[10] Thomas Henry Buscher. “Path-integral derivation of quantum duality in nonlinear sigma-models”. In: *Physics Letters B* 201.4 (1988), pp. 466–472.

[11] Ashoke Sen. “Strong–weak coupling duality in four-dimensional string theory”. In: *International Journal of Modern Physics A* 9.21 (1994), pp. 3707–3750.

[12] CM Hull. “String-string duality in ten dimensions”. In: *arXiv preprint hep-th/9506194* (1995).

[13] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. *Basic concepts of string theory*. Springer Science & Business Media, 2012.

[14] John H Schwarz. “An SL (2, Z) multiplet of type IIB superstrings”. In: *Physics Letters B* 360.1-2 (1995), pp. 13–18.

[15] Paul S Aspinwall. “Some relationships between dualities in string theory”. In: *Nuclear Physics B-Proceedings Supplements* 46.1-3 (1996), pp. 30–38.

[16] Eric Bergshoeff, Christopher M Hull, and Tomas Ortin. “Duality in the Type–II Superstring Effective Action”. In: *arXiv preprint hep-th/9504081* (1995).

[17] Katrin Becker, Melanie Becker, and John H Schwarz. *String theory and M-theory: A modern introduction*. Cambridge University Press, 2006.

[18] Ferdinando Gliozzi, J Scherk, D Olive, et al. “Supersymmetry, supergravity theories and the dual spinor model”. In: (1977).

[19] Alexander Kirillov Jr. *An introduction to Lie groups and Lie algebras*. Vol. 113. Cambridge University Press, 2008.

[20] Shoshichi Kobayashi. “Fixed points of isometries”. In: *Nagoya Mathematical Journal* 13 (1958), pp. 63–68.

[21] Shoshichi Kobayashi. *Transformation groups in differential geometry.* Springer Science & Business Media, 2012.

[22] Joseph J Rotman. *An introduction to algebraic topology.* Vol. 119. Springer Science & Business Media, 2013.

[23] Glen E Bredon. *Introduction to compact transformation groups.* Academic press, 1972.

[24] Peter Orlik and Frank Raymond. “Actions of the torus on 4-manifolds. I”. In: *Transactions of the American Mathematical Society* 152.2 (1970), pp. 531–559.

[25] Ronald Fintushel. “Classification of circle actions on 4-manifolds”. In: *Transactions of the American Mathematical Society* 242 (1978), pp. 377–390.

[26] Ronald Fintushel. “Circle actions on simply connected 4-manifolds”. In: *Transactions of the American Mathematical Society* 230 (1977), pp. 147–171.

[27] Gary W Gibbons and Christopher N Pope. “$\mathbb{CP}^2$ as a gravitational instanton”. In: *Communications in Mathematical Physics* 61.3 (1978), pp. 239–248.

[28] Peter Sie Pao. “The topological structure of 4-manifolds with effective torus actions. I”. In: *Transactions of the American Mathematical Society* 227 (1977), pp. 279–317.

[29] F Giani and M Pernici. “N= 2 supergravity in ten dimensions”. In: *Physical Review D* 30.2 (1984), p. 325.

[30] Jose Figueroa-O’Farrill and Joan Simon. “Supersymmetric Kaluza-Klein reductions of M2 and M5-branes”. In: *arXiv preprint hep-th/0208107* (2002).

[31] JF Sparks. “Kaluza-klein branes”. In: *arXiv preprint hep-th/0105209* (2001).

[32] Gary W Gibbons and Stephen W Hawking. “Classification of gravitational instanton symmetries”. In: *Communications in Mathematical Physics* 66.3 (1979), pp. 291–310.

[33] GW Gibbons and Malcolm J Perry. “New gravitational instantons and their interactions”. In: *Physical Review D* 22.2 (1980), p. 313.

[34] David J Gross and Malcolm J Perry. “Magnetic monopoles in Kaluza-Klein theories”. In: *Nuclear Physics B* 226.1 (1983), pp. 29–48.

[35] WB Bonnor. “An exact solution of the Einstein-Maxwell equations referring to a magnetic dipole”. In: *Zeitschrift für Physik* 190.4 (1966), pp. 444–445.

[36] Joseph Polchinski. “TASI lectures on D-branes”. In: *arXiv preprint hep-th/9611050* (1996).

[37] Gary T Horowitz and Andrew Strominger. “Black strings and P-branes”. In: *Nuclear Physics B* 360.1 (1991), pp. 197–209.

[38] Ashoke Sen. “Strong coupling dynamics of branes from M-theory”. In: *Journal of High Energy Physics* 1997.10 (1997), p. 002.

[39] Hongsu Kim. “Supergravity approach to tachyon potential in brane-antibrane systems”. In: *Journal of High Energy Physics* 2003.01 (2003), p. 080.

[40] Fay Dowker et al. “Decay of magnetic fields in Kaluza-Klein theory”. In: *Physical Review D* 52.12 (1995), p. 6929.

[41] Fay Dowker et al. “Nucleation of p-branes and fundamental strings”. In: *Physical Review D* 53.12 (1996), p. 7115.

[42] Michael Gutperle and Andrew Strominger. “Fluxbranes in string theory”. In: *Journal of High Energy Physics* 2001.06 (2001), p. 035.

[43] Miguel S Costa and Michael Gutperle. "The Kaluza Klein Melvin solution in M-theory". In: *Journal of High Energy Physics* 2001.03 (2001), p. 027.

[44] Jose Figueroa-O'Farrill and Joan Simon. "Generalised supersymmetric fluxbranes". In: *Journal of High Energy Physics* 2001.12 (2002), p. 011.

[45] Mael A Melvin. "Pure magnetic and electric geons". In: *Phys. Letters* 8 (1964).

[46] Roberto Emparan and Michael Gutperle. "From p-branes to fluxbranes and back". In: *Journal of High Energy Physics* 2001.12 (2002), p. 023.

[47] Roberto Emparan. "Black diholes". In: *Physical Review D* 61.10 (2000), p. 104009.

[48] David Garfinkle. "General relativistic strings". In: *Physical Review D* 32.6 (1985), p. 1323.

[49] Raúl Rabadán and Joan Simón. "M-theory lift of brane-antibrane systems and localised closed string tachyons". In: *Journal of High Energy Physics* 2002.05 (2002), p. 045.

[50] Bihn Zhou and Chuan-Jie Zhu. "The complete brane solution in D-dimensional coupled gravity system". In: *arXiv preprint hep-th/9904157* (1999).

[51] Philippe Brax, Gautam Mandal, and Yaron Oz. "Supergravity description of non-Bogomol'nyi-Prasad-Sommerfield branes". In: *Physical Review D* 63.6 (2001), p. 064008.

[52] Allan Adams, Joseph Polchinski, and Eva Silverstein. "Don't panic! Closed string tachyons in ALE spacetimes". In: *Journal of High Energy Physics* 2001.10 (2001), p. 029.

[53] Frederik Denef. "Les Houches lectures on constructing string vacua". In: *arXiv preprint arXiv:0803.1194* (2008).

[54] Elias Kiritsis. *String theory in a nutshell.* Vol. 21. Princeton University Press, 2019.

[55] Fabio Apruzzi et al. "All AdS 7 solutions of type II supergravity". In: *Journal of High Energy Physics* 2014.4 (2014), p. 64.

[56] Horatiu Nastase, Diana Vaman, and Peter van Nieuwenhuizen. "Consistent nonlinear KK reduction of 11d supergravity on AdS7× S4 and self-duality in odd dimensions". In: *Physics Letters B* 469.1-4 (1999), pp. 96–102.

[57] Horatiu Nastase, Diana Vaman, and Peter van Nieuwenhuizen. "Consistency of the AdS7× S4 reduction and the origin of self-duality in odd dimensions". In: *Nuclear Physics B* 581.1-2 (2000), pp. 179–239.

[58] K Pilch, P Van Nieuwenhuizen, and PK Townsend. "Compactification of d= 11 supergravity on S4 (or 11= 7+ 4, too)". In: *Nuclear Physics B* 242.2 (1984), pp. 377–392.

[59] M Cvetič et al. "S3 and S4 reductions of type IIA supergravity". In: *Nuclear Physics B* 590.1-2 (2000), pp. 233–251.

[60] Mirjam Cvetic et al. "Embedding AdS black holes in ten and eleven dimensions". In: *arXiv preprint hep-th/9903214* (1999).

[61] M Cvetič et al. "Domain-wall supergravities from sphere reduction". In: *Nuclear Physics B* 560.1-3 (1999), pp. 230–256.

[62] Mark JD Hamilton. *Mathematical gauge theory.* Springer, 2017.