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MASTER'S THESIS

THEORETICAL PHYSICS

A Field-Theoretic Approach to Fermionic Cold Dark Matter

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Abstract

Inspired by the novel theory developed by Friedrich and Prokopec based on scalar dark matter, we derive the corresponding theory for Dirac fermions. Even though much interest is recently shown in ultra-light scalar dark matter, very little is known about fermionic dark matter. While exhibiting similarities, the fermionic dark matter theory is much richer in possibilities, such as particle-antiparticle and helicity mixing. The theory finds applications in large-scale structure formation, and also allows for a description of dark matter from sub-dominant neutrinos. By starting from a fundamental Lagrangian theory, we take a heavy-particle limit and show that the resulting equations reduce to the classical dark matter description, while allowing for systematic treatment of gradient as well as relativistic corrections. We derive the integro-differential equations for fermionic phase-space densities, resulting in a generalized field-theoretic version of a Vlasov-Poisson equation.

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I very much doubt I would have gotten this close to the end of my studies if it wasn't for JK, whose trust and support throughout my master's were invaluable to me.

1 Introduction

Everything that we can see around us is made up of baryonic matter. Looking up in the sky, we see the sun, the moon and the stars, all because they either reflect or produce photons that we observe. It could therefore come as a surprise that, according to the most recent data, only 5% of the contents of the universe consists of baryonic matter [1]. In a ratio of five to one with this 'regular' matter, we then have about 25% dark matter. This type of matter (as the name suggests) does not interact with light, and is therefore not visible. While we cannot see dark matter directly, we can infer its existence from its gravitational effects. For example, we can observe that most of the luminous matter in a galaxy resides in its center. As a back-of-an-envolope estimate, the rotational velocity of a star at a distance r from the center of such a galaxy should drop off approximately like $1/\sqrt{r}$. What we see, however, is that this velocity stays about the same for a large range of r [2]. Assuming that Newtonian mechanics is correct on such large scales, that should mean that there is an enormous amount of matter that is invisible, forming a halo containing the galaxy. With dark and baryonic matter together, we still only have 30% of the energy contents of the universe; the remaining 70% is dark energy, which is assumed to be responsible for the accelerated expansion of our universe, permeating all of space.

The rotational velocity of stars in galaxies is but one of the pieces of evidence for dark matter [3]. Via gravitational lensing, the effect in which the path of light is bent under the influence of gravity, we can also infer the presence of dark matter. We observe that the paths are bent a lot more than we would expect from the gravitational pull of just the luminous matter that the light passes. Another sign of dark matter is the way that large-scale structure has formed, a central subject of this thesis which we will look at later. If we were to accept this evidence and assume the existence of dark matter, one could ask what it exactly is. People first looked for candidates in the standard model: it has been suggested that there exists a large amount of massive astrophysical compact halo objects (MACHOs), objects consisting of baryonic matter that emit very little to no light. However, local interactions of these objects with brighter stars would then give away their existence, and it has been found that MACHOs cannot be the dominant contribution to dark matter. Neutrinos, being part of the standard model, have also been proposed and were for a time the prime candidate, but measurements on their mass have also ruled these out as dominant contribution. Nevertheless, Dirac neutrinos can still be described by the theory we will develop, so in chapter 6 we take another look at neutrino dark matter and why it is subdominant. We see that standard model candidates of dark matter are effectively ruled out, so other candidates have been proposed. Fuzzy dark matter, consisting of ultralight scalars (10^{-22}eV), has shown great promise in solving some of the problems with the current cold dark matter models [4], and much interest has been shown in it recently. Such scalar dark matter (spin 0 bosons) was the focus of [5], where a possibility for fuzzy dark matter was also presented in a field-theoretic setting.

The focus of this thesis lies on cold (meaning nonrelativistic), fermionic dark matter. We know that all baryonic matter in the universe either consists of bosons (with integer spin, e.g. photons) or fermions (with half-integer spin, e.g. electrons). We therefore assume that dark matter does not consist of some exotic new kind of particles, and say that it is either of these types of matter. Furthermore, by its nature it only interacts via gravity. This thesis aims to do the same for Dirac fermions as was done for scalars in [5]: develop a field-theoretic approach, and moving to a classical particle limit. As there is no *a priori* reason for dark matter to be bosonic, developing a similar theory for dark matter as fermions will be as useful as the one for bosons. By Dirac fermions, we mean spin 1/2-fermions that has the possibility of having an antiparticle, as opposed to Majorana fermions, which are their own antiparticle. While it might be possible to construct some symmetry transformation between the two kinds of fermions, we will not investigate this in this thesis.

Fermionic dark matter might even be useful in solving current problems of the standard model of cosmology, such as the "core-cusp" problem (see [4]), as its mass is constrained by the exclusion principle, assuming there is only a single flavor ([6] constrains Milky Way fermionic dark matter particles at a mass of $\sim$ 30eV). This principle states that fermions obey different statistics than bosons: only one fermion can occupy a single quantum state, while there is no limit for the amount of bosons in such a state. This also means that the development of a fermionic dark matter theory is therefore not as trivial as slightly adjusting the theory for bosons. Only at scales where quantum effects become less and less relevant, the theories will start to exhibit similarities.

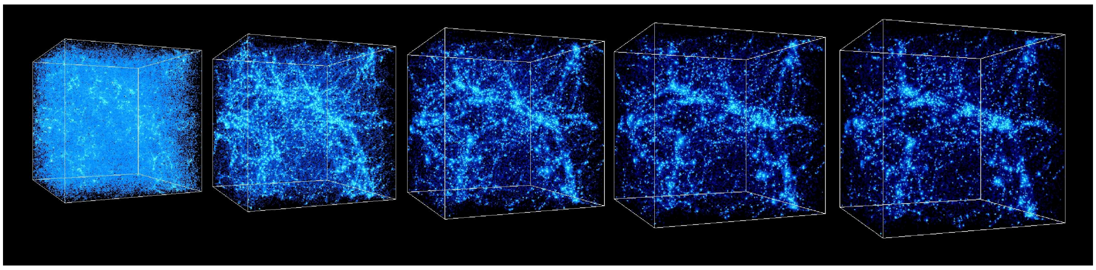

Figure 1.1: Clustering of dark matter into the large-scale structure of the universe. The box is 43MPc wide. Simulations from National Center for Supercomputer Applications by Andrey Kravtsov (The University of Chicago) and Anatoly Klypin (New Mexico State University).

On the largest scales where we can still discern meaningful structures (∼10-100MPc), the universe is structured in large filaments, as can be seen in the right-most box of figure 1.1. Understanding dark matter helps us understand how this structure came to be. According to the standard model of cosmology [4], the very early universe underwent a period of rapid inflation, after which a very hot, isotropic plasma emerged. Through inflation, quantum fluctuations from when the universe was small became the start for density fluctuations after inflation. These fluctuations are also responsible for the anisotropies in the temperature map of the cosmic microwave background (CMB), see figure 1.2. This background was formed when the universe cooled down enough for photons to freely travel, as opposed to being 'trapped' in the plasma by constantly scattering off ions comprising said plasma. More dense regions give off more photons, resulting in the colder and warmer parts of the CMB. The very tiny density fluctuations became the start for gravitational instabilities in the still very homogeneous universe. Dark matter, being $\sim 85\%$ of matter, acted as a catalyst for the collapse into itself at the more dense regions. A simulation of this process can be seen in figure 1.1. Through these instabilities, so-called dark matter halos formed, the centers of which are the cradles for galaxy formation.

In [7], there is already an overview of the current research concerning dark matter and field theory. To summarize, the classical regime of dark matter (i.e. not quantum, not relativistic) can be described by the Vlasov equation, which shows the dynamics of a phase-space density only affected by classical gravity. To get a more accurate description, one looks for higher-order corrections to this regime. This can for example be done by non-linear perturbation theory, like in [8]. A treatment for relativistic scalars reducing to this equation is also found in [9] for one-point functions. A model for fermions in curved spacetime is discussed in [10], but a reduction to the classical equation is not performed. For scalar field dark matter, a kinetic theory and its reduction to the classical Vlasov equation is extensively developed and discussed in [5] [7] [11], where the latter is also used in this thesis as a comparison.

The question then arises why we would go through the trouble of developing a (more complex) field-theoretic approach, when we already have several ways to model it. The approach taken in this thesis is more versatile: while the classical theory will have to simplify to the Vlasov equation, the corrections will naturally appear when making perturbative expansions of the underlying fundamental theory. This fundamental theory is based on the Lagrangian, and to extend our model we could in principle add more terms to this Lagrangian. Our initial starting point will be the Einstein equations, which come from a highly-accurate theory. When going to a classical limit, our theory will then lose predictiveness, but this can be done in a controlled way by choosing the amount of corrections one wants to keep. This ensures the resulting equations are as accurate as one wants it to be. The more accurate model that results from this approach can be used to identify signatures of dark matter, for which we can then look in astronomical observations. These signatures will help us to determine for example whether dark matter is fermionic, or bosonic (or perhaps both).

Thus, the goal is to find the dynamics of fermionic dark matter within a field-theoretic frame-work, set in a curved, expanding (FLRW) spacetime. We will start by a short review of relevant background theory in the next chapter. We will then look at the Dirac action, which describes the fermionic fields, in curved and expanding spacetime, rewriting it as a flat space action with corrections. We will also derive the energy-momentum tensor from it. This is similarly done for the Einstein-Hilbert action, which describes the gravitational part of the complete action. Combining the Dirac and Einstein-Hilbert actions gives our complete action in chapter 5, where we will choose a gauge and integrate out the weak gravitational potentials. After this, we develop the two-particle-irreducible (2PI) effective action truncated at two loops, which we use to find the equations of motion for the statistical two-point functions, and subsequently Wigner transform to find equations for the phase-space densities. These will then allow us to go to a classical particle limit of the fermionic dark matter, in which we find indeed the possibility of particles and antiparticles, and their classical dynamics with higher order corrections.

Accompanying the main body of the thesis are several appendices, whose usefulness ranges from checks with literature to essential mathematical identities.

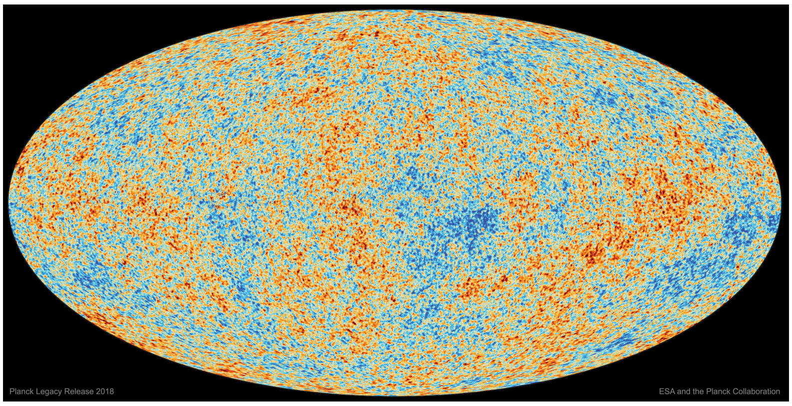

Figure 1.2: Temperature anisotropy map of the CMB. Temperature fluctuations are extremely small (of the order 10^{-4}-10^{-5}K), so the background is very isotropic [12].Source: ESA and Planck Collaboration.

Conventions

- All Greek letters, except for $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta, \pi$, are spacetime indices. These exceptions are the spinor indices.
- Latin letters are local Minkowski/tangent space indices in chapters 2, 3 and 4, and Keldysh indices in chapter 6 and appendix F.
- A mostly + signature is used, such that the Minkowski metric reads $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$
- I is an $n \times n$ identity matrix, it can often be inferred from the context what the value of n is.
- $\mathcal{O}(n)$ means 'terms of order n and higher', usually to indicate they are present but negligible.
- We set $c = \hbar = 1$.
- A regular time derivative is denoted $\dot{x} \equiv \frac{dx}{dt}$, a conformal time derivative $x' = \frac{\partial x}{\partial \eta}$
- The commutator is $[A, B] = AB - BA$, while the anticommutator is $\{A, B\} = AB + BA$.
- Antisymmetry in indices is shown as $\eta_{\mu[\nu}\eta_{\rho]\sigma} \equiv \frac{1}{2}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})$, and symmetry is shown as $\eta_{\mu(\nu}\eta_{\rho)\sigma} \equiv \frac{1}{2}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma})$
- $\epsilon_{\mu\nu\rho\sigma}$ is the fully anti-symmetric Levi-Civita symbol, with $\epsilon^{0123} \equiv 1$
- Summations over flat, spacetime and spinor indices are always implied (i.e. Einstein notation), Keldysh index summations are explicitly written down.
- The Heaviside step-function is defined as

$$\theta(t - t') \equiv \begin{cases} 1, & \text{if } t > t'. \\ \frac{1}{2} & \text{if } t = t'. \\ 0, & \text{if } t < t'. \end{cases} \tag{1.1}$$

- M^* is the complex conjugate of a matrix, while $M^\dagger \equiv (M^*)^T$ is the Hermitian conjugate of a matrix.

2 Relevant background theory

This chapter will give a short overview of the background needed to self-consistently understand this thesis. A fair amount of the contents can be found in [13], should one need further clarification.

2.1 The geometry of the universe

It is a well-established fact that our universe is expanding [1]. If our universe were dominated by a (nearly) homogeneous fluid, its metric would be represented by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, with line element

$$ds^2 = -dt^2 + a^2 d\vec{x}^2, \tag{2.1}$$

where $a = a(t)$ is called the scale factor. This is actually a fairly good approximation for our universe on very large scales, whose energy density is within error margin of the critical density [1]. The expansion rate compared to the scale factor is called the Hubble constant, which is defined by

$$H(t) = \frac{\dot{a}}{a}, \tag{2.2}$$

where the dot represents a derivative with respect to time. A simple example and the origin of this 'constant' lie in Hubble's law

$$r = Hv, \tag{2.3}$$

representing a linear relation between, for example, the distance between two galaxies (r), and their relative speed (v). So, if a galaxy is farther away, it is moving faster away from us. An ever-so-simple law, it was only discovered about 90 years ago by analyzing the redshift from far-away galaxies. Only 10 years before it, Einstein had incorporated the cosmological constant into his field equations to make the universe static; we will encounter it later.

We can also switch to conformal time, where we change our time coordinate as $dt = ad\eta$. This has the nice property that the line element is now proportional to a Minkowski metric, by a factor a^2: we see that the scale factor coincides with the conformal rescaling of the metric. We can represent this new metric as $g_{\mu\nu} = a^2\eta_{\mu\nu}$, so we have $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Note that by switching the time coordinate, the dependence of this scale factor on time has also changed $a(t) \to a(\eta)$, and we define a new 'conformal Hubble rate'

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a}\frac{da}{d\eta} = \frac{1}{a}\frac{dt}{d\eta}\frac{da}{dt} = aH, \tag{2.4}$$

with the prime denoting the derivative with respect to conformal time η. This representation of our universe will be used throughout this thesis. While the change of line element makes the incorporation of the expansion of the universe into calculations seem simple, the changes to the calculations are slightly more involved due to the universe being non-empty, as we will see it the next chapters.

The way we connect the contents of the universe to the curvature and expansion are through the Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.5}$$

On the right-hand side, beside some constants, we see the energy momentum-tensor, which is derived for fermions in curved spacetime in the next chapter. It describes the energy-aspects of a system, and in the next section we will go into more detail. On the right-hand side we find the Einstein tensor $G_{\mu\nu}$. It is constructed from the metric and its first derivatives, and we go through an extensive derivation for the tensor in curved, expanding spacetime in chapter 4. What is more important here is that the tensor describes the curvature of spacetime, meaning gravity. We say that energy and matter, which we describe with the energy-momentum tensor, provide a source for gravity. What we also see on the left-hand side is the cosmological constant Λ, which we alluded to before. It was used by Einstein to make the universe static, i.e. neither expanding nor collapsing. Willem de Sitter pointed out that such a solution, in which gravitational collapse balances with expansion, is unstable, and the constant later found another use as dark energy: as said earlier this chapter, we know the universe is expanding, and Λ can be conveniently used to add a constant acceleration to the dynamics of the equation. The simplest test of this formula is a vacuum with $\Lambda = 0$: when there is no energy and matter ($T_{\mu\nu} = 0$), there is no curvature ($G_{\mu\nu} = 0$), and hence no gravity. We have seen before that in a curvatureless universe, we have a metric proportional to the Minkowski metric.

In general, however, we find ourselves in a non-empty universe. This makes things more interesting, but also more complicated. The metric is not only non-Minkowskian, but also depends on the the point in spacetime where it is evaluated (assuming the distribution of matter is not perfectly homogeneous). Spacetime is then represented as a four-dimensional manifold $\mathcal{M}$, which has the property that it locally looks like $\mathbb{R}^{3,1}$. Vector fields on such a manifold are defined as mappings to the tangent space. As the manifold can be curved, we cannot choose a set of globally orthonormal basis vectors, as their relative angles will change. As the metric varies, we find a so-called coordinate change for vectors

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}. \tag{2.6}$$

The matrix $\frac{\partial x^{\mu'}}{\partial x^{\mu}}$ is an element of $GL(4)$, the group of invertible 4×4 matrices. A (proper) subgroup of this is $SO(3,1)$, which preserves the spacetime interval ds^2 and is associated with boosts and rotations of Minkowski spacetime. $GL(4)$ does not have finite dimensional spinorial representations, which is a fancy way of saying that we cannot define the notion of fermions in every spacetime. But $SO(3,1)$ does have these representations. So we have found a problem: we want to describe fermions, which are represented by spinors, on curved spacetime (coordinate transforms in $GL(4)$), while this can only happen on special cases of spacetime (coordinate transforms in $SO(3,1)$). The way to solve this problem is by vierbeins (or tetrads) $e^{\mu}_{a}(x)$, which use the property that locally the manifold looks like $\mathbf{R}^{3,1}$, which has a tangent space with orthonormal basis vectors. Transformations are then elements of $SO(3,1)$ where we *can* have finite dimensional representations. This is best illustrated by the following relation,

$$g_{\mu\nu}(x) = e^{a}_{\mu}(x) e^{b}_{\nu}(x) \eta_{ab}. \tag{2.7}$$

The original metric $g_{\mu\nu}$ gives rise to non-orthonormal basis vectors, while the vierbeins indeed act to obtain a Minkowski metric, which *does* give orthonormal basis vectors. All vectors and tensors with a 'curved' spacetime index will therefore get such vierbeins. Raising and lowering of indices now depends on the curved or flat indices: a 'curved index' μ is lowered by $g_{\mu\nu}$, a 'flat index' b by η_{bc}. Logically, the vierbeins will be orthonormal, so $e^{\mu}_{a} e^{b}_{\mu} = \delta^{b}_{a}$. We have already seen that a metric in a conformally expanding spacetime can be represented as proportional to the Minkowski metric. We can infer the scaling of the vierbeins from it,

$$g_{\mu\nu} = e^{b}_{\mu} e^{c}_{\nu} \eta_{bc} = a^2 \eta_{\mu\nu} \implies e^{b}_{\mu} = a \delta^{b}_{\mu}. \tag{2.8}$$

When the background is only approximately conformally flat, we will see higher-order corrections to the delta function, as in the next chapter. A much-seen vector when dealing with fermions is the one consisting of the gamma matrices, 4×4 matrices which obey the Clifford algebra:

$$\gamma^{\mu}_{\alpha\beta} = e^{\mu}_{b} \gamma^{b}_{\alpha\beta}, \qquad \{\gamma^b, \gamma^c\} = -2\eta^{cd}. \tag{2.9}$$

The left equation has the so-called spinor indices explicitly written down, though they are often left implicit, like we see in the right equation. This way, we can write down equations extremely compactly: on one hand, the spacetime index chooses one of the matrices, while the spinor indices then allow for matrix multiplication with spinors. Appendix A shows different representations of the gamma matrices, and some useful identities we will use in the thesis. There is also the fifth gamma matrix, which anticommutes with all other gamma matrices

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d, \qquad \{\gamma^5, \gamma^a\} = 0. \tag{2.10}$$

We can now define a set $\Gamma \equiv \{I, \gamma^5, \gamma^a, \gamma^5\gamma^a, \sigma^{ab}\}$, where I is a 4×4 identity matrix, and $\sigma^{ab} \equiv \frac{i}{4}[\gamma^a, \gamma^b]$ is the commutator of gamma matrices. The different σ^{ab} are the generators of the Lorentz algebra in the spinor representation, which generate boosts and rotations for fermion fields [14]. Γ consists of $1 + 1 + 4 + 4 + 6 = 16$ independent elements forming a complete basis, which means that any 4×4 matrix can be decomposed into these elements. It turns out that fermionic two-point propagators, the central elements of this thesis, can also be decomposed into this basis, see appendix B. We find that this is very useful in seeing how their different parts behave, as we will see in chapter 6. Depending on their behaviour under Lorentz transformations when combined with fermions, we call the different elements the scalar, pseudoscalar, vector, axial vector and tensor parts of Γ, in the order they were written down in the definition.

Another object that depends on the geometry of spacetime is the covariant derivative for spinors, which reads

$$D_\mu\psi = \big(\partial_\mu + \Gamma_\mu\big)\psi, \qquad D_\mu\bar{\psi} = \overline{D_\mu\psi} = \partial_\mu\bar{\psi} - \bar{\psi}\Gamma_\mu. \tag{2.11}$$

(Note that the object Γ_μ is different from the set of gamma matrices Γ.) ψ and $\bar{\psi} \equiv \psi^\dagger\gamma^0$ are the fermion field and its conjugate transpose times γ^0, which is present to make the Lorentz scalar $\bar{\psi}_\alpha\psi_\alpha$ Lorentz invariant, see e.g. [14] (similarly, one can construct more of these invariants by considering $\bar{\psi}\Gamma\psi$). The so-called spinor index α over which the scalar is summed shows that the fermion fields are actually four-component spinors. Usually, these are summed over and omitted.

On curved space, taking a derivative of a fermion field is not just $\partial_\mu\psi$, as we need the object $D_\mu\psi$ to be invariant under spinor transformations L, so

$$\psi \to L\psi, \quad D_\mu\psi \to LD_\mu\psi \;\;, \quad \bar{\psi} \to \bar{\psi}L^{-1}, \quad D_\mu\bar{\psi} \to \big(D_\mu\bar{\psi}\big)L^{-1} \quad \Longrightarrow \quad \Gamma_\mu \to L\Gamma_\mu L^{-1} - (\partial_\mu L)L^{-1}, \tag{2.12}$$

which is not hard to verify. In [15], it is derived that Γ_μ is written in terms of vierbeins, metrics and gamma matrices as

$$\Gamma_\mu = \frac{i}{2}\omega_{\mu cd}\sigma^{cd}, \qquad \omega_{\mu cd} = e_c^{\nu}(\partial_\mu e_{\nu d} - \Gamma^\rho_{\mu\nu}e_{\rho d}), \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{2.13}$$

The tensor $\omega_{\mu cd}$ is called the spin connection, and $\Gamma^\rho_{\mu\nu}$ the Chistoffel symbol. An important remark about the latter is that we assume it is symmetric in its lower indices, meaning that our theory is torsion-free. This will tremendously simplify some parts of our calculations, but one has to realize that dark matter might not be torsion-free. The inclusion of torsion is left to later works. The covariant derivative is now explicitly written as

$$D_\mu\psi = \partial_\mu\psi + \frac{i}{2}\omega_{\mu cd}\sigma^{cd}\psi, \qquad D_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \frac{i}{2}\bar{\psi}\omega_{\mu cd}\sigma^{cd}, \tag{2.14}$$

We have already remarked in that σ^{cd} plays a role in Lorentz transformations, so its appearance in the covariant derivative should come as no surprise. The covariant derivative clearly depends on vierbeins, and in the Dirac action it is also multiplied by a gamma matrix, giving even more dependence on the vierbein. Chapter 3 will explicitly deal with the effects of both curvature and the expansion of the universe of these quantities.

2.1.1 Classical kinetic equation in expanding spacetime

Now that we know that an expanding background enters in the theory via a conformal transformation, we will look at what kinetic equation we are supposed to find when taking the classical limit. In the introduction, we alluded to the fact that the classical modeling of cold dark matter is done by presuming it is a homogeneous plasma, and then using the Vlasov equation to describe the evolution of a phase-space density $f(\eta, \vec{x}, \vec{p})$. When the regular Vlasov equation is adjusted for an expanding background, we find the following [5]

$$\frac{df(\eta,\vec{x},\vec{p})}{d\eta} = \left[\frac{\partial}{\partial\eta} + \frac{p^k}{ma}\frac{\partial}{\partial x^k} - ma\frac{\partial\Phi_N(\eta,\vec{x})}{\partial x^k}\frac{\partial}{\partial p_k}\right]f(\eta,\vec{x},\vec{p}) = \mathrm{Coll}[f]. \tag{2.15}$$

For a more intuitive interpretation of the phase-space density, one can integrate $f(\eta, \vec{x}, \vec{p})$ over momentum $\vec{p}$, obtaining the number density. For scalars, this is for example no more than the particle number per given volume. The equation tells us the dynamics of the phase space density at different points in (conformal) phase space, and accounts for example for the gravitational force. This force is represented by the spatial derivative of the Newtonian potential $\Phi_N(\eta, \vec{x})$, obeying the Poisson equation

$$\frac{\Delta}{a^2}\Phi_N = 4\pi G\rho, \qquad \rho = \frac{m}{a^3}\int d^3p f(\eta,\vec{x},\vec{p}), \tag{2.16}$$

where the Laplacian is defined as $\Delta \equiv \vec{\nabla}^2$. We see that the Vlasov-Poisson system closes, as the Poisson equation also depends on the phase space density. If a system has interactions (which influence the evolution), we also get a collision term on the right-hand side, Coll$[f]$. As dark matter is generally collisionless and only interacts via gravity, the collisionless Vlasov equation works fairly well for modeling it.

In this thesis, we are assuming that the background is not perfectly homogeneous, as inhomogeneities might arise from quantum fluctuations. Even when disregarding the cause of the inhomogeneities, we would like to have a theory that incorporates them, and has the ability to isolate higher order relativistic and gradient corrections. The framework that is exceptionally useful when looking such corrections is field theory, with the fundamental Lagrangian as the starting point. By making appropriate perturbative expansions, corrections will automatically follow as the higher-order terms, making this approach very powerful. Another feature of the Lagrangian is that we could always add more terms to it, making the theory very rich and versatile. As noted, this framework already exists for scalars [5], and much interest has recently been shown in ultralight scalars called fuzzy dark matter (see e.g. [16]). Less interest exists for fermionic dark matter: initially neutrinos (spin 1/2) were proposed as a candidate as they are very hard to measure, but we now know that their mass is very tiny, and combined with the exclusion principle, they are ruled out as the dominant contribution for dark matter in galaxies. They could be heated up to get to the appropriate energy density, but then their velocities will exceed the escape velocity of large structures, making the formation of large scale structure by these particles impossible. We also know that precisely due to the ultrarelativistic velocities of neutrinos, they have stayed in equilibrium in the early universe far too long to be able to even form structure. This thesis will assume nonrelativitic (cold), nonequilibrium fermions to be able to explain dark matter, which as a consequence finds that the classical theory still holds to a good approximation. The non-homogeneous background is represented by assuming the gravitational potential is sourced by perturbations in said background,

$$\frac{\Delta}{a^2}\Phi(\eta,\vec{x}) = 4\pi G\delta\rho, \qquad \delta\rho = \frac{m}{a^3}\left(\int d^3p f(\eta,\vec{x},\vec{p}) - n_0(\eta)\right), \tag{2.17}$$

where n_0 is the number density of the averaged background, which we define now by

$$n_0(\eta) \equiv \frac{\int \frac{d^3p d^3x}{(2\pi)^3} f(\eta,\vec{x},\vec{p})}{V}, \tag{2.18}$$

where we see V as the volume over which the $\vec{x}$-integral extends. Its time dependence follows from the expanding background, as the density obviously drops for an expanding volume. The Vlasov-Poisson system that follows will become an integro-differential equation when moving the Δ to the right-hand side in (2.17),

$$\frac{df(\eta,\vec{x},\vec{p})}{d\eta} = \left[\frac{\partial}{\partial\eta} + \frac{p^k}{ma}\frac{\partial}{\partial x^k} - 4\pi Gm\vec{\nabla}_x\Delta^{-1}\left(m\int d^3p f(\eta,\vec{x},\vec{p}) - mn_0(\eta)\right)\cdot\partial_{\vec{p}}\right] f(\eta,\vec{x},\vec{p}) = 0. \tag{2.19}$$

This is the equation we set out to find in chapter 6. Notice that, due to n_0 being constant in space, its contribution is actually irrelevant to the force term $\vec{\nabla}\Phi$. By integration of the phase space densities over momentum and space up to the boundaries of the observable universe, we can actually set the particles that enter or leave the volume to 0 and find that classically, we have a fixed particle number. This has as a consequence that the mass is globally fixed, something we will use when deriving the classical particle limit.

2.2 Energy and pressure

In the previous section, we have already seen a glimpse of the energy-momentum tensor, which contains the 'energy information' about a system. The starting point for any such system in a field theory is the Lagrangian density (often just called 'the Lagrangian'), denoted $\mathcal{L}$. The Lagrangian in this thesis consists of the gravitational part from which the Einstein tensor can be derived (chapter 4), and a matter part which will be the Dirac Lagrangian minimally coupled to gravity. From it, we can construct an action S_m, which is in turn used to derive the energy-momentum tensor,

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}, \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.20}$$

In curved space, where the metric deviates from the Minkowski metric, we account for the curvature by adjusting the measure of integration by the term $\sqrt{-g}$, with g the determinant of the metric. In general, the energy-momentum tensor can be very complicated, depending on the metric-dependence of the action. However, in the case of a perfect fluid, we can decompose it very easily as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{2.21}$$

where the four-velocity u_μ has the normalization $g_{\mu\nu}u^\mu u^\nu = -1$. The component ρ is the energy density, and the component P is the pressure of the fluid, and one defines the equation of state as $P = w\rho$, with w the equation-of-state parameter. We can make the decomposition even easier by assuming a nonrelativistic fluid, where the temporal part of the four-velocity is so large compared to the spatial part that we can say $u_\mu \approx -a\delta^0_\mu$. We then have, if we have a scaling $g_{\mu\nu} = a^2\eta_{\mu\nu}$

$$T_{00} = a^2\rho + a^2 P - a^2 P = a^2\rho, \qquad T_{0i} = 0, \qquad T_{ij} = a^2 P\delta_{ij}, \qquad\qquad i, j = 1, 2, 3. \tag{2.22}$$

Of course, the simplifying assumptions here are huge, but the perfect fluid description does not deviate much from the dark matter in this thesis. Cold dark matter, be it bosonic or fermionic, is very well described by a nonrelativistic, collisionless plasma, when considering large enough scales (> 10MPc). The point of going to field theory was to find higher order corrections to this description, and we therefore introduce small perturbations,

$$T_{\mu\nu} = \langle T_{\mu\nu}\rangle + \delta T_{\mu\nu}, \qquad T_{00} = a^2\langle\rho\rangle + a^2\delta\rho, \qquad T_{ij} = a^2\langle P\rangle\delta_{ij} + a^2\delta P\delta_{ij}. \tag{2.23}$$

We have introduced the expectation value, a self-explanatory name, as the mean value for ρ and P. In quantum field theory, one uses the single particle state for defining the expectation value, but in this thesis we are looking at a large collection of states averaged in the so-called density operator, consisting of i states [17],

$$\hat{\rho} = \sum_i |i\rangle\hat{\rho}_i\langle i|, \qquad \sum_i \hat{\rho}_i = 1 \implies \mathrm{Tr}[\hat{\rho}] = 1, \qquad \langle\hat{A}\rangle \equiv \mathrm{Tr}[\hat{\rho}\hat{A}], \tag{2.24}$$

where $\hat{A}$ is any Hermitian operator. $\langle\rho\rangle$ and $\langle P\rangle$ are the so-called homogeneous background contributions, and as dark matter is assumed to be pressureless, we can even assume $\langle P\rangle = 0$. We will not set the pressure to 0, but we will assume that $\frac{\langle P\rangle}{\langle\rho\rangle} \sim \frac{v^2}{c^2}$, which for a nonrelativistic fluid means that the pressure is very small compared to the energy density. If we consider the plasma frame, the plasma will be isotropic. This means that the vector component of the energy-momentum tensor, T_{0i}, will also be much smaller than the other contributions we see in (2.23). The energy-momentum tensor for the fermions in a curved, expanding background will be derived in chapter 3.

What we already see in this chapter is that we make simplifying assumptions based on the relevance of the terms we want to keep. This is exactly the point of what we described in the introduction: we drop terms in a controlled way to get to a relevant description, but we can always go back, and keep more terms to get more corrections if we want, that will be less relevant. This is the versatility of starting from very general, but very accurate field-theoretic equations.

3 Interactions of fermions and gravitons

In the standard model, interactions are represented by the exchange of a force carrier between several particles. A well-known example of this is the electromagnetic force, in particular the collision of two electrons, where a virtual photon (spin 1 boson) carries over momentum and energy from one electron (spin $1/2$ fermion) to another electron. When the particles and force carriers are seen as excitations of fields, we enter the realm of field theory. It turns out that the rough sketch of electron collision is just the surface of what happens, called tree level in terms of Feynman diagrams. Higher order corrections enter in the form of loop diagrams. These corrections become progressively less important, which is why the expansion is often truncated at some loop order. If we also want to describe fermions affected by gravity as fields exchanging gravitons, we will have to reformulate the theory of fermions in curved spacetime. This chapter aims to do that, by deriving the three- and four-vertex for the diagrams. What follows is a computation of the cubic and quartic fermion-graviton interaction terms. We check the computation with [18].

3.1 Conformal invariance

As promised in the previous chapter, we will begin with the matter part of our action, which is the Dirac action in curved spacetime,

$$S[\psi, \psi^*] = \int d^4x \sqrt{-g}\left[\frac{i}{2}\Big(\bar{\psi} e^\mu_b \gamma^b D_\mu \psi - (D_\mu \bar{\psi}) e^\mu_b \gamma^b \psi\Big) - \bar{\psi} M \psi\right]. \tag{3.1}$$

All information about expansion and curvature is encoded in this action, and our goal is to extract it and find the relevant terms. The way the fermions couple to gravity is found in multiple terms in this action: the vierbein appears for the gamma matrix, we see coupling implicitly in the covariant derivative we wrote down in (2.14), and there is the determinant of the metric next to the measure.

It makes sense to contemplate why we chose this action here. This is in fact the most simple action coupling to gravity we can think of for Dirac fermions only interacting with gravity. If we were to consider Majorana fermions, which are their own antiparticles, we would have to choose a different starting Lagrangian. The only free parameter seems to be the mass, which actually consists of two parameters by decomposing into a scalar and pseudoscalar contribution as

$$M_{\alpha\beta} \equiv m_R \delta_{\alpha\beta} + i m_I \gamma^5_{\alpha\beta}, \qquad m^2 \equiv |M|^2 = m_R^2 + m_I^2, \tag{3.2}$$

where the imaginary mass plays a roll in e.g. CP-symmetry breaking [19]. Notice that

$$(\gamma^5)^n = \begin{cases} I, & \text{if } n \text{ is even,} \\ \gamma^5, & \text{if } n \text{ is odd,} \end{cases} \implies M_{\alpha\beta} = m\big(\delta_{\alpha\beta}\cos\alpha + i\gamma^5_{\alpha\beta}\sin(\alpha)\big) = m e^{i\gamma^5\alpha}. \tag{3.3}$$

It is not hard to verify that the first half of (3.1), the kinetic part, is invariant under a global rotation of the spinors by $\psi \to \psi e^{i\gamma^5\theta}$, $\bar{\psi} \to \bar{\psi} e^{i\gamma^5\theta}$, as the γ^5 anticommutes with the gamma matrices (and consequently commutes with σ^{ab}). The second part, the mass term, will then pick up a global rotation by $e^{2i\gamma^5\theta}$, as γ^5 commutes with itself. We therefore see that

$$\bar{\psi} M \psi \to m \bar{\psi}\psi e^{i\gamma^5(2\theta+\alpha)}. \tag{3.4}$$

So with a global rotation by θ, we can always remove the imaginary part of the mass, and simply write m instead of M. It is very important to note, however, that this breaks down when $\alpha = \alpha(x)$, which happens when the mass is not globally fixed, e.g. when mass is generated by scalar condensation, like in the Higgs mechanism. Such a condensate will then be affected locally by gravity, and develop a spacetime dependence (the Higgs boson is in fact so heavy that it will not get a meaningful dependence). Consequently, we would have to choose a spacetime-dependent θ, which will not commute with the derivative operator anymore. As the imaginary mass term does not complicate the equations at this point, we will keep it, but later on we will assume such a globally fixed mass. So we see that in that case, the mass parameter is the only one left in this action. However, the strength of the Lagrangian description is that we can always add more terms or allow for different fields. One could think of allowing for different flavors of fermions (which will also remove the constraints of the mass, as different flavors of fermions can occupy the same state). Another possibility is sourcing the mass via a complex scalar, which will then have its own kinetic part. These scalars could then self-couple via a scalar mass, adding a paramater to the model. Even more exotic perhaps is considering the possibility of dark charge, another parameter, to which so-called dark photons couple (do note that there exist constraints on dark matter charge, see e.g. [20]). These photons would not couple to the standard model, but do act in a similar way to electromagnetism.

We see that the possibilities on choosing the starting matter Lagrangian are practically endless, as the ones named here are among a large number of possibilities. However, as to our knowledge this will be the first time fermionic dark matter will be approached field-theoretically, in order to derive corrections to the classical equation, we will keep it simple and leave further extensions to other works. The task is now to analyze this action (3.1).

We assume an approximately conformally flat background. In the introduction, we have already seen how this works for an exactly conformally flat metric. Here we will represent the approximately flat metrics and vierbeins by a tilde, to remind us that the curvature is still present, so $g_{\mu\nu} = a^2\tilde{g}_{\mu\nu}$ (with $a = a(\eta)$), implying

$$g^{\mu\nu} = a^{-2}\tilde{g}^{\mu\nu}, \qquad \sqrt{-g} = a^4\sqrt{-\tilde{g}}, \qquad \tilde{g}_{\mu\nu} = \tilde{e}^a_\mu \tilde{e}^b_\nu \eta_{ab} \implies e^a_\mu = a\tilde{e}^a_\mu, \qquad e^\mu_b = a^{-1}\tilde{e}^\mu_b. \tag{3.5}$$

The Christoffel symbol $\Gamma^\rho_{\mu\nu}$ will change as (using $g^{\mu\nu}g_{\nu\rho} = \tilde{g}^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^\mu_\rho$)

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}a^{-2}\tilde{g}^{\rho\sigma}[\partial_\mu(a^2\tilde{g}_{\nu\sigma}) + \partial_\nu(a^2\tilde{g}_{\sigma\mu}) - \partial_\sigma(a^2\tilde{g}_{\mu\nu})] = a^{-1}[\delta^\rho_\nu\partial_\mu a + \delta^\rho_\mu\partial_\nu a - \tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu}\partial_\sigma a] + \tilde{\Gamma}^\rho_{\mu\nu}. \tag{3.6}$$

$\tilde{\Gamma}^\rho_{\mu\nu}$ looks the same as in (2.13), but now with tildes on the metrics. We see that we have isolated the part of the Christoffel symbol that is invariant under conformal scaling, as there are no scale factor multiplying or present in $\tilde{\Gamma}^\rho_{\mu\nu}$, and a part that now depends on derivatives of the scale factor. $\omega_{\mu cd}$ also changes, where we use the normalization of the (rescaled) metrics ($\tilde{g}^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^\mu_\rho$):

$$\begin{aligned}
\omega_{\mu cd} &= a^{-1}\tilde{e}^\nu_c[\partial_\mu(a\tilde{e}_{\nu d}) - \Gamma^\rho_{\mu\nu}a\tilde{e}_{\rho d}] \\
&= a^{-1}\tilde{e}^\nu_c\Big[\tilde{e}_{\nu d}\partial_\mu a + a\partial_\mu\tilde{e}_{\nu d} - (\delta^\rho_\nu\partial_\mu a + \delta^\rho_\mu\partial_\nu a - \tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu}\partial_\sigma a)\tilde{e}_{\rho d} - a\tilde{\Gamma}^\rho_{\mu\nu}\tilde{e}_{\rho d}\Big] \\
&= a^{-1}\tilde{e}^\nu_c\tilde{e}_{\nu d}\partial_\mu a + \tilde{e}^\nu_c\partial_\mu\tilde{e}_{\nu d} - a^{-1}\tilde{e}^\rho_c(\partial_\mu a)\tilde{e}_{\rho d} - a^{-1}\tilde{e}^\nu_c(\partial_\nu a)\tilde{e}_{\mu d} + a^{-1}\tilde{e}_{\mu c}\tilde{e}^\sigma_d(\partial_\sigma a) - \tilde{e}^\nu_c\tilde{\Gamma}^\rho_{\mu\nu}\tilde{e}_{\rho d} \\
&= a^{-1}\Big[\tilde{e}^\nu_c\tilde{e}_{\nu d}\partial_\mu a - \tilde{e}^\rho_c\tilde{e}_{\rho d}\partial_\mu a - \tilde{e}^\nu_c\tilde{e}_{\mu d}\partial_\nu a + \tilde{e}_{\mu c}\tilde{e}^\sigma_d\partial_\sigma a\Big] + \tilde{\omega}_{\mu cd} \\
&= -a^{-1}\Big[\tilde{e}^\nu_c\tilde{e}_{\mu d} - \tilde{e}^\nu_d\tilde{e}_{\mu c}\Big]\partial_\nu a + \tilde{\omega}_{\mu cd}.
\end{aligned} \tag{3.7}$$

In the third line, we have obtained a conformally invariant part $\tilde{\omega}_{\mu cd}$ by combining the second and last term. In the fourth line, we note that the first two terms drop out after relabeling indices. Looking back at (3.1), we see that when we use the rescaled quantities, and rescale the fermion fields as $\psi = a^{-3/2}\chi$, we get

$$\begin{aligned}
S[\chi,\bar{\chi}] &= \int d\eta\, d^3x\, a^4\sqrt{-\tilde{g}}\Big[\frac{i}{2}\Big(a^{-3/2}\bar{\chi}a^{-1}\tilde{e}^\mu_b\gamma^b D_\mu(a^{-3/2}\chi) - D_\mu(a^{-3/2}\bar{\chi})a^{-1}\tilde{e}^\mu_b\gamma^b a^{-3/2}\chi\Big) - a^{-3}\bar{\chi}M\chi\Big] \\
&= \int d\eta\, d^3x\sqrt{-\tilde{g}}\Big[\frac{i}{2}a^{3/2}\Big(\bar{\chi}\tilde{e}^\mu_b\gamma^b D_\mu(a^{-3/2}\chi) - D_\mu(a^{-3/2}\bar{\chi})\tilde{e}^\mu_b\gamma^b\chi\Big) - \bar{\chi}Ma\chi\Big].
\end{aligned} \tag{3.8}$$

Focusing solely on the part in round parentheses, we obtain

$$\begin{aligned}
\bar{\chi}\tilde{e}^\mu_b\gamma^b D_\mu\big(a^{-3/2}\chi\big) - D_\mu\big(a^{-3/2}\bar{\chi}\big)\tilde{e}^\mu_b\gamma^b\chi &= \\
&\bar{\chi}\tilde{e}^\mu_b\gamma^b\partial_\mu\big(a^{-3/2}\chi\big) - \partial_\mu\big(a^{-3/2}\bar{\chi}\big)\tilde{e}^\mu_b\gamma^b\chi + \frac{i}{2}a^{-3/2}\bar{\chi}\tilde{e}^\mu_b\omega_{\mu cd}\{\gamma^b,\sigma^{cd}\}\chi \\
&= a^{-3/2}\Big[\bar{\chi}\tilde{e}^\mu_b\gamma^b\partial_\mu\chi - (\partial_\mu\bar{\chi})\tilde{e}^\mu_b\gamma^b\chi\Big] + \frac{i}{2}a^{-3/2}\bar{\chi}\tilde{e}^\mu_b\omega_{\mu cd}\{\gamma^b,\sigma^{cd}\}\chi.
\end{aligned} \tag{3.9}$$

In the second equality we see terms with a common factor $\frac{3}{2}a^{-5/2}(\partial_\mu a)$ disappear. In the last term we still need to expand the term $\omega_{\mu cd}$, giving (using $\tilde{e}^\mu_b \tilde{e}^c_\mu = \delta^c_b$ and $\gamma_d \gamma^d = -4I$)

$$\begin{aligned}\tilde{e}^\mu_b \omega_{\mu cd}\{\gamma^b, \sigma^{cd}\} =& -2a^{-1}\tilde{e}^\mu_b \tilde{e}^\nu_c \tilde{e}_{\mu d}(\partial_\nu a)\{\gamma^b, \sigma^{cd}\} + \tilde{e}^\mu_b \tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\} \\ =& -\frac{i}{2}a^{-1}\tilde{e}^\nu_c(\partial_\nu a)\Big(\gamma_d\gamma^c\gamma^d - \gamma_d\gamma^d\gamma^c + \gamma^c\gamma^d\gamma_d - \gamma^d\gamma^c\gamma_d\Big) + \tilde{e}^\mu_b \tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\} \\ =& \tilde{e}^\mu_b \tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\}.\end{aligned} \tag{3.10}$$

These cancellations, together with the fact that the prefactor $a^{-3/2}$ nicely cancels in the Dirac action (3.8), lets us write down the conformally rescaled version of the action,

$$\begin{aligned}S[\chi, \bar{\chi}] =& \int d\eta\, d^3x \sqrt{-\tilde{g}}\left[\frac{i}{2}\Big(\bar{\chi}\tilde{e}^\mu_b\gamma^b\partial_\mu\chi - (\partial_\mu\bar{\chi})\tilde{e}^\mu_b\gamma^b\chi + \frac{i}{2}\bar{\chi}\tilde{e}^\mu_b\tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\}\chi\Big) - \bar{\chi}Ma\chi\right] \\ =& \int d\eta\, d^3x \sqrt{-\tilde{g}}\left[\frac{i}{2}\Big(\bar{\chi}\tilde{e}^\mu_b\gamma^b\tilde{D}_\mu\chi - (\tilde{D}_\mu\bar{\chi})\tilde{e}^\mu_b\gamma^b\chi\Big) - \bar{\chi}Ma\chi\right],\end{aligned} \tag{3.11}$$

where we chose to define $\tilde{D}_\mu\chi \equiv \partial_\mu\chi + \frac{i}{2}\tilde{\omega}_{\mu cd}\sigma^{cd}\chi$ and $\tilde{D}_\mu\bar{\chi} \equiv \partial_\mu\bar{\chi} - \frac{i}{2}\bar{\chi}\tilde{\omega}_{\mu cd}\sigma^{cd}$. These are the rescaled versions of the covariant derivative, and it shows that the kinetic part of the Dirac action is conformally invariant, given an appropriate rescaling of the fields. Only the mass term sees the expansion of the universe: either we have a mass growing in time, or a fixed mass with expansion in the Lagrangian. Were the background exactly flat, we would see not the scaling in (3.5), but simply a Minkowski metric for $\tilde{g}_{\mu\nu}$, and a Kronecker delta for $\tilde{e}^\mu_a$. The derivatives of the vierbein and the metric vanish, and consequently the spin connection, reducing the covariant derivative to the regular derivative, which is exactly what we would expect for flat space. Now that we have rescaled the conformally approximately flat background, we have the rescaled metric $\tilde{g}_{\mu\nu}$, which is then approximately flat. The focus of the next chapter is to use this to write this metric as Minkowski with perturbations.

3.2 The metric perturbations

Because our action at the start of the chapter was conformally flat (up to the mass term), and we did a conformal scaling of the metric, we see that our rescaled metric can be approximated by a comoving Minkowski metric: $\tilde{g}_{\mu\nu} = \eta^{\rm co}_{\mu\nu} + \kappa h_{\mu\nu}$, where we give a sub- or superscript 'co' (whatever is not in the way of the indices) to denote a comoving flat metric. Here we have

$$\kappa^2 \equiv 16\pi G, \tag{3.12}$$

which has the role of the coupling strength or loop counting paramater; this will become apparent shortly. The metric $g_{\mu\nu}$ is by definition symmetric in its spacetime indices, so it follows that the perturbations will have the same property. Based on the order of loops we will consider relevant in this thesis, we are interested in the cubic and quartic vertices, so all terms of order κ^3 and higher will be neglected. The determinant and inverse of the rescaled metric and the rescaled vierbein with its inverse can be expanded as (using $h = \eta^{\mu\nu}_{\rm co}h_{\mu\nu}$)

$$\begin{aligned}\sqrt{-\tilde{g}} &= 1 + \frac{\kappa}{2}h + \frac{1}{8}\kappa^2 h^2 - \frac{1}{4}\kappa^2 h^{\mu\nu}h_{\mu\nu} + \mathcal{O}(\kappa^3), \quad \tilde{g}^{\mu\nu} = \eta^{\mu\nu}_{\rm co} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\sigma h^{\sigma\nu} + \mathcal{O}(\kappa^3), \\ \tilde{e}^b_\mu &= \delta^b_\mu + \frac{\kappa}{2}h^b_\mu - \frac{1}{8}\kappa^2 h^{b\rho}h_{\rho\mu} + \mathcal{O}(\kappa^3), \quad \tilde{e}^\mu_b = \delta^\mu_b - \frac{\kappa}{2}h^\mu_b + \frac{3}{8}\kappa^2 h^{\mu\rho}h_{\rho b} + \mathcal{O}(\kappa^3).\end{aligned} \tag{3.13}$$

We can now calculate the relation between the comoving and regular flat metric, using the vierbeins

$$a^{-2}g_{\mu\nu} = \tilde{g}_{\mu\nu} = \eta^{\rm co}_{\mu\nu} + \kappa h_{\mu\nu} = a^{-2}e^a_\mu e^b_\nu \eta_{ab} = \tilde{e}^a_\mu \tilde{e}^b_\nu \eta_{ab} \approx (\delta^a_\mu\delta^b_\nu + \frac{\kappa}{2}h^a_\mu\delta^b_\nu + \frac{\kappa}{2}h^b_\nu\delta^a_\mu)\eta_{ab} = \delta^a_\mu\delta^b_\nu\eta_{ab} + \kappa h_{\mu\nu} \tag{3.14}$$

We see that the vierbeins do exactly what the are supposed to do: they relate curved space to flat space, and we see that the flat metric with Latin indices is no more than the comoving flat metric.

We also get a new expression for the conformally invariant part of the Christoffel symbol,

$$\begin{aligned}\tilde{\Gamma}^\rho_{\mu\nu} &= \frac{1}{2}(\eta^{\rho\sigma} - \kappa h^{\rho\sigma} + \kappa^2 h^\mu_\sigma h^{\sigma\nu})\Big[\partial_\mu(\eta_{\nu\sigma} + \kappa h_{\nu\sigma}) + \partial_\nu(\eta_{\sigma\mu} + \kappa h_{\sigma\mu}) - \partial_\sigma(\eta_{\mu\nu} + \kappa h_{\mu\nu})\Big] + \mathcal{O}(\kappa^3) \\ &= \Big(\frac{\kappa}{2}\eta^{\rho\sigma} - \frac{\kappa^2}{2}h^{\rho\sigma}\Big)(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(\kappa^3),\end{aligned} \tag{3.15}$$

with which we can then expand the conformally invariant part of the spin connection,

$$\begin{aligned}
\tilde{\omega}_{\mu cd} =& \tilde{e}_c^\nu \partial_\mu \tilde{e}_{\nu d} - \tilde{e}_c^\nu \tilde{\Gamma}^\rho_{\mu\nu} \tilde{e}_{\rho d} \\
=& \Big(\delta_c^\nu - \frac{\kappa}{2} h_c^\nu\Big)\partial_\mu\Big(\frac{\kappa}{2}h_{\nu d} - \frac{1}{8}\kappa^2 h_d^\sigma h_{\sigma\nu}\Big) \\
&- \Big(\delta_c^\nu - \frac{\kappa}{2}h_c^\nu\Big)\Big(\frac{\kappa}{2}\eta^{\rho\sigma} - \frac{\kappa^2}{2}h^{\rho\sigma}\Big)(\eta_{\rho d} + \frac{\kappa}{2}h_{\rho d})(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(\kappa^3) \\
=& \frac{\kappa}{2}\partial_\mu h_{cd} - \frac{\kappa^2}{4} h_c^\nu \partial_\mu h_{\nu d} - \frac{\kappa^2}{8}(h_d^\sigma \partial_\mu h_{\sigma c} + h_c^\sigma \partial_\mu h_{\sigma d}) \\
&- \Big(\frac{\kappa}{2}\delta_c^\nu \delta_d^\sigma - \frac{\kappa^2}{4}(h_d^\sigma \delta_c^\nu + h_c^\nu \delta_d^\sigma)\Big)(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(\kappa^3) \\
=& \frac{\kappa}{2}(\partial_d h_{\mu c} - \partial_c h_{\mu d}) + \frac{\kappa^2}{4}h_d^\sigma\Big(\frac{1}{2}\partial_\mu h_{\sigma c} + \partial_c h_{\sigma\mu} - \partial_\sigma h_{\mu c}\Big) - \frac{\kappa^2}{4}h_c^\sigma\Big(\frac{1}{2}\partial_\mu h_{\sigma d} + \partial_d h_{\sigma\mu} - \partial_\sigma h_{\mu d}\Big) \\
&+ \mathcal{O}(\kappa^3).
\end{aligned} \tag{3.16}$$

We notice the obvious antisymmetry in the cd indices in $\tilde{\omega}_{\mu cd}$, such that we can add terms when combining with the antisymmetric σ^{cd} term,

$$\tilde{\omega}_{\mu cd}\sigma^{cd} = \kappa(\partial_d h_{\mu c})\sigma^{cd} + \frac{\kappa^2}{2}h_d^\sigma\Big(\frac{1}{2}\partial_\mu h_{\sigma c} + \partial_c h_{\sigma\mu} - \partial_\sigma h_{\mu c}\Big)\sigma^{cd} + \mathcal{O}(\kappa^3). \tag{3.17}$$

To keep things tidy, we separately look at the most complicated part of (3.11), up to second order in κ,

$$\begin{aligned}
&\sqrt{-\tilde{g}}\tilde{e}_b^\mu \tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\} \approx \\
&\quad \Big(1 + \frac{\kappa}{2}h\Big)\Big(\delta_b^\mu - \frac{\kappa}{2}h_b^\mu\Big)\Big[\kappa(\partial_d h_{\mu c}) + \frac{\kappa^2}{2}h_d^\sigma\Big(\frac{1}{2}\partial_\mu h_{\sigma c} + \partial_c h_{\sigma\mu} - \partial_\sigma h_{\mu c}\Big)\Big]\{\gamma^b, \sigma^{cd}\} \\
&= \Big[\kappa(\partial_d h_{bc}) + \frac{\kappa^2}{2}\Big(h(\partial_d h_{bc}) - h_b^\mu(\partial_d h_{\mu c}) + \frac{1}{2}h_d^\sigma(\partial_b h_{\sigma c}) + h_d^\sigma(\partial_c h_{\sigma b}) - h_d^\sigma(\partial_\sigma h_{bc})\Big)\Big]\{\gamma^b, \sigma^{cd}\} \\
&= \Big[\kappa(\partial_d h_{bc}) - \frac{\kappa^2}{4}\Big(2\partial_d(h_b^\sigma h_{\sigma c}) + h_c^\sigma(\partial_b h_{\sigma d}) + 2h_d^\sigma(\partial_\sigma h_{bc}) - 2h(\partial_d h_{bc})\Big)\Big]\{\gamma^b, \sigma^{cd}\}.
\end{aligned} \tag{3.18}$$

The final rearrangement, in which we used the antisymmetry in c and d, is used to obtain similarity with the result in [18]. We can now look at the expanded, conformally rescaled Dirac action (3.11)

$$\begin{aligned}
S[\chi, \bar{\chi}] =& \int d\eta\, d^3x \sqrt{-\tilde{g}}\Big[\frac{i}{2}\Big(\bar{\chi}\tilde{e}_b^\mu \gamma^b \partial_\mu \chi - (\partial_\mu \bar{\chi})\tilde{e}_b^\mu \gamma^b \chi + \frac{i}{2}\bar{\chi}\tilde{e}_b^\mu \tilde{\omega}_{\mu cd}\{\gamma^b, \sigma^{cd}\}\chi\Big) - \bar{\chi}Ma\chi\Big] \\
=& \int d\eta\, d^3x \Big\{\Big(1 + \frac{\kappa}{2}h + \frac{1}{8}\kappa^2 h^2 - \frac{1}{4}\kappa^2 h^{\lambda\tau}h_{\lambda\tau}\Big)\Big[\frac{i}{2}\bar{\chi}(\delta_b^\mu - \frac{\kappa}{2}h_b^\mu + \frac{3}{8}\kappa^2 h^{\mu\rho}h_{\rho b})\gamma^b \partial_\mu \chi \\
&- \frac{i}{2}(\partial_\mu \bar{\chi})(\delta_b^\mu - \frac{\kappa}{2}h_b^\mu + \frac{3}{8}\kappa^2 h^{\mu\rho}h_{\rho b})\gamma^b \chi - \bar{\chi}Ma\chi\Big] \\
&- \frac{1}{4}\Big[\kappa(\partial_d h_{bc}) - \frac{\kappa^2}{4}\Big(2\partial_d(h_b^\sigma h_{\sigma c}) + h_c^\sigma(\partial_b h_{\sigma d}) + 2h_d^\sigma(\partial_\sigma h_{bc}) - 2h(\partial_d h_{bc})\Big)\Big]\bar{\chi}\{\gamma^b, \sigma^{cd}\}\chi\Big\} \\
=& \int d\eta\, d^3x \Big\{\frac{i}{2}\bar{\chi}\overrightarrow{\not{\partial}}\chi - \frac{i}{2}\bar{\chi}\overleftarrow{\not{\partial}}\chi - \bar{\chi}Ma\chi \\
&+ \frac{\kappa}{2}\Big[h\Big(\frac{i}{2}\bar{\chi}\overrightarrow{\not{\partial}}\chi - \frac{i}{2}\bar{\chi}\overleftarrow{\not{\partial}}\chi - \bar{\chi}Ma\chi\Big) - \frac{i}{2}h_b^\mu\big(\bar{\chi}\gamma^b\partial_\mu\chi - (\partial_\mu\bar{\chi})\gamma^b\chi\big) - \frac{1}{2}(\partial_d h_{bc})\bar{\chi}\{\gamma^b, \sigma^{cd}\}\chi\Big] \\
&+ \frac{\kappa^2}{8}\Big[(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\Big(\frac{i}{2}\bar{\chi}\overrightarrow{\not{\partial}}\chi - \frac{i}{2}\bar{\chi}\overleftarrow{\not{\partial}}\chi - \bar{\chi}Ma\chi\Big) + \frac{i}{2}(3h^{\mu\rho}h_{\rho b} - 2hh_b^\mu)\big(\bar{\chi}\gamma^b\partial_\mu\chi - (\partial_\mu\bar{\chi})\gamma^b\chi\big) \\
&+ \frac{1}{2}\Big(2\partial_d(h_b^\sigma h_{\sigma c}) + h_c^\sigma(\partial_b h_{\sigma d}) + 2h_d^\sigma(\partial_\sigma h_{bc}) - 2h(\partial_d h_{bc})\Big)\bar{\chi}\{\gamma^b, \sigma^{cd}\}\chi\Big]\Big\}.
\end{aligned} \tag{3.19}$$

In the last equality we introduce Feynman slash notation: $\not{p} = \gamma^\mu p_\mu$, with arrows indicating the field on which the derivative acts. Notice that the terms zeroth order in κ form the flat rescaled Dirac action, which is exactly what we would expect. At several places, we used the defining equation of the gamma matrices to interchange them, with the addition of a Minkowski metric. Notice that $\gamma^b\sigma^{cd} \notin \Gamma$. It forms a 4×4 matrix, so we can use the identities in appendix A to decompose it. We find that it is written as a product of $\gamma^5\gamma^a$ and ϵ^{abcd}, the last of which is the fully antisymmetric Levi-Civita symbol. The terms with symmetry in any of these indices therefore drop out, simplifying the second and the last line of (3.19). We can now write the action in the following short way, to group similar terms:

$$S[\chi,\bar\chi] = S^{(0,2)}[\chi,\bar\chi] + S^{(1,2)}[\chi,\bar\chi] + S^{(2,2)}[\chi,\bar\chi] + \mathcal{O}\big((h_{\mu\nu})^3\big), \tag{3.20}$$

where the superscript denotes (*# of gravitational perturbations , # of spinors*), and

$$S^{(0,2)}[\chi,\bar\chi] = \int d\eta\, d^3x\Big(\frac{i}{2}\bar\chi\overrightarrow{\partial\!\!\!/}\chi - \frac{i}{2}\bar\chi\overleftarrow{\partial\!\!\!/}\chi - \bar\chi Ma\chi\Big) = \int d\eta\, d^3x\mathcal{L}_0, \tag{3.21}$$

$$\begin{aligned}S^{(1,2)}[\chi,\bar\chi] =&\frac{\kappa}{2}\int d\eta\, d^3x\Big[h\Big(\frac{i}{2}\bar\chi\overrightarrow{\partial\!\!\!/}\chi - \frac{i}{2}\bar\chi\overleftarrow{\partial\!\!\!/}\chi - \bar\chi Ma\chi\Big) - \frac{i}{2}h^\mu_b\big(\bar\chi\gamma^b\partial_\mu\chi - (\partial_\mu\bar\chi)\gamma^b\chi\big)\Big]\\ =&\frac{\kappa}{2}\int d\eta\, d^3x(h\mathcal{L}_0 - h^{\mu\nu}\mathcal{L}_{\mu\nu}),\end{aligned} \tag{3.22}$$

$$S^{(2,2)}[\chi,\bar\chi] =\frac{\kappa^2}{8}\int d\eta\, d^3x\Big((h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\mathcal{L}_0 + (3h^{\mu\rho}h^\nu_\rho - 2hh^{\mu\nu})\mathcal{L}_{\mu\nu} - \frac{1}{2}h^\rho_\mu(\partial_\sigma h_{\rho\nu})\bar\chi\epsilon^{\tau\sigma\mu\nu}\gamma_\tau\gamma^5\chi\Big), \tag{3.23}$$

where we defined $\mathcal{L}_0$ as the flat space invariant lagrangian, and $\mathcal{L}_{(\mu\nu)}$ as a 'non-Lorentz-covariant' term to increase readability where needed. Notice we also replaced all Latin indices by Greek, as a distinction is no longer needed: curved space is now represented by flat space perturbations. Indices are therefore also raised and lowered by a Minkowski metric. Observe that the final term in (3.23) is parity-breaking [14]. The equation (3.20) accompanied by (3.21), (3.22) and (3.23), is the central result of this chapter, as it will serve as our rescaled and expanded matter action in chapter 5. Given the importance of this part of the action, we check it with literature in appendix C.

3.3 Energy-momentum tensor

With (3.20) being the action we were after at the start of the chapter, we can now calculate the energy-momentum tensor, defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{3.24}$$

We want to write this in terms of $h_{\mu\nu}$, but we see there are higher order terms arising from (3.13) for $g^{\mu\nu}$. By making use of $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\lambda\nu}\delta g_{\rho\lambda}$, $(-g)^{-1/2} \approx a^{-4}(1-\frac{\kappa}{2}h)$, and the fact that the variation of the scale factor can be neglected on account of an adiabatically slowly expanding universe,

$$\mathcal{O}\left(\frac{\mathcal{H}^2}{(ma)^2}, \frac{\mathcal{H}'}{(ma)^2}\right) \ll 1, \tag{3.25}$$

we can write for the energy-momentum tensor,

$$T_{\mu\nu} =g_{\rho\mu}g_{\lambda\nu}\frac{2}{\sqrt{-g}}\frac{\delta S[\chi,\bar\chi]}{\delta g_{\rho\lambda}} = a^{-2}(\eta_{\rho\mu}+\kappa h_{\rho\mu})(\eta_{\lambda\nu}+\kappa h_{\lambda\nu})\Big(1-\frac{\kappa}{2}h\Big)\frac{2}{\kappa}\frac{\delta S[\chi,\bar\chi]}{\delta h_{\rho\lambda}} + \mathcal{O}\big((h_{\mu\nu})^2\big). \tag{3.26}$$

We obtain the following terms for (3.22) and (3.23), which we call the A and B terms, up to linear order in κ

$$\begin{aligned}a^2T^A_{\mu\nu} =&\Big(\eta_{\rho\mu}\eta_{\lambda\nu} + \kappa\Big[h_{\lambda\nu}\eta_{\rho\mu} + h_{\rho\mu}\eta_{\lambda\nu} - \frac{h}{2}\eta_{\rho\mu}\eta_{\lambda\nu}\Big]\Big)\big(\eta^{\rho\lambda}\mathcal{L}_0 - \mathcal{L}^{(\rho\lambda)}\big) + \mathcal{O}\big((h_{\mu\nu})^2\big),\\ =&\eta_{\mu\nu}\mathcal{L}_0 - \mathcal{L}_{(\mu\nu)} + \kappa\Big[\Big(2h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}\Big)\mathcal{L}_0 + \frac{h}{2}\mathcal{L}_{(\mu\nu)} - \big(h^{(\tau}_\nu\delta^{\sigma)}_\mu + h^{(\tau}_\mu\delta^{\sigma)}_\nu\big)\mathcal{L}_{\tau\sigma}\Big] + \mathcal{O}((h_{\mu\nu})^2)\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
a^2T^B_{\mu\nu} =&\eta_{\rho\mu}\eta_{\lambda\nu}\kappa\Big[\Big(\frac{h}{2}\eta^{\rho\lambda}-h^{\rho\lambda}\Big)\mathcal{L}_0+\frac{3}{4}\big(h^{\tau(\lambda}\eta^{\rho)\sigma}+h^{\sigma(\lambda}\eta^{\rho)\tau}\big)\mathcal{L}_{\tau\sigma}-\frac{1}{2}\big(\eta^{\rho\lambda}h^{\tau\sigma}+h\eta^{\rho(\tau}\eta^{\sigma)\lambda}\big)\mathcal{L}_{\tau\sigma}\\
&-\frac{1}{8}\big(\eta^{\kappa(\rho}\delta^{\lambda)}_{\phi}\partial_\sigma h_{\kappa\omega}-\eta^{\kappa(\rho}\delta^{\lambda)}_{\omega}\partial_\sigma h_{\kappa\phi}\big)\bar{\chi}\epsilon^{\tau\sigma\phi\omega}\gamma_\tau\gamma^5\chi\Big]+\mathcal{O}((h_{\mu\nu})^2)\\
=&\kappa\Big[\Big(\frac{h}{2}\eta_{\mu\nu}-h_{\mu\nu}\Big)\mathcal{L}_0+\frac{3}{4}\big(h^{(\tau}_{\mu}\delta^{\sigma)}_{\nu}+h^{(\tau}_{\nu}\delta^{\sigma)}_{\mu}\big)\mathcal{L}_{\tau\sigma}-\frac{1}{2}\big(h^{\tau\sigma}\eta_{\mu\nu}+h\delta^{(\tau}_{\mu}\delta^{\sigma)}_{\nu}\big)\mathcal{L}_{\tau\sigma}\\
&+\frac{1}{4}(\partial_\sigma h_{\phi(\mu}\eta_{\nu)\omega})\bar{\chi}\epsilon^{\tau\sigma\phi\omega}\gamma_\tau\gamma^5\chi\Big]+\mathcal{O}((h_{\mu\nu})^2),
\end{aligned}
\tag{3.28}
$$

where we note that upon varying the last line of (3.23), boundary terms drop out on account of antisymmetry. It is important to see that $T^A_{\mu\nu}$ and $T^B_{\mu\nu}$ have no physical meaning by themselves, however, adding the two contributions yields the complete energy-momentum tensor up to quadratic order in gravitational perturbations,

$$
\begin{aligned}
a^2T_{\mu\nu} =&a^2T^A_{\mu\nu}+a^2T^B_{\mu\nu}\\
=&\eta_{\mu\nu}\mathcal{L}_0-\mathcal{L}_{(\mu\nu)}+\kappa\Big[h_{\mu\nu}\mathcal{L}_0-\frac{1}{2}\eta_{\mu\nu}h^{\tau\sigma}\mathcal{L}_{\tau\sigma}-\frac{1}{2}h^{\tau}_{(\mu}\delta^{\sigma}_{\nu)}\mathcal{L}_{(\tau\sigma)}+\frac{1}{4}(\partial_\sigma h_{\phi(\mu}\eta_{\nu)\omega})\bar{\chi}\epsilon^{\tau\sigma\phi\omega}\gamma_\tau\gamma^5\chi\Big]\\
&+\mathcal{O}((h_{\mu\nu})^2).
\end{aligned}
\tag{3.29}
$$

We check this energy-momentum tensor with literature in appendix C. Notice that

$$
S^{(1,2)}[\chi,\bar{\chi}] =\frac{\kappa}{2}\int d\eta\ d^3x\ a^2\Big[h^{\mu\nu}-\kappa\Big(2h^{\mu\rho}h^{\nu}_{\rho}-\frac{1}{2}hh^{\mu\nu}\Big)\Big]T^A_{\mu\nu}, \tag{3.30}
$$

$$
S^{(2,2)}[\chi,\bar{\chi}] =\frac{\kappa}{2}\int d\eta\ d^3x\frac{a^2}{2}h^{\mu\nu}T^B_{\mu\nu}. \tag{3.31}
$$

3.4 Dirac equation

We can also take a look at the Dirac equation, which is the equation of motion for ψ, and see how it progresses through the processes of conformal rescaling and perturbatively expanding. The Dirac equation and its complement are obtained by varying the action (3.1) with respect to the fields ψ and $\bar{\psi}$, and setting the variation to 0 [21]:

$$
\big(ie^{\mu}_{b}\gamma^b\overrightarrow{D}_\mu-M\big)\psi =0, \tag{3.32}
$$

$$
\bar{\psi}\big(i\overleftarrow{D}_\mu e^{\mu}_{b}\gamma^b+M\big) =0. \tag{3.33}
$$

If we now apply (3.5) and (3.7) again, and rescale the fermion field in the same way as we did before, we get

$$
\begin{aligned}
\big(ie^{\mu}_{b}\gamma^bD_\mu-m\big)\psi =&0=\Big(ia^{-1}\tilde{e}^{\mu}_{b}\gamma^b\Big[\partial_\mu+\frac{i}{2}(-2a^{-1}\tilde{e}^{\nu}_{c}\tilde{e}_{\mu d}\partial_\nu a+\tilde{\omega}_{\mu cd})\sigma^{cd}\Big]-M\Big)a^{-3/2}\chi\\
=&\Big(ia^{-5/2}\tilde{e}^{\mu}_{b}\gamma^b\tilde{D}_\mu-a^{-3/2}M-\frac{3}{2}ia^{-7/2}\tilde{e}^{\mu}_{b}\gamma^b\partial_\mu a+a^{-7/2}\tilde{e}^{\nu}_{c}\gamma_d\sigma^{cd}\partial_\nu a\Big)\chi\\
=&a^{-5/2}\big(i\tilde{e}^{\mu}_{b}\gamma^b\tilde{D}_\mu-Ma\big)\chi+\frac{i}{4}a^{-7/2}(\partial_\mu a)\Big(\tilde{e}^{\mu}_{c}[\gamma_d\gamma^c\gamma^d-\gamma_d\gamma^d\gamma^c]-6\tilde{e}^{\mu}_{b}\gamma^b\Big)\chi,\\
\Longrightarrow&\ \big(i\tilde{e}^{\mu}_{b}\gamma^b\tilde{D}_\mu-Ma\big)\chi=0.
\end{aligned}
\tag{3.34}
$$

For the gamma matrices, we used again $\gamma_d\gamma^d=-4I$, and also $\gamma_d\gamma^c\gamma^d=2\gamma^c$. For (3.33), the derivation proceeds in a similar manner, taking into account the different covariant derivative, and the placement of the gamma matrix with respect to it. So we see that the Dirac equation (unsurprisingly) also holds up when conformally rescaling, and simply yields the variations of (3.11) with respect to the new fields.

Expanding these derived equations then gives us, using (3.13),

$$
\begin{aligned}
0 =&(i\tilde{e}^{\mu}_{b}\gamma^{b}\tilde{D}_{\mu}-Ma)\chi=\Big(i\tilde{e}^{\mu}_{b}\gamma^{b}\Big[\partial_{\mu}+\frac{i}{2}\tilde{\omega}_{\mu cd}\sigma^{cd}\Big]-Ma\Big)\chi\\
=&\bigg[i\Big(\delta^{\mu}_{b}-\frac{\kappa}{2}h^{\mu}_{b}+\frac{3}{8}\kappa^{2}h^{\mu\rho}h_{\rho b}\Big)\gamma^{b}\Big(\partial_{\mu}+\frac{i}{2}\Big[\kappa(\partial_{d}h_{\mu c})\sigma^{cd}+\frac{\kappa^{2}}{2}h^{\sigma}_{d}\Big(\frac{1}{2}\partial_{\mu}h_{\sigma c}+\partial_{c}h_{\sigma\mu}-\partial_{\sigma}h_{\mu c}\Big)\sigma^{cd}\Big]\Big)\\
&-Ma\bigg]\chi\\
=&(i\partial\!\!\!/-Ma)\chi-\frac{\kappa}{2}\Big(ih^{\mu}_{b}\gamma^{b}\partial_{\mu}+(\partial_{d}h_{bc})\gamma^{b}\sigma^{cd}\Big)\chi+i\frac{3\kappa^{2}}{8}h^{\mu\sigma}h_{\sigma b}\gamma^{b}\partial_{\mu}\chi\\
&+\frac{\kappa^{2}}{4}\Big(h^{\sigma}_{b}\partial_{d}h_{\sigma c}-\frac{1}{2}h^{\sigma}_{d}\partial_{b}h_{\sigma c}-h^{\sigma}_{d}\partial_{c}h_{\sigma b}+h^{\sigma}_{d}\partial_{\sigma}h_{bc}\Big)\gamma^{b}\sigma^{cd}\chi\\
=&(i\partial\!\!\!/-Ma)\chi-\frac{\kappa}{2}(ih^{\mu\nu}\gamma_{\nu}\partial_{\mu}+(\partial^{\mu}h^{\rho\sigma})\gamma_{\rho}\sigma_{\sigma\mu})\chi+\frac{\kappa^{2}}{4}\Big(\partial^{\mu}(h^{\rho\tau}h^{\sigma}_{\tau})-\frac{1}{2}h^{\tau\mu}\partial^{\rho}h^{\sigma}_{\tau}+h^{\mu\tau}\partial_{\tau}h^{\rho\sigma}\Big)\gamma_{\rho}\sigma_{\sigma\mu}\chi\\
&+i\frac{3\kappa^{2}}{8}h^{\mu\tau}h^{\nu}_{\tau}\gamma_{\nu}\partial_{\mu}\chi.
\end{aligned}
\tag{3.35}
$$

In the last line, we again replaced all Latin indices by Greek ones. If we now write σ_{cd} fully, we get

$$
\begin{aligned}
0 =&(i\partial\!\!\!/-Ma)\chi-i\frac{\kappa}{2}\Big[h^{\mu\nu}\gamma_{\nu}\partial_{\mu}+\frac{1}{4}(\partial^{\mu}h^{\rho\sigma}-\partial^{\sigma}h^{\rho\mu})(\gamma_{\rho}\gamma_{\sigma}\gamma_{\mu}-\gamma_{\rho}\eta_{\sigma\mu})\Big]\chi\\
&+i\frac{\kappa^{2}}{8}\Big[\partial^{\mu}(h^{\rho\tau}h^{\sigma}_{\tau})-\frac{1}{2}h^{\tau\mu}\partial^{\rho}h^{\sigma}_{\tau}+h^{\mu\tau}\partial_{\tau}h^{\rho\sigma}-(\mu\leftrightarrow\sigma)\Big](\gamma_{\rho}\gamma_{\sigma}\gamma_{\mu}-\gamma_{\rho}\eta_{\sigma\mu})\chi+i\frac{3\kappa^{2}}{8}h^{\mu\tau}h^{\nu}_{\tau}\gamma_{\nu}\partial_{\mu}\chi\\
=&(i\partial\!\!\!/-Ma)\chi-i\frac{\kappa}{2}\Big[h^{\mu\nu}\gamma_{\nu}\partial_{\mu}+\frac{1}{4}(\partial^{\mu}h^{\rho\sigma}-\partial^{\sigma}h^{\rho\mu})(-\eta_{\rho\sigma}\gamma_{\mu}+\eta_{\rho\mu}\gamma_{\sigma})\Big]\chi\\
&+i\frac{\kappa^{2}}{8}\Big[\partial^{\mu}(h^{\rho\tau}h^{\sigma}_{\tau})-\frac{1}{2}h^{\tau\mu}\partial^{\rho}h^{\sigma}_{\tau}+h^{\mu\tau}\partial_{\tau}h^{\rho\sigma}-(\mu\leftrightarrow\sigma)\Big](\gamma_{\rho}\gamma_{\sigma}\gamma_{\mu}-\gamma_{\rho}\eta_{\sigma\mu})\chi+i\frac{3\kappa^{2}}{8}h^{\mu\tau}h^{\nu}_{\tau}\gamma_{\nu}\partial_{\mu}\chi\\
=&(i\partial\!\!\!/-Ma)\chi-i\frac{\kappa}{2}\Big[h^{\mu\nu}\gamma_{\nu}\partial_{\mu}+\frac{1}{2}(\partial_{\rho}h^{\rho\sigma}-\partial^{\sigma}h)\gamma_{\sigma}\Big]\chi\\
&+i\frac{\kappa^{2}}{8}\Big[\partial^{\mu}(h^{\rho\tau}h^{\sigma}_{\tau})-\frac{1}{2}h^{\tau\mu}\partial^{\rho}h^{\sigma}_{\tau}+h^{\mu\tau}\partial_{\tau}h^{\rho\sigma}-(\mu\leftrightarrow\sigma)\Big](\gamma_{\rho}\gamma_{\sigma}\gamma_{\mu}-\gamma_{\rho}\eta_{\sigma\mu})\chi+i\frac{3\kappa^{2}}{8}h^{\mu\tau}h^{\nu}_{\tau}\gamma_{\nu}\partial_{\mu}\chi.
\end{aligned}
\tag{3.36}
$$

Multiplying this by the expanded version of $\sqrt{-g}$ gives the same result as varying (3.20) with respect to $\bar{\chi}$, as one would expect.

4 Einstein-Hilbert action

In the previous chapter, we looked at the matter part of our action, from which the energy-momentum tensor can be derived. While we already saw gravity appearing in the different fermion-graviton interaction terms, we also need a part of the action that deals with how gravity evolves by itself, called the Einstein-Hilbert action. We briefly mentioned the Einstein field equations (2.5) in chapter 2, where we saw the Einstein tensor appearing. Similar to how the energy-momentum tensor was found by varying the matter action, we can find the Einstein tensor from varying the Einstein-Hilbert action. The action is

$$S_g = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}(\mathcal{R}-2\Lambda), \tag{4.1}$$

which looks deceptively more simple than the Dirac action (3.1). Λ is the cosmological constant, which is the contribution from dark energy. This is a constant, homogeneous contribution to the background density, and it is therefore easy to include in the formalism. For generality, we will keep it in the equations, but drop it later for brevity (notice that we do not disregard dark energy, but acknowledge that the dynamics we are interested in do not come from the cosmological constant).

The Ricci scalar $\mathcal{R}$ contains information about curvature, which will become clear from the calculations. In a flat, constantly expanding universe, it will for example be 0. It is obtained from the Ricci tensor $\mathcal{R}_{\mu\nu}$, and the Ricci tensor is obtained from the Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}$, in the following way

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}, \qquad \mathcal{R}_{\mu\nu} = \mathcal{R}^{\rho}_{\mu\rho\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}. \tag{4.2}$$

To be able to rescale (4.2) in the same way we did in the previous chapter, we rewrite it in terms of the vierbeins and $\omega_{\mu cd}$. First, we note that the latter is antisymmetric in its Latin indices, making use of the identities

$$\begin{aligned} \partial_\sigma\delta^\mu_\rho = \partial_\sigma(g^{\mu\nu}g_{\rho\nu}) = g^{\mu\nu}\partial_\sigma g_{\rho\nu} + g_{\rho\nu}\partial_\sigma g^{\mu\nu} = 0 \implies g_{\rho\nu}\partial_\sigma g^{\mu\nu} = -g^{\mu\nu}\partial_\sigma g_{\rho\nu}, \\ \partial_\sigma\delta^\mu_\rho = \partial_\sigma(e^c_\rho e^\mu_c) = e^\mu_c\partial_\sigma e^c_\rho + e^c_\rho\partial_\sigma e^\mu_c = 0 \implies e^\mu_c\partial_\sigma e^c_\rho = -e^c_\rho\partial_\sigma e^\mu_c. \end{aligned} \tag{4.3}$$

The antisymmetry then becomes apparent in $\omega_{\mu cd}$ by looking at

$$\begin{aligned} \omega_{\mu dc} &= e^\nu_d(\partial_\mu e_{\nu c}) - e^\nu_d e_{\rho c}\Gamma^\rho_{\mu\nu} = -e_{\nu c}\partial_\mu(g^{\nu\rho}e_{\rho d}) - e_{\rho c}e_{\tau d}g^{\tau\nu}\left(\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\right) \\ &= -e_{\nu c}e_{\rho d}\partial_\mu g^{\rho\nu} - e^\rho_c\partial_\mu e_{\rho d} - e_{\rho c}e_{\tau d}\left(\frac{1}{2}g^{\rho\nu}g^{\tau\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\sigma g_{\nu\mu} - \partial_\nu g_{\mu\sigma})\right) \\ &= -e_{\nu c}e_{\rho d}\partial_\mu g^{\rho\nu} - e^\nu_c\partial_\mu e_{\nu d} + e^\nu_c e_{\tau d}\left(\frac{1}{2}g^{\tau\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) - g^{\tau\sigma}\partial_\mu g_{\sigma\nu}\right) \\ &= -e^\nu_c(\partial_\mu e_{\nu d} - e_{\rho d}\Gamma^\rho_{\mu\nu}) - e_{\nu c}e_{\rho d}\partial_\mu g^{\rho\nu} + e_{\sigma c}e_{\tau d}\partial_\mu g^{\tau\sigma} \\ &= -\omega_{\mu cd}. \end{aligned} \tag{4.4}$$

In the second line, we interchanged the summed-over indices ν and σ in the last term. In the third line, the most important step happens: we arrange the derivatives of the metric in such a way, that we obtain a new Christoffel symbol $\Gamma^\tau_{\mu\nu}$, at the cost of the extra term in the large parentheses. We then see in the fourth line that this extra term exactly cancels against the metric derivative we obtained in the first line.

The point of showing this antisymmetry in $\omega_{\mu cd}$ is to use it in re-expressing the Christoffel symbol, which we do as follows

$$\begin{aligned} e^{d\lambda}e^c_\sigma\omega_{\mu cd} = e^{d\lambda}\delta^\nu_\sigma\big(\partial_\mu e_{\nu d} - \Gamma^\rho_{\mu\nu}e_{\rho d}\big) = e^{d\lambda}\partial_\mu e_{\sigma d} - \Gamma^\lambda_{\mu\sigma} \\ \implies \Gamma^\rho_{\nu\mu} = e^{d\rho}\big(\partial_\nu e_{\mu d} - e^c_\mu\omega_{\nu cd}\big) = e^{c\rho}\big(\partial_\nu e_{\mu c} + e^d_\mu\omega_{\nu cd}\big). \end{aligned} \tag{4.5}$$

In the last equality, we changed $d \to c$ in the first term, and $c \leftrightarrow d$ at the cost of a minus sign in the second due to the new-found antisymmetry in $\omega_{\mu cd}$. We can now rewrite (4.2):

$$\begin{aligned}\mathcal{R}_{\mu\nu} =&(\partial_\rho e^{c\rho})(\partial_\nu e_{\mu c}) + e^{c\rho}(\partial_\rho\partial_\nu e_{\mu c}) + (\partial_\rho e^{c\rho}) + e^{c\rho}(\partial_\rho e^d_\mu)\omega_{\nu cd} + e^{c\rho}e^d_\mu(\partial_\rho\omega_{\nu cd}) - (\partial_\nu e^{c\rho})(\partial_\rho e_{\mu c})\\ &- e^{c\rho}(\partial_\nu\partial_\rho e_{\mu c}) - (\partial_\nu e^{c\rho}) - e^{c\rho}(\partial_\nu e^d_\mu)\omega_{\rho cd} - e^{c\rho}e^d_\mu(\partial_\nu\omega_{\rho cd}) + e^{c\rho}e^{a\lambda}(\partial_\rho e_{\lambda c})(\partial_\nu e_{\mu a})\\ &+ e^{c\rho}e^{a\lambda}(\partial_\rho e_{\lambda c})e^b_\mu\omega_{\nu ab} + e^{c\rho}e^{a\lambda}e^d_\lambda\omega_{\rho cd}(\partial_\nu e_{\mu a}) + e^{c\rho}e^{a\lambda}e^d_\lambda e^b_\mu\omega_{\rho cd}\omega_{\nu ab} - e^{c\rho}e^{a\lambda}(\partial_\nu e_{\lambda c})(\partial_\rho e_{\mu a})\\ &- e^{c\rho}e^{a\lambda}(\partial_\nu e_{\lambda c})e^b_\mu\omega_{\rho ab} - e^{c\rho}e^{a\lambda}e^d_\lambda\omega_{\nu cd}(\partial_\rho e_{\mu a}) - e^{c\rho}e^{a\lambda}e^d_\lambda e^b_\mu\omega_{\nu cd}\omega_{\rho ab}\\ =&e^{c\rho}\Big(e^d_\mu\partial_\rho\omega_{\nu cd} + e^b_\mu\eta^{ad}\omega_{\rho cd}\omega_{\nu ab} - (\rho\leftrightarrow\nu)\Big).\end{aligned} \tag{4.6}$$

In the final line, using $(\rho\leftrightarrow\nu)$ will significantly reduce the amount of terms we have to write out. Should there be any confusion: the interchange of these indices does not apply to $e^{c\rho}$ outside the parentheses. If we now expand the different terms according to (3.5) and (3.7), we find

$$\begin{aligned}\mathcal{R}_{\mu\nu} = \tilde{\mathcal{R}}_{\mu\nu} + \tilde{e}^{c\rho}\Big\{&\tilde{e}^d_\mu\partial_\rho\big[a^{-1}(\tilde{e}^\sigma_d\tilde{e}_{\nu c} - \tilde{e}^\sigma_c\tilde{e}_{\nu d})\partial_\sigma a\big] + \tilde{e}^b_\mu\eta^{ad}\Big[a^{-2}(\tilde{e}^\sigma_d\tilde{e}_{\rho c} - \tilde{e}^\sigma_c\tilde{e}_{\rho d})(\tilde{e}^\tau_b\tilde{e}_{\nu a} - \tilde{e}^\tau_a\tilde{e}_{\nu b})(\partial_\sigma a)(\partial_\tau a)\\ &+ a^{-1}(\tilde{e}^\sigma_d\tilde{e}_{\rho c} - \tilde{e}^\sigma_c\tilde{e}_{\rho d})(\partial_\sigma a)\tilde{\omega}_{\nu ab} + a^{-1}(\tilde{e}^\tau_b\tilde{e}_{\nu a} - \tilde{e}^\tau_a\tilde{e}_{\nu b})(\partial_\tau a)\tilde{\omega}_{\rho cd}\Big] - (\rho\leftrightarrow\nu)\Big\}.\end{aligned} \tag{4.7}$$

$\tilde{\mathcal{R}}_{\mu\nu}$ is the conformally invariant part of the Ricci tensor, so the same but with tildes on all terms, similar to what we did with the Christoffel symbol and the spin connection. Expanding up to quadratic order in perturbations, we find

$$\begin{aligned}\tilde{\mathcal{R}}_{\mu\nu} &= \tilde{e}^{c\rho}\tilde{e}^d_\mu\partial_\rho\tilde{\omega}_{\nu cd} + \tilde{e}^{c\rho}\tilde{e}^b_\mu\eta^{ad}\tilde{\omega}_{\rho cd}\omega_{\nu ab} - (\rho\leftrightarrow\nu)\\ &=\Big(\eta^{c\rho}\partial^d_\mu + \frac{\kappa}{2}(\eta^{c\rho}h^d_\mu - \delta^d_\mu h^{c\rho})\Big)\Big(\partial_\rho\Big[\frac{\kappa}{2}\partial_d h_{\nu c} + \frac{\kappa^2}{4}h^\sigma_d\big(\frac{1}{2}\partial_\nu h_{\sigma c} + \partial_c h_{\sigma\nu} - \partial_\sigma h_{\nu c}\big) - (c\leftrightarrow d)\Big] - (\rho\leftrightarrow\nu)\Big)\\ &\quad+\frac{\kappa^2}{4}\Big(\eta^{c\rho}\delta^b_\mu\eta^{ad}(\partial_d h_{\rho c} - \partial_c h_{\rho d})(\partial_b h_{\nu a} - \partial_a h_{\nu b}) - (\rho\leftrightarrow\nu)\Big) + \mathcal{O}(\kappa^2)\\ &=\frac{\kappa}{2}\Big[\partial_\rho\partial_\mu h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\Big]\\ &\quad+\frac{\kappa^2}{4}\Big[2h^{\rho\sigma}\partial_\rho\partial_\sigma h_{\mu\nu} + 2h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - 2h^{\rho\sigma}\partial_\rho\partial_\mu h_{\nu\sigma} - 2h^{\sigma\rho}\partial_\nu\partial_\sigma h_{\rho\mu} - 2(\partial_\nu h_{\sigma\mu})(\partial_\rho h^{\sigma\rho})\\ &\qquad+(\partial^\rho h^\sigma_\mu)(\partial_\rho h_{\sigma\nu}) + (\partial_\nu h^\sigma_\mu)(\partial_\sigma h) - 2(\partial_\rho h^\sigma_\mu)(\partial_\sigma h^\rho_\nu) + 2(\partial_\rho h^{\rho\sigma})(\partial_\sigma h_{\mu\nu}) + (\partial_\nu h^{\rho\sigma})(\partial_\mu h_{\rho\sigma})\\ &\qquad-2(\partial_\rho h^{\rho\sigma})(\partial_\mu h_{\sigma\nu}) + (\partial_\sigma h)(\partial_\mu h^\sigma_\nu) - (\partial_\sigma h)(\partial^\sigma h_{\mu\nu})\Big].\end{aligned} \tag{4.8}$$

We evaluate the other terms in (4.7) step-by-step. Before we do that, we first note that the spacetime indices of the rescaled vierbeins can be raised and lowered by the rescaled metric, i.e.

$$\tilde{g}_{\mu\nu}\tilde{e}^\mu_b = a^{-2}g_{\mu\nu}ae^\mu_b = a^{-1}e_{\nu b} = \tilde{e}_{\nu b} \implies \tilde{e}_{\nu b}\tilde{e}^b_\sigma = \tilde{g}_{\mu\nu}\delta^\mu_\sigma = \tilde{g}_{\sigma\nu}. \tag{4.9}$$

The implication shows that a summation over two lower Greek indices of two vierbeins gives us a rescaled metric. We will also use (4.3), which works for the rescaled metrics and vierbeins too. When a vierbein has two lower or upper Greek indices, we will use

$$\partial_\sigma\tilde{g}^{\mu\rho} = \partial_\sigma(\tilde{e}^\mu_c\tilde{e}^{\rho c}) = \tilde{e}^\mu_c\partial_\sigma\tilde{e}^{\rho c} + \tilde{e}^{\rho c}\partial_\sigma\tilde{e}^\mu_c. \tag{4.10}$$

As the derivative of a Minkowski metric is 0, we can still use (4.3) when we have two lower or upper Latin indices. Using the previous few relations, we can now evaluate the first term in the curly brackets of (4.7),

$$\begin{aligned}&\tilde{e}^{c\rho}\Big(\tilde{e}^d_\mu\partial_\rho\big[a^{-1}(\tilde{e}^\sigma_d\tilde{e}_{\nu c} - \tilde{e}^\sigma_c\tilde{e}_{\nu d})\partial_\sigma a\big] - (\rho\leftrightarrow\nu)\Big)\\ &=a^{-2}\Big[2(\partial_\mu a)(\partial_\nu a) + \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}(\partial_\rho a)(\partial_\sigma a)\Big] + a^{-1}\Big[-2(\partial_\mu\partial_\nu a) - \tilde{e}^{c\rho}(\partial_\nu\tilde{e}_{c\rho})(\partial_\mu a) - \tilde{e}^d_\mu(\partial_\nu\tilde{e}^\sigma_d)(\partial_\sigma a)\\ &\quad+\tilde{g}^{\rho\sigma}(\partial_\nu\tilde{g}_{\rho\mu})(\partial_\sigma a) - \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\partial_\rho\partial_\sigma a + \tilde{e}^{c\rho}(\partial_\rho\tilde{e}_{\nu c})(\partial_\mu a) - \tilde{g}^{\rho\sigma}\tilde{e}^d_\mu(\partial_\rho\tilde{e}_{\nu d})(\partial_\sigma a) - \tilde{g}_{\mu\nu}\tilde{e}^{c\rho}(\partial_\rho\tilde{e}^\sigma_c)(\partial_\sigma a)\Big].\end{aligned} \tag{4.11}$$

While we could expand the different terms in terms of perturbations, this would complicate the expression even more. We will first evaluate the other terms, and for the second term in the curly brackets of (4.7) (which is much easier to evaluate as there are no derivatives on a vierbein) we get,

$$\begin{aligned} &a^{-2}\tilde{e}^{c\rho}\Big(\tilde{e}^b_\mu\eta^{ad}(\tilde{e}^\sigma_d\tilde{e}_{\rho c}-\tilde{e}^\sigma_c\tilde{e}_{\rho d})(\tilde{e}^\tau_b\tilde{e}_{\nu a}-\tilde{e}^\tau_a\tilde{e}_{\nu b})(\partial_\sigma a)(\partial_\tau a)-(\rho\leftrightarrow\nu)\Big)\\ &=2a^{-2}(\partial_\mu a)(\partial_\nu a)-2a^{-2}\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}(\partial_\rho a)(\partial_\sigma a). \end{aligned} \tag{4.12}$$

For the third and fourth terms, we will also not expand $\tilde{\omega}_{\mu cd}$, but we will write it in terms of rescaled vierbeins and Christoffel symbols, so for the third term we have,

$$\begin{aligned} &a^{-1}\tilde{e}^{c\rho}\Big(\tilde{e}^b_\mu\eta^{ad}(\tilde{e}^\sigma_d\tilde{e}_{\rho c}-\tilde{e}^\sigma_c\tilde{e}_{\rho d})(\partial_\sigma a)(\tilde{e}^\tau_a\partial_\nu\tilde{e}_{\tau b}-\tilde{e}^\tau_a\tilde{\Gamma}^\lambda_{\nu\tau}\tilde{e}_{\lambda b})-(\rho\leftrightarrow\nu)\Big)\\ &=2a^{-1}\tilde{g}^{\rho\sigma}\tilde{e}^b_\mu(\partial_\nu\tilde{e}_{\rho b})(\partial_\sigma a)+a^{-1}\tilde{g}^{\rho\sigma}\tilde{e}^b_\mu(\partial_\rho\tilde{e}_{\nu b})(\partial_\sigma a)-3a^{-1}\tilde{g}_{\lambda\mu}g^{\sigma\tau}\tilde{\Gamma}^\lambda_{\nu\tau}(\partial_\sigma a). \end{aligned} \tag{4.13}$$

In the fourth term, we will find a rescaled Christoffel symbol with repeating indices, which can be evaluated as

$$\tilde{\Gamma}^\sigma_{\nu\sigma}=\frac{1}{2}\tilde{g}^{\sigma\rho}(\partial_\nu\tilde{g}_{\sigma\rho}+\partial_\sigma\tilde{g}_{\rho\nu}-\partial_\rho\tilde{g}_{\nu\sigma})=\frac{1}{2}\tilde{g}^{\sigma\rho}\partial_\nu\tilde{g}_{\sigma\rho}=\frac{1}{2}\tilde{e}^{c\sigma}\tilde{e}^\rho_c\partial_\nu(\tilde{e}^b_\sigma\tilde{e}_{b\rho})=\tilde{e}^{\sigma c}\partial_\nu\tilde{e}_{\sigma c}, \tag{4.14}$$

and we find

$$\begin{aligned} &a^{-1}\tilde{e}^{c\rho}\Big(\tilde{e}^b_\mu\eta^{ad}(\tilde{e}^\tau_b\tilde{e}_{\nu a}-\tilde{e}^\tau_a\tilde{e}_{\nu b})(\partial_\tau a)\tilde{\omega}_{\rho cd}-(\rho\leftrightarrow\nu)\Big)\\ &=a^{-1}\Big[\tilde{g}^{\rho\sigma}\tilde{e}^d_\nu(\partial_\rho\tilde{e}_{\sigma d})(\partial_\mu a)-\tilde{g}^{\rho\sigma}\tilde{g}_{\nu\lambda}\tilde{\Gamma}^\lambda_{\rho\sigma}(\partial_\mu a)+\tilde{g}_{\mu\nu}\tilde{e}^\rho_d(\partial_\rho\tilde{e}^{\tau d})(\partial_\tau a)+\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\tau_{\rho\sigma}(\partial_\tau a)\\ &\quad+\tilde{e}^{\tau d}(\partial_\nu\tilde{e}_{\mu d})(\partial_\tau a)-\tilde{\Gamma}^\tau_{\mu\nu}(\partial_\tau a)\Big]. \end{aligned} \tag{4.15}$$

We can now substitute the contributions (4.11), (4.12), (4.13) and (4.15) into (4.7),

$$\begin{aligned} \mathcal{R}_{\mu\nu}=&\tilde{\mathcal{R}}_{\mu\nu}+a^{-2}\Big[4(\partial_\mu a)(\partial_\nu a)-\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}(\partial_\rho a)(\partial_\sigma a)\Big]-a^{-1}\Big[2\partial_\mu\partial_\nu a+\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\partial_\rho\partial_\sigma a\Big]\\ &+a^{-1}\Big[\tilde{g}^{\rho\sigma}\tilde{e}^d_\nu(\partial_\rho\tilde{e}_{\sigma d})+\tilde{e}^{c\rho}(\partial_\rho\tilde{e}_{\nu c})-\tilde{e}^{c\rho}(\partial_\nu\tilde{e}_{c\rho})-\tilde{g}^{\rho\sigma}\tilde{g}_{\nu\lambda}\tilde{\Gamma}^\lambda_{\rho\sigma}\Big](\partial_\mu a)\\ &+a^{-1}\Big[2\tilde{g}^{\rho\sigma}\tilde{e}^b_\mu(\partial_\nu\tilde{e}_{\rho b})+\tilde{e}^{\sigma d}(\partial_\nu\tilde{e}_{\mu d})-\tilde{e}^d_\mu(\partial_\nu\tilde{e}^\sigma_d)+\tilde{g}^{\rho\sigma}(\partial_\nu\tilde{g}_{\rho\mu})-3\tilde{g}_{\lambda\mu}g^{\sigma\tau}\tilde{\Gamma}^\lambda_{\nu\tau}\\ &\quad+\tilde{g}^{\rho\lambda}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\sigma_{\rho\lambda}-\tilde{\Gamma}^\sigma_{\mu\nu}\Big](\partial_\sigma a), \end{aligned} \tag{4.16}$$

where we have already added and subtracted several terms, and grouped similar terms together by changing summed-over indices. By writing out the rescaled Christoffel symbol, as well as (4.10) and the last two equalities of (4.14), one can see that the second line drops out entirely. We can also use these equations to see that the first three terms of the third line are equal to $2\tilde{g}^{\rho\sigma}\partial_\nu\tilde{g}_{\rho\mu}$. We can finally look at

$$3\tilde{g}^{\rho\sigma}\partial_\nu\tilde{g}_{\rho\mu}-3\tilde{g}_{\lambda\mu}\tilde{g}^{\sigma\tau}\tilde{\Gamma}^\lambda_{\nu\tau}=\frac{3}{2}\tilde{g}^{\rho\sigma}(\partial_\mu\tilde{g}_{\rho\nu}+\partial_\nu\tilde{g}_{\mu\rho}-\partial_\rho\tilde{g}_{\mu\nu})=\tilde{\Gamma}^\sigma_{\mu\nu}, \tag{4.17}$$

and find that the Ricci tensor is

$$\begin{aligned} \mathcal{R}_{\mu\nu}=&\tilde{\mathcal{R}}_{\mu\nu}+a^{-2}\Big[4(\partial_\mu a)(\partial_\nu a)-\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}(\partial_\rho a)(\partial_\sigma a)\Big]-a^{-1}\Big[2\partial_\mu\partial_\nu a+\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\partial_\rho\partial_\sigma a\Big]\\ &+a^{-1}\Big[2\tilde{\Gamma}^\sigma_{\mu\nu}+\tilde{g}^{\rho\lambda}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\sigma_{\rho\lambda}\Big](\partial_\sigma a). \end{aligned} \tag{4.18}$$

From here, getting the Ricci scalar is fairly straightforward: we contract with $g^{\mu\nu}=a^{-2}\tilde{g}^{\mu\nu}$, so

$$\mathcal{R}=a^{-2}\tilde{\mathcal{R}}-6a^{-3}\tilde{g}^{\mu\nu}\big(\partial_\mu\partial_\nu a-\tilde{\Gamma}^\sigma_{\mu\nu}\partial_\sigma a\big), \tag{4.19}$$

where up to quadratic order in perturbations,

$$\begin{aligned} \tilde{\mathcal{R}}\equiv&\tilde{g}^{\mu\nu}\tilde{\mathcal{R}}_{\mu\nu}=(\eta^{\mu\nu}-\frac{\kappa}{2}h^{\mu\nu})\tilde{\mathcal{R}}_{\mu\nu}+\mathcal{O}(\kappa^3)\\ =&\kappa\big[\partial^\mu\partial^\nu h_{\mu\nu}-\partial^2h\big]+\kappa^2\Big[\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma})+h^{\sigma\nu}(\partial^2h_{\sigma\nu}-2\partial_\mu\partial_\sigma h^\mu_\nu)+(\partial_\mu h^{\sigma\mu})(\partial_\sigma h)\\ &+h^{\sigma\mu}(\partial_\mu\partial_\sigma h)-\frac{1}{4}(\partial^\sigma h)(\partial_\sigma h)-(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma)-\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)\Big]+\mathcal{O}(\kappa^3). \end{aligned} \tag{4.20}$$

The conformal scaling agrees with [18], the perturbative expansion of the Ricci scalar agrees (up to a boundary term and an extension to 4 dimensions) with [22]. It is clear how the Ricci scalar and tensor are dependent on an expanding universe, and on perturbations around flat space. $\tilde{\mathcal{R}}_{\mu\nu}$ and $\tilde{\mathcal{R}}$ are both conformally invariant, and they both only contain first and second derivatives of the perturbations. If the perturbations were 0, these quantities would also be 0, as expected for a flat universe. For $\mathcal{R}$ and $\mathcal{R}_{\mu\nu}$ we also see a dependence on the first and second spacetime derivatives of the scale factor. If the factor is constant (no expansion), these terms will all be 0. What we also see is that a conformally flat universe (so no perturbations), the rescaled Christoffel symbol will vanish, and we only have a second derivative of a. The Ricci scalar and tensor will therefore be 0, corresponding to no curvature, only when the universe is FLRW with constant expansion.

As we have already seen at the start of the chapter, the Ricci scalar is the main contribution to the Einstein-Hilbert action. With the explicit form of this scalar given in (4.19) and (4.20), and using (3.5), we find

$$\begin{aligned}
S_g =&\frac{1}{\kappa^2}\int d^4x\sqrt{-g}(\mathcal{R}-2\Lambda)\\
=&\frac{1}{\kappa^2}\int d\eta\, d^3x\,\sqrt{-\tilde{g}}\Big\{a^2\kappa(\partial_\mu\partial^\nu h^\mu_\nu-\partial^2 h)+a^2\kappa^2\Big[\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma})+h^{\sigma\nu}(\partial^2 h_{\sigma\nu}-2\partial_\mu\partial_\sigma h^\mu_\nu)\\
&+(\partial_\mu h^{\sigma\mu})(\partial_\sigma h)+h^{\sigma\mu}(\partial_\mu\partial_\sigma h)-\frac{1}{4}(\partial^\sigma h)(\partial_\sigma h)-(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma)-\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)\Big]\\
&-6a\tilde{g}^{\mu\nu}(\partial_\mu\partial_\nu a-\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a))-2a^4\Lambda\Big\}+\mathcal{O}(\kappa)\\
=&\int d\eta\, d^3x\,\Big\{a^2\frac{1}{\kappa}(\partial_\mu\partial^\nu h^\mu_\nu-\partial^2 h)+a^2\Big[\frac{h}{2}(\partial_\mu\partial^\nu h^\mu_\nu-\partial^2 h)-h^{\sigma\nu}(2\partial_\mu\partial_\sigma h^\mu_\nu-\partial^2 h_{\sigma\nu})+h^{\sigma\mu}(\partial_\mu\partial_\sigma h)\\
&-(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma)-\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)-\frac{1}{4}(\partial^\sigma h)(\partial_\sigma h)+(\partial_\sigma h)(\partial_\mu h^{\sigma\mu})+\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma})\Big]\\
&-6a\Big(\frac{1}{\kappa^2}+\frac{1}{2\kappa}h-\frac{1}{4}h^{\lambda\sigma}h_{\lambda\sigma}+\frac{1}{8}h^2\Big)\tilde{g}^{\mu\nu}(\partial_\mu\partial_\nu a-\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a))-\frac{2}{\kappa^2}a^4\sqrt{-\tilde{g}}\Lambda\Big\}+\mathcal{O}(\kappa).
\end{aligned}\tag{4.21}$$

We did not expand the metric determinant that multiplies the cosmological constant, as it will not add new information at this point, but will take up more space in the equations. We can make the quadratic part look nicer at the cost of a total derivative, by applying the Leibniz identity to the following terms:

$$\begin{aligned}
a^2h\partial_\mu\partial^\nu h^\mu_\nu &= \partial_\mu(a^2h\partial^\nu h^\mu_\nu)-a^2(\partial_\mu h)(\partial^\nu h^\mu_\nu)-2a(\partial_\rho a)h\partial^\nu h^\rho_\nu,\\
-a^2h\partial^2 h &= -\partial_\mu(a^2h\partial^\mu h)+a^2(\partial_\mu h)(\partial^\mu h)+2a(\partial_\rho a)h\partial^\rho h,\\
-a^2h^{\sigma\nu}\partial_\mu\partial_\sigma h^\mu_\nu &= -\partial_\mu(a^2h^{\sigma\nu}\partial_\sigma h^\mu_\nu)+a^2(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu)+2a(\partial_\rho a)h^{\sigma\nu}\partial_\sigma h^\rho_\nu,\\
a^2h^{\nu\sigma}\partial^2 h_{\nu\sigma} &= \partial_\mu(a^2h^{\nu\sigma}\partial^\mu h_{\nu\sigma})-a^2(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma})-2a(\partial_\rho a)h^{\nu\sigma}\partial^\rho h_{\nu\sigma},\\
a^2h^{\sigma\mu}(\partial_\mu\partial_\sigma h) &= \partial_\mu(a^2h^{\sigma\mu}\partial_\sigma h)-a^2(\partial_\mu h^{\sigma\mu})(\partial_\sigma h)-2a(\partial_\rho a)h^{\sigma\rho}\partial_\sigma h.
\end{aligned}\tag{4.22}$$

We have already relabeled $\mu\to\rho$ in all last terms to attain similarity with other such terms in (4.21). The equation now becomes

$$\begin{aligned}
S_g =&\int d\eta\, d^3x\,\Big\{a^2\frac{1}{\kappa}(\partial_\mu\partial^\nu h^\mu_\nu-\partial^2 h)\\
&+a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu)+\frac{1}{2}(\partial_\mu h)(\partial^\mu h)-\frac{1}{4}(\partial^\sigma h)(\partial_\sigma h)-(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma})+\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma})\\
&+2(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu)-\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)-(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma)+(\partial_\sigma h)(\partial_\mu h^{\sigma\mu})-(\partial_\mu h^{\sigma\mu})(\partial_\sigma h)\Big]\\
&+a\Big[-h\partial^\nu h^\rho_\nu+h\partial^\rho h+4h^{\sigma\nu}\partial_\sigma h^\rho_\nu-2h^{\nu\sigma}\partial^\rho h_{\nu\sigma}-2h^{\sigma\rho}\partial_\sigma h+6\Big(\frac{1}{\kappa^2}+\frac{1}{2\kappa}h\Big)\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}\Big](\partial_\rho a)\\
&-6a\Big(\frac{1}{\kappa^2}+\frac{1}{2\kappa}h-\frac{1}{4}h^{\lambda\sigma}h_{\lambda\sigma}+\frac{1}{8}h^2\Big)\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu a\\
&+\partial_\mu\Big[a^2h\partial^\nu h^\mu_\nu-a^2h\partial^\mu h-a^2h^{\sigma\nu}\partial_\sigma h^\mu_\nu+a^2h^{\nu\sigma}\partial^\mu h_{\nu\sigma}+a^2h^{\sigma\mu}\partial_\sigma h\Big]-a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Big\}+\mathcal{O}(\kappa).
\end{aligned}\tag{4.23}$$

We have changed the positions of some terms to make it obvious they can be added. Also, looking at the last term of the second line, we see that if we rewrite it (again using the Leibniz rule, twice)

$$\begin{aligned}-a^2(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma) =& -\partial_\mu\big(a^2 h^{\sigma\mu}(\partial_\nu h^\nu_\sigma)\big) + 2a(\partial_\mu a)h^{\sigma\mu}(\partial_\nu h^\nu_\sigma) + a^2 h^{\sigma\mu}(\partial_\nu\partial_\mu h^\nu_\sigma)\\ =&\partial_\mu\big(a^2 h^{\sigma\nu}(\partial_\nu h^\mu_\sigma) - a^2 h^{\sigma\mu}(\partial_\nu h^\nu_\sigma)\big) + a\big[2h^{\sigma\rho}(\partial_\nu h^\nu_\sigma) - 2h^{\sigma\nu}(\partial_\nu h^\rho_\sigma)\big](\partial_\rho a)\\ &- a^2(\partial_\nu h^{\sigma\mu})(\partial_\mu h^\nu_\sigma),\end{aligned} \tag{4.24}$$

we get another term that is similar to others. Writing (4.23) with these terms added yields

$$\begin{aligned}S_g = \int d\eta\, d^3x\, \Big\{& a^2\frac{1}{\kappa}(\partial_\mu\partial^\nu h^\mu_\nu - \partial^2 h)\\ &+ a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu)\Big]\\ &+ a\Big[-h\partial^\nu h^\rho_\nu + h\partial^\rho h + 4h^{\sigma\nu}\partial_\sigma h^\rho_\nu - 2h^{\nu\sigma}\partial^\rho h_{\nu\sigma} - 2h^{\sigma\rho}\partial_\sigma h + 2h^{\sigma\rho}(\partial_\nu h^\nu_\sigma) - 2h^{\sigma\nu}(\partial_\nu h^\rho_\sigma)\\ &+ 6\big(\frac{1}{\kappa^2} + \frac{1}{2\kappa}h\big)\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}\Big](\partial_\rho a) - 6a\Big[\frac{1}{\kappa^2} + \frac{1}{2\kappa}h - \frac{1}{4}h^{\lambda\sigma}h_{\lambda\sigma} + \frac{1}{8}h^2\Big]\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu a\\ &+ \partial_\mu\Big[a^2 h\partial^\nu h^\mu_\nu - a^2 h\partial^\mu h - a^2 h^{\sigma\nu}\partial_\sigma h^\mu_\nu + a^2 h^{\nu\sigma}\partial^\mu h_{\nu\sigma} + a^2 h^{\sigma\mu}\partial_\sigma h + a^2 h^{\sigma\nu}(\partial_\nu h^\mu_\sigma) - a^2 h^{\sigma\mu}(\partial_\nu h^\nu_\sigma)\Big]\\ &- a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Big\} + \mathcal{O}(\kappa).\end{aligned} \tag{4.25}$$

Looking at $\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}$, when making a perturbative expansion, we write it as

$$\begin{aligned}\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu} &= \frac{1}{2}(\eta^{\mu\nu} - \kappa h^{\mu\nu})(\kappa\eta^{\rho\sigma} - \kappa^2 h^{\rho\sigma})\big(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\big) + \mathcal{O}(\kappa^3)\\ &= \kappa(\partial_\mu h^{\mu\rho} - \frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h) - \kappa^2\Big(h^{\mu\nu}\Big[\partial_\mu h^\rho_\nu - \frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{\mu\nu}\Big] + h^{\rho\sigma}\Big[\partial_\mu h^\mu_\sigma - \frac{1}{2}\partial_\sigma h\Big]\Big) + \mathcal{O}(\kappa^3),\end{aligned} \tag{4.26}$$

$$\begin{aligned}\implies 6\Big(\frac{1}{\kappa^2} + \frac{1}{2\kappa}h\Big)\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu} =& \frac{6}{\kappa}\partial_\mu h^{\mu\rho} - \frac{3}{\kappa}\eta^{\rho\sigma}\partial_\sigma h - 6h^{\mu\nu}\partial_\mu h^\rho_\nu + 3h^{\mu\nu}\eta^{\rho\sigma}\partial_\sigma h_{\mu\nu} - 6h^{\rho\sigma}\partial_\mu h^\mu_\sigma\\ &+ 3h^{\rho\sigma}\partial_\sigma h + 3h\partial_\mu h^{\mu\rho} - \frac{3}{2}h\eta^{\rho\sigma}\partial_\sigma h + \mathcal{O}(\kappa).\end{aligned} \tag{4.27}$$

The term with double derivative on the scale factor a can also be expanded,

$$\begin{aligned}-\frac{6}{\kappa^2}a\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu a =& -\frac{6}{\kappa^2}a\Big(1 + \frac{\kappa}{2}h - \frac{1}{4}\kappa^2 h^{\lambda\sigma}h_{\lambda\sigma} + \frac{1}{8}\kappa^2 h^2\Big)(\eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\rho h^{\rho\nu})\partial_\mu\partial_\nu a + \mathcal{O}(\kappa)\\ =& -a\Big(\frac{6}{\kappa^2}\eta^{\mu\nu} + \frac{1}{\kappa}(3h\eta^{\mu\nu} - 6h^{\mu\nu})\\ &+ (6h^{\mu\rho}h^\nu_\rho - 3h^{\mu\nu}h - \frac{3}{2}\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma} + \frac{3}{4}\eta^{\mu\nu}h^2)\Big)\partial_\mu\partial_\nu a + \mathcal{O}(\kappa).\end{aligned} \tag{4.28}$$

We use the Leibniz rule once again to get some more similarity with the other terms in (4.25)

$$\begin{aligned}-a\eta^{\mu\nu}\partial_\mu\partial_\nu a &= -\partial_\mu\big(a\eta^{\mu\nu}\partial_\nu a\big) + \eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a),\\ -ah\eta^{\mu\nu}\partial_\mu\partial_\nu a &= -\partial_\mu\big(ah\eta^{\mu\nu}\partial_\nu a\big) + h\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) + a\eta^{\mu\rho}\partial_\mu h(\partial_\rho a),\\ ah^{\mu\nu}\partial_\mu\partial_\nu a &= \partial_\mu\big(ah^{\mu\nu}\partial_\nu a\big) - h^{\mu\nu}(\partial_\mu a)(\partial_\nu a) - a\partial_\mu h^{\mu\rho}(\partial_\rho a),\\ -ah^{\mu\rho}h^\nu_\rho\partial_\mu\partial_\nu a &= -\partial_\mu\big(ah^{\mu\rho}h^\nu_\rho\partial_\nu a\big) + h^{\mu\rho}h^\nu_\rho(\partial_\mu a)(\partial_\nu a) + a(\partial_\mu h^{\mu\nu})h^\rho_\nu(\partial_\rho a) + ah^{\mu\nu}(\partial_\mu h^\rho_\nu)(\partial_\rho a),\\ ah^{\mu\rho}h^\nu_\rho\partial_\mu\partial_\nu a &= \partial_\mu\big(ah^{\mu\nu}h\partial_\nu a\big) - h^{\mu\nu}h(\partial_\mu a)(\partial_\nu a) - a(\partial_\mu h^{\mu\rho})h(\partial_\rho a) - ah^{\mu\rho}(\partial_\mu h)(\partial_\rho a),\\ a\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}\partial_\mu\partial_\nu a &= \partial_\mu\big(a\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}\partial_\nu a\big) - h^{\lambda\sigma}h_{\lambda\sigma}\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) - 2ah^{\lambda\sigma}\eta^{\mu\rho}(\partial_\mu h_{\lambda\sigma})(\partial_\rho a),\\ -a\eta^{\mu\nu}h^2\partial_\mu\partial_\nu a &= -\partial_\mu\big(a\eta^{\mu\nu}h^2\partial_\nu a\big) + h^2\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) + 2a\eta^{\mu\rho}h(\partial_\mu h)(\partial_\rho a).\end{aligned} \tag{4.29}$$

We again renamed or switched some indices. Putting these back into (4.28)

$$
\begin{aligned}
-\frac{6}{\kappa^2}a\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu a =&\Big(\frac{6}{\kappa^2}\eta^{\mu\nu}+\frac{3}{\kappa}h\eta^{\mu\nu}-\frac{6}{\kappa}h^{\mu\nu}+6h^{\mu\rho}h^\nu_\rho-3h^{\mu\nu}h-\frac{3}{2}\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}+\frac{3}{4}h^2\eta^{\mu\nu}\Big)(\partial_\mu a)(\partial_\nu a)\\
&+a\Big(\frac{3}{\kappa}\eta^{\mu\rho}\partial_\mu h-\frac{6}{\kappa}\partial_\mu h^{\mu\rho}+6(\partial_\mu h^{\mu\nu})h^\rho_\nu+6h^{\mu\nu}(\partial_\mu h^\rho_\nu)-3(\partial_\mu h^{\mu\rho})h-3h^{\mu\rho}(\partial_\mu h)\\
&-3h^{\lambda\sigma}\eta^{\mu\rho}(\partial_\mu h_{\lambda\sigma})+\frac{3}{2}\eta^{\mu\rho}h(\partial_\mu h)\Big)(\partial_\rho a)\\
&+\partial_\mu\Big(-a\eta^{\mu\nu}\partial_\nu a-ah\eta^{\mu\nu}\partial_\nu a+ah^{\mu\nu}\partial_\nu a-ah^{\mu\rho}h^\nu_\rho\partial_\nu a+ah^{\mu\nu}h\partial_\nu a\\
&+a\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}\partial_\nu a+\frac{3}{4}h^2\eta^{\mu\nu}\Big)+\mathcal{O}(\kappa)\\
=&\Big(\frac{6}{\kappa^2}\eta^{\mu\nu}+\frac{3}{\kappa}h\eta^{\mu\nu}-\frac{6}{\kappa}h^{\mu\nu}+6h^{\mu\rho}h^\nu_\rho-3h^{\mu\nu}h-\frac{3}{2}\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}+\frac{3}{4}\eta^{\mu\nu}h^2\Big)(\partial_\mu a)(\partial_\nu a)\\
&-6a\Big(\frac{1}{\kappa^2}+\frac{1}{2\kappa}h\Big)\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a)\\
&+\partial_\mu\Big(-a\eta^{\mu\nu}\partial_\nu a-ah\eta^{\mu\nu}\partial_\nu a+ah^{\mu\nu}\partial_\nu a-ah^{\mu\rho}h^\nu_\rho\partial_\nu a+ah^{\mu\nu}h\partial_\nu a\\
&+a\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}\partial_\nu a-\frac{3}{4}a\eta^{\mu\nu}h^2\partial_\nu a\Big)+\mathcal{O}(\kappa).
\end{aligned}
\tag{4.30}
$$

We see in the second to last line that the $\tilde{\Gamma}^\rho_{\mu\nu}$ term will cancel with the same term with opposite sign in (4.25). This action now is

$$
\begin{aligned}
S_g=\int d\eta\, d^3x\,\Big\{&\frac{6}{\kappa^2}\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a)+\frac{1}{\kappa}\Big[a^2\partial_\mu\partial^\nu h^\mu_\nu-a^2\partial^2 h+(3h\eta^{\mu\nu}-6h^{\mu\nu})(\partial_\mu a)(\partial_\nu a)\Big]\\
&+a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu)+\frac{1}{4}(\partial_\mu h)(\partial^\mu h)-\frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma})+\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu)\Big]\\
&+a\Big[-h\partial^\nu h^\rho_\nu+h\partial^\rho h+2h^{\sigma\nu}\partial_\sigma h^\rho_\nu-2h^{\nu\sigma}\partial^\rho h_{\nu\sigma}-2h^{\sigma\rho}\partial_\sigma h+2h^{\sigma\rho}(\partial_\nu h^\nu_\sigma)\Big](\partial_\rho a)\\
&+\Big[6h^{\mu\rho}h^\nu_\rho-3h^{\mu\nu}h-\frac{3}{2}\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}+\frac{3}{4}h^2\eta^{\mu\nu}\Big](\partial_\mu a)(\partial_\nu a)\\
&+\partial_\mu\Big[a^2h\partial^\nu h^\mu_\nu-a^2h\partial^\mu h-a^2h^{\sigma\nu}\partial_\sigma h^\mu_\nu+a^2h^{\nu\sigma}\partial^\mu h_{\nu\sigma}+a^2h^{\sigma\mu}\partial_\sigma h+a^2h^{\sigma\nu}(\partial_\nu h^\mu_\sigma)\\
&-a^2h^{\sigma\mu}(\partial_\nu h^\nu_\sigma)-a\eta^{\mu\nu}\partial_\nu a-ah\eta^{\mu\nu}\partial_\nu a+ah^{\mu\nu}\partial_\nu a-ah^{\mu\rho}h^\nu_\rho\partial_\nu a+ah^{\mu\nu}h\partial_\nu a\\
&+a\eta^{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma}\partial_\nu a-\frac{3}{4}a\eta^{\mu\nu}h^2\Big]-a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Big\}+\mathcal{O}(\kappa).
\end{aligned}
\tag{4.31}
$$

In the last three lines we have a total derivative. With the integration, we can therefore invoke Stokes' theorem and evaluate the term in square brackets at the boundary. Of course, we have to determine (or make assumptions) about these boundaries. The spatial boundary can, for example, be taken outside the observable universe, such that it makes no difference if we set the term to 0. For the timelike term, since we only evaluate the perturbations up to quadratic order, we can also assume it to be 0 (or at least be absorbed into an overall constant, which we then absorb in the cosmological constant and set to 0). Therefore, we drop any total derivatives from now on. The $\mathcal{O}(\frac{1}{\kappa})$ term in the first line of (4.31) can alternatively be written by using

$$
\begin{aligned}
a^2\partial_\mu\partial^\nu h^\mu_\nu &= \partial_\mu(a^2\partial_\nu h^{\mu\nu})-2a\partial_\nu h^{\mu\nu}\partial_\mu a\\
&=\partial_\mu(a^2\partial_\nu h^{\mu\nu})-2\partial_\nu(ah^{\mu\nu}\partial_\mu a)+2h^{\mu\nu}(\partial_\mu a)(\partial_\nu a)+2ah^{\mu\nu}\partial_\mu\partial_\nu a,\\
a^2\partial^2 h &= \partial_\mu(a^2\partial^\mu h)-2a\partial^\mu h\partial_\mu a\\
&=\partial_\mu(a^2\partial^\mu h)-2\partial_\nu(ah\eta^{\mu\nu}\partial_\mu a)+2h\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a)+2ah\eta^{\mu\nu}\partial_\mu\partial_\nu a.
\end{aligned}
\tag{4.32}
$$

Ignoring the boundary terms, we substitute these terms and we simplify the third line in (4.31)

$$\begin{aligned}S_g = \int d\eta\, d^3x\, \Bigg\{ &\frac{6}{\kappa^2}\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) + \frac{1}{\kappa}\Big[(h\eta^{\mu\nu} - 4h^{\mu\nu})(\partial_\mu a)(\partial_\nu a) + 2a(h^{\mu\nu} - h\eta^{\mu\nu})\partial_\mu\partial_\nu a\Big] \\ &+ a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{\sigma\rho}\partial_\sigma h(\partial_\rho \mathrm{ln} a)\Big] \\ &+ a\Big[-(\partial_\sigma h^{\rho\sigma} h) + \frac{1}{2}\eta^{\rho\sigma}(\partial_\sigma h^2) + 2(\partial_\sigma h^{\sigma\nu}h^\rho_\nu) - \eta^{\rho\sigma}\partial_\sigma h^{\nu\lambda}h_{\nu\lambda}\Big](\partial_\rho a) \\ &+ \Big[6h^{\mu\rho}h^\nu_\rho - 3h^{\mu\nu}h - \frac{3}{2}h^{\lambda\sigma}h_{\lambda\sigma} + \frac{3}{4}h^2\eta^{\mu\nu}\Big](\partial_\mu a)(\partial_\nu a) - a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Bigg\} + \mathcal{O}(\kappa),\end{aligned} \tag{4.33}$$

$$\begin{aligned}S_g = \int d\eta\, d^3x\, \Bigg\{ &\frac{6}{\kappa^2}\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) + \frac{1}{\kappa}\Big[(h\eta^{\mu\nu} - 4h^{\mu\nu})(\partial_\mu a)(\partial_\nu a) + 2a(h^{\mu\nu} - h\eta^{\mu\nu})\partial_\mu\partial_\nu a\Big] \\ &+ a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{\sigma\rho}\partial_\sigma h(\partial_\rho \mathrm{ln} a)\Big] \\ &+ a\partial_\sigma\Big[-h^{\rho\sigma} h + \frac{1}{2}\eta^{\rho\sigma}h^2 + 2h^{\sigma\nu}h^\rho_\nu - \eta^{\rho\sigma}h^{\nu\lambda}h_{\nu\lambda}\Big](\partial_\rho a) \\ &+ \Big[6h^{\sigma\nu}h^\rho_\nu - 3h^{\rho\sigma}h - \frac{3}{2}\eta^{\rho\sigma}h^{\lambda\nu}h_{\lambda\nu} + \frac{3}{4}h^2\eta^{\sigma\rho}\Big](\partial_\sigma a)(\partial_\rho a) - a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Bigg\} + \mathcal{O}(\kappa).\end{aligned} \tag{4.34}$$

Now applying Leibniz' rule to the third line in (4.34) to be able to add it to the fourth gives

$$\begin{aligned}S_g = \int d\eta\, d^3x\, \Bigg\{ &\frac{6}{\kappa^2}\eta^{\mu\nu}(\partial_\mu a)(\partial_\nu a) + \frac{1}{\kappa}\Big[(h\eta^{\mu\nu} - 4h^{\mu\nu})(\partial_\mu a)(\partial_\nu a) + 2a(h^{\mu\nu} - h\eta^{\mu\nu})\partial_\mu\partial_\nu a\Big] \\ &+ a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{\sigma\rho}\partial_\sigma h(\partial_\rho \mathrm{ln} a)\Big] \\ &- a\Big[-h^{\rho\sigma} h + \frac{1}{2}\eta^{\rho\sigma}h^2 + 2h^{\sigma\nu}h^\rho_\nu - \eta^{\rho\sigma}h^{\nu\lambda}h_{\nu\lambda}\Big](\partial_\sigma\partial_\rho a) \\ &+ \Big[4h^{\sigma\nu}h^\rho_\nu - 2h^{\rho\sigma}h - \frac{1}{2}\eta^{\rho\sigma}h^{\lambda\nu}h_{\lambda\nu} + \frac{1}{4}h^2\eta^{\sigma\rho}\Big](\partial_\sigma a)(\partial_\rho a) - a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda\Bigg\} + \mathcal{O}(\kappa).\end{aligned} \tag{4.35}$$

We dropped a boundary term, as mentioned we would. The scale factor is dependent on conformal time, so we then see that $\partial_\mu a = \delta^0_\mu \partial_0 a = \delta^0_\mu \partial_\eta a \equiv \delta^0_\mu a'$, which we then express as the conformal Hubble rate as

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a}\frac{dt}{d\eta}\frac{da}{dt} = \dot{a} = aH, \qquad \mathcal{H}' = \frac{d}{d\eta}\frac{a'}{a} = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 \implies \partial_0\partial_0 a = a'' = a(\mathcal{H}' + \mathcal{H}^2). \tag{4.36}$$

In the first equation, we also gave the relation of the conformal Hubble rate to the regular Hubble parameter H. Looking at the last term of the second line in (4.33), we see that we chose to write a peculiar term

$$a^2 h^{\rho\sigma}(\partial_\sigma h)(\partial_\rho \mathrm{ln} a) = a h^{0\sigma}(\partial_\sigma h)\partial_0 a = a^2 h^{0\sigma}(\partial_\sigma h)\mathcal{H}, \tag{4.37}$$

which is a term linear in the Hubble rate and a first derivative in perturbations. In analogy with for example a scalar field in inflation [13], we identify this term as a Hubble damping term, acting as a friction term for fast-moving oscillations in the perturbations.

Rewriting the other derivatives of scale factors makes (4.35) look like

$$
\begin{aligned}
S_g = \int d\eta\, d^3x \, \Big\{ & - \frac{6}{\kappa^2} a^2 \mathcal{H}^2 + \frac{1}{\kappa} a^2 \Big[- (h + 4h^{00}) \mathcal{H}^2 + 2(h^{00} + h)(\mathcal{H}' + \mathcal{H}^2) \Big] \\
& + a^2 \Big[- \frac{1}{2} (\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4} (\partial_\mu h)(\partial^\mu h) - \frac{1}{4} (\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2} (\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{0\sigma} (\partial_\sigma h) \mathcal{H} \Big] \\
& - a^2 \Big[- h^{00} h - \frac{1}{2} h^2 + 2 h^{0\nu} h^0_\nu + h^{\nu\lambda} h_{\nu\lambda} \Big] (\mathcal{H}' + \mathcal{H}^2) \\
& + a^2 \Big[4 h^{0\nu} h^0_\nu - 2 h^{00} h + \frac{1}{2} h^{\lambda\nu} h_{\lambda\nu} - \frac{1}{4} h^2 \Big] \mathcal{H}^2 - a^4 \frac{2}{\kappa^2} \sqrt{-\tilde{g}} \Lambda \Big\} + \mathcal{O}(\kappa), \\
= \int d\eta\, d^3x \, \Big\{ & - \frac{6}{\kappa^2} a^2 \mathcal{H}^2 + \frac{1}{\kappa} a^2 \Big[(h - 2h^{00}) \mathcal{H}^2 + 2(h^{00} + h) \mathcal{H}' \Big] \\
& + a^2 \Big[- \frac{1}{2} (\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4} (\partial_\mu h)(\partial^\mu h) - \frac{1}{4} (\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2} (\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{0\sigma} (\partial_\sigma h) \mathcal{H} \Big] \\
& + a^2 \Big[h^{00} h + \frac{1}{2} h^2 - 2 h^{0\nu} h^0_\nu - h^{\nu\lambda} h_{\nu\lambda} \Big] \mathcal{H}' + a^2 \Big[2 h^{0\nu} h^0_\nu - h^{00} h - \frac{1}{2} h^{\nu\lambda} h_{\nu\lambda} + \frac{1}{4} h^2 \Big] \mathcal{H}^2 \\
& - a^4 \frac{2}{\kappa^2} \sqrt{-\tilde{g}} \Lambda \Big\} + \mathcal{O}(\kappa).
\end{aligned}
\tag{4.38}
$$

Once again, we split the action into several parts for convenience, writing some parts more concisely

$$
S_g = S_g^{(0,0)} + S_g^{(1,0)} + S_g^{(2,0)} - \int d\eta\, d^3x \, a^4 \frac{2}{\kappa^2} \sqrt{-\tilde{g}} \Lambda + \mathcal{O}\big((h_{\mu\nu})^3\big), \tag{4.39}
$$

$$
S_g^{(0,0)} = - \frac{6}{\kappa^2} \int d\eta\, d^3x \, a^2 \mathcal{H}^2, \tag{4.40}
$$

$$
S_g^{(1,0)} = \frac{1}{\kappa} \int d\eta\, d^3x \, a^2 \Big(h(\mathcal{H}^2 + 2\mathcal{H}') + 2 h^{\mu\nu} \delta^0_\mu \delta^0_\nu (\mathcal{H}' - \mathcal{H}^2) \Big), \tag{4.41}
$$

$$
\begin{aligned}
S_g^{(2,0)} = \int d\eta\, d^3x \Big\{ & a^2 \Big[- \frac{1}{2} (\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4} (\partial_\mu h)(\partial^\mu h) - \frac{1}{4} (\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) + \frac{1}{2} (\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) \Big] \\
& - a^2 h^{\mu\sigma} (\partial_\sigma h) \delta^0_\mu \mathcal{H} + a^2 \Big[\frac{1}{4} h^2 - \frac{1}{2} h^{\nu\lambda} h_{\nu\lambda} \Big] (\mathcal{H}^2 + 2\mathcal{H}') + a^2 \big[h h^{\mu\nu} - 2 h^{\mu\sigma} h^\nu_\sigma \big] \delta^0_\mu \delta^0_\nu (\mathcal{H}' - \mathcal{H}^2) \Big\}.
\end{aligned}
\tag{4.42}
$$

This action is the main result of this chapter, and because of its importance for the theory, we have included a series of checks with literature in appendix D. Together with (3.20), it will form the total action in the next chapter.

4.1 Einstein tensor

The Einstein tensor can be found in two ways: it is directly defined in terms of a Ricci scalar and tensor, but it is also (by construction) found by varying the Einstein-Hilbert action with respect to the metric. Up to linear order in $h_{\mu\nu}$, we obtain for the definition of the Einstein tensor

$$
\begin{aligned}
G_{\mu\nu} \equiv & \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \\
\approx & - \big[\eta_{\mu\nu} (\mathcal{H}^2 + 2\mathcal{H}') + 2 \delta^0_\mu \delta^0_\nu (\mathcal{H}' - \mathcal{H}^2) \big] \\
& + \kappa \Big[\frac{1}{2} \big(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu + \eta_{\mu\nu} \partial^2 h - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} \big) \\
& - \big(h_{\mu\nu} + \eta_{\mu\nu} h_{00} \big) (\mathcal{H}^2 + 2\mathcal{H}') + \big(\partial_0 h_{\mu\nu} - \partial_\mu h_{\nu 0} - \partial_\nu h_{\mu 0} - \eta_{\mu\nu} (2 \partial_\rho h^{\rho 0} + \partial_0 h) \big) \mathcal{H} \Big].
\end{aligned}
\tag{4.43}
$$

Another check for the action (4.38) would then be to vary it with respect to $h_{\mu\nu}$, which we will do in the same manner as for the energy-momentum tensor,

$$
G_{\mu\nu} \equiv \frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_g}{\delta g^{\mu\nu}} = - g_{\rho\mu} g_{\lambda\nu} \frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_g}{\delta g_{\rho\lambda}} = - a^{-2} (\eta_{\rho\mu} + \kappa h_{\rho\mu})(\eta_{\lambda\nu} + \kappa h_{\lambda\nu}) \Big(1 - \frac{\kappa}{2} h \Big) \kappa \frac{\delta S_g}{\delta h_{\rho\lambda}} + \mathcal{O}\big((h_{\mu\nu})^2\big).
\tag{4.44}
$$

We can do this again separately for $S_g^{(1,0)}$ and $S_g^{(2,0)}$, up to linear order in $h_{\mu\nu}$.

$$\begin{aligned} G^A_{\mu\nu} =& -\Big(\eta_{\rho\mu}\eta_{\lambda\nu} + \kappa\Big[h_{\lambda\nu}\eta_{\rho\mu} + h_{\rho\mu}\eta_{\lambda\nu} - \frac{h}{2}\eta_{\rho\mu}\eta_{\lambda\nu}\Big]\Big)\Big(\eta^{\rho\lambda}(\mathcal{H}^2 + 2\mathcal{H}') + 2\eta^{\rho 0}\eta^{\lambda 0}(\mathcal{H}' - \mathcal{H}^2)\Big) + \mathcal{O}\big((h_{\mu\nu})^2\big) \\ \approx& -\Big[\eta_{\mu\nu}(\mathcal{H}^2 + 2\mathcal{H}') + 2\delta^0_\mu\delta^0_\nu(\mathcal{H}' - \mathcal{H}^2)\Big] \\ &+ \kappa\Big[\Big(\frac{h}{2}\eta_{\mu\nu} - 2h_{\mu\nu}\Big)(\mathcal{H}^2 + 2\mathcal{H}') + (h\delta^0_\mu\delta^0_\nu - 4h^0_{(\mu}\delta^0_{\nu)})(\mathcal{H}' - \mathcal{H}^2)\Big]\Big). \end{aligned} \tag{4.45}$$

We see that the linear part indeed corresponds to (4.43). For varying $S_g^{(2,0)}$, we would get several boundary terms, which we disregard again on account of previously mentioned arguments.

$$\begin{aligned} G^B_{\mu\nu} \approx& -a^{-2}\eta_{\rho\mu}\eta_{\lambda\nu}\kappa\Big(a\Big[\eta^{\rho\lambda}\partial_\tau h^{\tau\sigma} + \eta^{\tau(\rho}\eta^{\lambda)\sigma}\partial_\tau h - \eta^{\rho\lambda}\partial^\sigma h + \partial^\sigma h^{\rho\lambda} - 2\partial^{(\rho}h^{\lambda)\sigma}\Big](\partial_\sigma a) \\ &+ \frac{1}{2}a^2\Big[\eta^{\rho\lambda}\partial_\tau\partial_\sigma h^{\tau\sigma} + \partial^\rho\partial^\lambda h - \eta^{\rho\lambda}\partial^2 h + \partial^2 h^{\rho\lambda} - 2\partial_\sigma\partial^{(\rho}h^{\lambda)\sigma}\Big] \\ &+ \eta^{\rho\lambda}\Big[h^{\tau\sigma}(\partial_\tau a)(\partial_\sigma a) + a(\partial_\tau h^{\tau\sigma})(\partial_\sigma a) + ah^{\tau\sigma}(\partial_\tau\partial_\sigma a)\Big] - \eta^{\tau(\rho}\eta^{\lambda)\sigma}a(\partial_\tau h)(\partial_\sigma a) \\ &+ a^2\Big[\frac{1}{2}\eta^{\rho\lambda}h - h^{\rho\lambda}\Big](\mathcal{H}^2 + 2\mathcal{H}') + a^2\Big[\eta^{\rho\lambda}h^{\tau\sigma}\delta^0_\tau\delta^0_\sigma + \eta^{\rho 0}\eta^{\lambda 0}h - 4\eta^{\tau(\rho}h^{\lambda)\sigma}\delta^0_\tau\delta^0_\sigma\Big](\mathcal{H}' - \mathcal{H}^2)\Big) \\ =&\kappa\Big(\Big[\partial_0 h_{\mu\nu} - 2\partial_{(\mu}h_{\nu)0} - \eta_{\mu\nu}(2\partial_\tau h^{\tau 0} + \partial_0 h)\Big]\mathcal{H} + \Big[h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \eta_{\mu\nu}h_{00}\Big](\mathcal{H}^2 + 2\mathcal{H}') \\ &+ \frac{1}{2}\Big[2\partial_\sigma\partial_{(\mu}h^\sigma_{\nu)} + \eta_{\mu\nu}(\partial^2 h - \partial_\tau\partial_\sigma h^{\tau\sigma}) - \partial_\mu\partial_\nu h - \partial^2 h_{\mu\nu}\Big] - \big[h\delta^0_\mu\delta^0_\nu - 4h^0_{(\mu}\delta^0_{\nu)}\big](\mathcal{H}' - \mathcal{H}^2)\Big). \end{aligned} \tag{4.46}$$

In appendix D, we show that this variation of the quadratic part agrees with the Lichnerowicz operator mentioned in [23]. We obtain for the complete Einstein tensor, again up to quadratic order in perturbations

$$\begin{aligned} G_{\mu\nu} =& G^A_{\mu\nu} + G^B_{\mu\nu} \\ =& -\Big(\eta_{\mu\nu}(\mathcal{H}^2 + 2\mathcal{H}') + 2\delta^0_\mu\delta^0_\nu(\mathcal{H}' - \mathcal{H}^2)\Big) \\ &+ \kappa\Big(\frac{1}{2}\Big[2\partial_\sigma\partial_{(\mu}h^\sigma_{\nu)} + \eta_{\mu\nu}(\partial^2 h - \partial_\tau\partial_\sigma h^{\tau\sigma}) - \partial_\mu\partial_\nu h - \partial^2 h_{\mu\nu}\Big] - \big[h_{\mu\nu} + \eta_{\mu\nu}h_{00}\big](\mathcal{H}^2 + 2\mathcal{H}') \\ &+ \big[\partial_0 h_{\mu\nu} - 2\partial_{(\mu}h_{\nu)0} - \eta_{\mu\nu}(2\partial_\tau h^{\tau 0} + \partial_0 h)\big]\mathcal{H}\Big) + \mathcal{O}\big((h_{\mu\nu})^2\big), \end{aligned} \tag{4.47}$$

which indeed concurs with (4.43). Again, the parts $G^A_{\mu\nu}$ and $G^B_{\mu\nu}$ have no physical meaning by themselves. A minor check with literature for this Einstein tensor is ignoring perturbations ($g_{\mu\nu} = a^2\eta_{\mu\nu}$) and taking the trace of (4.47)

$$\tilde{G} = g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}\mathcal{R}_{\mu\nu} - 2\mathcal{R} = -\mathcal{R} = -6(\mathcal{H}^2 + \mathcal{H}') = -6(2H^2 + \dot{H}) = -6\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right). \tag{4.48}$$

If we were to calculate $\mathcal{R}$ from the Christoffel symbols obtained from $g_{\mu\nu}$ (see e.g. [13], p. 333), we would get this trace of the Einstein tensor.

Notice, in a similar manner as for the energy-momentum tensor, that we can write

$$S_g^{(1,0)} = -\frac{1}{\kappa}\int d\eta\, d^3x\, a^2\big(h^{\mu\nu} - \kappa(h^{\mu\rho}h^\nu_\rho - hh^{\mu\nu})\big)G^A_{\mu\nu} \tag{4.49}$$

$$S_g^{(2,0)} = -\frac{1}{\kappa}\int d\eta\, d^3x \frac{a^2}{2}h^{\mu\nu}G^B_{\mu\nu}, \tag{4.50}$$

where for $S_g^{(2,0)}$, several integrations by parts were performed, like in appendix D.

With the Einstein tensor and the energy-momentum tensor both expanded up to an order that we deem relevant, we have all the ingredients to write the Einstein equation, from which we can infer the relation between for example the Hubble rate and the pressure and energy density. We can also finally add the matter and gravity actions to get our complete action, which is the subject of the next chapter.

5 Integrating out gravitational fields

In the previous chapters, we computed a rescaled Dirac action and a rescaled Einstein-Hilbert action. We can now look at the total rescaled action, which is up to quartic interaction and quadratic gravitational contribution

$$\begin{aligned} S &= S_g + S[\chi,\bar{\chi}] \\ &= \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\mathcal{R} + \int d^4x\sqrt{-g}\Big[\frac{i}{2}\Big(\bar{\psi}e^\mu_b\gamma^b D_\mu\psi - (D_\mu\bar{\psi})e^\mu_b\gamma^b\psi\Big) - \bar{\psi}M\psi\Big] \\ &= S^{(0)}_{tot} + S^{(1)}_{tot} + S^{(2)}_{tot} + \mathcal{O}((h_{\mu\nu})^3), \end{aligned} \tag{5.1}$$

where we now only count the number of gravitational perturbations in the superscript. The different contributions are of course the sums of the individual Dirac and Einstein-Hilbert terms. We once again mention that we have defined $\mathcal{L}_0$ and $\mathcal{L}_{(\mu\nu)}$ in (3.21) and (3.22), as

$$\mathcal{L}_0 \equiv \frac{i}{2}(\bar{\chi}\overrightarrow{\partial\!\!\!/}\chi - \bar{\chi}\overleftarrow{\partial\!\!\!/}\chi) - \bar{\chi}Ma\chi, \qquad \mathcal{L}_{(\mu\nu)} \equiv \frac{i}{2}\bar{\chi}\gamma_{(\nu}\partial_{\mu)}\chi - \frac{i}{2}(\partial_{(\mu}\bar{\chi})\gamma_{\nu)}\chi. \tag{5.2}$$

For completeness, we now show the different total contributions:

$$S^{(0)}_{tot} = S^{(0,2)}[\chi,\bar{\chi}] + S^{(0,0)}_g = \int d\eta\, d^3x\left(\mathcal{L}_0 - \frac{6}{\kappa^2}a^2\mathcal{H}^2\right), \tag{5.3}$$

$$\begin{aligned} S^{(1)}_{tot} &= S^{(1,2)}[\chi,\bar{\chi}] + S^{(1,0)}_g \\ &= \int d\eta\, d^3x\left\{\frac{\kappa}{2}\Big[h\mathcal{L}_0 - h^{\mu\nu}\mathcal{L}_{\mu\nu}\Big] + \frac{1}{\kappa}a^2\Big[h(\mathcal{H}^2 + 2\mathcal{H}') + 2h^{\mu\nu}\delta^0_\mu\delta^0_\nu(\mathcal{H}' - \mathcal{H}^2)\Big]\right\}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} S^{(2)}_{tot} &= S^{(2,2)}[\chi,\bar{\chi}] + S^{(2,0)}_g \\ &= \int d\eta\, d^3x\Big\{\frac{\kappa^2}{8}\Big[(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\mathcal{L}_0 + (3h^{\mu\rho}h^\nu_\rho - 2hh^{\mu\nu})\mathcal{L}_{\mu\nu} - \frac{1}{2}h^\rho_\mu(\partial_\sigma h_{\rho\nu})\bar{\chi}\epsilon^{\tau\sigma\mu\nu}\gamma_\tau\gamma^5\chi\Big] \\ &\quad + \frac{1}{2}a^2\Big[-(\partial_\mu h)(\partial_\nu h^{\mu\nu}) + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(\partial^\mu h^{\nu\sigma})(\partial_\mu h_{\nu\sigma}) + (\partial^\mu h^{\nu\sigma})(\partial_\sigma h_{\mu\nu}) - 2h^{\sigma 0}(\partial_\sigma h)\mathcal{H}\Big] \\ &\quad + a^2\Big[\frac{1}{4}h^2 - \frac{1}{2}h^{\nu\lambda}h_{\nu\lambda}\Big](\mathcal{H}^2 + 2\mathcal{H}') + a^2\big[hh^{\mu\nu} - 2h^{\mu\sigma}h^\nu_\sigma\big]\delta^0_\mu\delta^0_\nu(\mathcal{H}' - \mathcal{H}^2)\Big\}. \end{aligned} \tag{5.5}$$

Note that we now chose to set $\Lambda = 0$, to focus solely on the effects of fermions on gravity and vice versa. However, the cosmological constant could always be added later. In this chapter, we aim to integrate out the gravitons to get an effective field theory: the gravitational effects of the fluctuations in the homogeneous dark matter background are small and their relevance is negligible when going to classical scales. By integrating these fields out, we obtain a nonlocal action. This already gives a glimpse of classical Newtonian gravity, which also behaves nonlocally. Before we do this, we will look at some properties of the action such as being able to derive the Einstein field equations, and check for the absence of graviton mass.

5.1 Einstein Equations

The mentioned expressions should, by construction, yield the classical Einstein field equations

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{\kappa^2}{2}T_{\mu\nu}. \tag{5.6}$$

This is easily determined for the linear part, i.e.

$$\frac{\delta S}{\delta h^{\mu\nu}}\Big|_{h\to 0} = \frac{\delta S^{(1)}_{tot}}{\delta h^{\mu\nu}} = \frac{\delta}{\delta h^{\mu\nu}}\big(S^{(1,2)}[\chi,\bar{\chi}] + S^{(1,0)}_g\big) = \frac{a^2}{\kappa}\Big(\frac{\kappa^2}{2}T_{\mu\nu} - G_{\mu\nu}\Big) = 0, \tag{5.7}$$

where we used (3.24) and (4.45) to identify both contributions. Notice also that we have

$$S_{tot}^{(1)} = \frac{1}{\kappa}\int d^4x\; a^2 h^{\mu\nu}\left[\frac{\kappa^2}{2}T_{\mu\nu} - G_{\mu\nu}\right]_{h_{\mu\nu}\to 0}, \tag{5.8}$$

by looking at the linear parts of (3.29) and (4.47).

These equations hold in the classical regime, and will work fine if we were only interested in that regime. But the energy-momentum tensor we calculated was based on a quantum field-theoretic action, and we have to account for fluctuations in this tensor,

$$T_{\mu\nu} = \langle T_{\mu\nu}\rangle + \delta T_{\mu\nu}. \tag{5.9}$$

This is also what we did in chapter 2: for the scales we are looking at (on the order of ∼10-100MPc) we have a homogeneous background contribution, which is averaged into the expectation value, defined in (2.24). These fluctuations will source gravitational effects from the expanding background. However, we know that they are very small, and assume the field equations only to hold for a classical FLRW-background, so a classical homogeneous fluid,

$$G_{\mu\nu} = \frac{\kappa^2}{2}\langle T_{\mu\nu}\rangle. \tag{5.10}$$

This equation can also be inferred from the fact that the Einstein tensor is spatially invariant, while the perturbations $\delta T_{\mu\nu}$ are not. The higher order corrections to this equation are the topic of interest here. To give some indication of the smallness of the potentials sourced by these fluctuations, we look at the escape velocity from a mass M, which can be approximated by

$$\frac{mv^2}{2} = \frac{GMm}{r} = \Phi_N m, \tag{5.11}$$

i.e. the potential is proportional to the squared escape velocity. In units of c, the escape velocity from the event horizon of the black hole would be 1, while for example the Milky way (with $v_{\mathrm{esc}} \approx 5.5\cdot 10^5$m/s) will then get

$$\frac{\Phi}{c^2} \sim \mathcal{O}(10^{-6}). \tag{5.12}$$

We see that even linear perturbation theory, like (5.9), will have only tiny corrections, and higher orders will be immensely suppressed. Especially with the curvature being small (i.e. much larger than the Planck length $l_p = \sqrt{G\hbar}$), introducing the expectation value of the energy-momentum tensor in the Einstein equations is only logical if we want to focus on the relevant contributions. Since we are treating cold dark matter, these assumptions are justified, as we have no extremes in curvature or fluctuations as would occur for e.g. black holes or relativistic speeds. We therefore don't find that $S_{tot}^{(1)}$ drops out entirely (on account of (5.7)), but leaves us with small fluctuations in the energy-momentum tensor. For $S_{tot}^{(2)}$, we find something similar in the next section.

An important observation to be made is that 'just' taking the expectation value of the energy-momentum tensor will yield divergences. We therefore have to add a suitable counter-term to cancel this divergence, and we will find a renormalized vacuum contribution. This procedure is outlined in appendix E. Fortunately, this contribution is constant and can therefore be absorbed into the cosmological constant (which we keep at 0), so it will not alter our current calculations.

5.2 Looking at the graviton mass

As a result of (5.9) and (5.10), the deviation from the background is can be found through the Einstein field equations,

$$\frac{\kappa^2}{2}T_{\mu\nu} - G_{\mu\nu} = \frac{\kappa^2}{2}\delta T_{\mu\nu}. \tag{5.13}$$

On a classical level, the graviton mass is 0, as expected for a gauge boson. Clearly we see that a mass can now be sourced by these deviations, which we accept as just a consequence of the framework of semiclassical gravity. We have seen that the action is in fact proportional to the Einstein equation, like in (5.8). From it, we can group some terms:

$$\begin{aligned} S^{(1)}_{tot} =& \frac{1}{\kappa}\int d\eta\, d^3x \left\{ h\Big[\frac{\kappa^2}{2}\mathcal{L}_0 + a^2(\mathcal{H}^2+2\mathcal{H}')\Big] + h^{\mu\nu}\Big[2a^2\delta^0_\mu\delta^0_\nu(\mathcal{H}'-\mathcal{H}^2) - \frac{\kappa^2}{2}\mathcal{L}_{\mu\nu}\Big]\right\} \\ =& \frac{\kappa}{2}\int d\eta\, d^3x\, a^2 h^{\mu\nu}\delta T^{(0)}_{\mu\nu}. \end{aligned} \tag{5.14}$$

The significance of this split becomes clear when we consider the expectation value of $T_{\mu\nu}$, zeroth order in gravitational perturbations (see (2.21) and (3.29))

$$a^2\langle T^{(0)}_{\mu\nu}\rangle = \eta_{\mu\nu}\langle\mathcal{L}_0\rangle - \langle\mathcal{L}_{(\mu\nu)}\rangle = -\langle\mathcal{L}_{(\mu\nu)}\rangle = a^4(\langle\rho\rangle + \langle P\rangle)\delta^0_\mu\delta^0_\nu + a^4\eta_{\mu\nu}\langle P\rangle. \tag{5.15}$$

On account of being in the rest frame of a nonrelativistic plasma modeled by a perfect fluid with perturbations (i.e. the dark matter), the pressure is very small $\langle P\rangle/\langle\rho\rangle \propto v^2/c^2 \propto \Phi$. The expectation value of the Lagrangian $\mathcal{L}_0$ is 0 by the equations of motion of χ and $\bar\chi$, which is the Dirac equation obtained in (3.36) and its conjugate. We see that,

$$a^4\rho = -\mathcal{L}_0 - \mathcal{L}_{00}, \tag{5.16}$$

$$\delta_{ij}a^4 P = \delta_{ij}\mathcal{L}_0 - \mathcal{L}_{ij}. \tag{5.17}$$

If we use this information in the Einstein equations, again up to zeroth order, we get

$$\begin{aligned} a^2 G^{(0)}_{00} &= 3a^2\mathcal{H}^2 = a^4\frac{\kappa^2}{2}\langle\rho\rangle, \\ a^2 G^{(0)}_{ij} &= -\delta_{ij}a^2(\mathcal{H}^2+2\mathcal{H}') = a^4\frac{\kappa^2}{2}\langle P\rangle\delta_{ij} \\ \Longrightarrow 2a^2(\mathcal{H}'-\mathcal{H}^2) &= -\frac{1}{3}\delta^{ij}a^2G^{(0)}_{ij} - a^2G^{(0)}_{00} = -a^4\frac{\kappa^2}{2}(\langle\rho\rangle + \langle P\rangle). \end{aligned} \tag{5.18}$$

We see indeed that the split into the two parts of (5.14) leaves us only with perturbations in $T_{\mu\nu}$, as expected. If we now try to obtain this split in the part quadratic in perturbations, we get

$$\begin{aligned} S^{(2)}_{tot} = \int d\eta\, d^3x \Big\{ & -\frac{1}{2}h^\rho_\mu(\partial_\sigma h_{\rho\nu})\bar\chi\epsilon^{\tau\sigma\mu\nu}\gamma_\tau\gamma^5\chi - \frac{\kappa^2}{8}h^{\mu\rho}h^\nu_\rho\mathcal{L}_{\mu\nu} - a^2h^{\sigma0}(\partial_\sigma h)\mathcal{H} \\ &+ \frac{1}{4}(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\Big[\frac{\kappa^2}{2}\mathcal{L}_0 + a^2(\mathcal{H}^2+2\mathcal{H}')\Big] + \frac{1}{4}(2hh^{\mu\nu} - 4h^{\mu\rho}h^\nu_\rho)\Big[2a^2\delta^0_\mu\delta^0_\nu(\mathcal{H}'-\mathcal{H}^2) - \frac{\kappa^2}{2}\mathcal{L}_{\mu\nu}\Big] \\ &+ \frac{1}{2}a^2\Big[-(\partial_\mu h)(\partial_\nu h^{\mu\nu}) + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(\partial^\mu h^{\nu\sigma})(\partial_\mu h_{\nu\sigma}) + (\partial^\mu h^{\nu\sigma})(\partial_\sigma h_{\mu\nu})\Big]\Big\}. \end{aligned} \tag{5.19}$$

The third line consists of massless graviton propagators. The first line has a parity breaking axial vector term, which will not induce a meaningful graviton mass either (moreover, it will turn out to be gauge dependent), and a (again, gauge dependent) Hubble damping term. For the other terms, we find

$$-\frac{\kappa^2}{8}h^{\mu\rho}h^\nu_\rho\mathcal{L}_{\mu\nu} = -\frac{\kappa^2}{8}h^{0\rho}h^0_\rho\big[\langle\mathcal{L}_{00}\rangle + \delta\mathcal{L}_{00}\big] - \frac{\kappa^2}{8}h^{0\rho}h^i_\rho\big[2\delta\mathcal{L}_{0i}\big] - \frac{\kappa^2}{8}h^{i\rho}h^j_\rho\left[\frac{\langle\delta^{kl}\mathcal{L}_{kl}\rangle}{3}\delta_{ij} + \delta\mathcal{L}_{ij}\right], \tag{5.20}$$

$$\frac{1}{4}(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\frac{\kappa^2}{2}\mathcal{L}_0 + a^2(\mathcal{H}^2+2\mathcal{H}') = \frac{\kappa^2}{8}(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\Big[\delta\mathcal{L}_0 + \frac{\langle\delta^{ij}\mathcal{L}_{ij}\rangle}{3}\Big], \tag{5.21}$$

$$\begin{aligned} \frac{1}{4}(2hh^{\mu\nu} - 4h^{\mu\rho}h^\nu_\rho)\Big[2a^2\delta^0_\mu\delta^0_\nu(\mathcal{H}'-\mathcal{H}^2) - \frac{\kappa^2}{2}\mathcal{L}_{\mu\nu}\Big] =& \frac{\kappa^2}{8}(2hh^{00} - 4h^{0\rho}h^0_\rho)\left[\frac{\langle\delta^{ij}\mathcal{L}_{ij}\rangle}{3} - \delta\mathcal{L}_{00}\right] \\ &- \frac{\kappa^2}{8}(2hh^{0i} - 4h^{0\rho}h^i_\rho)\delta\mathcal{L}_{0i} \\ &- \frac{\kappa^2}{8}(2hh^{ij} - 4h^{i\rho}h^j_\rho)\left[\delta_{ij}\frac{\langle\delta^{kl}\mathcal{L}_{kl}\rangle}{3} + \delta\mathcal{L}_{ij}\right]. \end{aligned} \tag{5.22}$$

The perturbations are allowed to generate a mass, so adding the expectation values, we find

$$-\frac{\kappa^2}{8}h^{0\rho}h^0_\rho\langle\mathcal{L}_{00}\rangle + \frac{\kappa^2}{8}\left[h_{00}^2 - 3h_{0i}^2 + 2\delta^{ij}h_{ij}h_{00} - (\delta^{ij}h_{ij})^2 + h_{ij}^2\right]\frac{\langle\delta^{kl}\mathcal{L}_{kl}\rangle}{3}. \tag{5.23}$$

The first term, and the first four terms in square brackets can all be removed by choosing a gauge, for example the traceless transverse gauge. However, we would then be left with a problematic term $-h_{ij}^2 a^4\langle P\rangle$, which would mean that the fermionic dark matter fluid sources a graviton mass via its pressure. Although regular dark matter is assumed to be pressureless [24], the fact that we found this possibility is still troublesome: the graviton mass should be 0. Either we have found ground-breaking new physics, or there is a mistake in the calculations; we expect the latter. Nevertheless, in the next section these troublesome terms will turn out to be negligible for our purposes (they turn out to be $\mathcal{O}(\Phi^3)$), and any mistake in these terms will therefore not propagate throughout the thesis. The final results will still be valid.

5.3 Newtonian Gauge

In this section we will integrate out the gravitational fields, which will turn out to proceed in a very similar way as in [5] (implicitly, we are dealing with path integrals, hence the name 'integrating out'). The goal is to remove degrees of freedom that are irrelevant to the length and energy scales we are interested in. To do this, we first expand the perturbations $h_{\mu\nu}$ in the following way, as done in [13],

$$\begin{aligned}\kappa h_{00} = -2\Phi, \qquad \kappa h_{0i} = w_{0i}, \qquad \kappa h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}\\ \Psi = -\frac{1}{6}\delta^{ij}\kappa h_{ij}, \qquad s_{ij} = \frac{1}{2}(\kappa h_{ij} - \frac{1}{3}\delta^{lm}\kappa h_{lm}\delta_{ij}).\end{aligned} \tag{5.24}$$

It is a clear way of breaking up the ten independent components of the perturbations into scalar (Φ, Ψ), vector (w_{0i}) and tensor (s_{ij}) perturbations. As we have gauge freedom, we can set some of these terms to 0 depending on the setting we are working in, as already done in the previous section yielding the massless gravitons. For our dark matter model, we've already made the assumption that we are observing in its rest frame, so the vector perturbations are negligible on account of rotational invariance. We also assume no strong fields which generate tensor perturbations, so they are set to 0 too. Such fields are encountered near black holes, for example. Of course, as also noted in [7], the vector and tensor perturbations can only be set to 0 on the initial spatial hypersurface, but may still be generated through non-linear dynamics. However, for the sake of simplicity we assume them to stay at least much smaller than the potential contributions, and note that vector and tensor perturbations could always be included in later works. The rescaled metric becomes

$$a^{-2}g_{\mu\nu} = \tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} = \mathrm{diag}(-(1+2\Phi), 1-2\Psi, 1-2\Psi, 1-2\Psi) \tag{5.25}$$

This is called the weak field limit, where the quantities Φ and Ψ are the gravitational potentials. We assume these to be small, and introduce a perturbation parameter

$$\mathcal{O}(\Phi, \Psi) = \varepsilon_g \sim \mathcal{O}(10^{-6}) \ll 1, \tag{5.26}$$

to indicate to what order of these perturbations we are working. Only for extremal situations like black holes it would be relevant to include $\mathcal{O}(\varepsilon_g^3)$. We will neglect these for now. In chapter 2, we mentioned that $P/\rho \sim v^2/c^2$, which is itself proportional to Φ, so pressure terms ($\propto \langle\delta^{ij}\mathcal{L}_{ij}\rangle$) will actually be one order higher in ε_g.

We obtain for the matter action parts (3.22) and (3.23),

$$S^{(1,2)}[\chi,\bar{\chi}] = \frac{\kappa}{2}\int d\eta\, d^3x(h\mathcal{L}_0 - h^{\mu\nu}\mathcal{L}_{\mu\nu}) = \int d\eta\, d^3x\Big((\Phi - 3\Psi)\mathcal{L}_0 + \Phi\mathcal{L}_{00} + \Psi\delta^{ij}\mathcal{L}_{ij}\Big), \tag{5.27}$$

$$\begin{aligned}S^{(2,2)}[\chi,\bar{\chi}] =& \frac{\kappa^2}{8}\int d\eta\, d^3x\Big\{(h^2 - 2h^{\lambda\tau}h_{\lambda\tau})\mathcal{L}_0 + (3h^{\mu\rho}h^\nu_\rho - 2hh^{\mu\nu})\mathcal{L}_{\mu\nu} - \frac{1}{2}h^\rho_\mu(\partial_\sigma h_{\rho\nu})\bar{\chi}\epsilon^{\tau\sigma\mu\nu}\gamma_\tau\gamma^5\chi\Big\}\\ =& \frac{1}{2}\int d\eta\, d^3x\Big\{\big[(\Phi - 3\Psi)^2 - 2\Phi^2 - 6\Psi^2\big]\mathcal{L}_0 + \big[-3\Phi^2 + (2\Phi - 6\Psi)\Phi\big]\mathcal{L}_{00}\\ &+ \big[3\Psi^2 + (2\Phi - 6\Psi)\Psi\big]\delta^{ij}\mathcal{L}_{ij}\Big\}\\ =& \int d\eta\, d^3x\Big\{-\frac{\Phi^2}{2}(\mathcal{L}_0 + \mathcal{L}_{00}) - \Phi\Psi(3\mathcal{L}_0 + 3\mathcal{L}_{00} - \delta^{ij}\mathcal{L}_{ij}) + \frac{\Psi^2}{2}(3\mathcal{L}_0 - \delta^{ij}\mathcal{L}_{ij}) - \Psi^2\delta^{ij}\mathcal{L}_{ij}\Big\}.\end{aligned} \tag{5.28}$$

To keep things readable, we refrain from writing out the different $\mathcal{L}$-terms. We see that the parity-breaking term is not present in this gauge, as it is set to 0 by the forced equality of the μ and ν indices.

For the Einstein-Hilbert action, we get up to quadratic order in ε_g,

$$\begin{aligned}S_g =&\frac{1}{\kappa^2}\int d\eta\, d^3x\, a^2\Big\{-6\mathcal{H}^2+(2\Phi-6\Psi)(\mathcal{H}^2+2\mathcal{H}')-4\Phi(\mathcal{H}'-\mathcal{H}^2)\\&+(2\Phi'-6\Psi')\Phi'+(2\vec{\nabla}\Phi-6\vec{\nabla}\Psi)\cdot\vec{\nabla}\Psi-(\Phi'-3\Psi')^2+(\vec{\nabla}\Phi-3\vec{\nabla}\Psi)^2\\&+\Phi'^2+3\Psi'^2-(\vec{\nabla}\Phi)^2-3(\vec{\nabla}\Psi)^2-2\Phi'^2+2(\vec{\nabla}\Psi)^2+2\Phi(2\Phi'-6\Psi')\mathcal{H}\\&+\Big[(\Phi-3\Psi)^2-2\Phi^2-6\Psi^2\Big](\mathcal{H}^2+2\mathcal{H}')+\big[-2(2\Phi-6\Psi)\Phi+8\Phi^2\big](\mathcal{H}'-\mathcal{H}^2)\Big\}\\=&\frac{1}{\kappa^2}\int d\eta\, d^3x\, a^2\Big\{-6\mathcal{H}^2+6\Phi\mathcal{H}^2-6\Psi(\mathcal{H}^2+2\mathcal{H}')-4(\vec{\nabla}\Phi)\cdot(\vec{\nabla}\Psi)-6\Psi'^2+2(\vec{\nabla}\Psi)^2\\&+(4\Phi\Phi'-12\Phi\Psi')\mathcal{H}+(-5\Phi^2-18\Phi\Psi+3\Psi^2)\mathcal{H}^2+(2\Phi^2+6\Psi^2)\mathcal{H}'\Big\},\end{aligned}\tag{5.29}$$

where $(\vec{\nabla}\Psi)^2\equiv(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Psi)$, with $\vec{\nabla}$ the spatial derivative. Observe that

$$4a^2\Phi\Phi'\mathcal{H}=2aa'(\partial_0\Phi^2)=2\partial_0(aa'\Phi^2)-2\Phi^2a'^2-2\Phi^2aa''=2\partial_0(aa'\Phi^2)-2a^2\Phi^2(2\mathcal{H}^2+\mathcal{H}'),\tag{5.30}$$

so neglecting the boundary term,

$$\begin{aligned}S_g =&\frac{1}{\kappa^2}\int d\eta\, d^3x\, a^2\Big\{-6\mathcal{H}^2+6\Phi\mathcal{H}^2-6\Psi(\mathcal{H}^2+2\mathcal{H}')-4(\vec{\nabla}\Phi)(\vec{\nabla}\Psi)-6\Psi'^2+2(\vec{\nabla}\Psi)^2\\&-12\Phi\Psi'\mathcal{H}-(9\Phi^2+18\Phi\Psi)\mathcal{H}^2+3\Psi^2(\mathcal{H}^2+2\mathcal{H}')\Big\}.\end{aligned}\tag{5.31}$$

To make these equations a bit easier to handle, we define the following quantities

$$E_0\equiv\frac{\kappa^2}{2a}(\mathcal{L}_0+\mathcal{L}_{00})+3a\mathcal{H}^2=\frac{\kappa^2}{2a}\big(\delta\mathcal{L}_0-a^4\langle\rho\rangle+\delta\mathcal{L}_{00}+a^4\langle\rho\rangle\big)=\frac{\kappa^2}{2a}\delta(\mathcal{L}_0+\mathcal{L}_{00}),\tag{5.32}$$

$$E_1\equiv\frac{\kappa^2}{2a}(3\mathcal{L}_0-\delta^{ij}\mathcal{L}_{ij})+3a(\mathcal{H}^2+2\mathcal{H}')=\frac{\kappa^2}{2a}\delta(3\mathcal{L}_0-\delta^{ij}\mathcal{L}_{ij}),\tag{5.33}$$

in which we have used (5.18). These quantities are clearly on the order of the perturbations in the background, and we therefore introduce another perturbation parameter, $\varepsilon_E=\mathcal{O}(E_0,E_1)$. We define now $\tilde{S}$, which is the action up to quadratic order in ε_g without the zeroth order contributions, as they do not play a role in integrating out the gravitational fields,

$$\begin{aligned}\tilde{S}\equiv S^{(1)}_{tot}+S^{(2)}_{tot}=&\frac{2}{\kappa^2}\int d\eta\, d^3x\Big\{a\Phi E_0-a\Psi E_1-\frac{\Phi^2}{2}aE_0-3\Phi\Psi aE_0+\frac{\Psi^2}{2}aE_1\\&+(\Phi\Psi-\Psi^2)\delta^{ij}\mathcal{L}_{ij}+2a^2\Phi\Delta\Psi-3a^2\Psi'^2-a^2\Psi\Delta\Psi-6a^2\Phi\Psi'\mathcal{H}-3a^2\Phi^2\mathcal{H}^2\Big\}.\end{aligned}\tag{5.34}$$

Note that integration by parts was performed for the spatial derivatives (dropping boundary terms), which is easily done as the conformal Hubble rate and the scale factor do not depend on space. From it follows the Laplacian, which is defined as usual as $\Delta\equiv\vec{\nabla}^2$. If we now want integrate out the fields Φ and Ψ, we first vary with respect to Φ, and then demand that $\delta\Phi=0$ at the boundary. Afterwards, we will substitute back the resulting equation of motion back into the action. We find the following

$$\frac{\kappa^2}{2}\frac{\delta\tilde{S}}{\delta\Phi}=aE_0(1-3\Psi)+\frac{\kappa^2}{2}\delta^{ij}\mathcal{L}_{ij}\Psi+2a^2\Delta\Psi-6a^2\mathcal{H}\Psi'-a\Phi\big(E_0+6a\mathcal{H}^2\big)=0,\tag{5.35}$$

$$\Phi=\frac{E_0(1-3\Psi)+\frac{\kappa^2}{2a}\delta^{ij}\mathcal{L}_{ij}\Psi+2a\Delta\Psi-6a\mathcal{H}\Psi'}{6a\mathcal{H}^2+E_0}.\tag{5.36}$$

Observe that the equation of motion for Φ contains no time derivative on Φ: it is a constraint equation. We could immediately substitute this back into the action, making a mess, but we can also look at the action once more and find

$$\begin{aligned}\tilde{S} =& \frac{2}{\kappa^2}\int d\eta\, d^3x \Big\{ a\Phi\big[(1-3\Psi)E_0 + \Psi\frac{\kappa^2}{2a}\delta^{ij}\mathcal{L}_{ij} + 2a\Delta\Psi - 6a\Psi'\mathcal{H}\big] - a\frac{\Phi^2}{2}\big(E_0 + 6a\mathcal{H}^2\big) \\ & - a\Psi(1-\frac{\Psi}{2})E_1 - \frac{\kappa^2}{2}\delta^{ij}\mathcal{L}_{ij}\Psi^2 - 3a^2\Psi'^2 - a^2\Psi\Delta\Psi\Big\} \\ =& \frac{2}{\kappa^2}\int d\eta\, d^3x \Big\{ a\frac{\Phi^2}{2}\big(E_0 + 6a\mathcal{H}^2\big) - a\Psi(1-\frac{\Psi}{2})E_1 - \frac{\kappa^2}{2}\delta^{ij}\mathcal{L}_{ij}\Psi^2 - 3a^2\Psi'^2 - a^2\Psi\Delta\Psi\Big\}\end{aligned} \tag{5.37}$$

We only have Φ^2-terms, which we can evaluate separately to avoid writing out terms which we are not using anyway. We look at the following:

$$\begin{aligned} a\Phi^2\big(E_0 + 6a\mathcal{H}^2\big) =& a\frac{\big[E_0(1-3\Psi) + \frac{\kappa^2}{2a}\delta^{ij}\mathcal{L}_{ij}\Psi + 2a\Delta\Psi - 6a\mathcal{H}\Psi'\big]^2}{6a\mathcal{H}^2 + E_0} \\ =& \frac{1}{6\mathcal{H}^2}\big[E_0(1-3\Psi) + \frac{\kappa^2}{2a}\delta^{ij}\mathcal{L}_{ij}\Psi + 2a\Delta\Psi - 6a\mathcal{H}\Psi'\big]^2\Big(1 - \frac{E_0}{6a\mathcal{H}^2} + \Big(\frac{E_0}{6a\mathcal{H}^2}\Big)^2 + ...\Big),\end{aligned} \tag{5.38}$$

where we have expanded the denominator. We do this to illustrate the contribution E_0 has to the denominator, which becomes smaller with each term of the expansion. This greatly reduces the number of terms we eventually have to write out. We define P_0 and write,

$$P_0 \equiv \frac{\kappa^2}{2a^2}\delta^{ij}\mathcal{L}_{ij}, \tag{5.39}$$

$$\begin{aligned}\int d\eta\, d^3x\; a\Phi^2\big(E_0 + 6a\mathcal{H}^2\big) =& \int d\eta\, d^3x\, \Big\{ \frac{E_0^2}{6\mathcal{H}^2} + \frac{a^2}{6\mathcal{H}^2}P_0^2\Psi^2 + \frac{2a^2}{3\mathcal{H}^2}(\Delta\Psi)^2 + 6a^2\Psi'^2 + \frac{2a}{3\mathcal{H}^2}E_0\Delta\Psi \\ & - \frac{2a}{\mathcal{H}}E_0\Psi' - \frac{4a^2}{\mathcal{H}}\Psi'\Delta\Psi + \frac{a}{3\mathcal{H}^2}P_0E_0\Psi + \frac{2a^2}{3\mathcal{H}^2}P_0\Psi\Delta\Psi - \frac{2a^2}{\mathcal{H}}P_0\Psi\Psi'\Big\} \\ & + \mathcal{O}(\varepsilon_g\varepsilon_E^2,\ \varepsilon_g^2\varepsilon_E) \\ =& \int d\eta\, d^3x\, \Big\{ \frac{E_0^2}{6\mathcal{H}^2} + \frac{2a^2}{3\mathcal{H}^2}\Psi\Delta^2\Psi + 6a^2\Psi'^2 \\ & + a\mathcal{H}^{-2}\Big(\frac{2}{3}\Delta E_0 + 2(\mathcal{H}^2 - \mathcal{H}')E_0 + 2\mathcal{H}E_0' + \frac{P_0}{3}E_0\Big)\Psi \\ & + a^2\mathcal{H}^{-2}\Big[\frac{P_0^2}{6} + (2\mathcal{H}^2 - \mathcal{H}')P_0 + \mathcal{H}P_0'\Big]\Psi^2 \\ & + 2a^2\mathcal{H}^{-2}(\frac{P_0}{3} + 2\mathcal{H}^2 - \mathcal{H}')\Psi\Delta\Psi\Big\} + \mathcal{O}(\varepsilon_g\varepsilon_E^2,\ \varepsilon_g^2\varepsilon_E).\end{aligned} \tag{5.40}$$

In the second step, we performed some integrations by parts, dropping boundary terms again. Most notably, we split $4\Psi'\Delta\Psi = 2\Psi'\Delta\Psi + 2\Psi'\Delta\Psi$, and then applied Leibniz to one of the terms a few times. We also used the fact that $\partial_\eta\Psi^2 = 2\Psi\Psi'$. P_0 can be written differently, cf. (5.33),

$$P_0 = \frac{\kappa^2}{2a^2}\delta^{ij}\mathcal{L}_{ij} = \frac{\kappa^2}{2a^2}\mathcal{L}_0 + 3(\mathcal{H}^2 + 2\mathcal{H}') - \frac{E_1}{a} = \frac{\kappa^2}{2a^2}\delta\mathcal{L}_0 + 3(\mathcal{H}^2 + 2\mathcal{H}') - \frac{E_1}{a}, \tag{5.41}$$

where we again used the fact that the expectation value of the tree-level Lagrangian is 0. This can be used to see that, when working up to quadratic order in ε_E, ε_g and $\mathcal{O}(\delta\mathcal{L}_0)$, P_0 depends only on conformal time. We will therefore treat this term as such, and neglect any spatial derivatives on this term that could appear due to integration by parts.

We can finally substitute back into (5.37), so

$$\begin{aligned}\tilde{S} =&\frac{2}{\kappa^2}\int d\eta\, d^3x\bigg\{\frac{E_0^2}{12\mathcal{H}^2}+\frac{a^2}{3\mathcal{H}^2}\Psi\Delta^2\Psi+a\mathcal{H}^{-2}\Big(\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0\Big)\Psi\\&+a^2\mathcal{H}^{-2}\Big(\frac{P_0^2}{6}+(2\mathcal{H}^2-\mathcal{H}')P_0+\mathcal{H}P_0'\Big)\frac{\Psi^2}{2}+3a^2\Psi'^2+a^2\mathcal{H}^{-2}\Big(\frac{P_0}{3}+2\mathcal{H}^2-\mathcal{H}'\Big)\Psi\Delta\Psi\\&-a\Psi E_1-a^2P_0\Psi^2-3a^2\Psi'^2-a^2\Psi\Delta\Psi\bigg\}+\mathcal{O}(\varepsilon_g\varepsilon_E^2,\ \varepsilon_g^2\varepsilon_E)\\=&\frac{2}{\kappa^2}\int d\eta\, d^3x\bigg\{\frac{E_0^2}{12\mathcal{H}^2}+\frac{a^2}{3\mathcal{H}^2}\Psi\Delta^2\Psi+a\mathcal{H}^{-2}\Big(\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0-\mathcal{H}^2E_1\Big)\Psi\\&+a^2\mathcal{H}^{-2}\Big(\frac{P_0}{6}-\mathcal{H}'P_0+\mathcal{H}P_0'\Big)\frac{\Psi^2}{2}+a^2\mathcal{H}^{-2}\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)\Psi\Delta\Psi\bigg\}+\mathcal{O}(\varepsilon_g\varepsilon_E^2,\ \varepsilon_g^2\varepsilon_E).\end{aligned}\tag{5.42}$$

One thing to note is that the terms with temporal derivatives on Ψ have completely dropped out, such that the equation of motion for this potential will, similar to that for Φ, be nondynamical: another constraint equation. This is essential for this procedure, as we could not go to an effective theory if these equations were dynamical. We have also written the action in such a way that we can now easily vary with respect to the potential Ψ,

$$\begin{aligned}\frac{\kappa^2}{2}\mathcal{H}^2\frac{\delta\tilde{S}}{\delta\Psi}=&\frac{2a^2}{3}\Delta^2\Psi+2a^2\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)\Delta\Psi+a^2\Big(\frac{P_0^2}{6}-\mathcal{H}'P-0+\mathcal{H}P_0'\Big)\Psi\\&+a\Big(\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}-\mathcal{H}^2E_1\Big)=0,\end{aligned}\tag{5.43}$$

$$\implies \Delta_{\mathcal{H}}^2\Psi=-\frac{3}{2}a^{-1}\Big(\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0-\mathcal{H}^2E_1\Big).\tag{5.44}$$

Here we have defined the following operator

$$\Delta_{\mathcal{H}}^2\equiv\Delta^2+3\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)\Delta+\frac{3}{2}\Big(\frac{P_0^2}{6}-\mathcal{H}'P_0+\mathcal{H}P_0'\Big).\tag{5.45}$$

Substituting this back into (5.42) gives (dropping the higher order terms)

$$\begin{aligned}\tilde{S}\approx&\frac{2}{\kappa^2}\int d\eta\, d^3x\bigg\{\frac{E_0^2}{12\mathcal{H}^2}+\frac{a}{2\mathcal{H}^2}\Big[\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0-\mathcal{H}^2E_1\Big]\Psi\bigg\}\\=&\frac{2}{\kappa^2}\int d\eta\, d^3x\bigg\{\frac{E_0^2}{12\mathcal{H}^2}-\frac{3}{4\mathcal{H}^2}\Big[\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0-\mathcal{H}^2E_1\Big]\\&\times\Delta_{\mathcal{H}}^{-2}\Big[\frac{1}{3}\Delta E_0+(\mathcal{H}^2-\mathcal{H}')E_0+\mathcal{H}E_0'+\frac{P_0}{6}E_0-\mathcal{H}^2E_1\Big]\bigg\}.\end{aligned}\tag{5.46}$$

For the inversion of (5.45), we look at a suitable approximation to limit the number of terms. We take the sub-Hubble limit, where we define yet another small parameter

$$\mathcal{O}\left(\frac{\mathcal{H}^2}{|\Delta|},\frac{\mathcal{H}'}{|\Delta|}\right)=\varepsilon_{\mathcal{H}}\ll 1.\tag{5.47}$$

We can infer this limit from e.g. the de Broglie-wavelength of the dark matter, assuming it is much smaller than the Hubble radius, which is reasonable for nonrelativistic particles that have masses of the order of electronvolts. We find $\lambda_{dB}=h/p\ll R_H\implies p\ll H$, where h is Planck's constant, $\vec{p}=-i\hbar\vec{\nabla}$ is the momentum, and $R_H\approx c/H$ is the Hubble radius. Inverting the operator gives

$$\begin{aligned}\Delta_{\mathcal{H}}^2\Delta_{\mathcal{H}}^{-2}=I=&\Big[1+3(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}')\Delta^{-1}+\frac{3}{2}[\frac{P_0^2}{6}-\mathcal{H}'P_0+\mathcal{H}P_0']\Delta^{-2}\Big]\Delta^2\Delta_{\mathcal{H}}^{-2}\\\implies\Delta_{\mathcal{H}}^{-2}=&\Big[1-3(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}')\Delta^{-1}-\frac{3}{2}[\frac{P_0^2}{6}-\mathcal{H}'P_0+\mathcal{H}P_0']\Delta^{-2}\\&+9\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)^2\Delta^{-2}\Big]\Delta^{-2}+\mathcal{O}(\varepsilon_{\mathcal{H}}^3).\end{aligned}\tag{5.48}$$

In the first equality, I is some identity operator. If we work to leading order in $\varepsilon_{\mathcal{H}}^2$ we have

$$
\begin{aligned}
\tilde{S} =&\frac{2}{\kappa^2}\int d\eta\, d^3x\Big\{\frac{E_0^2}{12\mathcal{H}^2}-\frac{3}{4\mathcal{H}^2}\Big(\frac{1}{9}E_0^2-\frac{1}{3}\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)E_0\Delta^{-1}E_0+\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)^2E_0\Delta^{-2}E_0\\
&-\frac{1}{6}\Big[\frac{P_0^2}{6}-\mathcal{H}'P_0+\mathcal{H}P_0'\Big]E_0\Delta^{-2}E_0\\
&+\frac{2}{3}\Big[(\mathcal{H}^2-\mathcal{H}')E_0+\frac{P_0}{6}E_0+\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Delta^{-1}E_0\\
&-2\Big[\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)E_0-\frac{P_0}{6}E_0+\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)\Delta^{-2}E_0\\
&+\Big[\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)E_0-\frac{P_0}{6}E_0+\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\\
&\times\Delta^{-2}\Big[\Big(\frac{P_0}{3}+\mathcal{H}^2-\mathcal{H}'\Big)E_0-\frac{P_0}{6}E_0+\mathcal{H}E_0'-\mathcal{H}^2E_1\Big)\Big\}\\
&+\mathcal{O}(\varepsilon_{\mathcal{H}}^3)\\
=&\frac{2}{\kappa^2}\int d\eta\, d^3x\Big\{\frac{E_0^2}{12\mathcal{H}^2}-\frac{3}{4\mathcal{H}^2}\Big(\frac{1}{9}E_0^2+\frac{1}{3}(\mathcal{H}^2-\mathcal{H}')E_0\Delta^{-1}E_0+\frac{1}{6}\Big[\mathcal{H}'P_0-\mathcal{H}P_0'\Big]E_0\Delta^{-2}E_0\\
&+\frac{2}{3}\Big[\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Delta^{-1}E_0-\frac{P_0}{3}\Big[\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Delta^{-2}E_0+\Big[\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Delta^{-2}\Big[\mathcal{H}E_0'-\mathcal{H}^2E_1\Big]\Big)\Big\}\\
&+\mathcal{O}(\varepsilon_{\mathcal{H}}^3).
\end{aligned}
\tag{5.49}
$$

We have again used integration by parts, dropping boundary terms. While it may also seem we assumed Leibniz' rule to hold for Δ^{-1}, we actually used the following

$$
\int d^3xE_0\Delta^{-1}E_1=\int d^3x(\Delta[\Delta^{-1}E_0])(\Delta^{-1}E_1)=\int d^3x(\Delta^{-1}E_0)(\Delta[\Delta^{-1}E_1])=\int d^3x(\Delta^{-1}E_0)E_1,
\tag{5.50}
$$

where we again integrated by parts in the second step, and dropped the spatial boundary terms. For E_0', we can also use this relation to obtain

$$
2\int d^3x\; E_0'\Delta^{-1}E_0=\int d^3x\;\Big(E_0'\Delta^{-1}E_0+E_0\Delta^{-1}E_0'\Big)=\int d^3x\;\partial_\eta\Big(E_0\Delta^{-1}E_0\Big),
\tag{5.51}
$$

after which we can move the temporal derivative to any conformal-time-dependent prefactor. Applying this, we find for (5.49)

$$
\begin{aligned}
\tilde{S} =&\frac{2}{\kappa^2}\int d\eta\, d^3x\Big\{-\frac{1}{4\mathcal{H}^2}(\mathcal{H}^2-\mathcal{H}')E_0\Delta^{-1}E_0-\frac{1}{8\mathcal{H}^2}\Big[\mathcal{H}'P_0-\mathcal{H}P_0'\Big]E_0\Delta^{-2}E_0\\
&+\frac{1}{2}\Big[E_1-\mathcal{H}^{-1}E_0'\Big]\Delta^{-1}E_0+\frac{P_0}{4}\mathcal{H}^{-1}E_0'\Delta^{-2}E_0-\frac{P_0}{4}E_1\Delta^{-2}E_0\\
&-\frac{3}{4}\Big[E_0'-\mathcal{H}E_1\Big]\Delta^{-2}\Big[E_0'-\mathcal{H}E_1\Big]\Big\}+\mathcal{O}(\varepsilon_{\mathcal{H}}^3)\\
=&\frac{1}{2\kappa^2}\int d\eta\, d^3x\Big\{\Big[2E_1-E_0\Big]\Delta^{-1}E_0-P_0E_1\Delta^{-2}E_0-3\Big[E_0'-\mathcal{H}E_1\Big]\Delta^{-2}\Big[E_0'-\mathcal{H}E_1\Big]\Big\}+\mathcal{O}(\varepsilon_{\mathcal{H}}^3).
\end{aligned}
\tag{5.52}
$$

The result we have found here has the nonlocality encoded in the inverse Laplacian, which when evaluated as the Green's function of the Laplacian will give a clear spatial $1/||\vec{x}||$ dependence (we will see this explicitly in the next chapter). We have evaluated up to second order in $\varepsilon_{\mathcal{H}}$, and it is interesting to see that the same result (but up to linear order) is also found in [5]. Of course, the main difference between the result here and in the cited paper is that our E_0 and E_1 are dependent on fermionic fields. We see that the combinations we defined for these quantities were not arbitrary:

$$
E_0=\frac{\kappa^2}{2a}(\delta\mathcal{L}_0+\delta\mathcal{L}_{00})=-\frac{\kappa^2}{2a}\delta(a^2T_{00})=-\frac{\kappa^2}{2}a^3\delta\rho,
\tag{5.53}
$$

$$
E_1=\frac{\kappa^2}{2a}(3\delta\mathcal{L}_0-\delta^{ij}\delta\mathcal{L}_{ij})=\frac{\kappa^2}{2a}\delta(a^2\delta^{ij}T_{ij})=3\frac{\kappa^2}{2}a^3\delta P,
\tag{5.54}
$$

the different contributions are the (rescaled) energy density and pressure of the fluctuations in the homogeneous background. Even though we are looking at the variations, it is still instructive to look at the following equation

$$\partial_\eta(a^3\langle\rho\rangle) = 3\partial_\eta(a\mathcal{H}^2) = 3\mathcal{H}(a\mathcal{H}^2 + 2a\mathcal{H}') = -3\mathcal{H}a^3\langle P\rangle. \tag{5.55}$$

The resemblance between E_0' and $\mathcal{H}E_1$ should not go unnoticed, which could serve as another reason besides being second order in $\varepsilon_{\mathcal{H}}$ to drop the second term in (5.52).

For completeness, the final result of this chapter is the following action, assuming a weak gravitational field, the sub-Hubble limit, adiabatic expansion and small perturbations,

$$\begin{aligned} S_\chi[\bar{\chi},\chi] \equiv S_{tot} - S_g^{(0,0)} = \int d\eta\, d^3x \Big\{ &\mathcal{L}_0 + \frac{1}{2\kappa^2}\big(2E_1 - E_0\big)\Delta^{-1}E_0 - \frac{1}{4a^2}\delta^{ij}\mathcal{L}_{ij}E_1\Delta^{-2}E_0 \\ &- \frac{3}{2\kappa^2}(E_0' - \mathcal{H}E_1)\Delta^{-2}(E_0' - \mathcal{H}E_1)\Big\}. \end{aligned} \tag{5.56}$$

This is the final result of this chapter, which is the action that only contains parts relevant for the equations of motion for the fermion propagator. We will develop the 2PI effective action from it in the next chapter, and obtain the equations of motion for the fermionic two point function from it. These can be linked to the phase space densities we have already looked at in the introduction. The procedure of integrating out the gravitational potentials has given us a new action: the now nonlocal effect of gravity is encoded in the four-fermion interaction vertices constructed from the E_0, E_1 contributions.

6 2PI effective action in the large-mass limit

The goal of this chapter is to obtain the equations of motion for dark matter fermions. We will make several more approximations to make sure these equations are as relevant as needed; for example, the equations still allow for a large range of masses, while there exists a theoretical lower bound on fermionic dark matter mass. And in the case of Weyl or Dirac fermions, the equations allow for particles and antiparticles. For standard model particles, we know that particles and antiparticles annihilate each other upon collision, leaving only radiation (at least up to tree level). This way all antimatter was annihilated in the early universe, leaving us only with a small fraction of regular matter due to asymmetry [25]. We will assume the fermions of which dark matter consists are Dirac fermions (and therefore use the Dirac representation of the gamma matrices, see appendix E), just like the bulk of the standard model fermions. We then also have particles and antiparticles, and will also want to assume that the antiparticle number has been set to 0. A large chemical potential will then make the creation of antiparticles highly unlikely, which also suppresses particle-antiparticle mixing. We see that we will lose a large amount of generality, for the purpose of extracting relevant equations, as we aimed to do.

Starting from the final result of the last chapter, we can write it as follows

$$\begin{aligned} S_\chi = \int d\eta\, d^3x \Big\{ &\bar{\chi}(i\partial\!\!\!/ - Ma)\chi - \frac{1}{2\kappa^2}\big(E_0 - E_1(2 - 3(\mathcal{H}^2 + 2\mathcal{H}')\Delta^{-1})\big)\Delta^{-1}E_0 \\ &- \frac{3}{2\kappa^2}(E_0' - \mathcal{H}E_1)\Delta^{-2}(E_0' - \mathcal{H}E_1)\Big\} + \mathcal{O}((\bar{\chi}\chi)^3). \end{aligned} \tag{6.1}$$

We have partially integrated parts of $\mathcal{L}_0$, dropping boundary terms. We have also used (5.33) to rewrite $\delta^{ij}\mathcal{L}_{ij}$, where we dropped the $\delta\mathcal{L}_0$ and E_1 contributions. This is done because we will evaluate the two particle irreducible effective action truncated at two loops in this chapter, making terms of higher order than $(\bar{\chi}\chi)^2$ irrelevant as they do not appear in two loop diagrams.

In the previous chapter we have already defined the sub-Hubble limit, so the terms with Δ^{-2} are obviously smaller than the other terms in (6.1). We can also make a simple approximation, namely

$$\varepsilon_m = \frac{|\vec{\nabla}|^2}{(ma)^2} \ll 1, \tag{6.2}$$

which we call the large-mass limit. It is a natural approximation for a theory where the particles are (relatively) heavy and slowly moving, as opposed to e.g. fuzzy dark matter, in which extremely light particles are assumed. Given that fermions obey the exclusion principle, a limited amount of particles can be in a given volume, constraining the mass of nonrelativistic dark matter fermions. This prevents them from having mass as low as fuzzy dark matter, which is typically of the order $\sim 10^{-22}$eV [26]. However, lower bounds on fermionic dark matter, for example in halos around dwarf spheroidal galaxies, have been set on ~ 190eV [27], and for the Milky way on ~ 30eV [6]. Notice that the corresponding de Broglie wavelength, assuming the latter mass and a rotational velocity of 220km/s, is on the order of micrometers, which is far exceeded by the cosmological scales of megaparsecs we are interested in (consequently making a particle description very accurate). Of course, the case could be made that lighter fermions (like neutrinos) in some volume can be heated up to increase the corresponding energy density. The problem then quickly arises that because speed relates to temperature by $mv^2/2 \approx k_BT$, such light particles will have to obtain ultrarelativistic speeds when heated up, and it is shown that structure formation, if any, will not happen in the way we observe it [28]. An upper limit on the speeds are actually set by the escape velocities of the structures, which are at most ~ 1000km/s, which is very nonrelativistic, thereby ruling out light fermions. The only way the bound could be lowered by assuming multiple flavors of fermionic dark matter, but we will not do that here and take on these constraints.

Given the slowly moving and weakly interacting dark matter particles, a first consequence of the approximation is

$$\frac{|E_1|}{|E_0|} = \frac{|\delta P|}{|\delta\rho|} \sim v_{\text{sound}} \ll c. \tag{6.3}$$

The ratio between pressure and energy density will be comparable to the speed of sound, which will be of the order of the velocities of the constituent particles, very nonrelativistic. This means that the interaction term in (6.1) is brought down to the $E_0\Delta^{-1}E_0$-term only. We can then define the following quantity and find for E_0

$$n_0(\eta) \equiv 3a\mathcal{H}^2\frac{2}{\kappa^2 m} = a^3\frac{\langle\rho\rangle}{m} \approx \frac{1}{ma}\left\langle -\frac{i}{2}(\bar{\chi}\vec{\gamma}\cdot\vec{\nabla}\chi + (\vec{\nabla}\bar{\chi})\cdot\vec{\gamma}\chi) + \bar{\chi}Ma\chi\right\rangle \xrightarrow{\langle P\rangle\to 0} \frac{1}{m}\langle\bar{\chi}M\chi\rangle\,, \tag{6.4}$$

$$E_0 = \frac{\kappa^2}{2a}\Big[\frac{i}{2}(\bar{\chi}\vec{\gamma}\cdot\vec{\nabla}\chi - (\vec{\nabla}\bar{\chi})\cdot\vec{\gamma}\chi) - \bar{\chi}Ma\chi + man_0\Big]. \tag{6.5}$$

We will neglect the spatial derivative terms in E_0. We do this because the n_0 term includes the expectation value of these terms, meaning that what is left are the perturbations in these terms. If we then also assume that the pressure is very small (6.3), to which these terms are linked, and that the mass terms are very large on account of (6.2), it is reasonable to leave these out. From hereon, we therefore only consider the massive part of n_0. In the matter era of the universe, where ρ scales as a^3, and assuming the expectation value of the pressure will be extremely small for dark matter (at least compared to the expectation value of the energy density), n_0 can be interpreted as the expectation value of the comoving number density of the homogeneous background (counting the number of particles minus the number of antiparticles). We would also need to have $m = m_R$, as the mass mixing induced by the $m_I\gamma^5$ term will not affect $\langle\rho\rangle$ in such a way that simply dividing by $m = \sqrt{m_R^2 + m_I^2}$ gives a meaningful quantity. One should therefore be careful in giving physical meaning to n_0, and here it is just a geometric quantity dependent on the expansion rate of the universe. We make no statements about the contents of the universe at this point, but simply use the quantity as a way of writing our expressions concisely. Notice that $n_0(\eta)$ is defined in terms of constants and $a(\eta)$, $\mathcal{H}(\eta)$, and therefore only depends on conformal time; its value throughout space is constant, as expected for a homogeneous background value. We now write for the action

$$\begin{aligned} S_\chi &\approx \int d\eta\, d^3x\Big\{\bar{\chi}(i\partial\!\!\!/ - Ma)\chi - \frac{1}{2\kappa^2}E_0\Delta^{-1}E_0\Big\} \\ &\approx \int d\eta\, d^3x\Big\{\bar{\chi}(i\partial\!\!\!/ - Ma)\chi + \frac{\kappa^2}{32\pi}\int d\tilde{\eta}\int d^3y\frac{\big(\bar{\chi}(x)M\chi(x) - mn_0\big)\big(\bar{\chi}(y)M\chi(y) - mn_0\big)}{||\vec{x}-\vec{y}||}\delta(\eta-\tilde{\eta})\Big\}. \end{aligned} \tag{6.6}$$

To evaluate Δ^{-1}, we already noted that it is by definition a Green's function for the Laplacian. Given that we assume an isotropic universe, one can easily solve for it by taking spherical coordinates and neglecting the angular derivatives. We will frequently rewrite the inverse Laplacian to save space, so note that

$$4\pi\Delta_y^{-1}(\cdot) = -\int\frac{d^3x}{||\vec{x}-\vec{y}||}(\cdot). \tag{6.7}$$

Notice also that, as the definition of n_0 includes m^{-1}, the quantity mn_0 appearing in (6.6) is in fact independent of m, but only dependent on M.

The expectation values are taken with respect to an in- and an out-state. Of course, to do this, one needs to know what these states are, and for an equilibrium process it is not hard to define these states and make calculations. However, for nonequilibrium processes, e.g. dark matter in an expanding universe, it is generally not known what the out-state exactly is. A clever 'trick' by Schwinger [29] bypasses this problem by introducing a closed time path, such that the in- and out-states are the same, see figure 6.1. This closed time path formalism is also called in-in formalism, or as we call it here, the Schwinger-Keldysh formalism, as Keldysh developed Schwinger's theory further.

What we see in figure 6.1 is the evolution from a time η_0, going through η_+, η'_+ to a finite time η, and then moving back through η'_- and η_- to η_0 again. This defines a contour $\mathcal{C}$, for which we construct the Keldysh propagator [31]

$$iS_{\alpha\beta}(x;y) = \langle T_{\mathcal{C}}[\chi_\alpha(x)\bar{\chi}_\beta(y)]\rangle = \begin{pmatrix} iS^{++}_{\alpha\beta}(x;y) & iS^{+-}_{\alpha\beta}(x;y) \\ iS^{-+}_{\alpha\beta}(x;y) & iS^{--}_{\alpha\beta}(x;y)\end{pmatrix}. \tag{6.8}$$

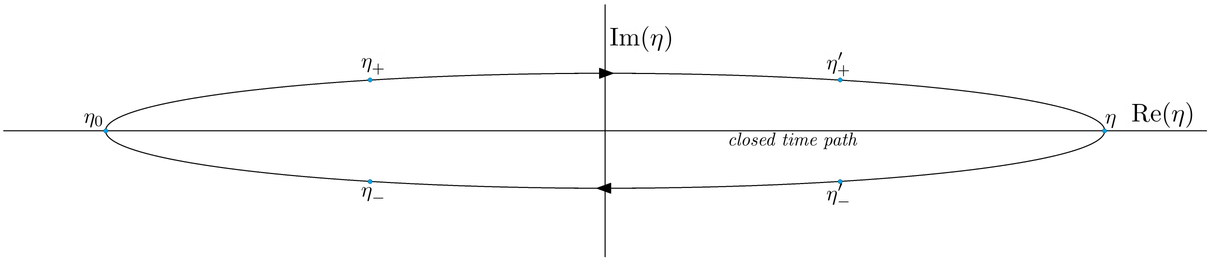

Figure 6.1: The closed time path, adapted from [30] for conformal time. The times $\eta_\pm$ are meant to indicate a small imaginary shift from the positive to negative curves, to distinguish between Feynman/Dyson and Wightman functions (see also appendix F).

$T_\mathcal{C}$ denotes time ordering along the contour, and the matrix consists of the following propagators

$$iS^{++}_{\alpha\beta}(x;y) = \langle T[\chi_\alpha(x)\bar{\chi}_\beta(y)]\rangle \equiv \theta(\eta-\eta')\langle\chi_\alpha(x)\bar{\chi}_\beta(y)\rangle - \theta(\eta'-\eta)\langle\bar{\chi}_\beta(y)\chi_\alpha(x)\rangle, \tag{6.9}$$

$$iS^{--}_{\alpha\beta}(x;y) = \langle \bar{T}[\chi_\alpha(x)\bar{\chi}_\beta(y)]\rangle \equiv \theta(\eta'-\eta)\langle\chi_\alpha(x)\bar{\chi}_\beta(y)\rangle - \theta(\eta-\eta')\langle\bar{\chi}_\beta(y)\chi_\alpha(x)\rangle, \tag{6.10}$$

$$iS^{-+}_{\alpha\beta}(x;y) = \langle\chi_\alpha(x)\bar{\chi}_\beta(y)\rangle, \tag{6.11}$$

$$iS^{+-}_{\alpha\beta}(x;y) = -\langle\bar{\chi}_\beta(y)\chi_\alpha(x)\rangle, \tag{6.12}$$

where $\theta(t)$ is the Heaviside step-function, T denotes time ordering and $\bar{T}$ denotes anti-time ordering. The first two equations define the Feynman and Dyson propagators, while the last two define the positive and negative frequency Wightman functions (the minus in (6.12) comes from the anticommutation of fermionic fields). It should not go unnoticed that these last two functions can be used to construct the Feynman and Dyson propagators. Usually we will drop the spinor indices, and the Latin indices (which we do not use anymore for a distinction in flat and curved space) to now indicate the positive and negative branches of the contour.

6.1 Effective action for large-mass limit

In appendix F, we give a review of the relevant parts of [32] [33] [34] [35] we need to derive the 2PI effective action for any theory. We first rewrite (6.6) to make variations easier,

$$\begin{aligned} S_\chi = \int d^4x\, d^4y \Big\{ &-(i\partial\!\!\!/_{x,\beta\alpha} - M_{\beta\alpha}a - \frac{\kappa^2 m M_{\beta\alpha}}{4} n_0 \Delta_x^{-1})\delta^4(x-y)\chi_\alpha(x)\bar{\chi}_\beta(y) \\ &+ \frac{\kappa^2}{32\pi}\frac{M_{\beta\alpha}M_{\delta\gamma}\bar{\chi}_\alpha(x)\chi_\beta(x)\bar{\chi}_\gamma(y)\chi_\delta(y) + m^2 n_0^2}{||\vec{x}-\vec{y}||}\delta(\eta-\eta')\Big\}. \end{aligned} \tag{6.13}$$

We have introduced $d^4x = d\eta d^3x$ to save space, where from now on we have conformal time integration and not the 'regular time' integration measure we have seen at the start of chapter 3. From the way we have written (6.13), we can easily derive the derivative operator by using (F.16):

$$\mathcal{D}_{\beta\alpha}(y,x) = \Big[i\partial\!\!\!/_{x,\beta\alpha} - M_{\beta\alpha}a - \frac{\kappa^2 m M_{\beta\alpha}}{4} n_0 \Delta_x^{-1}\Big]\delta^4(y-x). \tag{6.14}$$

The subscript tells us on which field an operator acts. The spinor indices on the slashed derivative operator belong to the gamma matrices. The 2PI effective action in Schwinger-Keldysh formalism is then

$$\Gamma[iS^{cd}_{\alpha\beta}] = \sum_{c,d=\pm}\int d^4x d^4y\Big(-c\mathcal{D}_{\beta\alpha}(y,x)\delta_{cd}iS^{cd}_{\alpha\beta}(x;y) + i\mathrm{Tr}\big[\log\big(iS^{cd}_{\alpha\beta}(x;y)\big)\big] + \Gamma_2[iS^{cd}_{\alpha\beta}]\Big). \tag{6.15}$$

In the appendix we already derived the Schwinger-Dyson equation, now we will do it again and multiply by $cS^{cb}_{\alpha\gamma}(x;z)$ to get an equation of motion for the exact propagator, in Schwinger-Keldysh formalism

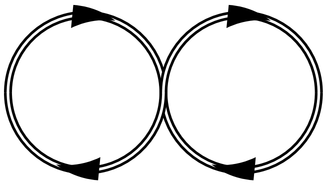

Figure 6.2: Diagram with 4-vertex

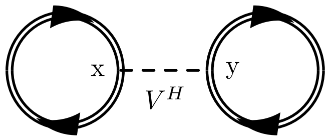

Figure 6.3: Diagram with Hartree vertex

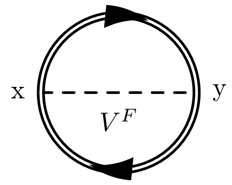

Figure 6.4: Diagram with Fock vertex

[29]

$$\frac{\delta\Gamma[iS]}{\delta S^{cd}_{\alpha\beta}(x,y)} = -ci\mathcal{D}_{\beta\alpha}(y,x)\delta_{cd} + i\mathrm{Tr}\big[\big(S^{cd}_{\alpha\beta}(x;y)\big)^{-1}\big] + \frac{\delta\Gamma_2[iS]}{\delta S^{cd}_{\alpha\beta}(x;y)} = 0, \tag{6.16}$$

$$\begin{aligned}\Longrightarrow \Big[i\partial\!\!\!/_{y,\beta\alpha} - M_{\beta\alpha}a - \frac{\kappa^2 m M_{\beta\alpha}}{4}n_0\Delta_y^{-1}\delta_{\beta\alpha}\Big] iS^{db}_{\alpha\gamma}(y;z) = \\ ia\delta^{db}\delta_{\beta\gamma}\delta^4(y-z) + \sum_{c=\pm} c\int d^4x\Sigma^{dc}_{\beta\alpha}(y,x,iS)iS^{cb}_{\alpha\gamma}(x,z).\end{aligned} \tag{6.17}$$

With the implication, we have also defined the self-energy,

$$\Sigma^{dc}_{\beta\alpha}(y,x,iS) \equiv -icd\frac{\delta\Gamma_2[iS]}{\delta S^{cd}_{\alpha\beta}(x;y)}. \tag{6.18}$$

We have integrated over x and summed over c, and we used that

$$\sum_{c=\pm}\int d^4x\big(S^{cd}_{\alpha\beta}(x;y)\big)^{-1}S^{cb}_{\alpha\gamma}(x,z) = \delta_{\beta\gamma}\delta^{bc}\delta^4(y-z). \tag{6.19}$$

The part $\Gamma_2[iS]$ is the collection of Feynman diagrams that are two-particle irreducible, meaning that we can make a cut in two fermion propagators and still have a connected diagram, or similarly, we can make a single cut and get self-energy (i.e. loop) diagrams. As mentioned in appendix F, their general form is still given by the Wick expansion made in (F.12), but the tree level propagators will be adjusted by the 2PI-source $-iK$.

The split of the mass into real and imaginary parts will introduce several more terms than just the four-point interaction we have seen in appendix D. The derivation of the other terms is similar [30],

$$\langle \Gamma_{\beta\alpha}\bar{\chi}_\alpha\chi_\beta\tilde{\Gamma}_{\delta\gamma}\bar{\chi}_\gamma\chi_\delta\rangle = \mathrm{Tr}[iS\Gamma]\,\mathrm{Tr}[iS\tilde{\Gamma}] - \mathrm{Tr}[iS\Gamma iS\tilde{\Gamma}], \tag{6.20}$$

where we use Γ to represent any element it contains. Including the nonlocal interaction from Δ_x^{-1}, we have

$$\begin{aligned}&\sum_{c=\pm} c\frac{\kappa^2}{32\pi}\int d^4x_1...d^4x_4\frac{\delta(x_1-x_2)\delta(x_3-x_4)\delta(\eta_1-\eta_3)}{||x_1-x_3||}M_{\beta\alpha}M_{\delta\gamma}\langle T^a\left[\bar{\chi}_\delta(x_4)\chi_\gamma(x_3)\bar{\chi}_\beta(x_2)\chi_\alpha(x_1)\right]\rangle \\ &\quad= \sum_{c=\pm} c\frac{\kappa^2}{32\pi}\int d^4x_1...d^4x_4\frac{\delta(x_1-x_2)\delta(x_3-x_4)\delta(\eta_1-\eta_3)}{||x_1-x_3||} \\ &\qquad\qquad\times\Big\{M_{\beta\alpha}M_{\delta\gamma}iS^{cc}_{\alpha\beta}(x_1,x_2)iS^{cc}_{\gamma\delta}(x_3,x_4) - M_{\beta\alpha}M_{\delta\gamma}iS^{cc}_{\alpha\delta}(x_1,x_4)iS^{cc}_{\gamma\beta}(x_3,x_2)\Big\} \\ &\quad\to\frac{1}{2}\sum_{c=\pm}\int d^4x_1...d^4x_4\; iS^{cc}_{\alpha\beta}(x_1,x_2)iS^{cc}_{\gamma\delta}(x_3,x_4)(V^{H,c}_{\delta\gamma\beta\alpha}(x_1,...,x_4) - V^{F,c}_{\delta\alpha\gamma\beta}(x_1,...,x_4)).\end{aligned} \tag{6.21}$$

In the last line, we have renamed $x_2 \leftrightarrow x_4$, $\beta \leftrightarrow \delta$ for the second contribution, and defined the following Hartree and Fock (or exchange) vertices:

$$V^{H,c}_{\delta\gamma\beta\alpha}(x_1,...,x_4) \equiv \frac{\kappa^2}{16\pi}M_{\delta\gamma}M_{\beta\alpha}\frac{c\delta^4(x_1-x_2)\delta^4(x_3-x_4)\delta(\eta_1-\eta_3)}{||\vec{x}_1-\vec{x}_3||}, \tag{6.22}$$

$$V^{F,c}_{\delta\alpha\beta\gamma}(x_1,...,x_4) \equiv \frac{\kappa^2}{16\pi}M_{\delta\alpha}M_{\beta\gamma}\frac{c\delta^4(x_1-x_4)\delta^4(x_3-x_2)\delta(\eta_1-\eta_3)}{||\vec{x}_1-\vec{x}_3||}. \tag{6.23}$$

We note that the minus sign ensures that when the two propagators have the same spacetime and spinor indices, the contributions cancel, which is a manifestation of the exclusion principle: interactions between two fermions in the same state are non-existent. In figure 6.2, we see a diagram with a simple 4-vertex, which is what we observe on-shell. What we can see in figures 6.3 and 6.4, is that there are two ways of introducing the nonlocal interaction into the diagrams: either the loops get separated, giving two loops connected by the interaction, which can have separate spinor indices. This gives a trace on each propagator. The other way is making a single fermion loop with the interaction in itself, so we have a spinor trace over both propagators. The minus sign for the Fock diagram can also be inferred from the single fermion loop in it, whereas the Hartree diagram has two loops, giving plus.

The Hartree and Fock diagrams have different interpretations. The closed propagator loops can be seen as localized fermion densities $\sim \langle\bar{\chi}(x)Ma\chi(x)\rangle$, such that the Hartree diagram represents the nonlocal Newtonian interactions between two such densities. The Fock diagram is a quantum correction to the mean free field, which we therefore do not see in the classical theory; again an interesting consequence of the field-theoretic approach. It turns out to be negligible compared to the Hartree contribution at early times [5] for scalars, and we will neglect it later for the fermions, but leave its investigation to other works. What we have found is simply the 2-loop contribution, so our complete action is

$$\begin{aligned}\Gamma[iS^{cd}_{\alpha\beta}] = &\sum_{c,d=\pm}\int d^4x d^4y\Big(-c\mathcal{D}_{\beta\alpha}(y,x)\delta_{cd}iS^{cd}_{\alpha\beta}(x;y) + i\mathrm{Tr}\big[\log\big(iS^{cd}_{\alpha\beta}(x;y)\big)\big]\Big)\\ &+\frac{1}{2}\sum_{c=\pm}\int d^4x_1...d^4x_4\; iS^{cc}_{\alpha\beta}(x_1,x_2)iS^{cc}_{\gamma\delta}(x_3,x_4)\big(V^{H,c}_{\delta\gamma\beta\alpha}(x_1,...,x_4) - V^{F,c}_{\delta\alpha\beta\gamma}(x_1,...,x_4)\big).\end{aligned} \tag{6.24}$$

We see that the n_0^2-term, which was present in (6.13), does not return in the effective action. This is because it gives a large, but constant contribution, which therefore does not alter the effective dynamics of the system (this would also have happened to the $S_g^{(0,0)}$ contribution we had already neglected in the previous chapter). Thus, it is sensible that it gets removed in the procedure of getting the effective action. The cross term, i.e. the $n_0\Delta_x^{-1}$ part of the derivative operator, does enter in the effective action, and it is an addition to the usual Dirac equation. As mentioned in [5], this addition can be seen as part of the interactions, as it removes the homogeneous contributions to the nonlocal interaction of the vertices. This ensures we are looking at only the interactions between fluctuations, as we would expect from the structure of (6.6). With this final contribution, we can calculate the self-energy, yielding

$$\Sigma^{dc}_{\beta\alpha}(y,x,iS) = \int d^4x_1\; d^4x_2\; iS^{cc}_{\gamma\delta}(x_1,x_2)\delta^{cd}\big(V^{H,c}_{\delta\gamma\beta\alpha}(x_1,x_2,x,y) - V^{F,c}_{\delta\alpha\beta\gamma}(x_1,x_2,x,y)\big), \tag{6.25}$$

where we renamed some indices to add different contributions. This can be visualized by looking at the figures: in the Hartree diagram of figure 6.3, 'cutting' either the left or the right propagator will give us two topologically identical diagrams, and the same is true for cutting the upper or lower propagator in figure 6.4. Looking at the equation for the exact propagator (F.22), schematically, we find

$$\begin{aligned}iS = \big((iS_0)^{-1} + i\Sigma\big)^{-1} = iS_0\big(1 + iS_0 i\Sigma\big)^{-1} =& iS_0 - iS_0(i\Sigma)iS_0 + iS_0(i\Sigma)iS_0(i\Sigma)iS_0 - ...\\ =& iS_0 - iS_0 i\Sigma\big(iS_0 - iS_0(i\Sigma)iS_0 + ...\big)\\ =& iS_0 - iS_0(i\Sigma)iS.\end{aligned} \tag{6.26}$$

and we have defined $S_0^{-1} \equiv -iD_0 - iK$, as it is the tree level inverse propagator. The minus signs in front of the terms with an uneven number of $i\Sigma$'s can be inferred from the Feynman diagrams (figure 6.5) looking at the fermion loops again: by cutting a propagator, we lost a loop in both diagrams.

The last line of (6.26) is the important result, as it is the exact Dyson equation. It shows that the kinetic equation actually resums the series up to infinity, and we obtain an iterative expansion: by substituting the equation 'into itself', we actually generate all possible self-energy diagrams. This result is shown diagrammatically in figure 6.5, which is exact.

We are now in a position to get the full equation of motion for the propagator by substituting the

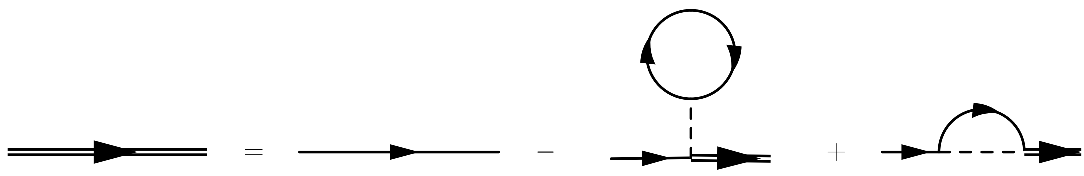

Figure 6.5: Diagrammatic representation of the exact propagator (6.26). By iteration, all relevant diagrams can be generated from this equation.

self-energy (6.25) into (6.17),

$$\begin{aligned}&\Big[i\partial\!\!\!/_{y,\beta\alpha} - M_{\beta\alpha}a - \frac{\kappa^2 m M_{\beta\alpha}}{4} n_0 \Delta_y^{-1}\Big] iS^{cd}_{\alpha\gamma}(y;z)\\ &\quad - c\int d^4x_1\, d^4x_2\, d^4x\, \big(V^{H,c}_{\delta\epsilon\beta\alpha}(x_1,x_2,x,y) - V^{F,c}_{\delta\alpha\beta\epsilon}(x_1,x_2,x,y)\big) iS^{cc}_{\epsilon\delta}(x_1,x_2) iS^{cd}_{\alpha\gamma}(x,z)\\ &= ic\delta^{cd}\delta_{\beta\gamma}\delta^4(y-z).\end{aligned} \tag{6.27}$$

This equation of motion is the main result of this section, so it is sensible to compare our results with the real scalar in [5]. We see that we have followed an almost equivalent derivation, and we have therefore also found a similar result. A difference is of course the anticommutation in the fields, which has resulted in a negative Hartree diagram in the equation of motion, an expected result for fermions. This anticommutation is also the reason we do not have the so-called squeezing contributions, which would be $\langle\chi\chi\rangle$ and $\langle\bar{\chi}\bar{\chi}\rangle$. For scalars, one has to have physical grounds on which to discard these contributions, but for fermions, we can simply assume them to be 0 on the initial hypersurface. If we subsequently do not Wick contract these contributions, they will never be dynamically generated, and they can therefore be ignored.

6.1.1 Note on renormalization

Notice that n_0 involves an expectation value that is proportional to $\langle T_{00}\rangle$, and will induce a divergence coming from the one-loop energy-momentum tensor. The way to renormalize it is fairly standard (see e.g. [21]) in which several schemes are discussed. One would want to use a regularization scheme in which the symmetries of the theories are kept like dimensional regularization, as opposed to e.g. introducing a cutoff scale. Besides it not being necessary at this point, we also note that if we would want to use the dimensional regularization scheme, we would have to have written our action in D dimensions from the start. It could still be instructive to have examples of counterterm diagrams that would appear for the self energy diagrams in 6.5. In [30], the renormalization of a similar 2PI effective action is discussed, in which counterterms appear for the cosmological constant, the Ricci scalar (squared), the Weyl tensor and the mass. The counterterm diagram for the self-energy is also shown there.

The objects we are interested in are the Hadamard two-point functions, subject of the next section. We will assume that their form is not affected by the counterterms for the propagators, and that their equation of motion is the same as derived from a renormalized effective action. A case in which this is not true can be found in [36], so one should be careful in this assumption. Nevertheless, the equations we are eventually interested in will be classical and should not be affected by renormalization; any such higher-order field-theoretic corrections will be negligible compared to e.g. first-order relativistic corrections we will find.

6.2 The Hadamard two-point function

To extract real (physical) results from equations, it is necessary to be working with Hermitian objects. However, the Dirac equation is not Hermitian, as is the equation of motion we have found in the previous section. The goal of this section is firstly to rewrite the equation of motion (6.27) in terms of objects that have clear Hermiticity properties (as opposed to the propagators of the previous section), called the Hadamard functions. After this is done, we will use the decomposition in appendix B to obtain a better grasp on the information contained in the spinor indices.

As mentioned, the Wightman functions (6.11) and (6.12) are not Hermitian by themselves. Multiplying them by γ^0 will give Hermiticity in the spinor indices, and exchanges the spacetime coordinates upon Hermitian conjugation. Noting the following properties,

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i \implies \gamma^0\gamma^\mu = (\gamma^\mu)^\dagger\gamma^0, \qquad (\gamma^5)^\dagger = \gamma^5 \implies M^\dagger_{\alpha\beta} = M^*_{\beta\alpha}, \tag{6.28}$$

which follow from the definition (2.10), we obtain for the Feynman, Dyson and Wightman functions,

$$\begin{aligned}\left(i\gamma^0_{\delta\beta}S^{\pm\mp}_{\beta\gamma}(x,y)\right)^\dagger &= \left\langle \gamma^0_{\delta\beta}\chi^\pm_\beta(x)\chi^{\dagger\mp}_\alpha(y)\gamma^0_{\alpha\gamma}\right\rangle^\dagger = \left\langle \gamma^0_{\gamma\beta}\chi^\mp_\beta(y)\chi^{\dagger\pm}_\alpha(x)\gamma^0_{\alpha\delta}\right\rangle^\dagger = i\gamma^0_{\gamma\beta}S^{\mp\pm}_{\beta\delta}(y,x),\\ \left(i\gamma^0_{\delta\beta}S^{\pm\pm}_{\beta\gamma}(x,y)\right)^\dagger &= i\gamma^0_{\gamma\beta}S^{\pm\pm}_{\beta\delta}(y,x).\end{aligned} \tag{6.29}$$

The second line follows from the first, and in the first line we made use of the notation in (F.10) to keep track of the χ and $\bar{\chi}$. As stated there, the plus and minus indicate a slight positive or negative deviation from the closed time path, such that we can meaningfully define the Wightman function. Notice also the following: if we take the spinor trace of $i\gamma^0 S$, we find

$$\mathrm{Tr}[i\gamma^0 S] = i\gamma^0_{\alpha\beta}S_{\beta\alpha} = \langle\gamma^0_{\alpha\beta}\chi_\beta\chi^\dagger_\delta\gamma^0_{\delta\alpha}\rangle = -\langle\chi^\dagger_\alpha\chi_\alpha\rangle, \tag{6.30}$$

which has the interpretation of (minus) the expectation value of the total particle number density at some point in space and time (as opposed to n_0, which only depends on time), and is obviously Hermitian.

We will not be using the Wightman functions on their own, but a combination of them. Using the fact that $\theta(t)+\theta(-t)=1$, we can immediately see that

$$-i\gamma^0 S^{++}(x;y) - i\gamma^0 S^{--}(x;y) = -i\gamma^0 S^{-+}(x;y) - i\gamma^0 S^{+-}(x;y) \equiv 2F(x,y) = -\langle[\chi(x),\bar{\chi}(y)]\rangle, \tag{6.31}$$

where we have defined the Hadamard (or statistical) two-point function $F(x,y)$. The minus signs are there to get positive number densities, see (6.30). As the Wightman functions transform into each other, their sum is obviously Hermitian, given an exchange of spacetime indices as $(F_{\alpha\beta}(x,y))^\dagger = F_{\beta\alpha}(y,x)$. We note that the Hadamard function is a fundamental object in this thesis, as it is composed of the fermionic number densities and allows for the derivation of the classical particle limit: exactly what we intended at the start.

Another two-point function worth mentioning is the Pauli-Jordan (or causal, spectral) function, defined as

$$i\rho(x;y) \equiv -i\gamma^0 S^{-+}(x;y) + i\gamma^0 S^{+-}(x;y) = -\langle\{\chi(x),\bar{\chi}(y)\}\rangle. \tag{6.32}$$

An important consequence of going to the classical particle limit is that all relevant information about the system, like the particle density and consequently the von Neumann entropy, will be completely determined by only the Hadamard two-point function. The Pauli-Jordan two-point function contains information about the quantum features of the system, and loses its relevance in this limit [37]. We will therefore derive our result (6.27) in terms of the Hadamard functions, using the relations in (6.31).

To this end, we will need the $ac = \pm\mp$ case of (6.27), which is

$$\begin{aligned}&\left[i\gamma^0_{\beta\alpha}\partial_\eta + i\vec{\gamma}_{\beta\alpha}\cdot\vec{\nabla}_y - M_{\beta\alpha}a - \frac{\kappa^2 m}{16\pi}M_{\beta\alpha}\int d^3x\frac{n_0}{||\vec{x}-\vec{y}||}\right]iS^{\pm\mp}_{\alpha\gamma}(\eta,\vec{y};\eta',\vec{z})\\ &\qquad -\frac{\kappa^2}{16\pi}\int d^3x\Bigg\{\frac{M_{\alpha\epsilon}M_{\beta\zeta}iS^{\pm\pm}_{\epsilon\alpha}(\eta,\vec{x};\eta,\vec{x})}{||\vec{x}-\vec{y}||}iS^{\pm\mp}_{\zeta\gamma}(\eta,\vec{y};\eta',\vec{z})\\ &\qquad\qquad -\frac{M_{\beta\zeta}M_{\alpha\epsilon}iS^{\pm\pm}_{\zeta\alpha}(\eta,\vec{y};\eta,\vec{x})}{||\vec{x}-\vec{y}||}iS^{\pm\mp}_{\epsilon\gamma}(\eta,\vec{x};\eta',\vec{z})\Bigg\} = 0.\end{aligned} \tag{6.33}$$

If we now multiply this equation with $\gamma^0_{\delta\beta}$, (anti)commuting it through different operators to combine it with propagators, we find (using $\gamma^0 M = M^*\gamma^0$, as γ^5 anticommutes with γ^0)

$$\begin{aligned}&\left[i\gamma^0_{\delta\beta}\partial_\eta - i\vec{\gamma}_{\delta\beta}\cdot\vec{\nabla}_y - M^*_{\delta\beta}a - \frac{\kappa^2 m}{16\pi}M^*_{\delta\beta}\int d^3x\frac{n_0}{||\vec{x}-\vec{y}||}\right]i\gamma^0_{\beta\alpha}S^{\pm\mp}_{\alpha\gamma}(\eta,\vec{y};\eta',\vec{z})\\ &\qquad -\frac{\kappa^2}{16\pi}\int d^3x\Bigg\{\frac{M_{\alpha\epsilon}M^*_{\delta\beta}iS^{\pm\pm}_{\epsilon\alpha}(\eta,\vec{x};\eta,\vec{x})}{||\vec{x}-\vec{y}||}i\gamma^0_{\beta\zeta}S^{\pm\mp}_{\zeta\gamma}(\eta,\vec{y};\eta',\vec{z})\\ &\qquad\qquad - M^*_{\delta\beta}M_{\alpha\theta}\frac{i\gamma^0_{\beta\epsilon}S^{\pm\pm}_{\epsilon\alpha}(\eta,\vec{y};\eta,\vec{x})}{||\vec{x}-\vec{y}||}\gamma^0_{\theta\zeta}i\gamma^0_{\zeta\pi}S^{\pm\mp}_{\pi\gamma}(\eta,\vec{x};\eta',\vec{z})\Bigg\} = 0,\end{aligned} \tag{6.34}$$

where we have used $(\gamma^0)^2 = I$ in the Fock term. We see that the $iS^{\pm\pm}$ are both at time coincidence, which makes makes it particularly easy to rewrite it as the Feynman and Dyson propagator are equal, i.e.

$$-i\gamma^0_{\alpha\delta}S^{++}_{\delta\beta}(\eta,\vec{x};\eta,\vec{y}) = -i\gamma^0_{\alpha\delta}S^{--}_{\delta\beta}(\eta,\vec{x};\eta,\vec{y}) = -\frac{1}{2}\big(i\gamma^0_{\alpha\delta}S^{++}_{\delta\beta}(\eta,\vec{x};\eta,\vec{y}) + i\gamma^0_{\alpha\delta}S^{--}_{\delta\beta}(\eta,\vec{x};\eta,\vec{y})\big) \tag{6.35}$$

$$= F_{\alpha\beta}(\eta,\vec{x};\eta,\vec{y}). \tag{6.36}$$

We then only have to add the $+-$ and $-+$ version of (6.34), which we then divide by 2 and multiply by a minus, so

$$\begin{aligned}\Big[i\gamma^0_{\delta\beta}\partial_\eta - i\vec{\gamma}_{\delta\beta}\cdot\vec{\nabla}_y - M^*_{\delta\beta}a\Big]F_{\beta\gamma}(\eta,\vec{y};\eta',\vec{z}) \\ + \frac{\kappa^2}{16\pi}M^*_{\delta\beta}\int d^3x\Big\{\frac{M_{\alpha\epsilon}\gamma^0_{\epsilon\zeta}F_{\zeta\alpha}(\eta,\vec{x};\eta,\vec{x}) - mn_0}{||\vec{x}-\vec{y}||}F_{\beta\gamma}(\eta,\vec{y};\eta',\vec{z}) \\ - \frac{M_{\alpha\epsilon}F_{\beta\alpha}(\eta,\vec{y};\eta,\vec{x})}{||\vec{x}-\vec{y}||}\gamma^0_{\epsilon\zeta}F_{\zeta\gamma}(\eta,\vec{x};\eta',\vec{z})\Big\} = 0.\end{aligned} \tag{6.37}$$

We have also placed the n_0-contribution in such a way that its use of removing the homogeneous contributions will become more obvious, such that the gravitational potential from the Hartree contribution is only sourced by density perturbations. In [5], the Hermitian conjugate of this equation was subtracted to get an equation that is fully anti-Hermitian. We note that this is also possible here, by a shift in $\vec{x}$ and commuting several terms that have spinor indices. It will not be necessary to make this subtraction in this thesis, and we will only look at Hermitian and anti-Hermitian parts when we have decomposed the Hadamard functions.

The shift in $\vec{x}$ will have the advantage of making the equations cleaner later on, so we will perform it here too. We will also go to the coincidence limit, $\eta' \to \eta$, which is the equivalent of 'gluing' the dashed lines in figures 6.3 and 6.4 together, or going on shell. For non-equilibrium problems, it is also useful to define an average or 'center-of-mass' coordinate $\vec{X}$, and a relative coordinate $\vec{r}$,

$$\vec{X} \equiv \frac{\vec{y}+\vec{z}}{2}, \qquad \vec{r} \equiv \vec{y}-\vec{z} \qquad\qquad \Longrightarrow\ F(\eta,\vec{y};\eta',\vec{z})\big|_{\eta'\to\eta} = F(\eta,\vec{X}+\frac{\vec{r}}{2},\vec{X}-\frac{\vec{r}}{2}) \equiv F(\eta,\vec{X},\vec{r}). \tag{6.38}$$

The average coordinate vanishes when going to equilibrium, but for a non-equilibrium situation it evaluates the midpoint between two coordinates. The relative coordinate has a self-explanatory name, and finds great use in going to the phase-space representation later, by Wigner transforming. But first, we rewrite (6.37) as (shifting $\vec{x} \to \vec{x}+\vec{y}$)

$$\begin{aligned}\Big[i\gamma^0_{\delta\beta}\partial_\eta - i\vec{\gamma}_{\delta\beta}\cdot\vec{\nabla}_y - M^*_{\delta\beta}a\Big]F_{\beta\gamma}(\eta,\vec{y};\eta,\vec{z}) \\ + \frac{\kappa^2}{16\pi}M^*_{\delta\beta}\int\frac{d^3x}{||\vec{x}||}\Big\{\big[M_{\alpha\epsilon}\gamma^0_{\epsilon\zeta}F_{\zeta\alpha}(\eta,\vec{x}+\vec{y};\eta,\vec{x}+\vec{y}) - mn_0\big]F_{\beta\gamma}(\eta,\vec{y};\eta,\vec{z}) \\ - M_{\alpha\epsilon}F_{\beta\alpha}(\eta,\vec{y};\eta,\vec{x}+\vec{y})\gamma^0_{\epsilon\zeta}F_{\zeta\gamma}(\eta,\vec{x}+\vec{y};\eta,\vec{z})\Big\} = 0.\end{aligned} \tag{6.39}$$

$$\begin{aligned}\Big[i\gamma^0_{\delta\beta}\partial_\eta - i\vec{\gamma}_{\delta\beta}\cdot(\vec{\nabla}_{\vec{X}}/2 + \vec{\nabla}_{\vec{r}}) - M^*_{\delta\beta}a\Big]F_{\beta\gamma}(\eta,\vec{X},\vec{r}) \\ + \frac{\kappa^2}{16\pi}M^*_{\delta\beta}\int\frac{d^3x}{||\vec{x}||}\Big\{\big[M_{\alpha\epsilon}\gamma^0_{\epsilon\zeta}F_{\zeta\alpha}(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - mn_0\big]F_{\beta\gamma}(\eta,\vec{X},\vec{r}) \\ - M_{\alpha\epsilon}F_{\beta\alpha}(\eta,\vec{X}+\frac{\vec{r}+\vec{x}}{2},-\vec{x})\gamma^0_{\epsilon\zeta}F_{\zeta\gamma}(\eta,\vec{X}+\frac{\vec{x}}{2},\vec{r}+\vec{x})\Big\} = 0.\end{aligned} \tag{6.40}$$

The shortcoming in the aesthetics of this equation is made up by what it describes: this will be the most compact way we write down the dynamics of the fermionic dark matter. It still contains both Hermitian and anti-Hermitian parts, and the Hadamard two-point function in fact describes sixteen different densities, so we are still a long way off describing the Vlasov-Poisson system we want to derive. To make further progress, we will decompose the Hadamard function into relevant densities (what they describe, will be a central item in the next sections). In appendix B, we show how to decompose the propagator (B.8) in terms of traces multiplied by elements of the set Γ,

$$iS^{cd}_{\beta\gamma} = \frac{1}{4}iS^{cd}_{\alpha\alpha}\delta_{\beta\gamma} + \frac{1}{4}iS^{cd}_{\alpha\delta}\gamma^5_{\delta\alpha}\gamma^5_{\beta\gamma} - \frac{1}{4}iS^{cd}_{\alpha\delta}\gamma_{\mu,\delta\alpha}\gamma^\mu_{\beta\gamma} + \frac{1}{4}iS^{cd}_{\alpha\delta}(\gamma_\mu\gamma^5)_{\delta\alpha}(\gamma^\mu\gamma^5)_{\beta\gamma} + 2iS^{cd}_{\alpha\delta}\sigma_{\mu\nu,\delta\alpha}\sigma^{\mu\nu}_{\beta\gamma}. \tag{6.41}$$

We can rewrite this easily in terms of the Hadamard two-point function by considering $cd = \pm\mp$ and putting an extra γ^0 in the traces, e.g. $iS_{\alpha\alpha} \to i\gamma^0_{\alpha\beta}S_{\beta\alpha}$,

$$\begin{aligned} F_{\alpha\beta}(\eta,\vec{X},\vec{r}) = \frac{1}{4}\Big(& f_S(\eta,\vec{X},\vec{r})\delta_{\alpha\beta} + f_P(\eta,\vec{X},\vec{r})\gamma^5_{\alpha\beta} - f_{V,\mu}(\eta,\vec{X},\vec{r})\gamma^\mu_{\alpha\beta} \\ & + f_{A,\mu}(\eta,\vec{X},\vec{r})\big(\gamma^5\gamma^\mu\big)_{\alpha\beta} + 2f_{\sigma,\mu\nu}(\eta,\vec{X},\vec{r})\sigma^{\mu\nu}_{\alpha\beta}\Big). \end{aligned} \tag{6.42}$$

For clarity, we write down the definitions we have made for the scalar, pseudoscalar, vector, axial vector and spin functions, respectively,

$$\begin{aligned} &2f_S \equiv \mathrm{Tr}[-i\gamma^0 S^{-+} - i\gamma^0 S^{+-}], \qquad 2f_P \equiv \mathrm{Tr}[(-i\gamma^0 S^{-+} - i\gamma^0 S^{+-})\gamma^5], \\ &2f_V^\mu \equiv \mathrm{Tr}[(-i\gamma^0 S^{-+} - i\gamma^0 S^{+-})\gamma^\mu], \qquad 2f_A^\mu \equiv \mathrm{Tr}[(-i\gamma^0 S^{-+} - i\gamma^0 S^{+-})\gamma^\mu\gamma^5], \\ &2f_\sigma^{\mu\nu} \equiv \mathrm{Tr}[(-i\gamma^0 S^{-+} - i\gamma^0 S^{+-})\sigma^{\mu\nu}], \end{aligned} \tag{6.43}$$

where the trace is over the spinor indices. For further reference: we will collectively write f_Γ.

Keeping the imaginary mass at this point will complicate our calculations tremendously, as we had anticipated in section 3.1. A quick look at the Fock terms, for example, shows that we decompose the Hadamard functions into five parts, and each mass into two parts, seemingly giving around one hundred different terms. If the mass is generated by a scalar condensate with a steep potential, we find that gravity to a high degree does not affect the vacuum expectation value, while a low potential would in fact make this this value spacetime dependent. If such a mechanism were to apply, we will assume the former situation, meaning the mass matrix is globally constant. We can then rotate the imaginary mass away, so $M_{\delta\gamma} \to m\delta_{\delta\gamma}$. As already mentioned in chapter 2, the classical Vlasov equation presumes a fixed particle number over a large enough volume, also meaning mass is globally constant.

The last line of (6.40), the Fock terms, contains a product of Hadamard two-point functions with unequal spinor indices, is the most complicated to evaluate. We will reduce a lot of terms to elements of the set Γ, for which we refer to appendix A. The product of Levi-Civita symbols is evaluated by a matrix determinant,

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\lambda\tau\kappa\xi} = -\begin{vmatrix} \delta^\mu_\lambda & \delta^\mu_\tau & \delta^\mu_\kappa & \delta^\mu_\xi \\ \delta^\nu_\lambda & \delta^\nu_\tau & \delta^\nu_\kappa & \delta^\nu_\xi \\ \delta^\rho_\lambda & \delta^\rho_\tau & \delta^\rho_\kappa & \delta^\rho_\xi \\ \delta^\sigma_\lambda & \delta^\sigma_\tau & \delta^\sigma_\kappa & \delta^\sigma_\xi \end{vmatrix}. \tag{6.44}$$

Using these given relations, we have for a general combination of two Hadamard functions and a γ^0 matrix (moving the factor $(1/4)^2$ to the other side),

$$\begin{aligned}
&16F_{\delta\alpha}\gamma^0_{\alpha\epsilon}(a)F_{\epsilon\gamma}(b) = \\
&\quad \big[f_S(a)f_V^0(b) + f_V^0(a)f_S(b) + f_A^0(a)f_P(b) - f_P(a)f_A^0(b) \\
&\qquad + 2if_{V,\mu}(a)f_\sigma^{0\mu}(b) + 2if_\sigma^{0\mu}(a)f_{V,\mu}(b) + f_{A,\mu}(a)f_{\sigma,\lambda\tau}(b)\epsilon^{\mu0\lambda\tau} + f_{\sigma,\mu\nu}(a)f_{A,\lambda}(b)\epsilon^{\lambda0\mu\nu}\big]\delta_{\delta\gamma} \\
&\quad + \big[f_A^0(a)f_S(b) - f_S(a)f_A^0(b) + f_P(a)f_V^0(b) + f_V^0(a)f_P(b) \\
&\qquad + f_{V,\lambda}(a)f_{\sigma,\mu\nu}(b)\epsilon^{\lambda0\mu\nu} + f_{\sigma,\mu\nu}(a)f_{V,\lambda}(b)\epsilon^{\mu\nu\lambda0} + 2if_{A,\mu}(a)f_\sigma^{\mu0}(b) + 2if_\sigma^{0\mu}(a)f_{A,\mu}(b)\big]\gamma^5_{\delta\gamma} \\
&\quad + \big[f_S(a)f_S(b)\delta^0_\mu + 2if_S(a)f_{\sigma,\mu\nu}(b)\eta^{0\nu} - 2if_{\sigma,\mu\nu}(a)f_S(b)\eta^{0\nu} - f_P(a)f_P(b)\delta^0_\mu \\
&\qquad + f_P(a)f_\sigma^{\lambda\tau}(b)\epsilon_{0\mu\lambda\tau} - f_\sigma^{\lambda\tau}(a)f_P(b)\epsilon_{0\mu\lambda\tau} + f_{V,\lambda}(a)f_{V,\tau}(b)(-2\eta^{0(\tau}\delta^{\lambda)}_\mu + \eta^{\lambda\tau}\delta^0_\mu) \\
&\qquad\quad + if_V^\lambda(a)f_A^\tau(b)\epsilon_{0\mu\lambda\tau} + if_A^\lambda(a)f_V^\tau(b)\epsilon_{\mu0\lambda\tau} + f_{A,\lambda}(a)f_{A,\tau}(b)(-2\eta^{0(\tau}\delta^{\lambda)}_\mu + \eta^{\lambda\tau}\delta^0_\mu) \\
&\qquad\qquad + 2f_{\sigma,\rho\kappa}(a)f_{\sigma,\lambda\tau}(b)(2\eta^{0\kappa}\eta^{\rho\tau}\delta^\lambda_\mu + 2\eta^{0\tau}\eta^{\lambda\kappa}\delta^\rho_\mu + \eta^{\kappa\lambda}\eta^{\tau\rho}\gamma^0_\mu)\big]\gamma^\mu_{\delta\gamma} \\
&\quad + \big[\big(f_S(a)f_P(b) - f_P(a)f_S(b)\big)\delta^0_\mu + f_S(a)f_\sigma^{\lambda\tau}(b)\epsilon_{\mu0\lambda\tau} + f_\sigma^{\lambda\tau}(a)f_S(b)\epsilon_{\mu0\lambda\tau} \\
&\qquad + 2if_P(a)f_{\sigma,\nu\mu}(b)\eta^{0\nu} + 2if_{\sigma,\nu\mu}(a)f_P(b)\eta^{0\nu} + if_V^\lambda(a)f_V^\tau(b)\epsilon_{0\mu\lambda\tau} \\
&\qquad\quad - f_{V,\lambda}(a)f_{A,\tau}(b)(2\eta^{0(\lambda}\delta^{\tau)}_\mu) - \eta^{\lambda\tau}\delta^0_\mu) - f_{A,\lambda}(a)f_{V,\tau}(b)(2\eta^{0(\lambda}\delta^{\tau)}_\mu - \eta^{\lambda\tau}\delta^0_\mu) \\
&\qquad\qquad + if_A^\lambda(a)f_A^\tau(b)\epsilon_{\mu0\lambda\tau} + 2if_\sigma^{\kappa\rho}(a)f_\sigma^{\lambda\tau}(b)(\eta_{0\rho}\epsilon_{\kappa\mu\lambda\tau} + \epsilon_{\rho\kappa0\tau}\eta_{\lambda\mu})\big](\gamma^\mu\gamma^5)_{\delta\gamma} \\
&\quad + \big[2if_S(a)f_{V,\nu}(b)\delta^0_\mu + f_{V,\mu}(a)f_S(b)\delta^0_\nu + f_S(a)f_A^\lambda(b)\epsilon_{\lambda0\mu\nu} + f_A^\lambda(a)f_S(b)\epsilon_{\lambda0\mu\nu} \\
&\qquad + f_P(a)f_V^\lambda(b)\epsilon_{0\lambda\mu\nu} + f_V^\lambda(a)f_P(b)\epsilon_{\lambda0\mu\nu} + 2if_P(a)f_{A,\nu}(b)\delta^0_\nu + 2if_{A,\nu}(a)f_P(b)\delta^0_\nu \\
&\qquad\; + 2f_{V,\rho}(a)f_{\sigma,\lambda\tau}(b)(2\eta^{0\lambda}\delta^\tau_\mu\delta^\rho_\nu + 2\eta^{\rho\tau}\delta^\lambda_\mu\delta^0_\nu + \eta^{\rho0}\delta^\tau_\mu\delta^\lambda_\nu) \\
&\qquad\quad + 2f_{\sigma,\lambda\tau}(a)f_{V,\rho}(b)(2\eta^{0\tau}\delta^\lambda_\mu\delta^\rho_\nu + 2\eta^{\rho\lambda}\delta^\tau_\mu\delta^0_\nu + \eta^{\rho0}\delta^\lambda_\mu\delta^\tau_\nu) \\
&\qquad\quad\; + 2if_A^\rho(a)f_\sigma^{\tau\lambda}(b)(\eta_{0\tau}\epsilon_{\lambda\rho\mu\nu} + \eta_{\rho\mu}\epsilon_{\nu0\lambda\tau}) + 2if_\sigma^{\tau\lambda}(a)f_A^\rho(b)(\eta_{0\tau}\epsilon_{\lambda\rho\mu\nu} + \eta_{\rho\nu}\epsilon_{\mu0\lambda\tau})\big]\sigma^{\mu\nu}_{\delta\gamma}.
\end{aligned} \tag{6.45}$$

Where the a, b are shorthand notation, $a = \eta, \vec{X} + \frac{\vec{r}+\vec{x}}{2}, -\vec{x}$, and $b = \eta, \vec{X} + \frac{\vec{x}}{2}, \vec{r} + \vec{x}$ for the first term of the third line of (6.40).

The second most complicated line in (6.40) is the first one, and again we will thankfully make use of appendix A, and it reads when decomposed,

$$\begin{aligned}
&\Big[i\gamma^0_{\delta\beta}\partial_\eta - i\vec{\gamma}_{\delta\beta}\cdot(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2) - ma\delta_{\delta\beta}\Big]F_{\beta\gamma}(\eta, \vec{X}, \vec{r}) = \\
&\quad [i\gamma^0\partial_\eta - i\vec{\gamma}\cdot(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2) - ma]f_S(b) \\
&\quad + [i\gamma^0\gamma^5\partial_\eta - i(\vec{\gamma}\gamma^5)\cdot(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2) - ma\gamma^5]f_P(b) \\
&\quad - [i(-\eta^{0\lambda} - 2i\sigma^{0\lambda})\partial_\eta - i(-\eta^{i\lambda} - 2i\sigma^{i\lambda})(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2) - ma\gamma^\lambda]f_{V,\lambda}(b) \\
&\quad + [i(-\eta^{0\lambda}\gamma^5 + \epsilon^{\mu\nu0\lambda}\sigma_{\mu\nu})\partial_\eta - i(-\eta^{i\lambda}\gamma^5 + \epsilon^{\mu\nu i\lambda}\sigma_{\mu\nu})(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2) - ma\gamma^\lambda\gamma^5]f_{A,\lambda}(b) \\
&\quad + 2[i\frac{i}{2}(2\eta^{0\nu}\gamma^\mu + i\epsilon^{\lambda0\mu\nu}\gamma_\lambda\gamma^5)\partial_\eta - i\frac{i}{2}(2\eta^{i\nu}\gamma^\mu + i\epsilon^{\lambda i\mu\nu}\gamma_\lambda\gamma^5)(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2) - ma\sigma^{\mu\nu}]f_{\sigma,\mu\nu}(b) \\
&= [i\partial_\eta f_V^0(b) - i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot\vec{f}_V(b) - maf_S(b)] \\
&\quad + [-i\partial_\eta f_A^0(b) + i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot\vec{f}_A(b) - maf_P(b)]\gamma^5 \\
&\quad + [i\delta^0_\mu\partial_\eta f_S(b) - i\delta^i_\mu(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2)f_S(b) - 2\eta^{0\nu}\partial_\eta f_{\sigma,\mu\nu}(b) + 2\eta^{i\nu}(\partial^i_{\vec{r}} + \partial_{\vec{X}}/2)f_{\sigma,\mu\nu}(b) + maf_{V,\mu}(b)]\gamma^\mu \\
&\quad + [i\delta^0_\mu\partial_\eta f_P(b) - i\delta^i_\mu(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2)f_P(b) + i\epsilon_{\mu0\lambda\tau}\partial_\eta f_\sigma^{\mu\nu}(b) + i\epsilon_{\mu i\lambda\tau}(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2)f_\sigma^{\mu\nu}(b) - maf_{A,\mu}(b)]\gamma^\mu\gamma^5 \\
&\quad + [-2\delta^0_\mu\partial_\eta f_{V,\nu}(b) + 2\delta^i_\mu(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2)f_{V,\nu}(b) - i\epsilon_{\mu\nu0\lambda}\partial_\eta f_A^\lambda(b) - i\epsilon_{\mu\nu i\lambda}(\partial^i_{\vec{r}} + \partial^i_{\vec{X}}/2)f_A^\lambda(b) \\
&\qquad\qquad - 2maf_{\sigma,\mu\nu}(b)]\sigma^{\mu\nu},
\end{aligned} \tag{6.46}$$

where $b = \eta, \vec{X}, \vec{r}$. In the next section, we will adjust our notation to be able to remove the Levi-Civita symbols.

The Hartree terms, i.e. the second line of (6.40), is significantly less complicated than the other parts of the equation, and is written as,

$$\gamma^0_{\alpha\epsilon}\big[F_{\epsilon\alpha}(a) - F_{\epsilon\alpha}(c)\big]F_{\delta\gamma}(b) = [f^0_V(a) - f^0_V(c)]F_{\delta\gamma}(b). \tag{6.47}$$

Here, $a = \eta, \vec{X} + \frac{\vec{r}}{2} + \vec{x}, 0$, $\quad b = \eta, \vec{X}, \vec{r}$ and $c = \eta, \vec{X} - \frac{\vec{r}}{2} + \vec{x}, 0$ for (6.40). We see that γ^0 picks out the temporal component of the vector contribution. We will see that it describes the sum of the positive frequency shell and negative frequency shell solutions. Before moving on to looking at the separate equations, we first look at the relevant combinations of f_Γ we can make.

6.2.1 Projecting out particles

In an effort to extract relevant contributions from (6.42), we will use a projector, which obeys the following properties in Dirac representation of the gamma matrices (see appendix A),

$$P_\pm \equiv \frac{1 \pm \gamma^0}{2}, \qquad P_\pm^2 = \frac{1 \pm 2\gamma^0 + (\gamma^0)^2}{4} = P_\pm, \qquad P_\pm P_\mp = \frac{1 \pm \gamma^0}{2}\frac{1 \mp \gamma^0}{2} = 0. \tag{6.48}$$

Depending on how a part of the decomposition of $F_{\alpha\beta}$ commutes with γ^0, it will either drop out or contribute. We find (suppressing the spacetime coordinates for brevity)

$$P_{\pm,\gamma\alpha}F_{\alpha\beta}P_{\pm,\beta\delta} = \frac{1}{4}P_{\pm,\gamma\beta}\big(f_S\delta_{\beta\delta} - f_{V,0}\gamma^0_{\beta\delta} + f_{A,i}(\gamma^i\gamma^5)_{\beta\delta} + 2f_{\sigma,ij}\sigma^{ij}_{\beta\delta}\big). \tag{6.49}$$

The other terms anticommute with γ^0 and drop out on account of the third equation in (6.48). We are left with 1+1+3+3=8 degrees of freedom, so we have removed half, which will turn out to correspond to particle-antiparticle mixing terms. Moreover, $P_\pm$ each take half of these degrees, so choosing one projection leaves us with 4 degrees of freedom.

In the Dirac representation, the four-component spinors can be represented as a 'bispinor of bispinors', i.e. $\chi_\alpha = (\chi_{\alpha_1} \;\; \chi_{\alpha_2})^T$ and $\bar{\chi}_\beta = (\chi^\dagger_{\beta_1} \;\; -\chi^\dagger_{\beta_2})$, where $\alpha_1, \beta_1 = 0, 1$ and $\alpha_2, \beta_2 = 2, 3$. The propagator then forms a block matrix, on which the projector P_+ acts as

$$i\gamma^0\alpha\delta S_{\delta\beta} = \begin{pmatrix} \langle\chi_{\alpha_1}\chi^\dagger_{\beta_1}\rangle & \langle\chi_{\alpha_1}\chi^\dagger_{\beta_2}\rangle \\ \langle\chi_{\alpha_2}\chi^\dagger_{\beta_1}\rangle & \langle\chi_{\alpha_2}\chi^\dagger_{\beta_2}\rangle \end{pmatrix}, \tag{6.50}$$

$$\Longrightarrow \; P_{+,\alpha\zeta}i\gamma^0_{\zeta\gamma}S_{\gamma\delta}P_{+,\delta\beta} = \begin{pmatrix} \langle\chi_{\alpha_1}\chi^\dagger_{\beta_1}\rangle & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_{-,\alpha\zeta}i\gamma^0_{\zeta\gamma}S_{\gamma\delta}P_{-,\delta\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \langle\chi_{\alpha_2}\chi^\dagger_{\beta_2}\rangle \end{pmatrix}. \tag{6.51}$$

If we interpret $\chi_{\alpha_{1(2)}}\chi^\dagger_{\beta_{1(2)}}$ as the (anti)particle solutions, we see that the projector only leaves the 'regular' particle part of the propagator, while the anti-particle and the mixing parts are set to 0. The 16 degrees of freedom of the matrix have indeed been reduced to $2 \times 2 = 4$ for each projection.

Looking at the individual contributions in (6.49), we find

$$I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad -\gamma^0 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \qquad \gamma^i\gamma^5 = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad 2\sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}. \tag{6.52}$$

P_+ (P_-) takes the upper-left (lower-right) block of these matrices, so we find the following combinations (with $f_{V,0} = -f^0_V$),

$$P_+F = 2(f^0_V + f_S)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + 2(f_{A,k} + \epsilon_{ijk}f^{ij}_\sigma)\begin{pmatrix} \sigma^k & 0 \\ 0 & 0 \end{pmatrix} \tag{6.53}$$

$$P_-F = 2(f^0_V - f_S)\begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} + 2(f_{A,k} - \epsilon_{ijk}f^{ij}_\sigma)\begin{pmatrix} 0 & 0 \\ 0 & -\sigma^k \end{pmatrix}. \tag{6.54}$$

It is now made explicit that we have 4 degrees of freedom for each equation: 1 each from $f^0_V \pm f_S$, and 3 each from $f_{A,k} \pm \epsilon_{ijk}f^{ij}_\sigma$. When we have found the equations of motion for the separate f_Γ, we can use the information from this subsection to combine them into equations of motion for these quantities. Another interesting consequence of the projections is that we have found that $f^{\mu\nu}_\sigma$ can be divided into two contributions. In fact, taking inspiration from the field strength tensor in electrodynamics, we can look at the $0i$ and ij components separately. Not only will this simplify our notation, but we will find these combinations to be more natural to work with,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad \longrightarrow \qquad f_\sigma^{\mu\nu} = \begin{pmatrix} 0 & f_E^1 & f_E^2 & f_E^3 \\ -f_E^1 & 0 & f_B^3 & -f_B^2 \\ -f_E^2 & -f_B^3 & 0 & f_B^1 \\ -f_E^3 & f_B^2 & -f_B^1 & 0 \end{pmatrix}. \tag{6.55}$$

We emphasize that this division in E and B is only notational, and holds no relation to the physical electric and magnetic fields. We find that we have $\epsilon^{ijk} f_\sigma^{ij} = 2 f_B^k$, which is clearly easier in notation than the product of a Levi-Civita tensor with f_σ.

For completeness and perhaps even more clarity, we will also give the quantities f_P, f_V^l, f_A^0 and $2f_E^l = f_\sigma^{0l}$ in Dirac representation,

$$f_P = \frac{1}{2}\operatorname{Tr}\left[(i\gamma^0 S^{+-} + i\gamma^0 S^{-+})\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\right], \qquad f_A^0 = \frac{1}{2}\operatorname{Tr}\left[(i\gamma^0 S^{+-} + i\gamma^0 S^{-+})\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\right], \tag{6.56}$$

$$f_V^l = \frac{1}{2}\operatorname{Tr}\left[(i\gamma^0 S^{+-} + i\gamma^0 S^{-+})\begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0\end{pmatrix}\right], \quad 2f_E^l = \frac{i}{2}\operatorname{Tr}\left[(i\gamma^0 S^{+-} + i\gamma^0 S^{-+})\begin{pmatrix} 0 & \sigma^l \\ \sigma^l & 0\end{pmatrix}\right]. \tag{6.57}$$

These form the off-diagonal terms of $F_{\alpha\beta}$, and one can see that they induce mixing between the particles and antiparticles.

6.3 Equations with dominant Hartree contribution

As can be seen in [5], at earlier times, the Hartree interaction terms dominate the equation. The transition time η_{trans}, at which we switch to a Fock-dominated equation, depends on the temperature of the dark matter, which lowers due to the expansion of the universe. In fact, for bosons, the Fock terms will be suppressed by a factor $\mathrm{e}^{-ip\Delta x}$, where p is momentum and Δx the seperation of points between which the nonlocal action acts. However, for fermions, we have additional Pauli blocking from the exclusion principle, and we cannot simply assume that the regimes in which the Fock terms are suppressed hold for fermions too. This issue will have to be investigated, but for now we simply assume that there is at least some regime in which we have Hartree-dominated equations (meaning we disregard the Fock terms). Looking at (6.40), we see that this equation is solved in the homogeneous and isotropic limit for a Hadamard function that is constant in time, the relative and the average coordinate. We can actually see that, when taking the relative coordinate to 0 for such a function, we find (compare to (6.30))

$$f_V^0(\eta, \vec{X}, \vec{r}) \to f_{V,hom}^0(\eta, \vec{X}, 0) = \frac{1}{2}\operatorname{Tr}[-iS_{hom}^{-+}(\eta, \vec{X}, 0) - iS_{hom}^{+-}(\eta, \vec{X}, 0)] = n_0, \tag{6.58}$$

as similarly found in [5]. The functions at spatial infinity will obey this relation at all times, which is the condition that the perturbations will go to 0. This way we can safely neglect the boundary terms when partially integrating. We also see that exactly due to (6.58), the n_0 term in (6.40) will indeed remove the homogeneous contribution, making sure we are only dealing with fluctuations in the background.

In the last two lines of (6.46), we see several Levi-Civita symbols appearing. For it to be nonzero, we logically have to get 3 different spatial indices and 1 temporal index, which we used to give values to summed-over indices. We have defined the symbol to obey $\epsilon^{0123} = 1$, meaning that when we move the upper temporal index to be the first, we can simply use $\epsilon^{0ijk} = \epsilon^{ijk}$. We also note that we can write the cross product in terms of a Levi-Civita tensor as $\epsilon^{ijk}\partial^j A^k = (\vec{\nabla} \times \vec{A})^i$, and have the new f_E and f_B. We will now insert (6.46) and (6.47) back into (6.40), neglect the Fock terms, and similar to the procedure in appendix B, we multiply the equations by elements of Γ and take the spinor trace, yielding,

$$i\partial_\eta f_V^0(b) - i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot \vec{f}_V(b) - maf_S(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_S(b), \quad (\boldsymbol{K_S})$$

$$-i\partial_\eta f_A^0(b) + i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot \vec{f}_A(b) - maf_P(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_P(b), \quad (\boldsymbol{K_P})$$

$$i\partial_\eta f_S(b) - 2(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot \vec{f}_E(b) - maf_V^0(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_V^0(b), \quad (\boldsymbol{K_V^0})$$

$$i(\partial_{\vec{r}}^l + \partial_{\vec{X}}^l/2)f_S(b) - 2\partial_\eta f_E^l(b) - 2\Big((\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times \vec{f}_B(b)\Big)^l - maf_V^l(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0)f_V^l(b), \quad (\boldsymbol{K_V^l})$$

$$-i\partial_\eta f_P(b) + 2i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot \vec{f}_B(b) - maf_A^0(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_A^0(b), \quad (\boldsymbol{K_A^0})$$

$$-i(\partial_{\vec{r}}^l + \partial_{\vec{X}}^l/2)f_P(b) + 2i\partial_\eta f_B^l(b) - 2i\big((\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times \vec{f}_E(b)\big)^l - maf_A^l(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_A^l(b), \quad (\boldsymbol{K_A^l})$$

$$\partial_\eta f_V^l(b) - (\partial_{\vec{r}}^l + \partial_{\vec{X}}^l/2)f_V^0(b) - i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times \vec{f}_A(b)\big)^l - 2maf_E^l(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)2f_E^l(b), \quad (\boldsymbol{K_E^l})$$

$$\big((\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times \vec{f}_V(b)\big)^l + i\partial_\eta f_A^l(b) - i(\partial_{\vec{r}}^l + \partial_{\vec{X}}^l/2)f_A^0(b) - 2maf_B^l(b) = -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)2f_B^l(b), \quad (\boldsymbol{K_B^l})$$

with $b = \eta, \vec{X}, \vec{r}$ (unless otherwise stated in this section, b will stay this shorthand notation). These are 16 equations, and by taking the Hermitian conjugate, we gain 16 more. These could be reduced to 16 dynamical equations and 16 constraints. The latter 16 are the on-shell relations, which are automatically satisfied in the coincidence limit. These equations obviously couple to each other, and our goal will be to get separate equations by plugging these equations into each other, or to identify contributions as corrections to the classical limit. From section 6.2.1, we have the combinations

$$f_\pm(b) = \frac{1}{2}\big(f_V^0(b) \pm f_S(b)\big), \qquad \big(f_\pm(\eta,\vec{X},\vec{r})\big)^\dagger = f_\pm(\eta,\vec{X},-\vec{r}), \tag{6.59}$$

$$\vec{f}_\pm(b) = \frac{1}{2}\big(\vec{f}_A(b) \pm 2\vec{f}_B(b)\big), \qquad \big(\vec{f}_\pm(\eta,\vec{X},\vec{r})\big)^\dagger = \vec{f}_\pm(\eta,\vec{X},-\vec{r}), \tag{6.60}$$

$$g_\pm(b) = \frac{1}{2}\big(f_P(b) \pm f_A^0(b)\big), \qquad \big(g_\pm(\eta,\vec{X},\vec{r})\big)^\dagger = g_\mp(\eta,\vec{X},-\vec{r}), \tag{6.61}$$

$$\vec{g}_\pm(b) = \frac{i}{2}\big(2\vec{f}_E(b) \pm i\vec{f}_V(b)\big), \qquad \big(\vec{g}_\pm(\eta,\vec{X},\vec{r})\big)^\dagger = \vec{g}_\mp(\eta,\vec{X},-\vec{r}). \tag{6.62}$$

These Hermiticity relations can be inferred from the gamma matrices in the separate f_Γ contributions, and using (6.28). Choosing either the + or − projection is reasonable when there exists a large chemical potential μ for the particles (and opposite potential for antiparticles). To illustrate the meaning of this, we take a look at a thermal equilibrium situations, in which the densities $f_\pm$ will reduce to [38],

$$f_+ \to n_f = \frac{1}{\mathrm{e}^{(E-\mu)/k_BT}+1}, \qquad f_- \to n_{\bar{f}} = \frac{1}{\mathrm{e}^{(E+\mu)/k_BT}+1}, \tag{6.63}$$

where $(\bar{f})$ f represent fermionic (anti)particles. In the case of a fixed particle number and nonrelativistic speeds, the conditions that apply here are that the energy is comparable to the chemical potential, and that $\mu \gg k_BT$

$$\frac{n_{\bar{f}}}{n_f} = \frac{\mathrm{e}^{(E-\mu)/k_BT}+1}{\mathrm{e}^{(E+\mu)/k_BT}+1} \to 0. \tag{6.64}$$

We see that this implies that the number of antiparticles is negligible compared to the number of particles. It is then reasonable to disregard the antiparticle projection.

From section 2.2.1 we can infer what the g densities will look like: either an upper-right or lower-left off-diagonal block matrix, mixing the diagonal entries. But we haven't yet looked explicitly at the form of the f, which read

$$f_\pm = \frac{1}{2}\,\mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}\right] \pm \frac{1}{2}\,\mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right] = \mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} I\pm I & 0 \\ 0 & -I\pm I\end{pmatrix}\right], \tag{6.65}$$

$$\vec{f}_\pm = \frac{1}{2}\,\mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma}\end{pmatrix}\right] \pm \frac{1}{2}\,\mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}\right] = \mathrm{Tr}\left[F_{\alpha\beta}\begin{pmatrix} \vec{\sigma}\pm\vec{\sigma} & 0 \\ 0 & -\vec{\sigma}\pm\vec{\sigma}\end{pmatrix}\right]. \tag{6.66}$$

The particle and antiparticle solutions are situated on the diagonal, and if we interpret the lower-right block matrix as the antiparticles, they obtain a minus sign. The Hadamard two point function gives the average of the expectation values of $-\chi_\alpha\chi_\beta^\dagger$ and $\chi_\beta^\dagger\chi_\alpha$, and in the section on projectors we have motivated that we divide it into a 2×2 block matrix of products of bispinors. These can then be interpreted as having spin up or down. We see that $f_\pm$ simply counts the contributions from all particles having either spin up or down. $\vec{f}_\pm$ puts a Pauli matrix in the expectation values: it measures the spin densities of particles in some x-y-z-coordinate basis. Comparing to the classical equations, we would expect to find only two relevant degrees of freedom: spin up and spin down, with respect to some fixed z-direction. Here we have four, and it appears to be a consequence of not being able to define such a fixed coordinate basis, as we are on curved spacetime. Due to time constraints, we will not be able to investigate this further in this thesis.

(6.59) and (6.59) were already known, (6.61) and (6.62) will turn out to be convenient when rewriting the equations, but inconvenient when taking real and imaginary parts of the equations, so we will use the last two less often. Rewriting $f_V^0 = f_+ + f_-$ will now give us, by taking appropriate combinations of ($\boldsymbol{K_S}$)-($\boldsymbol{K_B^l}$),

$$\begin{aligned}
&\pm i\partial_\eta f_\pm(b) + i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot\vec{g}_\pm(b) - maf_\pm(b) = \\
&\qquad -\frac{\kappa^2 m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}\Big(f_+(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) + f_-(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)f_\pm(b),
\end{aligned} \tag{6.67}$$

$$\begin{aligned}
&\pm i\partial_\eta \vec{f}_\pm(b) - (\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times\vec{g}_\pm(b) - i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)g_\pm(b) - ma\vec{f}_\pm(b) = \\
&\qquad -\frac{\kappa^2 m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}\Big(f_+(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) + f_-(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)\vec{f}_\pm(b),
\end{aligned} \tag{6.68}$$

$$\begin{aligned}
&\mp i\partial_\eta g_\pm(b) + i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\cdot\vec{f}_\pm(b) - mag_\pm(b) = \\
&\qquad -\frac{\kappa^2 m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}\Big(f_+(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) + f_-(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)g_\pm(b),
\end{aligned} \tag{6.69}$$

$$\begin{aligned}
&\mp i\partial_\eta \vec{g}_\pm(b) + (\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)\times\vec{f}_\pm(b) - i(\vec{\nabla}_{\vec{r}} + \vec{\nabla}_{\vec{X}}/2)f_\pm(b) - ma\vec{g}_\pm(b) = \\
&\qquad -\frac{\kappa^2 m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}\Big(f_+(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) + f_-(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x},0) - n_0\Big)\vec{g}_\pm(b).
\end{aligned} \tag{6.70}$$

The right-hand sides of these equations will be referred to as the collision terms. These equations can be reduced to the Vlasov-Poisson system we are after. However, $f_\pm$, $\vec{f}_\pm$, $g_\pm$ and $\vec{g}_\pm$ are still in position space, and we want to get equations for the phase space densities. We will therefore first look at the Wigner transform that will make it possible to move to phase space.

6.3.1 Wigner transform and gradient expansion

We will use a Fourier transform for the relative coordinate $\vec{r}$ to go to phase space, also called a Wigner transform. The f_Γ in position space will then be transformed into proper phase-space densities, exactly the objects of interest. The Wigner transform is defined as

$$f_\Gamma(\eta,\vec{X},\vec{p}) = \frac{1}{(2\pi)^3}\int d^3r\mathrm{e}^{-i\vec{p}\cdot\vec{r}}f_\Gamma(\eta,\vec{X},\vec{r}), \tag{6.71}$$

which has an inverse

$$f_\Gamma(\eta,\vec{X},\vec{r}) = \int d^3p\,\mathrm{e}^{i\vec{p}\cdot\vec{r}}f_\Gamma(\eta,\vec{X},\vec{p}). \tag{6.72}$$

With these definitions, it is fairly simple to transform the left-hand side of ($\boldsymbol{K_S}$)-($\boldsymbol{K_B^l}$). However, we find that the collision terms depend on the product of two $f_\Gamma(\eta,\vec{X},\vec{r})$, whose transformation needs more care. We first want to extract the relative coordinate $\vec{r}$ from the argument of $f_V^0(\eta,\vec{X}+\frac{\vec{r}}{2}+\vec{x})$, by using a Taylor expansion in $\vec{X}$ around $\vec{X}-\frac{\vec{r}}{2}$ to write the following,

$$f(\vec{X}+\frac{\vec{r}}{2}+\vec{x},0)=\sum_{n=0}^{\infty}\frac{(\vec{\nabla}_{\vec{X}})^n f(\vec{X}+\vec{x},0)}{n!}\Big(\frac{\vec{r}}{2}\Big)^n = f(\vec{X}+\vec{x},0)\mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_{\vec{X}}\cdot\vec{r}}. \tag{6.73}$$

We also take note of the convolution theorem, which states that the Fourier transform of a convolution is the product of Fourier transforms of the convoluted functions. A corollary to this is that the Fourier transform of a product of functions (which is what we have in the collision term) will then be the convolution of the Fourier transforms of the individual functions, written as

$$\int\frac{d^3r}{(2\pi)^3}g(\vec{r})f(\vec{r})\mathrm{e}^{-i\vec{p}\cdot\vec{r}}=\int d^3q\; g(\vec{p}-\vec{q})f(\vec{q}). \tag{6.74}$$

To use this corollary more easily, we now define a function

$$g(\partial_{\vec{X}},\vec{r})\equiv\mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_{\vec{X}}\cdot\vec{r}}, \tag{6.75}$$

such that we can write for the Wigner transform of the collision terms,

$$\begin{aligned}
\int\frac{d^3r}{(2\pi)^3}\int\frac{d^3x}{||\vec{x}||}&f_V^0(\eta,\vec{X}+\vec{x},0)g(\partial_{\vec{X}},\vec{r})f_\Gamma(\eta,\vec{X},\vec{r})\mathrm{e}^{-i\vec{p}\cdot\vec{r}}=\\
&\int d^3q\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)g(\partial_{\vec{X}},\vec{p}-\vec{q})f_\Gamma(\eta,\vec{X},\vec{q})\\
&=\int d^3q\frac{d^3r}{(2\pi)^3}\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)g(\partial_{\vec{X}},\vec{r})\mathrm{e}^{-i(\vec{p}-\vec{q})\cdot\vec{r}}f_\Gamma(\eta,\vec{X},\vec{q})\\
&=\int d^3q\frac{d^3r}{(2\pi)^3}\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)\mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{q}}}\mathrm{e}^{-i(\vec{p}-\vec{q})\cdot\vec{r}}f_\Gamma(\eta,\vec{X},\vec{q})\\
&=\int d^3q\frac{d^3r}{(2\pi)^3}\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)\mathrm{e}^{-i(\vec{p}-\vec{q})\cdot\vec{r}}\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{q}}}f_\Gamma(\eta,\vec{X},\vec{q})\\
&=\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}}f_\Gamma(\eta,\vec{X},\vec{p}).
\end{aligned} \tag{6.76}$$

We use the convolution theorem in the first equality, write the Wigner transform of $g(\partial_{\vec{X}},\vec{r})$ in the second and rewrite $\vec{r}$ as $-i\partial_{\vec{q}}$ in the exponent in the third equality (which is more obvious if the exponent is expanded). In the fourth equality, we partially integrate infinitely many times in $\partial_{\vec{q}}$; a simpler version of this reads,

$$\begin{aligned}
\int dx\; f(x)\mathrm{e}^{\partial_x}g(x)\approx\int dx\; f(x)\big(1+\partial_x+\partial_x^2/(2!)\big)g(x)&=\int dx\; g(x)\big(1-\partial_x+\partial_x^2/(2!)\big)f(x)+\text{B.T.}\\
&\approx\int dx\; g(x)\mathrm{e}^{-\partial_x}f(x)+\text{B.T.}.
\end{aligned} \tag{6.77}$$

Clearly, we have assumed the boundary terms (B.T.) to vanish. The final step of (6.76) is finding a Dirac delta function after integrating over $\vec{r}$, after which integration over $\vec{q}$ becomes very easy. If we now use the identity $\mathrm{e}^{ix}=\cos(x)+i\sin(x)$, we can extract the real and imaginary parts of the collision terms, so we define

$$\mathcal{C}(f_\Gamma)\equiv\frac{\kappa^2m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta,\vec{X}+\vec{x},0)\cos\Big(\frac{\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}}{2}\Big)-n_0\Big)f_\Gamma(\eta,\vec{X},\vec{p}), \tag{6.78}$$

$$\mathcal{S}(f_\Gamma)\equiv\frac{\kappa^2m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)\sin\Big(\frac{\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}}{2}\Big)f_\Gamma(\eta,\vec{X},\vec{p}). \tag{6.79}$$

The derivation when the argument is $\vec{X}-\frac{\vec{r}}{2}+\vec{x}$ will clearly be very similar, and will result in $\mathcal{C}$ and $\mathcal{S}$ which have $\partial_{\vec{p}}\to-\partial_{\vec{p}}$. This means that $\mathcal{C}$ stays the same, and $\mathcal{S}$ will obtain a minus sign. If we were to expand these cosine and sine, we obtain terms even and odd in $\partial_{\vec{X}} \cdot \partial_{\vec{p}}$, hence the name 'gradient expansion'. Had we not set $\hbar = 1$, there would be a factor $\hbar$ appearing for each dot product of gradients, which is a first indication higher order terms will vanish on account of the smallness of the reduced Planck's constant.

For the spatial gradient expansion to actually apply, we will need the higher order terms to be suppressed. This can be seen by assuming that the de Broglie wavelenght λ_{dB} is much smaller than the scales we are interested in, $|\vec{X}| \sim L \sim 10 - 100\text{MPc}$ [7], a reasonable assumption when we are already in the classical (nonrelativistic, large mass) particle limit. We find

$$L \gg \lambda_{dB} = h/|\vec{p}| \sim h\partial_{\vec{p}} \implies h\partial_{\vec{X}} \cdot \partial_{\vec{p}} \ll 1. \tag{6.80}$$

Only when assuming extremely light particles (like fuzzy dark matter, 10^{-22}eV), which we have already excluded by assuming single-flavored fermions, the de Broglie wavelength could be of such sizes. We see that $\partial_{\vec{p}}$ acts on the phase space density. By taking the spatial gradient expansion to be true, we exclude sudden large changes in $\vec{p}$ of the phase-space densities, which could happen during particle-pair creation. As a consequence, we therefore disallow particle-pair creation by taking the expansion to hold. Given that the Vlasov equation presumes a fixed particle number, as seen in chapter 2, the exclusion of pair creation goes well with going to the classical limit.

The right-hand sides of ($\boldsymbol{K_S}$)-($\boldsymbol{K_B^l}$), which we will call collision terms $\text{Coll}_K[f_\Gamma]$, can now be decomposed into a real and imaginary part by using (6.76) and subsequently (6.78) and (6.79), after Wigner transforming,

$$\begin{aligned}\text{Coll}_K[f_\Gamma] =& -\mathcal{C}(f_\Gamma) - i\mathcal{S}(f_\Gamma)\\ =& -\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||}\Big(f_V^0(\eta, \vec{X}+\vec{x}, 0)\big[\cos(\overleftarrow{\partial}_{\vec{X}} \cdot \partial_{\vec{p}}/2) + i\sin(\overleftarrow{\partial}_{\vec{X}} \cdot \partial_{\vec{p}}/2)\big] - n_0\Big) f_\Gamma(\eta, \vec{X}, \vec{p}).\end{aligned} \tag{6.81}$$

6.3.2 Reduction to the Vlasov-Poisson system for $f_\pm$

We first take note of the equations ($\boldsymbol{K_P}$), ($\boldsymbol{K_V^l}$), ($\boldsymbol{K_A^0}$) and ($\boldsymbol{K_E^l}$), multiplying the final three by i, Wigner transforming in the relative coordinate and taking the real part, also writing f_S, f_V^0, $\vec{f}_A$ and $\vec{f}_B$ in terms of (6.59)-(6.62):

$$-\partial_\eta(if_A^0(c)) - \vec{p}\cdot(\vec{f}_+(c) + \vec{f}_-(c)) - maf_P(c) = -\mathcal{C}(f_P), \tag{6.82}$$

$$-\partial_\eta(2i\vec{f}_E(c)) - \frac{\vec{\nabla}_{\vec{X}}}{2}(f_+(c) - f_-(c)) + \vec{p}\times(\vec{f}_+(c) - \vec{f}_-(c)) - ma(i\vec{f}_V(c)) = -\mathcal{C}(i\vec{f}_V), \tag{6.83}$$

$$\partial_\eta f_P(c) - \frac{\vec{\nabla}_{\vec{X}}}{2}\cdot(\vec{f}_+(c) - \vec{f}_-(c)) - ma(if_A^0(c)) = -\mathcal{C}(if_A^0), \tag{6.84}$$

$$\partial_\eta(i\vec{f}_V(c)) + \vec{p}(f_+(c) + f_-(c)) + \frac{\vec{\nabla}_{\vec{X}}}{2}\times(\vec{f}_+(c) + \vec{f}_-(c)) - ma(2i\vec{f}_E(c)) = -\mathcal{C}(2i\vec{f}_E(c)), \tag{6.85}$$

with $c = \eta, \vec{X}, \vec{p}$ (unless otherwise stated in this section, c will stay this shorthand notation). If we now multiply by $-i$ and take the real part of (6.67), where we write $\vec{g}_\pm$ as it was defined in terms of f_Γ, we obtain

$$\pm\partial_\eta f_\pm(c) + \mathcal{S}(f_\pm) = \pm\frac{\vec{p}}{2}\cdot(i\vec{f}_V(c)) - \frac{\vec{\nabla}_{\vec{X}}}{2}\cdot(i\vec{f}_E(c)). \tag{6.86}$$

To get a new expression for the terms on the right-hand side, we take the dot product of (6.83) with $\vec{p}/(ma)$ and of (6.85) with $\vec{\nabla}_{\vec{X}}/(2ma)$ and divide by 2,

$$\begin{aligned}\pm\frac{\vec{p}}{2}\cdot(i\vec{f}_V(c)) - \frac{\vec{\nabla}_{\vec{X}}}{4}\cdot 2i\vec{f}_E =& -\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{2ma}f_\pm(c) - \frac{\partial_\eta}{2ma}\Big[\pm\vec{p}\cdot 2i\vec{f}_E(c) + \frac{\vec{\nabla}_{\vec{X}}}{2}\cdot i\vec{f}_V(c)\Big]\\ &\pm\frac{\vec{p}}{2ma}\cdot\mathcal{C}(i\vec{f}_V) - \frac{\vec{\nabla}_{\vec{X}}}{2ma}\cdot\mathcal{C}(2i\vec{f}_E).\end{aligned} \tag{6.87}$$

such that we find for (6.86)

$$\begin{aligned}\pm\partial_\eta f_\pm(c)+\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{2ma}f_\pm(c)+\mathcal{S}(f_\pm)=&\\ &-\frac{\partial_\eta}{2ma}\Big[\pm\vec{p}\cdot 2i\vec{f}_E(c)+\frac{\vec{\nabla}_{\vec{X}}}{2}\cdot i\vec{f}_V(c)\Big]\pm\frac{\vec{p}}{2ma}\cdot\mathcal{C}(i\vec{f}_V)-\frac{\vec{\nabla}_{\vec{X}}}{2ma}\cdot\mathcal{C}(2i\vec{f}_E).\end{aligned}\tag{6.88}$$

The terms on the left-hand side will construct the Vlasov equation to first order in gradients, where the $\pm$ in front of the time derivative counteracts the minus signs on the lower diagonal matrix in (6.65) and (6.66). The minus sign is kept in terms where there is a momentum or derivative of the momentum, which we interpret as antiparticles having opposite momentum. To explicitly construct the force term from the sine, we look at they way the classical Newtonian potential is related to the collision terms,

$$\frac{\kappa^2m^2}{16\pi}\int\frac{d^3x}{||\vec{x}-\vec{X}||}f_V^0(\eta,\vec{x},0)=-ma(4\pi G)a^2\Delta_{\vec{X}}^{-1}\Big(ma^{-3}\sum_\pm f_\pm(\eta,\vec{x},0)\Big)=-ma\Phi_N(\eta,\vec{X}).\tag{6.89}$$

The minus sign in the large round parentheses comes from (6.7). In the second part we have restored the scale factors (which cancel out), to see a clearer comparison to (2.16). An even better comparison is made when realizing that, using a Wigner transform,

$$f(\eta,\vec{X},0)=\int d^3p\,\mathrm{e}^{i\vec{p}\cdot\vec{0}}f(\eta,\vec{X},\vec{p})=\int d^3p f(\eta,\vec{X},\vec{p}).\tag{6.90}$$

With this, we can construct from the $\mathcal{S}$-term,

$$\begin{aligned}\mathcal{S}(f_\pm)=&\frac{1}{2}\vec{\nabla}_{\vec{X}}\Big(\frac{\kappa^2m^2}{16\pi}\int\frac{d^3x}{||\vec{x}||}f_V^0(\eta,\vec{X}+\vec{x},0)\Big)\cdot\vec{\nabla}_{\vec{p}}f_\pm(\eta,\vec{X},\vec{p})+\mathcal{O}(|\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}|^3)\\ \approx&-\frac{1}{2}ma\frac{\partial\Phi_N(\eta,\vec{X})}{\partial X^k}\frac{\partial}{\partial(p_k)}f_\pm(\eta,\vec{X},\vec{p}),\end{aligned}\tag{6.91}$$

up to a factor 1/2 exactly the final term on the left-hand side of (2.15), and indeed keeping the minus sign on the lower diagonal as there is a momentum term. To see what corrections we are dealing with, we take a dot product of $\vec{p}/(ma)$ with (6.85) and $\vec{\nabla}_{\vec{X}}/(2ma)$ with (6.83), which will allow us to write

$$\begin{aligned}\pm\vec{p}\cdot 2i\vec{f}_E(c)+\frac{\vec{\nabla}_{\vec{X}}}{2}\cdot i\vec{f}_V(c)=&\frac{\partial_\eta}{ma}\Big[\pm\vec{p}\cdot i\vec{f}_V(c)-\frac{\vec{\nabla}_{\vec{X}}}{2}\cdot 2i\vec{f}_E(c)\Big]\pm\frac{p^2}{ma}(f_+(c)+f_-(c))\\ &-\frac{\vec{\nabla}^2_{\vec{X}}}{4ma}(f_+(c)-f_-(c))\pm\frac{\vec{p}}{ma}\cdot\big(\vec{\nabla}_{\vec{X}}\times\vec{f}_\mp(c)\big)\pm\frac{\vec{p}}{ma}\cdot\mathcal{C}(2i\vec{f}_E)\\ &+\frac{\vec{\nabla}_{\vec{X}}}{2ma}\cdot\mathcal{C}(i\vec{f}_V).\end{aligned}\tag{6.92}$$

We see that the new term in square brackets is up to a factor 1/2 the same as (6.87). This way, we can substitute it again and get more corrections in terms of $f_\pm$ and $\vec{f}_\pm$, with either $\vec{\nabla}_{\vec{X}}/(2ma)$ or $\vec{p}/(ma)$ turning up in each iteration. The $\mathcal{C}$-terms show a similar structure: we use $\partial^i_{\vec{p}}\,p^k=\delta^{ik}$, and find

$$\begin{aligned}\cos(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})\big(\vec{p}f(\vec{p})\big)=&\big(1-\frac{1}{2!}(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})^2+...\big)\big(\vec{p}\cdot\vec{f}(\vec{p})\big)\\ =&\vec{p}\big(1-\frac{1}{2!}(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})^2+...\big)\cdot\vec{f}(\vec{p})-\big(\frac{2}{2!}\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}\overleftarrow{\partial}_{\vec{X}}+...\big)\cdot\vec{f}(\vec{p})\\ =&\big[\vec{p}\cos(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})-\overleftarrow{\partial}_{\vec{X}}\sin(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})\big]\cdot\vec{f}(\vec{p})\\ \Longrightarrow\;\vec{p}\cdot\mathcal{C}(i\vec{f}_V)=&\mathcal{C}(\vec{p}\cdot i\vec{f}_V)+\mathcal{S}(\overleftarrow{\partial}_{\vec{X}}\cdot i\vec{f}_V),\end{aligned}\tag{6.93}$$

such that we obtain

$$\pm\frac{\vec{p}}{2ma}\cdot\mathcal{C}(i\vec{f}_V)-\frac{\vec{\nabla}_{\vec{X}}}{2ma}\cdot\mathcal{C}(2i\vec{f}_E)=\frac{1}{2ma}\Big[\mathcal{C}(\pm\vec{p}\cdot i\vec{f}_V-\vec{\nabla}_{\vec{X}}\cdot 2i\vec{f}_E)\pm\mathcal{S}(\overleftarrow{\partial}_{\vec{X}}\cdot i\vec{f}_V)-\mathcal{C}(\overleftarrow{\partial}_{\vec{X}}\cdot 2i\vec{f}_E)\Big].\tag{6.94}$$

The terms within the first $\mathcal{C}$ in the square brackets again have the same structure as (6.87), and we can iterate. To evaluate the remaining terms, it will make sense to look at the scales we are dealing with to see how the different terms contribute, and we have

$$\lambda_{dB} = h/|\vec{p}| \ll |\vec{X}| \implies |\vec{\nabla}_{\vec{X}}| \ll p, \qquad \frac{|\vec{p}|}{ma} \sim \frac{v}{c} \propto \sqrt{\Phi_N}, \tag{6.95}$$

where $c = 1$ is light speed. For the same reason we can apply the spatial gradient expansion, we have found the inequality. The proportionality shows us that we are dealing with relativistic corrections, proportional to the square root of the classical potential. In the previous chapter, we have already neglected higher order terms than ε_g^2, so any corrections we obtain that are higher order than this cannot be trusted. This gives us a clear upper limit on the amount of reliable corrections we can find. What's left is to interpret the terms containing $\partial_\eta/(ma)$. We know that the phase space densities can change in time, either by the expansion of the universe expressed in $\mathcal{H}$, or using the continuity equation and re-expressing the change per time with the flow from outside the volume under consideration, expressed in $\vec{\nabla}_{\vec{X}}$. The former is small from the adiabatic limit (3.25), the latter from the large mass limit (6.2), so either way the time derivatives will be suppressed in higher orders.

The relation in (6.89) will help us to identify terms that are higher order than ε_g^2. Through the iteration structure, we find that the each time we go to a higher iteration, we gain terms that are suppressed by the adiabatic or large mass limit, the nonrelativistic limit or quantum suppressed:

$$\mathcal{O}\Big(\frac{\partial_\eta}{ma}\Big) \ll \mathcal{O}\Big(\frac{\mathcal{H}}{ma}\Big) \cup \mathcal{O}\Big(\frac{|\vec{\nabla}_{\vec{X}}|}{ma}\Big) \ll 1, \qquad \mathcal{O}\Big(\frac{|\vec{\nabla}_{\vec{X}}|}{ma}\Big) \ll \mathcal{O}\Big(\frac{|\vec{p}|}{ma}\Big) \ll 1. \tag{6.96}$$

We now have a clear way of obtaining corrections, interpreting what kind of corrections they are, and to what order they are relevant. We are then only left with the terms in (6.94). The $\mathcal{S}$-term is second order in $\vec{\nabla}_{\vec{X}}$ at best and will be suppressed compared to the $\mathcal{C}$-term. The final $\mathcal{C}$-term has to lowest order the structure

$$\frac{1}{ma}\vec{\nabla}_{\vec{X}}\Big(\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}||} f_V^0(\eta, \vec{X}+\vec{x}, 0)\Big) = -\frac{\vec{\nabla}_{\vec{X}}\Phi_N(\eta, \vec{X})}{ma}, \tag{6.97}$$

which is a force suppressed by the mass term. We have already established that the first $\mathcal{C}$-term on the right-hand side had the combination that can be iterated, gaining again $\vec{\nabla}_{\vec{X}}/(ma)$, $\vec{p}/(ma)$ and $\partial_\eta/(ma)$ corrections, or to lowest order in gradients

$$\frac{1}{ma}\mathcal{C}(f_\Gamma(\eta, \vec{X}, \vec{p})) \approx \frac{1}{ma}\frac{\kappa^2 m^2}{16\pi}\int \frac{d^3x}{||\vec{x}-\vec{X}||}\big[f_V^0(\eta, \vec{x}, 0) - n_0(\eta)\big] f_\Gamma(\eta, \vec{X}, \vec{p}) = -\frac{\Phi_{N,\delta\rho}(\eta, \vec{X})}{ma} f_\Gamma(\eta, \vec{X}, \vec{p}), \tag{6.98}$$

where we see that this is the only time the n_0 term actually shows up as there is to $\vec{\nabla}_{\vec{X}}$ to remove it. We see that we can now explicitly see that the potential is generated by the perturbations, and we obtain a term that is a potential suppressed by the mass, which again becomes smaller upon iteration.

This section has shown that we can obtain the classical Vlasov-Poisson system for the particle and antiparticle densities from the equations of motion we have derived from the 2PI effective action. The higher-order corrections follow from an iterative structure, yielding corrections of at most $\mathcal{O}(\varepsilon_g)$ from each iteration, and we have indicated at what order the iterations cannot be trusted anymore. The next section shows we can do the same for $\vec{f}_\pm$, albeit slightly more heavy on calculations for the corrections, yet the procedure stays the same.

6.3.3 Reduction to the Vlasov-Poisson system for $\vec{f}_\pm$

To get to the correct equation for $\vec{f}_\pm$, we look at (6.68) and extracting the relevant real parts. Multiplying it by $-i$ and Wigner transforming yields

$$\pm\partial_\eta \vec{f}_\pm(b) - (\vec{p} - i\vec{\nabla}_{\vec{X}}/2) \times \vec{g}_\pm(b) - (i\vec{p} + \vec{\nabla}_{\vec{X}}/2) g_\pm(b) + ima\vec{f}_\pm(b) = -i\mathrm{Coll}_K[\vec{f}_\pm(b)], \tag{6.99}$$

and taking the real part gives

$$\pm\partial_\eta \vec{f}_\pm(c) + \mathcal{S}(\vec{f}_\pm(c)) = \pm\frac{\vec{\nabla}_{\vec{X}}}{4} \times (i\vec{f}_V(c)) + \frac{\vec{p}}{2} \times (2i\vec{f}_E(c)) \pm \frac{\vec{p}}{2} i f_A^0(c) + \frac{\vec{\nabla}_{\vec{X}}}{4} f_P(c). \tag{6.100}$$

Just like in the previous section, we will now evaluate the terms on the right-hand side by taking a cross product of $\vec{p}/(2ma)$ with (6.85), a cross product of $\vec{\nabla}_{\vec{X}}/(4ma)$ with (6.83), taking the gradient $\vec{\nabla}_{\vec{X}}/(4ma)$ with (6.82) and multiplying (6.84) with $\vec{p}/(2ma)$, respectively,

$$\frac{\partial_\eta}{2ma}\vec{p}\times i\vec{f}_V(c)+\frac{\vec{\nabla}_{\vec{X}}}{4ma}\vec{p}\cdot(\vec{f}_+(c)+\vec{f}_-(c))-\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{4ma}(\vec{f}_+(c)+\vec{f}_-(c))-\frac{\vec{p}}{2}\times 2i\vec{f}_E(c)=-\frac{\vec{p}}{2ma}\times\mathcal{C}(2i\vec{f}_E), \tag{6.101}$$

$$-\frac{\partial_\eta}{2ma}\frac{\vec{\nabla}_{\vec{X}}}{2}\times 2i\vec{f}_E(c)+\frac{\vec{p}}{4ma}\vec{\nabla}_{\vec{X}}\cdot(\vec{f}_+(c)-\vec{f}_-(c))-\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{4ma}(\vec{f}_+(c)-\vec{f}_-(c))-\frac{\vec{\nabla}_{\vec{X}}}{4}\times i\vec{f}_V(c)=-\frac{\vec{\nabla}_{\vec{X}}}{4ma}\times\mathcal{C}(i\vec{f}_V), \tag{6.102}$$

$$-\frac{\partial_\eta}{2ma}\frac{\vec{\nabla}_{\vec{X}}}{2}if_A^0(c)-\frac{\vec{\nabla}_{\vec{X}}}{4ma}\vec{p}\cdot(\vec{f}_+(c)+\vec{f}_-(c))-\frac{\vec{\nabla}_{\vec{X}}}{4}f_P(c)=-\frac{\vec{\nabla}_{\vec{X}}}{4ma}\mathcal{C}(f_P), \tag{6.103}$$

$$\frac{\partial_\eta}{2ma}\vec{p}f_P(c)-\frac{\vec{p}}{4ma}\vec{\nabla}_{\vec{X}}\cdot(\vec{f}_+(c)-\vec{f}_-(c))-\frac{\vec{p}}{2}if_A^0(c)=-\frac{\vec{p}}{2ma}\mathcal{C}(if_A^0). \tag{6.104}$$

Substituting these terms shows that we again get the familiar $\vec{p}\cdot\vec{\nabla}_{\vec{X}}$ term,

$$\begin{aligned}\pm\partial_\eta\vec{f}_\pm(c)+\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{2ma}\vec{f}_\pm(c)+\mathcal{S}(\vec{f}_\pm(c))=&\frac{\partial_\eta}{2ma}\Big[\mp\frac{\vec{\nabla}_{\vec{X}}}{2}\times 2i\vec{f}_E(c)+\vec{p}\times i\vec{f}_V(c)\pm\vec{p}f_P(c)-\frac{\vec{\nabla}_{\vec{X}}}{2}if_A^0(c)\Big]\\&\pm\frac{\vec{\nabla}_{\vec{X}}}{4ma}\times\mathcal{C}(i\vec{f}_V)+\frac{\vec{p}}{2ma}\times\mathcal{C}(2i\vec{f}_E)\pm\frac{\vec{p}}{2ma}\mathcal{C}(if_A^0)+\frac{\vec{\nabla}_{\vec{X}}}{4ma}\mathcal{C}(f_P).\end{aligned} \tag{6.105}$$

The $\mathcal{S}$-term is evaluated in the same way as in the previous subsection, so we now look at the corrections on the right-hand side. For the combination in the square brackets, we find

$$\begin{aligned}&\mp\frac{\vec{\nabla}_{\vec{X}}}{2}\times 2i\vec{f}_E(c)+\vec{p}\times i\vec{f}_V(c)\pm\vec{p}f_P(c)-\frac{\vec{\nabla}_{\vec{X}}}{2}if_A^0(c)=\\&\quad\frac{\partial_\eta}{ma}\Big[\mp\frac{\vec{\nabla}_{\vec{X}}}{2}\times i\vec{f}_V(c)-\vec{p}\times 2i\vec{f}_E(c))\mp\vec{p}\,if_A^0(c)-\frac{\vec{\nabla}_{\vec{X}}}{2}f_P(c)\Big]\\&\quad+\frac{\vec{p}}{ma}\times\big(\vec{p}\times(\vec{f}_+(c)-\vec{f}_-(c))\big)\mp\frac{\vec{p}}{ma}\vec{p}\cdot(\vec{f}_+(c)+\vec{f}_-(c))\mp\frac{\vec{\nabla}_{\vec{X}}\times\vec{p}}{2ma}(f_+(c)+f_-(c))\\&\quad\mp\frac{\vec{\nabla}_{\vec{X}}}{4ma}\times\big(\vec{\nabla}_{\vec{X}}\times(\vec{f}_+(c)+\vec{f}_-(c))\big)+\frac{\vec{\nabla}_{\vec{X}}}{4ma}\vec{\nabla}_{\vec{X}}\cdot(\vec{f}_+(c)-\vec{f}_-(c))-\frac{\vec{p}\times\vec{\nabla}_{\vec{X}}}{2ma}(f_+(c)-f_-(c))\\&\quad\pm\frac{\vec{\nabla}_{\vec{X}}}{2ma}\times\mathcal{C}(2i\vec{f}_E)-\frac{\vec{p}}{ma}\times\mathcal{C}(i\vec{f}_V)\mp\frac{\vec{p}}{ma}\mathcal{C}(f_P)+\frac{\vec{\nabla}_{\vec{X}}}{2ma}\mathcal{C}(if_A^0)\\&=\frac{\partial_\eta}{ma}\Big[\mp\frac{\vec{\nabla}_{\vec{X}}}{2}\times i\vec{f}_V(c)-\vec{p}\times 2i\vec{f}_E(c))\mp\vec{p}\,if_A^0(c)-\frac{\vec{\nabla}_{\vec{X}}}{2}f_P(c)\Big]\pm\frac{\vec{p}}{ma}\times(\vec{\nabla}_{\vec{X}}f_\mp(c))\\&\quad\mp 2\frac{\vec{p}}{ma}(\vec{p}\cdot\vec{f}_\mp(c))-\frac{p^2}{ma}(\vec{f}_+(c)-\vec{f}_-(c))\mp\frac{\vec{\nabla}_{\vec{X}}}{2ma}(\vec{\nabla}_{\vec{X}}\cdot\vec{f}_\mp(c))\pm\frac{\vec{\nabla}_{\vec{X}}^2}{4ma}(\vec{f}_+(c)+\vec{f}_-(c))\\&\quad\pm\frac{\vec{\nabla}_{\vec{X}}}{2ma}\times\mathcal{C}(2i\vec{f}_E)-\frac{\vec{p}}{ma}\times\mathcal{C}(i\vec{f}_V)\mp\frac{\vec{p}}{ma}\mathcal{C}(f_P)+\frac{\vec{\nabla}_{\vec{X}}}{2ma}\mathcal{C}(if_A^0).\end{aligned} \tag{6.106}$$

We see again a wide range of corrections in terms of $f_\pm$ and $\vec{f}_\pm$ appearing, and we have some familiar terms in the square brackets of the last equality, so we can use (6.101)-(6.104) again to iterate. For the cosine terms in (6.105) we find

$$\begin{aligned}\vec{\nabla}_{\vec{X}} \times \big(f(\vec{X})\cos(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})\vec{f}(\vec{X},\vec{p})\big) =& f(\vec{X})\cos(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})[\vec{\nabla}_{\vec{X}}\times\vec{f}(\vec{X},\vec{p})] \\ &+\vec{\nabla}_{\vec{X}}[f(\vec{X})\cos(\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}})]\times\vec{f}(\vec{X},\vec{p}),\end{aligned} \tag{6.107}$$

$$\begin{aligned}\implies \vec{\nabla}_{\vec{X}}\times\mathcal{C}(i\vec{f}_V) =& \mathcal{C}(\vec{\nabla}_{\vec{X}}\times i\vec{f}_V)+\mathcal{C}(\overleftarrow{\partial}_{\vec{X}}\times i\vec{f}_V), \\ \vec{p}\times\mathcal{C}(2i\vec{f}_E) =& \mathcal{C}(\vec{p}\times 2i\vec{f}_E)+\mathcal{S}(\overleftarrow{\partial}_{\vec{X}}\times 2i\vec{f}_E),\end{aligned} \tag{6.108}$$

$$\vec{p}\,\mathcal{C}(if^0_A) = \mathcal{C}(\vec{p}if^0_A)+\mathcal{S}(\overleftarrow{\partial}_{\vec{X}}if^0_A), \tag{6.109}$$

$$\vec{\nabla}_{\vec{X}}\mathcal{C}(f_P) = \mathcal{C}(\vec{\nabla}_{\vec{X}}f_P)+\mathcal{C}(\overleftarrow{\partial}_{\vec{X}}f_P), \tag{6.110}$$

and we get

$$\begin{aligned}&-\frac{1}{2ma}\Big[\pm\frac{\vec{\nabla}_{\vec{X}}}{2}\times\mathcal{C}(i\vec{f}_V)+\vec{p}\times\mathcal{C}(2i\vec{f}_E)\pm\vec{p}\,\mathcal{C}(if^0_A)+\frac{\vec{\nabla}_{\vec{X}}}{2}\mathcal{C}(f_P)\Big]= \\ &\qquad -\frac{1}{2ma}\Big[\mathcal{C}\Big(\pm\frac{\vec{\nabla}_{\vec{X}}}{2}\times(i\vec{f}_V(c))+\vec{p}\times(2i\vec{f}_E(c))\pm\vec{p}if^0_A(c)+\frac{\vec{\nabla}_{\vec{X}}}{2}f_P(c)\Big) \\ &\qquad\qquad +\mathcal{C}\Big(\pm\overleftarrow{\partial}_{\vec{X}}\times i\vec{f}_V+\overleftarrow{\partial}_{\vec{X}}f_P\Big)+\mathcal{S}\Big(\overleftarrow{\partial}_{\vec{X}}\times 2i\vec{f}_E\pm\overleftarrow{\partial}_{\vec{X}}if^0_A\Big)\Big].\end{aligned} \tag{6.111}$$

The second line can again be iteratively evaluated by (6.101)-(6.104), while the last line is, at best, $\Phi_N/(ma)$-suppressed, like in the previous subsection.

We see that the structure of iteration also applies to $\vec{f}_\pm$, and with it we have constructed (up to a factor $1/2$ on some terms) the classical Vlasov equations with higher order corrections. The equations (6.88) and (6.105) are the fundamental results of this thesis, so we will write them now again, gradient expanding to see their explicit form,

$$\begin{aligned}\Big[\pm\partial_\eta+\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{2ma}-\frac{ma}{2}\vec{\nabla}_{\vec{X}}\big(\Phi_N(\eta,\vec{X})\big)\cdot\partial_{\vec{p}}\Big]f_\pm(c) =& \mathrm{Corr}_f(f_\pm,\vec{f}_\pm) \\ &+\mathcal{O}\left(\Big(\frac{\partial_\eta}{ma}\Big)^2,\Big(\frac{|\vec{\nabla}_{\vec{X}}|}{ma}\Big)^2,\Big(\frac{|\vec{p}|}{ma}\Big)^2,|\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}|^2\right),\end{aligned} \tag{6.112}$$

$$\begin{aligned}\Big[\pm\partial_\eta+\frac{\vec{p}\cdot\vec{\nabla}_{\vec{X}}}{2ma}-\frac{ma}{2}\vec{\nabla}_{\vec{X}}\big(\Phi_N(\eta,\vec{X})\big)\cdot\partial_{\vec{p}}\Big]\vec{f}_\pm(c) =& \mathrm{Corr}_{\vec{f}}(f_\pm,\vec{f}_\pm) \\ &+\mathcal{O}\left(\Big(\frac{\partial_\eta}{ma}\Big)^2,\Big(\frac{|\vec{\nabla}_{\vec{X}}|}{ma}\Big)^2,\Big(\frac{|\vec{p}|}{ma}\Big)^2,|\overleftarrow{\partial}_{\vec{X}}\cdot\partial_{\vec{p}}|^2\right).\end{aligned} \tag{6.113}$$

Where $\mathrm{Corr}(f_\pm,\vec{f}_\pm)$ are the corrections obtained by iterations in (6.87), (6.92) for $f_\pm$, and (6.101)-(6.104), (6.106) for $\vec{f}_\pm$.

We have shown that by taking an action with an effective four-point interaction, we could construct a 2PI effective action truncated at two loops. From it, we could construct the equations of motion for the Hadamard two-point functions, showing us the effective dynamics of fermions influenced by gravity. This two-point function allowed us to derive the equation of motion for phase-space densities corresponding to the classical particle limit, with centrally the scalar densities $f_\pm$, and the spin densities $\vec{f}_\pm$, with plus and minus indicating particles and antiparticles. We find that these densities couple through their higher order corrections, but in the classical theory, their equations separate.

7 Conclusions and discussion

In this thesis, a new formalism was derived for the classical particle limit of a field-theoretic action of minimally coupled Dirac fermions. This formalism explicitly allows for nonequilibrium fermionic systems, such as far from thermal equilibrium fermions or free Fermi liquid theory. To our knowledge, this is the first time such an approach is taken to fermionic dark matter. We started with the simplest possible action in curved spacetime, consisting of a Dirac action (3.20), in which the fermion fields had a minimal coupling to gravity, and an Einstein-Hilbert action (4.39). By conformally rescaling the action, and expanding around an FLRW-background, we were able to perturbatively describe the effects of gravity on fermionic density pertubations of a homogeneous background. We then chose to describe these fermionic perturbations in the longitudinal gauge, where vector and tensor perturbations are considered negligible. This was possible because we linearized in gravity in (5.25), and the vector and tensor perturbations are only dynamically generated nonlinearly. This linearization was justified by the smallness of the potentials of the longitudinal gauge. We found these potentials to be nondynamical, so we were able to integrate these out to obtain an effective four-point interaction, as seen in (6.6). From the resulting action, we could derive the 2PI effective action truncated at two loops (6.24), in which the central objects were the Hadamard two-point functions. We found no dissipative corrections of the self-energy of the two-point functions, so we have gotten closure for their dynamics, similarly found for scalars in [5]. The two-point functions are necessary to derive a classical particle limit for fermions, as they do not allow for a classical field description. By finding the equations of motion for the two-point functions and assuming a nonrelativistic limit, we could derive the equations of motion for the scalar (6.112) and spin (6.113) densities, which classically coincide with the known Vlasov-Poisson system, while also keeping relativistic and gradient corrections to this description. These corrections result from an iterative process through which a large amount of them can be created, we have therefore also determined a bound on the validity of the corrections. We have found that these equations also allow for separate particle and antiparticle descriptions.

We have constructed a highly-constrained model, where we found a classical limit with corrections to it. We have obtained this limit from a fundamental Lagrangian description, which automatically generates these corrections. To identify these corrections, we had to make certain physically justified assumptions for the particles and its background. To summarize, we assumed a heavy mass (6.2) and nonrelativistic (6.95) limit, together with disallowing multiple flavors of fermions. This was then justified by seeing that structure formation requires nonrelativistic particle velocities, and by the exclusion principle, the particle mass is bounded below due to the limit on the amount of nonrelativistic particles in a volume. This in turn gave us cause to assume that the de Broglie wavelength of these particles is much smaller than the Hubble radius and also the scales of large-scale structure formation, which we are interested in. Corresponding to these assumptions, we could take the sub-Hubble limit (5.47) and make a gradient expansion (6.80). The nonrelativistic velocity corresponding to structure formation is the escape velocity, which in terms of $v_{\rm esc}^2/c^2$ is proportional to the gravitational potentials. This has shown us that a weak field limit could be assumed in (5.26). Lastly, we have assumed a mass much large than the Hubble rate, which is the limit of adiabatic expansion. While all these assumptions are very constraining, they have resulted in the derivation of the classical Vlasov-Poisson system, while also giving a scale to the corrections we have found (6.96).

For the nonlocal action with effective interactions (5.56), we found corrections E_1 and E_0' which were an order $(v_{\rm sound}/c)^2$ smaller than the leading-order effective interaction. This speed of sound in the dark matter fluid is of the order of particle velocities, which is constrained by the escape velocity of the structures. We therefore found these corrections to be at least one order in gravitational potential smaller than the leading order contribution, and disregarded these, although they should be included when higher-order corrections in gravitational potential are investigated.

We found that the derivations of this thesis proceeded in a similar way as for the scalars in [5], but we have also found consequences of taking fermions as the particles of interest. We have for example excluded the Fock terms in (6.40), but found that unlike for the scalar, it has to be investigated further whether this is allowed. Due to time constraints, this was not possible here. We have also found that unlike scalars, Dirac fermions can describe a particle/antiparticle description, leading us to separate equations for the particle and antiparticle solutions. By assuming a large energy gap created by a chemical potential, one can discard either contribution as being negligible to the other, finding the classical description of fermions in gravity. Moreover, due to fermions having spin, we have found an additional equation for the spin (or vector) densities (6.113), which to our knowledge describes densities of fermions having spin with respect to some coordinate basis. Because this basis is not fixed in curved spacetime, this leads together with the scalar density (6.112) to four degrees of freedom, two more than usually found in the classical description, where a simple 'spin up', 'spin down' split is assumed. Again, due to time constraints, we were not able to fully investigate the meaning of the vector density, but leave it to other works to investigate this interesting phenomenon. Another consequence of the fermion theory is allowing for particle/antiparticle mixing densities, which also have scalar (6.69) and vector (6.70) components. As their equations are similar to those for the particle and antiparticle densities, it seems we could similarly derive Vlasov-Poisson-like equations for the mixing densities. We have not investigated this here, and instead chose to focus on the particle densities. The possibility for helicity mixing is also present, but was also not investigated. We have noted that, because standard model neutrinos are Dirac fermions, they are also described by this theory, but make up a subdominant contribution to dark matter.

Unfortunately, we have found some discrepancies in our derivations, which seem to be the result of calculational error. In chapter 5.2, we have found a graviton mass proportional to the dark matter pressure, which is highly unlikely to be correct for a gauge boson mediating a long-range force. This result was fortunately of higher order in gravitational potentials than we were interested in, so it did not affect our further calculations. Another deviation from what we expected to find in the classical theory was a factor 2 difference in the time derivative of the Vlasov equation. A simple physical explanation for this was not found, and it seems again a consequence of calculational error. While much effort was made to resolve these issues, they still persist.

As described, we have only considered the most simple Lagrangian of fermions minimally coupling to gravity. This was mainly because this is the first time this approach was derived, but extensions to the theory we have found here can now also be included very easily. As one starts with this fundamental Lagrangian description, adding more terms to the Lagrangian will allow for richer, yet more complicated theories. One could think of including for example dark photons, which couple (in a similar way to electromagnetism) to the fermions via the covariant derivative. We have also mentioned the possibility of adding a scalar field, which could find use in generating the fermion mass via a Yukawa-like term. Of course, Majorana fermions could also be considered, but the starting Lagrangian (3.1) should be changed to allow for these. Perhaps it is possible to find a symmetry transformation between the particle and antiparticle densities to get to the Majorana description while keeping the results of this thesis, but this should also be investigated.

A Gamma matrix identities

Unless a gamma matrix representation is chosen at the start, a theory containing fermions will be littered with gamma matrices. This thesis is no exception, so we will list a number of useful identities. Some identities follow straightforwardly from the defining identity (2.10), others contain lengthy proofs, which can e.g. be found in [39]. All identities are representation independent. In this appendix, we assume flat spacetime.

A.1 commutation identities

$$\{\gamma^\rho, I\} = 2\gamma^\rho, \qquad [\gamma^\rho, I] = 0, \tag{A.1}$$

$$\{\gamma^\rho, \gamma^5\} = 0, \qquad [\gamma^\rho, \gamma^5] = 2\gamma^\rho\gamma^5, \tag{A.2}$$

$$\{\gamma^\rho, \gamma^\mu\} = -2\eta^{\rho\mu}, \qquad [\gamma^\rho, \gamma^\mu] = 4i\sigma^{\mu\rho}, \tag{A.3}$$

$$\{\gamma^\rho, \gamma^\mu\gamma^5\} = -4i\gamma^5\sigma^{\rho\mu} = 2\epsilon^{\lambda\tau\rho\mu}\sigma_{\lambda\tau}, \qquad [\gamma^\rho, \gamma^\mu\gamma^5] = -2\eta^{\rho\mu}\gamma^5, \tag{A.4}$$

$$\{\gamma^\rho, \sigma^{\mu\nu}\} = -\epsilon^{\lambda\rho\mu\nu}\gamma_\lambda\gamma^5, \qquad [\gamma^\rho, \sigma^{\mu\nu}] = 2i\eta^{\rho[\nu}\gamma^{\mu]}. \tag{A.5}$$

A.2 Trace identities

$$\begin{gathered}\mathrm{Tr}[I_4] = 4, \quad \mathrm{Tr}[\gamma^5] = 0, \quad \mathrm{Tr}[\gamma^\mu] = 0, \quad \mathrm{Tr}[\gamma^5\gamma^\mu] = 0, \quad \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0, \quad \mathrm{Tr}[\gamma^5\gamma^\rho\gamma^\mu\gamma^\nu] = 0, \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4\eta^{\mu\nu}, \quad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0, \quad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}).\end{gathered} \tag{A.6}$$

A.3 Identities to reduce to elements of Γ

As any 4×4 matrix can be decomposed in terms of elements of Γ, we can reduce a product of any number of gamma matrices to (a linear combination of) elements of this set. We note the following identities:

$$\begin{gathered}\gamma^\mu\gamma^\nu = -2i\sigma^{\mu\nu} - \eta^{\mu\nu}, \qquad \gamma^\rho\gamma^\mu\gamma^\nu = -\eta^{\rho\mu}\gamma^\nu - \eta^{\mu\nu}\gamma^\rho + \eta^{\rho\nu}\gamma^\mu + i\epsilon^{\lambda\rho\mu\nu}\gamma_\lambda\gamma^5, \\ \sigma^{\lambda\tau}\gamma^5 = \frac{i}{2}\epsilon^{\mu\nu\lambda\tau}\sigma_{\mu\nu}, \qquad \sigma^{\mu\nu}\gamma^\rho = \frac{i}{2}(2\eta^{\rho[\mu}\gamma^{\nu]} + i\epsilon^{\lambda\rho\mu\nu}\gamma_\lambda\gamma^5), \qquad \gamma^\rho\sigma^{\mu\nu} = \frac{i}{2}(2\eta^{\rho[\nu}\gamma^{\mu]} + i\epsilon^{\lambda\rho\mu\nu}\gamma_\lambda\gamma^5).\end{gathered} \tag{A.7}$$

A.4 Gamma matrix representations

Noting the definition of the Pauli matrices, and their commutation relation ($\epsilon^{123} = 1$),

$$\sigma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k, \tag{A.8}$$

we have for the different representations,

Dirac
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

Weyl
$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

Majorana
$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}.$$

B Decomposition of Spinor Propagator

To get a better grasp on the versatility of the propagator $\langle\psi_\alpha\bar{\psi}_\beta\rangle$, this appendix aims to decompose it into the elements of Γ as follows:

$$\langle\psi_\alpha\bar{\psi}_\beta\rangle = f\delta_{\alpha\beta} + h\gamma^5_{\alpha\beta} + l_a\gamma^a_{\alpha\beta} + m_a(\gamma^a\gamma^5)_{\alpha\beta} + n_{ab}\sigma^{ab}_{\alpha\beta}. \tag{B.1}$$

This decomposition is representation independent. The objects f, h, l_a, m_a, n_{ab} are determined by combining the propagator with elements of Γ and taking the trace (i.e. summing over α). Using the identities in appendix A, together with cyclicity of the trace ($\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB]$), allow us to set a great number of terms to 0, so we can find the terms in the decomposition.

$$\begin{aligned}\langle\psi_\alpha\bar{\psi}_\alpha\rangle =& f\delta_{\alpha\alpha} + h\gamma^5_{\alpha\alpha} + l_a\gamma^a_\alpha + m_a(\gamma^a\gamma^5)_{\alpha\alpha} + n_{ab}\sigma^{ab}_{\alpha\alpha}\\ =&4f,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\langle\psi_\alpha(\bar{\psi}\gamma^5)_\alpha\rangle =& f\gamma^5_{\alpha\alpha} + h\delta_{\alpha\alpha} + l_a(\gamma^a\gamma^5)_{\alpha\alpha} + m_a(\gamma^a)_{\alpha\alpha} + n_{ab}(\gamma^5\sigma^{ab})_{\alpha\alpha}\\ =&4h,\end{aligned} \tag{B.3}$$

$$\begin{aligned}\langle\psi_\alpha(\bar{\psi}\gamma^c)_\alpha\rangle =& f\gamma^c_{\alpha\alpha} + h(\gamma^c\gamma^5)_{\alpha\alpha} + l_a(\gamma^c\gamma^a)_{\alpha\alpha} + m_a(\gamma^c\gamma^a\gamma^5)_{\alpha\alpha} + n_{ab}(\gamma^c\sigma^{ab})_{\alpha\alpha}\\ =&-4l^c,\end{aligned} \tag{B.4}$$

$$\begin{aligned}\langle\psi_\alpha(\bar{\psi}\gamma^c\gamma^5)_\alpha\rangle =& f(\gamma^c\gamma^5)_{\alpha\alpha} + h(\gamma^c)_{\alpha\alpha} + l_a(\gamma^c\gamma^5\gamma^a)_{\alpha\alpha} + m_a(\gamma^c\gamma^5\gamma^a\gamma^5)_{\alpha\alpha} + n_{ab}(\gamma^c\gamma^5\sigma^{ab})_{\alpha\alpha}\\ =&4m^c,\end{aligned} \tag{B.5}$$

$$\begin{aligned}\langle\psi_\alpha(\bar{\psi}\sigma^{cd})_\alpha\rangle =& \frac{i}{4}f((\gamma^c\gamma^d)_{\alpha\alpha} - (\gamma^d\gamma^c)_{\alpha\alpha}) + \frac{i}{4}h((\gamma^c\gamma^d\gamma^5)_{\alpha\alpha} - (\gamma^d\gamma^c\gamma^5)_{\alpha\alpha})\\ &+ \frac{i}{4}l_a((\gamma^c\gamma^d\gamma^a)_{\alpha\alpha} - (\gamma^d\gamma^c\gamma^a)_{\alpha\alpha}) + \frac{i}{4}m_a((\gamma^c\gamma^d\gamma^a\gamma^5)_{\alpha\alpha} - (\gamma^d\gamma^c\gamma^a\gamma^5)_{\alpha\alpha})\\ &- \frac{1}{16}n_{ab}(\gamma^c\gamma^d\gamma^a\gamma^b - \gamma^d\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^d\gamma^b\gamma^a + \gamma^d\gamma^c\gamma^b\gamma^a)_{\alpha\alpha}\\ =&-\frac{1}{4}n_{ab}(\eta^{cd}\eta^{ab} - \eta^{ca}\eta^{db} + \eta^{cb}\eta^{da} - \eta^{dc}\eta^{ab} + \eta^{da}\eta^{cb} - \eta^{db}\eta^{ca}\\ &- \eta^{cd}\eta^{ba} + \eta^{cb}\eta^{da} - \eta^{ca}\eta^{db} + \eta^{dc}\eta^{ba} - \eta^{db}\eta^{ca} + \eta^{da}\eta^{cb})\\ =&(n^{cd} - n^{dc}) = 2n^{[cd]}.\end{aligned} \tag{B.6}$$

In the second part of (B.6), we note that due to symmetry of the Minkowski metric ($\eta^{ba} = \eta^{ab}$), some terms drop out, while others add, simplifying the expression greatly. In the last part, we observe that the commutator forces n^{cd} to be antisymmetric, so we get a sign change on exchange of indices. Lowering indices, we now see that the objects are

$$\begin{aligned}&f = \frac{1}{4}\langle\psi_\alpha\bar{\psi}_\alpha\rangle, \quad h = \frac{1}{4}\langle\psi_\alpha(\bar{\psi}\gamma^5)_\alpha\rangle, \quad l_a = -\frac{1}{4}\langle\psi_\alpha(\bar{\psi}\gamma_a)_\alpha\rangle,\\ &m_a = \frac{1}{4}\langle\psi_\alpha(\bar{\psi}\gamma_a\gamma^5)_\alpha\rangle, \quad n_{[ab]} = \frac{1}{2}\langle\psi_\alpha(\bar{\psi}\sigma_{ab})_\alpha\rangle.\end{aligned} \tag{B.7}$$

The full decomposition for our propagator in terms of elements of Γ is then

$$\begin{aligned}\langle\psi_\alpha\bar{\psi}_\beta\rangle =& \frac{1}{4}\langle\psi_\delta\bar{\psi}_\delta\rangle\delta_{\alpha\beta} + \frac{1}{4}\langle\psi_\delta(\bar{\psi}\gamma^5)_\delta\rangle\gamma^5_{\alpha\beta} - \frac{1}{4}\langle\psi_\delta(\bar{\psi}\gamma_a)_\delta\rangle\gamma^a_{\alpha\beta}\\ &+ \frac{1}{4}\langle\psi_\delta(\bar{\psi}\gamma_a\gamma^5)_\delta\rangle(\gamma^a\gamma^5)_{\alpha\beta} + \frac{1}{2}\langle\psi_\delta(\bar{\psi}\sigma_{ab})_\delta\rangle\sigma^{ab}_{\alpha\beta}.\end{aligned} \tag{B.8}$$

C Checks for the matter action

This appendix serves as a check for the results of chapter 3: we first check the matter action (3.20), then we will check the energy-momentum tensor (3.29).

C.1 Checking the Dirac action

A check we can use is in [40], where a computer program was used to calculate the terms in the action. However, the further simplification to the Levi-Civita symbol was not made, so we write (3.23) without simplifying the product of three gamma matrices. We have the following combination

$$\begin{aligned}\frac{1}{2}h^\sigma_c\partial_b h_{\sigma d}\{\gamma^b,\sigma^{cd}\} =&\frac{1}{4}(h^\sigma_c\partial_b h_{\sigma d}-h^\sigma_d\partial_b h_{\sigma c})\{\gamma^b,\sigma^{cd}\}\\ =&\frac{i}{4}(h^\sigma_c\partial_b h_{\sigma d}-h^\sigma_d\partial_b h_{\sigma c})\gamma^b\gamma^c\gamma^d+\frac{i}{4}(h^\sigma_c\partial_b h_{\sigma d}-h^\sigma_d\partial_b h_{\sigma c})(2\gamma^d\eta^{bc})\\ =&\frac{i}{4}(h^\sigma_c\partial_b h_{\sigma d}-h^\sigma_d\partial_b h_{\sigma c})\gamma^b\gamma^c\gamma^d+\frac{i}{2}(h^{\sigma b}\partial_b h_{\sigma d}-h_{\sigma d}\partial_b h^{\sigma b})\gamma^d.\end{aligned} \tag{C.1}$$

In the first line, we again wrote the antisymmetry in the h terms. The second line is obtained by rearranging gamma matrices, where we use the antisymmetry in cd to set a term to 0, and adding others. We now find

$$\begin{aligned}S^{(2,2)}[\chi,\bar\chi] =&\frac{\kappa^2}{8}\int d\eta\, d^3x\big((h^2-2h^{\lambda\tau}h_{\lambda\tau})\mathcal{L}_0+(3h^{\mu\rho}h^\nu_\rho-2hh^{\mu\nu})\mathcal{L}_{\mu\nu}\\ &+\frac{i}{4}(h^{\mu\rho}\partial^\sigma h^\nu_\rho-h^{\nu\rho}\partial^\sigma h^\mu_\rho)\bar\chi\gamma_\sigma\gamma_\mu\gamma_\nu\chi+\frac{i}{2}(h^{\rho\sigma}\partial_\sigma h^\mu_\rho-h^{\mu\rho}\partial_\sigma h^\sigma_\rho)\bar\chi\gamma_\mu\chi,\end{aligned} \tag{C.2}$$

exactly as in [40].

C.2 Checking the energy-momentum tensor

All expansions are up to linear order in $h_{\mu\nu}$

We can do a check for the energy-momentum tensor by expanding the one given in [41], which is directly calculated from (3.1) (ignoring the $(\bar\psi\psi)^2$ term mentioned in that paper, and noting a difference in minus-sign due to the metric signature),

$$T_{\mu\nu}=\Big[\frac{i}{2}\left(\bar\psi\gamma^\rho D_\rho\psi-(D_\rho\bar\psi)\gamma^\rho\psi\right)-m\bar\psi\psi\Big]g_{\mu\nu}-\frac{i}{2}\left(\bar\psi\gamma_{(\nu}D_{\mu)}\psi-(D_{(\mu}\bar\psi)\gamma_{\nu)}\psi\right). \tag{C.3}$$

Again, we rescale the fields and use (3.5) and (3.13), but not before noting that in the square brackets we simply have the Lagrangian density leading to (3.11), which in turn yields

$$\begin{aligned}\frac{i}{2}\left(\bar\psi\gamma^\rho D_\rho\psi-(D_\rho\bar\psi)\gamma^\rho\psi\right)-m\bar\psi\psi =&a^{-4}\Big[\frac{i}{2}\Big(\bar\chi\tilde e^\mu_b\gamma^b\tilde D_\mu\chi-(\tilde D_\mu\bar\chi)\tilde e^\mu_b\gamma^b\chi\Big)-ma\bar\chi\chi\Big]\\ =&a^{-4}\Big[\bar\chi(\delta^\mu_b-\frac{\kappa}{2}h^\mu_b)\gamma^b\partial_\mu\chi-\partial_\mu\bar\chi(\delta^\mu_b-\frac{\kappa}{2}h^\mu_b)\gamma^b\chi-ma\bar\chi\chi\Big]\\ =&a^{-4}\Big[\mathcal{L}_0-\frac{\kappa}{2}h^{\mu\nu}\mathcal{L}_{\mu\nu}\Big].\end{aligned} \tag{C.4}$$

Before we evaluate the other part of (C.3), we look at

$$\begin{aligned}e^b_{(\nu}\omega_{\mu)cd}\{\gamma_b,\sigma^{cd}\} =&\tilde e^b_{(\nu}\tilde e_{\mu)c}\tilde e^\rho_d(\partial_\rho a)\{\gamma_b,\sigma^{cd}\}+a\tilde e^b_{(\nu}\tilde\omega_{\mu)cd}\{\gamma_b,\sigma^{cd}\}\\ =&(\delta^b_{(\nu}\eta_{\mu)c}\delta^\rho_d+\frac{\kappa}{2}[h^b_{(\nu}\eta_{\mu)c}\delta^\rho_d+\delta^b_{(\nu}h_{\mu)c}\delta^\rho_d-\delta^b_{(\nu}\eta_{\mu)c}h^\rho_d])(\partial_\rho a)\{\gamma_b,\sigma^{cd}\}+a\tilde e^b_{(\nu}\tilde\omega_{\mu)cd}\{\gamma_b,\sigma^{cd}\}\\ =&a\tilde e^b_{(\nu}\tilde\omega_{\mu)cd}\{\gamma_b,\sigma^{cd}\}.\end{aligned} \tag{C.5}$$

In the second line, we see a symmetry in the bc indices, which was shown to give 0 when multiplied with $\{\gamma^b, \sigma^{cd}\}$. We then evaluate what is left of (C.3)

$$\begin{aligned}
-\frac{i}{2}\left(\bar{\psi}\gamma_{(\nu}D_{\mu)}\psi - (D_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right) =& -\frac{i}{2}\left(\bar{\psi}\gamma_b e^b_{(\nu}\partial_{\mu)}\psi - (\partial_{(\mu}\bar{\psi})e^b_{\nu)}\gamma_b\psi\right) - \frac{1}{4}\left(\bar{\psi}e^b_{(\nu}\omega_{\mu)cd}\{\gamma_b, \sigma^{cd}\}\psi\right) \\
=& -a^{-2}\Big[\frac{i}{2}\left(\bar{\chi}\gamma_b\tilde{e}^b_{(\nu}\partial_{\mu)}\chi - (\partial_{(\mu}\bar{\chi})\tilde{e}^b_{\nu)}\gamma_b\chi\right) - \frac{1}{4}\left(\bar{\chi}\tilde{e}_{b(\nu}\tilde{\omega}_{\mu)cd}\{\gamma^b, \sigma^{cd}\}\chi\right)\Big] \\
=& -a^{-2}\Big[\mathcal{L}_{(\mu\nu)} + \frac{\kappa}{2}h^{\tau}_{(\mu}\delta^{\sigma}_{\nu)}\mathcal{L}_{\tau\sigma} - \frac{\kappa}{8}(\partial_d h_{c(\mu}\eta_{\nu)b} - \partial_c h_{d(\mu}\eta_{\nu)b})\bar{\chi}\{\gamma^b, \sigma^{cd}\}\chi\Big].
\end{aligned} \tag{C.6}$$

Substituting both contributions into (C.3) finally gives (moving a^2 to the other side of the equation)

$$\begin{aligned}
a^2 T_{\mu\nu} =& \left(\mathcal{L}_0 - \frac{\kappa}{2}h^{\mu\nu}\mathcal{L}_{\mu\nu}\right)(\eta_{\mu\nu} + \kappa h_{\mu\nu}) - \mathcal{L}_{(\mu\nu)} - \frac{\kappa}{2}h^{\tau}_{(\mu}\delta^{\sigma}_{\nu)}\mathcal{L}_{\tau\sigma} + \frac{\kappa}{8}(\partial_d h_{c(\mu}\eta_{\nu)b} - \partial_c h_{d(\mu}\eta_{\nu)b})\bar{\chi}\{\gamma^b, \sigma^{cd}\}\chi \\
=& \eta_{\mu\nu}\mathcal{L}_0 - \mathcal{L}_{(\mu\nu)} + \kappa\big[h_{\mu\nu}\mathcal{L}_0 - \frac{1}{2}\eta_{\mu\nu}h^{\tau\sigma}\mathcal{L}_{\tau\sigma} - \frac{1}{2}h^{\tau}_{(\mu}\delta^{\sigma}_{\nu)}\mathcal{L}_{(\tau\sigma)} + \frac{1}{4}\partial_\sigma h_{\phi(\mu}\eta_{\nu)\omega}\bar{\chi}\epsilon^{\tau\sigma\phi\omega}\gamma_\tau\gamma^5\chi\big] \\
& + \mathcal{O}((h_{\mu\nu})^2).
\end{aligned} \tag{C.7}$$

where we used the total antisymmetry of the Levi-Civita symbol to switch some indices. This result is precisely what we found in (3.29).

D Checks for the Einstein-Hilbert action

In this appendix, a series of checks with relevant literature is conducted to verify the correct form of the Einstein-Hilbert action we have found in chapter 4.

D.1 Checking the action

We have not expanded the cosmological constant term up to now mostly to keep things more clean, but also because we can first look at some cases where it gets related to the Hubble parameter. From the Friedmann equations, we have (assume curvature $k = 0$)

$$H^2 = a^{-2}\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = \frac{\kappa^2}{6}\rho + \frac{\Lambda}{3}, \tag{D.1}$$

$$\dot{H} = a^{-2}(\mathcal{H}' - \mathcal{H}^2) = -4\pi G(\rho + p) = -\frac{\kappa^2}{4}(\rho + p). \tag{D.2}$$

Here, p is the pressure, and ρ is the density of a fluid. As a test case, we can take an empty universe such that $p = \rho = 0$, so $3\mathcal{H}^2 = a^2\Lambda = 3\mathcal{H}'$. The term in (4.38) with the cosmological constant now becomes

$$a^4\frac{2}{\kappa^2}\sqrt{-\tilde{g}}\Lambda = \frac{6}{\kappa^2}a^2\sqrt{-\tilde{g}}\mathcal{H}^2 = a^2\left(\frac{6}{\kappa^2} + \frac{3h}{\kappa} + \frac{3h^2}{4} - \frac{3}{2}h^{\mu\nu}h_{\mu\nu}\right)\mathcal{H}^2 + \mathcal{O}(\kappa). \tag{D.3}$$

and we have for the action

$$\begin{aligned} S_g = \int d\eta\, d^3x \,\Big(&-\frac{12}{\kappa^2}a^2\mathcal{H}^2 + a^2\Big[-\frac{1}{2}(\partial_\mu h)(\partial^\nu h^\mu_\nu) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) \\ &+ \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{\sigma\rho}\partial_\sigma h(\partial_\rho \text{ln}a)\Big]\Big) + \mathcal{O}(\kappa). \end{aligned} \tag{D.4}$$

The linear part drops out entirely, and we are only left with some term that will not affect the equations of motion, and a quadratic part that precisely agrees with [42].

D.1.1 Deriving the Lichnerowicz operator

The quadratic part from (D.4) can be written as (symmetrizing the first term)

$$\begin{aligned} S^{(4)}_{quad}[h_{\mu\nu}] = &\int d\eta d^3x\, a^2\Big(-\frac{1}{4}(\partial_\mu h)(\partial_\nu h^{\mu\nu}) - \frac{1}{4}(\partial_\nu h)(\partial_\mu h^{\mu\nu}) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) - \frac{1}{4}(\partial_\mu h^{\nu\sigma})(\partial^\mu h_{\nu\sigma}) \\ &+ \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\sigma h^\mu_\nu) - h^{\sigma\rho}\partial_\sigma h(\partial_\rho \text{ln}a)\Big) \\ =&\frac{1}{2}\int d\eta d^3x\,\Big(\frac{1}{2}a^2\Big[h\partial_\mu\partial_\nu h^{\mu\nu} + h^{\mu\nu}\partial_\mu\partial_\nu h - h\partial_\mu\partial^\mu h + h^{\nu\sigma}\partial_\mu\partial^\mu h_{\nu\sigma} - 2h^{\sigma\nu}\partial_\mu\partial_\sigma h^\mu_\nu\Big] \\ &+ a\Big[h\partial_\nu h^{\mu\nu} + h^{\mu\nu}\partial_\nu h - \eta^{\mu\nu}h\partial_\nu h + \eta^{\mu\nu}h^{\nu\sigma}\partial_\nu h_{\nu\sigma} - 2h^{\sigma\nu}\partial_\sigma h^\mu_\nu - 2h^{\mu\nu}\partial_\nu h\Big](\partial_\mu a)\Big) \\ =&\frac{1}{2}\int d\eta d^3x\,\Big(\frac{1}{2}a^2\Big[h\partial^\mu\partial^\nu h_{\mu\nu} + h_{\mu\nu}\partial^\mu\partial^\nu h - h\partial^2 h + h^{\nu\sigma}\partial^2 h_{\nu\sigma} - 2h^\nu_\sigma\partial^\mu\partial^\sigma h_{\nu\mu}\Big] \\ &+ a\Big[2\eta^{\rho\mu}h\partial^\nu h_{\rho\nu} - \eta^{\mu\nu}h\partial_\nu h + \eta^{\mu\rho}h^{\nu\sigma}\partial_\rho h_{\nu\sigma} - 2\eta^{\rho\mu}\eta^{\lambda\nu}h_{\sigma\nu}\partial^\sigma h_{\rho\lambda}\Big](\partial_\mu a) \\ &+ h^{\mu\nu}h(\partial_\nu a)(\partial_\mu a) + ah^{\mu\nu}h\partial_\mu\partial_\nu a\Big). \end{aligned} \tag{D.5}$$

We obviously applied Leibniz in the second equality, and wrote $a^2\partial_\rho \text{ln}a = a\partial_\rho a$. To be able to add the first, second and last term in the fourth line, we again applied Leibniz, yielding the terms in the last line. Boundary terms were dropped. We can now again use the fact that the scale factor only depends on (conformal) time, and that $\dot{H} = 0$. We write regular Hubble constants this time.

$$
\begin{aligned}
S^{(4)}_{quad}[h_{\mu\nu}] =& \frac{1}{2}\int d\eta d^3x \, \Big(\frac{1}{2}a^2\Big[h\partial^\mu\partial^\nu h_{\mu\nu} + h_{\mu\nu}\partial^\mu\partial^\nu h - h\partial^2 h + h^{\nu\sigma}\partial^2 h_{\nu\sigma} - 2h^\nu_\sigma\partial^\mu\partial^\sigma h_{\nu\mu}\Big] \\
&+ a^3\Big[-2h\partial^\nu h_{0\nu} + h\partial_0 h - h^{\nu\sigma}\partial_0 h_{\nu\sigma} + 2\eta^{\lambda\nu}h_{\sigma\nu}\partial^\sigma h_{0\lambda}\Big]H \\
&+ h^{00}ha^4H^2 + a^4h^{00}h(\dot{H} + 2H^2)\Big) \\
=& \frac{1}{2}\int d\eta d^3x \, \Big(\frac{1}{2}a^2\Big[h_{\mu\nu}\eta^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} + h_{\mu\nu}\eta^{\rho\sigma}\partial^\mu\partial^\nu h_{\rho\sigma} - h_{\mu\nu}\eta^{\mu\nu}\eta^{\rho\sigma}\partial^2 h_{\rho\sigma} + h_{\mu\nu}\eta^{\mu\rho}\eta^{\sigma\nu}\partial^2 h_{\rho\sigma} \\
&- 2h_{\mu\nu}\eta^{\nu\rho}\partial^\sigma\partial^\mu h_{\rho\sigma}\Big] + Ha^3\Big[-2h_{\mu\nu}\eta^{\mu\nu}\delta^\rho_0\partial^\sigma h_{\rho\sigma} + h_{\mu\nu}\eta^{\rho\sigma}\partial_0 h_{\rho\sigma} - h_{\mu\nu}\eta^{\mu\rho}\eta^{\sigma\nu}\partial_0 h_{\rho\sigma} \\
&+ 2h_{\mu\nu}\eta^{\sigma\mu}\delta^\rho_0\partial^\nu h_{\rho\sigma}\Big] + 3H^2a^4h_{\mu\nu}\eta^{\mu\nu}\delta^\rho_0\delta^\sigma_0 h_{\rho\sigma}\Big).
\end{aligned} \tag{D.6}
$$

The reason we write the action in this way is so we can put it in the following form:

$$
S^{(4)}_{quad}[h_{\mu\nu}] = \frac{1}{2\kappa^2}\int d^4x \; h_{\mu\nu}\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}. \tag{D.7}
$$

$\mathcal{D}^{\mu\nu\rho\sigma}$ is called the Lichnerowicz operator, which can also be obtained by varying the action with respect to the metric perturbation $h_{\mu\nu}$, but was here simply rewritten in this way. The operator is defined in [23], and it is used here to check the correctness of (4.38).

To show that it is similarly obtained from varying the action, we can use (4.46) with raised indices, subtracting the variation of (D.3) and setting $\dot{H} = a^{-2}(\mathcal{H}' - \mathcal{H}^2) = 0$,

$$
\begin{aligned}
\frac{\delta S^{(4)}}{\delta h_{\mu\nu}} =& -\eta^{\mu\rho}\eta^{\nu\lambda}\kappa^{-1}G^{(4)}_{\rho\lambda} - a^2(\frac{3}{2}h\eta^{\mu\nu} - 3h^{\mu\nu})\mathcal{H}^2 \\
=& a^2\big[-\partial_0 h^{\mu\nu} + 2\partial^{(\mu}h^{\nu)}_0 + \eta^{\mu\nu}(2\partial_\tau h^{\tau 0} + \partial_0 h)\big]\mathcal{H} + a^4\eta^{\mu\nu}h_{00}3\mathcal{H}^2 \\
&+ \frac{1}{2}a^2\big[-2\partial_\sigma\partial^{(\mu}h^{\nu)\sigma} - \eta^{\mu\nu}(\partial^2 h - \partial_\tau\partial_\sigma h^{\tau\sigma}) + \partial^\mu\partial^\nu h + \partial^2 h^{\mu\nu}\big] \\
=& \frac{1}{2}a^2\Big[\eta^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} + \eta^{\rho\sigma}\partial^\mu\partial^\nu h_{\rho\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\partial^2 h_{\rho\sigma} + \eta^{\mu\rho}\eta^{\sigma\nu}\partial^2 h_{\rho\sigma} - 2\eta^{\rho(\mu}\partial^{\nu)}\partial^\sigma h_{\rho\sigma}\Big] \\
&+ Ha^3\Big[-2\eta^{\mu\nu}\delta^\rho_0\partial^\sigma h_{\rho\sigma} + \eta^{\rho\sigma}\partial_0 h_{\rho\sigma} - \eta^{\mu\rho}\eta^{\sigma\nu}\partial_0 h_{\rho\sigma} + 2\delta^\rho_0\eta^{\sigma(\mu}\partial^{\nu)}h_{\rho\sigma}\Big] + 3H^2a^4\eta^{\mu\nu}\delta^\rho_0\delta^\sigma_0 h_{\rho\sigma} \\
=& \mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}.
\end{aligned} \tag{D.8}
$$

D.2 Direct computation Ricci scalar

If we were to contract the Ricci tensor with the metric already in (4.6), we would have found

$$
\begin{aligned}
\mathcal{R} &= g^{\mu\nu}\mathcal{R}_{\mu\nu} = e^{c\rho}e^{d\nu}(\partial_\rho\omega_{\nu cd}) - e^{c\rho}e^{d\nu}(\partial_\nu\omega_{\rho cd}) + e^{c\rho}e^{b\nu}\eta^{ad}\omega_{\rho cd}\omega_{\nu ab} - e^{c\rho}e^{b\nu}\eta^{ad}\omega_{\nu cd}\omega_{\rho ab} \\
&= (e^{\mu a}e^{\nu b} - e^{\mu b}e^{\nu a})(\partial_\mu\omega_{\nu ab} + \omega_{\mu ad}\eta^{cd}\omega_{\nu cb}),
\end{aligned} \tag{D.9}
$$

where we did some relabeling of terms to write it as concisely as possible. We could then go through the whole procedure of rescaling for the Ricci scalar itself. Here, it serves as a check to see that we find the same scalar, regardless of where in the process we contract with the inverse metric. We use (3.5) and (3.7),

$$
\begin{aligned}
\mathcal{R} =& a^{-2}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})\Big(\partial_\mu\big(-a^{-1}\big[\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a}\big]\partial_\rho a\big) + \partial_\mu\tilde{\omega}_{\nu ab} \\
&+ (-a^{-1}\big[\tilde{e}^\sigma_a\tilde{e}_{\mu d} - \tilde{e}^\sigma_d\tilde{e}_{\mu a}\big]\partial_\sigma a + \tilde{\omega}_{\mu ad})\eta^{cd}(-a^{-1}\big[\tilde{e}^\rho_c\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu c}\big]\partial_\rho a + \tilde{\omega}_{\nu cb})\Big) \\
=& a^{-2}\tilde{\mathcal{R}} + a^{-2}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})\Big(a^{-2}(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})(\partial_\mu a)(\partial_\rho a) - a^{-1}\partial_\mu(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})\partial_\rho a \\
&- a^{-1}(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})\partial_\mu\partial_\rho a + a^{-2}(\tilde{e}^\sigma_a\tilde{e}_{\mu d} - \tilde{e}^\sigma_d\tilde{e}_{\mu a})\eta^{cd}(\tilde{e}^\rho_c\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu c})(\partial_\sigma a)(\partial_\rho a) \\
&- a^{-1}(\tilde{e}^\sigma_a\tilde{e}_{\mu d} - \tilde{e}^\sigma_d\tilde{e}_{\mu a})(\partial_\sigma a)\eta^{cd}\tilde{\omega}_{\nu cb} - a^{-1}(\tilde{e}^\rho_c\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu c})(\partial_\rho a)\eta^{cd}\tilde{\omega}_{\mu ad}\Big).
\end{aligned} \tag{D.10}
$$

Notice that we combined a few terms into $\tilde{\mathcal{R}}$ in the second line. Up to this point, everything was exact; we will now expand again in orders of κ up to second order (also given in [18]), so we use (3.13), (3.15) and (3.16). There are a lot of terms in (D.10), so to keep things tidy, we simplify them on a term-by-term basis:

$$
\begin{aligned}
\tilde{\mathcal{R}} =&(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\partial_\mu \tilde{\omega}_{\nu ab} + \tilde{\omega}_{\mu ad}\eta^{cd}\tilde{\omega}_{\nu cb}) \\
=&2(\eta^{\mu a}\eta^{\nu b} - \frac{\kappa}{2}(\eta^{\mu a}h^{\nu b} + \eta^{\nu b}h^{\mu a}))\partial_\mu \tilde{\omega}_{\nu ab} + (\eta^{\mu a}\eta^{\nu b} - \eta^{\mu b}\eta^{\nu a})\tilde{\omega}_{\mu ad}\eta^{cd}\tilde{\omega}_{\nu cb} + \mathcal{O}(\kappa^3) \\
=&\kappa\eta^{\mu a}\eta^{\nu b}\partial_\mu(\partial_b h_{\nu a} - \partial_a h_{\nu b}) + \frac{\kappa^2}{2}\eta^{\mu a}\eta^{\nu b}\partial_\mu\Big(h^\sigma_b(\frac{1}{2}\partial_\nu h_{\sigma a} + \partial_a h_{\sigma\nu} - \partial_\sigma h_{\nu a}) \\
&- h^\sigma_a(\frac{1}{2}\partial_\nu h_{\sigma b} + \partial_b h_{\sigma\nu} - \partial_\sigma h_{\nu b})\Big) - \frac{\kappa^2}{2}(\eta^{\mu a}h^{\nu b} + \eta^{\nu b}h^{\mu a})\partial_\mu(\partial_b h_{\nu a} - \partial_a h_{\nu b}) \\
&+ \frac{\kappa^2}{4}(\eta^{\mu a}\eta^{\nu b} - \eta^{\mu b}\eta^{\nu a})(\partial_d h_{\mu a} - \partial_a h_{\mu d})\eta^{cd}(\partial_b h_{\nu c} - \partial_c h_{\nu b}) + \mathcal{O}(\kappa^3) \\
=&\kappa(\partial_\mu\partial^\nu h^\mu_\nu - \partial^2 h) + \kappa^2\Big[\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\sigma\nu}) - \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma) + h^{\sigma\nu}(\partial^2 h_{\sigma\nu}) - 2h^{\sigma\nu}(\partial_\mu\partial_\sigma h^\mu_\nu) \\
&- \frac{3}{2}(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma) + (\partial_\mu h^{\sigma\mu})(\partial_\sigma h) + h^{\sigma\mu}(\partial_\mu\partial_\sigma h) - h^{\mu\sigma}(\partial_\mu\partial^\nu h_{\nu\sigma}) + \frac{1}{2}h^{\mu\sigma}(\partial^2 h_{\mu\sigma}) + \frac{1}{2}(\partial_\mu\partial_\sigma h) \\
&+ \frac{1}{2}(\partial^\sigma h)(\partial^\nu h_{\nu\sigma}) - \frac{1}{4}(\partial^\mu h^\sigma_\mu)(\partial^\nu h_{\nu\sigma}) - \frac{1}{4}(\partial^\sigma h)(\partial_\sigma h) - \frac{1}{4}(\partial^\sigma h^\nu_\mu)(\partial^\mu h_{\nu\sigma}) + \frac{1}{4}(\partial^\sigma h^\nu_\mu)(\partial_\sigma h^\mu_\nu)\Big] \\
&+ \mathcal{O}(\kappa^3) \\
=&\kappa(\partial_\mu\partial^\nu h^\mu_\nu - \partial^2 h) + \kappa^2\Big[\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma}) + h^{\sigma\nu}(\partial^2 h_{\sigma\nu} - 2\partial_\mu\partial_\sigma h^\mu_\nu) + (\partial_\mu h^{\sigma\mu})(\partial_\sigma h) + h^{\sigma\mu}(\partial_\mu\partial_\sigma h) \\
&- \frac{1}{4}(\partial^\sigma h)(\partial_\sigma h) - (\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma) - \frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)\Big] + \mathcal{O}(\kappa^3).
\end{aligned}
\tag{D.11}
$$

A lot of terms drop out due to symmetry of $h_{\mu\nu}$, and because we work up to quadratic order, we can freely lower and raise spacetime indices for the κ^2 terms. For the other terms in (D.10), we find

$$
a^{-4}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})(\partial_\mu a)(\partial_\rho a) = a^{-4}6\tilde{g}^{\mu\rho}(\partial_\mu a)(\partial_\rho a) = 6a^{-4}\tilde{g}^{\rho\mu}(\partial_\rho a)(\partial_\mu a), \tag{D.12}
$$

$$
\begin{aligned}
-a^{-3}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})\partial_\mu(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})(\partial_\rho a) &= -2a^{-3}(\tilde{g}^{\mu\rho}\tilde{e}^{\nu a}\partial_\mu\tilde{e}_{\nu a} - \tilde{g}^{\rho\nu}\tilde{e}^{\mu a}\partial_\mu\tilde{e}_{\nu a} + 3\tilde{e}^{\mu a}\partial_\mu\tilde{e}^\rho_a)(\partial_\rho a) \\
&= -2a^{-3}(\tilde{g}^{\mu\rho}\tilde{e}^{\nu a}\partial_\mu\tilde{e}_{\nu a} + \tilde{e}^\rho_a\partial_\mu\tilde{e}^{\mu a} + 3\tilde{e}^{\mu a}\partial_\mu\tilde{e}^\rho_a)(\partial_\rho a) \\
&= -2a^{-3}(\tilde{g}^{\mu\rho}\tilde{e}^{\nu a}\partial_\mu\tilde{e}_{\nu a} + 2\tilde{e}^{\mu a}\partial_\mu\tilde{e}^\rho_a - \partial_\mu\tilde{g}^{\mu\rho})(\partial_\rho a).
\end{aligned}
\tag{D.13}
$$

Notice we used (4.3) and the product rule for $\tilde{g}^{\mu\rho} = \tilde{e}^\mu_b\tilde{e}^{\rho b}$ to make the answer more concise. The other terms are

$$
-a^{-3}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\tilde{e}^\rho_a\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu a})\partial_\mu\partial_\rho a = -6a^{-3}\tilde{g}^{\mu\rho}\partial_\mu\partial_\rho a, \tag{D.14}
$$

$$
a^{-4}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\tilde{e}^\sigma_a\tilde{e}_{\mu d} - \tilde{e}^\sigma_d\tilde{e}_{\mu a})\eta^{cd}(\tilde{e}^\rho_c\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu c})(\partial_\sigma a)(\partial_\rho a) = -6a^{-4}\tilde{g}^{\rho\sigma}(\partial_\sigma a)(\partial_\rho a), \tag{D.15}
$$

$$
\begin{aligned}
-a^{-3}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\tilde{e}^\sigma_a\tilde{e}_{\mu d} - \tilde{e}^\sigma_d\tilde{e}_{\mu a})(\partial_\sigma a)\eta^{cd}\tilde{\omega}_{\nu cb} &= -a^{-3}(-2\tilde{e}^{\nu b}\tilde{e}^{\sigma c} - \tilde{g}^{\nu\sigma}\eta^{cb})(\partial_\sigma a)\tilde{\omega}_{\nu cb} \\
&= 2a^{-3}\tilde{e}^{\nu b}\tilde{e}^{\sigma c}(\partial_\sigma a)\tilde{\omega}_{\nu cb},
\end{aligned}
\tag{D.16}
$$

$$
\begin{aligned}
-a^{-3}(\tilde{e}^{\mu a}\tilde{e}^{\nu b} - \tilde{e}^{\mu b}\tilde{e}^{\nu a})(\tilde{e}^\rho_c\tilde{e}_{\nu b} - \tilde{e}^\rho_b\tilde{e}_{\nu c})(\partial_\rho a)\eta^{cd}\tilde{\omega}_{\mu ad} &= a^{-3}(2\tilde{e}^{\mu a}\tilde{e}^{\rho d} + \tilde{g}^{\mu\rho}\eta^{ad})(\partial_\rho a)\tilde{\omega}_{\mu ad} \\
&= -2a^{-3}\tilde{e}^{\mu a}\tilde{e}^{\rho d}(\partial_\rho a)\tilde{\omega}_{\mu ad}.
\end{aligned}
\tag{D.17}
$$

Some immediate observations are that (D.12) and (D.15) will drop out against each other, and that (D.16) and (D.17) can be added after relabeling. Looking at the last two equations added gives

$$\begin{aligned}2a^{-3}(\tilde{e}^{\nu b}\tilde{e}^{\sigma c}-\tilde{e}^{\nu c}\tilde{e}^{\sigma b})\tilde{\omega}_{\nu cb}\partial_\sigma a =&2a^{-3}(\tilde{e}^{\nu b}\tilde{e}^{\sigma c}-\tilde{e}^{\nu c}\tilde{e}^{\sigma b})(\tilde{e}^\rho_c\partial_\nu\tilde{e}_{\rho b}-\tilde{e}^\rho_c\tilde{\Gamma}^\lambda_{\nu\rho}\tilde{e}_{\lambda b})(\partial_\sigma a)\\ =&2a^{-3}(\tilde{g}^{\rho\sigma}\tilde{e}^{\nu b}\partial_\nu\tilde{e}_{\rho b}-\tilde{g}^{\rho\nu}\tilde{e}^{\sigma b}\partial_\nu\tilde{e}_{\rho b}-\tilde{g}^{\rho\sigma}\delta^\nu_\lambda\tilde{\Gamma}^\lambda_{\nu\rho}+\tilde{g}^{\rho\nu}\delta^\sigma_\lambda\tilde{\Gamma}^\lambda_{\nu\rho})(\partial_\sigma a)\\ =&2a^{-3}(\tilde{e}^\nu_b\partial_\nu\tilde{e}^{\sigma b}-\tilde{e}^\sigma_b\partial_\nu\tilde{e}^{\nu b}-\tilde{g}^{\rho\sigma}\frac{1}{2}\tilde{g}^{\nu\mu}(\partial_\nu\tilde{g}_{\rho\mu}+\partial_\rho\tilde{g}_{\mu\nu}-\partial_\mu\tilde{g}_{\rho\nu})\\ &+\tilde{g}^{\rho\nu}\frac{1}{2}\tilde{g}^{\sigma\mu}(\partial_\nu\tilde{g}_{\rho\mu}+\partial_\rho\tilde{g}_{\mu\nu}-\partial_\mu\tilde{g}_{\nu\rho}))(\partial_\sigma a)\\ =&2a^{-3}(2\tilde{e}^{\nu b}\partial_\nu\tilde{e}^\sigma_b-\partial_\nu\tilde{g}^{\nu\sigma}-\frac{1}{2}\tilde{g}^{\rho\sigma}\tilde{g}^{\nu\mu}(\partial_\rho\tilde{g}_{\mu\nu})\\ &+\tilde{g}^{\rho\nu}\tilde{g}^{\sigma\mu}(\partial_\nu\tilde{g}_{\rho\mu}-\frac{1}{2}\partial_\mu\tilde{g}_{\nu\rho}))(\partial_\sigma a)\\ =&a^{-3}(4\tilde{e}^{\nu b}\partial_\nu\tilde{e}^\sigma_b+2\tilde{g}^{\rho\nu}\tilde{g}^{\sigma\mu}(\partial_\nu\tilde{g}_{\rho\mu})-2\partial_\nu\tilde{g}^{\nu\sigma}-2\tilde{g}^{\rho\sigma}\tilde{g}^{\nu\mu}(\partial_\rho\tilde{g}_{\mu\nu}))(\partial_\sigma a).\end{aligned}\tag{D.18}$$

In the third line, we used the Leibniz rule to make the second term resemble the first. The third term in the last line, as well as the last term in (D.13) can be rewritten as

$$\partial_\nu\tilde{g}^{\nu\sigma}=\partial_\nu(\tilde{g}^{\nu\rho}\tilde{g}_{\rho\mu}\tilde{g}^{\mu\sigma})=\delta^\nu_\mu\partial_\nu\tilde{g}^{\mu\rho}+\delta^\sigma_\rho\partial_\nu\tilde{g}^{\nu\rho}+\tilde{g}^{\nu\rho}\tilde{g}^{\mu\sigma}\partial_\nu\tilde{g}_{\rho\mu}\implies\partial_\nu\tilde{g}^{\nu\sigma}=-\tilde{g}^{\rho\nu}\tilde{g}^{\sigma\mu}\partial_\nu\tilde{g}_{\rho\mu}.\tag{D.19}$$

Adding (D.18) and (D.13) (with some relabeling),

$$\begin{aligned}&a^{-3}(6\tilde{g}^{\sigma\nu}\tilde{g}^{\rho\mu}(\partial_\nu\tilde{g}_{\sigma\mu})-2\tilde{g}^{\rho\sigma}\tilde{g}^{\nu\mu}\partial_\sigma\tilde{g}_{\mu\nu}-2\tilde{g}^{\mu\rho}\tilde{e}^{\nu a}\partial_\mu\tilde{e}_{\nu a})(\partial_\rho a)\\ &\qquad=6a^{-3}(\tilde{g}^{\sigma\nu}\frac{1}{2}\tilde{g}^{\rho\mu}(\partial_\nu\tilde{g}_{\sigma\mu})-\frac{1}{2}\tilde{g}^{\rho\sigma}\tilde{g}^{\nu\mu}\partial_\sigma\tilde{g}_{\mu\nu})(\partial_\rho a)\\ &\qquad=6a^{-3}\tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}(\partial_\mu\tilde{g}_{\sigma\nu}+\partial_\nu\tilde{g}_{\sigma\mu}-\partial_\sigma\tilde{g}_{\mu\nu})(\partial_\rho a)\\ &\qquad=6a^{-3}\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a).\end{aligned}\tag{D.20}$$

We finally obtain for $\mathcal{R}$

$$\begin{aligned}\mathcal{R}=&a^{-2}\tilde{\mathcal{R}}+6a^{-3}\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a)-6a^{-3}\tilde{g}^{\mu\rho}\partial_\mu\partial_\rho a=a^{-2}\tilde{\mathcal{R}}-6a^{-3}\tilde{g}^{\mu\nu}(\partial_\mu\partial_\nu a-\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a))\\ =&a^{-2}\kappa(\partial_\mu\partial^\nu h^\mu_\nu-\partial^2 h)+a^{-2}\kappa^2\Big[\frac{3}{4}(\partial_\mu h^{\sigma\nu})(\partial^\mu h_{\nu\sigma})+h^{\sigma\nu}(\partial^2 h_{\sigma\nu}-2\partial_\mu\partial_\sigma h^\mu_\nu)+(\partial_\mu h^{\sigma\mu})(\partial_\sigma h)\\ &+h^{\sigma\mu}(\partial_\mu\partial_\sigma h)-\frac{1}{4}(\partial^\sigma h)(\partial_\sigma h)-(\partial_\mu h^{\sigma\mu})(\partial_\nu h^\nu_\sigma)-\frac{1}{2}(\partial_\mu h^{\sigma\nu})(\partial_\nu h^\mu_\sigma)\Big]-6a^{-3}\tilde{g}^{\mu\nu}(\partial_\mu\partial_\nu a-\tilde{\Gamma}^\rho_{\mu\nu}(\partial_\rho a))\\ &+\mathcal{O}(\kappa^3).\end{aligned}\tag{D.21}$$

Which is exactly the same as we have found in (4.19) and (4.20).

E Renormalizing the energy-momentum tensor

The calculations in this section follow [43] very closely. We start from (C.3), and ignore expansion on account of an adiabatically slowly expanding universe, i.e.

$$\frac{\mathcal{O}(\mathcal{H}^2,\mathcal{H}')}{m^2} \ll 1 \implies D_\mu \to \partial_\mu. \tag{E.1}$$

The Feynman propagator for ψ and $\bar{\psi}$ is defined in this appendix as

$$iS^{++}_{\alpha\beta}(x;y) = \langle T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\rangle \tag{E.2}$$

After defining the equal-time quantization relation for fermions, we can construct the following equation of motion for the Feynman propagator

$$\left\{\psi_\alpha(t,\vec{x}),\psi^\dagger_\beta(t,\vec{y})\right\} = a^{1-D}\delta_{\alpha\beta}\delta^{D-1}(\vec{x}-\vec{y}) \implies (i\partial\!\!\!/ - m)_{\alpha\delta} iS^{++}_{\delta\beta}(x;y) = i\delta_{\alpha\beta}\delta^D(x-y), \tag{E.3}$$

which follows from the Dirac equation. The energy-momentum tensor is

$$\langle T_{\mu\nu}\rangle = \Big\langle T\Big[\frac{i}{2}\left(\bar{\psi}\gamma^\rho\partial_\rho\psi - (\partial_\rho\bar{\psi})\gamma^\rho\psi\right) - m\bar{\psi}\psi\Big]\Big\rangle g_{\mu\nu} - \frac{i}{2}\langle\bar{\psi}\gamma_{(\nu}\partial_{\mu)}\psi - (\partial_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\rangle, \tag{E.4}$$

where $T[...]$ again denotes time ordering, defined in terms of step functions in (6.9). We want to know how we can express these expectation values as various propagators, and what their values are. To this end, we firstly look at $i\Delta_F(x;y)$, defined by

$$iS^{++}_{\alpha\beta}(x;y) = (i\partial\!\!\!/ + m)_{\alpha\beta} i\Delta_F(x;y). \tag{E.5}$$

Notice that substituting this back into (E.3) and using $\partial\!\!\!/^2 = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = -\partial^2$ gives a Klein-Gordon-like equation for $i\Delta_F(x;y)$, of which the solutions are well known. We quote the result in D dimensions

$$i\Delta_F(x;y) = \frac{m^{D-2}}{(2\pi)^{D/2}}\frac{K_{\frac{D-2}{2}}\left(m\sqrt{\Delta x^2}\right)}{\left(m\sqrt{\Delta x^2}\right)^{\frac{D-2}{2}}} - \int\frac{d^{D-1}p}{(2\pi)^{D-1}}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{E_p}\frac{\cos[E_p(t-t')]}{e^{\beta E_p}+1}. \tag{E.6}$$

It should be noted that this result is valid for slowly varying background fields, which is applicable to fermionic dark matter. $\beta \equiv 1/k_BT$ and $K_\alpha(x)$ is a modified bessel-function of the second kind, which naturally solves the Klein-Gordon equation. The integral is the thermal part of the equation (it goes to 0 as $T \to 0K$), the minus sign in front of it and the factor $(e^{\beta E_p}+1)^{-1}$ in it are a result of the fermions. Also

$$\Delta x^2(x;y) \equiv -(|t-t'|-i\epsilon)^2 + ||\vec{x}-\vec{y}||^2, \tag{E.7}$$

where the $i\epsilon$ is used to make contour integration possible by shifting the poles. The coincident limit $(x \to y)$ of (E.6), which we quote, is

$$i\Delta_F(x;x) = \frac{m^{D-2}\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}} + \frac{1}{2\pi^2\beta^3 m}\left[\partial_z J_F(4,z)\right]_{z=\beta m}. \tag{E.8}$$

The second part is obtained as

$$\begin{aligned}
J_F(n,z) &\equiv \int_0^\infty dx\, x^{n-2}\ln\Big(1+e^{-\sqrt{x^2+z^2}}\Big),\\
\partial_z J_F(n,z) &= -\int_0^\infty dx\frac{x^{n-2}}{\sqrt{x^2+z^2}}\frac{ze^{-\sqrt{x^2+z^2}}}{1+e^{-\sqrt{x^2+z^2}}} = -\int_0^\infty dx\frac{x^{n-2}}{\sqrt{x^2+z^2}}\frac{z}{e^{\sqrt{x^2+z^2}}+1},\\
[\partial_z J_F(n,z)]_{z=\beta m} &= -m\beta^{n-1}\int_0^\infty dp\frac{p^{n-2}}{\sqrt{p^2+m^2}}\frac{1}{e^{\beta\sqrt{p^2+m^2}}+1} = -m\beta^{n-1}\int_0^\infty dp\frac{p^{n-2}}{E_p}\frac{1}{e^{\beta E_p}+1}.
\end{aligned} \tag{E.9}$$

In the last line, we set $x = \beta p$, and used $E_p = \sqrt{p^2+m^2}$. Looking at the thermal part of (E.6), for the coincident limit and $D=4$, we obtain

$$-\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\frac{1}{e^{\beta E_p}+1} = -\int_0^\infty\frac{dp}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^\pi d\theta\, p^2\sin\theta\frac{1}{E_p}\frac{1}{e^{\beta E_p}+1} = -\frac{1}{2\pi^2}\int dp\,\frac{p^2}{E_p}\frac{1}{e^{\beta E_p}+1}. \tag{E.10}$$

It is not hard to see that the thermal part of (E.8) is correct, for $n = 4$.

We are now in a position to evaluate the individual parts of $T_{\mu\nu}$. We have for the mass term

$$\left\langle T[\bar{\psi}(x)\psi(x)]\right\rangle = -\,(i\partial\!\!\!/ + m)_{\alpha\alpha} i\Delta_F(x;x) = -\left(\frac{m^{D-1}\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}} + \frac{1}{2\pi^2\beta^3}\left[\partial_z J_F(4,z)\right]_{z=\beta m}\right)\mathrm{Tr}[I_D], \tag{E.11}$$

where we used the fact that the mass picks the trace, such that the traceless gamma matrices drop out. The trace of a D-dimensional identity matrix is $2^{D/2}$, is the amount of fermionic degrees of freedom. (Half of) the kinetic term is

$$\left\langle T\big[\bar{\psi}(x)\big(-\frac{i}{2}\gamma_{(\mu}\partial_{\nu)}\big)\psi(x)\big]\right\rangle = -\,\frac{i}{2}\gamma^{\alpha\beta}_{(\mu}\partial^x_{\nu)}\left\langle T^*\left[\psi_\beta(x)\bar{\psi}_\alpha(x)\right]\right\rangle_{x\to y} \tag{E.12}$$

$$= -\,\frac{i}{2}\left[\gamma^{\alpha\beta}_{(\mu}\partial^x_{\nu)}(i\partial\!\!\!/^x + m)_{\beta\alpha} i\Delta_F(x;y)\right]_{x\to y} \tag{E.13}$$

$$= -\,\frac{1}{2}\left[\partial^y_{(\mu}\partial^x_{\nu)} i\Delta_F(x;y)\right]_{x\to y}\mathrm{Tr}[I_D]. \tag{E.14}$$

The superscripts on the derivatives indicate on which x they act. T^* means that the derivative commutes with the time-ordering symbol, and we again used the tracelessness of the gamma matrices to set the term with a single gamma to 0. Similarly, the product of two gamma matrices only yields a trace if they are equal, so $\mathrm{Tr}[i\gamma_\mu\partial_\nu i\partial\!\!\!/] = \partial_\mu\partial_\nu\mathrm{Tr}[I_D]$. In the last line, we used that we always have the combination $x-y$, a derivative acting on x can always act on y, at the cost of a minus sign. The complex conjugate of this equation yields an overall minus sign, such that the complete kinetic term is

$$\left\langle T\big[\bar{\psi}(x)\big(-\frac{i}{2}\gamma_{(\mu}\partial_{\nu)} + \frac{i}{2}\overleftarrow{\partial}_{(\mu}\gamma_{\nu)}\big)\psi(x)\big]\right\rangle = -\left[\partial^y_{(\mu}\partial^x_{\nu)} i\Delta_F(x;y)\right]_{x\to y}\mathrm{Tr}[I_D] \overset{vac}{=} -\eta_{\mu\nu}\frac{m^D\Gamma(-\frac{D}{2})}{2(2\pi)^{D/2}}. \tag{E.15}$$

This final part is again a quoted result, however, it appears from the expansion of the Bessel function, after which the double derivative on $m\sqrt{\Delta x^2}$ gives an extra factor m^2 in the vacuum contribution. We can use this to evaluate the first part of $\langle T_{\mu\nu}\rangle$ in vacuum

$$\left\langle T\left[\frac{i}{2}\left(\bar{\psi}\gamma^\rho\partial_\rho\psi - (\partial_\rho\bar{\psi})\gamma^\rho\psi\right) - m\bar{\psi}\psi\right]\right\rangle_{\mathrm{vac}} = \langle\mathcal{L}_\psi\rangle_{\mathrm{vac}} = 0. \tag{E.16}$$

That is, the expectation value of the vacuum Lagrangian is 0; a similar result was encountered before in section 5.2.

This means that we can look at (E.4) again, see that the first part drops out and we are left with the following vacuum and thermal contributions

$$\langle T_{\mu\nu}\rangle_{\mathrm{vac}} = -\,\frac{i}{2}\left\langle\bar{\psi}\gamma_{(\nu}\partial_{\mu)}\psi - (\partial_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right\rangle = -\eta_{\mu\nu}\frac{m^D\Gamma(-\frac{D}{2})}{2(2\pi)^{D/2}}, \tag{E.17}$$

$$\begin{aligned}\langle T_{00}\rangle_{\mathrm{ther}} &= -\,2^{D/2}\left[\partial^y_0\partial^x_0 i\Delta_F(x;y)\right]_{x\to y}\\ &= -\,2^{D/2}\Big[-\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{E_p}E_p^2\frac{\cos[E_p(t-t')]}{e^{\beta E_p}+1}\Big]_{x\to y}\\ &\overset{D\to4}{=} 4\Big[\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{E_p}\frac{1}{e^{\beta E_p}+1} + m^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\frac{1}{e^{\beta E_p}+1}\Big]\\ &= -\,\frac{2}{\pi^2\beta^5 m}[\partial_z J_F(6,z)]_{z=\beta m} - \frac{2m}{\pi^2\beta^3}[\partial_z J_F(4,z)]_{z=\beta m},\end{aligned} \tag{E.18}$$

$$\begin{aligned}\langle T_{ij}\rangle_{\mathrm{ther}} &= -\,2^{D/2}\left[\partial^y_i\partial^x_j i\Delta_F(x;y)\right]_{x\to y}\\ &= -\,2^{D/2}\Big[-\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{E_p}p_ip_j\frac{\cos[E_p(t-t')]}{e^{\beta E_p}+1}\Big]_{x\to y}\\ &\overset{D\to4}{=} \frac{1}{3}\delta_{ij}4\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{E_p}\frac{1}{e^{\beta E_p}+1}\\ &= -\,\frac{2\delta_{ij}}{3\pi^2\beta^5 m}[\partial_z J_F(6,z)]_{z=\beta m}.\end{aligned} \tag{E.19}$$

For the third line of $\langle T_{ij}\rangle_{\text{ther}}$, we used the fact that the integral on the second line should be proportional to a second-rank rotationally invariant tensor, which is δ_{ij}, i.e.

$$\int d^3p\; p_ip_j = \int d^3p\delta_{ij}f(p) \implies f(p) = \frac{1}{3}p^2, \tag{E.20}$$

where $f(p)$ can be found by taking the trace. A similar argument can be used to see that we have $\langle T_{i0}\rangle_{\text{ther}} = \langle T_{0i}\rangle_{\text{ther}} = 0$, as the only rotationally invariant vector is the zero vector.

For an isotropic, nonrelativistic fluid, we can write the thermal part as

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\text{ther}} =& F\delta^0_\mu\delta^0_\nu + G\eta_{\mu\nu} \implies G = \frac{1}{3}\delta^{ij}\langle T_{ij}\rangle_{\text{ther}}, \qquad F = \langle T_{00}\rangle_{\text{ther}} + \frac{1}{3}\delta^{ij}\langle T_{ij}\rangle_{\text{ther}},\\ \langle T_{\mu\nu}\rangle_{\text{ther}} =& -\frac{2\eta_{\mu\nu}}{3\pi^2\beta^5 m}[\partial_z J_F(6,z)]_{z=\beta m} - \delta^0_\mu\delta^0_\nu\Big(\frac{8}{\pi^2\beta^5 m}[\partial_z J_F(6,z)]_{z=\beta m} + \frac{2m}{\pi^2\beta^3}[\partial_z J_F(4,z)]_{z=\beta m}\Big).\end{aligned} \tag{E.21}$$

We see that we have evaluated the thermal parts at $D = 4$ without encountering any infinities. However, problems will arise for the vacuum part, since $\Gamma(-\frac{D}{2})$ is undefined at $D = 4$. Moreover, we want to introduce an arbitrary scale μ (which is not the chemical potential in this appendix) to obtain a correct mass dimension for the different terms that will arise.

Finding a suitable counterterm is done by expanding the different parts of (E.17) in a suitable way. We have

$$\begin{aligned}\Gamma(-\frac{D}{2}) =& -\frac{2}{D}\Gamma(1-\frac{D}{2}) = -(1-\frac{D}{2})^{-1}\frac{2}{D}\Gamma(2-\frac{D}{2}) = \frac{4}{D^2-2D}\Gamma(2-\frac{D}{2})\\ =& \frac{4}{D^2-2D}\big[\frac{1}{2-\frac{D}{2}} - \gamma_E + \mathcal{O}(D-4)\big],\end{aligned} \tag{E.22}$$

$$\begin{aligned}\frac{m^D}{(2\pi)^{D/2}} =& \mu^D\left(\frac{m^2}{2\pi\mu^2}\right)^{D/2} = \mu^{D-4}\frac{m^4}{4\pi^2}\left(\frac{m^2}{2\pi\mu^2}\right)^{D/2-2}\\ =& \mu^{D-4}\frac{m^4}{4\pi^2}\Big[1 + \frac{D-4}{2}\ln\left(\frac{m^2}{2\pi\mu^2}\right) + \mathcal{O}((D-4)^2)\Big],\end{aligned} \tag{E.23}$$

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\text{vac}} =& \eta_{\mu\nu}\mu^{D-4}\frac{m^4}{8\pi^2}\frac{4}{D^2-2D}\big[\frac{2}{D-4} + \gamma_E\big]\Big(1 + \frac{D-4}{2}\ln\left(\frac{m^2}{2\pi\mu^2}\right)\Big) + \mathcal{O}(D-4)\\ =& \eta_{\mu\nu}\frac{m^4}{8\pi^2}\frac{4}{D^2-2D}\Big(\frac{2\mu^{D-4}}{D-4} + \mu^{D-4}\gamma_E + \mu^{D-4}\ln\left(\frac{m^2}{2\pi\mu^2}\right)\Big) + \mathcal{O}(D-4).\end{aligned} \tag{E.24}$$

It is obvious now which term is troublesome and should be removed. Assuming a mass generated by a scalar field condensate, i.e. $m = y\phi_0$, a suitable counter-term in the action (in the minimal subtraction scheme) and its resulting addition to the energy-momentum tensor will be

$$S^{CT} = \int d^4x\sqrt{-g}\Big[-\frac{\phi^4}{4!}\frac{3y^4}{\pi^2}\frac{\mu^{D-4}}{D-4}\Big] \implies T^{CT}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S^{CT}}{g^{\mu\nu}} = -\frac{1}{8\pi^2}(y\phi)^4\frac{\mu^{D-4}}{D-4}g_{\mu\nu}, \tag{E.25}$$

which, for the slowly varying background, will clearly remove the divergence if we let $D \to 4$. The renormalized energy-momentum tensor then reads

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \eta_{\mu\nu}\frac{m^4}{16\pi^2}\Big(\gamma_E + \ln\left(\frac{m^2}{2\pi\mu^2}\right)\Big). \tag{E.26}$$

This does not include the $-\frac{3}{2}$ contribution mentioned in [43], however, we are interested in the counter-term. We see that the vacuum contribution is constant, which can therefore be absorbed into a cosmological constant.

F Deriving the 2PI effective action

A clear discussion on how to derive the 2PI effective action in terms of a scalar propagator is given in [34]. We start with a review of the path integral following [32], and extend this to the closed time path of figure 6.1 by using [33]. This will give a clear background when moving to the one- and two-particle irreducible graphs, where we follow [35].

F.1 Path integral formalism for the closed time path

The usual way of deriving an n-point function is from the generating functional $Z[\bar{J}, J]$. For the fermion theory, it is related to the functional $W[\bar{J}, J]$ by

$$\begin{aligned} e^{iW_0} = Z_0[\bar{J}, J] &= \int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\big(i\big[S^{(0,2)}[\bar{\chi},\chi] + \bar{J}\chi + \bar{\chi}J\big]\big) \\ &= \int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\big(-\bar{\chi}(-i)\mathcal{D}_0\chi + i\bar{J}\chi + i\bar{\chi}J\big), \end{aligned} \tag{F.1}$$

where we used a shorter notation for the following

$$\begin{aligned} \bar{\chi}\mathcal{D}_0\chi &= \int d\eta\; d^3x\; d\tilde{\eta}\; d^3y\; \bar{\chi}_\alpha(x)(i\not{\partial} - ma)\chi_\alpha(y), \\ \bar{J}\chi &= \int d\eta\; d^3x \bar{J}_\alpha(x)\chi_\alpha(x), \quad \bar{\chi}J = \int d\eta\; d^3x \bar{\chi}_\alpha(x)J_\alpha(x). \end{aligned} \tag{F.2}$$

We have chosen $\mathcal{D}_0$ for the derivative operator for the free theory. The generating functional is useful, as varying with respect to the right currents will give us the condensates and the free propagator,

$$\frac{1}{i}\frac{1}{Z_0}\frac{\delta Z_0[\bar{J},J]}{\delta \bar{J}_\alpha(x)} = \langle\chi_\alpha(x)\rangle = 0, \qquad -\frac{1}{i}\frac{1}{Z_0}\frac{\delta Z_0[\bar{J},J]}{\delta J_\alpha(x)} = \langle\bar{\chi}_\alpha(x)\rangle = 0, \tag{F.3}$$

$$\begin{aligned} -\left(\frac{1}{i}\right)^2 \frac{1}{Z_0}\frac{\delta^2 Z_0[\bar{J},J]}{\delta\bar{J}_\alpha(x)\delta J_\beta(y)}\Big|_{\bar{J}=J=0} &= \frac{\int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\chi_\alpha(x)\bar{\chi}_\beta(y)\exp\big(i\big[S^{(0,2)}[\bar{\chi},\chi] + \bar{J}\chi + \bar{\chi}J\big]\big)}{\int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\big(i\big[S^{(0,2)}[\bar{\chi},\chi] + \bar{J}\chi + \bar{\chi}J\big]\big)} \\ &= \langle 0|T\big[\chi_\alpha(x)\bar{\chi}_\beta(y)\big]|0\rangle = iS^{++}_{0\alpha\beta}(x;y), \end{aligned} \tag{F.4}$$

where the minus sign appears because the sources are also Grassmann valued, and we suppress the spinor indices. The condensates vanish in absence of sources, because $\chi, \bar{\chi}$ are not Lorentz invariant by themselves. We have also explicitly written that the correlation function is with respect to the vacuum of a free theory, and used that $\langle\chi\rangle = \langle\bar{\chi}\rangle = 0$ in absence of sources. One can easily complete the square in the exponent of (F.1) by shifting the fields, which does not change the path integration measure

$$\chi \to \chi - \mathcal{D}_0^{-1}J, \qquad \bar{\chi} \to \bar{\chi} - \bar{J}\mathcal{D}_0^{-1} \tag{F.5}$$

$$Z_0[\bar{J},J] = \int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\big(-\bar{\chi}(-i)\mathcal{D}_0\chi - i\bar{J}\mathcal{D}_0^{-1}J\big), \propto \exp\big(-i\bar{J}\mathcal{D}_0^{-1}J + \mathrm{Tr}\big[\log\big(-i\mathcal{D}_0\big)\big]\big). \tag{F.6}$$

By applying (F.4) again with this rewritten generating functional, we see that $iS_0^{++}(x;y)$ is the inverse of $-i\mathcal{D}_0$, i.e. $-i(i\not{\partial} - ma)iS_0^{++}(x;y) = \delta^4(x-y)$. The proportionality in (F.6) comes from a constant, which is a result of the Gaussian integral over the Grassmann variables; details can be found in e.g. [14]. For a free theory, this constant, together with the trace exponent, are set to 1 by normalizing with respect to the vacuum. Since our dark matter theory is not a free one, we have left the trace exponent in the equation, as we will need it in the next subsection. Note also that the proportionality does not matter in (F.4). We now substitute iS_0^{++} for $(-i\mathcal{D}_0)^{-1}$ in (F.6):

$$Z_0[\bar{J},J] \propto \exp\big(-\bar{J}iS_0^{++}J - \mathrm{Tr}\big[\log(iS_0^{++})\big]\big), \qquad \bar{J}iS_0^{++}J = \int d\eta\; d^3x\; d\tilde{\eta}\; d^3y\; \bar{J}_\alpha(x)iS^{++}_{0\alpha\beta}(x;y)J_\beta(y). \tag{F.7}$$

where the plus in front of the trace became a minus by taking the exponent of the propagator out of the logarithm.

To extend this to the loop contour of figure 6.1, we look at sources and fields along the positive and negative time paths separately

$$Z_0[\bar{J}^+, J^+, \bar{J}^-, J^-] = \int \mathcal{D}[\bar{\chi}^+]\mathcal{D}[\chi^+]\mathcal{D}[\bar{\chi}^-]\mathcal{D}[\chi^-]\exp\big(i\big[S^{(0,2)}[\bar{\chi}^+, \chi^+] + \bar{J}^+\chi^+ + \bar{\chi}^+J^+ \\ - S^{(0,2)}[\bar{\chi}^-, \chi^-] - \bar{J}^-\chi^- - \bar{\chi}^-J^-\big]\big). \tag{F.8}$$

A superscript '+' implies along the positive path, '−' along the negative path. We see these as a slight imaginary deviation from the closed time path, such that we can meaningfully define the Wightman functions. The negative path variables and functionals also get a minus sign, as we evaluate the time integration in the exponent backwards ($\int_\eta^{\eta_0} d\tilde{\eta} \to -\int_{\eta_0}^{\eta} d\tilde{\eta}$). The various propagators, evaluated at tree level, of (6.9)-(6.12) are then found by variations of $W_0[\bar{J}^\pm, J^\pm]$ with respect to the appropriate sources. A difference with (F.4) here is that we do not require $J = \bar{J} = 0$, but only that the sources along both paths are equal, $J^+ = J^-$, $\bar{J}^+ = \bar{J}^-$. The whole reason of writing (F.8) with the fields and sources in this way, is because varying with respect to the negative sources will produce anti-time-ordered products, which we could not do with (F.1). Completing the square can again be done, keeping in mind that there are '+−' and '−+' terms at the boundaries of the contour, so we have

$$Z_0[\bar{J}^\pm, J^\pm] \propto \exp\Big\{ -\begin{pmatrix}\bar{J}^+ & -\bar{J}^-\end{pmatrix} \begin{pmatrix} iS_0^{++} & iS_0^{+-} \\ iS_0^{-+} & iS_0^{--}\end{pmatrix} \begin{pmatrix} J^+ \\ -J^- \end{pmatrix} \Big\}\exp\big(i\big\{i\mathrm{Tr}\big[\log(iS_0)\big]\big\}\big). \tag{F.9}$$

We have already written (F.9) in terms of free propagators to make it useful for the interactions; varying $Z_0[\bar{J}^\pm, J^\pm]$ will give the different propagators [44]

$$iS^{cd} = \langle \chi^c \bar{\chi}^d \rangle = -cd\left(\frac{1}{i}\right)^2 \frac{1}{Z_0}\frac{\delta^2 Z_0[\bar{J}^\pm, J^\pm]}{\delta \bar{J}^c \delta J^d}\Big|_{\substack{J^+=J^- \\ \bar{J}^+=\bar{J}^-}}, \qquad c, d = \pm. \tag{F.10}$$

The standard way of introducing interactions for this path integral is using $S_{\text{tot}} = S_{\text{free}} + S_{\text{int}}$, and making S_{int} a functional of variations, which looks like

$$\exp\big(iW[\bar{J}^\pm, J^\pm]\big) = Z[\bar{J}^\pm, J^\pm] = \langle\Omega|e^{iS_{\text{int}}}|\Omega\rangle Z_0[\bar{J}^\pm, J^\pm] \approx \big(1 + i\left\langle S_{\text{int}}\left[\frac{\delta}{\delta\bar{J}^\pm}, \frac{\delta}{\delta J^\pm}\right]\right\rangle\big) Z_0. \tag{F.11}$$

Because we are in an interaction picture, the ground state is no longer $|0\rangle$, but $|\Omega\rangle$. The interacting functional can clearly generate time- and anti-time-ordered four-point correlation functions, as well as describe them in terms of propagators, by using the two ways of writing the generating functional (F.8), (F.9). Doing this is also known as Wick's theorem. A loop expansion is then made by expanding $e^{iS_{\text{int}}}$ up to a relevant order, like we did in (F.11). After performing the variations, the terms in parentheses can be exponentiated again to find $W[\bar{J}^\pm, J^\pm]$.

For the four-point correlation function, we write T^+ (T^-) for (anti-)time-ordering, and get

$$
\begin{aligned}
&\sum_{c=\pm} c\langle T^c\big[\bar{\chi}_\alpha(x_1)\chi_\alpha(x_2)\bar{\chi}_\beta(x_3)\chi_\beta(x_4)\big]\rangle \\
&\quad= \sum_{c=\pm} c\left(\frac{1}{i}\right)^4 \frac{1}{Z_0}\frac{\delta^4}{\delta J^c_\alpha(x_1)\delta\bar{J}^c_\alpha(x_2)\delta J^c_\beta(x_3)\delta\bar{J}^c_\beta(x_4)} Z_0[\bar{J}^\pm, J^\pm]\Big|_{\substack{J^+=J^-\\ \bar{J}^+=\bar{J}^-}} \\
&\quad= \sum_{c=\pm} c\frac{1}{Z_0}\frac{\delta^3}{\delta J^c_\alpha(x_1)\delta\bar{J}^c_\alpha(x_2)\delta J^c_\beta(x_3)}\Big[-iS^{cd}_{0\beta\gamma}(x_4;x)J^d_\gamma(x)Z_0[\bar{J}^\pm, J^\pm]\Big]_{\substack{J^+=J^-\\ \bar{J}^+=\bar{J}^-}} \\
&\quad= \sum_{c=\pm} c\frac{1}{Z_0}\frac{\delta^2}{\delta J^c_\alpha(x_1)\delta\bar{J}^c_\alpha(x_2)}\Big[-iS^{cc}_{0\beta\beta}(x_4;x_3)Z_0[\bar{J}^\pm, J^\pm] \\
&\qquad\qquad + iS^{cd}_{0\beta\gamma}(x_4;x)J^d_\gamma(x)\bar{J}^b_\delta(y)iS^{bc}_{0\delta\beta}(y;x_3)Z_0[\bar{J}^\pm, J^\pm]\Big]_{\substack{J^+=J^-\\ \bar{J}^+=\bar{J}^-}} \\
&\quad= \sum_{c=\pm} c\frac{1}{Z_0}\frac{\delta}{\delta J^c_\alpha(x_1)}\Big[iS^{cc}_{0\beta\beta}(x_4;x_3)iS^{cd}_{0\alpha\gamma}(x_2;x)J^d_\gamma(x)Z_0[\bar{J}^\pm, J^\pm] \\
&\qquad\qquad - iS^{cd}_{0\beta\gamma}(x_4;x)J^d_\gamma(x)iS^{cc}_{0\alpha\beta}(x_2;x_3)Z_0[\bar{J}^\pm, J^\pm] + \mathrm{irr}\Big]_{\substack{J^+=J^-\\ \bar{J}^+=\bar{J}^-}} \\
&\quad= \sum_{c=\pm} c\frac{1}{Z_0}\Big[iS^{cc}_{0\beta\beta}(x_4;x_3)iS^{cc}_{0\alpha\alpha}(x_2;x_1)Z_0[\bar{J}^\pm, J^\pm] - iS^{cc}_{0\beta\alpha}(x_4;x_1)iS^{cc}_{0\alpha\beta}(x_2;x_3)Z_0[\bar{J}^\pm, J^\pm] + \mathrm{irr}\Big]_{\substack{J^+=J^-\\ \bar{J}^+=\bar{J}^-}} \\
&\quad= \sum_{c=\pm} ciS^{cc}_{0\beta\beta}(x_4;x_3)iS^{cc}_{0\alpha\alpha}(x_2;x_1) - ciS^{cc}_{0\beta\alpha}(x_4;x_1)iS^{cc}_{0\alpha\beta}(x_2;x_3).
\end{aligned}
\tag{F.12}
$$

We use $b, d = \pm$ as dummy indices, and 'irr' denotes irrelevant terms that drop out when taking the sum and $J^+ = J^-, \bar{J}^+ = \bar{J}^-$ is imposed. The result we have found is in terms of free propagators, and is one-particle irreducible.

F.2 2PI action in Schwinger-Keldysh formalism

The previous section was mainly a look into the path integral formalism, explaining the Gaussian integral and showing Wick's theorem. However, we want a two-particle irreducible action, and to obtain it, we have to introduce one more source. We look at

$$
W[\bar{J}, J, K] = -i\log\Big[\int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\Big\{i\big(S_{\mathrm{tot}}[\bar{\chi},\chi] + \bar{J}\chi + \bar{\chi}J - K\chi\bar{\chi}\big)\Big\}\Big], \tag{F.13}
$$

$$
K\bar{\chi}\chi = \int d\eta\ d^3x\ d\tilde{\eta}\ d^3y\ K_{\beta\alpha}(y,x)\chi_\alpha(x)\bar{\chi}_\beta(y). \tag{F.14}
$$

It is not hard to see that this new source will immediately produce the exact propagator upon variation, i.e.

$$
-\frac{\delta W[\bar{J}, J, K]}{\delta K_{\beta\alpha}(y,x)} = -\frac{1}{i}\frac{1}{Z}\frac{\delta Z[\bar{J}, J, K]}{\delta K_{\beta\alpha}(y,x)} = iS^{++}_{\alpha\beta}(x;y). \tag{F.15}
$$

We now define a derivative operator in terms of the expectation values of the fields,

$$
\mathcal{D}_{\beta\alpha}(y,x) \equiv -\frac{\delta^2 S_{\mathrm{tot}}[\bar{\chi},\chi]}{\delta\bar{\chi}_\beta(y)\delta\chi_\alpha(x)}\Big|_{\substack{\chi\to\langle\chi\rangle\\ \bar{\chi}\to\langle\bar{\chi}\rangle}}. \tag{F.16}
$$

As these expectation values are 0, the interactions do not influence this operator. We see that the previous section is a special case of what we have here, with a free theory and $K = 0$.

The 1PI action, for any general $W[\bar{J}, J, K]$ up to one loop, is defined as the Legendre transform with respect to $\bar{J}, J$,

$$
\Gamma^{1\mathrm{PI}} \equiv W[\bar{J}, J, K] - \int d\eta\ d^3x\Big(\bar{J}_\alpha(x)\frac{\delta W[\bar{J}, J, K]}{\delta\bar{J}_\alpha(x)} - \frac{\delta W[\bar{J}, J, K]}{\delta J_\alpha(x)}J_\alpha(x)\Big) \approx -i\mathrm{Tr}\big[\log(-i\mathcal{D}_0 - iK)\big]. \tag{F.17}
$$

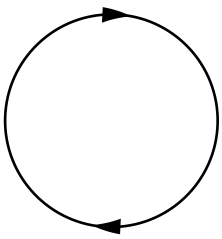

Figure F.1: 1 loop diagram contributing to the self energy Σ^{1PI}

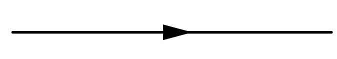

Figure F.2: After a single cut in figure F.1, we are left with a diagram that is still connected

We use (F.3) (only setting the condensates to 0 at the end), and see that it removes the $\bar{J}\chi + \bar{\chi}J$ terms in $W[\bar{J}, J, K]$. We then see that it is almost the same as (F.6), with an extra contribution from K. This is because it is similarly derived with a Gaussian integral. We note that there is also a 'contribution' $S_{\text{tot}}[\langle\bar{\chi}\rangle, \langle\chi\rangle]$, which then also vanishes on account of the condensates. The exact inverse propagator can be obtained from the effective action, which is

$$\frac{\delta^2\Gamma^{1\text{PI}}}{\delta\bar{\chi}_\alpha(x)\delta\chi_\beta(y)}\Big|_{\substack{\chi\to\langle\chi\rangle\\ \bar{\chi}\to\langle\bar{\chi}\rangle}} = (iS^{++})^{-1}_{\alpha\beta}(x;y) = -i\mathcal{D}_{0\alpha\beta}(x,y) - iK_{\beta\alpha}(y,x) + i\Sigma^{1\text{PI}}_{\alpha\beta}(x,y). \tag{F.18}$$

The first equality is true by definition, the second comes from varying (F.17). The self energy for the 1PI action is defined as consisting of contributions from one-particle irreducible Feynman diagrams, meaning diagrams that stay connected when you make a single 'cut' in a propagator, see figures F.1 and F.2. Notice that, in contrast to our results of the previous section, the (inverse) propagator is exact. It should be no surprise that the 2PI effective action is obtained from a second Legendre transform, with respect to $K_{\alpha\beta}$

$$\Gamma^{2\text{PI}} = \Gamma^{1\text{PI}} - \int d\eta\, d^3x\, d\tilde{\eta}\, d^3\tilde{x}\, \frac{\delta W[\bar{J}, J, K]}{\delta K_{\beta\alpha}(y,x)} K_{\alpha\beta}(x,y) = \Gamma^{1\text{PI}} + \text{Tr}\big[iS^{++}(x;y)K(x,y)\big], \tag{F.19}$$

with a trace over spinor and spacetime indices. The Schwinger-Dyson equations can be inferred from this equation: in absence of sources ($K = \bar{J} = J = 0$), varying the 2PI action with respect to the propagator yields 0. We obtain for the 2PI effective action

$$\begin{aligned}
\Gamma^{2\text{PI}}[iS^{++}] \approx& -i\text{Tr}\big[\log(-i\mathcal{D}_0 - iK)\big] + \text{Tr}\big[iS^{++}K\big] \\
=& -i\text{Tr}\big[\log\big((iS^{++})^{-1} + i\Sigma^{1\text{PI}}\big)\big] + \text{Tr}\big[iS^{++}\big((S^{++})^{-1} - \mathcal{D}_0 - \Sigma^{1\text{PI}}\big)\big] \\
=& i\text{Tr}\big[\log(iS^{++})\big] - i\text{Tr}\big[\log\big(1 + iS^{++}i\Sigma^{1\text{PI}}\big)\big] - \text{Tr}\big[\mathcal{D}_0 iS^{++}\big] - \text{Tr}\big[iS^{++}\Sigma^{1\text{PI}}\big] + \text{const.} \\
\approx& i\text{Tr}\big[\log(iS^{++})\big] - \text{Tr}\big[\mathcal{D}_0 iS^{++}\big] + \Gamma^{2\text{PI}}_2[iS^{++}] + \text{const.}.
\end{aligned} \tag{F.20}$$

The first step was just substituting (F.17), the second step was substituting (F.18) for K. In the third line, we split the logarithm, and took the -1 exponent out of the logarithm for the first term. We also note that $\text{Tr}\big[iSS^{-1}\big]$ =constant is irrelevant, as it is not contributing to the equations of motion. For the final step, we expand $\log(1+x) = x - x^2/2 + ...$, and see that the first term of the expansion drops out. The exact inverse propagator, now from the 2PI action truncated at 2 loops with indices restored, is

$$\frac{\delta\Gamma^{2\text{PI}}[iS^{++}]}{\delta S^{++}_{\alpha\beta}(x;y)} = i(S^{++})^{-1}_{\alpha\beta}(x;y) - i\mathcal{D}_{0\beta\alpha}(y,x) - iK_{\beta\alpha}(y,x) + \frac{\delta\Gamma^{2\text{PI}}_2[iS^{++}]}{\delta S^{++}_{\alpha\beta}(x;y)} = 0 \tag{F.21}$$

$$\begin{aligned}
\implies (iS^{++})^{-1}_{\alpha\beta}(x;y) =& -i\mathcal{D}_{0\beta\alpha}(y,x) - iK_{\beta\alpha}(y,x) + \frac{\delta\Gamma^{2\text{PI}}_2[iS^{++}]}{\delta S^{++}_{\alpha\beta}(x;y)} \\
=& -i\mathcal{D}_{0\beta\alpha}(y,x) - iK_{\beta\alpha}(y,x) + i\Sigma^{2\text{PI}}_{\beta\alpha}(y,x,iS^{++}).
\end{aligned} \tag{F.22}$$

The contribution $-iK$ comes from differentiation of the vanishing term $S_{\text{tot}}[\langle\bar{\chi}\rangle, \langle\chi\rangle]$ we mentioned before, and we note once more the definition of the self-energy

$$i\Sigma^{2\text{PI}}_{\beta\alpha}(y,x,iS^{++}) \equiv \frac{\delta\Gamma^{2\text{PI}}_2[iS^{++}]}{\delta S^{++}_{\alpha\beta}(x;y)}. \tag{F.23}$$

We can compare (F.18) and (F.22) and see that the self energies coincide, $\Sigma^{1PI} = \Sigma^{2PI}$. This implies that the 2PI self-energy only gets contributions from 1PI diagrams. If we see varying with respect to a propagator as a 'cut' in the diagram, we see that $\Gamma_2^{\rm 2PI}$ should consist of only those diagrams that are give us 1PI-diagrams when getting a cut, i.e. 2PI diagrams. This is exactly what we wanted to find. According to the prescription in [34], we can shift the fields $\chi \to \langle\chi\rangle + \chi$ and $\bar{\chi} \to \langle\bar{\chi}\rangle + \bar{\chi}$, then constructing Feynman diagrams with these new vertices and keeping only the diagrams that would yield "proper self-energy diagrams upon opening one line." The condensates are set to zero, so we are left with the original 4-vertex. The Wick expansion we performed in (F.12), while not exact, can still be used for the general form corresponding to these diagrams.

An extension to the Schwinger-Keldysh formalism is easily made. We realise we were already looking at the '++' sources and action, so like we did before, we include the negative action and sources,

$$
\begin{aligned}
W[\bar{J}^{\pm}, J^{\pm}, K^{\pm\pm}] = -i\log\Big[\int \mathcal{D}[\bar{\chi}]\mathcal{D}[\chi]\exp\Big\{ i\big(S_{\rm tot}[\bar{\chi}^{+},\chi^{+}] + \bar{J}^{+}\chi^{+} + \bar{\chi}^{+}J^{+} - K^{++}\chi^{+}\bar{\chi}^{+} \\
- S_{\rm tot}[\bar{\chi}^{-},\chi^{-}] - \bar{J}^{-}\chi^{-} - \bar{\chi}^{-}J^{-} - K^{--}\chi^{-}\bar{\chi}^{-}\big)\Big\}\Big].
\end{aligned}
\tag{F.24}
$$

The derivations are completely analogous, except for minus signs appearing for the negative time path. The 2PI effective action becomes

$$
\Gamma[iS^{cd}_{\alpha\beta}] = \sum_{a,b=\pm}\Big(-c\mathcal{D}_{0\beta\alpha}(y,x)\delta_{cd}iS^{ab}_{\alpha\beta}(x;y) + i\mathrm{Tr}\big[\log\big(iS^{cd}_{\alpha\beta}\big)\big] + \Gamma_2[iS^{cd}_{\alpha\beta}]\Big),
\tag{F.25}
$$

where the trace now works on spinor, spacetime and Schwinger-Keldysh indices, which we have chosen to write fully in the first term.

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