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Master thesis

Theta problem and asymptotic Hodge theory

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In girum imus nocte et consumimur igni

Abstract

In this thesis we study the recently proposed conjecture by Cumrum Vafa and Sergio Cecotti, about a possible solution of the theta problem in QCD as a result of Quantum Gravitational consistency of the theory in the UV [13] . The authors claim that one might be able to argue that the value of the theta angle can be fixed for theories corresponding to the Landscape of String theory. They support their claim by testing the theta angle corresponding to the graviphoton of $\mathcal{N} = 2$ Supergravity obtained by type IIB compactification on rigid Calabi-Yau manifolds. They find that indeed for most cases $\theta = 0$ (only around 50 such manifolds are known). We test the conjecture in a more general context. More precisely, we investigate the behaviour of the corresponding theta angles for the 4D low energy theory of type IIB strings compactified on Calabi-Yau threefolds with arbitrary hodge numbers near the boundaries of the complex structure moduli space. For this task, we employ several tools from degenerating variations of Hodge structures. Based on the data coming from the $Sl(2)$ orbit theorem we manage to find an electric-magnetic basis for the real threeforms which gives a vanishing theta angle in the strong coupling limit for every type of one modulus denegeration and enhancement between these. Moreover we investigate the weak coupling regime where we reproduce the known behaviour for the large volume-large complex structure point but also find similar expressions for any type of singular loci. This universal behaviour in each regime provides evidence for the restriction of the theta angle values due to Quantum Gravitational consistency for the first time since the original proposal.

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Chapter 1

Introduction

String Theory is a theory of Quantum Gravity in the sense that it incorporates Einstein Gravity as its low energy limit but also predicts quantum corrections to it. Moreover, it is also a theory which manages to unify gauge theories with gravity, since its spectrum includes apart from the graviton, a number of gauge fields, whose gauge groups can be chosen (almost) freely through the inclusion of extended objects such as D-branes [37]. Another attribute it possesses is the fact that self-consistency plays a huge role determining several parameters of the theory without the need of 'fine tuning', such as the dimension of the spacetime which is fixed to 10. However, due to the apparent gap between this number of dimensions and our real world, there is a huge freedom of choice for the remaining dimensions, which results to a huge number of low energy four dimensional theories since these correspond to the number of vacua one can construct. For many years people believed that in order to connect String Theory with the experimentally tested characteristics of our world, such as the Standard Model, one should find an extremely special manifold to compactify on and all the problems would be solved. Nevertheless, the discovery of dualities between different String Theories [21], made the search of a special compactification seem hopeless. The new way of thinking essentially made people believe that in the context of String Theory, any kind of four dimensional theory can be deduced provided that one makes an adequate choice of parameters and configurations. This idea leads oneto believe that, for example the cosmological constant problem could be solved using Weinberg's anthropic principle [43]. Weinberg supports that out of the plethora of possible vacua of a quantum gravity theory, the only one that can incorporate the existence of humans, is the one with the value of cosmological constant we live in. In other words, any value of the cosmological constant is fine in terms of consistency, but it happens that we live in a universe with this extremely small value. This idea evidently makes String Theory seem 'useless', since it looks like any effective field theory can be produced by some kind of wild compactification and therefore effective theories seem 'equivalent' to the full String Theory. Thankfully this turns out not to be the case. In particular, it is by now known that there are four dimensional effective field theories, which seem self-consistent (no anomalies for instance) , but which cannot be completed into String Theory in the UV. It is natural to actually investigate which theories cannot be completed into any Quantum Gravity theory in the UV, not just String Theory which as mentioned earlier is 'a' theory of Quantum Gravity. Then we give the following definition

The Swampland can be defined as the set of (apparently) consistent effective field theories that cannot be completed into quantum gravity in the ultraviolet [34].

In contrast, the theories which can be completed into QG in the UV correspond to the Landscape of QG (or String Theory). Evidently, this rather abstract concept gains a meaning only once one can come up with criteria to distinguish between the two sets. This is one of the tasks of the Swampland program, namely to understand what general properties QG has and how these reflect to their low energy effective theories. The tool that is mostly used, is String Theory but also more general arguments such as some coming from black hole physics are employed. Then the task is to be able to understand the properties of String Theory well enough, to be able to deduce their effect on the possible low energy theories, in order to distinguish between the Swampland and the Landscape. Recent work towards this aim has been made with the formulation of a number of conjectures [34] which are based on general properties observed.

The problem we put at test in this thesis is the Strong CP or theta angle problem. The issue arises when one notices that there is CP violating term in the Lagrangian of QCD, which however through experimental data is restricted to be $\theta_{exp} < 10^{-19}$ without any apparent reason. In other words the question is why CP symmetry in QCD is not badly violated? We gain our motivation to study the problem in the context of the Swampland from a recently proposed conjecture by Cumrum Vafa and Sergio Cecotti [13]. The authors propose that a possible solution to the problem could come by considering which values of θ are consistent with QG constraints. In this work we investigate type IIB compactifications on CY_3 near the boundaries of the moduli space using the theory of mixed Hodge structures. The resulting 4D theories are $\mathcal{N} = 2$ Supergravity theories where the vectors from the vector multiplets and the graviphoton from the gravity multiplet couple through matrices which depend on the complex structure moduli of CY_3. The gauge group is $U(1)^{h^{2,1}+1}$. The hypermultiplets coupling depend on the Kahler structure moduli space which will not interest us. The theta angles in this model are the corresponding topological couplings of the field strengths which correspond to the gauge group $U(1)^{h^{2,1}+1}$. The task is to find out whether these theta angles have a restricted behaviour when considered as low energy limits of String theory, or put differently, if QG consistency dictates a special behaviour.

1.1 Outline

We begin by introducing in chapter 2 in some detail the compactification procedure of type II strings and their resulting 4D theories. We then proceed with introducing how the theta problem arises in the QCD context [35]. Next, we give some details about the strategies developed through the Swampland program[34] regarding the restrictions put on EFTs as a result of String theory compactifications. Then, we discuss briefly the proposed conjecture and the toy model in which the authors test their claims [13]. After that, the necessary, for our purposes, mathematics of variation of Hodge structures is developed in 4. In 5 we compute the theta angles matrices for every type of one modulus and two moduli limits by suitable choice of basis vectors and demonstrate their behaviour in the strong and weak coupling regime. Finally we give an outlook of the work and point to new directions for the future.

Chapter 2

String Theory compactification

2.1 Introduction

We know that in order to describe String theory with fermions and bosons in a consistent way the strings should propagate in a ten dimensional spacetime [1]. This inevitably creates a gap between the theory and our current understanding of our universe, since we believe that the latter is four dimensional. However the idea of having a more fundamental and higher dimensional theory is not new. Kaluza and Klein, trying to incorporate in a single theory electromagnetism and General Relativity, developed a five dimensional model which was supposed to reproduce what was already known from GR and EM in four dimensions. Their model was not successful but the basic concepts in the procedure is still used. The general way of thinking is that the extra dimensions can be considered as some compact space with small enough size that is not detectable yet. In the next section we will see how this comes about for a higher dimensional field theory.

2.2 Kaluza Klein theory

2.2.1 Dimensional reduction on the circle

Let us start by a $4+1$ dimensional field theory of gravity $\hat{g}_{MN}$ with some additional scalars $\hat{\Phi}$ where the total space is $M_4 \times S^1$ and has coordinates $\hat{x}^M$. By x^μ, we denote the Minkowski coordinates and by y the coordindate on the circle which has radius R. Now assume that the five dimensional metric can be written as

$$\hat{g}_{MN} \rightarrow \begin{pmatrix} g_{\mu\nu} & V_\mu \\ V_\mu & \phi \end{pmatrix} \tag{2.1}$$

where V_μ is some massless vector and ϕ some scalar field. Now, the equations of motion for the $4+1$ dimensional scalar field can be written as

$$\nabla_5 \hat{\Phi}(\hat{x}) = \dots \tag{2.2}$$

[1] This requirement comes from the vanishing of the total central charge of the CFT on the worldsheet, which ensures the absense of Weyl anomaly in the quantum case. This can also be solves with remaining to four dimensions but considering an additional SCFT for the remaining degrees of freedom. Such approaches are usually called non-critical string theories and we will not discuss them in this work.

However, since our space is essentially split into to, so will the nabla operator. Therefore, we can expand $\hat{\Phi}$ into eigenfunctions of the nabla operator on the circle as $\hat{\Phi}(\hat{x}) = \sum_n \Phi_n(x) e^{in\frac{y}{R}}$ and obtain

$$\nabla_5 \hat{\Phi}(\hat{x}) = \sum_n \left(\nabla_4 - \frac{n^2}{R^2} \right) \Phi_n = \dots \tag{2.3}$$

Note that the effect of extra dimension, is viewed by the four dimensional theory simply as producing massive modes with masses $\frac{n^2}{R^2}$. These modes are called Kaluza-Klein modes and the idea is that since we want the size of the circle to be undetectable sending $R \to 0$ [2] makes these modes massive enough to not be probed in the low energy theory. The issue of whether this is consistent becomes much more involved for more complicated theories and more complicated spaces and we will give more details later. The important thing to observe is that what we do is expand our fields into harmonic functions of the internal space, and then ignore all the massive modes in the low energy theory. Notice that in this case the field $\hat{\Phi}$ does not depend on the internal coordinate y. Moreover, observe that there is a $U(1)$ gauge symmetry on the four dimensional field Φ. This is a general feature of Kaluza-Klein reductions, namely the isometries of the internal manifold are viewed as gauge symmetries from the four dimensional point of view. This is used many times to build various gauge theories with a desired gauge group. Next, we focus on reducing a higher dimensional theory on a general internal compact manifold.

2.2.2 Dimensional reduction on general internal manifold

After giving the most trivial example of dimensional reduction we want to describe the general procedure one needs to follow. Assume we start from a $d+4$ dimensional theory with gravity $\hat{g}_{MN}$ and fields $\hat{\Phi}$ with action

$$\int d^{d+4}\hat{x}\sqrt{-\hat{g}}\hat{R} + \dots \tag{2.4}$$

Next, we expand the higher dimensional fields around their vacuum expectation values, assuming that they exhibit 'spontaneous compactification' by which we mean that the metric decomposes as [17]

$$\langle \hat{g}_{MN} \rangle = \begin{pmatrix} \mathring{g}_{\mu\nu}(x) & 0 \\ 0 & \mathring{g}_{mn}(y) \end{pmatrix} \tag{2.5}$$

and it describes the space $M_4 \times M_d$ with coordinates x^μ and y^m correspondingly. [3]. After determining the vacuum expectation values, we need to expand our fields around them as follows

$$\hat{g}_{MN} = \langle \hat{g}_{MN} \rangle + \hat{h}_{MN} \tag{2.6}$$

$$\hat{\Phi} = \langle \hat{\Phi} \rangle + \hat{\phi} \tag{2.7}$$

In order to obtain the four dimensional theory we must substitute these fields in their equations of motion. Keeping only linear terms and maybe fixing a gauge, these can be written as

$$\mathcal{O}_4 \phi^{mn\cdots}_{\mu\nu\cdots} + \mathcal{O}_d \phi^{mn\cdots}_{\mu\nu\cdots} = 0 \tag{2.8}$$

[2] More precisely, we assume it to be around the planck scale.

[3] This is not the most general assumption since it is also common to include wrap factors. However for this work this will not be the case.

where $\mathcal{O}_4, \mathcal{O}_d$ are differential operators of order p ($p = 2$ for bosons and $p = 1$ for fermions). We have generally written the field ϕ as $\phi^{mn\cdots}_{\mu\nu\cdots}$ to demonstrate any possible indices it has. Now we must expand as in the previous example the higher dimensional fields, in terms of the eigenfunctions $Y_a^{\mu\nu\cdots}(y)$ of the internal operator $\mathcal{O}_d$

$$\phi^{mn\cdots}_{\mu\nu\cdots}(x,y) = \sum_a \phi_{\mu\nu\cdots}(x) Y_a^{\mu\nu\cdots}(y) \tag{2.9}$$

Now we observe as before, that since $O_d Y^{amn\cdots} = \lambda_a Y^{amn\cdots}$ again the eigenvalues of this operator will correspond in terms of the four dimensional theory to the masses of the resulting fields. The bosonic fields that we will have at hand in this work, will obey equations of motions where the operators $\mathcal{O}$ will be the ∇ operator. For such operators on compact spaces we know that the eigenvalues scale as $\frac{n}{S}$ where S is the size of the compact space [29] and there is only one eigenfunction with zero eigenvalue, the constant one. In this case ignoring again the 'massive modes' the only remaining fields are $\phi_{0\mu\nu\cdots}$ which are a result of expanding the full field around harmonic functions $Y^{0mn\cdots}$ of the internal manifold. For example a p-form $\hat{B}_p$ decomposes as

$$\hat{B}_p = B_p^{i_0}\omega_{i_0} + B_{p-1}^{i_1}\omega_{i_1} + \ldots + \ldots B_0^{i_p}\omega_{i_p} \tag{2.10}$$

where $B_k^{i_{p-k}}$ denotes i_{p-k} four dimensional k forms and ω_{i_k} denotes i_k harmonic k-forms. It is important at this point to highlight again that the consistency when ignoring all the Kaluza-Klein modes is not guarenteed. The reason is that one needs to replace this expansion in the original action of the theory. While this is done, it is possible that there are terms which include interactions between massive and massless modes which do not become small after taking the size of the internal manifold to become small [19]. The particular way to obtain a consistent truncation depends on the case at hand and we will come back to it later.

So far we have only treated bosonic fields. Thereofore, one might wonder what happens with the fermionic ones. In this case the operator $\mathcal{O}$ is the Dirac operator. The number of zero modes for such an operator for an internal manifold is treated by topological methods such as index theorems. For the particular case of our interest, namely when the internal space is a Calabi-Yau manifold, there are only two zero modes, the covariantly constant spinors ϵ_+, ϵ_-. We will come back to their importance later on [19].

The story so far might seem rather convincing and simple. However, it is important to remember that what we have shown so far only works for field theories. We have assumed that we have a particular action to integrate over the whole spacetime. Nevertheless, our aim is to compactify a ten dimensional superstring theory. Thankfully there is a way to obtain a low energy field theory out of the full string theory by looking at length scales large enough such that the string length is not important. This will be the topic of the next paragraph.

2.3 Low energy effective action of type II strings

As pointed out earlier, the Kaluza-Klein procedure requires the existence of some field theory action. In order to derive such an action from the full superstring theory there are several ways to proceed. First, one must keep in mind that since String theory describes the smallest possible scales (Planck scale), any field theory action will simply be an approximation of the full String theory. In particular, we are interested in peturbation in powers of the string scale $\alpha^{'}$ which is usually taken to be of order of the Planck scale. The first step is to derive the massless spectrum of the superconformal field theory on the worldsheet. Depending on the particular choice of conventions this leads to different types of spacetime supersymmetry. The cases that we will be interested in correspond to $\mathcal{N} = 2$ spacetime supersymmetry and the massless spectrum of the closed superstring is given in the following table

Type	Masless bosonic spectrum
IIA	g_{MN}, B_2, ϕ, A_1, C_3
IIB	g_{MN}, B_2, ϕ, C_0, C_2, A_4

We have not given the fermionic modes here since they will not be relevant in the rest of the discussion. More details can be found in [2]. Notice here that both types have a common part in the massless spectrum, namely the metric g_{MN} the so called NS 2 form B_2 and the dilaton ϕ. Type IIA also contains a one form A_1 and a threeform C_3. In type IIB we find a scalar C_0 usually called the axion, a two form C_2 and a four form A_4 whose field strength is self dual, meaning that $\star_{10} F_5 = F_5$. The rest of the spectrum in both cases contains states whose masses scale as $m \sim (\alpha^{'})^{-\frac{1}{2}}$ [4]. These are evidently extremely massive and an effective description of the theory is not expected to describe such states. We only want to focus on the massless spectrum. There are now two ways to proceed.

The first way is to write down a Polyakov action for the worldsheet where these fields have some non zero background configurations. Then one must require that the superconformal invariance, survives upon quantization since this is actually a gauge symmetry of the worldsheet and it cannot be broken by quantum corrections. This amounts to require that the beta functions for all the fields at hand vanish identically. This requirement comes about as an expansion in the $\alpha^{'}$ parameter indeed. The interpretation then is that the equations we obtain will correspond to the equations of motion of the massless fields. Then one writes down an action which reproduces these equations of motion. This final action corresponds to the low energy theory of the full string.

The second approach is through scattering amplitudes. The idea is that one can use the vertex operators which correspond to the superconformal field theory at hand to compute n-point functions. Then, the task is to find a field theory which reproduces these scattering amplitudes and contains the same field content. This matching then fixes the spacetime effective action. In particular, to zeroth order in the $\alpha^{'}$ parameter it is known that the effective description correspond to supergravity theories. The two methods have been used to derive a correction to the Einstein-Hilbert action in [27],[28] and they have been shown to match in [20].

The result of this procedure gives us two types of supergravity actions which can be found in [36]

$$S_{IIA} = \int e^{-2\hat{\phi}} \left(-\frac{1}{2}\hat{R} * \mathbf{1} + 2d\hat{\phi} \wedge * d\hat{\phi} - \frac{1}{4}\hat{H}_3 \wedge * \hat{H}_3 \right) \tag{2.11}$$

$$- \frac{1}{2}\int \left(\hat{F}_2 \wedge * \hat{F}_2 + \hat{F}_4 \wedge * \hat{F}_4 \right) + \frac{1}{2}\int \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 \tag{2.12}$$

where

$$\hat{H}_3 = d\hat{B}_2, \quad \hat{F}_2 = d\hat{A}_1, \quad \hat{F}_4 = d\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3$$

[4] This is because $\alpha = \ell_s^2 \simeq \ell_p^2$ where ℓ_s is the string length and ℓ_p the Planck length.

and

$$S_{IIB} = \int e^{-2\hat{\phi}} \left(-\frac{1}{2}\hat{R} * \mathbf{1} + 2d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{4}\hat{H}_3 \wedge *\hat{H}_3 \right) \tag{2.13}$$

$$- \frac{1}{2}\int \left(d\hat{C}_0 \wedge *d\hat{C}_0 + \hat{F}_3 \wedge *\hat{F}_3 + \frac{1}{2}\hat{F}_5 \wedge *\hat{F}_5 \right) \tag{2.14}$$

$$- \frac{1}{2}\int \hat{A}_4 \wedge \hat{H}_3 \wedge d\hat{C}_2 \tag{2.15}$$

where

$$\hat{F}_3 = d\hat{C}_2 - \hat{C}_0\hat{H}_3$$

$$\hat{F}_5 = d\hat{A}_4 - d\hat{B}_2 \wedge \hat{C}_2$$

We have used a hat symbol to illustrate that these fields are ten dimensional for later convenience. Once we obtain our low energy effective action, we can proceed and perform the Calabi-Yau compactification.

2.3.1 Why Calabi-Yau?

There are several reasons for which people are motivated to consider that the extra dimensions of our universe predicted by String Theory, should be Calabi Yau manifolds. A first indication is that we want our background fields to obey the classical equations of motion. In particular for the metric this means that the manifolds under consideration should be Ricci flat. This condition is indeed satisfied by Calabi-Yau manifolds. Another motivation is the surviving number of supercharges in the four dimensional theory. Ideally, we would like to keep some supersymmetry since we know that this has nice phenomenological implications and solves some long-lasting problems of the Standard model such as the hierarchy problem[32] .[5] Most of the models constructed have $\mathcal{N} = 1$ supersymmetry. However, this is not restrictive enough and therefore relatively hard to work with, that is why many people usually start working with $\mathcal{N} = 2$ supersymmetry and only later break it to $\mathcal{N} = 1$. It turns out that $\mathcal{N} = 2$ theories have small enough amount of supersymmetry to not be completely trivial, but large enough to allow for convenient treatment. Our motivation is therefore to end up with $\mathcal{N} = 2$ in the four dimensional theory. Recall that we also have $\mathcal{N} = 2$ in the ten dimensional Supergravity we start with. The general requirements are given in [6] to be

- The geometry to be of the form $M_4 \times M_6$, where M_4 is a maximally symmetric [6]
- There should be an unbroken supersymmetry in four dimensions.[7]
- The gauge group and fermion spectrum should be realistic.[8]

[5]In short this problems comes from the fact that between the electroweak scale and the Planck scale there is a huge window of energy scales which seperates them without any good reason.

[6]This essentially comes from the observation that our universe is homogeneous and isotropic.

[7]The main motivation comes from the solutions SUSY provides for issues such as the hierarchy problem which was already mentioned.

[8]Evidently, we would like at certain energy scales to be able to reproduce the Standard model.

Let's consider now that we split the total space as outlined in the previous section in the following way $M_{10} = M_4 \times M_6$, namely we have our usual Minkowski space-time times a six dimensional manifold. We want the two supercharges in the ten dimensional theory Q_1, Q_2 to transform in a suitable way, such that their four dimensional components also correspond to supercharges. In general since we have a split space the ten dimensional spinor representation we start with, will decompose into a direct product of a four dimensional and a six dimensional representation $Q_i^{10} = Q_i^4 \otimes \epsilon_i^6$. We would like to have an internal manifold such that under parallel transport the component ϵ_i^6 is not altered. We know that in general this vector is a $Spin(6) \simeq SU(4)$ representation. A non-zero vector ψ in $\mathbb{C}^4$ can be written without loss of generality as $\psi = (z, 0, 0, 0)^T$. Therefore, keeping this vector constant for any kind of path is equivalent to demanding that the holonomy group of the internal manifold is restrcited to $SU(3)$ [44]. The intuition behind this, should be that for each supercharge to remain a spinor in the four dimensional theory, it should be unaffected by rotations of the internal manifold, otherwise its definition would be inconsistent. There is another equivalent way to argue about why the Holonomy group should be restricted to be $SU(3)$ in [6]. It is well known that manifolds with such Holonomy are called Calabi-Yau manifolds and their properties are outlined in the appendix.

2.3.2 Calabi Yau compactification of type IIA supergravity

After briefly motivating the choice of Calabi-Yau manifolds for our six dimensional space we are ready to perform the dimensional reduction from ten to four dimensions. We will only focus on the bosonic sector since the fermionic one is fixed by supersymmetry. We will take all the background values of our fields to be zero, except for the metric which obeys (2.5) As highlighted previously, the task is to expand the fields in terms of the harmonic forms of the internal manifold. We first start from the metric. We pick a background hermitian metric for the compact space and consider the deformations thereof, as explained in the appendix. The intuition should be that we have one topologically distinct manifold which allows for different kind of extra structures, such as the Kahler and complex structure. We want to consider all the possible such structures at once. We have the following expressions

$$g_{\alpha\beta} = 0 + \bar{z}^a \left(\bar{b}_a\right)_{\alpha\beta} \tag{2.16}$$

$$g_{\alpha\bar{\alpha}} = g^0_{\alpha\bar{\alpha}} - iv^i \left(\omega_i\right)_{\alpha\bar{\alpha}} \tag{2.17}$$

For the matter part we have

$$\phi(\hat{x}) = \phi(x) \tag{2.18}$$

$$\hat{A}_1(\hat{x}) = A^0(x), \tag{2.19}$$

$$\hat{C}_3(\hat{x}) = C_3(x) + A^i(x) \wedge \omega_i(y) + \xi^A(x)\alpha_A(y) + \tilde{\xi}_A(x)\beta^A(y) \tag{2.20}$$

$$\hat{B}_2(\hat{x}) = B_2(x) + b^i(x)\omega_i(y) \tag{2.21}$$

where from the four dimensional perspective $A^0, A^i, i = 1, \dots, h^{1,1}, \quad B_2$ are one-forms and $u^i, i = 1, \dots, h^{1,1}, \quad \bar{z}^a, a = 1, \dots, h^{2,1}, \quad b^i, \xi^A, \tilde{\xi}_A, A = 0, \dots, h^{2,1}$ are scalars. On the other hand from the six dimensional point of view ω_i is a basis of harmonic (1,1) forms, $\bar{b}_a$ is a basis of (2,1) forms and α_A, β^A is a basis of harmonic 3-forms. The fields are organized into supersymmetry multiplets as shown in the next table

gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vector multiplets	$h^{(1,1)}$	(A^i, v^i, b^i)
hypermultiplets	$h^{(2,1)}$	$\left(z^a, \xi^a, \tilde{\xi}_a\right)$
tensor multiplet	1	$\left(B_2, \phi, \xi^0, \tilde{\xi}_0\right)$

(2.22)

The next step is to introduce this decomposition in the low energy effective action, use the properties of the Calabi-Yau manifold to integrate over the internal space, and obtain an expression for the four dimensional theory. We will only give the result of this procedure without giving any details which can be found in [29]. The four dimensional effective action is given by the expression

$$S_{IIA} = \int \Big[-\frac{1}{2} R * \mathbf{1} - g_{ij} dt^i \wedge * d\bar{t}^j - h_{uv} dq^u \wedge * dq^v \tag{2.23}$$

$$+\frac{1}{2} \operatorname{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \operatorname{Re} \mathcal{N}_{IJ} F^I \wedge F^J \Big] \tag{2.24}$$

where $t^i = b^i + iu^i$ are the complexified Kahler deformations, h_{uv} is the sigma model metric for the hypermultiplets given by

$$h_{uv} dq^u \wedge * dq^v = d\phi \wedge * d\phi + g_{ab} dz^a \wedge * d\bar{z}^b \tag{2.25}$$

$$+ \frac{e^{4\phi}}{4} \left[da + \left(\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A \right) \right] \wedge^* \left[da + \left(\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A \right) \right] \tag{2.26}$$

$$- \frac{e^{2\phi}}{2} \left(\operatorname{Im} \mathcal{M}^{-1} \right)^{AB} \left[d\tilde{\xi}_A + \mathcal{M}_{AC} d\xi^C \right] \wedge^* \left[d\tilde{\xi}_B + \overline{\mathcal{M}}_{BD} d\xi^D \right] \tag{2.27}$$

where $\mathcal{M}$ is defined in the appendix and a is the dual of B_2 which we refer to as an axion and we have defined $F^A = (dA^0, dA^a)$.

2.3.3 Calabi Yau compactification of type IIB supergravity

The situation is very similar with the previous chapter except for the matter content. In this case we have the following decompositions

$$\phi(\hat{x}) = \phi(x) \tag{2.28}$$

$$\hat{C}_0(\hat{x}) = C_0(x) \tag{2.29}$$

$$\hat{B}_2(\hat{x}) = B_2(x) + b^i(x)\omega_i(y), \quad i = 1, \ldots, h^{(1,1)} \tag{2.30}$$

$$\hat{C}_2(\hat{x}) = C_2(x) + c^i(x)\omega_i(y) \tag{2.31}$$

$$\hat{A}_4(\hat{x}) = D_2^i(x) \wedge \omega_i(y) + \rho_i(x) \wedge \tilde{\omega}^i(y) + V^A(x) \wedge \alpha_A(y) - U_A(x) \wedge \beta^A(y), \quad A = 1, \ldots, h^{(2,1)} \tag{2.32}$$

where D_2, B_2, C_2 are two forms, V^A, U_A are 1-forms and b^i, ρ_i, c^i are scalars from the four dimensional point of view. These fields are arranged in the following multiplets

gravity multiplet	1	$(g_{\mu\nu}, V^0)$
vector multiplets	$h^{(2,1)}$	(V^a, z^a)
hypermultiplets	$h^{(1,1)}$	(v^i, b^i, c^i, ρ_i)
tensor multiplet	1	(B_2, C_2, ϕ, C_0)

(2.33)

Notice that we did not mention all the fields in the previous table. The reason is that the self duality of the field strength imposes conditions which leaves only a few independent fields. We choose them to be ρ_i and V^A. The resulting action is given by the following expression

$$S_{\mathrm{IIB}} = \int \Big[-\frac{1}{2} R * \mathbf{1} + \frac{1}{2}\,\mathrm{Re}\,\mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2}\,\mathrm{Im}\,\mathcal{M}_{KL} F^K \wedge * F^L \tag{2.34}$$

$$-g_{ab} dz^a \wedge * d\bar{z}^b - h_{pq} d\tilde{q}^p \wedge * d\tilde{q}^q \Big] \tag{2.35}$$

where

$$h_{pq} d\tilde{q}^p \wedge * d\tilde{q}^q = g_{ij} dt^i \wedge * d\bar{t}^j - d\phi \wedge * d\phi \tag{2.36}$$

$$- \frac{e^{2\phi}}{8\mathcal{K}} g^{-1ij} \left(d\rho_i - \mathcal{K}_{ikl} c^k db^l \right) \wedge * \left(d\rho_j - \mathcal{K}_{jmn} c^m db^n \right) \tag{2.37}$$

$$- 2\mathcal{K} e^{2\phi} g_{ij} \left(dc^i - C_0 db^i \right) \wedge * \left(dc^j - C_0 db^j \right) - \frac{1}{2} \mathcal{K} e^{2\phi} dC_0 \wedge * dC_0 \tag{2.38}$$

$$\begin{aligned} &- \tfrac{1}{2\mathcal{K}} e^{2\phi} \left(dh_1 - b^i d\rho_i \right) \wedge * \left(dh_1 - b^j d\rho_j \right) \\ &- \mathrm{e}^{4\phi} D\tilde{h} \wedge * D\tilde{h} \end{aligned} \tag{2.39}$$

with

$$D\tilde{h} = dh_2 + l dh_1 + \left(c^i - l b^i \right) d\rho_i - \frac{1}{2} \mathcal{K}_{ijk} c^i c^j db^k \tag{2.40}$$

and we have dualized the two forms C_2, B_2 into the scalars h_1, h_2. The symbols $\mathcal{K}, \mathcal{K}_{ijk}$ are explained in the appendix. Moreover we have defined $F^A = dV^A$.

2.3.4 Consistency of Calabi-Yau compactification

In the previous section we argued that in order for the truncation of the massive modes to be consistent, one must take the volume of the internal manifold to be small enough such that these modes are not probed in the energy scales we are interested in. To be more precise, the idea is to impose the equations of motion into the original action and then find a way to isolate the y dependend terms and integrate them out[9]. If this is not possible either because this splitting is not possible or because higher derivative terms arise, the truncation is not consistent. This procedure is however not practical for Calabi-Yau compactification since one would need to know the internal metric, which is generally not known for arbitrary CY spaces. That is for general type II compactifications on CY_3 more involved arguments are developed in the litarature to ensure consistency [16]. A final remark follows. We mentioned earlier that in order to neglect the massive modes, one must take the volume of the manifold to be small enough, more precisely we take it to be around the Planck scale. However there is an additional source of corrections one needs to take care of. The expressions for the effective action derived above, are only valid when the supergravity limit of String theory is a good approximation. This requires that the momenta we have at hand are sufficiently bounded, or in other words, when the characteristic length of the internal manifold is large enough. When this is not the case, $\alpha^{'}$ corrections become important as well as instanton corrections [5]. This seems to contradict with the requirement for consistent truncation. One must be very careful with the assumptions and the involved energy scales upon compactification. More specifically we have to consider Calabi-Yau manifolds whose 'average radius' l_Y fullfills $\frac{1}{p} \ll l_Y \ll \sqrt{\alpha^{'}}$, where p is the characteristic momentum

[9]This should be also done in the supersymmetry transformations otherwise massless modes become massive and vice versa.

of the lower-dimensional fields. This ensures that we do not get higher derivatives corrections to our supergravity approximation and on the mean time that our energy scale is not enough to probe any Kaluza Klein massive modes. More details can be found in [30].

2.4 Mirror symmetry and quantum corrections

Looking closely at the previously derived relations, one can get a first feeling for a hidden symmetry. It seems like somehow there should be a way to obtain the same low energy effective action compactifying on two different Calabi-Yau manifolds, provided that their complex structure and Kahler structure are reversed. This is because one can observe that in type IIB for instance, the vectors in the vector multiple couple through the matrices $\mathcal{M}$ which is the same matrix involved in the couplings of the hypermultiplets of the type IIA action. Also if one looks at the other way around, the triple intersections numbers are involved in the coupling of the hypermultiplets of type IIB theory, and they appear again in the coupling of the vectors in the vector multiplet of type IIA through the matrix $\mathcal{N}$. The task of the next section will be to make this observation more precise.

Mirror symmetry is a deep mathematical result connecting pairs of Calabi-Yau manifolds. The rough idea is that for each Calabi-Yau manifold CY_3 with a certain complex and Kahler structure, there is a mirror manifold $\hat{CY}_3$ whose complex and Kahler structure are reversed. The first strong indications came from the papers [23],[8] (and others). In the first paper the existence of the duality from the String theory point of vie w (superconformal field theory on the worldsheet) is almost trivial to observe. In the second paper the authors construct the mirror of the quintic threefold and they show that the duality holds when one looks at the large volume-large complex structure point of the pair. This property of Calabi-Yau manifolds is not proven yet, it remains a conjecture, however many mirror paris are constructed and the community if very confident about the validity of the conjecture. Our interest will be mostly be in the formulation of mirror symmetry in physics. This states that there exist two kinds of topologically distinct Calabi-Yau manifolds CY_3,$\hat{CY}_3$ with exchanged hodge numbers $h^{1,1}(CY_3) \leftrightarrow h^{2,1}(\hat{CY}_3)$ such that the resulting SCFTS are equivalent. The result of this statement is that indeed compactifying type IIA on CY_3 gives the same low energy theory as compactifying type IIB on $\hat{CY}_3$. The particular mapping of fields of one action to the other can be found in [5]. This symmetry (even if yet remains a conjecture) can be very useful in calculating quantum corrections of our low energy effective theories by alternating between type IIB and type IIA expressions [10]. For example, we know that the field which dictates the loop expansion of string amplitudes is the dilaton, which in the type IIB case sits in the $N = 2$ hypermultiplet. Moreover we know that the moduli space of Calabi Yau manifolds locally splits into $\mathcal{M}^{CS} \times \mathcal{M}^{KS}$. Since the hypermultiplet couples through expressions on M^{KS} these will not give loop corrections to the vector multiplets couplings. The $\alpha^{'}$ corrections as mentioned earlier are controlled by the Kahler moduli which are related to the volume of CY_3 and they also sit in the hypermultiplet in type IIB case. Therefore the vector multiplets coupling dictated by M^{CS} is classically excact while M^{KS} receives both $\alpha^{'}$ and g_s corrections . On the other hand in type IIA the Kahler moduli sit in the vector multiplet and therefore give $\alpha^{'}$ corrections to M^{CS} while the dilaton sits on the hypermultiplet

[10] Recall that from the previous discussion, in order for both low energy theories to be consistent, we must be near the large volume-large complex structure point on the corresponding moduli spaces.

resulting in g_s corrections to M^{KS}. This fact can be used to compute M^{CS} α'-corrections for type IIA vector multiplets couplings, using the mirror picture in type IIB which is excact [38].

Chapter 3

Theta angle and the Swampland

3.1 Strong CP problem

The strong CP problem (or otherwise called theta problem) in QCD is one of the puzzles of theoretical physics, sometimes argued to be the most underated one [40]. The issue arises from the fact that there is a mismatch between an aspect of the theory which is believed to explain strong interactions, QCD, and experimental observations. In particular the QCD action in the limit of massless quarks takes the form

$$\mathcal{S}_{QCD} = \frac{1}{g^2}\int \mathrm{Tr}\{F \wedge \star F\} \qquad F = dA + i[A, A] \quad \text{with} \quad A \in \mathfrak{su}(3) \otimes \Omega^1(M^{3,1}). \tag{3.1}$$

To be more precise, A is an $SU(3)$ gauge field taking values in the Lie algebra of $\mathfrak{su}(3)$ and F is the corresponding field strength and the star symbol is the usual Hodge-star operator. However, it is by now known that there are certain allowed configurations for the gauge fields (instantons) such that an additional term of the following form contributes

$$\mathcal{S}_{\theta} = \frac{\theta}{32\pi^2}\int \mathrm{Tr}\{F \wedge F\}. \tag{3.2}$$

This is a topological term and gives rise to time invariance violation (the hodge star in the previous term protects T invariance). At this step, the Standard model has a saying, since it is strongly believed that CPT symmetry, meaning charge conjugation, parity and time reversal, is a symmetry of all interactions in nature. Therefore, the theta term violates CP symmetry. From the QCD perspective there is no good argument to fix the value of theta, however experimentally it is observed to be extremelly small, in particular $\theta_{exp} < 10^{-9}$ [1]. This roughly constitutes the strong CP problem of the Standard model. There is a big number of proposed solutions but none of them has been verified yet, therefore this constitutes one of the puzzles that Physics beyond the Standard Model has to tackle.

Let us briefly dive into the details of how such a term comes about in the context of QCD. It is known that the QCD Lagrangian for N flavours, in the limit of vanishing quark masses, has a global $U(N)_V \times U(N)_A$ symmetry. Now, since $m_u, m_d \ll \Lambda_{QCD}$, where m_u, m_d are the up and down quark masses while Λ_{QCD} is the cut-off scale beyond which one cannot trust the theory, it is natural to expect that this global symmetry mentioned earlier should hold true at least approximately. However, experimentally one finds that while the vector symmetry is indeed a good approximation symmetry, the axial one is broken spontaneously. This problem was called by Weinberg as the $U(1)_A$ problem, suggesting that there is no such symmetry in the strong interactions. The answer to this question originally came from t' Hooft, who realised that the solution should be found to the more complicated structure of the vacuum of QCD [42]. This vacuum structure is closely related to the θ angle mentioned earlier. Therefore, the solution to the $U(1)_A$ problem induces a new problem into the theory, namely that since this term is violating time reversal symmetry, QCD ends up violating CP symmetry. However, since as mentioned before, the value of the theta angle is extremelly small, CP is not badly violated. Let us now quantify what we said before in order to see how the violating term comes about. One should start with the axial current J_5^μ associated with $U(1)_A$. We know that there is a chiral anomaly, namely that quantum corrections prohibits the symmetry from surviving quantization. This is demonstrated by considering the current divergence

$$\partial_\mu J_5^\mu = \frac{g^2 N}{32\pi^2} F_a^{\mu\nu} (\tilde{F^a_{\mu\nu}}) \tag{3.3}$$

where $\tilde{F^a_{\mu\nu}} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F_a^{\alpha\beta}$. This divergence effectively introduces the following term in the action

$$\delta W = \frac{g^2 N}{32\pi^2} \int d^4x \partial_\mu K^\mu = \frac{g^2 N}{32\pi^2} \int d\sigma_\mu K^\mu \tag{3.4}$$

where

$$K^\mu = \epsilon^{\mu\alpha\beta\gamma} A_{a\alpha} \left[F_{a\beta\gamma} - \frac{g}{3} f_{abc} A_{b\beta} A_{c\gamma} \right] \tag{3.5}$$

Now, depending on the boundary conditions one chooces to use, this integral might be non zero. The most general ones, correspond to assuming that the gauge field must be gauge equivalent of zero, translate to vanishing of the field strength at the boundary. Let's investigate for simplicity the case where the gauge group is $SU(2)$. Demanding that A is a pure gauge means that

$$A = \frac{1}{2}\tau^a A_a \to \frac{i}{g} \nabla^i \Omega \Omega^{-1} \tag{3.6}$$

where τ^a are the Pauli matrices, $\Omega \in SU(2)$ and we work in the A_a^0 gauge where the zeroth component of the gauge field vanishes. Such configurations are classified by maps from the boundary, which can be assumed to be S_3 to the gauge group, which is $SU(2) \simeq S^3$ in this case. The effect of their existence, is that they effectively add an additional term in the Lagrangian of QCD which is proportional to the theta angle as follows

$$S_{\text{eff}}[A] = S_{\text{QCD}}[A] + \theta \frac{g^2}{32\pi^2} \int d^4x F_a^{\mu\nu} \tilde{F}_{a\mu\nu} \tag{3.7}$$

where

$$L_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}. \tag{3.8}$$

The interpretation of this term is that one should not assume necessarily that in the vacuum the gauge field is zero, and therefore expand the path integral around this configuration to obtain the quantum corrections. The full vacuum structure also requires settings where there is a non-trivial configuration to expand around. These correspond to non-perturbative effects of the quantum theory and they are extremely important for the full description of the physics involved.

To summarise, one should keep in mind that the theta angle in QCD, is naturally generated, when one wants to get rid of the $U(1)_A$ problem related to axial current conservation. Then the puzzle consists of the fact that its value cannot be predicted by any theoretical arguments. There are several approaches developed to explain this phenomenon such as the introduction of new particles but none of them has been verified experimentally yet and therefore the problem remains open [35].

3.2 The swampland program

The new line of thought about QG developed through the Swampland program introduces several new lines of work for the physics community. Evidently, a first step must be the determination of the rules behind the separation of the Swampland and the Landscape. This task requires a better understanding on the implications that QG theories impose on their low energy counterparts. After understanding these general properties one should formulate them ,at first, as conjectures which should hold true for every low energy theory. There is a number of conjectures already stated which vastly differ in terms of their validity and rigour as well as the importance in terms of implications they impose. An equally important task is to provide evidence (or even prove) in as general as possible context or to discard them by finding counter examples. The usual way to test the conjectures is by studying the already known possible vacua that one can produce through String Theory compactifications. The issue is that these vacua have different levels of rigour which prohibits some of them from being trustable 'experimental' checks for the conjectures [34]. For instance, there are vacua of the full string worldhseet description, where one picks a simple manifold for the compact space which are called *string derived*. These vacua can be fully trusted when one wants to test general properties of the low energy theories. On the contrary, there are vacua which are valid only after specifying a number of assumptions which are called *string inspired*. These latter ones provide weaker arguments since they are not enough rigorous to be fully trusted. A different line of work is using generally known QG properties such as its holographic nature to determine the effect on EFTs. This approach is much stronger in theoretical terms but the explicit derivations are not as clear as in the context of string theory. Given these ideas, one might argue that the Swampland program can be summarized in the following tasks

- Understand the fundamental properties of QG and determine their imprints on their EFTs.
- Construct the correct criteria to determine which theories belong to the Swampland in the form of conjectures.
- Test the criteria in as general context as possible, preferably through general QG arguments (not just string theory).
- Apply these criteria to already known problems of EFTs (such as the Standard Model) and determine which ones can be accepted as part of the Landscape

Let's see now how Quantum Gravity consistency can effect an apparently acceptable QFT. We start from class of QFTs with a cutoff scale Λ_{QFT}. Then we want to couple these to gravity, with finite gravitational strength given by a finite value of M_p. The effect of the coupling to gravity should be that the theory has to be modified (for example by introducing new particles), at some energy scale we call Λ_{Swamp}. The idea is that if the theory is not modified at that scale, then one will not be able to complete it into a QG theory in the UV. Now, depending on the specifics of the QFT, the relation between the different energy scales we refer to, might change. For instance if $\Lambda_{Swamp} < \Lambda_{QFT}$ then the theory is not modified at all, but in case $\Lambda_{Swamp} > \Lambda_{QFT}$ then the theory is strongly affecting by the coupling to gravity. Lastly, it is also possible that Λ_{Swamp} ends up being smaller than any energy scale of the theory, a situation which hints that the QFT is in the Swampland. These cases are summarised in figure 3.1 .

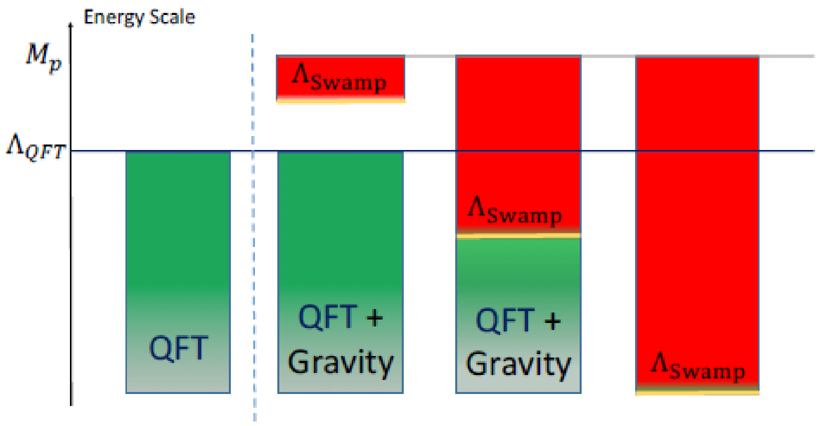


Figure 3.1: The figure demonstrates how different energy scales are present when one couples a QFT to gravity [34].

The conclusion we would like to highlight is that this new way of thinking can push the work of people working on the field in a number of directions. A first task, which has already drawn a lot of attention, is coming up with the correct universal properties of Quantum Gravity, which will lead to the essential conjectures about the low energy theories which correspond to the Swampland. For this task, the most fruitful context so far has been the properties of String Theory, but general QG arguments such as ones from the holographic nature of it or black holes are used. Another line of work should be the test of the already existing conjectures in the most rigorous and precise possible way. To state only a few, the Swampland distance conjecture, the Weak gravity conjecture and the de Sitter conjecture have already been tested in a quite big extend already. Finally, once the rules of the game to determine which theories belong to the Swampland and which belong to the Landscape are well set, the next step is to apply them in the already known low energy theories in order get a better connection to experimental evidence as well. This phenomenological interest is one of the most important hopes of the program, since any Quantum Gravity corrections which are relevant around the Planck scale are far away from the energies that experimentalists can produce. However, the Swampland program might work as a connection between QG and low energy physics experiments without the neccesity of studying processes around the Planck scale.

For the purposes in this work in particular, we would like to find some evidence for the value of the theta angle in relation to the ability of QCD to be a low energy limit of String Theory. In particular, we would like to test a recently proposed conjecture by Vafa and Cecotti [13] ,where the authors claim that the value of θ might be fixed from QG consistency, in a more general context. Before that, we investigate their proposal in more detail.

3.3 The Vafa-Cecotti proposal

As stated in the previous section, up to now there are no strong arguments for fixing the theta angle of QCD to a certain value due to consistency reasons. Let us now introduce the problem by looking at how an action of our world would look like and whether we can say anything about how much it should be constrained. More precisely, in the IR limit, the only massless fields in our world are the graviton $g_{\mu\nu}$ and the photon A_μ (low energy limit does not allow for any other type of interactions). Here, the photon is simply a $U(1)$ gauge field, and the IR lagrangian is given by [1]

$$\mathcal{S}_{IR} = \int \sqrt{-g}\left(\frac{1}{2}R - \frac{1}{4e^2}F\wedge \star F + \frac{\theta}{32\pi^2}F\wedge F\right) \tag{3.9}$$

Notice that usually the topological term for a $U(1)$ gauge theory vanishes(since we usually work in Minkowski), but for a general gravitational background, there might be some configuration in which this is not the case. The question now is to determine the restrictions that would exist for the coupling and the theta angle, if this action was considered as the low energy effective action of a full QG theory in the UV. To do so, we first define the parameter which controls the possible theories of the above form as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{3.10}$$

Note that the theory enjoys an $SL(2,Z)$ electric-magnetic duality transformation of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.11}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. The choice of a particular τ physically corresponds to fixing an electric-magnetic frame, i.e. specifying which fields are seen as electric or magnetic. In order to have a frame-invariant object to speak about, one defines the inequivalent actions to be parametrized by the modular function

$$j(\tau) = q^{-1} + 744 + \sum_{n\geq 1} c_n q^n \quad q = e^{2\pi i\tau}, c_n \in \mathbb{Z} \tag{3.12}$$

Then the space of all possible actions is $\mathbb{P}^1$. On the other hand, a general feature of quantum gravity theories, which is also formulated as a conjecture in [34] is that low energy effective theories which complete to QG in the UV, can not have free parameters. In particular it is expected that all the parameters correspond either to v.e.v. of light fields or to v.e.v of massive fields in the UV theory which however do not appear in the effective description and thus give a frozen v.e.v. The set of permitted actions from a QG perspective can then be called $\mathcal{S}$ and we are interested in determining $\mathcal{S} \subset \mathbb{P}^1$. In order to determine this set, we need a QG 'lab' where we can test what possible types of theories such as (3.9) one can produce in this context. The authors in [13] argue that the real world action given in (3.9) is quite hard to be reproduced in the context of String Theory, therefore they test a simpler problem, that of $\mathcal{N} = 2$ Supergravity. More precisely, in the long wavelength limit this system also has a graviton $g_{\mu\nu}$ and a $U(1)$ gauge field, which is the superpartner of the graviton, called the graviphoton [2]. A natural 'lab' to check the restrictions that this theory should have, is type IIB string theory compactified on Calabi Yau threefold. In that context, if one picks a rigid Calabi-Yau manifold for the compact space (no complex structure deformations), then there is

[1] The cosmological constant is set to zero for simplicity.

[2] Here the fermionic sector is ignored.

no vector multiplet (since the scalars in the vector multiplets correspond to moduli fields of complex structure deformations) [3]. Then the only gauge field is the graviphoton and the coupling τ must be a field dependent complex number depending on the particular choice of the internal space. More precisely, for such a Calabi Yau manifold X, one has $dim(H^3(X,\mathbb{Z})) = 2$ and there is always an integral basis of 3-cycles γ_1, γ_2 such that we obtain the following decomposition on the Poincare dual homology basis

$$H_3(X,Z) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 \tag{3.13}$$

A choice of such a basis fixes the electric-magnetic frame, resulting to the following expressions for the coupling

$$\tau = -\frac{\int_{\gamma_2}\Omega}{\int_{\gamma_1}\Omega} \tag{3.14}$$

where Ω is the unique holomorphic threeform of the CY. Given these data, the authors now argue that using a mathematical conjecture, and testing all the cases of about 50 known rigid CY manifolds, the value of the theta angle is in most of them 0 and only in a handful of them π. Even if only in one of these cases the theta angle was zero it would still be very impressive since it would mean that out of all allowed universes in terms of QG consistency, we simply live in one these universes where $\theta = 0$. These values for the theta angle physically correspond to theories which preserve time reversal symmetry (and therefore CP symmetry). After these indications they proceed to argue that probably this is the case for all rigid CY manifolds (which are expected to be finite in number). Their conclusion is that if our world was fully described by $\mathcal{N} = 2$ Supergravity without vector multiplets, then the theta angle would have a frozen value to 0 or π enjoying time reversal symmetry explained only by QG consistency.

In this project our aim is to investigate whether this conjecture can be tested in a more general context. In particular we will consider general Calabi Yau manifolds (with general complex and Kahler structure) and see whether QG consistency dictates any special values for the corresponding 'theta angle' of the resulting low energy theory. Evidently, there is a huge gap between restricting the theta angle to this context, and to saying something about our real world. However, this work should be seen more like a new way of thinking for generally determining properties of effective field theories, than actually fully solving the theta problem.

[3] The details about why this is the case can be found in 2.

Chapter 4

Variation of Hodge structures

Our aim in this section is to introduce in a physicists-friendly way, the most central tools that we will need for our work, related to VHS. The context in which we work is type IIB compactification on a CY_3. In particular, we are interested in the moduli space of complex structure deformations, which in terms of physics in 4D, represents the scalars in the vector multiplets of $\mathcal{N} = 2$ Supergravity. It has been shown in the past that smooth variation of Hodge structure on Calabi Yau manifolds are in one to one correspondence with $\mathcal{N} = 2$ supergravity in four dimensions [12]. The geometric data of the moduli space provides us with the couplings of these vectors.

Before introducing the mathematical machinery of VHS it is important to clarify the geometric meaning of what one calls moduli space of CY_3. As the name suggests, it is related to the following problem: I have a class of objects X_t and some kind of equivalence relation between them C. I am interested in the space T that parametrizes the class of objects that I have modulo some equivalence relation I want to impose. If I manage to specify the space T I will end up with some kind of 'fiber bundle' with base T where at each point over $t \in T$ I will have a class of equivalent objects X_t. For our purposes the class of objects will be Calabi Yau manifolds with fixed topology (fixed hodge numbers), the equivalence relation will be the complex structure and the space of parameters is the moduli space of complex structures. Recalling the definition of complex manifolds, one remembers that they are defined as spaces which look locally like $\mathbb{C}^n$, where on overlaps the transition functions are holomorphic. The choice of transition functions and charts fixes the complex structure. Now if one decides to give a different complex strucure on the same topological space, but there is a biholomorphic map between the transition functions of the two choices, then one says that these spaces as complex manifolds are equivalent. Note that two manifolds can be diffeomorphic as real manifolds but still be equipped with inequivalent complex structures. It turns out that there is a manifold parametrizing these different possible complex structures which is finite dimensional and of Special Kähler type. Moreover, it has been constantly observed that these spaces exhibit singularities and it is these singularities which highlight the interesting physics of the theory.

In most areas of physics, if not all, it is usually customary to work with spaces which are smooth. When this is not the case we know that many ambiguities appear as for example the description of black holes in General Relativity. The striking difference of String Theory is that it encorporates a consistent description of spaces with singularities. Some of the most striking examples are the existence of orbifold singularities which lead to smooth string S-matrix or the conifold singularity [41]. This second one is actually a singularity in the moduli space, which is a result of a mistreatment of the low energy effective action and can be resolved by keeping the correct degrees of freedom during the Kaluza Klein reduction. These observations motivate people to study further the structure of theories resulting from String Theory, near such singularities. The VHS is an effort towards this direction.

4.1 Monodromy and period map

As mentioned in the introduction our aim is to explore the consequences of having a singularity in our moduli space. We are working in the complex structure moduli space which is a Kähler manifold with Kähler potential [1]

$$K(z,\bar{z}) = -\log\left[\mathrm{i}\int_{Y_3}\Omega\wedge\bar{\Omega}\right] \equiv -\log\left[\mathrm{i}\bar{\Pi}^T\eta_{\mathcal{I}\mathcal{J}}\Pi^{\mathcal{J}}\right] \tag{4.1}$$

In the last step we have chosen a certain real integral basis $\gamma_{\mathcal{I}}, \mathcal{I}=1,\ldots,2h^{2,1}+2$ for $H^3(CY_3,\mathbb{Z})$ such that

$$\Omega = \Pi^{\mathcal{I}}\gamma_{\mathcal{I}}, \quad \eta_{\mathcal{I}\mathcal{J}} = -\int_{CY_3}\gamma_{\mathcal{I}}\wedge\gamma_{\mathcal{J}} \tag{4.2}$$

In this context the vectors $\Pi^{\mathcal{I}}$ are called the periods of Ω and they are shown to be holomorphic functions of the complex structure moduli space. The assignement to each point in the moduli space, of a period vector, is essentially what mathematicians call a period map. Moreover, $\eta_{\mathcal{I}\mathcal{J}}$ is antisymmetirc and can be used to obtain a symplectic inner product on the space of threeforms as follows

$$S(v,w) \equiv S(\mathbf{v},\mathbf{w}) = \mathbf{v}^{\mathrm{T}}\eta\mathbf{w} \equiv -\int_{CY_3} v\wedge w \tag{4.3}$$

where $u,w\in H_3(CY_3,\mathbb{Z})$ and $\mathbf{u},\mathbf{w}$ are their coefficients in the chosen basis. This product is called symplectic since the transformations of the group which preserves the structure is $Sp(2h^{2,1}+2,\mathbb{R})$ meaning that

$$M^{\mathrm{T}}\eta M = \eta, \quad M\in \mathrm{Sp}\left(2h^{2,1}+2,\mathrm{R}\right) \tag{4.4}$$

and therefore

$$S(\mathbf{v},\mathbf{w}) = S(M\mathbf{v},M\mathbf{w}) \tag{4.5}$$

The way that singularities occur, is when one takes certain limits in the moduli space which makes the Calabi-Yau manifold singular. This can be done for example as one shrinks some cycle to zero length and a cartoon representation is given in figure 4.1. The lines in the figure correspond to the points in the moduli space where singularities emerge, and are called discriminant loci. It is a known result in mathematics that these loci, appear on divisors of the manifold and moreover these divisors can be made to intersect normally. This is of technical importance such that the rest of the formulation works but otherwise we will not be too much concerned about it. These lines, which as mentioned correspond to divisors, will be called $\Delta_{k_1,\ldots,k_l} = \Delta_{k_1}\cup\ldots\cup\Delta_{k_l}$ where by this symbol we refer to actually l simultaneously intersecting divisors. The interest part now comes when one considers a rotation of the period vector along these divisors. Since these correspond to singularities, it turns out that the vectors $\mathbf{\Pi}$ [2] are multivalued, which means that after a full rotation around a

[1] For more details about complex geometry consider A,B.2.

[2] By bold here we refer to the components of the monodromy matrix under the chosen basis.

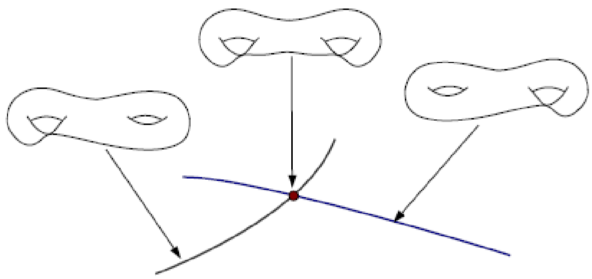


Figure 4.1: Pictorial representaiton of degenerating manifolds appearing as one moves along discriminant loci on the moduli space [24]

divisor they change, through a transformation which we call monodromy. More precisely we have the following expression encircling the k-th divisor Δ_k by sending $z^k \to e^{2\pi i} z^k$ [3]

$$\mathbf{\Pi}\left(\ldots, e^{2\pi i} z_k, \ldots\right) = T_k^{-1} \mathbf{\Pi}\left(\ldots, z^k, \ldots\right) \tag{4.6}$$

We have also introduced coordinates on the moduli space as in the appendix, such that the loci Δ_i are located at $z^i = 0$ for $i \in \{1, \ldots, h^{2,1}\}$. Now, these matrices turn out to have some very interesting properties. First, monodromy matrices of different loci commute, namely $[T_k, T_l] = 0$ for loci Δ_k, Δ_l. Second they are quasi-unipotent which means that there are some integers n, m such that $(T_k^{m_k} - \mathrm{Id})^{n_k+1} = 0$. This allows us to define the logarithm of these matrices which will be nilpotent as follows $N_k = \frac{1}{m_k} \log(T_k^{m_k}) \equiv \log\left(T_k^{(u)}\right)$ where $T_k^{(u)}$ is the unipotent part of the monodromy matrix. This definition ensures that there is an integer n_k such that $N_k^{n_k+1} = 0$. This fact is very important and will play a major role to what comes next. To make our life easy we can perform suitable rescalings on our coordinates to get rid of the integer m and always have unipotent monodromy matrices [4]. Finally collecting all the different monodromy matrices for all possible loci one forms the monodromy group Γ which preserves the η pairing since $\Gamma \subset \mathrm{Sp}\left(2h^{2,1} + 2, \mathbb{R}\right)$.

4.2 Nilpotent orbits

In the previous section we introduced the main actors of our story and now we want to see how these behave when we move to discriminant loci. More precisely, we want to ask the question whether approaching the previously mentioned divisors, there is a simple (or at least simpler) expression for the period vectors in terms of the coordinates of the moduli space. We already know that a general expression of $\mathbf{\Pi}$ for the whole moduli space is extremely complicated, however the answer to the previous question turns out to be positive. We want a local approximation therefore we have to specify a particular patch to consider called $\mathcal{E}$ such that

$$\mathcal{E} = (\mathbb{D}^*)^{n_\mathcal{E}} \times \mathbb{D}^{h^{2,1} - n_\mathcal{E}} \tag{4.7}$$

[3]The inverse appears depending on whether one wants to act on the vectors or on the basis therefore it is simply a convention.

[4]This rescaling hides some types of singularities. We will not be concerned with this case, more details can be found in [24].

where $\mathbb{D}^*$ is a punctured disc and $\mathbb{D}$ is a unit disc. The point under consideration is located "on the puncture" and it is evident that we are actually considering $n_{\mathcal{E}}$ discriminant divisors $\Delta_i, i = 1, \ldots, n_{\mathcal{E}}$. Our approximation will be valid as long as we are near these divisors but away from any other additional ones. On this patch, the coordinates can be decomposed as the ones we intend to put the divisors on, and the rest which do not play a major role as follows $z^I = \left(z^i, \zeta^K\right)$ where the divisors are located at $z^i = 0, i = 1, \ldots, n_{\mathcal{E}}$. Given these assumptions the Nilpotent orbit theorem proved by Schmid give the following expression for the period vectors

$$\mathbf{\Pi}(z,\zeta) = \exp\left[\sum_{j=1}^{n_E} -\frac{1}{2\pi \mathrm{i}}\left(\log z^j\right) N_j\right]\mathbf{A}(z,\zeta) \tag{4.8}$$

$$\equiv \exp\left[\sum_{j=1}^{n_\ell} -t^j N_j\right]\mathbf{A}\left(e^{2\pi \mathrm{i} t},\zeta\right) \tag{4.9}$$

where $\mathbf{A}(z,\zeta)$ is a holomorphic function of z, ζ and we have made the following redefinition [5]

$$t^j \equiv x^j + \mathbf{i}y^j = \frac{1}{2\pi\mathbf{i}}\log z^j \tag{4.10}$$

But since $\mathbf{A}(z,\zeta)$ is a holomorphic function it admits the following series expansion

$$\mathbf{A}(z,\zeta) = \mathbf{a}_0(\zeta) + \mathbf{a}_j(\zeta)z^j + \mathbf{a}_{jl}(\zeta)z^j z^l + \mathbf{a}_{jlm}(\zeta)z^j z^l z^m + \ldots \tag{4.11}$$

where each $\mathbf{a}_0, \mathbf{a}_j, \ldots$ are holomorphic functions of the remaining coordinates ζ. At this point the nilpotent orbit theorem comes to establish that the periods in the limit $z^i \to 0$ or $t^i \to i\infty$ are well approximated by [9]

$$\mathbf{\Pi}_{\text{nil}} = \exp\left[\sum_{j=1}^{n_{\mathcal{E}}} -\frac{1}{2\pi\mathbf{i}}\left(\log z^j\right) N_j\right]\mathbf{a}_0(\zeta) \equiv \exp\left[\sum_{j=1}^{n_{\mathcal{E}}} -t^j N_j\right]\mathbf{a}_0(\zeta) \tag{4.12}$$

This approximation is good as long the remaining exponential terms from the expansion of $\mathbf{A}$ become small enough. More precisely the terms we ignore are

$$\mathbf{\Pi}(t,\zeta) = \underbrace{\exp\left[\sum_{j=1}^{n_{\mathcal{E}}} -t^j N_j\right]\left(\mathbf{a}_0(\zeta)\right.}_{\text{nilpotent orbit }\Pi_{\text{nil}}} \left. + \mathcal{O}\left(e^{2\pi i t}\right)\right) \tag{4.13}$$

A first observation we can make from the above theorem is that the information about the behaviour of the periods and therefore the behaviour of Ω is encoded in the log monodromy matrices N_j. In the next section we will make this observation more specific.

4.3 Hodge filtrations

In the previous section the nilpotent orbit seemed to hint that there is a crucial role to be played in our description by the matrices N_j. Let us quantify this statement. Recall from A that the third cohomology group for complex manifold with fixed complex structure can be decomposed as

$$H^3\left(CY_3,\mathbb{C}\right) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \tag{4.14}$$

[5] This change of coordinates corresponds to putting the degenerating loci at $t^i \to i\infty$. Then we can think of these locations as the boundaries of the moduli space.

This decomposition is called a pure Hodge structure of weight 3 [6]. However, one expects this decomposition to be altered when one changes the complex structure. That is indeed the case, namely moving in the moduli space results in changing the notion of what (2,1) or (3,0) form means. This can easily be seen by the formula of Kodaira which gives the variation of the holomorphic (3,0) form as in (B.2) where we observe that a (3,0) form becomes a combination of (3,0) and (2,1) forms. This motivates the definition of the following spaces

$$F^3 = H^{3,0} \tag{4.15}$$
$$F^2 = H^{3,0} \oplus H^{2,1} \tag{4.16}$$
$$F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \tag{4.17}$$
$$F^0 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \tag{4.18}$$

The elements of these spaces are all holomorphic functions of the moduli space coordinates z^I. It is now possible to introduce a connection on this space, in order to determine how they vary as one moves in $\mathcal{M}^{2,1}$. One choice is a flat connection which is known as Gauss-Manin connection and it has the property $\nabla_I F^p \subset F^{p-1}$. For the special case of Calabi Yau threefolds it is known that all the elements of the higher F's can be obtained by taking appropriate derivatives of elements of the lower F's. This means that one can start from the highest one, where the holomorphic threeform Ω is the only element, and by taking derivatives span the rest of the spaces. This is very useful because we already have a well approximated expression for Ω given by the nilpotent orbit and we can work as follows

$$\mathbf{\Pi}_{\text{nil}} \xrightarrow{\partial_{t^i}} N_i\mathbf{\Pi}_{\text{nil}} \xrightarrow{\partial_{t^j}} N_iN_j\mathbf{\Pi}_{\text{nil}} \to \dots \tag{4.19}$$

However, if one observes carefully the expressions for the nilpotent orbit, it is clear that the above expressions diverge once we take the limits $t^i \to i\infty$. Nevertheless, it has been shown that it is possible to capture the different types of singularities by forgetting about the divergent prefactor and focusing only on the action of the log monodromy matrices to the vectors $\mathbf{a}_0$ as follows

$$\mathbf{a}_0 \longrightarrow N_i\mathbf{a}_0 \longrightarrow N_iN_j\mathbf{a}_0 \to \dots \tag{4.20}$$

Given the previous observations we want now to be more precise and introduce the objects that will play the major role in the rest of the discussion. Our aim will be to come up with a structure in a neighbourhood very close to the degenerating loci and to see how we can incorporate the importance of the matrices N_i in this structure. Let V be a rational vector space and $V_\mathbb{C} = V \otimes \mathbb{C}$ its complexification. A pure Hodge structure of weight w on this vector space provides the following decomposition

$$V_\text{C} = \mathcal{H}^{w,0} \oplus \mathcal{H}^{w-1,1} \oplus \dots \oplus \mathcal{H}^{1,w-1} \oplus \mathcal{H}^{0,w} \tag{4.21}$$

with subspaces satisfying $\mathcal{H}^{p,q} = F^p \cap \bar{F}^q$ with $p+q=w$. Given this decomposition we can define the Hodge filtration as $F^p = \oplus_{i\geq p}\mathcal{H}^{i,w-i}$ satistying

$$V_\text{C} = F^0 \supset F^1 \supset \dots \supset F^{w-1} \supset F^w = \mathcal{H}^{w,0} \tag{4.22}$$

such that $\mathcal{H}^{p,q} = F^p \cap \bar{F}^q$. The decomposition into $\mathcal{H}^{p,q}$ and the Hodge filtration are equivalent and they define a pure Hodge structure on $V_\mathbb{C}$ of weight w. A polarized pure Hodge structure has the

[6] The interested reader might want to take a look at [18] for a detailed introduction to Hodge structures.

additional property of being equipped with a bilinear form $S(\cdot,\cdot)$ on $V_{\mathbb{C}}$ such that

$$S\left(\mathcal{H}^{p,q},\mathcal{H}^{r,s}\right)=0 \quad \text{for } p\neq s, q\neq r \tag{4.23}$$

$$\mathbf{i}^{p-q}S(v,\bar{v})>0 \quad \text{for } u\neq 0, u\in\mathcal{H}^{p,q} \tag{4.24}$$

We now want to include the nilpotent matrices N into the game. We define [11] the monodromy weight filtration W_i to be the unique filtration of V

$$W_{-1}\equiv 0\subset W_0\subset W_1\subset\ldots\subset W_{2w-1}\subset W_{2w}=V \tag{4.25}$$

such that the following properties hold true

- $NW_i\subset W_{i-2}$
- $N^j: Gr_{w+j}\to Gr_{w-j}$ is an isomorphism , with $Gr\equiv W_j/W_{j-1}$

These properties can be shown to fully determine the filtration. We can combine all the above into what is called a mixed Hodge structure given by the data $(V,W(N),F)$. The defining property of this structure is that the spaces Gr_j admit an induced Hodge filtration

$$F^pGr_j^C\equiv\left(F^p\cap W_j^C\right)/\left(F^p\cap W_{j-1}^C\right) \tag{4.26}$$

with $Gr_j^{\mathbb{C}}=Gr_j\otimes\mathbb{C}$ and $W_i^{\mathbb{C}}=W_i\otimes\mathbb{C}$. More precisely, we have

$$Gr_j=\bigoplus_{p+q=j}\mathcal{H}^{p,q},\quad \mathcal{H}^{p,q}=F^pGr_j\cap\overline{F^qGr_j}, \tag{4.27}$$

meaning that the F^p induce a pure hodge structure of weight w on the Gr_j. Taking into account how N acts on F^p we have $NGr_j\subset Gr_{j-2}$ and $N\mathcal{H}^{p,q}\subset\mathcal{H}^{p-1,q-1}$. However, our aim is to extract the behaviour of the filtrations, once we approach the singular loci. To do so we consider the following limit (the notation is schematic but hopefully obvious)

$$F^p\left(\Delta_k^\circ\right)=\lim_{t^i\to\mathrm{i}\infty}\exp\left[-\sum_{i=1}^{k}t^iN_i\right]F^p \tag{4.28}$$

where by Δ_k° we mean the interesection of excactly k divisors, in a region away from any other divisor. It turns out that the expression defined above is well behaved, and only depends (in a smooth way) on the $h^{2,1}-k$ remaining coordinates on the moduli space not send to infinity.

We know have all the data to define a more useful splitting once one wants to focus on the singular loci of the moduli space. This splitting is called the Deligne splitting and is defined as

$$I^{p,q}=F_\Delta^p\cap W_{p+q}\cap\left(\bar{F}_\Delta^q\cap W_{p+q}+\sum_{j\geq 1}\bar{F}_\Delta^{q-j}\cap W_{p+q-j-1}\right) \tag{4.29}$$

What we effectively do is to replace the information encoded in the objects $(F_\Delta^p,\sum_i N_i)$ [7] by $I^{p,q}$. The above definition might not seem handy at first glance, but the usefulness lies on the fact that it

[7] At this point the choice of the nilpotent matrix for a multivariable degenerating locus with respect to each nilpotent matrix for each variable might seem ambiguous. However it turns out that any choice among the $\sum_i a_iN_i$ with $a_i>0$ is equally good and gives the same filtration.

is the unique filtration satisfying the relations [8]

$$F_\Delta^p = \bigoplus_{r\geq p}\bigoplus_{s} I^{rs}, \quad W_l = \bigoplus_{p+q\leq l} I^{p,q}, \quad \overline{I^{p\cdot q}} = I^{q,p} \bmod \bigoplus_{r<q,s<p} I^{r,s}. \tag{4.30}$$

The Deligne splitting $I^{p,q}$ satisfies some nice properties. Firstly, it indeed provides with a splitting of 3-forms for the Calabi Yau manifold.

$$H^3(CY_3,\mathbb{C}) = \bigoplus_{p+q=0}^{3} I^{p,q}. \tag{4.31}$$

Moreover the nilpotent matrix acts on them in the following way

$$NI^{p,q} \subset I^{p-1,q-1}. \tag{4.32}$$

This is a very important fact because it means that starting from the highest $I^{p,q}$ we can not generate all the lowest weight spaces by acting with N. In contrast, there are subspaces which are not in the image of N, these are called primitive parts and are defined as

$$P^{p,q} = I^{p,q} \cap \ker N^{p+q-2} \tag{4.33}$$

which obviously means that all the information about the splitting is encoded in these spaces. More precisely we have the further decomposition of the Deligne splitting into primitive subspaces

$$I^{p,q} = \bigoplus_{i\geq 0} N^i(P^{q+i,p+i}). \tag{4.34}$$

Another important fact that will be handy later on is that these spaces obey certain orthogonality relations, coming from the fact that only top forms can be integrated and any other integral automatically vanishes

$$S\left(P^{p,q}, N^l P^{r,s}\right) = 0 \text{ for } p+q = r+s = l+3 \text{ and } (p,q) \neq (s,r) \tag{4.35}$$

$$\mathbf{i}^{p-q} S\left(v, N^{p+q-3}\bar{v}\right) > 0 \text{ for } v \in P^{p,q}, v \neq 0. \tag{4.36}$$

Before proceeding to the next chapter regarding the Sl_2 orbit theorem let us summarise what we have done so far. We want to study the moduli space of complex structure deformations. It has been proven that is problem is identical with studying the different filtrations occuring as one moves in this moduli space.[9] Our interest lies on what happens when we approach a point lying on a degenerating locus (or multiple loci). For that purpose we collect the information lying on the limiting filtration and the nilpotent matrix N associated with the monodromy on the desired locus, to obtain the Deligne splitting. Evidently, this splitting crucially depends on the point under consideration, meaning that for instance by moving from a single locus to a multiple loci we expect a different adequate Deligne splitting. This procedure will be further discussed later when we introduce the enhancements of singularities.

[8]Pay attention in the last property. This essentially means that the subspaces are not 'symmetric' under complex conjugation, there is a dependence on all the lower subspaces. This feature is not pleasant and we will make particular effort in the next section to get rid of it.

[9]This identification is called the Torelli problem in general but specifically holds for Calabi-Yau manifolds [33].

4.4 $Sl(2)$ orbit theorem

It should be clear by now that the nilpotent orbit theorem together with the Deligne splitting give us very useful tools to study the singular points of the complex structure deformation moduli space. However, it turns out that there is some further structure one can use to investigate these properties. The first important point of the $Sl(2)$ orbit theorem is that it provides us with a better approximation of the period vectors where the corrections are suppressed as a power law and not exponentially, however with the cost of path dependence. Moreover it gives us a recipe to construct a Deligne splitting which is $\mathbb{R}$-split. This property means that complex conjugation of some subspace does not depend on the rest of the subspaces with smaller weight and therefore $\overline{I^{p,q}} = I^{q,p}$ for all p, q.

Let us now see how the theorem is established. The situation is the same as before, we consider $n_{\mathcal{E}}$ intersecting divisors $\Delta_i, \quad i = 1, \dots, n_{\mathcal{E}}$ with monodromy logarithms $N_i, \quad i = 1, \dots n_{\mathcal{E}}$ with an ordering $N_1, \dots, N_{n_{\mathcal{E}}}$ and everything will depend on this ordering. For different orderings one just has to rearrange things. Given this data the $Sl(2)$ orbit theorem states, that there are associated $\mathfrak{sl}(2, \mathbb{C})_i$ commuting algebras acting on the period vectors [9]. Each of these algebras is generated by three elements (N_i^-, N_i^+, Y_i) which satisfy the usual $\mathfrak{sl}_2$ commutation relations $\left[Y_i, N_i^{\pm}\right] = \pm 2N_i^{\pm}$ and $\left[N_i^+, N_i^-\right] = Y_i$. These triples are pairwise commuting, meaning that each operator in the i-th triple commutes with any other in the j-th triple. This is a crucial feature as we will later see when we discuss the different types of singularity enhancements. In order to get more comfortable with these operators, one can think of them like lowering,raising and eigenvalue operators. The theorem then states that the vectors $\mathbf{a_0}$ can be decomposed into eigenspaces of one of these operators. This will become more precise later though. In order for this action to be established though, since the period vectors are representations of the $\mathfrak{sp}(2h^{2,1} + 2)$ we also need a Lie algreba homomorphism associated with each triple

$$\rho_* : \bigoplus_i \mathfrak{sl}(2, \mathbf{C})_i \longrightarrow \mathfrak{sp}\left(2h^{2,1} + 2, \mathbf{C}\right). \tag{4.37}$$

The central statement of the theorem, is that one can replace the data coming from $(N, \mathbf{a_0})$ with $(N^-, \tilde{\mathbf{a}}_0)$ such that it is N^- that is part of the $\mathfrak{sl}(2)$ triple and the new vector $\tilde{\mathbf{a_0}} = e^{-\zeta} e^{-i\delta} \mathbf{a_0}$ decomposes under the action of the triple into subspaces. The theorem also states that this new vector, obtained after the action with these special matrices ζ, δ can be used to define a new type of filtration $\hat{F} = e^{-\zeta} e^{-i\delta} F$ such that the Deligne splitting obtained from the data $(\hat{F}, N_i^-)$ denoted as $\tilde{I}^{p,q}$ satisfies the relation $\overline{\tilde{I}^{p,q}} = \tilde{I}^{q,p}$ making it $\mathbb{R} - split$. The splitting provided by the data $\hat{F}, N_i^-$ is called the $Sl(2)$ splitting of (F, W). The new splitting is decomposed into eigenspaces of $Y_{(i)}$ as

$$Y_{(k)} \tilde{I}^{p,q}\left(\Delta^o_{1,..i}\right) = (p + q - 3) \tilde{I}^{p,q}\left(\Delta^\circ_{1,\dots,i}\right) \tag{4.38}$$

where $Y_{(i)} = Y_1 + \dots + Y_i$ and $\tilde{I}^{p,q}(\Delta^\circ_{1\dots k})$ is the one associated to the intersection loci $\Delta^\circ_{1\dots k}$. More details about how these are constructed can be found in the appendix D, in this work we will assume that the procedure is already done and we have the $Sl(2)$ split already at hand. Observe here that the above relation tells us that we essentially get a very nice decomposition of the 3-Cohomology provided by the eigenspaces of the Y operators as following

$$H^3\left(Y_3, \mathbb{R}\right) = \bigoplus_{\ell \in \mathcal{E}} V_{\vec{\ell}}, \quad \vec{\ell} = (\ell_1, \dots, \ell_n) \tag{4.39}$$

where $l_i \in \{0,\ldots,6\}$ are integers representing the eigenvalues of $Y_{(k)} = Y_1 + \ldots + Y_k$ meaning

$$v_{\vec{\ell}} \in V_{\vec{\ell}} \implies Y_{(k)} v_{\vec{\ell}} = (\ell_i - 3)\, v_{\vec{\ell}} \tag{4.40}$$

Above, by $\mathcal{E}$ we have denoted the possible vectors $\vec{\ell}$ which consist of the eigenvalues of the operators $(Y_{(1)}, Y_{(2)}, \ldots Y_{(i)})$ with associated eigenspaces $V_{\vec{\ell}}$. This decomposition is quite remarkable, since now one manages to decompose the cohomology elements into independent vector spaces that one can imagine as spins. In particular these spaces satisfy the following relations

$$\dim V_{\vec{\ell}} = \dim V_{\vec{6}-\vec{\ell}} \tag{4.41}$$

$$\langle V_{\vec{\ell}}, V_{\vec{r}} \rangle = 0 \quad \text{unless} \quad \vec{\ell} + \vec{r} = \vec{6} \tag{4.42}$$

where we denote $\vec{6} = (6,\ldots,6)$.

The first of the above relations provides us with an identification between basis vectors of $V_{\vec{\ell}}$ and $V_{\vec{6}-\vec{l}}$. The second one is an orthogonality relation between these vector spaces which will be very much used later. Both these properties hint that these eigenspaces can be properly identified with a real symplectic basis for the complex structure moduli space as in (B.17). An important caveat at this point is the vector spaces under consideration, are real vector spaces but we know that $\tilde{\mathbf{a}}_\mathbf{0}$ is a complex vector. The statement we make is that $\tilde{\mathbf{a}}_\mathbf{0}$ will live in one of the complexified versions of these spaces and only the real and imaginary parts lie in the original spaces, more precisely

$$\tilde{\mathbf{a}}_\mathbf{0} \in V_{\vec{3}+\vec{d}} \otimes \mathbb{C} \tag{4.43}$$

where later we will specify what is the significance of $\vec{d}$.

As promised earlier, the $Sl(2)$-orbit theorem apart from the very nice decomposition that it provides us with, it also gives an approximation of the period vector which is valid up to exponential suppressed terms. This approximation is called the $Sl(2)$ orbit and is given by the following expression

$$\Pi_{\mathrm{Sl}(2)}(y,\zeta) = e^{-iy^i N_i^-}\, \tilde{\mathbf{a}}_\mathbf{0}(\zeta) \tag{4.44}$$

where $t^i = x^i + iy^i$ and ζ denotes the rest of the coordinates not taken to inifinity. The orbit is a good approximation as long as $\frac{y^1}{y^2}, \ldots, \frac{y^{n-1}}{y^n}, y^n \to \infty$ and $x^i = 0$. A first observation we can make here is that the validity of the approximation only holds once one specifies a particular path. In other words, the order under which each modulus goes to infinity, is chosen to be the same order one choses for the nilpotent matrix $N_{(i)}$ and only then the approximation is correct. If one wishes to pick another path then the above expression must be suitably changed by rearranging the fields y_i. The second observation is that it seems arbitrary to set all x^i to zero and one might expect that switching them on might be essential. However one can recall from the physical interpretation of coordinates that they correspond to scalars in the vector multiplets of the $\mathcal{N} = 2$ four dimensional theory. In the mirror picture they will be mapped to the coordinates of the Kahler moduli fields. But the real parts of them b^A can be interpreted as axions, since the gauge invariance of the $NS-NS$ two form induces a shift symmetry on them [10]. Therefore setting x^i to zero is a sensible thing to do if one looks on how this translates under mirror symmetry. Moreover from now on, due to this property and general supersymmetry arguments we will refer to $\mathrm{Re}(t^i)$ as axions and $\mathrm{Im}(t^i)$ as saxions. This is justified if one looks at the scalars of the superfield formulation of $\mathcal{N} = 2$ SUSY. We will investigate the dependence on the axions in the next section.

[10] To be more precise, in the classical picture the constants are real but quantum corrections break the symmetry to integer shifts.

4.5 Weil operator and axion dependence

In this section we want to study the asymptotic behaviour of the Hodge star operator. Most of the ideas presented here follow the line of thought developed in [25]. As one moves in the complex structure moduli space we expect the Hodge star operator to change as well and our aim would be obtain a some kind of simplification for its action just like before, when we move towards a degenerating locus. This is a very important task because as we know the expressions for the real and imaginary parts of $\mathcal{M}$ depend on it as one observes from the appendix B. Let us make a quick recap on how the hodge-star operator looks like for a smooth geometry. We have the usual decomposition

$$H^3\left(CY_3, \mathbb{C}\right) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \tag{4.45}$$

Away from singular points we know that the action of the hodge-star is $\star u^{p,q} = (-1)^{p-q} u^{p,q}$ for $u^{p,q} \in H^{p,q}$. It turns out that that using the nilpotent orbit theorem, the expression for the hodge norm of a general element of $v \in H^3(CY_3, \mathbb{C})$ can be approximated as

$$\|v\|^2 = \int_{Y_3} \mathbf{v} \wedge \star \bar{\mathbf{v}} = \langle C_{nil}\mathbf{v}, \bar{\mathbf{v}}\rangle + \mathcal{O}(e^{2\pi i t^j}) \tag{4.46}$$

where C_{nil} is called the Weil operator provided by the data of the nilpotent orbit theorem and by bold symbols we refer to the components of the chosen element in the integral basis we mentioned earlier. Moreover, by manipulating the results of the nilpotent orbit theorem, we can isolate the axion dependence [11] of the above expression, by defining the following new Weil operator [9]

$$C_{\text{nil}}(t,\zeta) = e^{\phi^i N_i} \hat{C}_{\text{nil}} e^{-\phi^i N_i}, \quad \hat{C}_{\text{nil}} \equiv C_{\text{nil}}\left(\phi^i = 0\right) \tag{4.47}$$

Given the above definition and also defining $\rho(u,\phi) = e^{-\phi N_i} u$ [12] we can obtain an expression to approximate the hodge norm, valid up to exponentially suppressed corrections and all the axion dependence captured by $\rho(u,\phi)$

$$\|v\|^2 = \int_{Y_3} \mathbf{v} \wedge \star \bar{\mathbf{v}} = \langle \hat{C}_{nil}\rho, \rho\rangle + \mathcal{O}(e^{2\pi i t^j}) \tag{4.48}$$

The next natural step to make, is to check what kind of approximation we obtain from the other theorem we discussed, the $Sl(2)$-orbit theorem. We expect again that the expressions will be valid approximations up to terms suppressed as power law. The important part of the expression is the fact that the power law suppression is strongly related to the components that the element u has in the eigenspaces V_ℓ of the operators $Y_{(i)}$. In particular we have that if $u = \sum_{\vec{\ell}\in\mathcal{E}} u^{\vec{\ell}}$ then the hodge norm is given by

$$\|u\|^2 \sim \|u\|^2_{\mathrm{Sl}(2)} = \sum_{\vec{\ell}\in\mathcal{E}} \left(\frac{s^1}{s^2}\right)^{\ell_1-3} \cdots \left(\frac{s^{\hat{n}-1}}{s^{\hat{n}}}\right)^{\ell_{\hat{n}-1}-3} \left(s^{\hat{n}}\right)^{\ell_{\hat{n}}-3} \left\|\rho_\ell\left(u,\phi\right)\right\|^2_\infty \tag{4.49}$$

this approximation is valid where we have also introduced the notation $\|u\||_\infty = \langle C_\infty u, u\rangle$ which gives C_∞ the interpretation of Weil operator at the the singular point (not just near it). This operator

[11] From now one we will give the symbol ϕ for axions and s for saxions.

[12] Recall at this point that since N_i are elements of $\mathfrak{sp}(2h^{2,1}+2)$ then $e^{-\phi N_i}$ is an element of $Sp(2h^{2,1}+2$ and therefore it does not change the symplectic pairing.

gives us a nice orthogonoality property between V_ℓ of different weight, which originates from the fact that we can only integrate top forms

$$\langle C_\infty V_{\vec{\ell}}, V_{\vec{r}} \rangle = 0, \qquad \text{for } \vec{r} \neq \vec{\ell} \tag{4.50}$$

Moreover recalling (4.41) we conclude that

$$C_\infty : V_{\vec{\ell}} \to V_{6-\vec{\ell}} \tag{4.51}$$

An important remark here is again that in order to obtain the $Sl(2)$ approximation one has to determine a path to approach the limiting point, or otherwise said, a growth sector given by

$$\mathcal{R}_{12\cdots\hat{n}} = \left\{ t^j = \phi^j + is^j | \frac{s^1}{s^2} > \gamma, \ldots, \frac{s^{\hat{n}-1}}{s^{\hat{n}}} > \gamma, s^{\hat{n}} > \gamma, \phi^j < \delta \right\} \tag{4.52}$$

with $\gamma \gg 1$. Then the terms we drop are corrections which vary as power law s^i/s^{i+1}. We can summarise the different types of approximations and regimes of validity in the next table

Regime of validity:	Asymptotic s^i large	Strict asymptotic $\mathcal{R}_{1\cdots\hat{n}}$ with $\gamma \gg 1$	At boundary $s^i = \infty$
Approx. Hodge-operator:	C_{nil}	$C_{\text{sl}(2)}$	C_∞
Corrections dropped:	drop $\mathcal{O}\left(e^{2\pi i t^j}\right)$	drop sub-leading $\frac{s^i}{s^{i+1}}$ -polys	t^i -independent

Before closing this chapter let us recap on the main results, because they will be later heavily used in our 'theta-angle' analysis. We managed to obtain a simplified expression for the norm of our cohomology vectors in different validity regimes, representing different kinds of approximations as one moves towards the bounary point. Moreover, at the boundary point we observe that a new structure emerges, as we also saw in the previous chapter, since the cohomology decomposes into vector spaces orthogonal to each other, and to which one can act with 'creation' (N^+) 'annihilation' (N^-) operators and represent eigenspaces of different eigenvalue under Ys. Moreover, we managed to isolate the dependence of the axions in our expressions. We also obtain an approximation of the norms at the boundary points, where the expressions are moduli-independnt. However, reestablishing them corresponds to looking at points not at the boundary but slightly away from it. This will be crucial later, because we know that at the boundary the effective description of our theory can not be trusted anymore and therefore we can not use it to deduce any general features of the consistency of the low energy theory with quantum gravity. This is connected with the swampland distance conjecture which states that at infinite distances, an infinite tower of massless states emerges, invalidating the effective description of String Theory. We will come back to this point later.

4.6 Singularity classification

In this chapter the aim will be to classify the possible ways that a Calabi Yau threefold can degenerate based on [31]. It turns out that there is only a finite number of ways that this can be done and also there are only a few operations one needs to make in order to characterize the degenerations. As pointed out earlier, the basic structure of the singularity is encoded in the Hodge-Deligne decomposition. The dimension of these subspaces can be used to construct a so called Deligne diamond

$$\begin{array}{ccccccc} & & & i^{3,3} & & & \\ & & i^{3,2} & & i^{2,3} & & \\ & i^{3,1} & & i^{2,2} & & i^{1,3} & \\ i^{3,0} & & i^{2,1} & & i^{1,2} & & i^{0,3} \\ & i^{2,0} & & i^{1,1} & & i^{0,2} & \\ & & i^{1,0} & & i^{0,1} & & \\ & & & i^{0,0} & & & \end{array}\ , \quad \dim_{\mathbb{C}} I^{p,q} = i^{p,q} \tag{4.53}$$

Then the first type of classification comes from the possible Deligne diamonds that one can write down. The possibilities are the ones that obey the following properties

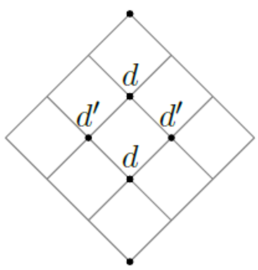
$$h^{p,3-p} = \sum_{q=0}^{3} i^{p,q}, \quad p = 0, \ldots, 3 \tag{4.54}$$

$$i^{p,q} = i^{q,p} = i^{3-q,3-p}, \quad \text{for all } p, q \tag{4.55}$$

$$i^{p-1,q-1} \leq i^{p,q}, \quad \text{for } p+q \leq 3 \tag{4.56}$$

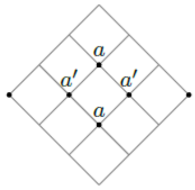
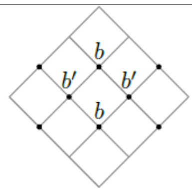
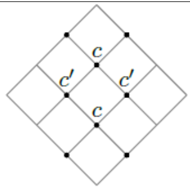
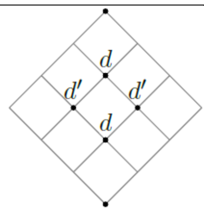
These properties are true for general variations of hodge structures, however for our purposes Calabi-Yau threefolds are our only interest. We know from the appendix A that these have $h^{3,0} = 1$. This gives us four possible cases $i^{3,d} = 1$ for $d = 0, 1, 2, 3$. These four cases label the different types of degenerations and they are labeled by a latin letter correspondingly as I,II,III,IV,V. Moreover, due to symmetry properties of the diamonds we have only two independent numbers on each diamonds, namely $i^{2,1}, i^{2,2}$ which we depict as follows for example in the case $d = 3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & i^{2,2} & & 0 & \\ 1 & & i^{2,1} & & i^{2,1} & & 1 \\ & 0 & & i^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \simeq$$



$$d = i^{2,2}, d' = i^{2,1} \tag{4.57}$$

Actually, even these two numbers are not independent, since they satisfy $i^{2,1} + i^{2,2} = h^{2,1}$ and then the diagramms can be classified as $\mathrm{I}_{i^{2,2}}, \mathrm{II}_{2,2}, \mathrm{III}_{2,2}, \mathrm{IV}_{i^{2,2}}, \mathrm{V}_{i^{2,2}}$. However, this is not the whole story, there is a further classification of the types of singularities related to the properties of the nilpotent matrix N. We are not going to focus on these properties here though, therefore the interested reader can find details in [24]. The classification is given in the following table.

singularity type	Hodge-Deligne diamond
I_a	
II_b	
III_c	
IV_d	

(4.58)

4.7 Singularity enhancement patterns

The classification given in the previous section is general for any number of degenerating loci. However a natural question arises from the intuition of singularities getting 'worse'. What happens if one moves from a singular locus to a more complicating degeneration? Are there any restrictions on the types of singularities in such patterns?

To answer these questions we investigate a situation such as the one in 4.1 but the results are valid for any kind of degeneration pattern. Moreover we assume that on the locus Δ_1 the singularity is of type $\mathrm{Type}_a(\Delta_1)$. The problem at hand is to specify what can $\mathrm{Type}_{a'}(\Delta_{12})$ be when we move to Δ_{12}. The possible enhancements will be denoted by an arrow as follows

$$\mathrm{Type}_a(\Delta_1) \to \mathrm{Type}_{a'}(\Delta_{12}) \tag{4.59}$$

The conditions under which this is possible are rigorously given in [31]. Here we will only give the final criteria but not prove them. More details can also be found in the appendix C. The analysis for the possibility of any enhancements crucially depends on the primitive parts Deligne splitting as defined in (4.33). As mentioned earlier these spaces encode the whole information about the decomposition and we can write any Hodge-Deligne diamond as expressed in terms of these spaces as follows

$$\begin{array}{ccccccc}
 & & & P^{3,3} & & & \\
 & & P^{3,2} & & P^{2,3} & & \\
 & P^{3,1} & & P^{2,2}\oplus NP^{3,3} & & P^{1,3} & \\
P^{3,0} & & P^{2,1}\oplus NP^{3,2} & & P^{1,2}\oplus NP^{2,3} & & P^{0,3} \\
 & NP^{3,1} & & NP^{2,2}\oplus N^2P^{3,3} & & NP^{1,3} & \\
 & & N^2P^{3,2} & & N^2P^{2,3} & & \\
 & & & N^3P^{3,3} & & &
\end{array} \tag{4.60}$$

Then the primitive subspaces are

$$\begin{array}{ll} P^6 = P^{3,3}, & P^5 = P^{3,2}\oplus P^{2,3} \\ P^4 = P^{3,1}\oplus P^{2,2}\oplus P^{1,3}, & P^3 = P^{3,0}\oplus P^{2,1}\oplus P^{1,2}\oplus P^{0,3} \end{array} \tag{4.61}$$

The crucial feature of these spaces in relation to the enhancements is that each P^j with $j = 3,\ldots,6$ defines a pure Hodge structure of weight j. Now, in order to investigate what kind of primitive spaces are probed when we move to a more complicated locus, the idea is to start from these pure Hodge structures which split into a mixed Hodge structure. Therefore the picture one should have in mind is the following

$$P^j\left(\Delta_1\right) \quad \longrightarrow \quad \left[I^{p,q}\right]^j\left(\Delta_{12}\right) \text{ with } 0 \le p+q \le 2j \tag{4.62}$$

Then the question is whether we can reassemble the spaces $I^{p,q}$ to form a diamond of a certain type. If that is the case then the enhancement is possible. A particular example of how this is tested is given in the appendix C. The results of this analysis are given in the following table

starting singularity type	enhance singularity type
I_a	$\mathrm{I}_{\hat a}$ for $a \le \hat a$ $\mathrm{II}_{\hat b}$ for $a \le \hat b, a < h^{2,1}$ $\mathrm{III}_{\hat c}$ for $a \le \hat c, a < h^{2,1}$ $\mathrm{IV}_{\hat d}$ for $a < \hat d, a < h^{2,1}$
II_b	$\mathrm{II}_{\hat b}$ for $b \le \hat b$ $\mathrm{III}_{\hat c}$ for $2 \le b \le \hat c + 2$ $\mathrm{IV}_{\hat d}$ for $1 \le b \le \hat d - 1$
III_c	$\mathrm{III}_{\hat c}$ for $c \le \hat c$ $\mathrm{IV}_{\hat d}$ for $c + 2 \le \hat d$
IV_d	$\mathrm{IV}_{\hat d}$ for $d \le \hat d$

(4.63)

There are several remarks to make here. First we notice that the singularity type as we move to more complicated loci, can only get 'worse', we can only go to higher types of degeneration through consecutive enhancements which fits ones intuition. Another point to make is that these results are not geometric. This means that the rules we obtain about possible types of singularity and enhancements thereof do not guarantee us that there is a geometric situation where they occur. Everything is derived using abstract algebraic methods and actually it is not currently known if every one of these can occur for some Calabi-Yau threefold. In some sense one might argue that testing our claims for all these cases might be overcounting, since at the end of the day we want to speak about what happens for geometric situations. However, the machinery is so powerful and gives us such great simplification tools that any possible overcounting is not an issue.

Chapter 5

Computation of $\mathrm{Re}\,\mathcal{M}$

5.1 One modulus case general analysis

In this chapter we want to develop a general analysis for obtaining a basis of states of certain eigenvalue under the action of the Y operator of the $\mathfrak{sl}(2,\mathbb{C})$ triple on $H^3(CY_3)$. To do so we will use some general facts about representation theory [39]. Firstly, we recall that given an $\mathfrak{sl}(2,\mathbb{C})$-algebra with generators $\{N^-, Y, N^+\}$ every (finite dimensional) irreducible representation is isomorphic to a vector space generated by a highest weight vector $\hat{a}^{p+4}$, defined by demanding that $(N^-)^p\,\hat{a}^{p+4} \neq 0$ while $(N^-)^{p+1}\,\hat{a}^{p+4} = 0$ and its images under N^-. In other words the irreducible representation can be written as

$$\mathrm{span}_C\left\{\hat{a}^{l+4}, N^-\hat{a}^{l+4}, \ldots, \left(N^-\right)^l \hat{a}^{l+4}\right\} \tag{5.1}$$

A general representation of this $\mathfrak{sl}(2,\mathbb{C})$ -algebra is then given by a direct sum of irreducible representations. Therefore it suffices to specify a set of highest weight vectors to fix a representation of the $\mathfrak{sl}(2,\mathbb{C})$-algebra. The highest weight vectors essentially correspond to what was before called primitive cohomology elements $P^{p,q}$ in the Hodge-Deligne diamond. In order to construct our basis we will assume that we have at hand some highest weight vectors spanning these primitive parts, and then everything else can be obtained by consecutive actions with the lowering operator N^-. Moreover,we have the following polarization conditions for limiting mixed hodge structures

$$S\left(P^{p,q}, N^l P^{r,s}\right) = 0 \quad \text{for } p+q = r+s = l+3 \text{ and } (p,q) \neq (s,r) \tag{5.2}$$

$$\mathbf{i}^{p-q} S\left(v, N^{p+q-3}\bar{v}\right) > 0 \quad \text{for } v \in P^{p,q}, v \neq 0 \tag{5.3}$$

We start by assuming the existence of highest weight states $\hat{a}^{m+2a}_{k_m}$, where k_m denotes that we probably have multiple highest weight states of the same weight m. Then the following set of vectors span the eigenspace of Y with eigenvalue labeled by m [1]

$$\{u^m_{j_m}\}_{j_m=1}^{\dim V_m} = \{(N^-)^a \hat{a}^{m+2a}_{k_m}\} \tag{5.4}$$

Using the commutation relation of the algebra one can indeed show that these are subspaces of fixed eigenvalue under Y. These are just general facts for any representation. However, in our discussion we want to make contact with the information obtained by the limiting mixed hodge structure theory.

[1] In this expression we pick the first vectors to correspond to highest weight ones, which means that for $j_m = 1, \ldots, k_m$ we have highest weight vectors, and the rest are non-primitive ones.

For that reason recall that the space of threeforms is decomposed in the following way into primitive and non-primitive parts

$$\begin{array}{ccccccc} & & & P^{3,3} & & & \\ & & P^{3,2} & & P^{2,3} & & \\ & P^{3,1} & & P^{2,2}\oplus NP^{3,3} & & P^{1,3} & \\ P^{3,0} & & P^{2,1}\oplus NP^{3,2} & & P^{1,2}\oplus NP^{2,3} & & P^{0,3} \\ & NP^{3,1} & & NP^{2,2}\oplus N^2P^{3,3} & & NP^{1,3} & \\ & & N^2P^{3,2} & & N^2P^{2,3} & & \\ & & & N^3P^{3,3} & & & \end{array} \tag{5.5}$$

This is very helpful because the possible weights for the basis vectors are evident from the decomposition. However, our upshot is to contstruct a real symplectic basis for the real cohomology group, namely for $H^3(CY_3,\mathbb{R})$ thus we need real vector spaces. Then, the decomposition will have the following data

$$H^3(CY_3,\mathbb{R}) = \bigoplus_m V_m, \qquad V_m = \text{span}_{\mathbb{R}}\{(N^-)^a \hat{a}_k^{m+2a}\} \tag{5.6}$$

Now we want to identify these basis vectors with real forms α_I, β^J $I,J = 0,\ldots,h^{2,1}$ such that (B.17) is satisfied. More details about the interpretation of this basis can be found in appendix B.2 and in [14]. To do so we start by identifying first the α_I in the following way [2,3]

$$\{\alpha_I\}_{I=0,\ldots,h^{2,1}} \equiv \{u^m_{j_m}\}^{dimV_m}_{j_m=1}|_{m>3} \oplus \text{Re}\{(N^-)^b a_l^{3+2b}\} \tag{5.7}$$

By the real and imaginary parts of these spaces, we mean that for the vectors with eigenvalue 3, we chose as the eigenspace the one given by the (p,q) decomposition above. Then for example one highest weight vector of weight 3 which is identified with α_I is $\frac{a^{3,0}+a^{0,3}}{2}$ and one non primitive vector of the same weight is $\frac{Na^{3,2}+Na^{2,3}}{2}$. The reason is that otherwise we can not comply with the symplectic pairing given by (B.17). Then, we chose the β^J such that (B.17) is satisfied taking into account how these equations translate to (5.2). This gives the following identification

$$\{\beta^J\}_{J=0,\ldots,h^{2,1}} \equiv \{u^m_{j_m}\}^{dimV_m}_{j_m=1}|_{m<3} \oplus \text{Im}\{(N^-)^b a_l^{3+2b}\} \tag{5.8}$$

Note that the normalization of these vectors is not automatically satisfied, one needs to include minus signs in a case by case analysis in order to achieve that. The important fact is that essentially we have the following decomposition

$$H^3(CY_3,\mathbb{R}) = \underbrace{V_{heavy} \oplus \text{Re}\, V_{rest}}_{\alpha_I} \oplus \underbrace{\text{Im}\, V_{rest} \oplus V_{light}}_{\beta^J} \tag{5.9}$$

[2]We pick the identification such that the first elements correspond to primitive vectors and the rest to non primitive.

[3]Note that at this point there is no difference in chosing α or β to be the heavy vectors. However the different choices will correspond to different physics, once we introduce saxion dependence later on.

where by heavy we mean $m > 3$, by light $m < 3$ and by rest $m = 3$. The reason for which we must treat the V_{rest} separately lies on (4.41). This shows us that in order for (B.17) to be satisfied, one needs simply to pick all the α in V_{heavy} and then β will be in V_{light} automatically. This is not the case for V_{rest} though since the inner product of elements in this subspace does not vanish automatically. The ambiguity comes when one considers the highest weight states of weight three. The rest of the vectors in V_{rest} which are non-primitive, do not give extra restrictions because one can use the fact that $\langle ., N. \rangle = -\langle N., . \rangle$ to connect their inner products, with the ones corresponding to elements in V_{heavy} and V_{light}. For example for $u, w \in \text{span}_{\mathbb{R}}(N^- \hat{a}_k^5)$ we have

$$\langle u, w \rangle = \langle N^- \hat{a}_k^5, N^- \hat{a}_l^5 \rangle = -\langle (N^-)^2 \hat{a}_k^5, \hat{a}_l^5 \rangle \tag{5.10}$$

which does not give an extra condition indeed.

The next step is to consider the Hodge norms approximations. We will begin by considering the approximation valid at the boundary point, where the corresponding Weil operator is given by C_∞. Looking again at (4.51) we obtain the following relations

$$\langle u^m_{i_m}, C_\infty u^n_{j_n} \rangle = 0 \quad \text{for} \quad m \neq n \tag{5.11}$$

$$\langle u^m_{i_m}, C_\infty u^m_{j_m} \rangle = k^{(i)}_{m,6-m} \delta_{ij} \tag{5.12}$$

These statements are obvious if one does not consider elements in V_{rest} since the action of C_∞ is such, that only inner products between elements with the same weight are non-zero. Therefore it essentially sends all β to α and the other way around. The V_{rest} case is special and one must be more careful since both α and β lie there. However, there is again a decoupling and the reason is that since C_∞ operator is used to define a positive definitive norm as in (4.49) it must map the previously defined real part of the V_{rest} to the imaginary part and the other way around as we can also infer from (5.2).

Let us explain this a bit further with an example. Take $u = \frac{a^{p,q} + a^{q,p}}{2}$ with $a^{p,q} \in P^{p,q}, p + q = 3$. Evidently, u belongs to V_{rest}. We would like to know to which vector this is mapped under the action of C_∞. The only information we know is that $\|v\|\|_\infty = \langle v, C_\infty v \rangle \geq 0$. Moreover from (5.2) we can write

$$i^{p,q} \langle a^{p,q}, \overline{a^{p,q}} \rangle > 0 \tag{5.13}$$

and

$$\langle a^{p,q}, C_\infty a^{p,q} \rangle > 0. \tag{5.14}$$

We know use the fact that we have an $\mathbb{R}$-split filtration such that $\overline{I^{p,q}} = I^{q,p}$ and make the identification

$$C_\infty a^{p,q} = i^{p,q} a^{q,p} \tag{5.15}$$

. Combining these results we finally conclude that

$$C_\infty \frac{a^{p,q} + a^{q,p}}{2} = \frac{i}{2}(a^{p,q} - a^{q,p}) \tag{5.16}$$

which verified that indeed C_∞ maps α vectors into β and vice versa. The normalization factors are not important for this argument. At this point we would like the reader to appreciate the importance of the $\mathbb{R}$-splitness of the decomposition. If this tool was not available, then taking the complex conjugate of a vector $a^{p,q}$ would generate a number of vectors of lower weight as we can see from (4.30). The issue is that in that context, these vectors would not be identified with a specific α or β making the computations harder. In the $\mathbb{R}$-split case C_∞ indeed maps between basis vectors identified with the symplectic basis.

Finally the hodge norm approximations at the boundary point are given by the following expressions [4]

$$\int \beta^I \wedge \star\beta^J = \langle u^n_{i_n}, C_\infty u^{n'}_{j_{n'}}\rangle = k^{(i)}_{n,6-n}\delta_{n,n'}\delta_{ij} \tag{5.17}$$

$$\int \alpha_I \wedge \star\beta^J = \langle u^m_{i_m}, C_\infty u^n_{j_n}\rangle = 0 \tag{5.18}$$

These equations also hold for the V_{rest} case based on the previous arguments.

We now want to slightly move away from the boundary point through introducing the approximation $e^{\phi N^-}C_\infty e^{-\phi N^-}$ for the Weil operator and later by introducing saxion dependence. The highest weight state basis is very helpful in this task because the lowering operator acts on it canonically. Moreover, the weight 3 primitive vectors vanish as N^- acts on them which is also useful as we later see. The idea here is take advantage of the way that N^- behaves in the inner product in order to obtain the following expression for $u, v \in H^3(CY_3, \mathbb{R})$

$$\langle u, e^{\phi N^-}C_\infty e^{-\phi N^-}v\rangle = \langle e^{-\phi N^-}u, C_\infty e^{-\phi N^-}v\rangle \tag{5.19}$$

This action can then be seen as a basis rotation and we have the following expression with the new Weil operator approximation

$$e^{-\phi N^-}u^m_{j_m} = e^{-\phi N^-}\{(N^-)^a \hat{a}^{m+2a}_k\} = \sum_{l^m=0}^{l^m_{max}} \frac{(-\phi)^{l^m}}{l^m!} u^{m-2l^m}_{j_{m-2l_m}} \tag{5.20}$$

where l_{max} depends on each particular vector considered. Moreover, due to the fact that the action of N^- on any vector, can only produce vectors of lower weight which are non primitive, in the expression above not all the possible vectors in the lower subspaces are enhanced. We demonstrate this fact by labeling the rotated vectors with red indices. More precisely we have that $j_{m-2l_m} = k_{m-2l_m}, \dots, \dim V_{m-2l_m}$. Using this expression, which is identical also for the V_{light} vectors, we obtain the following expressions

$$\int \beta^I \wedge \star\beta^J = \langle e^{-\phi N^-}u^n_{i_n}, C_\infty e^{-\phi N^-}u^{n'}_{j_{n'}}\rangle = \sum_{l^n,l^{n'}=0}^{l^n_{max},l^{n'}_{max}} \frac{(-\phi)^{l^n+l^{n'}}}{(l^n)!(l^{n'})!}\langle u^{n-2l^n}_{i_{n-2l^n}}, C_\infty u^{n'-2l^{n'}}_{i_{n'-2l^{n'}}}\rangle . \tag{5.21}$$

[4]From now on we will use the symbol n for values corresponding to β, and m for values corresponding to α.

Moreover we also have [5]

$$\int \alpha_I \wedge \star\beta^J = \langle e^{-\phi N^-} u^m_{i_m}, C_\infty e^{-\phi N^-} u^n_{j_n}\rangle = \sum_{l^m,l^n=0}^{l^m_{max},l^n_{max}} \frac{(-\phi)^{l^m+l^n}}{(l^m)!(l^n)!}\langle u^{m-2l^m}_{i_{m-2l^m}}, C_\infty u^{n-2l^n}_{i_{n-2l^n}}\rangle \tag{5.22}$$

To conclude which terms are non-zero in the above expression, one must see what kind of vectors of the lower subspaces are enhanced and then match their inner products based on (5.11)-(5.12). The idea is that we know how C_∞ acts on our initial basis and since the new basis is written as a linear combination of the old basis we can write everything in terms of the right hand side of elemets as the one in (5.12). However there is an additional issue here. Due to the way we defined the basis vectors for V_{rest} and the identification with the elements of the sympletic basis, one needs for each Hodge-Deligne diamond to see how the previously defined inner products are related, in order to conclude which terms vanish and which do not. The reason for that caveat is the fact that the basis of V_{rest} is not a natural image of actions of higher weight spaces and primitive vectors of weight 3, we have mixed such terms into the identifications with the α, β basis.

We will not demonstrate a general expression for the theta angle matrix here because we believe that the expression will look messy. In contrast, after giving the extention of this procedure to two moduli, we proceed with considering particular examples to demonstrate our technique explicitly.

5.2 Two moduli general case

For two moduli fields sent to infinity we have at hand two copies of the $\mathfrak{sl}(2,\mathbb{C})$ algebra. This means that we will be working with the simulteneous eigenspaces of the operators $Y_{(1)}, Y_{(2)}$. As before, given highest weight vectors $\hat{a}^{m,n}_{k_{m,n}}$ we can form the following spaces, which will be eigenspaces of $Y_{(1)}, Y_{(2)}$ at the same time

$$V_{mn} = \{u^{m,n}_{j_{m,n}}\}|^{\dim V_{mn}}_{j_{m,n}=1} = \{(N_1^-)^a (N_2^-)^b \hat{a}^{m+2a,n+2a+2b}_{k_{m+2a,n+2a+2b}}\} \tag{5.23}$$

We can make the following identifications

$$\{\alpha_I\}_{I=0,\dots h^{2,1}} \equiv \{u^{m,n}_{j_{m,n}}\}|_{m>3} \oplus \mathrm{Re}(\{(N_1^-)^a (N_2^-)^b \hat{a}^{3+2a,3+2a+2b}_{k_{3+2a,3+2a+2b}}\}) \tag{5.24}$$

Then by demanding that the symplectic pairing is satisfied, as in the one modulis case, we obtain the following identification for β^J

$$\{\beta^J\}_{J=0,\dots,h^{2,1}} \equiv \{u^{\hat{m},\hat{n}}_{j_{\hat{m},\hat{n}}}\}|_{m<3} \oplus \mathrm{Im}(\{(N_1^-)^a (N_2^-)^b \hat{a}^{3+2a,3+2a+2b}_{k_{3+2a,3+2a+2b}}\}) \tag{5.25}$$

Then taking into account the way that N_1^-, N_2^- act, namely

$$N_i^- V_{\vec{\ell}} \subseteq V_{\vec{\ell}'}\,, \qquad \vec{\ell}' = (\ell_1, \ell_2, \dots, \ell_i - 2, \dots, \ell_n - 2) \tag{5.26}$$

[5] An important remark to make here is that the allowed values for l^m, l^n are determined by the specific Hodge-Deligne diamond that one has at hand.

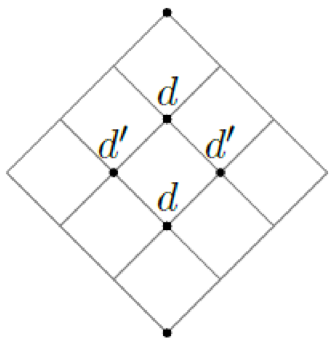
for n moduli taken to infinity, we obtain similar relations with the one modulus case [6]

$$\int \alpha_I \wedge \star\beta^J \equiv \langle e^{-\phi^i N_i^-} u^{m,n}_{j_{m,n}}, C_\infty(e^{-\phi^i N_i^-} u^{\hat{m},\hat{n}}_{j_{\hat{m},\hat{n}}})\rangle = \tag{5.27}$$

$$\sum_{l^m,l^n,l^{\hat{m}},l^{\hat{n}}=0}^{l^m_{MAX},l^n_{MAX},l^{\hat{m}}_{MAX},l^{\hat{n}}_{MAX}} \frac{(-\phi^1)^{l^m+l^{\hat{m}}}(-\phi^2)^{l^n+l^{\hat{n}}}}{(l^m)!(l^n)!(l^{\hat{m}})!(l^{\hat{n}})!}\langle u^{m-2l^m,n-2l^m-2l^n}_{j_{m-2l^m,n-2l^n}}, C_\infty u^{\hat{m}-2l^{\hat{m}},\hat{n}-2l^{\hat{m}}-2l^{\hat{n}}}_{j_{\hat{m}-2l^{\hat{m}},\hat{n}-2l^{\hat{n}}}}\rangle \tag{5.28}$$

Again the particular terms that survive, depend on the types of vector spaces we have at hand as before, which additionally here depend on the possible enhancements that are allowed. However, as before, only non-primitive vectors are enhanced through the action of the lowering operators.

One modulus example : type IV We first show the procedure for one degenerating modulus in the IV case. The only data we will need about the type of degeneration is the corresponding Hodge-Deligne diamond. From this diamond we can read off the corresponding spaces V_ℓ based on their



eigenvalues under Y which are just the height of each element in the diamond. The dots represent the positions where the dimension of the corresponding subspace is non zero. The letters correspond to the dimensions of these subspaces and where there is no letter the dimension is 1. Moreover for the above diamond we have $d + d^{'} = h^{2,1}, 1 \le d \le m$. We read off the following subspaces

$$V_6 = \text{span}_{\mathbb{C}}\{\tilde{\mathbf{a}}_0\} \qquad \dim(V_6) = 1 \tag{5.29}$$
$$V_4 = P^{2,2}(N^-) \oplus \text{span}_{\mathbb{C}}\{N^-\tilde{\mathbf{a}}_0\} \qquad \dim(V_4) = d \tag{5.30}$$
$$V_3 = P^3(N^-) \qquad \dim(V_3) = 2d^{'} \tag{5.31}$$
$$V_2 = P^{1,1}(N^-) \oplus \text{span}_{\mathbb{C}}((N^-)^2\tilde{\mathbf{a}}_0) \qquad \dim(V_2) = d \tag{5.32}$$
$$V_0 = \text{span}_{\mathbb{C}}\{(N^-)^3\tilde{\mathbf{a}}_0\} \qquad \dim(V_0) = 1 \tag{5.33}$$

At this point we already assume that we have made the required steps to obtain an $\mathbb{R}$-split decomposition such that $\bar{I}^{p,q} = I^{q,p}$. Therefore all but V_3 automatically admit real bases. More precisely we write [7]

$$P^{3,3} = \text{span}_{\mathbb{R}}\{e^6\} \tag{5.34}$$
$$NP^{3,3} = \text{span}_{\mathbb{R}}\{e^2_i\} \qquad i = 1, \dots d \tag{5.35}$$
$$N^2P^{3,3} = \text{span}_{\mathbb{R}}\{e^4_i\} \qquad i = 1, \dots d \tag{5.36}$$
$$N^3P^{3,3} = \text{span}_{\mathbb{R}}\{e^0\} \tag{5.37}$$

[6] By hat indices we refer to vectors identified with β and the indices without hat refer to α.

[7] We assume that we have done the procedure outlined previously in the general analysis to obtain a basis coming from highest weight states, and then we simply write this basis in a nicer notation.

As for V_3 we will write down a basis for the real part as $e^3_m, \quad m = 1, \ldots d'$ and for the imaginary part as $e^{\bar{3}}_m, \quad m = 1, \ldots d'$. Then the decomposition of the real threeforms according to (5.6) is

$$H^3(CY_3, \mathbb{R}) = V_6 \oplus V_4 \oplus \mathrm{Re}(V_3) \oplus \mathrm{Im}(V_3) \oplus V_2 \oplus V_0 \tag{5.38}$$

Given this decomposition we can now make the following identifications for the real symplectic basis α_I, β^J such that (B.17)

$$\alpha_0 \simeq e^6, \quad \alpha_1, \ldots \alpha^{d+1} \simeq e^4_i, \quad \alpha^{d+2} \ldots \alpha^{h^{2,1}} \simeq e^3_m \tag{5.39}$$

$$\beta^0 \simeq e^0, \quad \beta^1, \ldots \beta^{d+1} \simeq e^2_i, \quad \beta^{d+2} \ldots \beta^{h^{2,1}} \simeq e^{\bar{3}}_m \tag{5.40}$$

Recall that both the identifications we make and the particular choice of basis for the subspaces is at our hands. The only requirement is that (B.17) is satisfied. This means that we can always make suitable rescalings and permutations as long as these equations remain valid. The next step is to calculate the terms involving the Hodge-star operator. We will start from the theory at the boundary where there is no ϕ or s dependence and the valid approximation is C_∞ and then introduce the axion dependence through alternating to the operator $e^{\phi N} C_\infty e^{-\phi N}$ and later through introducing saxion dependence as well. Recall from (4.51) that $C_\infty : V_\ell \to V_{6-\ell}$. We will rearrange our basis such that this operator maps vectors with the same index, for example we chose our basis such that

$$C_\infty e^4_i = k^{(i)}_{42} e^2_i, \quad C_\infty e^3_m = k^{(m)}_{3\bar{3}} e^{\bar{3}}_m \tag{5.41}$$

One can check that once this choice is made the identification of the symplectic basis gives us

$$\int_{CY_3} \alpha_I \wedge \star \beta^J = 0, \quad \int_{CY_3} \beta^I \wedge \star \beta^J = diag(\ldots) \tag{5.42}$$

Looking again at equations (B.24)-(B.26) we conclude that at the boundary point the 'would be' $\mathrm{Im}\,\mathcal{M}$ is diagonal giving $\mathrm{Re}\,\mathcal{M} = 0$. Recall that the zero in the right hand side is true up to corrections which get suppressed once we set all the moduli fields to infinity. The usage of the words 'would be' is made because exactly at the boundary point the effective theory is not valid and these integrals lose their physical meaning. This is due to the fact that at infinite distances, there are towers of state which become massless invalidating the effective theory description . Recall that in order to obtain the 4D theory in chapter 3 we ignored all the massive KK modes by making the volume of the internal manifold small enough. However moving infinite distance in the complex structure moduli space, produces states whose masses are small enough to be probed. This procedure is investigated by the Swampland distance conjecture, for which one might find details in [24],[34]. In order to retain the 4D physics, we must slightly move away from the boundary point. This is done by the introduction of the axions as we said earlier. First we need to see how N^- maps between subspaces.

In the chosen basis, we have [8]

$$N^- = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & \delta_{i1} & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & \delta_{j1} & 0
\end{array}\right), \quad (N^-)^2 = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & \delta_{i1} & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & \delta_{j1} & 0 & 0 & 0 & 0
\end{array}\right), \tag{5.43}$$

$$(N^-)^3 = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{5.44}$$

Then we conclude that since $e^{-\phi N^-} = \mathbf{1} - \phi N^- + \frac{\phi^2}{2}(N^-)^2 - \frac{\phi^3}{6}(N^-)^3$, we have

$$e^{-\phi N^-} = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & -\phi\delta_{i1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \hline
d & \frac{\phi^2}{2}\delta_{i1} & -\phi\mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \hline
1 & -\frac{\phi^3}{6} & \frac{\phi^2}{2}\delta_{j1} & 0 & 0 & -\phi\delta_{j1} & 1
\end{array}\right) \tag{5.45}$$

Inserting these into the new definition of the Weil operator is equivalent to rotating our chosen basis. Therefore the new basis vectors are

$$e^{6'} = e^6 - \phi e_4^1 + \frac{\phi^2}{2} e_2^1 - \frac{\phi^3}{6} e_0 \tag{5.46}$$

$$e_i^{4'} = e_i^4 - \phi e_2^i + \frac{\phi^2}{2} e_0 \delta_{i1} \tag{5.47}$$

$$e_m^{3'} = e_3^m \tag{5.48}$$

$$e_n^{\bar{3}'} = e_n^{\bar{3}} \tag{5.49}$$

$$e_i^{2'} = e_i^2 - \phi e_0 \delta_{i1} \tag{5.50}$$

$$e^{0'} = e^0 \tag{5.51}$$

[8] Each row and each line in the following matrices is $(1, d, d', d', d, 1)$ dimensional following the decomposition of real threeforms. This notation will be used in the rest of the thesis.

We now compute the inner products between α, β on this new basis

$$\int_{CY_3} \alpha_I \wedge \star\beta^J = \left(\begin{array}{c|c|c|c|c|c|c} & 1 & d & d' & d' & d & 1 \\ \hline 1 & 0 & 0 & 0 & & c\delta_{i1} & \frac{\phi^3}{6}k_{06} \\ \hline d & 0 & 0 & 0 & 0 & \phi k^i_{24}\delta_{ij} + \frac{\phi^2}{2}\delta_{i1}\delta_{j1} & \frac{\phi^2}{2}k_{06}\delta_{i1} \\ \hline d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d' & & & & 0 & 0 & 0 \\ \hline d & & & & 0 & 0 & 0 \\ \hline 1 & & & & 0 & 0 & 0 \end{array}\right) \tag{5.52}$$

$$\int_{CY_3} \beta^I \wedge \star\beta^J = \left(\begin{array}{c|c|c|c|c|c|c} & 1 & d & d' & d' & d & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d' & 0 & 0 & 0 & k^m_{33}\delta_{mn} & 0 & 0 \\ \hline d & 0 & 0 & 0 & 0 & k^i_{24}\delta_{ij} & \phi k_{06}\delta_{i1} \\ \hline 1 & 0 & 0 & 0 & 0 & \phi k_{06}\delta_{j1} & k_{06} \end{array}\right) \tag{5.53}$$

Note that due to the identification of the real basis, the first matrix has components only on the top right and low left side and the second one only in the low right side. We only show the elements that will be important in the calculation of $\mathrm{Re}\,\mathcal{M}$ here. Given this computation we then have from (B.24)-(B.26)

$$\mathrm{Re}\,\mathcal{M}_{IJ} = \left(\int_{CY_3} \alpha_I \wedge \star\beta^K\right)\left(\int_{CY_3} \beta^K \wedge \star\beta^J\right)^{-1} \tag{5.54}$$

$$= \left(\begin{array}{c|c|c|c|c|c|c} & 1 & d & d' & d' & d & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & \left(ca^{(1)} + \frac{\phi^3}{6}k_{06}\beta^2\right)\delta_{i1} & c\beta^1 + \frac{\phi^3}{6}k_{06}a^{(d+1)} \\ \hline d & 0 & 0 & 0 & 0 & \left((C_1)_{11}a^{(1)} + \frac{\phi^2}{2}k_{06}\beta^2\right)\delta_{i1}\delta_{j1} & 0 \\ \hline d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{5.55}$$

where we have defined the following constants

$$c = -\frac{\phi^2}{2}k^1_{24} - \frac{\phi^4}{6}k_{06} \tag{5.56}$$

$$C_1 = \phi k^i_{24}\delta_{ij} + \frac{\phi^3}{2}k_{06}\delta_{i1}\delta_{j1} \tag{5.57}$$

$$a^{(1)} = \frac{1}{k^1_{24} - \phi^2 k_{06}} \tag{5.58}$$

$$\beta^2 = \frac{-\phi}{k^1_{24} - \phi^2 k_{06}} \tag{5.59}$$

$$a^{(d+1)} = -\frac{k^1_{24}}{\phi^2 k_{06} - k^1_{24}k_{06}} \tag{5.60}$$

$$\beta^1 = \frac{\phi}{\phi^2 k_{06} - k^1_{24}} \tag{5.61}$$

It is important to note that none of the constants k can be zero.

Two moduli example $\mathrm{I}_a \rightarrow \mathrm{III}_c$ **enhancement** The first step is to investigate the way that this enhancement takes places. The idea is that this enhancement describes the situation where we

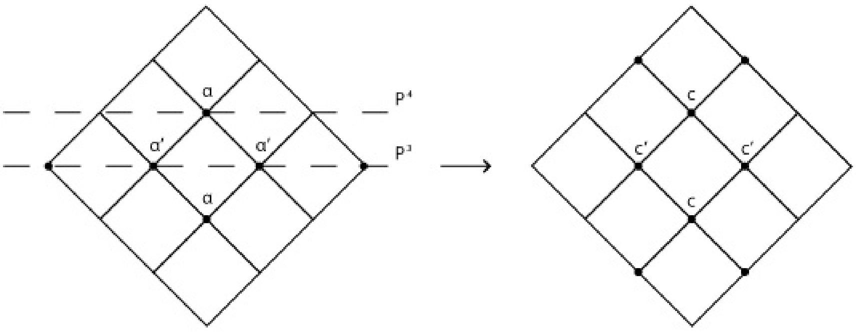
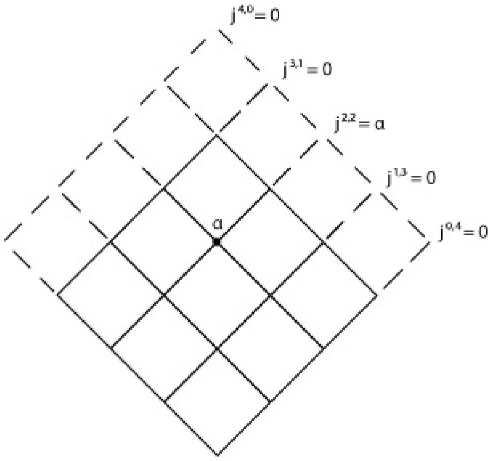


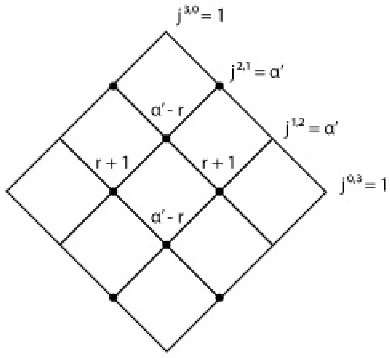
Figure 5.1: The dimensions above obey $a+a^{'} = h^{2,1}, 0 \leq a \leq h^{2,1}, \quad c+c^{'} = h^{2,1}-1, 0 \leq c \leq h^{2,1}-2$

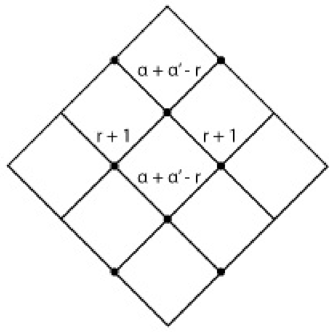
first move to the first degenerating limit, and therefore the diamond is induced by the action of N_1^- and then we move to the interesection of two degenerating limits. This diamond can be split into primitive spaces of the form $(N_1^-)^a P^b (N_1^-)$. The result of the second limit is that $N_{(2)}^-$ induces a mixed hodge structure on each one of these primitive subspaces. The enhancement is possible if the resulting diamonds coming from a type I degeneration can be reassembled to form a diamond of type III degeneration, in our case . More details about determining the allowed enhancements can be found in [31] for the mathematically inclined reader, or in Appendix C,[24] for a physicist. We will mostly be interested in the eigenspaces of Y operators. The eigenvalues of these vectors correspond to the height that they have on the diamond. A vector space V_{54} for instance corresponds to the vectors belong to height 5 in the diamond coming from N_1^- and to height 4 in the diamond induced by N_2^-. Recall that the fact the the $\mathfrak{sl}_2$ triplets are commuting is very crucial here, since that means that there is a simultanesouly diagonalizing basis. Let us now investigate the example at hand to find out what kind of eigenspaces occur. First we must decompose the primitive spaces that are shown in the previous diagram. We have the primitive spaces P^4, P^3. The weights of the induced mixed hodge structures are $(0,0,a,0,0)$ and $(0,1,a^{'},a^{'},1)$ respectively, with diamonds At this point we recall that in order for the enhancement to be possible the following equation must hold true

$$\diamond\,(F_3, N_3) = \sum_{\substack{3\leq k\leq 6 \\ 0\leq a\leq k-3}} \diamond\left(F_k^{'}, N_k^{'}\right)[a] \tag{5.62}$$

The right hand side is the following diamond We conclude that in order for the enhancement to be possible we must have $r+1>0$ which gives $a<c$ for the enhancement to be possible. We can now







read off the following eigenspaces of $Y_{(1)}, Y_{(2)}$

$$V_{44} = P^{2,2}(N_1^-) \qquad \dim V_{44} = a \tag{5.63}$$
$$V_{35} = \mathrm{span}_{\mathbb{C}}\{\tilde{\mathbf{a}}_0, \bar{\tilde{\mathbf{a}}}_0\} \qquad \dim V_{35} = 2 \tag{5.64}$$
$$V_{34} = P^3(N_1^-) \cap P^{2,2}(N_{(2)}^-) \qquad \dim V_{34} = a^{'} - r \tag{5.65}$$
$$V_{33} = P^3(N_{(2)}^-) \cup \mathrm{span}_{\mathbb{C}}\{N_{(2)}^-\tilde{\mathbf{a}}_0, N_{(2)}^-\bar{\tilde{\mathbf{a}}}_0\} \qquad \dim V_{33} = 2(r+1) \tag{5.66}$$
$$V_{32} = N_2^-(P^3(N_1^-) \cap P^{2,2}(N_{(2)}^-)) \qquad \dim V_{32} = a^{'} - r \tag{5.67}$$
$$V_{22} = N_1^- P^{2,2}(N_1^-) \qquad \dim V_{22} = a \tag{5.68}$$
$$V_{31} = \mathrm{span}_{\mathbb{C}}\{(N_2^-)^2\tilde{\mathbf{a}}_0, (N_2^-)^2\bar{\tilde{\mathbf{a}}}_0\} \qquad \dim V_{31} = 2 \tag{5.69}$$

Based on these vector spaces we can find a real basis each of these vector spaces except for V_{33} which will be split into real and imaginary parts. More precisely we make the following identifications

$$\alpha^0, \alpha^1 \simeq e_{35}^1, e_{35}^2 \qquad \beta^0, \beta^1 \simeq e_{31}^1, e_{31}^2 \tag{5.70}$$
$$\alpha^2, \dots, \alpha^{1+a} \simeq e_{44}^i \qquad \beta^2, \dots, \beta^{1+a} \simeq e_{22}^i \tag{5.71}$$
$$\alpha^{2+a}, \dots, \alpha^{1+a+a^{'}-r} \simeq e_{34}^k \qquad \beta^{2+a}, \dots, \beta^{1+a+a^{'}-r} \simeq e_{32}^k \tag{5.72}$$
$$\alpha^{2+a+a^{'}-r}, \dots, \alpha^{h^{2,1}} \simeq \mathrm{Re}(e_{33}^m) \qquad \beta^{2+a+a^{'}-r}, \dots, \beta^{h^{2,1}} \simeq \mathrm{Im}(e_{33}^m) \tag{5.73}$$

With this identifiation we essentially have the following decomposition of real threeforms

$$H^3(CY_3, \mathbb{R}) = V_{35} \oplus V_{44} \oplus V_{34} \oplus \mathrm{Re}\, V_{33} \oplus \mathrm{Im}\, V_{33} \oplus V_{32} \oplus V_{22} \oplus V_{31} \tag{5.74}$$

This identification can be shown to satisfy (B.17). The next step is to see how N_1^-, $N_{(2)}^-$ maps between subspaces.

$$N_1^- = \left(\begin{array}{c|c|c|c|c|c|c|c|c} & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{5.75}$$

$$N_2^- = \left(\begin{array}{c|c|c|c|c|c|c|c|c} & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & \delta_{j1}\delta_{m1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & \delta_{j2}\delta_{n1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & \mathbf{1} & 0 & 0 & \delta_{i1}\delta_{m1} & \delta_{j2}\delta_{n1} & 0 & 0 & 0 \end{array}\right) \tag{5.76}$$

$$(N_2^-)^2 = \left(\begin{array}{c|c|c|c|c|c|c|c|c} & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{5.77}$$

Then we compute

$$e^{-\phi^i N_i^-} = \left(\begin{array}{c|c|c|c|c|c|c|c|c} & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline 2 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \hline r+1 & -\phi^2\delta_{j1}\delta_{m1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline r+1 & -\phi^2\delta_{j2}\delta_{n1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \hline a^{'}-r & 0 & 0 & -\phi^2\mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ \hline a & 0 & -\phi^{\mathbf{1}} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \hline 2 & \frac{(\phi^2)^2}{2}\mathbf{1} & 0 & 0 & -\phi^2\delta_{i1}\delta_{m1} & -\phi^2\delta_{j2}\delta_{n1} & 0 & 0 & \mathbf{1} \end{array}\right) \tag{5.78}$$

Then in the new basis, just like before, we compute the matrices

$$
\int_{CY_3} \alpha_I \wedge \star \beta^J = \left(\begin{array}{c|c|c|c|c|c|c|c|c}
 & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline
2 & 0 & 0 & 0 & 0 & c_2\delta_{i2}\delta_{m2} & 0 & 0 & c_1 \\ \hline
a & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 \\ \hline
a^{'}-r & 0 & 0 & 0 & 0 & 0 & C_4 & 0 & 0 \\ \hline
r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
r+1 & & & & & 0 & 0 & 0 & 0 \\ \hline
a^{'}-r & & & & & 0 & 0 & 0 & 0 \\ \hline
a & & & & & 0 & 0 & 0 & 0 \\ \hline
2 & & & & & 0 & 0 & 0 & 0
\end{array} \right) \tag{5.79}
$$

$$
\int_{CY_3} \beta^I \wedge \star \beta^J = \left(\begin{array}{c|c|c|c|c|c|c|c|c}
 & 2 & a & a^{'}-r & r+1 & r+1 & a^{'}-r & a & 2 \\ \hline
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
a^{'}-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
r+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
r+1 & 0 & 0 & 0 & 0 & k_{33}^{33}\delta_{mn} & 0 & 0 & 0 \\ \hline
a^{'}-r & 0 & 0 & 0 & 0 & 0 & k_{32}^{34(k)}\delta_{kl} & 0 & \phi^2 k_{31}^{35(2)}\delta_{m1}\delta_{i2} \\ \hline
a & 0 & 0 & 0 & 0 & 0 & 0 & k_{22}^{44(i)}\delta_{ij} & 0 \\ \hline
2 & 0 & 0 & 0 & 0 & 0 & \phi^2 k_{31}^{35(2)}\delta_{m1}\delta_{i2} & 0 & k_{31}^{35(i)}\delta_{ij}
\end{array} \right) \tag{5.80}
$$

Which in turn give [9]

$$
\begin{aligned}
\mathrm{Re}\,\mathcal{M}_{IJ} &= \left(\int_{CY_3} \alpha_I \wedge \star \beta^K \right) \left(\int_{CY_3} \beta^K \wedge \star \beta^J \right)^{-1} \\
&= \left(\begin{array}{c|c|c|c|c}
 & r+1 & a^{'}-r & a & 2 \\ \hline
2 & c_2 a^{(1)} + (C_1)_{11}\beta^2\delta_{i2}\delta_{m1} & 0 & 0 & c_2\beta^1 + (C_1)_{11}a^{(d+1)}\delta_{i2}\delta_{j1} \\ \hline
a & 0 & 0 & \phi^1\mathbf{1} & 0 \\ \hline
a^{'}-r & 0 & \phi^2\mathbf{1} & 0 & 0 \\ \hline
r+1 & 0 & 0 & 0 & 0
\end{array} \right)
\end{aligned} \tag{5.81}
$$

[9]Here we only demonstrate the right upper part of the theta angle matrix while the parts we do not show vanish.

Moreover we have defined the following constants and matrices

$$\begin{aligned}
C_1 &= -\frac{(\phi^2)^2}{2} k_{31}^{35(j)} \delta_{ij} \\
c_2 &= \phi^2 k_{\bar{3}\bar{3}}^{33(1)} - (\phi^2)^3 k_{31}^{35(2)} \\
a^{(1)} &= \frac{1}{k_{\bar{3}\bar{3}}^{33(1)} - (\phi^2)^2 k_{31}^{34(2)}} \\
\beta^2 &= \frac{-\phi^2}{k_{\bar{3}\bar{3}}^{33(1)} - (\phi)^2 k_{31}^{35(2)}} \\
\beta^1 &= \frac{\phi^2}{(\phi^2)^2 k_{31}^{35(2)} - k_{\bar{3}\bar{3}}^{33(1)}} \\
a^{(d+1)} &= \frac{k_{\bar{3}\bar{3}}^{33(1)}}{(\phi^2)^2 k_{31}^{35(2)} - k_{\bar{3}\bar{3}}^{33(1)} k_{31}^{35(2)}}
\end{aligned} \tag{5.82}$$

We observe that in both cases we discuss in this paragraph it is evident that the inclusion of the axion dependence rotates our basis such that Re $\mathcal{M}$ does not vanish identically. This corresponds as mentioned earlier to slightly moving away from the boundary point. In this case one obtains an expression for Re $\mathcal{M}$ whose dependence on the powers of ϕ differs on each singularity type one considers. In particular it turns out that the types which corresponds to monodromies of higher order, also give higher order corrections in terms of ϕ. This is true for both one modulus and two moduli cases and also fits to the intuition that singularities of higher order, correspond to 'worse' degenerations. Moreover, one obviously notices that setting $\phi = 0$ in the previous expressions gives Re $\mathcal{M} = 0$ as expected. This might seem as a trivial statement given the way we approach the problem. After all, the ϕ dependence is there only once we introduce it, therefore setting the axions to zero tautologically gives Re $\mathcal{M} = 0$. However one needs to always keep in mind what is the task we have at hand. We want to show that there is a way to obtain an electric magnetic basis corresponding to α, β such that for all the limits in the moduli space, there is a universal behaviour of the theta angle matrix. The hope is that once we include saxion dependence of our expressions, approaching the boundary, this dependence will be present only in subleading terms and the leading behaviour is at most allowed to depend on the axions. The reason is that it is generally known that one can not stabilize saxions. However there is a way to stabilize the axions. This is done by turning on fluxes for the ten dimensional fields we started with because in the four dimensional theory they introduce a potential for axions. The minimum of this potential determines the vacuum expectation value of the moduli. More details can be found in [15],[22]. The result of this procedure is that, indeed the theta angle is not a free parameter, complying with a relevant Swampland conjecture in [4], but will be determined by the vev of the axions. We will come back to this point later on.

The next task is to see what happens when one introduces saxion dependence. The hope is that once we approximate the Weil operator with C_{sl2}, which is the one obtained by the Sl_2 orbit theorem, there should be a choice of electric magnetic basis α, β such that the dependence on the saxions relies only on suppressed terms. In the next chapter we will identify such a basis for any type of limit for one and two moduli cases.

5.3 Saxion dependence

The procedure to obtain an expression for $\mathrm{Re}\,\mathcal{M}$ is very similar with the one explained before. An important difference is that since we want to find a basis where all the elements decay, and we know that $\mathrm{Re}\,\mathcal{M}$ scales as the inverse of the norms of the β^I, we should pick β^I to lie in the V_{heavy} since this will lead to a decaying contribution. We will only show two examples here one for one modulus and one for two moduli but the results we mention are tested for all kinds of limits. In the two moduli case we also need to determine the path which we consider for the degeneration. We also give a general expression for one modulus case based on the results of the previous chapter.

One modulus case Let us see how the general expressions (5.21),(5.22) once we introduce the saxions, provided that we now identify β^I with the heavy states and α_J with the light ones. Note that saxions are introduced as in [25], by using the operator $e(s)e^{-\phi N^-}C_\infty$ to approximate the Weil operator. The $e(s)$ operator is defined by the following expression

$$e(s) := \prod_{j=1}^{\hat{n}} \exp\left\{\frac{1}{2}\log\left(s^j\right)Y_j\right\} \tag{5.83}$$

where $\hat{n}$ is the number of fields becoming large. Moreover, it acts as follows on each subspace $V_{\vec{\ell}}$

$$e(s)u_{\vec{\ell}} = \left(\frac{s^1}{s^2}\right)^{\frac{l_1-4}{2}} \cdots \left(\frac{s^{n-1}}{s^n}\right)^{\frac{l_{n-1}-4}{2}} (s^n)^{\frac{l_n-4}{2}} u_{\vec{\ell}}, \quad u_{\vec{\ell}} \in V_{\vec{\ell}} \tag{5.84}$$

We then get the following equations in the new approximation

$$\begin{aligned}\int \beta^I \wedge \star\beta^J &= \langle e^{-\phi N^-}u^n_{i_n}, C_\infty e^{-\phi N^-}u^{n'}_{j_{n'}}\rangle = \\ &\sum_{l^n,l^{n'}=0}^{l^n_{max},l^{n'}_{max}} \frac{(-\phi)^{l^n+l^{n'}}}{(l^n)!(l^{n'})!}\left(s^{\frac{n+n'-2l^n-2l^{n'}-6}{2}}\right)\langle u^{n-2l^n}_{i_{n-2l^n}}, C_\infty u^{n'-2l^{n'}}_{i_{n'-2l^{n'}}}\rangle\end{aligned}. \tag{5.85}$$

where since $n,n' \geq 3$ we have $n+n' \geq 0$ this will be the leading order term. Note that in this expression there will also be terms which actually decay as long as the exponent becomes negative. Moreover we can also generally write

$$\begin{aligned}\int \alpha_I \wedge \star\beta^J &= \langle e^{-\phi N^-}u^m_{i_m}, C_\infty e^{-\phi N^-}u^n_{j_n}\rangle = \\ &\sum_{l^m,l^n=0}^{l^m_{max},l^n_{max}} \frac{(-\phi)^{l^m+l^n}}{(l^m)!(l^n)!}\left(s^{\frac{m+n-2l^m-2l^n-6}{2}}\right)\langle u^{m-2l^m}_{i_{m-2l^m}}, C_\infty u^{n-2l^n}_{i_{n-2l^n}}\rangle\end{aligned} \tag{5.86}$$

where in this case we have $m \leq 3, n \geq 3$ giving indeed only decaying terms.

IV degeneracy Let us now proceed to a concrete example, the type IV degeneracy. We make the following choice of basis. Note that this is the reverse identification from the previous chapter:

$$\beta^0 \simeq e^6, \qquad \beta^i \simeq e^4_i, \qquad \beta^m \simeq e^3_m \tag{5.87}$$

$$\alpha_0 \simeq e^0, \qquad \alpha_i \simeq e^2_i, \qquad \alpha^m \simeq e^{\bar{3}}_m \tag{5.88}$$

where similar conventions as in the previous chapter are used. Given this identification we find the following rotated vectors

$$e'^6 = e(s)e^{-\phi N}e^6 = s^{\frac{3}{2}}e^6 - s^{\frac{1}{2}}e_1^4 + \frac{\phi^2}{2}s^{-\frac{1}{2}}e_1^2 - s^{-\frac{3}{2}}\frac{\phi^3}{6}e_0 \tag{5.89}$$

$$e_i'^4 = e(s)e^{-\phi N}e_i^4 = s^{\frac{1}{2}}e_i^4 - \phi s^{-\frac{1}{2}}e_i^2 + s^{-\frac{3}{2}}\frac{\phi^2}{2}e_0\delta_{i1} \tag{5.90}$$

$$e_m'^3 = e_m^3 \tag{5.91}$$

$$e_m'^{\bar{3}} = e_m^{\bar{3}} \tag{5.92}$$

$$e_i^2 = s^{-\frac{1}{2}}e_i^2 - \phi s^{-\frac{3}{2}}e_0\delta_{i1} \tag{5.93}$$

$$e'^0 = s^{-\frac{3}{2}}e^0 \tag{5.94}$$

Given this data we in turn compute

$$\int \alpha_I \wedge \star\beta^J = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & 0 & 0 & 0 & & & \\ \hline
d & 0 & 0 & 0 & & & \\ \hline
d' & 0 & 0 & 0 & & & \\ \hline
d' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
d & B_2\delta_{i1} & B_3 & 0 & 0 & 0 & 0 \\ \hline
1 & B_1 & B_4\delta_{j1} & 0 & 0 & 0 & 0
\end{array}\right) \int \beta^I \wedge \star\beta^J = \left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & d & d' & d' & d & 1 \\ \hline
1 & C_1 & C_2\delta_{i1} & 0 & 0 & 0 & 0 \\ \hline
d & C_2\delta_{j1} & C_3 & 0 & 0 & 0 & 0 \\ \hline
d' & 0 & 0 & C_4 & 0 & 0 & 0 \\ \hline
d' & 0 & & 0 & 0 & 0 & 0 \\ \hline
d & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{5.95}$$

where we have defined

$$B_1 = -s^{-3}\frac{\phi^3}{6}k_{06} \tag{5.96}$$

$$B_2 = \frac{\phi^2}{2}s^{-1}k_{24}^1 + \frac{\phi^4}{6}s^{-3}k_{06} \tag{5.97}$$

$$B_3 = -\phi s^{-1}k_{24}^i\delta_{ij} - \frac{\phi^3}{2}s^{-3}k_{06}\delta_{i1}\delta_{j1} \tag{5.98}$$

$$B_4 = s^{-3}\frac{\phi^2}{2}k_{06} \tag{5.99}$$

$$C_1 = -s^3k_{60} - sk_{42}^1 + \frac{\phi^4}{4}s^{-1}k_{24}^1 + s^{-3}\frac{\phi^6}{36}k_{06} \tag{5.100}$$

$$C_2 = -sk_{42}^1 \tag{5.101}$$

$$C_3 = -sk_{42}^i\delta_{ij} + \phi^2 s^{-1}k_{24}^i\delta_{ij} + s^{-3}\frac{\phi^4}{4}k_{06}\delta_{i1}\delta_{j1} \tag{5.102}$$

$$C_4 = k_{3\bar{3}}^m\delta_{mn} \tag{5.103}$$

We now observe that in all cases taking $s \to \infty$ we obtain

$$\boxed{\operatorname{Re}\mathcal{M}^{I,II,III,IV}|_{s\to\infty} = 0} \tag{5.104}$$

This is obviously expected since we have decaying contribution from $(\int \beta_I \wedge \star\beta_J)^{-1}$ as well as from the α's. However, one should keep in mind that the above expression is correct up to exponentially decaying terms of the form $s^{-n}, n > 0$. The result is still surprising, it tells us that indeed there is suitable choice of basis, which makes the theta angles vanish near the boundaries of the moduli space.

Two moduli We again demonstrate one example and then comment on the general behaviour of the expressions at interest. We give the example of the enhancement $\mathrm{I}_a \to \mathrm{II}_\mathrm{b}$ and we follow a similar procedure as the one in the previous section, therefore many details are ignored. We make the following basis identification

$$\beta^0, \dots, \beta^{a-1} \simeq e^i_{44} \quad \alpha_0, \dots \alpha_{a-1} \simeq e^j_{22} \tag{5.105}$$

$$\beta^\alpha, \dots, \beta^{a+a^{'}-r+2} \simeq e^k_{34} \quad \alpha_a, \dots, \alpha_{a+a'-r+2} \simeq e^l_{32} \tag{5.106}$$

$$\beta^{a+a^{'}-r+3}, \dots, \beta^{h^{2,1}} \simeq e^m_{33} \quad \alpha_{a+a'-r+3}, \dots, a^{h^{2,1}} \simeq e^m_{\bar{3}\bar{3}} \tag{5.107}$$

Moreover we compute

$e^{-\phi^i N_i^-} =$ (5.108)

	a	$a^{'}-r$	$r+1$	$r+1$	$a^{'}-r$	a
a	$\mathbf{1}$	0	0	0	0	0
$a^{'}-r$	0	$\mathbf{1}$	0	0	0	0
$r+1$	0	0	$\mathbf{1}$	0	0	0
$r+1$	0	0	0	$\mathbf{1}$	0	0
$a^{'}-r$	0	$-\phi^2\mathbf{1}$	0	0	$\mathbf{1}$	0
a	$-\phi^1\mathbf{1}$	0	0	0	0	$\mathbf{1}$

The new basis we get based on the $Sl(2)$ approximation of the Weil operator in the spirit of the procedure in the previous section will be

$$e'^i_{44} = e(s)e^{-\phi N_i}e^i_{44} = (s^1)^{\frac{1}{2}}(s^2)^{\frac{1}{2}}e^i_{44} - \phi^1(s^1)^{-\frac{1}{2}}(s^2)^{-\frac{1}{2}}e^i_{22} \tag{5.109}$$

$$e'^k_{34} = (s^2)^{\frac{1}{2}}e^k_{34} - \phi^2(s^2)^{-\frac{1}{2}} \tag{5.110}$$

$$e'^m_{33} = e^m_{33} \tag{5.111}$$

$$e'^m_{\bar{3}\bar{3}} = e^m_{\bar{3}\bar{3}} \tag{5.112}$$

$$e'^l_{32} = (s^2)^{-\frac{1}{2}}e^l_{32} \tag{5.113}$$

$$e'^j_{22} = (s^1)^{-\frac{1}{2}}(s^2)^{-\frac{1}{2}}e^j_{22} \tag{5.114}$$

with this identification we in turn obtain the following expressions

$\int \alpha_I \wedge \star\beta^J =$ (5.115)

	a	$a^{'}-r$	$r+1$	$r+1$	$a^{'}-r$	a
a	0	0	0			
$a^{'}-r$	0	0	0			
$r+1$	0	0	0			
$r+1$	0	0	0		0	0
$a^{'}-r$	0	A_2	0	0	0	0
a	A_1	0	0	0	0	0

$\int \beta^I \wedge \star\beta^J =$ (5.116)

	a	$a^{'}-r$	$r+1$	$r+1$	$a^{'}-r$	a
a	b_1	0	0	0	0	0
$a^{'}-r$	0	b_2	0	0	0	0
$r+1$	0	0	b_3	0	0	0
$r+1$	0	0	0		0	0
$a^{'}-r$	0	0	0	0	0	0
a	0	0	0	0	0	0

where the parameteres above are defined as

$$b_1 = (s^1)(s^2)^{-1}k_{44}^{22(i)}\delta_{ij} + (\phi^1)^2(s^1)^{-1}(s^2)^{-1}k_{22}^{44(j)}\delta_{ij} \tag{5.117}$$

$$b_2 = (s^2)k_{34}^{32(k)}\delta_{kl} + (\phi^2)^2(s^2)^{-1}k_{32}^{34(k)}\delta_{kl} \tag{5.118}$$

$$b_3 = k_{33}^{\bar{3}\bar{3}(m)}\delta_{mn} \tag{5.119}$$

$$A_1 = -\phi^1(s^1)^{-1}(s^2)^{-1}k_{22}^{44(i)}\delta_{ij} \tag{5.120}$$

$$A_2 = -\phi^2(s^2)^{-1}k_{34}^{32(k)}\delta_{kl} \tag{5.121}$$

Now in order to determine the asymptotic behaviour of the theta angles, we must chose a path. In this case we will pick the following path towards the degeneration

$$\mathcal{R}_{12} = \left\{ \left(t^1, t^2\right) | \frac{s^1}{s^2} > \gamma, s^2 > \gamma, \phi^i < \delta \right\} \tag{5.122}$$

where γ, δ are positive constants. With this path choice taking the limit $\frac{s^1}{s^2} \to \infty, s^2 \to \infty$ we observe that

$$\boxed{\mathrm{Re}\,\mathcal{M}|_{s\to\infty} \to 0} \tag{5.123}$$

where again one ignores subleading polynomial corrections. More impressive is the fact that for any kind of enhancement making the same kind of identifications for our choice of basis, one finds a vanishing theta angle matrix. At this point it might seem that we are proving nothing. Afterall, it looks like we start from the boundary theory, we chose a basis adapted to the data provided by the $Sl(2)$ orbit theorem, then we move away from the boundary point through introducing axion and saxion dependence of the Weil operator and thus the Hodge norm. In the final step, taking $s \to \infty$ one expects that we are now considering again the boundary point. However this is not true at all. The boundary theory is not obtained simply in a continuous way from the 'bulk' theory. This can be seen in a number of ways. The first obvious hint is that, as mentioned earlier, at the boundary, the would be theta angle vanishes identically, since $\mathrm{Im}\,\mathcal{M}$ is diagonal and $\int \alpha_I \wedge \star\beta^J$ is zero, making $\mathrm{Re}\,\mathcal{M}$ vanishing exactly. However, in a case by case analysis one can observe that the non-diagonal terms in $\mathrm{Im}\,\mathcal{M}$ do not vanish (they actually grow as a power law) after making the saxions very large failing to reproduce the boundary theory result. Moreover in the next section we show that there is another choice of basis which gives again different result compared to the boundary theory. This fact demonstrates that there is something special about the theta angles in this context, their values can be chosen to be as small as one wants through moving close enough to the boundary point of the moduli space.

There is an important remark to make here. The choice of basis that we make does not correspond to the usual theory that one studies once moving towards the boundaries of the moduli space as for instance in [26]. This can be noticed from the fact that this basis actually corresponds to a strong coupling limit of the theory. This can be seen from the fact that the gauge coupling matrix decays, which means that the corresponding couplings (which are the inverse of this matrix) grow. The idea is then that this different choice, corresponds to a magnetic description of the theory. Therefore this result should be thought of as a strong coupling one. In the next section we clarify what is the corresponding result for a weakly coupled theory.

5.4 The weak coupling limit

As mentioned earlier, identifying β with V_{heavy} corresponds to a strongly coupled regime of the theory. In that context one finds that approaching the boundaries of the moduli space, all the components of the theta angles vanish up to power law suppresed terms. We would like to also consider the weakly coupled regime for the type IV case. This type as already mentioned corresponds to the large complex structure point, which gets mapped to large volume point of type IIA theory on the mirror manifold. Therefore, mirror symmetry tells us that considering type IV case here, we should reproduce the result of (B.16). More precisely we expect that close to the boundary, the theta angles' leading behaviour corresponding to the real part of $\mathcal{M}$ will be a polynomial in the axions ϕ. Let us see what we find making that analysis. We only demonstrate the result here, the choice of basis is the one that matches α with V_{heavy} and β with V_{light} as we did earlier in (5.39)

$$\mathrm{Re}\,\mathcal{M}|^{IV}_{s\to\infty} = \left(\begin{array}{c|c|c|c} & 1 & d-1 & 1 \\ \hline 1 & \frac{\phi^2}{2}k_{06} & 0 & \phi-\frac{\phi^3}{6}k_{06} \\ \hline 1 & \phi & 0 & \frac{\phi^2}{k_{06}}+\frac{\phi^3}{2} \\ \hline d-1 & 0 & -\phi\mathbf{1} & 0 \end{array}\right) \tag{5.124}$$

Again, this expression is valid up to power law suppressed terms. The behaviour that we find is exactly the expected one. The real part of $\mathcal{M}$ close to the large volume-large complex structure point, does not depend on the saxions to leading order. In contrast all the non zero terms are polynomials of maximum order 3 as expected. Note that this is indeed the same qualitatively as in (B.16). However the formalism we develop goes beyond the already known behaviour at type IV degenerations. In particular we find that in all kinds of one modulus cases, the leading behaviour of the theta angle matrix is still a polynomial of maximum order the number defined by the type of degeneration. More precisely we have

$$\boxed{\mathrm{Re}\,\mathcal{M}_{IJ}|^{I,II,III}_{s\to\infty} = a_1\phi + a_2\phi^2} \tag{5.125}$$

where a_1 and a_2 are constants related to the norms of the vectors at infinity as defined in (5.12) . This expression means that all the leading terms of this matrix are polynomials in terms of the axions ϕ without a constant where the subleading terms decay as power law expressions in terms of the saxions. This is a remarkable fact. Not only we manage to reproduce the known behaviour for the large volume-large complex structure case, but we also find explicit expressions for all types of possible ways that the manifold degenerates. This demonstrates the power and the generality of the techniques we use. In this context, it is clear that stabilizing the axions is the next natural step as we have already mentioned. However the interesting fact is that also in this basis, we manage to find a universal behaviour of the theta angle matrices which reflects something very special about it in the context of Calabi Yau compactifications. The special property it has is that in both choices of basis corresponding to strongly and weakly coupled EFT, there is a universal behaviour as we approach the boundary. In the strongly coupled case every element of the matrix decays to 0 while in the weakly coupled one the only remaining leading terms depend polynomially on the vev of the axion and there is no constant.

Chapter 6

Summary and comments on the results

In this section we would like to give the outline of the work done in this thesis and propose possible extensions for the future. The main task was investigating the proposal from Vafa and Cecotti regarding a possible connection between that the theta problem and the Swampland. The claim is that quantum gravitational consistency puts restrictions to the values that the theta angle can take, and these restrictions might be exactly the values that preserve CP symmetry. We investigated this claim in the context of CY_3 compactification of Type IIB strings. Focusing on the boundaries of the moduli space, using the data from $Sl(2)$ orbit theorem we test the behaviour of $\mathrm{Re}\,\mathcal{M}$ as one moves towards the boundary. To do so we start from the boundary point and gradually move away from it through introducing axion and saxion dependence. We find that in the choice of basis which corresponds the strong coupling regime of the EFT, there is a universal behaviour on any type of one modulus singularity and the enhancements between these. This universal behaviour is that every element of the matrix decays as a power law expression. Moreover, we also test for one degenerating modulus the behaviour on the weak coupling basis. In this case there is also a universal behaviour for any type of singularity, since in the limit of large saxions, we find that the leading behaviour is a polynomial expression without a constant term, in ϕ and the saxions appear only in subleading terms. This fact, apart from matching with the already known expressions of type IIA in the mirror picture, gives us a generalization thereof, since the machinery developed here works for any type of singularity not only the large volume-large complex structure one. In some sense these results indeed verify the hopes of the conjecture since indeed the theta angles in this context have a restricted behaviour at least once we approach the boundaries of the moduli space which, depending on the basis, corresponds to either weak or strong coupling limit of the theory. Note that this is the best that one can do, since having a 'continuous' moduli space here where the moduli fields are not restricted by any potential, the only essential limits to consider are the ones we do. For a general point of a moduli space it is not expected to obtain any special behaviour. This is in contrast to the original test of Vafa and Cecotti where they only consider a small number of rigid Calabi-Yau manifolds (no complex structure moduli space) and therefore test around 50 cases. In this work the approach we follow treats any topologically distinct Calabi-Yau threefold which corresponds to thousands of possibilities. Finally it is important to remember from the discussion in section 2.4 that the vector multiplet sector which is related with the complex structure moduli space in type IIB compactification, is classically exact, there are no quantum corrections neither in α' nor in the string coupling g_s.

It is fair to say that the work done in this thesis, should be seen as a first step towards the investigation of the nature of the theta angles in terms of the Swampland. The aim is mostly to shed light in the ideas of the Swampland, as well as the usefulness of the machinery coming from asymptotic Hodge theory in this context. However, the results we obtain give us clear directions regarding the next steps that one should make in the future. More precisely, as already mentioned, in this context there is nothing to restrict the vevs of the axions. Therefore an immediate question is what would happen if we turned on fluxes, which generate a potential for the axions and therefore fix their vevs. The hope is that it will be possible to fix these vevs at $\langle \phi \rangle = 0$. Then the theta angles, even in the weak coupling limit will vanish, since they depend only on powers of ϕ. Another unsatisfactory feature of our setup is the gauge group under consideration. The EFT we have is a gauge theory containing a bunch of abelian gauge fields and therefore the total gauge group is $U(1)^{h^{2,1}+1}$. This is obviously very different from the QCD gauge group which is $SU(3)$. Therefore, another step we would like to make in the future is to enhance gauge groups which are closer to the QCD onw. This can be done by the introduction of extended objects in the theory such as D-branes [37]. Finally, we believe that it is possible to reproduce the results of this thesis in a general context. Namely we believe that one can come up with general expressions without referring to every different type of singularity separately. This task is also left for future work.

Appendix A

Kähler and Calabi Yau Manifolds

In this appendix the basic properties that we need about Kähler and Calabi Yau manifolds will be presented. The basis however for these, is the definition of a complex manifold. More details can be found in [3].

Definition A.0.1. *Let M be a $2m$ dimensional real manifold. We define an almost complex structure J on M to be a smooth tensor field $J \in \Gamma(TM \otimes T^{\star}M)$ on M such that fiberwise it squares to the identity $J : V \to V$ with $J \circ J = -1$.*

Then naturally we call a manifold which admits an almost complex structure, an almost complex manifold.

Definition A.0.2. *We define the Nijenhuis tensor N_J on an almost complex manifold to be the tensor field acting on $(u, w) \in \Gamma(TM) \otimes \Gamma(TM)$ as*

$$N_J(u, w) = [u, w] + J[u, Jw] + J[Ju, w] - [Ju, Jw] \tag{A.1}$$

Now we can finally define what a complex manifold is.

Definition A.0.3. *Let M be a $2m$ dimensional real manifold and J an almost complex structure on M. If $N_J \equiv 0$ we call J a complex structure on M. A complex manifold is defined by the data (M, J) where J is a complex structure on M.*

This definition might seem too technical. After all our intuition would be that as a real m dimensional manifold looks locally like $\mathbb{R}^m$ then an m dimensional complex manifold should locally look like $\mathbb{C}^n$. Actually there is a theorem which states that this is indeed the case, in particular one can define complex manifolds similarly to real manifolds but with holomorphic transition functions. However, the given definition here is more useful when one wants to speak about more complicated constructions such as Kähler manifolds. The intuition behind the definition of a complex manifold is similar to the definition of complex number. The complex structure J gives us a decomposition of the complexified tangent space of the real manifold into two parts, the ones that have eigenvalue i w.r.t the J action and the ones that have eigenvalue $-i$ under the J action, more precisely $TM \otimes \mathbf{C} = T^{(0,1)}M \oplus T^{(0,1)}M$. Next, we want to define Kähler manifolds and the following definition is required.

Definition A.0.4. *Let (M, J) be a complex manifold, and let g be a Riemannian metric on M. We call g an Hermitian metric if the following condition holds*

$$g(u, w) = g(Ju, Jw) \qquad \textit{where } (u, w) \in \Gamma(TM) \otimes \Gamma(TM) \tag{A.2}$$

In other words, one intuitively understands that an Hermitian metric is a possitive definite inner product $T^{(1,0)}M \otimes T^{(0,1)}$ at every point of the complex manifold M.

Given these data one can define an Hermitian form ω such that $\omega(u, w) = g(Ju, w)$ [1] for any vector fields u, w in M. One can show that this is indeed an antisymmetric expression and in particular that ω constructed in this way is a $(1, 1)$ form.

Definition A.0.5. *Let (M, J) be a complex manifold and g an Hermitian metric on M with Hermitian form ω. g is called a Kähler metric if $d\omega = 0$. In this case ω is called a Kähler form and we call the complex manifold (M, J) endowed with a Kähler metric, a Kähler manifold.*

Notice that essentially a Kähler manifold simultaneously has a symplectic structure (because ω is closed) and a complex structure (provided by J). Another important remark is that one can show that any (1,1) form on a Kähler manifold can be locally written as $\omega = dd^cK$ where K is called the Kähler potential and d^c acts on the space of complexified forms sending a k form to a k+1 form.

We are now ready to define Calabi Yau manifolds. There is big number of equivalent ways in which one can give a definition, with different choices reflecting different properties one is interested in.

Definition A.0.6. *A Calabi-Yau manifold of real dimension 2m is a compact Kähler manifold (M, J, g)*

- *with zero Ricci form*
- *with vanishing first class*
- *with $Hol(g) = SU(m)$ (or $Hol(g) \subseteq SU(m)$)*
- *with trivial canonical bundle*
- *that admits a globally defined nowhere vanishing holomorphic m-form.*

For our purposes only the first, the third and the fifth property are important but all the previous definitions are equivalent. Given these properties and defining $h^{p,q} = dimH^{p,q}_{\bar{\partial}}(CY_3, \mathbb{C})$ where $H_{\bar{\partial}}$ is the Dolbeault Cohomology one can construct the hodge diamond of the Calabi Yau manifold which is simply a represenation of the dimensions of its cohomology groups

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Figure A.1: Dimensions of cohomology groups of Calabi Yau threefolds

[1]The definition of ω implies that it is essentially just the complex structure with one index lowered. However it turns out that the Kähler form contains the same information as the Hermitian metric, which can be seen by picking an eigenbasis for the complex structure. After all , if one chooses to rewrite everything in real coordinates then ω only depends on the metric.

Appendix B

Moduli space of CY_3

In the context of Calabi Yau compactifications, one always has to deal with massless fields, namely fields which do not have any potential to restrict their behaviour. It turns out that these fields, which we call moduli, have a geometric significance, they parametrise the different inequivalent Calabi Yau manifolds with fixed topology. We want to present the basics of the space of parameters for CY_3 which we call moduli space, mostly based on [7]. A very exciting mathematical conjecture, whose proof was partly also initiated by its importance in physics, is the Calabi conjecture, stated by Calabi (1954,1957) and proved by Yau (1977,1978). The conjecture (nowadays a theorem) states that for given a Kähler manifold (M, J_0) where J_0 the Kähler form, there is a unique Ricci flat metric for M whose associated Kähler form J is in the same Cohomology class J_0. This means that the changes we are allowed to make can be parametrized by the different possible complex structures, and the corresponding possible Ricci flat metrics. . Looking at figure A it is not hard to suspect that the only possible parameters one should keep unchanged is the dimensions of the Cohomology groups. It actually turns out that for CY manifolds with $H^{2,0}CY = 0$ the two type of deformations are independent. Now, recall that one of the properties of such manifolds is that they are Ricci flat. Therefore a deformation thereof should satisfy the following equation

$$R_{mn}(g) = 0, \qquad R_{mn}(g + \delta g) = 0 \tag{B.1}$$

where the indices run from $m, n = 0, \ldots, 6$.
However we want to make sure that we only consider non-coordinate transformations . To do so we impose the additional condition $\nabla^n \delta g_{mn} = 0$. This gives us the following condition on the deformations known as Lichnerowicz equation

$$\nabla^l \nabla_l g_{mn} - \big[\nabla^l, \nabla_m\big] \delta g_{ln} - \big[\nabla^l, \nabla_n\big] \delta g_{lm} = 0 \tag{B.2}$$

The above equation now splits into two, one with mixed indices(real-complex) $\delta g_{\mu\bar{\nu}}$ and one with pure ones $\delta g_{\mu\nu}$ (or $\delta g_{\bar{\mu}\bar{\nu}}$), since all these components satisfy the equation independently. It turns out that the first type of variations are related to changes in the Kähler structure (metric), and are parametrised by elements in $H^{1,1}(CY_3)$ while the latters are connected with the complex structure deformations and are parametrized by $H^1(TCY_3) \simeq H^{(2,1)}(CY_3)$. More practically, this means that the mixed and pure index deformations of the metric, belong to the aforementioned Cohomology groups and can therefore be expanded in a basis consisting of (1,1) and (2,1) forms respectively. The coefficients of these decompositions will correspond to coordinates on the space of deformations, which turns out to be itself a manifold. This manifold decomposes as a product of complex structure and Kähler structure deformations $\mathcal{M} = \mathcal{M}^{2,1} \times \mathcal{M}^{1,1}$. Each of these manifolds turn to be Special

Kähler which means that apart from being Kähler they have an additional interesting property to which we will refer later. From the physics point of view, after compactification the coordinates of these manifolds correspond to massless scalar fields from a 4 dimensional point of view and are called moduli fields.

B.1 Kähler structure moduli space

As mentioned earlier the Kähler structure deformations turn out to be parametrized by $H^{1,1}(CY_3)$. By a small abuse of notation we will denote J as the Kähler form and expand it in a (real) basis ω_i of $H^{1,1}(CY_3)$

$$J = \sum_{i=1}^{h^{1,1}} u^i \omega_i \tag{B.3}$$

However in String theory we know that there is another two form which does not come from the geometry but is a result of the field content, that is the NS-NS form B_2. This means that actually the variations we can consider without chaging the topology should be parametrized by $B_2 + iJ$. The moduli fields are then given by $t^i = b^i + iu^i$. Note that the previous types of deformations, need to result in a possitive definite metric. This set of such deformations are called the complexified Kähler cone. With these at hand we can define the following quantities

$$\mathcal{K} = \frac{1}{6}\int_Y J \wedge J \wedge J \tag{B.4}$$

$$\mathcal{K}_i = \int_Y \omega_i \wedge J \wedge J \tag{B.5}$$

$$\mathcal{K}_{ij} = \int_Y \omega^i \wedge \omega^j \wedge J \tag{B.6}$$

$$\mathcal{K}_{ijk} = \int_Y \omega^i \wedge \omega^j \wedge \omega^k \tag{B.7}$$

It is evident that these objects satisfy the following identities

$$\mathcal{K}_{ijk} v^k = \mathcal{K}_{ij} \tag{B.8}$$

$$\mathcal{K}_{ij} v^j = \kappa_i \tag{B.9}$$

$$\mathcal{K}_i v^i = 6\mathcal{K} \tag{B.10}$$

Next we want to define a metric on the complexified Kähler cone as follows

$$g_{ij} = \frac{1}{4\mathcal{K}} \int \omega_i \wedge *\omega_j = -\frac{1}{4\mathcal{K}}\left(\mathcal{K}_{ij} - \frac{1}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j\right) \tag{B.11}$$

As mentioned earlier, the moduli space of Kähler deformations is itself a Kähler manifold, for which we define the following Kähler potential K

$$e^{-K} = 8\mathcal{K} \tag{B.12}$$

We also mentioned that it is not only Kähler, but in fact it is special Kähler. This in short means that there is always a choice of symplectic basis $(X_I, \mathcal{F}_I)^T$ with $X^I = (1, t^i)$ such that there is a function $\mathcal{F}$ satisfying

$$e^{-K} = i\left(\bar{X}^I \mathcal{F}_I - X^I \overline{\mathcal{F}}_I\right), \quad \mathcal{F}_I \equiv \frac{\partial}{\partial X^I}\mathcal{F} \tag{B.13}$$

with

$$\mathcal{F} = -\frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0} \tag{B.14}$$

More details about this geometry can be found in [14].
The vectors in the vector multiplets of type IIA and the scalars in the hypermultiplets of type IIB couple through the following matrix

$$\mathcal{N}_{IJ} = \overline{\mathcal{F}}_{IJ} + \frac{2i}{X^P \operatorname{Im}\mathcal{F}_{PQ}X^Q}\operatorname{Im}\mathcal{F}_{IK}X^K \operatorname{Im}\mathcal{F}_{JL}X^L \tag{B.15}$$

More precisely we distinguish between the imaginary and real part of this matrix, which is physically identified in the type IIA case with the gauge couplings and theta angles of our U(1) fields as follows

$$\begin{aligned}
\operatorname{Re}\mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k, \operatorname{Im}\mathcal{N}_{00} = -\mathcal{K} + \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^ib^j \\
\operatorname{Re}\mathcal{N}_{i0} &= \frac{1}{2}\mathcal{K}_{ijk}b^jb^k, \operatorname{Im}\mathcal{N}_{i0} = -\left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^j \\
\operatorname{Re}\mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k, \operatorname{Im}\mathcal{N}_{ij} = \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)
\end{aligned} \tag{B.16}$$

B.2 Complex structure moduli space

We now turn to the complex structure deformations which as mentioned earlier are parametrized by elements in $H^{(2,1)}$. Recalling that the only other three forms on a Calabi Yau manifold are given by the unique holomorphic (3,0) form Ω and its complex conjugate $\bar{\Omega}$ we conclude that probably a basis of real threeforms would be useful in the description of this moduli space. To see that this is indeed the case we can start from a canonical homology basis of $H_3(CY_3, \mathbb{Z})$, (A^A, B_B), $a, b = 0, \ldots, h^{2,1}$ and the Poincaré dual real cohomology basis α_A, β^B such that

$$\int_{A^b}\alpha_A = \int_{CY_3}\alpha_A \wedge \beta^B = \delta_A^B, \quad \int_{B_A}\beta^B = \int_{CY_3}\beta^B \wedge \alpha_A = -\delta_A^B \tag{B.17}$$

The above choice of basis is invariant under the symplectic group $Sp(2(h^{2,1}+1), \mathbb{Z})$. The different possible choices of the basis vectors physically correspond to defining what electric and magnetic states mean [14]. Given these, and the decomposition of the real three forms $H^3(M) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ we can expand the holomorphic (3,0) form as

$$\Omega = Z^A\alpha_A - \mathcal{F}_A\beta^A \tag{B.18}$$

where we have defined

$$Z^A = \int_{A^a}\Omega, \quad \mathcal{F}_A = \int_{B_a}\Omega \tag{B.19}$$

Moreover, observing that Ω is homogenous of degree one, means that a rescaling does not alter the definition of Z^A thus we conclude that only $h^{2,1}$ of them are independent. This can either be formulated by saying that they are projective coordinates $Z^A \in \mathbb{P}^{2,1}$ or by writing $Z^A = (1, z^a)$. These can be seen as coordinates for the moduli space. However, because only the Z's are enough to represent coordinates of the manifold, we conclude that the F_A should be functions of $Z's$. In particular one can derive that

$$\mathcal{F}_A = \frac{\partial \mathcal{F}}{\partial Z^A} \tag{B.20}$$

where $\mathcal{F}$ is again a homogenous function of degree two called the prepotential. Using these coordinates we can moreover derive the famous formula of Kodaira

$$\frac{\partial}{\partial z^a}\Omega = c_a \Omega + i\eta_a \tag{B.21}$$

where $\eta_a \in H^{2,1}$ and c_a are constants w.r.t. Calabi Yau coordinates.

With this expression at hand one can now define the metric on the moduli space of complex structure deformations

$$g_{a\bar{b}} = -\frac{i\int_{CY_3} \eta_a \wedge \bar{\eta}_b}{\int_{CY_3} \Omega \wedge \bar{\Omega}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial z^b}\left(-\ln\left(i\int_{CY_3} \Omega \wedge \bar{\Omega}\right)\right) = \partial_a \partial_{\bar{b}} K \tag{B.22}$$

By this expression one concludes that the Kähler potential for the moduli space of complex structure deformations is given by

$$e^{-K} = i\int_{CY_3} \Omega \wedge \bar{\Omega} = i\left(\bar{Z}^A G_A - Z^A \bar{G}_A\right) \tag{B.23}$$

Finally we want to give the expressions which appear as couplings of the vectors in the vector multiplets of type IIB and scalars of hypermultiplets of type IIA

$$\int \alpha_A \wedge \star\alpha_B = -\left(\mathrm{Im}\,\mathcal{M} + (\mathrm{Re}\,\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}(\mathrm{Re}\,\mathcal{M})\right)_{AB} \tag{B.24}$$

$$\int \beta^A \wedge \star\beta^B = -((\mathrm{Im}\,\mathcal{M})^{-1})^{AB} \tag{B.25}$$

$$\int \alpha_A \wedge \star\beta^B = -\left((\mathrm{Re}\,\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}\right)_A^{\;B} \tag{B.26}$$

Appendix C

Deriving enhancement patterns

In this section we give the general analysis of determining the allowed enhancement patterns. The analysis will be based on [31]. The first step is to consider a function which quantifies the HD (Hodge-Deligne) diamonds one has at hand and allows for manipulations thereof. Given a HD diamond with hodge number $\{i^{p,q}\}$ we define an integer valued function $\diamond(p,q) := i^{p,q}$ on the lattice $\mathbb{Z}\times\mathbb{Z}$. Then in turn, a diamond of a variation of weight-w Hodge structure polarized by some nilpotent matrix N with hodge numbers $(h^{w,0}, h^{w-1,1}, \ldots, h^{0,w})$ is abstractly defined as any integer valued function $\diamond(p,q)$ on the lattice $\mathbb{Z}\times\mathbb{Z}$ such that

$$\sum_{q=0}^{w} \diamond(p,q) = h^{p,w-p}, \text{ for all p} \tag{C.1}$$

and satisfying the symmetry properties

$$\diamond(p,q) = \diamond(q,p) = \diamond(w-q, w-p), \text{ for all } p, q \tag{C.2}$$

$$\diamond(p-1, q-1) \leq \diamond(p,q), \text{ for } p+q \leq w \tag{C.3}$$

It will be also useful to define a summation as well as a shift of this function. The sum of two diamonds $\diamond_1 + \diamond_2$ corresponding to variation of the same Hodge structure (not neccessarily of the same weight) is naturally defined as the pointwise sum, giving the diamond

$$\diamond(p,q) = \diamond_1(p,q) + \diamond_2(p,q) \tag{C.4}$$

Moreover, the shifted Hodge-Deligne diamond $\diamond[a]$ which correspond to $[a]$ shifts of the diamond $\diamond$ is defined as

$$\diamond[a](p,q) = \diamond(p+a, q+q) \tag{C.5}$$

Having defined all the required objects we are now ready to answer the question we are interested in. Assume we have two nilpotent orbits at hand (F_1, N_1) and (F_2, N_2) with corresponding diamonds $\diamond(F_1,N_1), \diamond(F_2,N_2)$ and we are interested in whether the degeneration $(F_1,N_1) \to (F_2,N_2)$ is possible. In order to answer this question one must look at the primitive subspaces $P^k(N_1), 3 \leq k \leq 6$ of the limiting hodge structure $(F_1, W(N_1))$ corresponding to (F_1, N_1). These subspaces turn out to form a pure Hodge structure of weight-k since $P^k(N_1) = \oplus_{p+q=k} P^{p,q}(N_1)$ and hodge numbers $j_1^{p,q} = dim_{\mathbb{C}} P^{p,q}(N_1)$. When one wants to consider the enhancement mentioned before, what happens is that N_2^- induces a mixed Hodge structure on the primitive subspaces. It is very important that the matrices N_1^-, N_2^- are matrices corresponding to commuting $\mathfrak{sl}_2$ triplets, which allows for the positions of vectors on each subspace to be independent since the operators commute. Then denoting $(F_k^{'}, N_k^{'})$ these induced mixed Hodge structures, the enhancement is possible when the following equation holds

$$\diamond\,(F_2, N_2) = \sum_{\substack{3\leq k\leq 6\\ 0\leq a\leq k-3}} \diamond\,(F_k', N_k')\,[a] \tag{C.6}$$

The intuition behind this statement is that after inducing the mixed hodge structure on the primitive subpsaces of $(F_1, W(N_1))$, if one can reassemble the resulting diamonds into the ones corresponding to $(F_2, W(N_2))$ then the enhancement is possible. One can see it in a different way by looking at the eigenspaces of the Y operators from the $\mathfrak{sl}(2,\mathbb{C})$-triplet. More precisely, when it is possible to find A basis for H^3CY_3, which simultanesouly diagonalizes Y_1 and Y_2 then the enhancement is possible. This is actually a statement that is very useful for this work.

Appendix D

$Sl(2)$-splitting data

We have extensively used the results of the Sl(2) orbit theorem regarding the possibility of obtaining an $\mathbb{R}$ split filtration $\hat{F}$ as well as a number of commuting $\mathfrak{sl}_2$-triples acting on $H_3(CY_3)$. These two properties are very important for the present work and for that reason we would like to expand in this appendix on the way that one constructs these objects given only the data coming from the MHS $(V, F, W(N))$. The mathematically rigorous details and proofs can be found in [10] but we will mostly follow the appendix of [24].

D.1 $\mathbb{R}$-split decomposition

The central statement we need to investigate is that given a MHS with data $(V, F, W(N))$, there are operators δ, ζ, such that the MHS with data $(V, \hat{F}, W(N))$, where $\hat{F} = e^{\zeta} e^{-i\delta} F$, splits over $\mathbb{R}$. The difference between the initial decomposition and the one which is $\mathbb{R}$-split lies on the fact that the corresponding Deligne splitting in the first case satisfies the following relation

$$\overline{I^{p,q}} \equiv I^{q,p} \bmod \bigoplus_{\substack{r \le q \\ s < p}} I^{r,s}. \tag{D.1}$$

The difference therefore with the second situation lies on the dependence on the lower weight subspaces. Therefore we should find a proper rotation to eliminate those. We start from the initial data $(V, F, W(N)$ where we have the decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=l} I^{p,q}$. We know that associated to this splitting there is a semi-simple operator T which acts on $I^{p,q}$ as multiplication by $l = p + q$. More details about its nature can be found in [10]. The complex conjugate of this operator is defined as

$$\overline{T}(v) := \overline{T(\bar{v})}, \quad v \in V_{\mathbb{C}} \tag{D.2}$$

Then $\overline{T}$ and T should be related by some conjugation of the form

$$\bar{T} = e^{-2i\delta} T e^{2i\delta} \tag{D.3}$$

where the operator δ acts as follows

$$\delta\left(I^{p,q}\right) \subset \bigoplus_{\substack{r<p \\ s<q}} I^{r,s}, \text{ for all } p, q \tag{D.4}$$

The reason for this relation between the grading operator and its complex conjugate is more involved than we demonstrate but the details are not in our interest at this point. What we are interested in is how to compute δ. The strategy should be to use the (D.4) as an ansantz, plug it in (D.3) and solve the equation. It turns out actually that because δ preserves the polarisation, namely $\delta^T\eta + \eta\delta = 0$, since it commutes with N, the solution to the equation (D.3) is actually unique.

The next task is to find ζ. We start with the new filtration $\tilde{F} = e^{-i\delta}F$ at hand. We now have the following Deligne decomposition for the $\mathbb{R}$-split filtration $\tilde{F}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=l} \tilde{I}^{p,q} \tag{D.5}$$

This decomposition induces a further decomposition for the operators themselves, where each component correspond to the action on each of the constituents of the above direct sum. Given this, we can write

$$\delta = \sum_{p,q>0} \delta_{-p,-q}. \tag{D.6}$$

Each component acts as follows

$$\delta_{-p,-q}\left(\tilde{I}^{r,s}\right) \subset \tilde{I}^{r-p,s-q}, \text{ for all } r, s. \tag{D.7}$$

A similar decomposition can be constructed for the ζ operator as well

$$\zeta = \sum_{p,q>0} \zeta_{-p,-q}. \tag{D.8}$$

The relation between the two operators can be found in Lemma 6.60 of [10] to be given by the following expression

$$e^{\mathrm{i}\delta} = e^{\zeta}\left(\sum_{k\geq 0} \frac{(-\mathrm{i})^k}{k!}\,\mathrm{ad}_N^k\left(\tilde{g}_k\right)\right) \tag{D.9}$$

where $\mathrm{ad}_N = [N,\cdot]$ is the usual adjoint action. Moreover, $\tilde{g}_k$ is a real operator which preserves the polarization η of the real vector space V but whose details are not important here. The Lemma 6.60 also tells us that there is a way to write ζ as a polynomial in terms of the components $\delta_{-p,-q}$ and commutators thereof. The important statement in this section is that there is always a way to obtain an $\mathbb{R}$-split decomposition for the threeforms even if there might be practical difficulties. This fact is very crucial in the identifications we make with the real basis of threeforms in the main text, since in particular for the V_{rest} vectors, without an $\mathbb{R}$-split decomposition the choice of basis would become extremelly hard and unclear. The existence of such a filtration is very surprising because it creates some kind of decoupling of the corresponding vector spaces $I^{p,q}$ which allows for many applications and simplifies the expressions. However, the physical interpretation of the rotations we need to perform in order to achieve this, is not yet clear.

D.2 $Sl(2)$-triples

In this section we would like to give an idea about how one constructs the $\mathfrak{sl}_2$ triplets which come about through the $Sl(2)$ orbit theorem. The main tool we have at hand is evidently the filtration data $(V, F, W(N))$, and then triplets $(N^-_{(i)}, Y_{(i)}, N^+_i)$ should be constructed based on this tool. The starting point will be the limiting mixed Hodge structure $(F_\infty, W^{n_\varepsilon})$ associated to the intersection

$\Delta_{1,\ldots,n_\varepsilon}$ as predicted from the nilpotent orbit theorem and $W^{n_\varepsilon} = W(N_1 + N_2 + \ldots + N_\varepsilon)$. Then in order to construct the n_ε triplets one has to make n_ε iterations as follows. For clarity we will denote the input filtration on each iteration k as (F', W^k), therefore the first iteration is for $k = n_\varepsilon$. The next step is to compute the corresponding Sl(2) splitting of (F', W^k) which we will denote (F_k, W^k). Associated to the $Sl(2)$ splitting we also compute the Deligne splitting such that

$$V_{\mathrm{C}} = \bigoplus_{p,q} I^{p,q}_{\left(F_k, W^k\right)} \tag{D.10}$$

We know that there is a semisimple grading operator $Y_{(k)}$ acting on these subspaces as follows

$$Y_{(k)}v = (p+q-3)v, \text{ for every } v \in I^{p,q}_{\left(F_k, W^k\right)} \tag{D.11}$$

The last step is to set $(e^{iN_k}F_k, W^{k-1})$ as the input of the next iteration and the loop stops at $k = 0$ and we set $Y_{(0)} = 0$. This procedure provides us with a series of grading operator Y_{n_ε} and corresponding $\mathbb{R}$-split Hodge structures $(F_{n_\varepsilon}, W^{n_\varepsilon}), \ldots, (F_1, W^1)$. Next we want to construct the lowering operators N_i^-. These are obtained in two steps. First we diagonalize the adjoint action of $Y_{(i-1)}$ on N_i through the decomposition

$$N_i = \sum_\alpha N_i^\alpha \tag{D.12}$$

such that each N_i^α satisfies

$$\left[Y_{(i-1)}, N_i^\alpha\right] = \alpha N_i^\alpha \tag{D.13}$$

Then the lowering operators are defined as the ones corresponding to the components with zero eigenvalues, namely $N_i^- := N_i^0$. The grading operators attached to each triple are set to be

$$Y_i = Y_{(i)} - Y_{(i-1)} \tag{D.14}$$

Then the only remaining object to compute is the raising operators N_i^+. They satisfy the usual $sl(2)$ commutator relations

$$\left[Y_i, N_i^+\right] = 2N_i^+, \quad \left[N_i^+, N_i^-\right] = Y_i \tag{D.15}$$

They can be uniquely determined by solving the above equations given the fact that they should also preserve the polarisation such that

$$\left(N_i^+\right)^{\mathrm{T}} \eta + \eta N_i^+ = 0 \tag{D.16}$$

This concludes the data of the $\mathfrak{sl}_2$ triples $\left(N_i^-, N_i^+, Y_i\right)$ for $i = 1, \ldots, n_\varepsilon$. The detailed statement of the theorem and its proof can be found in theorem 4.20 of [10]. Note that for one modulus, since $Y_{(0)} = 0$ we have $N = N^-$.

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