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The Knowability Paradox: A Step Towards Consensus

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Abstract

The knowability paradox is concerned with difficulties for anyone willing to accept the formalisation of the knowability principle as $p \to \Diamond Kp$ by showing that, under some assumptions, the absurd $p \to Kp$ follows. Many approaches to the paradox have been proposed, but little consensus on its plausibility and consequences has been reached. This thesis selects logical and philosophical criteria by which these approaches can be compared in a standardized way. After applying these criteria to the intuitionistic, paraconsistency and situations approaches, it is argued that the situations approach is the strongest with respect to the selected criteria.

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1 Introduction

One of the fundamental debates in philosophy is concerned with the relation between knowledge and truth. An important view, realism, says that truth is something that need not be mind-dependent, e.g. mathematical truths. Although some truths, such as “I am feeling nostalgic”, are mind-dependent, the realist argues that truth is not completely dependent on our ideas and concepts. Because of this partial independence, the realist thinks that there are statements that are true, but that we cannot possibly come to know. For example, we could not possibly come to know statements like ‘a flower bloomed in exactly this location exactly one billion years ago’.

Contrarily, many philosophical theories claim that all truths are knowable.[1] This view, often referred to as the *moderate anti-realist view*, can be formalized by the knowability principle:

(KP) $$\forall p(p \to \Diamond Kp)$$

Intuitively (KP) seems reasonable, but it was shown in (Fitch, 1963)[2] that it has problematic consequences. Using some reasonable modal and epistemic axioms, Fitch showed that if one accepts (KP), then one is also obliged to accept that any truth is known. This result[3] is what I will refer to as the *hard anti-realist* consequence:

(CONS) $$\forall p(p \to Kp)$$

Most people agree that there are truths that we do not know. An instance of such unknown truth could be that the number of leaves on the tree in my mother’s garden was odd at the time of writing. It could also be that it was even, but in any case one of these two statements is an unknown, but knowable, truth as nobody every bothered to count them. Therefore, it is mostly agreed upon that (CONS) is an absurd theorem. So in showing that the knowability principle entails this *hard anti-realist* consequence, Fitch has made it hard for the anti-realist to maintain the knowability principle.

The knowability principle is an essential part of many *anti-realist* philosophical theories, and thus rejecting (KP) to prevent accepting (CONS) is something the anti-realist is not so easily willing to do. Furthermore, intuitively it feels incorrect that accepting that all truths are knowable collapses into accepting that all truths are known. This is the reason that this problem is often referred to as Fitch’s Paradox, or the knowability paradox, and it has been a subject of debate ever since its publishing.

As many feel that the consequences of Fitch’s proof are unacceptable, there are several kinds of approaches to handling these consequences:

i. Reject the knowability principle and argue that the *anti-realist view* need not accept it;
ii. Accept (CONS) and argue why it is not an absurd theorem;
iii. Reject one or more of the modal and epistemic axioms used in the proof;
iv. Argue for the replacement of classical logic;
v. Syntactically or semantically restrict the statements for which (KP) holds.

In my thesis I will focus on the logically oriented side of the debate, so the first two approaches are outside of this scope. Approach (iii) has proven not to be fruitful, and in (Williamson, 1993) it is even argued that this approach has no prospect. I will discuss two approaches of the

[1] I will interpret ‘K’ as ‘it is known now that’. However, in the literature ((Williamson, 1982), (Dummett, 2009), (Percival, 1990), (Beall, 2000), (Priest, 2009), (Edgington, 1985)) it is often interpreted as ‘it was, is or will be known that’. It seems to go unnoticed that the latter interpretation would make knowledge lose its factivity in general, a property that is essential to Fitch’s proof. The matter of temporality will not be further discussed in my thesis, but is worth further research.

[2] Although his name is tied to the proof, Fitch attributes this proof to an anonymous referee report, which later turned out to be (Church, 2009).

[3] A detailed proof is given in section 2.

fourth kind and one of the last kind. The first discussed approach, the 'intuitionistic approach', replaces classical logic with intuitionistic logic ((Williamson, 1982), (Dummett, 2009), (Percival, 1990), (Proietti, 2012)). Secondly, I will discuss the 'paraconsistency approach' that removes some fundamental axioms from classical logic so that *reductio* will no longer be valid ((Beall, 2000), (Priest, 2009), (Wansing, 2002)). The third approach restricts the knowability principle using an 'Actuality'-operator to block the proof right at the start, and is known as the 'situations approach' ((Edgington, 1985), (Williamson, 1987), (Rabinowicz & Segerberg, 1994)).

There are many different approaches to the knowability paradox that have been argued about for over half a century. And although there have been many contributions to the debate, there does not seem to be a consensus on the plausibility and the consequences of Fitch's proof. The goal of my thesis is to carefully select criteria by which to compare different approaches. These criteria will then be applied to the intuitionistic, paraconsistency and situations approaches to get a better insight into what they have to offer to the debate. Providing these criteria can bring us a step further towards a consensus regarding one of the fundamental problems in epistemology and artificial intelligence.

By applying the selected criteria I will seek for an answer to the main research question of my thesis:

"What are the logical and philosophical strengths and weaknesses of the intuitionistic, paraconsistency and situations approaches to the knowability paradox?"

For each approach we must answer the following two questions, to get a concrete answer to the main question:

1. *"What is this approach's logical plausibility?"*
2. *"What is this approach's philosophical plausibility"*

Plausibility should be interpreted as a measure of how appealing an approach is from a logical or philosophical perspective. It is important that all approaches will be compared by the same standards, something that the selected criteria will be able to provide. Some criteria might be inherently less appealing to a certain approach, therefore the grounds on which they will be selected needs to be well-chosen.

Intelligence, consciousness, knowledge, truth and their relations are fundamental concepts in the field of artificial intelligence. Any research that is concerned with these concepts and their relations can be regarded as fundamental research in artificial intelligence. As the debate surrounding the knowability paradox is concerned with the relation between knowledge and truth, my thesis can be regarded as a contribution to fundamental research in artificial intelligence. However, one could also think of more pragmatic uses of a consensus in the realism versus anti-realism debate. For example, anti-realism often is interpreted using intuitionistic logic, a logic whose type theory also has been used as the basis for logic programming[4], similarly with paraconsistent logic[5]. This debate could therefore provide philosophical motivation for these programming styles. Also, if an AI system knows that all truths are knowable, it could use this as a heuristic. For example, if it knows that some statement is not knowable, for example within a reasonable time limit, it might assume its negation, or stop looking for a proof, for pragmatic uses.[6]

To get a better understanding of the reasoning used in the knowability paradox, the paradox itself will first be discussed in section 2. Using articles that are central to this debate, section 3 discusses three different ways Fitch's proof is called to a stop. This insight in the intuitionistic approach (section 3.1), the paraconsistency approach (section 3.2) and the situations approach (section 3.3) is essential to get a grasp of their logical and philosophical strengths and weaknesses. Section 4.1 will discuss several possible criteria to answer the research questions and

[4]See (Hodas & Miller, 1994).

[5]See (Blair & Subrahmanian, 1989), (Belnap, 2019)

[6]This is similar to the interpretation of negation as 'negation-as-failure', discussed in (Pearce, 1999).

argues that the selected criteria are optimal. These criteria will be applied to the approaches under discussion in section 4.2, after which the thesis will be concluded in section 5.

2 Knowability Paradox

The *moderate anti-realist view*, formalised by (KP), is maintained by many different philosophical theories, but for lack of a better term I will call someone that maintains (KP) a *verificationist*. The well known proof first published in (Fitch, 1963) starts with this principle combined with the non-omniscience principle. We suppose that we, as humans, are not omniscient, and thus that there exist unknown truths. This principle can be formalised as:

(NonO) $$\exists p(p \wedge \neg Kp)$$

An example of an instance of this principle is the 'garden'-example discussed in section 1. Next to these principles, the proof uses the two epistemic axioms that knowledge can be distributed over conjunction and that knowledge is factive. They can respectively be formalised as:

(K) $$K(p \wedge q) \rightarrow (Kp \wedge Kq)$$

(Fact) $$Kp \rightarrow p$$

Furthermore, the two modal axioms used are *Necessitation* and *Duality*. Respectively:

(Nec) $$\text{If } \vdash p, \text{ then } \vdash \Box p$$

(Dual) $$\Box \neg p \vdash \neg \Diamond p$$

If (NonO) is true, then so is an instance of it:

(1) $$p \wedge \neg Kp$$

If we apply this truth to (KP), we get:

(2) $$(p \wedge \neg Kp) \rightarrow \Diamond K(p \wedge \neg Kp)$$

Using Modus Ponens we get the following from (1) and (2):

(3) $$\Diamond K(p \wedge \neg Kp)$$

We now continue by making an assumption for *reductio* and applying the modal and epistemic axioms to it.[7]

(4)	$K(p \wedge \neg Kp)$	Assumption [for reductio]
(5)	$Kp \wedge K\neg Kp$	from (4) by (K)
(6)	$Kp \wedge \neg Kp$	from (5) by (Fact)
(7)	$\neg K(p \wedge \neg Kp)$	from (4)-(6) by *reductio*
(8)	$\Box \neg K(p \wedge \neg Kp)$	from (7) by (Nec)
(9)	$\neg \Diamond K(p \wedge \neg Kp)$	from (8) by (Dual)

[7] At this point one can also postulate the negation of (3) to claim that it is not the case that it is possible to know that some statement is both true and unknown. It is this postulation that some advocates of the paraconsistency approach (found in section 3.2) do not agree with.

First we have found that (3) follows from (NonO), but now we find that (3) is contradicted by (9), something that followed from the axioms alone. We therefore apply *reductio* to our first assumption, (NonO), to get its negation:

$$\neg\exists p(p \wedge \neg Kp) \tag{10}$$

Using rules that are valid in both intuitionistic and classical logic, this can be rewritten as:

$$\forall p(p \rightarrow \neg\neg Kp) \tag{11}$$

Using the classically valid, but intuitionistically invalid, double negation elimination we get to the dreaded *hard anti-realist* consequence:

$$\forall p(p \rightarrow Kp) \tag{12}$$

So from the assumption that all truths can be known, (KP), we derive that instead all truths are known, (12). This is what the verificationist is obliged to accept if he accepts (KP), Fitch's proof and classical logic.[8] However, as we have seen at the beginning of this section, there are many truths that we think will never be known, although they are possible to know.

Because of the discrepancy between what many feel to be intuitively the case and what Fitch's proof has shown, the problem just explained is seen as paradoxical and thus referred to as the knowability paradox. To be able to maintain the knowability principle without accepting the rather absurd theorem that all truths are known, the derivation from (KP) to (12) has to be blocked.

3 Approaches

3.1 Intuitionistic Approach

In the beginning of the 20th century a new philosophy of mathematics was founded by the Dutch mathematician L.E.J. Brouwer. This philosophy, called intuitionism, has a constructive view that sees mathematics as a creation of the mind. According to the intuitionist a mathematical statement can only be true, if in principle there is a construction possible that proves it to be true. Such a construction might be a proof or a pattern of reasoning that is conceived in the mind of an ideal reasoner. This means that all truths can in principle be known, an anti-realist view that can be formalized by the knowability principle. It is important to note that intuitionism originally was about mathematical truths, although it has since been argued that it cannot be seen apart from a more common discourse.[9] However, this remains a subject of debate and is a possible objection to the application of intuitionism to a paradox of knowledge and truth in the broader sense.

Intuitionistic logic is the formalization of intuitionism based on the *Brouwer-Heyting-Kolmorogov-interpretation*, a proof-theoretic interpretation of the connectives. This definition is very important, since in intuitionism knowing that a certain statement is true, means that one has a proof of that statement. This constructive view on the truth makes intuitionistic logic fundamentally different from classical logic. Most important for the current discussion are the interpretation of the negation and the lack of double negation elimination.

In intuitionism, the negation $\neg A$ of a statement A is proven by showing that there is a construction from A to an absurdity. So in other words, it has to be shown that $A \rightarrow \bot$. This way it can be proven that there cannot exist a proof of A, thus it proves the negation $\neg A$. Given this interpretation it becomes clear why there is no double negation elimination, $\neg\neg A \vdash A$, in

[8] Most realists are classicist and would agree with this reasoning. However, anti-realists are typically intuitionists, and would not agree to this so readily. This is further discussed in section 3.

[9] See (Dummett, 1975).

intuitionistic logic. The double negation means 'it cannot be shown that any proof of A can be be constructed into a proof of an absurdity', or in other words there is something in principle in the way of a proof that A cannot be proven. This is not equivalent to having a proof of A, thus double negation elimination is omitted from the axioms of the logic nor is it derivable in the system.

The use of intuitionistic logic to solve knowability paradox was first advocated in (Williamson, 1982). Although not an anti-realist himself, Williamson argues that the anti-realist should accept intuitionistic logic because its treatment of the negation blocks Fitch's proof. Specifically, the lack of double negation elimination makes it impossible to get to $p \to Kp$ in step (12) from $p \to \neg\neg Kp$ in step (11).[10] Using intuitionistic logic prevents that accepting the knowability principle forces one to accept the hard anti-realist principle, but instead makes one accept

(11) $$p \to \neg\neg Kp$$

Williamson claims that this statement is not absurd, although on first sight it might seem to make a threat. Statement (11) forbids the intuitionist to claim an instance of a truth that is not known, as now p implies that it is not unknown. Surely, the intuitionist would not want to claim that all truths are in fact known, so this seems problematic. However, the intuitionist can still claim that not all truths are known using $\neg\forall p(p \to Kp)$, which does not intuitionistically entail $\exists p\neg(p \to Kp)$. We could never be in a position to say that we know all truths, so under the intuitionistic interpretation of the quantifiers we can claim its negation, $\neg\forall p(p \to Kp)$. But to accept $\exists p\neg(p \to Kp)$ we would need to produce an instance of a statement that is true, but not known. It is now easily seen that the latter of these conditions is not implied by the former, thus there is no entailment from $\neg\forall p(p \to Kp)$ to $\exists p\neg(p \to Kp)$. Since the intuitionist can still claim that not all truths are known, the intuitionistically valid derived statement (11) is not absurd according to Williamson.[11]

Not only is (11) not absurd, (Dummett, 2009) argues that the intuitionist should be happy to accept it. Dummett even goes further to say that the intuitionist might prefer (11) over (KP) as the formalization of the knowability principle. Given the intuitionistic interpretation of negation, $\neg\neg Kp$ might be read as 'there is an obstacle in principle in our way of being able to deny that it is ever known that p', or equivalently 'the possibility to know p remains open forever'. This is exactly what the anti-realist view on the relation between knowledge and truth is, and thus should this view be formalized by (11) according to Dummett.

Unfortunately, not everybody agrees that (11) is perfectly suitable, as (Percival, 1990) shows that its acceptance has some problematic consequences. In intuitionistic logic $p \to \neg\neg Kp$ can be rewritten as

(NFact) $$\neg Kp \to \neg p$$

Since knowledge implies truth, we have $\neg p \to \neg Kp$ as a theorem by contraposition. Combining this with our intuitionistically valid statement (NFact) gives $\neg Kp \leftrightarrow \neg p$. In other words, $\neg Kp$ and $\neg p$ are logically equivalent. But this cannot be the case, let p for example be a true mathematical statement. In that case $\neg p$ is non-contingent, while $\neg Kp$ is contingent. Percival argues that it is hard to accept that two statements are logically equivalent, but are neither both contingent nor both non-contingent. Also, we have previously established that there are true statements, p, that are not known, $\neg Kp$; violating one direction of this equivalence.

Furthermore, Percival gives an additional reason why (NFact) is not generally true. If one assumes $\neg Kp \land \neg K\neg p$ and applies (NFact) to it twice using modus ponens, the contradiction $\neg p \land \neg\neg p$ is derived. Therefore, if one accepts (NFact),

[10] As is standard in the literature, I will omit the quantifiers unless they are relevant. If no quantifier is given it is to be interpreted as its universal closure.

[11] Williamson notes that for decidable Kp, $\neg\neg Kp$ entails Kp. Therefore, for p of which Kp is decidable it would still hold that $p \to Kp$. This is a rather problematic consequence that has seemed to go unnoticed by Williamson.

(NUnd) $\neg(\neg Kp \wedge \neg K\neg p)$

is a theorem, meaning that no statement remains forever undecided. But since there are many statements that remain undecided forever, we conclude that (NUnd) is not generally true. Given that (NUnd) is a consequence of (NFact), we also conclude that (NFact) is not generally true.

Percival has argued that (NFact) leads to some problematic consequences, and thus that the intuitionistic approach has not solved the knowability paradox. On the other hand, (DeVidi & Solomon, 2001) argue, and would be endorsed by Dummett, that these problems arise due to a classical, rather than intuitionistic, reading of the negation, a debate that will not be further discussed. Percival suggests that the strongest form of defence for the intuitionistic approach is to give an independently plausible semantics for the connectives validating both intuitionistic logic and (NFact).

Such a semantics for the connectives and the K-operator is given in (Proietti, 2012). He uses standard intuitionistic Kripke models (IKMs) that define a reflexive, transitive accessibility relation between information states. By being monotonic these models ensure that once something is true, it is true forever. Furthermore, the knowledge operator K is informally defined such that $K\varphi$ means 'φ is the case in all accessible states that have a history isomorphic to the present state'. This definition of K ensures perfect recall of the order in which information is obtained and represents the knowledge of an ideal reasoner.

To show that (11) can hold in these semantics while (12) does not, and thus showing the non-entailment, Proietti defines a class of models, $\mathtt{MAK}'$, in which (11) is valid. He then gives a counter-model in $\mathtt{MAK}'$ to (12), thus showing that (11) need not entail (12). The class $\mathtt{MAK}'$ is defined as the set of models that satisfy:

$$\forall w \forall v(wR_{\leq}v \rightarrow \exists z(vR_{\leq}z \wedge \forall z'(zR_K z' \rightarrow z' \equiv z)))$$ [12]

This condition implies that from any state w another state z is accessible, because of transitivity, from which no states are epistemically accessible that give different truth-values to atomic propositions. This condition is very strict, and although it blocks the paradox in the general sense, the paradox might still hold in all cases outside of $\mathtt{MAK}'$. Furthermore, this condition is not necessary for (11) to hold. As the intuitionist is committed to (11), it is important for them to find a *necessary* condition in which (11) is valid and (12) is not. Therefore, the current condition is not without its problems. Together with the monotonicity and the knowledge operator representing the ideal reasoner, this makes one likely to object that these semantics might block the proof but do not provide an appropriate semantics for common discourse.

To conclude, the intuitionistic approach argues for the use of intuitionistic logic, rather than classical logic, as its lack of double elimination negation blocks Fitch's proof at the penultimate step. Williamson argues that this urges the anti-realist to accept intuitionistic logic, and Dummett even thinks the intuitionistically derivable $p \rightarrow \neg\neg Kp$ should replace the current knowability principle. One of the objections to this approach argues that the penultimate step of Fitch's proof still has problematic consequences. The main philosophical objections are that the interpretation of the negation and the proposed semantics seem to be inappropriate in everyday reasoning involving empirical propositions, such as Kp.

3.2 Paraconsistency Approach

A logic is said to be *explosive* if from a contradiction any arbitrary statement follows. This rule of inference is called *ex contradictione quodlibet* and can, for example syntactically, be formalised as:

(ECQ) $A, \neg A \vdash B$

[12]Two states are informationally equivalent ($\equiv$) if they give the same truth-values to all atomic propositions.

Many logics, such as classical logic and intuitionistic logic, maintain this rule. Any logic that invalidates this rule is called a paraconsistent logic, as it allows an inconsistency without becoming trivial. Some behave like classical logic, boundary cases excepted, and some invalidate fundamental classical axioms or inference rules.

The motivation for paraconsistent logics is twofold, philosophically and more pragmatic. Several well known paradoxes, such as the Liar Paradox and the Knower Paradox, give us reason to believe that there might be statements that are both true and false. Also, in everyday use there are statements that are neither completely true nor completely false, or maybe neither true nor false at all. More pragmatically, human reasoning and information databases often contain contradictory elements, which does not mean that they believe, or should believe, anything to be true. So in order to formalise human reasoning, or optimize machine reasoning, a paraconsistent logic can be very useful, if not necessary.

Step (7) of the proof strongly rests on the assumption that no instance of $Kp \wedge \neg Kp$ can possibly exist. In other words, an unrestricted claim that it is not possible for any statement to be both known and unknown:

(NIgn) $$\neg\Diamond(Kp \wedge \neg Kp)$$

Since step (4) has made an assumption that leads to an instance that is contradicted by (NIgn), it therefore has to be false. However, (Beall, 2000) argues that there is a reason independent from the knowability paradox to doubt the validity of (NIgn), the infamous Knower Paradox.

Consider the following statement:

(κ) κ is unknown

Now suppose that κ is known, in which case factivity states it to be true. And since κ states of itself that it is unknown, it follows that both $K\kappa$ and $\neg K\kappa$ are true. If we suppose that κ is unknown, we see that κ is true. But by observing this we know that κ is true, so once again both $K\kappa$ and $\neg K\kappa$ are true.

Beall argues that κ produces an instance of a statement that is both known and unknown, which invalidates (NIgn) as an axiom. He therefore demands that the realist provides a clear answer to the knower paradox, before one can accept (NIgn) as an axiom used to contradict step (3).[13] Although some versions of the proof postulate (NIgn), and thereby omit step (4)-(8), we have seen in section 2 that the negation of (3) can also be syntactically proven using (K), (Fact), (Nec), and (Dual). Therefore, merely appealing to the possibility of both knowing and not knowing a certain statement does not seem to be sufficient. Nor does it give us reason to think that any unknown statement can possibly be both known and unknown, a problem that is discussed at the end of this section.

The rule of contraposition, or *reductio*, plays an essential role in Fitch's proof. It states that if an assumption leads to something that is contradicted by a theorem, then one can derive the negation of the assumption. Formally, we can write this in natural deduction as:

$$\frac{\begin{array}{c}\underline{[\beta]}\\ \vdots \\ \gamma\end{array} \qquad \neg\gamma}{\neg\beta}$$

Without this rule the negation of (3) cannot be syntactically proven, and thus (NIgn) has to be taken as an axiom to get something contradicting step (3).[14] But even when it is postulated and step (3) and (NIgn) lead to a contradiction, without contraposition the *reductio* for step (10) cannot be performed and the proof is blocked.

[13] If (3) were true, there would be some world in which knowledge is inconsistent. This is what is contradicted by (NIgn).

[14] Taking (NIgn) as an axiom is exactly what Beall has argued against using the Knower Paradox.

To block all versions of the proof (Priest, 2009) uses an extension of his paraconsistent logic LP, that does not have contraposition as a rule. Not only does it block Fitch's proof, Priest also defines a semantic for this logic and gives a counterexample that makes $p \to \Diamond Kp$ hold and $p \to Kp$ not hold. Priest's logic allows true contradictions, so that a certain statement might be true, false or both.[15] Also, it has a trivial world ∞, accessible by all other worlds, in which all statements are both true and false. The most relevant truth-conditions for a world w in this semantics are:

$$1 \in v_w(\Diamond A) \text{ iff for some } w' \text{ such that } wRw', 1 \in v_{w'}(A)$$

$$1 \in v_w(KA) \text{ iff for all } w' \text{ such that } wRw', 1 \in v_{w'}(A)$$

$$1 \in v_w(A \to B) \text{ iff for all } w' \text{ such that } wRw', \text{ if } 1 \in v_{w'}(A), \text{ then } 1 \in v_{w'}(B)$$

$$0 \in v_w(A \to B) \text{ iff for some } w' \text{ such that } wRw', 1 \in v_{w'}(A) \text{ and } 0 \in v_{w'}(B)$$

These truth-conditions combined with the trivial world make $\Diamond A$ true and $A \to B$ false in all worlds for arbitrary A and B. Therefore, $A \to \Diamond KA$ is trivially true in any world. Furthermore, Priest constructs a model in which a world w_0 that makes p true can access another world w_1 in which p is not true. This makes Kp and thereby $p \to Kp$ not true at w_0, providing a counterexample to Fitch's proof. Although this model has given a counterexample to Fitch's proof, the triviality of the $\Diamond$ and the $\to$ are not very satisfactory.

A different logic with a less trivial semantics, called RN4, is proposed in (Wansing, 2002). The negation used in this logic is a constructive strong negation, formalised as $\sim$. As discussed in section 3.1, under the normal constructive interpretation of the negation $\neg A$ is true just in case there cannot be a proof of A. Under the strong interpretation $\sim A$ is true just in case there is a construction that verifies the falsity of A, or, in other words, there is a disproof of A. As a construction from a proof of A into a proof of B does not guarantee the existence of a construction from a disproof of B into a disproof of A, the use of this negation makes contraposition fail. Wansing appeals to the paraconsistent logic N4 proposed in (Nelson, 1949), that uses the strong negation to treat truth and falsity in the same constructive manner.

However, Wansing notes that a mere paraconsistent logic is not enough. If one postulates, or derives, the strong analogue of (3), the following can be derived:

1.	$(p \wedge \sim Kp) \to \Diamond K(p \wedge \sim Kp)$	Instance of (KP)
2.	$\sim \Diamond K(p \wedge \sim Kp)$	Strong analogue of (3)
3.	$(p \wedge \sim Kp) \to \sim \Diamond K(p \wedge \sim Kp)$	2, empty implication introduction
4.	$(p \wedge \sim Kp) \to (\Diamond K(p \wedge \sim Kp) \wedge \sim \Diamond K(p \wedge \sim Kp))$	1, 3, conjunction introduction

This is problematic as something that is true and unknown intuitively need not imply that a statement is both verified and falsified. To avoid this problematic derivation, Wansing appeals to a relevant implication that does not permit empty implication introduction. The ternary relational semantics[16] for this implication is defined such that an implication is verified on the basis of some information state, t, if and only if the antecedent being verified on the basis of some information state, u, implies that the consequent is verified on the basis of an information state, s, that contains the combination t and u. This way it is not allowed to introduce any antecedent to a consequent that has already been derived. By appealing to RN4 in which both contraposition and empty implication introduction fail, Wansing blocks the proof at step (6).[17] The appeal to a combination of constructivism and a relevant implication is unusual. However, the knowability paradox deals with a problem concerning fundamental concepts of philosophical theories. So even though Wansing's approach is not motivated by an *existing* philosophical theory, it might urge the verificationist to embrace a combination of

[15] The truth valuation of a world is as a set that can be $\{1\}$ (true), $\{0\}$ (false) or $\{1, 0\}$ (both).
[16] See (Routley & Meyer, 1973).
[17] Or similarly at (3) if its negation is postulated.

constructivism, relevantism and paraconsistency. Whether this is desired, I leave up to the verificationist.

As Beall observes, a difficulty arises for the verificationist if he appeals to the paraconsistency approach. By blocking Fitch's proof at step (6), one has to accept that $p \wedge \neg Kp$ leads to the possibility of p being both known and not known. As the Knower Paradox seems to show, there are instances of truths that are both known and unknown. However, the verificationist now has to accept that *all* unknown truths, q, are both known and unknown in some possible world, which is a far less plausible statement. The verificationist could motivate this result, by constructing a paradoxical statement, λ, such that in some possible world knowing q is equivalent to knowing λ. In that way all unknown truths are both known and unknown in some possible world, but constructing such a statement might not be that easy and should be left to be constructed by the verificationist that appeals to paraconsistency.

To conclude, the paraconsistency approach argues for the use of paraconsistent logics to block Fitch's proof at the *reductio*. Beall claims that the Knower Paradox provides difficulties worrying enough for the proof, so that this paradox should be solved first before one can accept the proof. Priest appealed to a paraconsistent logic that allows for true contradictions and rejects contraposition, but at the cost of using a trivial model. Wansing used a paraconsistent logic with relevant implication and constructive strong negation to remove contraposition and empty implication introduction. The greatest philosophical objection to the paraconsistency approach is its obligation to accept that all unknown truths are both known and unknown in some possible world.

3.3 Situations Approach

Modal logic is often interpreted using a possible worlds semantics. A possible world is described by the truth value of all atomic statements, making two worlds different if they only differ in the truth value of one, possibly irrelevant, statement. For example, when I throw a die there is not just one possible world where I throw a six. But there is the possible world where I throw a six and a lorry driver in Japan makes a right turn, but there is also a possible world where I throw a six and the same lorry driver makes a left turn. On the other hand, when one talks about possible situations, there is just one possible situation in which I throw a six. As possible situations leave out most details about the world, knowledge of counterfactual situations is concerned with possible situations rather than possible worlds. Dorothy Edgington argues that this notion of possibility is essential for the approach advocated in (Edgington, 1985).

Edgington argues that there is a vital difference between knowing *in* a situation that something holds and knowing *of* a situation that something holds. To get a better understanding of her argument, it is important to get a grasp of what counterfactual situations are. A counterfactual is a statement of the form 'Had A not happened, then B would have been the case'. In possible situations that differ from our situation, there are possible thinkers that can have knowledge of counterfactual situations. These counterfactual situations might exactly describe our situations, and thus knowledge of our situation might be ascribed to these possible thinkers.[18] To formalise an example of such knowledge, Edgington appeals to the 'Actuality'-operator describing 'Actually it is the case that...'. This operator cannot be defined using the resources of basic modal language, and thus is introduced.[19] The truth condition of the 'A'-operator is '$A\varphi$ is true at any situation iff φ is true at the actual situation'.

For example, let there be an animal in a very excluded location that is last of a species that will go extinct in a short while. An expedition is set up to find out whether the animal does or does not howl at the moon in its last moments, let us call this truth p. As the expedition is very hard, it is not guaranteed that they will find the animal in time. However, it is granted that if they do, they get knowledge of p. Let us differentiate between two situations, one where the expedition succeeds, s_1, and the actual situation where it fails, s_2. In s_1 it is the case that Kp, but they also know that p still would have been true had they not succeeded. So in s_1

[18] An objection to this line of reasoning is discussed at the end of this section.

[19] Edgington gives several examples of the need for the 'Actuality'-operator.

there is counterfactual knowledge that in s_2 it is not known that p, but that p still is true. As s_2 is the actual situation, we can describe this knowledge in s_1 as $KA(p \wedge \neg Kp)$. Given that s_1 is a possible situation seen from s_2, we now have an example of $\Diamond KA(p \wedge \neg Kp)$ in s_2 while $p \wedge \neg Kp$ is also the case.

Edgington argues that using the A-operator and this notion of possibility there is no inconsistency between the knowability principle and the statement $p \wedge \neg Kp$. She therefore alters the knowability principle to:

(EKP) $$Ap \rightarrow \Diamond KAp$$

In words, if something is actually true, then there is a possible situation in which it is known that it is actually true. Now if we apply this principle to the infamous $p \wedge \neg Kp$ using Modus Ponens, we get $\Diamond KA(p \wedge \neg Kp)$. As we have shown an instance of this statement in the howling animal example, we see that, contrary to (KP), the new (EKP) does not yield an inconsistency when applied to $p \wedge \neg Kp$.

Although (EKP) is consistent with $p \wedge \neg Kp$, (Rabinowicz & Segerberg, 1994) noticed that there are still some problematic consequences. The problems arise when the normal truth-condition for the A-operator is combined with the truth-conditions for necessity, $\Box$, and knowledge, K. Rabinowicz and Segerberg showed that under these semantics $Ap \rightarrow \Box KAp$ is *strongly* valid: in all worlds[20] in all models it is the case that 'if a statement is actually true it is necessarily known that it is actually true'.[21] So the paradox does not seem to have been solved, and the problematic consequence has even gained necessity.

To prevent this return of the paradox, Rabinowicz and Segerberg appeal to a two-dimensional framework to interpret actuality and knowledge. Instead of the fixed actual world, this proposal uses a variable perspective. It is said that some formula φ is true in a world v from the perspective of some world w, or formally ${}^{w}\vdash_{v} \varphi$. The truth condition of the A-operator then is:

$${}^{w}\vdash_{v} A\varphi \text{ iff } {}^{w}\vdash_{w} \varphi$$

So we see that from the perspective of w, w is the actual world. Using an interpretation of the knowledge operator that has *variable perspective*, and thus does not demand the perspective to remain the same throughout a knowledge relation, the problem is avoided. In the two-dimensional framework proposed neither $p \rightarrow KAp$ nor $Ap \rightarrow \Box KAp$ is valid, while Edgington's $Ap \rightarrow \Diamond KAp$ is. So it seems that even though some problems arise in the regular semantics for actuality and knowledge, these problems can be avoided using a two-dimensional framework.

On the other hand, (Williamson, 1987) objects to Edgington's account of possible knowledge as there are cases that are not sufficiently described by it. One of the crucial notions of the situations approach is that the possible knower, that resides in the non-actual situation, knows

(CFK) *p would still have been true, had no one known that p.*

Edgington argues that, for statements for which (CFK) holds, knowledge of (CFK) and p is sufficient and necessary for the possible knower to have knowledge that, actually, $p \wedge \neg Kp$. On the other hand, Williamson argues that it is not sufficient by giving an example where in the possible situation s' there is knowledge of both p and (CFK), but it is not known that $(p \wedge \neg Kp)$ holds in the actual world s. Suppose that at some point in the past, before mankind evolved, a flower bloomed at a certain spot, we call this p. So in the actual situation, s, it is the case that $p \wedge \neg Kp$. For a counterfactual to hold, we look at situations that are most similar to s. One of the simplest situations s' would be one in which mankind evolved earlier and happened to

[20]It is important to note that (Rabinowicz & Segerberg, 1994) use possible worlds instead of possible situations. Whether the problem still arises in a possible situations semantics will not be further discussed, but might be an interesting topic for further research.

[21]If a statement φ is actually true, then $A\varphi$ is true in every world in the model. So for each world epistemically accessible from a world $A\varphi$ holds, making $KA\varphi$ hold in all worlds. This holds for all worlds, hence $\Box KA\varphi$ also holds for all worlds.

stumble upon this blooming flower. Now it is the case that in s' they know that (CFK) holds, but that is simply because there are situations far more similar to s' than s in which nobody knew p but it still obtained. Such a situation might be one where nobody went for a walk that day. So even though in s' they have counterfactual thought concerning p, it does not grant them knowledge that in s it holds that $p \wedge \neg Kp$.[22]

There are also statements p for which (CFK) does not hold. Edgington argues that for these statements for which (CFK) fails in s', such as 'I am in pain', it automatically holds in s that

(PK) *If p, then someone knows that p.*

Now Williamson lets us suppose a certain particle of which it can only be known what state it is in by interacting with it in a way that will unpredictably alter its state. Also suppose that in s it is an unknown truth, p, that this particle is in state k. Now in a similar situation, s', where p is known (CFK) does not hold, as not knowing the particle's state would not have altered it as it did in s'. However, neither does (PK) hold as p is the case and is not known in s. Thereby, Williamson has given two cases in which Edgington's account of knowledge has not been sufficient to ascribe knowledge of the actual situation to someone in a possible, non-actual situation.

Another objection to Edgington's account of knowledge concerns knowledge of the actual situation. If we go back to the howling animal example, we say that in s_1 there is knowledge that in s_2, the actual situation, a statement φ holds. However, as Edgington also notes, in s_1 this knowledge would not be described as knowledge of the actual situation, because they would argue that s_1 is the actual situation. It does not seem to be argued why knowing of some situation, that happens to be the actual situation, that φ holds ought to be equivalent to knowledge that actually φ holds, when the knower does not know that this possible situation is the actual situation. Without such an argument Edgington's argument does not seem to be valid.

To conclude, the situations approach alters the knowability principle so that the problematic consequence can no longer be derived. Edgington has argued for a new principle with the 'Actuality'-operator, as she feels that the crux is in the difference between knowing *in* a situation that p and knowing *of* a situation that p. Rabinowicz and Segerberg noticed that there are still some problematic consequences when regular semantics are used, and proposed a two-dimensional semantic framework. Williamson has argued that there are cases that are not described by Edgington's account of knowledge, and thus that this approach is not sufficient. Although this approach does not have many recent literature, it has an intuitive appeal and is worth more attention.

4 Criteria

4.1 Selecting the criteria

Although there are many different approaches to the knowability paradox, no consensus has been reached on the plausibility and consequences of the knowability paradox. To get one step further towards such consensus, I will discuss and select criteria on the grounds of which different approaches can be measured. Using these criteria I will be able to get a further insight in the logical and philosophical plausibility of the intuitionistic, paraconsistency and situations approaches, giving me an answer to the research question of this thesis. The research question has been divided in two separate questions concerning the logical and philosophical plausibility. Likewise, the criteria will also be divided in these two categories.

The logical criteria will provide a measure of the logical plausibility of the different approaches. Plausibility can be understood as a term describing how appealing an approach is

[22] Note that some verificationist do not think that statements about the past are meaningful or, as (Dummett, 2008) states, 'need very careful consideration'. For them, Williamson's first counterexample yields no problems for Edgington's account of possible knowledge.

from a certain perspective. One of the most important things that an approach to the knowability should have, is a proof that Fitch's proof is blocked when one uses the logic or principles appealed to in that approach. Usually, such a proof gives a version of Fitch's proof and shows that one of the steps is blocked. Even stronger would be a counterexample in which (KP) and (1) are true while (12) is not. So the first logical criterion is:

"Is it proven that the paradox is not derivable in the given logic?"

Even though the proof might be blocked from reaching (CONS), there might be other undesirable consequences that one still is obliged to accept. The strongest approach would block the proof and leave no undesirable consequences. This brings us to the second logical criterion:

"Are other undesirable consequences derivable?"

A further criterion to discuss concerns the semantics of the logic used. As semantics are used to interpret and model a logic, it is important for them to be able to express reasoning about everyday discourse. A semantics that provides uninteresting models or is only applicable under very strict conditions is less appealing than one that is widely applicable:

"In case of semantic solutions, is the semantics well-chosen?"

The philosophical criteria provide a way of discussing how appealing an approach is from a philosophical perspective. Many of the different approaches alter, add or reject axioms and inference rules that are used in Fitch's proof. As these form the background logic, they cannot be chosen solely for the purpose of blocking the paradox. The first philosophical criterion that the approaches will be measured along is:

"Is the background logic well-chosen?"

As the knowability paradox is a central part of a debate between philosophical theories, a proper solution should be embedded in this context. Without such embedding nobody might be willing to accept the foundation of the approach, and thus the paradox will remain standing. To discuss the context in which an approach is appealed to, the second philosophical criterion will be:

"Is this approach philosophically motivated?"

I have motivated several logical criteria concerning the blocking of the proof, related undesirable consequences and semantics. The selected philosophical criteria concern the background logic used and the philosophical motivation. I have carefully considered several criteria, and those not selected and discussed are either trivial or inherently favour one of the approaches.

4.2 Applying the criteria

To get an answer to the research question, I will discuss all three approaches one criterion at a time. After comparing them against these criteria, I can then conclude which approach comes out best, and where their logical and philosophical strengths and weaknesses lie.

"Is it proven that the paradox is not derivable in the given logic?".

The intuitionistic approach has blocked the proof at step (11). Proietti proposed an intuitionistic semantics in which, under strict conditions, a counterexample to the entailment from (11) to (12) can be given. Although this does block the versions of the proof that are currently known, it does not rule out that there might be another way of deriving the problematic (12) from (KP) and (1).

The paraconsistency approach rejected the rule of contraposition as to block the *reductio* steps of the proof. Both Priest and Wansing proposed a semantics in which a counterexample was given, making (KP) hold and (12) not hold.

The original proposal of the situations approach by Edgington gave an example of the situations analogue of (3), showing that (EKP) and (1) do not yield an inconsistency. This was done implicitly, without showing the exact step where the proof breaks down. However, Rabinowicz and Segerberg proposed a two-dimensional framework in which they did prove that (EKP) is valid, while (12) is not.

"Are other undesirable consequences derivable?"

The intuitionistic approach is committed to (11), as they only blocked the proof before the last step. Percival pointed out that from (11), using intuitionistic logic, two related undesirable consequences follow. On the other hand, DeVidi and Solomon argued that these consequences are only problematic under a classical reading of negation, not under the intuitionistic interpretation. So it seems that the intuitionistic approach has left us with some related consequences, but whether they are undesirable is up for debate.

As Beall pointed out, the paraconsistency approach's rejection of contraposition makes the verificationist obliged to accept that *all* unknown truths are both known and not known in some possible world. There is a possibility for the verificationist to motivate this result, although it would need a complex and unintuitive construction as discussed in section 3.2.

The situations approach blocks the proof at the first step by altering the knowability principle, and therefore blocks all intermediate steps that might be seen as undesirable. The undesirable consequences that the regular semantics would have are blocked by the framework provided by Rabinowicz and Segerberg, leaving no related undesirable consequences.

"In case of semantic solutions, is the semantics well-chosen?"

The intuitionistic semantics provided by Proietti represents an idealized situation in which the knower is an ideal reasoner. However, as Proietti himself points out, one of the main objections is that this semantics is not appropriate for everyday reasoning. Also, the counterexample given only holds under very strict conditions.

In the paraconsistency approach, two different semantics are proposed. The semantics of Priest's proposal has intuitive truth-conditions, but combined with the trivial world ∞ the $\Diamond$ and $\rightarrow$ get some unwanted, trivial properties. As we do not think that possibility and implication have these properties in the real world, these semantics are not very strong for common discourse. On the other hand, the semantics of Wansing's paraconsistent logic with relevant implication and strong negation has intuitive truth-conditions and a simple model for the counterexample. So all in all, the paraconsistency approach does have an appropriate semantics in Wansing's proposal.

Rabinowicz and Segerberg provided a two-dimensional framework without the problems of the regular semantics for the 'Actuality'-operator. Therefore, one would conclude that the answer to this criterion is positive.

"Is the background logic well-chosen?"

Intuitionistic logic rejects double negation elimination on the basis of its interpretation of the negation. As the intuitionistic approach appeals to the independently motivated intuitionistic logic, the rejection of this inference rule is well-chosen.

The most important rule of inference that the paraconsistency approach rejected is contraposition, motivated by the knower paradox. Both Beall and Priest appeal merely to this rejection, and thus it can be concluded that on their part the background logic is indeed well-chosen. On top of the rejection of contraposition, Wansing also appealed to a relevant implication and a strong negation. He has not argued why they should be part of his logic, other than that they help block the knowability paradox. Therefore, background logic in Wansing's proposal seems to be more *ad hoc* and not very well-chosen.

The situations approach appeals to the use of the 'Actuality'-operator, which has a strong intuitive appeal and is further argued for by Edgington. No other axioms or inference rules have been added and rejected, so it can be concluded that this criterion is positively answered.

"Is this approach philosophically motivated?"

Intuitioninism is one of the most common forms of anti-realism, making an appeal to intuitionistic logic a natural move to save the verificationist from the knowability paradox.[23] Therefore, the intuitionistic approach is philosophically motivated.

The paraconsistency approach seems to be two-sided on this criterion as well. Beall and Priest appeal to paraconsistent logics that are motivated by several well-known paradoxes and pragmatic reasons. On the other hand, Wansing also uses a combination of logical foundations usually affiliated with opposing philosophical theories. He uses a constructive strong interpretation of the negation, that is mostly used in constructivist theories. Also, a relevant implication is appealed to, something that defenders of relevance logics often use. As constructivism and relevantism differ in some fundamental ways, it is unusual to combine these two.

The situations approach is not motivated by an existing philosophical theory, but Edgington argues that her principle represents the *moderate anti-realist view* better. However, Williamson has argued that her account of knowledge is not sufficient for all cases, and thus concludes that this approach does not have sufficient philosophical motivation. Also, I have argued that her notion of knowledge of an actual situation is not without problems.

"What are the logical and philosophical strengths and weaknesses of the intuitionistic, paraconsistency and situations approaches to the knowability paradox?"

Having applied the logical and philosophical criteria to all three approaches, I am now able to answer the research question. I will shortly summarize the logical and philosophical weaknesses of these approaches.

The intuitionistic approach's logical strengths are that it blocked the current versions of the proof, and that there are no related undesirable consequences under the interpretation of DeVidi and Solomon. Its logical weaknesses are the lack of a proof of the non-entailment from (1) to (12), the inability of the semantics to describe everyday reasoning and the related undesirable consequences that appear under Percival's interpretation. The philosophical strengths of the intuitionistic approach are the strong motivation for the background logic, and the independent motivation of intuitionism. Philosophical weaknesses might lie in the debate surrounding intuitionism itself, but that is beyond the scope of my thesis. I therefore conclude that the intuitionistic approach has quite a few objections from a logical perspective, but is philosophically very strong.

The logical strengths of the paraconsistency approach are its counterexamples to the paradox, and the appropriate semantics by Wansing. The logical weaknesses are a related undesirable consequence, and the unwanted properties of $\Diamond$ and $\rightarrow$ in Priest's semantics. Both Beall and Priest contribute to the philosophical strengths by rejecting well-chosen inference rules and being philosophically motivated. The proposal by Wansing uses less well-chosen axioms and inference rules, and is not motivated by a single existing theory, showing some philosophical weaknesses of this approach. It can be concluded that the paraconsistency approach as advocated by Wansing is logically reasonably strong and philosophically weaker, while Beall and Priest have philosophical strength but are logically weaker.

The situations approach is logically strong in its proof that the paradox vanishes, lack of undesirable consequences and appropriate semantics. The philosophical strength lies in the intuitiveness of the 'Actuality'-operator, while the biggest philosophical weakness is exposed by Williamson's counterexamples. Therefore, the situations approach is logically very strong and philosophically reasonably strong, although that is debatable.

To conclude, applying the criteria selected in section 4.1 to the intuitionistic, paraconsistency and situations approaches enabled me to answer the research question. These results are visualised in table 1, where the intuitionistic approach (IA), the paraconsistency approach (PA)

[23] It should be noted that out of the approaches discussed, only intuitionists historically are committed to the knowability principle.

and the situations approach (SA) are given either a ✓or a X for each criterion. In cases which are very debatable a more neutral - is given.

	Proof	Conseq.	Semantics	Backgr. Logic	Phil. Motivated
IA	-	-	X	✓	✓
PA (Priest)	✓	X	X	✓	✓
PA (Wansing)	✓	X	✓	X	X
SA	✓	✓	✓	✓	-

Table 1: Criteria Applied

From a logical perspective, I can only conclude that the situations approach comes out on top. As seen from the philosophical side, the intuitionistic approach has the firmest basis and seems to be the best approach. Taking both perspectives into account, I argue that the situations approach has proven to be the strongest with respect to the criteria that I selected. As different criteria might be more important then others, it is possible to give different weights to them before concluding which approach comes out on top. I have used equal weights, but it must be noted that one is free to interpret table 1 using different weights.

5 Conclusion

The goal of my thesis was to select criteria on the grounds of which different approaches to the knowability paradox can be compared. The debate surrounding this paradox has many views, but little agreement concerning its plausibility and consequences. By setting up criteria in section 4.1 I have provided a standardized way of judging and comparing these approaches. The selected criteria then have been applied to the intuitionistic, paraconsistency and situations approaches in section 4.2.

Using these criteria I have answered my research question: *"What are the logical and philosophical strengths and weaknesses of the intuitionistic, paraconsistency and situations approaches to the knowability paradox?"*. The intuitionistic approach has several logical objections, but is very strong from a philosophical perspective. The paraconsistency approach advocated by Wansing is logically reasonably strong, but philosophically less motivated. Beall and Priest advocate a paraconsistency approach that is logically weaker, but philosophically more stable. The situations approach is logically very strong and philosophically reasonably strong, although the latter has some objections. From this I have argued that overall, the situations approach has proven to be the strongest of the approaches that I have discussed.

Although the situations approach has come out on top, it is not an approach that is widely represented in the literature. Further research on this topic might concern the application of possible situations semantics[24], rather than possible worlds semantics, to Edgington's proposal as this may represent the philosophical view of the situations approach better. Also, as discussed in footnote 1, the temporal interpretation of the K-operator is worth further investigation. Lastly, the application of the criteria that I proposed in this thesis to different approaches such as 'Modal Fallacies'[25], 'Cartesian Statements'[26] and 'Basic Statements'[27] could prove to be fruitful in finding agreement on the knowability paradox.

Having provided means of judging and comparing approaches to this problem, I hope to have made a step towards consensus.

[24]See (Humberstone, 1981).
[25]See (Kvanvig, 1995).
[26]See (Tennant, 2002).
[27]See (Dummett, 2001).

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