Universiteit Utrecht

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# The Distribution of Vacua in Asymptotic Flux Compactification of Type IIB / F-Theory

Master Thesis

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July 1, 2020

**Abstract**

Much of recent research in string theory has revolved around the complexity of the landscape of inequivalent vacua which may describe the world we observe. In this work, we review the emergence of this landscape from the perspective of flux compactifications of F-theory and discuss its statistical description in terms of the index density of supersymmetric flux vacua following the many works of Douglas *et al.* We then proceed to analyze the behaviour of the index density near singular loci in the complex structure moduli space using asymptotic Hodge theory. In all single parameter limits, we obtain a universal asymptotic behaviour of the index density which is integrable, providing evidence for the finiteness of string theory flux vacua. We are also able to extend our analysis to some multi-parameter limits and point the reader to possible methods to describe the full set of limits.

# Contents

# Introduction

Although the modern formulation of string theory has been around since 1974, it was not until the first superstring revolution, which began in 1984, that it was realized as a realistic description of elementary particles and their interactions. Indeed, using the newly discovered Green-Schwarz mechanism, Candelas, Horowitz, Strominger and Witten obtained a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry and a realistic gauge group by compactifying the heterotic string on a Calabi-Yau threefold [6]. However, the restrictions of supersymmetry and a realistic particle content were very stringent. For example, such compactifications must have a vanishing cosmological constant, a phenomenologically unacceptable feature.

The situation changed dramatically, however, after the second superstring revolution around 1994. In particular, the existence and necessity of D-branes, discovered by Polchinksi in 1995 [44], provided countless new structures to the theory and greatly evolved its cosmological aspects. The D-branes provide a natural source for the electric and magnetic Ramond-Ramond fields whose dynamics are closely related to the cosmological constant. Additionally, numerous brane constructions have been utilized to produce models of inflation and descriptions of black holes. Over the course of a few years, it was realized that the number of different low-energy theories that string theory can produce is huge. In 2003, this vast network of possible string vacua was coined the *string landscape* by Susskind [47]. During this time, a new method of analysing the string landscape and its implications was pioneered by Douglas. Rather than searching for specific compactifications which produce exactly the vacuum that describes the physics we see, emphasis was instead placed on the general statistical properties of the landscape. Essentially, this resulted in the search for a 'vacuum selection principle', a set of rules which we expect a 'natural string vacuum' to obey.

In a similar endeavour to improve our understanding of the landscape, Vafa instigated the *swampland program* in 2005 [49]. It consists of a number of conjectures that make very general statements about properties of string vacua, which are then conjectured to hold for any theory of quantum gravity. Somewhat jokingly, whenever some effective field theory does not obey such a conjecture, it are said to lie in the *swampland*. Over the years, the search for examples and proofs of these conjectures, as well as the link between them has provided us with various insights regarding the structure of the landscape. Additionally, their study has required the development and understanding of some remarkable mathematical constructions and properties that arise in string compactifications.

Our main motivation is the application of a particular instance of such a beautiful mathematical framework to generalize a known result for the distribution of supersymmetric flux vacua in certain asymptotic regions. In other words, we aim to investigate the statistical nature of the landscape following the work of Douglas and others by using modern techniques that are applied in tests of the swampland conjectures, most notably the distance conjecture and the de Sitter conjecture [28, 29]. More precisely, we will add to the work of Eguchi and Tachikawa, who investigated the behaviour of this distribution in 2005 [22]. It was found that, in the context of Type IIB compactification on Calabi-Yau threefolds, the distribution has a universal form in certain limits. In this work, we will generalize their results and show that this form persists in the context of F-theory compactification on Calabi-Yau fourfolds and for more general limits. This then serves as further evidence for the finiteness of string theory vacua.

*Outline of the Thesis*

In chapter 1 we start with a general description of the compactification of superstring theories and how this leads to effective four-dimensional physics. Most notably, we describe how different choices of curling up the unwanted dimensions of string theory can be parametrized by a so-called *moduli space*. We illustrate the various concepts involved using toroidal compactifications, before turning to the general situation of Calabi-Yau fourfolds. Next, we introduce F-theory as the twelve-dimensional rephrasing of Type IIB string theory and discuss its description in terms of the eleven-dimensional supergravity action of M-theory. Finally, we perform a quantitative analysis of the compactification of M-theory on a Calabi-Yau fourfold to illustrate the emergence of *moduli*, the fields parametrizing the shape and size of the internal dimensions. This also leads to the issue of *moduli stabilization* and is the prime motivation for considering *flux compactification*.

In chapter 2 we describe the emergence of the string landscape as a result of flux compactification of F-theory on a Calabi-Yau fourfold, which leads to a four-dimensional theory of $\mathcal{N} = 1$ supergravity. The vast freedom of possible shapes for the curled up dimensions and values of field strengths, each with their own energy, leads to a large number of different vacua in the resulting four-dimensional theory. Analysing the structure of this landscape from a more statistical point of view is the main content of this chapter. We do this by reproducing the result of Ashok and Douglas, who in 2004 derived an explicit expression for the distribution of supersymmetric flux vacua, the *AD-density*. We review their derivation along with the assumptions and estimations that are made, and discuss its interpretation and implication. In particular, we return to the question of whether the number of supersymmetric flux vacua in at all finite. In other words, we investigate the size of the string landscape and explore the range of predictability of string theory.

Subsequently, in chapter 3 we perform the analysis of the distribution of supersymmetric flux vacua in asymptotic regions of the moduli space. Here we call upon the intricate mathematical structures that arise in these limits, as such our discussion is mostly of mathematical and algebraic nature. However, we also highlight the relation between our methods and those used to study the various Swampland conjectures, one of the thriving topics in current string theory research. We spend quite some words on introducing the necessary mathematical concepts and constructions, most notably the mixed Hodge structure that arises and its description in terms of the Deligne splitting. Of central importance is the *nilpotent orbit theorem*, which was already proven in 1973 by Schmid. With the relevant tools in hand, we then return to the AD-density and reproduce and generalize the results of Eguchi and Tachikawa in all single-parameter limits.

Finally, in chapter 4 we make an attempt to generalize our results further to multi-moduli limits. To this end we deepen our understanding of the asymptotic regions of the moduli space by introducing the *SL(2)-orbit theorem*, derived by Cattani, Kaplan and Schmid in 1986.[1] In contrast to the results of chapter 3, we are only able to obtain the form of the AD-density in a particular set of multi-parameter limits. We end with a discussion on potential improvements of our methods and the challenges one faces in doing so. Additionally, we relate our findings with those found in the mathematics literature, in particular with regards to the finiteness of flux vacua.

[1] Note that it took around thirteen years to understand the intricacies of extending the results of Schmid to multi-parameter limits!

*The Three Questions*

To ease the reading experience we have formulated three questions which we aim to answer or at least address in this work. They are as follows:

Q1: How can the complex structure moduli be stabilized?

Q2: How are the F-theory flux vacua distributed over the moduli space?

Q3: Is the AD-density integrable near a given singular locus in the complex structure moduli space?

Questions Q1 and Q2 are answered in chapter 2 and constitute the main physical result of this thesis. Subsequently, question Q3 is answered in chapter 3 for single-parameter limits and partially in chapter 4 for multi-moduli limits. This comprises this works' core mathematical content. We also summarize our results in a similar set of three answers in the conclusion for quick reference.

# Chapter 1

# Type IIB/F-Theory Compactification

Even outside of the academic world of physics people are aware of this mysterious 26, 12, 11, or 10-dimensional theory that is called string theory. Although I had heard of the fact before, I could nevertheless barely contain my excitement when deriving the critical dimension $D = 26$ of bosonic string theory. After settling down, however, one quickly returns to the reality that our world is 4-dimensional, unfortunately. The goal of this chapter is to bridge the gap between the higher-dimensional string theories and ordinary 4-dimensional physics through a process called compactification. We will describe some specific aspects in the context of Type IIB string theory, or its more geometrical formulation F-theory, but the tools and main messages remain applicable to the other string theories as well. In a nutshell, the fact that string theories live in higher dimensions acts as a double-edged sword. On the one hand, there is a lot of freedom to construct nearly anything by using the numerous fields that are present in the theory, as well as D-branes and the various dualities between the different theories. On the other hand, we may have a bit too much freedom in our choices. This raises the question of which choice should be regarded as 'correct' or 'natural'. We will discuss this in more depth in chapter 2, when we cover statistical properties of the string landscape.

We start by introducing the concept of compactification in section 1.1. By considering the simplest example of a circle compactification, we see the emergence of lower dimensional fields and, in particular, moduli which parametrize the internal geometry. To go beyond this particular example, we first quickly review the content of Type IIB supergravity, the low-energy limit of Type IIB string theory, in section 1.2. From there we describe the more general Calabi-Yau compactification scheme in section 1.3 and use the torus as an important example. Here we obtain a 'moduli space' which parametrizes all the different kinds of tori one could use as compact dimensions. This leads us to the moduli space of general Calabi-Yau manifolds and its properties, most notably the natural metric it possesses. Ultimately, we are interested in compactifications of Calabi-Yau fourfolds, which leads us to F-theory. Its description as a geometrization of Type IIB and its definition in terms of M-theory will be discussed in section 1.4. Finally, we end this chapter with a computation of the compactification of the Ricci scalar in M-theory. We see that, as for the circle, one obtains a number of massless scalar fields parametrizing the shape of the internal manifold. This brings us to the issue of moduli stabilization, which will be solved in chapter 2.

## 1.1 Compactifying String Theory

There are five superstring theories, which are only consistent in exactly ten space-time dimensions. In this section, we will be concerned with the general features they all poses to motivate the idea of compactification and the resulting issue of moduli stabilization. First, we note that in each superstring theory, there are numerous dynamical fields which arise as excitations of the open/closed string of the superstring theory in question. We will be interested in the low-energy approximation, meaning that we only consider the massless modes of the superstring. A universally present field is a metric $g$ on the ten dimensional target space $\mathcal{M}_{10}$. The dynamics of $g$ are governed by an action, which is generally of the Einstein-Hilbert form

$$S = \int_{\mathcal{M}_{10}} R \star 1 + \cdots , \tag{1.1}$$

where $\star$ denotes the Hodge star on $\mathcal{M}_{10}$ and the dots contain kinetic terms for whatever fields are additionally present. Although the major part of this work revolves around F-theory, it suffices for now to keep the discussion general.

Of course, the aim of superstring theory is to describe the physics that we observe around us. However, the observable world is certainly not 10-dimensional, but rather 4-dimensional. To bridge this gap, we introduce the concept of Kaluza-Klein compactification. [1]. Our discussion is based on [3, Chapter 14]. The idea is to consider the full $d_c$-dimensional space $\mathcal{M}_{d_c}$ as a product manifold

$$\mathcal{M}_{d_c} = \mathcal{M}_{1,d-1} \times X_D, \quad d + D = d_c, \tag{1.2}$$

where $X_D$ is a compact manifold, called the *internal manifold*, and $\mathcal{M}_{1,d-1}$ is some lower dimensional Lorentzian manifold. Additionally, the metric decomposes as

$$ds^2_{d_c} = ds^2_d + ds^2_D, \tag{1.3}$$

where the metric on $\mathcal{M}_d$ is usually taken to be maximally symmetric (i.e. dS, Minkowski or AdS). Compactness of $X_D$ allows us to speak of its size and demand it to be small enough such that no particle detector has ever noticed its presence. However, as we will see, the theory observed in $\mathcal{M}_d$ certainly does depend on the nature of $X_D$.

*Compactifying on a Circle and the Emergence of Moduli*

Before delving into the details of the general construction, let us first discuss the simplest compactification there is. We consider a free massless scalar field theory on $\mathcal{M}_{1,4} = \mathcal{M}_{1,3} \times S^1$, where $\mathcal{M}_{1,3}$ is 3+1 dimensional Minkowski space-time and $S^1$ is the one-dimensional circle of radius $R$. The action is given by

$$S = \int d^5x \sqrt{-g}\, g^{MN} \partial_M \phi \partial^N \phi, \quad M, N = 0, \ldots, 4 \tag{1.4}$$

where

$$g_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + R^2 d\theta^2, \quad \mu, \nu = 0, \ldots, 3 \tag{1.5}$$

with $\eta$ the Minkowski metric on $\mathcal{M}_{1,3}$. The equation of motion for $\phi$ is that of a massless scalar field:

$$\Box\phi = 0 \implies \partial_\mu \partial^\mu \phi + R^{-2} \partial_\theta^2 \phi = 0, \tag{1.6}$$

[1] Note that Kaluza-Klein compactification is not the only way to make sense out of the higher dimensional theories. An alternative is provided by *brane-world* scenarios, where the visible Universe is considered to be a 3-brane, embedded in a higher-dimensional space, see e.g. [2, Chapter 10]

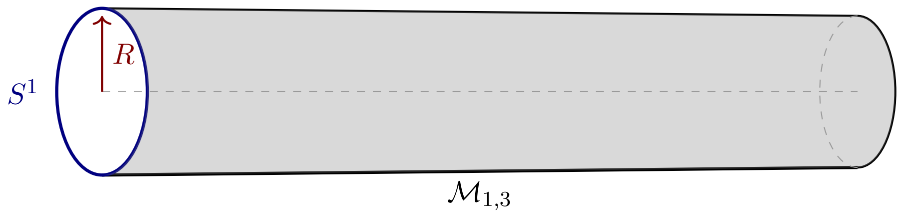



**Figure 1.1:** Compactification on the circle $S^1$, where $\mathcal{M}_{1,3}$ is represented as a line. The radius $R$ of the circle is taken to be small so that the masses of the KK-modes lie above the energy scale we can currently scope.

On the circle, we can perform a Fourier expansion of $\phi$ as follows:

$$\phi(x,\theta) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{in\theta}, \tag{1.7}$$

Inserting this into the equation of motion yields

$$\left(\partial_\mu \partial^\mu - \frac{n^2}{R^2}\right) \phi_n = 0. \tag{1.8}$$

In other words, we see that a single massless scalar $\phi$ in five dimensions gives rise to an infinite tower of scalars $\phi_n$ in four dimensions, whose masses are given by $n/R$. As promised, the properties of the internal manifold (here, the radius $R$) determine the resulting physics in the external manifold, namely the masses of the induced fields. In particular, in the limit $R \to 0$ all fields except for the massless field $\phi_0$ can be regarded as so massive that they are not relevant for low-energy physics. Besides the emergence of a tower of scalar fields, a perhaps more mysterious quantity is the radius $R$ of the internal circle. Is this a quantity we can measure? And is there some mechanism which would determine its value? These questions become even more intricate once one realizes that, in principle, $R$ could additionally depend on the external coordinates $x^\mu$. This gives rise to yet another scalar field $R(x)$ from the point of view of the external manifold and is our first example of a *modulus*. It is expected to be massless, since there is a priori no restriction on the values it can take (besides being positive).

More generally, compactification on $X_D$ will result in numerous moduli fields in the lower dimensional theory, which intuitively parametrize the 'shape' and 'size' of $X_D$ at each point in spacetime. See figure 1.2 for an example of two different shapes of the torus. Importantly, in this simple compactification scheme the moduli will always turn out to be massless. This is a major issue, since no massless scalar particles have been detected so far, and since they have no mass this cannot be due to the energy scales at which we are currently restricted to probe nature. This is know as the problem of *moduli stabilization*. How to solve this issue will be the topic of chapter 2. In this chapter, we will develop the necessary tools to understand how this problem arises quantitatively. This will require a good understanding of the dynamics of these moduli fields. However, before giving a more quantitative description of the moduli in the general compactification setting, we should first discuss the specific superstring theory we will consider, as well as the kinds of spaces $X_D$ that we will encounter in this context.

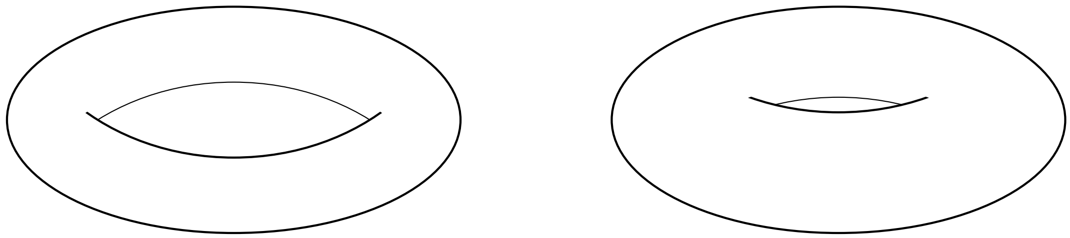


**Figure 1.2:** Two tori with a different shape. In relation to section 1.1, the torus on the left has complex structure modulus $\tau > 1$, whereas the torus on the right has $\tau \approx 1$.

## 1.2 A Review of Type IIB

For a large portion of this work we will consider F-theory, which for our purposes is a 12-dimensional geometrization of type IIB string theory. In this section we first give a short review of the latter. How type IIB is related to F-theory will be discussed in section 1.4. Type IIB string theory is one of the five possible superstring theories. It is a 10-dimensional chiral theory containing both open and closed strings, although we will mostly be concerned with the closed string sector. For the closed string, the massless bosonic modes in the NS-NS sector are given by

- A scalar field $\phi$, called the *dilaton*, which is related to the string coupling constant as $g_s = e^{\phi}$.
- A symmetric traceless (0,2) tensor $g$, which is interpreted as the metric on the target space where the superstring is embedded.
- A two-form $B_2$.

Moreover, the massless bosonic modes in the R-R sector are

- $p$-forms $C_p$, for $p = 0, 2, 4$.

We denote the corresponding field strengths by

$$F_p = dC_{p-1}, \quad H_3 = dB_2. \tag{1.9}$$

It is convenient to combine the fields in the following way

$$\tau = C_0 + ie^{-\phi}, \quad G_3 = F_3 - \tau H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \tag{1.10}$$

The equations of motion for these fields can be obtained by studying the corresponding $\beta$-functions, whose vanishing ensures that the Weyl symmetry of the superstring is not anomalous at the quantum level. In this manner one obtains the effective action for type IIB, given by

$$S_{\text{IIB}} = \frac{2\pi}{l_s^8}\int R \star 1 - \frac{1}{2}\frac{d\tau \wedge \star d\bar{\tau}}{(\text{Im}\,\tau)^2} + \frac{G_3 \wedge \star \bar{G}_3}{\text{Im}\,\tau} + \frac{1}{2}\tilde{F}_5 \wedge \star \tilde{F}_5 + C_4 \wedge H_3 \wedge F_3, \tag{1.11}$$

which has to be supplemented with the additional constraint $\star\tilde{F}_5 = \tilde{F}_5$. Here $l_s$ denotes the string length. Crucially, the action is manifestly invariant under the action of $\text{SL}(2,\mathbb{Z})$[2] given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z}): \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} F \\ H \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} F \\ H \end{pmatrix}, \tag{1.12}$$

which is known as S-duality. In section 1.3.2 we will see that this transformation behaviour is shared by the complex structure modulus of the torus, which is the prime inspiration for introducing F-theory in section 1.4.

[2]At the classical level the action enjoys the full $\text{SL}(2,\mathbb{R})$ symmetry, but this is broken at the semi-classical level due to the presence of D7 branes.

## 1.3 Calabi-Yau Moduli Spaces

Let us now return to the general picture of string compactifications. Naturally, there are some restrictions on $X_D$ in order to produce desired properties of the four dimensional physics after compactification. The main restriction is due to the fact we require $\mathcal{N} = 1$ supersymmetry in the lower-dimensional theory. Even though supersymmetry has not been observed in nature (yet), there are numerous motivations for considering supersymmetric field theories. These include the coupling constant unification, the fine-tuning of the Higgs mass and the fact that some supersymmetric particles are dark matter candidates.

Within the string theory context, there are several ways to obtain $\mathcal{N} = 1$ supersymmetry in four dimensions, these include [13, p. 9]

- Heterotic or type I superstring theory on a Calabi-Yau threefold.
- Type II superstring theory on a three-dimensional Calabi-Yau orientifold.
- F-theory on a Calabi-Yau fourfold.

For our purposes, this last one is of relevance. Hence we will be particularly interested in the properties of Calabi-Yau fourfolds. However, due to the general appearance of Calabi-Yau manifolds in string compactifications, we will discuss them in more generality.

### 1.3.1 Calabi-Yau Manifolds

By a (complex) $D$-dimensional Calabi-Yau manifold we mean a complex manifold $Y_D$ together with a Ricci-flat, Hermitian metric $g$ and a closed Kähler form $J$, which is given in terms of the metric as

$$J = \frac{i}{2} g_{a\bar{b}}\, dy^a \wedge d\bar{y}^{\bar{b}}, \quad a, b = 1, \ldots, D, \tag{1.13}$$

where $y^a, \bar{y}^{\bar{b}}$ denote complex coordinates on $Y_D$. A Calabi-Yau manifold is in particular a Kähler manifold, which implies that the metric can locally be expressed in terms of a Kähler potential as

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K. \tag{1.14}$$

For a more detailed account of Kähler manifolds, as well as some basic definitions of differential geometry and our conventions, we refer the reader to appendix A. Of utmost importance throughout this work are the de Rham cohomology groups $H^{p,q}(Y_D)$. Let $h^{p,q} := \dim H^{p,q}(Y_D)$ be the Hodge numbers. Then for Calabi-Yau $D$-folds these satisfy the following properties

$$\text{(a)}: \quad h^{p,q} = h^{q,p} \tag{1.15}$$

$$\text{(b)}: \quad h^{p,q} = h^{D-p,D-q} \tag{1.16}$$

$$\text{(c)}: \quad h^{p,0} = 0,\ 1 \leq p < D, \quad h^{D,0} = 1. \tag{1.17}$$

The Hodge numbers can be conveniently organized in a Hodge diamond, which for a Calabi-Yau fourfold is given by:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Then the first two relations of the Hodge numbers simply state that the diamond is symmetric in the horizontal and vertical axes. The final relation implies that $H^{4,0}(Y_4)$ and $H^{0,4}(Y_4)$ are both one-dimensional. It is customary to denote their generators by $\Omega$ and $\bar{\Omega}$, respectively. Finally, we mention that the four resulting Hodge numbers $h^{1,1}, h^{2,1}, h^{3,1}, h^{2,2}$ are not all independent. Indeed, using the index theorems one shows that [26, p. 21]

$$h^{2,1} = 22 + 2h^{1,1} + 2h^{3,1} - \frac{1}{2}h^{2,2}, \tag{1.18}$$

hence we are left with $h^{1,1}, h^{3,1}$ and $h^{2,2}$. We will see that $h^{1,1}$ counts the number of Kähler moduli and $h^{3,1}$ the number of complex structure moduli, which we introduce in subsections 1.3.4 and 1.3.5, respectively. These are precisely the fields which parametrize the 'size' and 'shape' of $Y_4$, as mentioned before. One readily sees that $H^4(Y_4)$, or more generally $H^D(Y_D)$, stands out as the most interesting cohomology group. This will remain true throughout this work. More specifically, in chapter 2 we will encounter 'fluxes' which are elements of $H^4(Y_4)$ (in F-theory compactifications) and in chapters 3 and 4 much of our interest lies in the degeneration of $H^D(Y_D)$ near singular points in the complex structure moduli space of $Y_D$.

Having stated the main properties of Calabi-Yau manifolds, our next step is to generalize the notion of moduli fields. To gain some intuition in the matter and moreover give an important example of a Calabi-Yau manifold, we next discuss the torus.

### 1.3.2 Example: The Torus

The torus $\mathbb{T}$ is the easiest example of a Calabi-Yau manifold, in fact it is the unique 1-dimensional one. Though our main goal here is to exemplify the concept of a complex structure modulus, this example will also be crucial in the F-theory description of Type IIB in section 1.4 and is referred to in the later chapters as well. Our discussion is based on [3, Chapter 6]. A convenient description of the torus is as a quotient of $\mathbb{C}$ by a lattice:

$$\mathbb{T} := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad \mathrm{Im}\,\tau > 0, \tag{1.19}$$

where we identify

$$z \sim z + n + m\tau, \quad z \in \mathbb{C},\ m, n \in \mathbb{Z}. \tag{1.20}$$

We see that different tori are parametrized by the Teichmüller parameter $\tau \in \mathbb{H}$, where $\mathbb{H}$ denotes the (strictly) upper-half plane. However, there are still some non-trivial transformations we can perform on $\tau$ which result in equivalent tori. They are generated by the Dehn twists:

$$\alpha : \tau \mapsto \tau + 1, \quad \beta : \tau \mapsto \frac{\tau}{\tau + 1}. \tag{1.21}$$

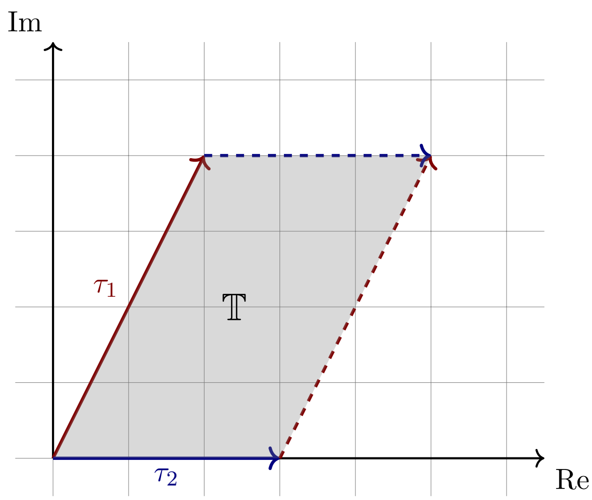



**Figure 1.3:** A geometrical interpretation of the complex structure modulus $\tau = \frac{\tau_1}{\tau_2}$ and the quotient construction in (1.19). We see that $\tau$ measures the ratio between the two sides of the parallelogram that makes up the torus. In particular, for Im $\tau$ much greater than 1, one obtains a thin torus as shown in figure 1.2.

These are global diffeomorphisms which are not connected to the identity. Indeed, one sees that the transformations $\alpha, \beta$ correspond to twists of the torus along the two canonical cycles over an angle of $2\pi$. These transformations generate the group $\mathrm{SL}(2, \mathbb{Z})$, which acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \tag{1.22}$$

Moreover, given a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$, we see that $A$ and $-A$ generate the same transformation. In other words, the modular group $\mathrm{PSL}(2, \mathbb{Z}) := \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$ preserves the complex structure of a torus. Therefore, we define the complex structure moduli space $\mathcal{M}_{\mathrm{cs}}$ of the complex 1-torus as

$$\mathcal{M}_{\mathrm{cs}} := \mathbb{H}/\mathrm{PSL}(2, \mathbb{Z}). \tag{1.23}$$

Then $\mathcal{M}_{\mathrm{cs}}$ precisely parametrizes those values of $\tau$ which yield tori with different complex structures. From a more geometrical point of view, as depicted in figure 1.3, it describes different *shapes* of the torus. For example, for large values of Im $\tau$, the torus will be very thin. This interpretation will be expanded upon in subsection 1.3.3, where the same will remain true for general Calabi-Yau manifolds.

For clarity and later reference, the fundamental domain $\mathscr{F}$ in $\mathbb{H}$ for the action of the modular group is

$$\mathscr{F} = \{-\frac{1}{2} \leq \mathrm{Re}\, \tau \leq 0, |\tau|^2 \geq 1\} \cup \{0 < \mathrm{Re}\, \tau < \frac{1}{2}, |\tau|^2 > 1\}, \tag{1.24}$$

see also figure 1.4. We also note that the action (1.22) is identical to the action of S-duality under which the Type IIB supergravity action (1.11) is invariant. In section 1.4 we will discuss how this gives rise to a geometrical interpretation of the parameter $\tau$ in F-theory.

As a concluding remark regarding the complex structure moduli space of the torus, we note that the action of the modular group is alternatively generated by the following maps:

$$T : \tau \mapsto \tau + 1, \quad S : \tau \mapsto -\frac{1}{\tau}, \tag{1.25}$$

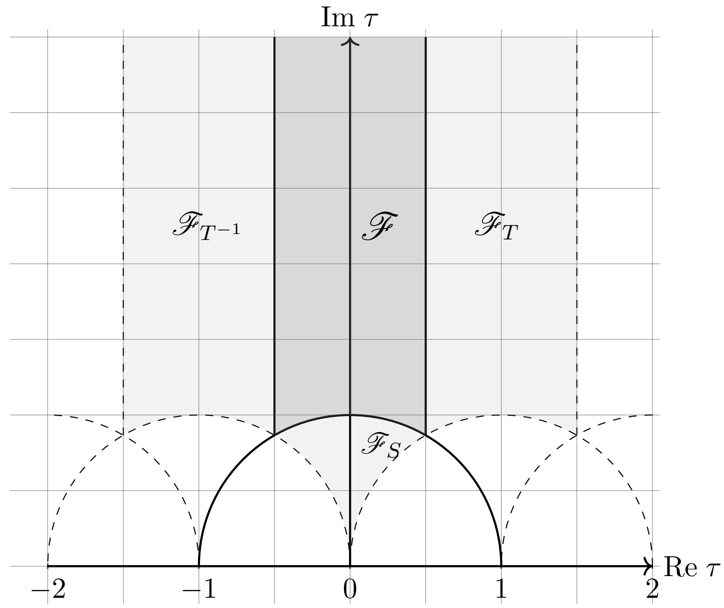



**Figure 1.4:** The fundamental domain $\mathscr{F}$ for the action of the modular group on the complex upper-half plane. This region described the values of $\tau$ which yield tori of different shapes, i.e. having different complex structures. The regions $\mathscr{F}_T$ and $\mathscr{F}_S$ denote the image of $\mathscr{F}$ under the action of the maps $T$ and $S$, respectively.

since $TST = \beta : \tau \mapsto \frac{\tau}{\tau+1}$. Crucially, the modular group does not act freely on $\mathbb{H}$. Indeed, we can identify at least two fixed points:

$$\begin{aligned} \tau_1 = i : &\quad S(\tau_1) = \tau_1 \\ \tau_2 = e^{2i\pi/3} : &\quad ST(\tau_2) = \tau_2 . \end{aligned}$$

As such the complex structure moduli space is not a smooth manifold, instead it is a so-called orbifold, which has singularities at precisely the fixed points. In chapters 3 and 4 we will devote much attention to the behaviour of moduli spaces near such singular points.

### 1.3.3 The Lichnerowicz Equation

Let us now turn to the discussion of the moduli spaces of a general Calabi-Yau fourfold $Y_4$. From the examples of the circle and torus, we have the intuition that the moduli should parametrize the 'size' and 'shape' of $Y_4$. Building further upon this picture, we note that the structure that defines the size and shape of a manifold is exactly that of a metric. Indeed, recall that for a path $\gamma$ connecting two points on a manifold $\mathcal{M}$ with metric $g$, its length is given simply by

$$\text{length of } \gamma = \int_\gamma ds, \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.26}$$

see also figure 1.5. In other words, we expect that deformations of the metric on $Y_4$ can be written in terms of the moduli. However, not all deformations should be allowed, since the resulting manifold must still be Calabi-Yau, i.e. have a Ricci flat metric. More precisely, given a Ricci flat background metric $g$ and a perturbation $g \mapsto g + \delta g$, we require that

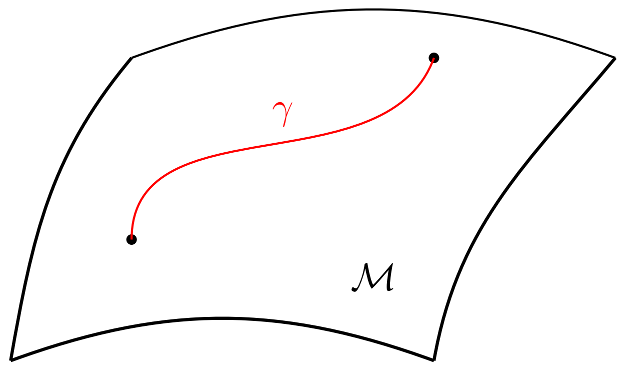



**Figure 1.5:** A path $\gamma$ on a manifold $\mathcal{M}$ connecting two points. Its length can be computed in terms of the metric using equation 1.26.

1. $R(g + \delta g) = 0$.
2. The transformation $g \mapsto g + \delta g$ is not generated by a change of coordinates, since from the physical point of view one cannot distinguish between the resulting spaces.

One can show that these two conditions on $\delta g$ are encapsulated by the Lichnerowicz equation [3, 25]

$$\nabla^C \nabla_C \delta g_{AB} + 2R_A{}^C{}_B{}^D \delta g_{CD} = 0, \tag{1.27}$$

where the indices can be both holomorphic and anti-holomorphic. Due to the index properties of the Riemann tensor on Calabi-Yau manifolds, we can make the following decomposition:

$$\delta g = \delta g_{ab} dy^a dy^b + \delta g_{a\bar{b}} dy^a d\bar{y}^{\bar{b}} + \text{c.c.} \tag{1.28}$$

where c.c. stands for the complex conjugate of the terms before it. Then $\delta g_{a\bar{b}}$ and $\delta g_{ab}$ will satisfy (1.27) separately. Due to the different index structure of $\delta g_{a\bar{b}}$ and $\delta g_{ab}$, these two perturbations will have different effects on the manifold. They will be described by the Kähler moduli and the complex structure moduli, respectively.

### 1.3.4 The Kähler Moduli Space

Let us first consider deformations of the form $\delta g_{a\bar{b}}$, these will correspond to deformations of the Kähler class of $Y_4$, which is readily seen from (A.7). The space of deformations of the Kähler class is called the Kähler moduli space, denoted by $\mathcal{M}_{\text{kähler}}$. Since $\delta g_{a\bar{b}}$ can be regarded as a (1,1)-form, one sees that (1.27) is equivalent to having $(\Delta g_{a\bar{b}}) = 0$, i.e. $\delta g_{a\bar{b}}$ is harmonic. To obtain a suitable parametrization of $\mathcal{M}_K$, let $\{\omega_I\}$, $I = 1, \ldots, h^{1,1}(Y_4)$, be an integral cohomology basis for $H^{1,1}(Y_4)$. Then we can expand the Kähler form $J$ on $Y_4$ as

$$J = v^I \omega_I. \tag{1.29}$$

The coefficients $v^I$ will then serve as the complex coordinates on $\mathcal{M}_{\text{kähler}}$, making it into a complex manifold. They are called the *Kähler moduli*. The precise connection between the metric deformations $\delta g_{a\bar{b}}$ and the Kähler moduli $v^I$ is then given by:

$$\boxed{i\delta g_{a\bar{b}} = (\omega_I)_{a\bar{b}} \delta v^I.} \tag{1.30}$$

Intuitively, the Kähler moduli parametrize the volume of $Y_4$ and the volume of its cycles. Indeed, note that the Kähler form can be used to construct a volume form, which then gives the volume $K_{\text{kähler}}$ of $Y_4$ by:

$$K_{\text{kähler}} = \frac{1}{4!}\int_{Y_4} J\wedge J\wedge J\wedge J. \tag{1.31}$$

Moreover, it turns out that the Kähler moduli space $\mathcal{M}_{\text{kähler}}$ can be equipped with a Kähler metric given by:

$$G_{IJ}(v) = -\frac{1}{2}\partial_I\partial_J \log K_{\text{kähler}}. \tag{1.32}$$

In particular, we see that the Kähler potential for this metric $G_{IJ}$ is determined solely by the volume $K_{\text{kähler}}$. Again, this indicates that $\mathcal{M}_{\text{kähler}}$ is naturally associated with the volume of $Y_4$.

### 1.3.5 The Complex Structure Moduli Space

Next, we consider the deformations of the form $\delta g_{ab}$. First, we note that since $\delta g_{ab}$ does not have mixed indices, the deformed metric is no longer Hermitian. However, it can be made into a Hermitian metric by a change of coordinates which is *not* holomorphic (since any holomorphic change of coordinates cannot affect the index structure of the metric). In other words, the deformations $\delta g_{ab}$ can be interpreted as deformations of the complex structure of the Calabi-Yau manifold. The space of all such deformations is called the *complex structure moduli space*, and is denoted by $\mathcal{M}_{\text{cs}}$. It will turn out that these deformations are in one-to-one correspondence with harmonic (3,1) forms. Indeed, let $\{\chi_i\},\ i=1,\dots h^{3,1}(Y_4)$, be a complex cohomology basis of $H^{3,1}(Y_4)$. We define

$$\left(\bar{b}_{\bar{\imath}}\right)_a^{\ \bar{b}} = \frac{i}{||\Omega||^2}\,(\bar{\chi}_{\bar{\imath}})_{a\bar{c}\bar{d}\bar{e}}\,\Omega^{\bar{c}\bar{d}\bar{e}\bar{b}}, \quad ||\Omega||^2 = \frac{1}{4!}\Omega_{abcd}\overline{\Omega^{abcd}}. \tag{1.33}$$

Then the complex structure deformations $\delta g_{ab}$ are related to changes in the complex structure moduli $\delta\bar{z}^{\bar{\imath}}$ by

$$\boxed{\delta g_{ab} = \left(\bar{b}_{\bar{\imath}}\right)_{ab}\delta\bar{z}^{\bar{\imath}}.} \tag{1.34}$$

Moreover, under this correspondence one can show that (1.27) is equivalent to $\bar{\chi}_{\bar{A}}$ being a harmonic (3,1)-form. Similar to the Kähler moduli space $\mathcal{M}_{\text{kähler}}$, the complex structure moduli space $\mathcal{M}_{\text{cs}}$ can also be equipped with a Kähler metric, defined by:

$$g_{i\bar{\jmath}} = \partial_i\bar{\partial}_{\bar{\jmath}}K_{\text{cs}}(z,\bar{z}), \tag{1.35}$$

where the complex structure Kähler potential $\mathcal{K}_{\text{cs}}$ is given by:

$$e^{-K_{\text{cs}}} = \int_{Y_4}\Omega\wedge\bar{\Omega}. \tag{1.36}$$

For the remainder of this work, we will mostly focus on the complex structure moduli, but occasionally the importance of the Kähler moduli will be discussed as well. Having discussed Calabi-Yau manifolds and their moduli spaces in detail, our next step will be to formulate the setting of F-theory in which we will work.

## 1.4 F-Theory via M-Theory

As we have mentioned in the previous sections, we would like to describe the Type IIB effective action in a more geometric way, by interpreting the parameter $\tau$ as the complex structure modulus of a torus. This is what is done in F-theory. In principle, the idea is to construct a 12-dimensional theory of supergravity which, when compactified on a torus, reduces precisely to Type IIB supergravity. Unfortunately, no 12-dimensional theories of supergravity exist. As such, we make a slight detour via the unique 11-dimensional supergravity theory known as M-theory. Its effective action is given by

$$\hat{S}^{(11)} = \int_{\mathcal{M}_{1,10}} \frac{1}{2}\hat{R} \star 1 - \frac{1}{4}\hat{F}_4 \wedge \star\hat{F}_4 - \frac{1}{12}\hat{C}_3 \wedge \hat{F}_4 \wedge \hat{F}_4, \tag{1.37}$$

where $\mathcal{M}_{1,10}$ is a general 11-dimensional Riemannian manifold with Lorentzian metric. Moreover $\hat{R}$ denotes the Ricci scalar associated to this metric, $\hat{C}_3$ is a 3-form and $\hat{F}_4 = d\hat{C}_3$ denotes its field strength. We now give a rough overview of how one can relate M-theory to Type IIB following [2, 51]. The main point is to compactify M-theory on $\mathcal{M}_{1,9} \times \mathbb{T}$, where we write the torus as

$$\mathbb{T} = S^1_A \times S^1_B, \tag{1.38}$$

where the circles have coordinates $x$ and $y$, respectively. It is a fact that compactifying M-theory on a circle with radius $R_A$ yields Type IIA in the limit $R_A \to 0$. Hence, interpreting $S^1_A$ as this circle, we obtain Type IIA on $\mathcal{M}_{1,8} \times S^1_B$ in this limit. More quantitatively, the relation between the M-theory and Type IIA metric is given by

$$ds^2_M = L^2 e^{4\chi/3}(dx + C_1)^2 + e^{-2\chi/3} ds^2_{\text{IIA}}, \tag{1.39}$$

where $C_1$ is the RR 1-form of Type IIA, $\chi$ is the Type IIA dilaton and $L$ sets a length scale for the M-theory circle. Next, we recall that Type IIA on a circle $S^1_B$ with radius $R_B$ is T-dual to Type IIB on a circle $\tilde{S}^1_B$ with radius $\tilde{R}_B = l_s^2/R_B$. Hence we obtain Type IIB on $\mathcal{M}_{1,8} \times \tilde{S}^1_B$. Finally then, by taking the decompactification limit $\tilde{R}_B \to \infty$, i.e. $R_B \to 0$, we recover Type IIB on $\mathcal{M}_{1,9}$. To be more precise, one can explicitly show that after performing the various dualities and compactifications, the complex structure modulus $\tau$ of $\mathbb{T}$ is equal to

$$\tau = C_0 + ie^{-\phi} \tag{1.40}$$

where we have put $C_0 := (C_1)_y$, i.e. the $y$-component of $C_1$ and $\phi$ is the Type IIB dilaton. In particular, we can indeed obtain the axio-dilaton field $\tau$ present in Type IIB from M-theory via these various dualities. It remains to construct the forms $C_2, C_4$ and $B_2$. These can all be obtained from the original M-theory 3-form $\hat{C}_3$ by setting

$$\hat{C}_3 = C_3' + B_2 \wedge L dx + C_2 \wedge L dy + B_1 \wedge L dx \wedge L dy \tag{1.41}$$

and putting $C_4^{(y)} = C_3' \wedge dy$. Then $B_1$ gives rise to the off-diagonal components of the Type IIB metric, i.e. $g_{iy} = (B_1)_i$, restoring full generality for the metric. In summary, we find the following duality:

$$\boxed{\text{M-Theory on } \mathcal{M}_{1,8} \times T^2_{\text{Vol}(T^2)\to 0} \iff \text{Type IIB on } \mathcal{M}_{1,9}} \tag{1.42}$$

The theory obtained on the LHS, by taking the volume of the torus to zero, is called *F-theory.*

Strictly speaking, in our current setting the parameter $\tau$ is a constant, hence not really a field. However, one can consider a more general setup where the complex structure of the torus is allowed to vary over $\mathcal{M}_{1,8}$, which corresponds to an elliptic fibration. To ensure $\mathcal{N} = 1$ supersymmetry in four dimensions, this elliptic fibration is chosen to be a Calabi-Yau fourfold, i.e. we compactify M-theory on $\mathcal{M}_{1,2} \times Y_4$, where $Y_4 \to B_3$ is a fibre bundle with $\mathbb{T}$ as fibres and base space $B_3$. By a similar procedure as outlined above, this is then dual to Type IIB on $\mathcal{M}_{1,3} \times B_3$. The upshot of this whole discussion is that instead of considering Type IIB compactified on a Calabi-Yau threefold, we will instead choose to consider M-theory on a Calabi-Yau fourfold. On the one hand this is advantageous, since the M-theory action is easier to work with. On the other hand, Calabi-Yau fourfolds are more complicated than threefolds. Indeed, our goal in the coming chapters is to generalize known results for threefolds to fourfolds within this setting.

*Comparing with Maxwell and Harmonic Forms*

Having discussed the need for M-theory, let us spend some words on the 11d action and make a comparison with the more familiar action for Maxwell theory. Recall that Maxwell theory is described by the following action:

$$S_{\text{Maxwell}} = \int_{\mathcal{M}_{1,3}} -\frac{1}{4} F_2 \wedge \star F_2 - \frac{1}{2} A_1 \wedge \star J_1, \tag{1.43}$$

where $A_1$ is a 1-form which denotes the electromagnetic potential, $F_2 = dA_1$ is the associated field strength, and $J_1$ is a 1-form describing the electromagnetic sources. Let us first consider the equations of motion for both the 11d supergravity and Maxwell actions, which are obtained by varying them with respect to $\hat{C}_3$ and $A_1$, respectively. This yields the following equations:

$$\begin{aligned} \text{11D SUGRA}: \quad & -d \star \hat{F}_4 = \star \hat{J}_3, \quad \hat{J}_3 := \frac{1}{2} \star (F_4 \wedge F_4) \\ \text{Maxwell}: \quad & -d \star F_2 = \star J_1. \end{aligned}$$

As expected, the equations are very similar. Let us consider the situation where there are no fluxes, i.e. we set $\hat{J}_3 = 0$. In chapter 2 we will relax this condition by considering flux compactifications. By analogy with Maxwell theory, this means that we are neglecting all sources, essentially considering the theory in a vacuum. The equation of motion for $\hat{C}_3$ simply becomes

$$d \star d\hat{C}_3 = 0. \tag{1.44}$$

Recalling the definition of the Laplacian, we have

$$-\Delta \hat{C}_3 = d \star d \star \hat{C}_3 + \star d \star d\hat{C}_3 = d \star d \star \hat{C}_3, \tag{1.45}$$

where the second term vanishes by the equations of motion. Moreover, note that both actions enjoy a gauge symmetry given by

$$\begin{aligned} \text{11D SUGRA}: \quad & \hat{C}_3 \to \hat{C}_3 + d\hat{\Lambda}_2 \\ \text{Maxwell}: \quad & A_1 \to A_1 + d\Lambda, \end{aligned}$$

for any 2-form $\hat{\Lambda}_2$ or scalar $\Lambda$. In particular, since the combination

$$\star d \star \hat{C}_3 \tag{1.46}$$

is a 2-form in 11 dimensions, we see that we can fix our gauge such that this term vanishes. In our analogy with Maxwell theory, this corresponds precisely to the Lorentz gauge. It follows that

$$\Delta \hat{C}_3 = 0,$$

As expected, the 3-form $\hat{C}_3$ obeys the Laplace equation, i.e. it is a harmonic form.[3] Moreover, since

$$\Delta \circ d = -d \star d \star d = d \circ \Delta, \tag{1.47}$$

it follows that $\hat{F}_4 = d\hat{C}_3$ is also a harmonic form.

## 1.5 Compactification of the Ricci Scalar

In this last section, we will use our knowledge of Calabi-Yau moduli spaces to derive the dynamics of the complex structure moduli by performing the compactification of the Ricci scalar. We follow the discussion in Appendix D of [33]. Consider the 11-dimensional manifold $\mathcal{M}_{1,10}$ as a product manifold:

$$\mathcal{M}_{1,10} = \mathcal{M}_{1,2} \times Y_4, \tag{1.48}$$

where $Y_4$ is a Calabi-Yau fourfold and $\mathcal{M}_{1,2}$ is a general Minkowskian manifold. The metric can then be decomposed as

$$g_{MN} dx^M dx^N = g_{\mu\nu}(x) dx^\mu dx^\nu + h_{a\bar{b}} dy^a d\bar{y}^{\bar{b}} + \bar{z}^{\bar{\imath}}(x)(\bar{b}_{\bar{\imath}})_{ab} dy^a dy^b. \tag{1.49}$$

Here $\{y^a, \bar{y}^{\bar{a}}\}_{a=1}^4$ denote the coordinates on $Y_4$ and $\{x^\mu\}_{\mu=0}^2$ denote the coordinates on $\mathcal{M}_{1,2}$. Moreover $h_{a\bar{b}}$ denotes a constant background metric on $Y_4$ and deviations from it are parametrized exactly by the complex structure moduli $z$. For the purpose of this derivation, we neglect the Kähler moduli, since their inclusion does not affect the resulting dynamics of the complex structure moduli. We are interested in the dimensional reduction of the Ricci scalar using this ansatz for the metric, whose computation is delegated to appendix B. There the following result is obtained (see equation B.9)

$$R_{11} = R_3 + \frac{1}{2}(b_i \cdot \bar{b}_{\bar{\jmath}}) \partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}}. \tag{1.50}$$

Here the contraction $b_i \cdot \bar{b}_{\bar{\jmath}}$ is defined using the background metric $h_{a\bar{b}}$, see appendix B for the exact relation. For convenience, we define

$$Q_{i\bar{\jmath}} := \frac{1}{2} \int_{Y_4} (b_i \cdot \bar{b}_{\bar{\jmath}}). \tag{1.51}$$

Then we obtain the following result for the compactification of the Ricci scalar action:

$$\int_{\mathcal{M}_{1,10}} d^{11}x \sqrt{-g_{11}}\, R_{11} = \int_{\mathcal{M}_{1,2}} d^3x \sqrt{-g_3} \left( \mathcal{K} R_3 + Q_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \right), \tag{1.52}$$

where $\mathcal{K}$ denotes the volume of $Y_4$. At this point, we already see that $R_{11}$ splits into the three-dimensional Ricci scalar $R_3$ plus an additional kinetic term for the moduli fields. To bring the action into the canonical Einstein-Hilbert form we perform a Weyl transformation, under which

$$g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu} \tag{1.53}$$

and additionally, in $d$ space-time dimensions [33, p. 70]

$$\int dx^d \sqrt{-g}\, \Omega^{d-2} R \to \int dx^d \sqrt{-g} \left( R + (d-1)(d-2) \partial_\mu \log \Omega \, \partial^\mu \log \Omega \right). \tag{1.54}$$

[3] We refer the reader to appendix A for the exact definition of harmonic forms and the Hodge decomposition theorem, which we will use in the next chapter.

and finally

$$\sqrt{-g_3}\partial^\mu \to \Omega^{-1}\sqrt{-g_3}\partial^\mu. \tag{1.55}$$

In particular, since $d = 3$, we take $\Omega = \mathcal{K}$ and obtain

$$\int_{\mathcal{M}_{1,10}} \hat{R} \star 1 = \int_{\mathcal{M}_{1,2}} \left( R \star 1 + 2d\log\mathcal{K} \wedge \star d\log\mathcal{K} + \mathcal{K}^{-1} Q_{i\bar{\jmath}}\, dz^i \wedge \star d\bar{z}^{\bar{\jmath}} \right). \tag{1.56}$$

Finally, we note that one can relate $Q_{i\bar{\jmath}}$ to the natural metric $g_{i\bar{\jmath}}$ on the complex structure moduli spaces as follows [5, 39]

$$Q_{i\bar{\jmath}} = -2\mathcal{K} g_{i\bar{\jmath}}. \tag{1.57}$$

Hence the final result for the compactification of the Ricci scalar can be written as

$$\boxed{\int_{\mathcal{M}_{1,10}} \frac{1}{2}\hat{R} \star 1 = \int_{\mathcal{M}_{1,2}} \frac{1}{2} R \star 1 - g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} + \cdots} \tag{1.58}$$

where we have only included the resulting terms depending on the complex structure moduli, since those are the ones we are interested in. We have succeeded in obtaining the dynamics of these moduli in the lower dimensions, and crucially we see that they are *massless*, since the above construction cannot give rise to a scalar potential. As alluded to in section 1.1, this poses a problem for creating realistic string theory compactifications, since no massless scalar fields are currently observed. Note also that generically one will obtain numerous such fields, with the exact number given by $h^{3,1}$ which can easily be of order 100. We therefore pose the following question

Q1: How can the complex structure moduli be stabilized?

Providing an answer will be the topic of the next chapter, but we will see that the proposed solution gives rise to many more interesting questions.

# Chapter 2

# Distributions of Flux Vacua

In chapter 1 we encountered the issue of moduli stabilization. We have seen that the compactification of the Ricci scalar to lower dimensions yields a number of massless scalar fields whose presence is not supported by physical observation. This is a general feature of all string compactifications, not just Type IIB/M/F-theory. However, propagation of other fields in the internal dimensions, known as fluxes, may give a potential and thereby yield a mass term for the complex structure moduli. These fluxes are governed by the field strength $F_4$ and can be thought of as higher-dimensional analogues of the electromagnetic fields of Maxwell theory. They are illustrated in figure 2.1 for a torus background. For a typical Calabi-Yau fourfold compactification, one can have as many as 10 to $10^4$ different flux components. In the specific setting of flux compactification of F-theory on a Calabi-Yau fourfold mthe resulting theory is known to be a 4-dimensional theory of $\mathcal{N} = 1$ supergravity. Such theories are elegantly described in terms of a Kähler potential and a superpotential, the latter of which is easily related to the flux. However, the many different choices of fluxes give rise to a rich landscape of possible string vacua, whose explicit construction is in general very complicated. As such our main concern in this chapter lies with more statistical properties of such vacua and, in particular, whether they are finite in number.

In section 2.1 we start by discussing the basics of flux compactification, its necessity due to the tadpole constraint and the effect fluxes have on the underlying geometry through warping. We quickly see that the inclusion of fluxes naturally leads to a potential for the complex structure moduli. Then in section 2.2 we introduce the standard setup of $\mathcal{N} = 1$ supergravity in which the

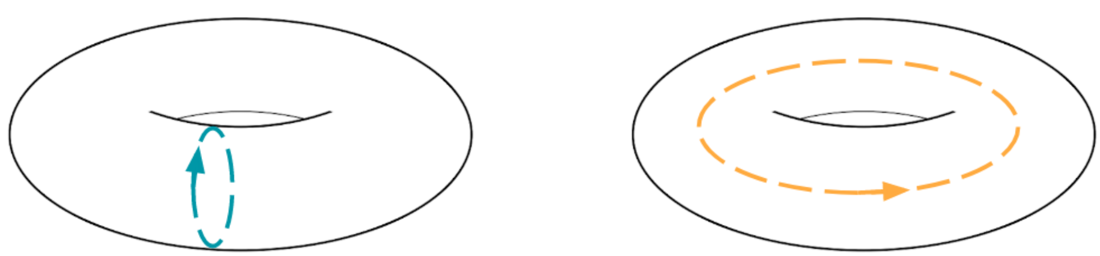


**Figure 2.1:** Different flux lines running around in the internal dimensions, here illustrated for the torus. Calabi-Yau fourfolds are expected to have numerous different components which can be turned on, depending on the sources which are present.

resulting theory is well described and the vacuum structure is given in terms of a superpotential. We then turn our attention to the statistical properties of such vacua in section 2.3 by deriving the Ashok-Douglas density and ultimately the index of supersymmetric vacua. The derivation can be seen as computing particular expectation values in Gaussian ensembles of superpotentials using constrained two-point functions. The predictions of the Ashok-Douglas density are compared with exact results for rigid compactifications on $Y_4 = X_3 \times \mathbb{T}$ and we spend some time discussing the details and assumptions that enter the derivation of the Ashok-Douglas density in section 2.4. Most notably we discuss the quantization of fluxes and the stabilization of the Kähler moduli through non-perturbative contributions to the superpotential. Finally, given a concrete expression for the Ashok-Douglas density, we question whether it is integrable over the moduli space, essentially asking whether the number of string vacua is finite. In section 2.5 we set the stage to answer this question by a suitable rewriting but leave the remainder of the discussion to chapters 3 and 4.

## 2.1 Flux Compactification

*The Tadpole Constraint, Fluxes and Warped Geometries*

In this section we review the basics of flux compactification, the essential properties of fluxes and how their inclusion generates a potential for the complex structure moduli. First, we recall the 11-dimensional supergravity action [1]

$$S = \int_{\mathcal{M}_{11}} \frac{1}{2} R \star 1 - \frac{1}{4} F_4 \wedge \star F_4 - \frac{1}{12} C_3 \wedge F_4 \wedge F_4, \quad F_4 = dC_3. \tag{2.1}$$

In chapter 1 we have derived the compactification of the Ricci scalar and saw that it resulted in a kinetic term for the complex structure moduli fields. There we essentially ignored the dynamics of $C_3$, effectively setting $F_4$ to zero. However, due higher order corrections to the 11-dimensional supergravity action we are actually forced to consider non-zero values of $F_4$. This can be seen as follows. One of the higher derivative correction terms is given by [13]

$$\delta S = \int C_3 \wedge I_8(R), \tag{2.2}$$

where $I_8$ is a polynomial of degree 4 in the curvature tensor. One way to obtain this result is to consider a one-loop scattering diagram in Type IIA involving four gravitons and the Kalb-Ramond field and lifting this to M-theory[2]. Including this correction, the equation of motion for $C_3$ in the possible presence of M2-branes[2] is given by

$$d \star F_4 = \frac{1}{2} F_4 \wedge F_4 - I_8(R) + \sum_i \delta_{\mathrm{M2}_i}, \tag{2.3}$$

where the sum runs over all positions of the M2 branes. Integrating both sides and noting that the LHS is a total derivative yields the *tadpole constraint*

$$N_{\mathrm{M2}} + \frac{1}{2} \int_{Y_4} F_4 \wedge F_4 = \frac{\chi(Y_4)}{4!}, \tag{2.4}$$

where $\chi$ denotes the Euler characteristic of $Y_4$ and $N_{\mathrm{M2}}$ denotes the number of M2 branes. Crucially, the tadpole constraint generically enforces a non-zero value of $F_4$ on the internal manifold.

[1] We now drop the hats in the notation of the 11-dimensional fields.

[2] Recall that M2 branes are charged under the $C_3$ field.

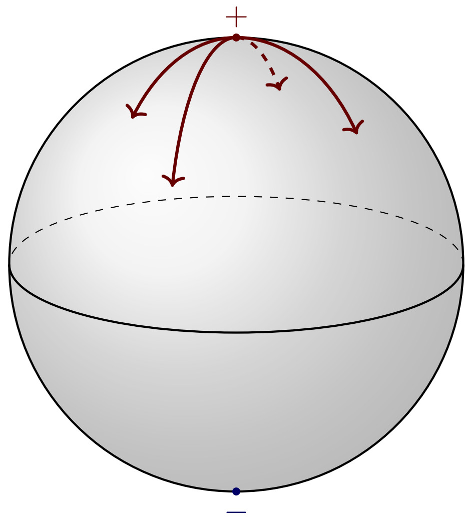



**Figure 2.2:** A 2-sphere together with a positive charge and a negative charge is depicted. Since the 2-sphere is compact the field lines emitted at the positive charge must end at the negative charge, they cannot run off to infinity.

Essentially, this constraint arises from the fact that the internal manifold is compact. Indeed, on a compact manifold the flux lines of e.g. an electric field cannot run off to infinity, see e.g. figure 2.2. As a result, the total charge on a compact manifold must vanish, or there must be some M2 branes on which the field lines can end. [3]

By a *flux* we will henceforth mean a non-zero choice of $F_4$ on the internal manifold, and emphasize the fact that it only has internal components by denoting it by $G_4$ instead. In section 1.4 we showed that $F_4$ is a harmonic form when imposing the vacuum equations of motion. Decomposing the Laplace operator as

$$\Delta = \Delta_{\mathcal{M}_{1,3}} + \Delta_{Y_4}, \tag{2.6}$$

we see that $G_4$ is a harmonic form on $Y_4$, i.e. $\Delta_{Y_4} G_4 = 0$. Using the properties of $Y_4$ the Hodge decomposition theorem[4] implies that we may uniquely associate $G_4$ with a cohomology class in $H^4(Y_4)$. Although the inclusion of a non-zero $G_4$ takes us outside of the vacuum, it nevertheless suffices to consider only the cohomology class of $G_4$, since for any 4-cycle $\gamma$ the value of the integral

$$\int_\gamma G_4 \tag{2.7}$$

only depends on the cohomology class $[G_4]$ by Stokes' theorem. In the following we will abuse the notation somewhat by using the same notation, putting $G_4 \in H^4(Y_4)$. One of the implications of including an internal flux is that the underlying geometry becomes *warped*. Indeed, because there is

[3] Following the discussion in [2, p. 484], this is made more precise by rephrasing the tadpole constraint in Type IIB language, where it takes the form

$$N_{\mathrm{D3}} + \int_{X_3} H_3 \wedge F_3 = \frac{\chi(Y_4)}{4!}, \tag{2.5}$$

where $X_3$ is the base of the elliptically fibred $Y_4$. In this setting the RHS can be interpreted as minus the D3-brane charge, which is induced by the curvature of possibly present D7-branes. As a result, the tadpole constraint can be interpreted as the vanishing of the total D3-brane charge when including all sources.

[4] Again, we refer the reader to Appendix A for details on the Hodge decomposition theorem

a non-trivial coupling between the internal metric and $G_4$ the resulting geometry is often no longer Ricci-flat. Rather, one finds a metric of the form

$$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} g_{a\bar{b}} dy^a d\bar{y}^{\bar{b}}, \tag{2.8}$$

where $g_{a\bar{b}}$ is Ricci-flat. In other words, the internal geometry is conformally Calabi-Yau [31].[5] Here $A$ is called the *warp factor*, and compactifications of this kind are known as *warped compactifications.* Throughout this thesis we will assume that the warping does not influence our results significantly and we will comment on this where appropriate.

Besides being constrained by the tadpole constraint, flux are additionally quantized due to Dirac quantization. More precisely, in terms of an integral basis $\gamma_\alpha$ of $H^4(Y_4, \mathbb{C})$, the *flux quanta*

$$N^\alpha := \int_{\gamma_\alpha} G_4 \tag{2.9}$$

are integers[6]. We will denote by $\mathbf{N}$ the vector of flux quanta, which has $b_4 = \dim H^4(Y_4)$ components.

*Constructing a Potential*

In analogy with Maxwell theory, a choice of $G_4$ corresponds to a (generalized) electromagnetic field which is present in the internal manifold only. Its kinetics are partly governed by the metric (present in the Hodge star), which in turn depends on the complex structure moduli. Performing the integration over the internal manifold in (2.1) therefore yields a potential for the complex structure moduli of the form

$$\boxed{V(z) = \frac{1}{\mathcal{V}_b^3}\left(\int_{Y_4} G_4 \wedge \star G_4 - \frac{\chi(Y_4)}{12}\right)} \tag{2.11}$$

which is quadratic in the fluxes [29, p. 5]. Here $\mathcal{V}_b$ denotes the volume of the base $B_3$ of the elliptic fibration $Y_4$. Note that the second term arises from the higher-derivative term in the action. By considering the extrema of this potential we will be able to stabilize the moduli. In other words, we have the following answer to the question posed at the end of chapter 1:

> A1: The complex structure moduli $z^i$ can be stabilized by extremizing the flux-induced scalar potential $V(z)$, i.e. by setting
>
> $$\frac{\partial V}{\partial z^i} = 0. \tag{2.12}$$

Of course this answer is incomplete, as we have not specified solutions to this equation yet. This is further complicated by the fact that the scalar potential can take very different forms, since (in general) there are many flux components that can be turned on. The goal of this chapter is to address this issue from a more statistical point of view, by asking the question

[5] This is the case for Type IIB and F-theory compactifications. More subtle complications arise when considering e.g. flux compactifications of the heterotic string, which requires a non-Kähler background. [2, p.458]

[6] Up to the shift of a half-integer. More precisely, in [52] it was shown that

$$G_4 - \frac{\lambda}{2} \in H^4(Y_4, \mathbb{Z}), \tag{2.10}$$

where $\lambda = p_1(Y_4)/2$ with $p_1(Y_4)$ the first Pontryagin class of $Y_4$.

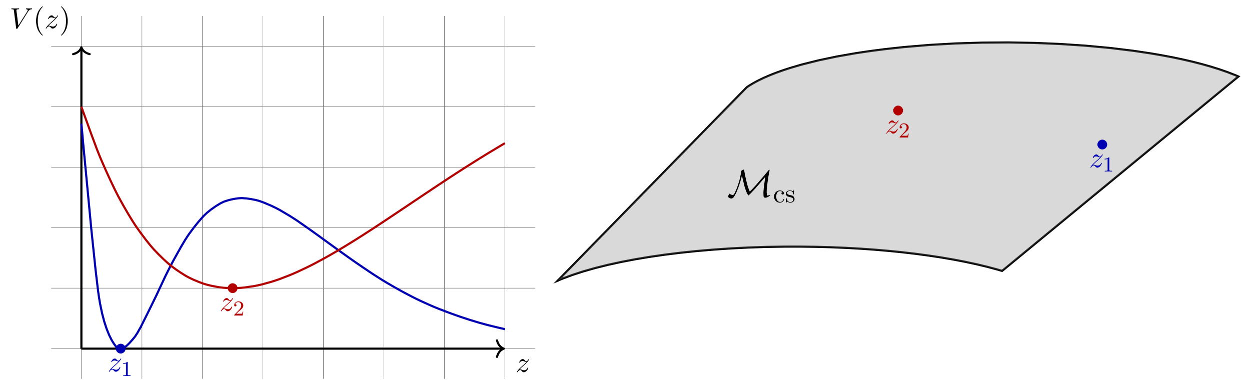



**Figure 2.3:** (Left) Two different minima $z_1$ and $z_2$ for two different flux induced scalar potentials. (Right) Schematic positions of the minima $z_1$ and $z_2$ in the complex structure moduli space.

> Q2: How are the F-theory vacua, i.e. the minima of $V(z)$, distributed over the moduli space?

See figure 2.3 for a graphical depiction of this question. Additionally, one might also wonder about the role of the Kähler moduli in this story and whether the total number of flux vacua is even finite. We will provide a small discussion regarding the Kähler moduli and other details in section 2.4. The question of finiteness will be formulated and addressed in chapters 3 and 4 and its answer will be the main result of this entire work.

## 2.2 $\mathcal{N} = 1$ Supergravity Formulation

All of the questions that we posed in the previous section are more easily tackled within the framework of $\mathcal{N} = 1$ supergravity. It has been shown in e.g. [27] that flux compactification of $F$-theory on a Calabi-Yau fourfold yields a 4-dimensional theory of $\mathcal{N} = 1$ supergravity. The $h^{3,1}(Y_4)$ complex structure moduli $z^i$ form the bosonic scalar fields of a chiral multiplet. Let us recall the main data that comprises an $\mathcal{N} = 1$ theory of supergravity. For a detailed account of supergravity theories, one may consult [23]. It consists of a triple $(\mathcal{C}, L, W)$, where

- $\mathcal{C}$ is a Kähler manifold, called the *configuration space*, with Kähler potential $K$ and metric determined by $g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$.
- $L \to \mathcal{C}$ is a holomorphic line bundle, whose first Chern class is given in terms of the Kähler form $\omega = \partial\bar{\partial}K$ of $\mathcal{C}$ as $c_1(L) = -\frac{1}{\pi}[\omega]$.
- $W$ is a holomorphic section of $L$, called the *superpotential*.

The superpotential induces a scalar potential $V$ which is given by[7]

$$V = e^K \left( g^{i\bar{j}} D_i W \overline{D_j W} - 3|W|^2 \right), \tag{2.13}$$

where

$$D_i W = \partial_i W + (\partial_i K) W, \tag{2.14}$$

[7]Here we have set the Planck mass $M_{\text{pl}} = 1$.

denotes the Kähler-Weyl covariant derivative. It accounts for the fact that the theory is invariant under the following Kähler-Weyl transformation

$$K(z,\bar{z}) \mapsto K(z,\bar{z}) + F(z) + \bar{F}(\bar{z}), \tag{2.15}$$

for any holomorphic function $F(z)$. Indeed, one readily sees that the metric $g_{i\bar{j}}$ is invariant under this transformation. Moreover, one can explicitly compute that

$$[D_i, \bar{D}_{\bar{j}}] = 2i\partial_i\bar{\partial}_{\bar{j}}K, \tag{2.16}$$

which implies that $c_1(L) = -\frac{1}{\pi}[\omega]$, where $\omega = \partial\bar{\partial}K$ is the Kähler form.

In the setting of F-theory flux compactification, the configuration space $\mathcal{C}$ is given by the complex structure moduli space $\mathcal{M} = \mathcal{M}_{\text{cs}}$ of the Calabi-Yau fourfold $Y_4$, with the Kähler potential given by $K = K_{\text{cs}}$. We emphasize that in the following we ignore the Kähler moduli completely, and discuss the effects of their inclusion in 2.4.2. By analysing the $\mathcal{N} = 1$ supersymmetry conditions in four dimensions, it was shown in [32] that the superpotential $W$ arising from a non-zero internal flux $G_4$ is explicitly given by

$$W = \int_{Y_4} G_4 \wedge \Omega. \tag{2.17}$$

Intuitively, this is indeed a holomorphic section of a line bundle $L$, since $\Omega$ depends holomorphically on the complex structure moduli $z$ and $H^{4,0}(Y_4)$ is one dimensional. Using 2.17 and the fact that

$$e^{-K} = \int_{Y_4} \Omega \wedge \bar{\Omega}. \tag{2.18}$$

it follows that $W$ transforms under Kähler-Weyl transformations as

$$W(z) \mapsto e^{-F(z)}W(z), \tag{2.19}$$

which in particular implies that $V$ is invariant. To find the vacuum solutions of $V$ in terms of $W$, we note that

$$\partial_i V = e^K \left(D_i D_j W - 2\delta_{ij}\overline{W}\right) D^j W. \tag{2.20}$$

In particular, the equation $\partial_i V = 0$ can be solved by

$$D_i W(z) = 0. \tag{2.21}$$

Incidentally, this condition results in a supersymmetric vacuum, which we will elucidate shortly. Note that there may be other solutions to $\partial_i V = 0$, which will generically break supersymmetry. Although these can be studied, see for example [15], analytical results are very rare and hence we restrict ourselves to the supersymmetric solutions.

*Supersymmetric Vacua*

Let us give a few comments on what we mean by supersymmetric vacua, and clear up a possible confusion which may arise. As with any symmetry, a solution which is supersymmetric is such that it is invariant under supersymmetry transformations. In four-dimensional $\mathcal{N} = 1$ supergravity this imposes a condition on the supersymmetry variation of the gravitino field $\psi^i$ which is of the form [9, p. 16]

$$\langle \delta\psi^i \rangle = \left[e^{G/2} g^{i\bar{j}}\partial_{\bar{j}}G\right]\epsilon \overset{!}{=} 0, \tag{2.22}$$

where $\epsilon$ parametrizes the supersymmetry transformation and $G$ is the Kähler covariant combination

$$G = K + \log |W|^2. \tag{2.23}$$

In particular, the supersymmetry condition $\langle \delta\psi^i \rangle = 0$ boils down to

$$\text{SUSY}: \quad \partial_i G = 0, \quad \text{or} \quad D_i W = 0 \tag{2.24}$$

exactly as stated before. Another way to investigate the supersymmetry properties of a given solution is by looking at the physical gravitino mass. Indeed, in a supersymmetric configuration particles in the same supermultiplet have the same mass. As such the gravitino, being the superpartner of the graviton, must be massless in order to preserve supersymmetry. In the Lagrangian of four-dimensional $\mathcal{N} = 1$ supergravity there appears a potential mass term for the gravitino with the Kähler-invariant mass parameter $|m_{3/2}|^2$ given by

$$|m_{3/2}|^2 = e^K |W|^2. \tag{2.25}$$

In particular, to ensure supersymmetry one might be tempted to additionally impose the condition $W = 0$. However, this is only true in a Minkowski background, as is further explained by considering the value of the scalar potential when $D_i W = 0$, which is

$$V = -3e^K |W|^2. \tag{2.26}$$

Hence for space-times with vanishing cosmological constant, we are also forced to set $W = 0$. The situation changes, however, when considering an AdS background. There it is known that $m_{3/2}$ does not describe the physical gravitino mass. The appropriate interpretation is provided in [17]. Upon further inspection, demanding that the supersymmetry variation of the gravitino vanishes results in a delicate balance between the curvature of the background and the gravitino mass, in such a way that

$$\Lambda = -3|m_{3/2}|^2, \tag{2.27}$$

whilst the physical gravitino mass remains zero, preserving supersymmetry. Here $\Lambda$ denotes the cosmological constant. We refer the reader to [12, section 2] for more details. The upshot of this discussion is that a supersymmetric vacuum is determined by $D_i W = 0$, without additional conditions imposed on $W$. As such we allow for both Minkowski ($W = 0$) and AdS ($W \neq 0$) vacua.

*Some Notation*

To close this section, we introduce some more convenient notation. Of utmost importance is the period vector $\mathbf{\Pi}$ whose components are defined by

$$\Pi_\alpha := \int_{\gamma_\alpha} \Omega, \tag{2.28}$$

where we recall that $\gamma_\alpha$ is a basis of $H^4(Y_4)$. Although it will not be apparent in this chapter, the period vector will be crucial when investigating certain limits in the moduli space as described in chapters 3 and 4. We also introduce the intersection form $\eta$ as

$$\eta_{\alpha\beta} := \int_{Y_4} \gamma_\alpha \wedge \gamma_\beta, \tag{2.29}$$

which is symmetric for fourfolds. Then the Kähler potential can be written as follows

$$e^{-K} = \int_{Y_4} \Omega \wedge \bar{\Omega} = \mathbf{\Pi}^\dagger \eta \mathbf{\Pi}. \tag{2.30}$$

Moreover, the superpotential is given by

$$W(z) = \mathbf{N}^T \cdot \mathbf{\Pi}(z), \tag{2.31}$$

where we stress that the $z$-dependence resides within the period vector. Finally the tadpole constraint takes the form

$$\mathbf{N}^T \eta \mathbf{N} = L_* - 2N_{\mathrm{M2}}, \quad L_* = \frac{\chi(Y_4)}{12}. \tag{2.32}$$

## 2.3 The Ashok-Douglas Density

Investigating the possible flux vacua arising from flux compactification is essential for string phenomenology. Indeed, quantities such as the cosmological constant and Yukawa couplings are expressed in terms of the complex structure moduli. As a result, the choice of vacuum induces, for instance, a particular value for the cosmological constant which may then be tested against reality. Unfortunately, to check what kind of vacua occur by going through all Calabi-Yau fourfolds individually is not feasible. Indeed, only in a handful of cases are we able to obtain an explicit expression for the periods by solving the Picard-Fuchs equations. As such, Ashok and Douglas instigated a more statistical approach to studying flux vacua. In particular, they derived the general formula [1]

$$\det(R + \omega) \tag{2.33}$$

for the density of supersymmetric flux vacua on the moduli space, expressed in terms of the curvature $R$ and Kähler two form $\omega$. In this section, we will reproduce their result in detail following the original work as presented in [1, 14]. We then check the predictions it makes for the flux vacua of rigid compactification on $Y_4 = \mathbb{T} \times X_3$. In the next section we comment on the range of validity of the result, the assumptions that are made and the interpretation. Finally, while our current setup involves F-theory compactifications on a Calabi-Yau fourfold, one can perform the exact same calculation for e.g. Type IIB on a Calabi-Yau orientifold. More precisely, the statements below depend only on the $\mathcal{N} = 1$ supergravity data, together with a whole set of flux quanta leading to an ensemble of superpotentials (as opposed to a single choice).

### 2.3.1 Counting Flux Vacua

We recall that a flux is characterized by a choice of $\mathbf{N}$ satisfying the tadpole constraint. By allowing for any number of M2 branes, the tadpole constraint can be interpreted as a bound

$$\mathbf{N}^T \eta \mathbf{N} \leq L_*, \tag{2.34}$$

where $L_*$ is determined by the topological properties of the internal manifold. For each choice of $\mathbf{N}$, we denote

$$L := \mathbf{N}^T \eta \mathbf{N}. \tag{2.35}$$

Let $N_{\mathrm{vac}}(L \leq L_*)$ denote the total number of flux vacua[8] satisfying (2.34). Then one can implement the tadpole constraint using the Heaviside $\theta$-function as follows

$$\begin{aligned} N_{\mathrm{vac}}(L \leq L_*) &= \sum_{\mathrm{vac}} \theta(L - L_*) \\ &= \frac{1}{2\pi i} \sum_{\mathrm{vac}} \int_C \frac{d\alpha}{\alpha} e^{\alpha(L_* - L)} \\ &= \frac{1}{2\pi i} \int_C \frac{d\alpha}{\alpha} e^{\alpha L_*} N(\alpha) \end{aligned}$$

[8] From here on out, whenever we say flux vacua we mean only the supersymmetric ones satisfying $D_i W = 0$.

where the second equality follows from the residue theorem and the contour $C$ runs over the imaginary axis, passing zero from the right. Here

$$N(\alpha) = \sum_{\text{vac}} e^{-\alpha \mathbf{N}^T \eta \mathbf{N}}, \tag{2.36}$$

is the weighted sum of all flux vacua, which are characterized by $D_i W = 0$. We can incorporate this constraint as follows. Recall that for a function $f$ of one variable, the number of zeroes of $f$ can be written as

$$\#\{x | f(x) = 0\} = \int dx \; \delta(f(x)) |f'(x)|, \tag{2.37}$$

here the factor $|f'(x)|$ is required to cancel the same factor which arises from the change of variables $f(x) \mapsto x$. In our setting we count the zeroes of $D_i W$, weighted by $\exp(-\alpha \mathbf{N}^T \eta \mathbf{N})$. Hence the proper generalization of this formula yields

$$N(\alpha) = \int_{\mathcal{M}_{\text{cs}}} d^2 z \int d\mathbf{N} \; e^{-\alpha \mathbf{N}^T \eta \mathbf{N}} \delta(D_i W) |\det D^2 W|, \tag{2.38}$$

where we approximated the sum over flux by an integral. Approximating the discrete sum over fluxes by an integral over the whole moduli space is quite non-trivial and we will comment on its validity in section 2.4.1. For now, we simply continue with this assumption. We have also defined

$$D^2 W := \begin{pmatrix} D_i D_j W & D_i \bar{D}_{\bar{j}} \overline{W} \\ \bar{D}_{\bar{i}} D_j W & \bar{D}_{\bar{i}} \bar{D}_{\bar{j}} \overline{W} \end{pmatrix}, \tag{2.39}$$

which can be interpreted as the fermionic mass matrix [14, p. 4]. By rescaling $\sqrt{\alpha}\mathbf{N} \mapsto \mathbf{N}$ we see that $N(\alpha)$ scales as $\alpha^{-b/2}$, where use the short-hand $b := b_4$ to denote the dimension of the flux vector $\mathbf{N}$. This gives

$$N_{\text{vac}}(L \leq L_*) = \frac{N(\alpha = 1)}{2\pi i} \int \frac{d\alpha}{\alpha} \alpha^{-b/2} e^{\alpha L_*}. \tag{2.40}$$

Expanding the exponential as a power series, we again use the residue theorem to see that only the term $(\alpha L_*)^{b/2}$ yields a contribution, i.e. [9]

$$N_{\text{vac}}(L \leq L_*) = \frac{L_*^{b/2}}{(b/2)!} N(\alpha = 1). \tag{2.41}$$

Finally, we note that $N(\alpha = 1)$ can be written as follows:

$$N(\alpha = 1) = \int_{\mathcal{M}_{\text{cs}}} d^2 z \int d\mu[W] \delta(D_i W) |\det D^2 W| \tag{2.42}$$

where

$$d\mu[W] = \int d\mathbf{N} \; \delta\left(W - \mathbf{N}^T \cdot \mathbf{\Pi}\right) e^{-\mathbf{N}^T \eta \mathbf{N}}, \tag{2.43}$$

carries the interpretation of the distribution function of a Gaussian ensemble of superpotentials, weighted by the intersection form $\eta$. In the following section we will investigate this ensemble in detail. As a final remark, we note that $N_{\text{vac}}$ behaves exponentially as a function of $L_*$, with exponent given in terms of the number of flux components that can be turned on. Hence in the limit of large $L_*$, we expect the prefactor to dominate over $N(\alpha = 1)$, if it is finite.

[9] Note that $b$ is defined as the real dimension of the complex vector space $H^4(Y_4)$, in particular $b$ is even.

### 2.3.2 Distributions of Flux Vacua

In the previous subsection we have argued that the total number of flux vacua can be expressed in terms of a Gaussian ensemble of superpotentials as in equation (2.43). The next step is to evaluate (2.42) explicitly. One complication is the the presence of the $|\det D^2W|$ term. To simplify the problem, we will instead consider the supersymmetric index defined by

$$I_{\rm vac}(L \leq L_*) = \frac{L_*^{b/2}}{(b/2)!} I(\alpha = 1). \tag{2.44}$$

where

$$I(\alpha = 1) = \int_{\mathcal{M}_{\rm cs}} d^2z \int d\mu[W] \delta(D_i W) \det D^2 W, \quad d\mu[W] = \int d^2\mathbf{N}\, \delta\left(W - \mathbf{N}^T \cdot \mathbf{\Pi}\right) e^{-\mathbf{N}^\dagger \eta \mathbf{N}}, \tag{2.45}$$

where for generality and convenience we allow $\mathbf{N}$ to take complex values[10]. The difference between $N_{\rm vac}$ and $I_{\rm vac}$ is that the latter does not contain the absolute values signs around det $D^2W$. As such $I_{\rm vac}$ will only give a lower bound on the total number of flux vacua, as it counts each vacuum with a sign. In section 2.4 we will comment in more detail on the relation between the two. The upshot is that the main features of $N_{\rm vac}$ are already described by $I_{\rm vac}$. Moreover, in this section we will be able to compute an explicit formula for $I_{\rm vac}$. We denote an expectation value of a quantity $X$ in this ensemble by

$$\langle X \rangle = \frac{1}{Z_0} \int d\mu[W] X, \quad Z_0 = \frac{\pi^b}{\det \eta} \tag{2.47}$$

In particular, the supersymmetric index density $d\mu[z]$ we are interested in can be written as

$$d\mu[z] = Z_0 \langle \delta(D_i W) \det D^2 W \rangle. \tag{2.48}$$

*The Two-Point Function and a Simpler Version of $d\mu$*

The first important quantity we should consider is the two-point function, defined by

$$G(z_1, \bar{z}_2) := \langle W(z_1) \overline{W}(\bar{z}_2) \rangle. \tag{2.49}$$

Since we are working in a Gaussian ensemble, it is possible to derive an explicit expression for the two-point function as follows

$$\begin{aligned} G(z_1, \bar{z}_2) &= \frac{1}{Z_0} \int d^2\mathbf{N}\, \delta\left(W - \mathbf{N}^T \cdot \mathbf{\Pi}\right) e^{-\mathbf{N}^\dagger \eta \mathbf{N}} W(z_1) \overline{W}(\bar{z}_2) \\ &= \Pi_\alpha(z_1) \overline{\Pi}_\beta(\bar{z}_2) \times \frac{1}{Z_0} \int d^2\mathbf{N}\, N^\alpha \bar{N}^\beta e^{-\mathbf{N}^\dagger \eta \mathbf{N}} \\ &= \Pi_\alpha(z_1) \overline{\Pi}_\beta(\bar{z}_2) (\eta^{-1})^{\alpha\bar{\beta}} \\ &= e^{\kappa K(z_1, \bar{z}_2)}, \end{aligned}$$

where $\kappa = -1$ and $K(z_1, \bar{z}_2)$ is precisely the Kähler potential on the moduli space, reinterpreted as a function of $z_1$ and $\bar{z}_2$. Here in the second line we performed the integral over $W$ and $\overline{W}$

[10] To be explicit, the measure $d^2\mathbf{N}$ is a shorthand for

$$d^2\mathbf{N} = \Pi_{\alpha=1}^b dN^\alpha d\bar{N}^\alpha. \tag{2.46}$$

using the $\delta$-function. The remaining integral in the third line is standard and yields a factor of $Z_0(\eta^{-1})^{\alpha\beta}$ giving the above result. We keep $\kappa$ general for now since it will aid us in computations by grouping terms in powers of $\kappa$. At the end we will set $\kappa = -1$. Crucially, we see that $G(z_1, \bar{z}_2)$ is completely determined by the Kähler potential $K$. Furthermore, by Wicks theorem we know that most expectation values can be reduced to an expression involving only $G(z_1, \bar{z}_2)$, which is now completely determined by the geometry of $\mathcal{M}_{\text{cs}}$. As a result, we expect the final answer to be elegantly expressed in terms of geometrical quantities.

As a first step towards computing $d\mu[z]$, let us first consider the simpler quantity $\langle \det D^2W \rangle$, but we let $W$ depend on $z_1$ and $\overline{W}$ on $\bar{z}_2$. Later we will consider the limit $z_1 = z_2$. We start by using a trick familiar from field theory calculations. Let $\{\theta^i, \bar{\theta}^{\bar{\imath}}\}$ and $\{\psi^i, \bar{\psi}^{\bar{\imath}}\}$, be two sets of Grassmann variables, collectively denoted by $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$. Then we can express the determinant of the matrix $D^2W$ as follows:

$$\det D^2W(z_1, \bar{z}_2) = \int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \, \exp\left[-\boldsymbol{\theta}^T D^2W(z_1, \bar{z}_2)\boldsymbol{\psi}\right] \tag{2.50}$$

or

$$\det D^2W(z_1, \bar{z}_2) = \int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \, \exp\left[-\theta^i\psi^j D_i D_j W(z_1) - \bar{\theta}^{\bar{\imath}}\psi^j \bar{D}_{\bar{\imath}} D_j W(z_1) + \text{c.c.}\right], \tag{2.51}$$

where c.c. denotes the complex conjugate of the terms before it, which depends only on $\bar{z}_2$. We can now evaluate the expectation value in the Gaussian ensemble as follows:

$$\langle \det D^2W \rangle = \frac{1}{Z_0} \int d^2\boldsymbol{N} d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \, \exp \underbrace{\left[-\boldsymbol{N}^\dagger \eta \boldsymbol{N} - \theta^i\psi^j \boldsymbol{N}^T \cdot D_i D_j \boldsymbol{\Pi}(z_1) - \bar{\theta}^{\bar{\imath}}\psi^j \boldsymbol{N}^T \cdot \bar{D}_{\bar{\imath}} D_j \boldsymbol{\Pi}(z_1) + \text{c.c.}\right]}_{(*)}, \tag{2.52}$$

where we have already performed the integral over $W$ using the delta-function. The next step is to complete the square in the exponential factor, which is achieved by the following expression

$$\begin{aligned}(*) = -&\left(\boldsymbol{N} + \theta^i\bar{\psi}^{\bar{\jmath}}\eta^{-1} D_i \bar{D}_{\bar{\jmath}} \bar{\boldsymbol{\Pi}}(\bar{z}_2) + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\eta^{-1} \bar{D}_{\bar{\imath}} \bar{D}_{\bar{\jmath}} \bar{\boldsymbol{\Pi}}(\bar{z}_2)\right)^T \eta \\ &\Big(\bar{\boldsymbol{N}} + \bar{\theta}^{\bar{k}}\psi^l \eta^{-1} \bar{D}_{\bar{k}} D_l \boldsymbol{\Pi}(z_1) + \theta^k\psi^l\eta^{-1} D_k D_l \boldsymbol{\Pi}(z_1)\Big) + (**)\end{aligned}$$

where $(**)$ denotes the four quadratic terms in $\boldsymbol{\Pi}$. An example of such a term is

$$\theta^i\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l (D_i \bar{D}_{\bar{\jmath}} \boldsymbol{\Pi}^\dagger(\bar{z}_2))\eta^{-1}(\bar{D}_{\bar{k}} D_d \boldsymbol{\Pi}(z_1)) = \theta^i\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l D_{1i}\bar{D}_{\bar{1}\bar{\jmath}}\bar{D}_{\bar{2}\bar{k}} D_{2l} G(z_1, \bar{z}_2), \tag{2.53}$$

where the equality follows directly from the definition of $G(z_1, \bar{z}_2)$. One easily sees that all terms in $(**)$ are expressed in terms of covariant derivatives of $G(z_1, \bar{z}_2)$, hence we introduce the following notation

$$\tilde{F}_{AB\cdots|MN\cdots}(z_1, \bar{z}_2) := e^{-\kappa K(z_1, \bar{z}_2)} D_{1A} D_{1B} \cdots D_{2M} D_{2N} \cdots G(z_1, \bar{z}_2), \tag{2.54}$$

where the indices can be both holomorphic and anti-holomorphic. Then the four quadratic terms are given by

$$(**) = e^{\kappa K(z_1, \bar{z}_2)} \left[\theta^i\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l \tilde{F}_{i\bar{\jmath}|\bar{k}l} + \theta^i\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l \tilde{F}_{i\bar{\jmath}|kl} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l \tilde{F}_{\bar{\imath}\bar{\jmath}|\bar{k}l} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l \tilde{F}_{\bar{\imath}\bar{\jmath}|kl}\right] \tag{2.55}$$

Having completed the square, we shift the $\mathbf{N}$ integrand in (2.52) and perform the (now trivial) integration over $\mathbf{N}$, which yields a factor $Z_0$. We are then left with an integration of the terms in (**) over the Grassmann variables as

$$\langle \det D^2W \rangle = \int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \, \exp\left[e^{\kappa K(z_1, \bar{z}_2)} \left(\theta^i\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l \tilde{F}_{i\bar{\jmath}|\bar{k}l} + \theta^i\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l \tilde{F}_{i\bar{\jmath}|kl} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l \tilde{F}_{\bar{\imath}\bar{\jmath}|\bar{k}l} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l \tilde{F}_{\bar{\imath}\bar{\jmath}|kl}\right)\right]. \tag{2.56}$$

*Imposing $D_iW = 0$ and onto the Real $d\mu$*

We now turn to the real supersymmetric index, which imposes the additional constraint $D_iW = 0$. As a result, instead of using the two-point function $G(z_1, \bar{z}_2)$, we should instead consider the constrained two-point function defined by

$$G_{z_0}(z_1, \bar{z}_2) := \langle W(z_1)\overline{W}(\bar{z}_2)\rangle_{D_iW(z_0)=\bar{D}\overline{W}(\bar{z}_0)=0}. \tag{2.57}$$

Let us compute this quantity. First, we note that we can express the constraint $D_iW(z) = 0$ in terms of a $\delta$-function as follows:

$$\begin{aligned} G_{z_0}(z_1, \bar{z}_2) &= \frac{1}{Z}\int d\mu[W]\ \delta^n(D_iW(z_0))\delta^n(\bar{D}\overline{W}(\bar{z}_0))W(z_1)\overline{W}(\bar{z}_2) \\ &= \frac{1}{Z}\int d^2\mathbf{N}\ \delta(W - \mathbf{N}^T\cdot\mathbf{\Pi})e^{-\mathbf{N}^T\eta\bar{\mathbf{N}}}\delta^n(D_iW(z_0))\delta^n(\bar{D}\overline{W}(\bar{z}_0))W(z_1)\overline{W}(\bar{z}_2), \end{aligned}$$

where instead of $Z_0$ we must now normalize by the constrained partition function given by

$$Z = \int d\mu[W]\delta^n(D_iW)\delta^n(\bar{D}\overline{W}) \tag{2.58}$$

Next, we use a trick. Recall the following integral expression for the $\delta$-function:

$$\delta(x) \sim \int d\lambda\ e^{ix\lambda}. \tag{2.59}$$

In our setting, we introduce pairs of Lagrange multipliers $\{\lambda^i, \bar{\lambda}^{\bar{\imath}}\}$, collectively denoted by $\lambda$ and write [11]

$$\delta^n(D_iW(z_0))\delta^n(\bar{D}\overline{W}(\bar{z}_0)) \sim \int d^2\lambda\ e^{i\lambda^iD_iW(z_0)+i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{\jmath}}\overline{W}(\bar{z}_0)}. \tag{2.60}$$

Performing the integration over the remaining $\delta$-function sets $W = \mathbf{N}^T\cdot\mathbf{\Pi}$, hence we obtain

$$G_z(z_1, \bar{z}_2) = \frac{1}{Z_0}\int d^2\mathbf{N}\ d^2\lambda\ \exp\left[-\mathbf{N}^T\eta\bar{\mathbf{N}} + i\lambda^i\mathbf{N}^T\cdot D_i\mathbf{\Pi}(z_0) + i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}^T(\bar{z}_0)\cdot\bar{N}\right]\mathbf{N}^T\cdot\mathbf{\Pi}(z_1)\bar{\mathbf{N}}^T\cdot\overline{\mathbf{\Pi}}(\bar{z}_2). \tag{2.61}$$

As before, we simplify the term in brackets by completing the square:

$$\begin{aligned} -\mathbf{N}^T\eta\bar{\mathbf{N}} + i\lambda^i\mathbf{N}^T\cdot D_i\mathbf{\Pi}(z_0) + i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}^T(\bar{z}_0)\cdot\bar{\mathbf{N}} = &-\left(\mathbf{N} - i\bar{\lambda}^{\bar{\jmath}}\eta^{-1}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}(\bar{z}_0)\right)^T\eta\left(\bar{\mathbf{N}} - i\lambda^i\eta^{-1}D_i\mathbf{\Pi}(z_0)\right) \\ &- \lambda^i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}^T(\bar{z}_0)\eta^{-1}D_i\mathbf{\Pi}(z_0). \end{aligned}$$

Note that this last term can be expressed in terms of the original two-point function:

$$\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}^T(\bar{z}_0)\eta^{-1}D_i\mathbf{\Pi}(z_0) = \bar{D}_{\bar{\jmath}}D_iG(z_0, \bar{z}_0) \tag{2.62}$$

We then shift the integrand

$$\mathbf{N} \mapsto \mathbf{N} + i\bar{\lambda}^{\bar{\jmath}}\eta^{-1}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}(\bar{z}_0). \tag{2.63}$$

In particular, we have

$$\mathbf{N}^T\cdot\mathbf{\Pi}(z_1) \mapsto \mathbf{N}^T\cdot\mathbf{\Pi}(z_1) + i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathbf{\Pi}}(\bar{z}_0)^T\eta^{-1}\cdot\mathbf{\Pi}(z_1) = \mathbf{N}^T\cdot\mathbf{\Pi}(z_1) + i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{\bar{0}\bar{\jmath}}G(z_1, \bar{z}_0), \tag{2.64}$$

[11] Since the overall normalization of the integral expression of the $\delta$-functions will drop out, we will ignore it here for notational convenience.

Inserting the above results into our expression for $G_{z_0}(z_1, \bar{z}_2)$, we obtain

$$G_{z_0}(z_1,\bar{z}_2) = \frac{1}{Z}\int d^2\mathbf{N}\, d^2\lambda \exp\left[-\mathbf{N}^T\eta\bar{\mathbf{N}} - \lambda^i\bar{\lambda}^{\bar{\jmath}} D_i\bar{D}_{\bar{\jmath}}G(z_0,z_0)\right] \\ \times \left(\mathbf{N}^T\cdot\mathbf{\Pi}(z_1) + i\bar{\lambda}^{\bar{\jmath}}\bar{D}_{0\bar{\jmath}}G(z_1,\bar{z}_0)\right)\times\left(\bar{\mathbf{N}}^T\cdot\bar{\mathbf{\Pi}}(z_2) + i\lambda^i D_{0i}G(z_0,\bar{z}_2)\right)$$

Noting that only terms even in $\mathbf{N}$ contribute and using the results above, this simplifies to:

$$G_{z_0}(z_1,\bar{z}_2) = G(z_1,\bar{z}_2) - \bar{D}_{0\bar{\jmath}}G(z_1,\bar{z}_0)D_{0i}G(z,\bar{z}_2)\frac{1}{Z}\int d^2\lambda\, \lambda^i\bar{\lambda}^{\bar{\jmath}}e^{-\lambda^i\bar{\lambda}^{\bar{\jmath}}D_i\bar{D}_{\bar{\jmath}}G(z_0,z_0)}\int d^2\mathbf{N}\, e^{-\mathbf{N}^\dagger\eta\mathbf{N}}. \tag{2.65}$$

Again, the integral over $d^2\lambda$ is standard and yields a factor $(D_i\bar{D}_{\bar{\jmath}}G(z,\bar{z}))^{-1}$ and $Z$. The final result for the constrained two-point function is

$$G_{z_0}(z_1,\bar{z}_2) = G(z_1,\bar{z}_2) - \bar{D}_{0\bar{\jmath}}G(z_1,\bar{z}_0)(D_i\bar{D}_{\bar{\jmath}}G(z_0,\bar{z}_0))^{-1}D_{0i}G(z_0,\bar{z}_2) \tag{2.66}$$

Finally then, the supersymmetric index can be written as follows:

$$d\mu[z_0] = \frac{Z_0}{Z}\int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \exp\left(e^{\kappa K(z_0,\bar{z}_0)}\left[\theta^i\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l F_{i\bar{\jmath}|\bar{k}l} + \theta^i\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l F_{i\bar{\jmath}|kl} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\bar{\theta}^{\bar{k}}\psi^l F_{\bar{\imath}\bar{\jmath}|\bar{k}l} + \bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l F_{\bar{\imath}\bar{\jmath}|kl}\right]\right), \tag{2.67}$$

where now

$$F_{AB\cdots|MN\cdots} := e^{-\kappa K(z_0,\bar{z}_0)}D_{1A}D_{1B}\cdots D_{2M}D_{2N}\cdots G_{z_0}(z_1,\bar{z}_2)\Big|_{z_0=z_1=z_2} \tag{2.68}$$

is expressed in terms of the constrained two-point function. The computation of these quantities is performed in the appendix D, here we state the results:

$$F_{i\bar{\jmath}|\bar{k}l} = \kappa^2 g_{i\bar{\jmath}}g_{\bar{k}l}, \quad F_{\bar{\imath}\bar{\jmath}|kl} = -\kappa R_{\bar{\imath}k\bar{\jmath}l} + \kappa^2\left(g_{\bar{\imath}k}g_{\bar{\jmath}l} + g_{\bar{\imath}l}g_{\bar{\jmath}k}\right), \quad F_{i\bar{\jmath}|kl} = F_{\bar{\imath}\bar{\jmath}|\bar{k}l} = 0. \tag{2.69}$$

Additionally, using the exact same procedure as done above for computing $G_{z_0}(z_1,\bar{z}_2)$, one can find the following expression for $Z$

$$Z = \pi^n\kappa^n e^{n\kappa K}\det g, \tag{2.70}$$

which is again elaborated upon in the appendix D. Inserting these results into $d\mu[z_0]$ and using the anti-commutativity of the Grassmann variables, we obtain

$$d\mu[z_0] = \frac{Z_0}{Z}\int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \exp\left(e^{\kappa K(z_0,\bar{z}_0)}\left[\bar{\theta}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\theta^k\psi^l(-\kappa R_{\bar{\imath}k\bar{\jmath}l} + \kappa^2 g_{\bar{\imath}k}g_{\bar{\jmath}l})\right]\right), \tag{2.71}$$

Let $\{e^i{}_\alpha, \bar{e}^{\bar{\jmath}}{}_{\bar{\beta}}\}$ be an orthonormal frame w.r.t. the metric $g$. Then we perform a change of variables

$$\theta^i \mapsto e^i{}_\alpha\theta^\alpha, \quad d^2\boldsymbol{\theta} \mapsto \det g\; d^2\boldsymbol{\theta}, \tag{2.72}$$

such that

$$\begin{aligned} d\mu[z_0] &= Z_0\pi^{-n}\kappa^{-n}e^{-nK(z_0,\bar{z}_0)}\int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \exp\left(e^{\kappa K(z_0,\bar{z}_0)}\left[\bar{\theta}^{\bar{\alpha}}\bar{\psi}^{\bar{\jmath}}\theta^\gamma\psi^l(-\kappa R_{\bar{\imath}k\bar{\jmath}l}\bar{e}^{\bar{\imath}}{}_{\bar{\alpha}}e^k{}_\gamma + \kappa^2\delta_{\bar{\alpha}\gamma}g_{\bar{\jmath}l})\right]\right) \\ &= Z_0\pi^{-n}\int d^2\boldsymbol{\theta} d^2\boldsymbol{\psi} \exp\left(\bar{\theta}^{\bar{\alpha}}\bar{\psi}^{\bar{\jmath}}\theta^\gamma\psi^l(-R_{\bar{\imath}k\bar{\jmath}l}\bar{e}^{\bar{\imath}}{}_{\bar{\alpha}}e^k{}_\gamma + \kappa\delta_{\bar{\alpha}\gamma}g_{\bar{\jmath}l})\right) \\ &= \frac{\pi^{b-n}}{\det\eta}\det(-R + \kappa\omega\cdot\mathbf{1}), \end{aligned}$$

where $\mathbf{1}$ denotes the unit matrix in $\text{End}(T\mathcal{M}_{\text{cs}})$ and we recall that $R$ and $\omega$ denote the curvature and Kähler two-form on $\mathcal{M}_{\text{cs}}$, respectively. Here a comment on the interpretation of the above formula is in order. Recall that the curvature two-form $R$ is a map

$$R : T\mathcal{M}_{\text{cs}} \times T\mathcal{M}_{\text{cs}} \to \text{End}(T\mathcal{M}_{\text{cs}}), \tag{2.73}$$

where $\text{End}(T\mathcal{M}_{\text{cs}})$ is the space of endomorphisms of the tangent bundle of $\mathcal{M}_{\text{cs}}$. For Kähler manifolds, it is given in terms of the metric $g$ by

$$R^{l}{}_{i\bar{j}k} = -\bar{\partial}_{\bar{j}} \left( g^{l\bar{m}} \partial_i g_{k\bar{m}} \right), \tag{2.74}$$

where $i, \bar{j}$ denote the two-form indices, and $l, k$ the matrix indices of $R$. On the other hand, $\omega$ is simply a two-form on $\mathcal{M}_{\text{cs}}$, which explains why it should be multiplied with a unit matrix. Hence the determinant is understood to act on the matrix indices of $R + \omega \cdot \mathbf{1}$. Moreover, we recall that $\omega$ is precisely the curvature of the line bundle $L$ of which $W$ is a section. Therefore $R + \omega$ is the curvature of the vector bundle $T\mathcal{M}_{\text{cs}} \otimes L$ of which $D_i W$ is a section. In fact, using the complex structure of $\mathcal{M}_{\text{cs}}$ one can show that [13, p. 104]

$$\pi^{-n} \det \left( R + \omega \cdot \mathbf{1} \right) = e(\nabla), \tag{2.75}$$

where $e(\nabla)$ is the Euler class of $T\mathcal{M}_{\text{cs}} \otimes L$. Though we will not use this specific identity, it highlights the fact that the distribution is determined by general geometrical properties of $\mathcal{M}_{\text{cs}}$.

*The Final Result*

Finally, we return to the the supersymmetric index. Inserting the above result for the index density, setting $\kappa = -1$ and returning to real variables yields:

$$\boxed{I_{\text{vac}}(L \le L_*) = \frac{1}{\sqrt{\det \eta}} \frac{(\pi L_*)^{b/2}}{(b/2)!} \pi^{-n} \int_{\mathcal{M}_{\text{cs}}} \det(-R - \omega \cdot \mathbf{1})} \tag{2.76}$$

Having obtained the final result, let us make some remarks regarding its interpretation

1. The factor $(\pi L_*)^{b/2}/(b/2)!$ can be interpreted as the volume of a $b/2$ dimensional sphere of radius $\sqrt{L_*}$ in flux space.

2. In regions where the curvature is negligible, one sees that $\det(R + \omega \cdot \mathbf{1})$ is simply the volume form on $\mathcal{M}_{\text{cs}}$. In this case, we have

$$I_{\text{susy vac}}(L \le L_*) \sim \text{Vol} \left( B^{b/2}(\sqrt{L_*}) \right) \times \text{Vol}(\mathcal{M}_{\text{cs}}) \tag{2.77}$$

   Very roughly, we can interpret this by saying that the supersymmetric index is given by multiplying the number of fluxes within the sphere $B^{b/2}(\sqrt{L_*})$, set by the tadpole constraint and the number of cycles on $Y_4$, with the size of the moduli space (this latter part is sensible because the flux vacua are generally believed to lie at isolated points, see [18, p. 28].

3. Finally, the factor $\det(R + \omega \cdot \mathbf{1})$ is not very surprising, on mathematical grounds. Indeed, we have already mentioned that $R + \omega \cdot \mathbf{1}$ is precisely the curvature of $T\mathcal{M}_{\text{cs}} \times L$, of which $D_i W$ is a section. Since we are imposing the constraint $D_i W = 0$, a natural top form to integrate over $\mathcal{M}_{\text{cs}}$ is the highest Chern class of $T\mathcal{M}_{\text{cs}} \times L$, which is precisely $\det(R + \omega \cdot \mathbf{1})$. In fact, if $\mathcal{M}_{\text{cs}}$ were smooth this would be precisely the statement of the Chern-Gauss-Bonnet theorem.

However, since $\mathcal{M}_{\text{cs}}$ is not smooth, it is interesting that a similar result is still obtained. This is likely due to the fact that we consider an ensemble of superpotentials, i.e. by averaging over all superpotentials the possible singular behaviour is smeared out and we obtain a result similar to the Chern-Gauss-Bonnet theorem.

Returning to our original question posed in section 2.2, we have obtained the following answer

A2: The distribution of supersymmetric vacua (counted with signs) over the complex structure moduli space is given by the Ashok-Douglas density

$$d\mu = \det(R + \omega \cdot \mathbf{1}). \tag{2.78}$$

Note that we slightly abuse the notation here by not including the constant pre-factor $\pi^{b-n}/\det\eta$. Investigating the above expression for the AD-density will be the main focus of the rest of the work, as we will also discuss in section 2.4.3. First, let us turn to a concrete example/application of the above results.

### 2.3.3 Example: The Torus

In section 1.3.2 we discussed the complex structure moduli space of the torus in detail. Morally, we can think of this example as appearing in a rigid compactification of the form $Y_4 = X_3 \times \mathbb{T}$, where all the moduli of $X_3$ are fixed and we only need to consider the torus. Within this setting we will contrast the abstract derivation of the Ashok-Douglas density above with a very concrete description of possible flux vacua on a torus. We will see that the result agree.

*The Exact Result*

For this derivation we closely follow the discussion in [1] First, we note that any point $z \in \mathbb{T}$ can be written as

$$z = \xi_1 + \tau\xi_2, \quad \xi_1, \xi_2 \in [0,1]. \tag{2.79}$$

In particular, the holomorpic (1,0) form is given by

$$\Omega = dz = d\xi_1 + \tau d\xi_2, \tag{2.80}$$

and the Kähler potential is

$$K(\tau,\bar\tau) = -\log\left[i\int_{\mathbb{T}} \Omega\wedge\bar\Omega\right] = -\log\left[i(\bar\tau - \tau)\right], \tag{2.81}$$

which satisfies

$$\partial_\tau K = \frac{1}{\bar\tau - \tau}, \quad \bar\partial_{\bar\tau} K = -\frac{1}{\bar\tau - \tau} \tag{2.82}$$

Next, we expand $\Omega$ in terms of the canonical cocycles $d\xi_i$. In terms of this basis, the period vector $\mathbf{\Pi}$ is simply given by

$$\mathbf{\Pi} = \begin{pmatrix} 1 \\ \tau \end{pmatrix}. \tag{2.83}$$

As a result, the superpotential becomes

$$W(\tau) = A + B\tau, \quad \mathbf{N} = \begin{pmatrix} A \\ B \end{pmatrix}, \tag{2.84}$$

for two integer complex numbers $A = a_1 + ia_2, B = b_1 + ib_2$. The intersection form $\eta$ in the chosen basis is given by

$$\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.85}$$

Although the Euler character of the torus vanishes, we can still implement the tadpole condition by hand by setting [12]

$$\frac{1}{2}\mathbf{N}^\dagger \eta \mathbf{N} = \mathrm{Im}(A\bar{B}) = L. \tag{2.86}$$

The condition for a supersymmetric vacuum $D_\tau W = 0$ then becomes

$$D_\tau W = 0 \iff \bar{\tau} = -\frac{A}{B}. \tag{2.87}$$

At this point we should recall the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the superpotential. Indeed, we can make two transformations

$$\tau \mapsto \tau + C, \quad \tau \mapsto D\tau, \tag{2.88}$$

for suitable choices of $C, D$ such that $a_2 = 0$ and $0 \le b_1 < a_1$. In particular, the tadpole constraint becomes

$$a_1 b_2 = L. \tag{2.89}$$

As a result, a choice of flux is specified by

1. An integer $a_1$ dividing $L$, which then determines $b_2$ via the tadpole condition.
2. An integer $b_1$ such that $0 \le b_1 < a_1$, hence taking $|a_1|$ possible values.

Moreover, the cases $a_1 > 0$ and $a_1 < 0$ are identical, hence the total number of flux vacua for a given $L$ is

$$N_{\text{vac}}(L) = 2\sum_{k|L} k =: 2\sigma(L), \tag{2.90}$$

where $\sigma(L)$ is the divisor function satisfying the asymptotic behaviour [34, p. 266]

$$\sum_{L \le L_*} \sigma(L) = \frac{\pi^2}{12} L_*^2 + \mathcal{O}(L_* \log L_*). \tag{2.91}$$

In particular, the total number of flux vacua with $L \le L_*$ is given by

$$N_{\text{vac}}(L \le L_*) = \frac{\pi^2}{6} L_*^2 + \mathcal{O}(L_* \log L_*). \tag{2.92}$$

*Comparing with the Ashok-Douglas Index*

We now compare this with the result from section 2.3.2. Using the expression for the Kähler potential, we have

$$g_{\tau\bar{\tau}} = \partial_\tau \bar{\partial}_{\bar{\tau}} K = -\frac{1}{(\tau - \bar{\tau})^2}, \tag{2.93}$$

which implies that

$$\omega = -\frac{i}{2}\frac{d\tau \wedge d\bar{\tau}}{(\tau - \bar{\tau})^2}, \quad R = -2\omega \tag{2.94}$$

[12] For convenience we have redefined $L \mapsto 2L$ in the tadpole constraint, note that this will also affect the formula for the index density 2.76.

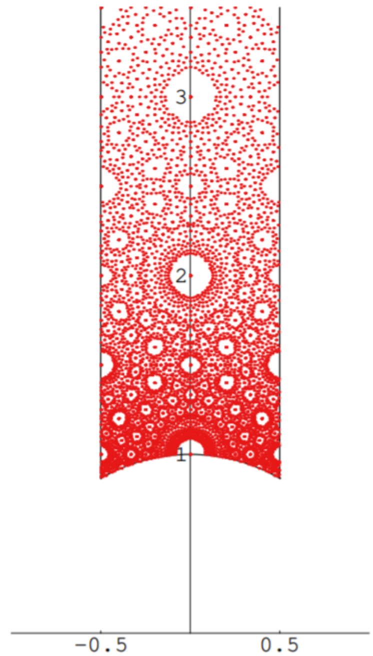



**Figure 2.4:** Solutions to $D_\tau W = 0$ for the torus, with $L_* = 150$. Figure taken from [14, p. 38].

Inserting these results into (2.76), we obtain

$$\begin{aligned} I_{\text{vac}}(L \leq L_*) &= \frac{(2\pi L_*)^2}{2!}\pi^{-1}\int_{\mathscr{F}} \frac{i}{2}\frac{d\tau \wedge d\bar{\tau}}{(\tau - \bar{\tau})^2} \\ &= 2\pi L_*^2 \int_{\mathscr{F}} \frac{d^2\tau}{(\tau - \bar{\tau})^2} \\ &= \frac{\pi^2}{6} L_*^2, \end{aligned}$$

where we have used the fact that

$$\int_{\mathscr{F}} \frac{d^2\tau}{(\tau - \bar{\tau})^2} = \frac{\pi}{12} \tag{2.95}$$

for the fundamental domain of the torus [1, p. 15]. We see that the index predicted by the Ashok-Douglas density coincides with (2.92).

To close this section, we point the reader to figure 2.4, where the exact solutions to $D_\tau W = 0$, i.e. the possible flux vacua, are plotted. The actual distribution shows a much richer behaviour than predicted by the Ashok-Douglas density. Nevertheless, the two agree on the total number of flux vacua when $L_*$ is large. Especially interesting is the structure of the distribution around integer points $\tau = ni$. At the centre of these points there is a large degeneracy, e.g. at $\tau = 2i$ there are 240 flux vacua, whereas in a small neighbourhood around them no vacua are present. This behaviour comes from the underlying discreteness of the fluxes, which we have so far neglected. In the next section we will discuss this in detail.

## 2.4 Discussion and Interpretation

In this section we discuss some of the assumptions we have made in the derivation of the index density, most notably ignoring the quantization of fluxes and the Kähler moduli, as well the interpretation one should give to index density and the predictions for numbers of vacua it provides.

### 2.4.1 Quantization of Fluxes

In the derivation of 2.38 we have ignored the fact that fluxes are quantized, by approximating the sum over fluxes by an integral. This can only be valid in certain regimes, which we discuss now, following [1, 14].

First, the sum in $\mathcal{N}(\alpha)$ runs over all supersymmetric flux vacua, unconstrained by the tadpole condition, i.e. satisfying $D_i W = 0$. This condition is scale-invariant, i.e. invariant under $\sqrt{L_*}\mathbf{N} \mapsto \mathbf{N}$. In particular, for a function $f$ depending on $\mathbf{N}$, we expect that

$$\sum_{\mathbf{N}\in\mathbb{Z}^b} f(\mathbf{N}) = \sum_{\mathbf{N}\in(\mathbb{Z}/\sqrt{L_*})^b} f\left(\frac{\mathbf{N}}{\sqrt{L_*}}\right), \tag{2.96}$$

see also [1, p. 24]. Therefore, for large $\sqrt{L_*}$ we expect that the integral provides the leading behaviour to the discrete sum, since the spacing between consecutive values of $\mathbf{N}$ becomes small. By large we mean large compared to the number of fluxes $b$ Indeed, suppose $L_* \sim b$, then many cycles will have either zero or one flux, hence the discreteness cannot be ignored. [13] In the more extreme case of $L_* \ll b$, we can readily see that the approximation breaks down, since then the expression in 2.76 essentially predicts no vacua at all, due to the $b!$ factor (in regions where the curvature is negligible).

*A Geometrical Picture*

We can make the above statements a bit more precise by considering the geometry of the flux space as a subset of $\mathbb{R}^b$. Denote by $S$ the subspace of $\mathbb{R}^b$ containing the vacua satisfying $\mathbf{N}^T \cdot D_i\mathbf{\Pi} = D_i W = 0$. Then by scale invariance $\lambda S = S$, so $S$ can be viewed as a cone. Denote by $S_* \subseteq S$ the subspace containing vacua which additionally satisfy the tadpole constraint. This acts as a positive definite[14] constraint on $S$, so $S_*$ is roughly the part of $S$ having radius $\sqrt{L_*}$. Having described $S_*$, the question now is whether its volume $V$ is a good approximation to the number of lattice points $N_*$ it contains. The leading correction is given by the surface area $A$ of $S_*$, i.e.

$$N_* = L_*^{b/2}\left(V + L_*^{-1/2}A + \mathcal{O}(L^{-\epsilon})\right), \quad \epsilon > 0, \tag{2.97}$$

see also [14, p. 37]. In particular, for $V$ to be a good approximation, we need two conditions:

$$L_*^{b/2}V \gg 1, \quad V \gg L_*^{-1/2}A. \tag{2.98}$$

In geometrical terms, these conditions amount to saying that the cone should be well-aligned with the lattice, and that the width of the cone should be larger than the spacing between the lattice points, which scales inversely with $\sqrt{L_*}$ as discussed above. This is also exemplified in figure 2.5

[13] Here we call upon the intuition that the tadpole constraint (hence $L_*$) gives an upper bound on the number of flux lines that can run through the compact space $Y_4$, which is determined by its topology through the Euler character $\chi$.

[14] This is a little tricky, since $\eta$ is an indefinite form. However, in [1], page 21, it is claimed that one can make a subtle argument that $\mathbf{N}^T\eta\mathbf{N} > 0$, so this statement makes sense.

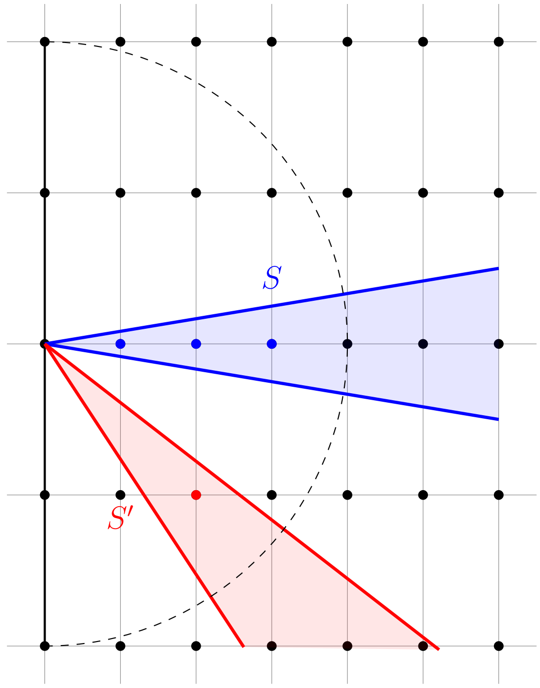



**Figure 2.5:** Two examples of cones $S$ and $S'$ in a $b = 2$ dimensional flux space. Even though the opening angles of the cones are identical, the amount of lattice points contained in $S$ and $S'$ within a circle of radius $L_* = 4$ differs (three versus one, respectively) due to the different alignments with the lattice.

for a two-dimensional flux space. We see that indeed for small $L_*$ the cone $S'$ contains significantly fewer lattice points that $S$ due to its misalignment with the lattice. Moreover, within the circle of radius $\sqrt{L_*} = 4$ the width of $S'$ is less than the spacing between the lattice points. Hence in this example neither of the conditions in (2.98) are satisfied.

It is now argued in [14] that by taking a sphere of radius $r$ in moduli space, one can show that

$$\frac{A}{V} \sim \frac{\sqrt{b}}{r}. \tag{2.99}$$

In other words, the second condition can be phrased in terms of the number of fluxes $b$ as

$$\boxed{L_* \gg \frac{b}{r^2}} \tag{2.100}$$

This is in agreement with our earlier discussion and additionally makes the region in moduli space under consideration explicit. We end this subsection by referring the reader to [21, Thm. 1.8], where an estimate for the error produced by passing from a sum to an integral in 2.38 is shown to be $\mathcal{O}(L_*^{-1/2})$, in accordance with 2.97.

### 2.4.2 The Kähler Moduli

In our derivation of the index density we have essentially ignored the presence of the Kähler moduli, in particular the fact that their presence changes the Kähler potential and therefore the expression for the scalar potential and its vacuum structure. The goal of this subsection is to argue that the inclusion of the Kähler moduli does not affect our analysis, provided that one introduces non-perturbative corrections to the superpotential which depend on the Kähler moduli. Moreover, as discussed in [1, 14, 18], these non-perturbative effects can be used to stabilize the Kähler moduli.

*Tree-level Kähler Moduli and No-Scale Structure*

For simplicity we consider a single Kähler modulus $\rho$, which can be interpreted as an overall (complexified) volume factor. Dimensional reduction to four dimensions produces a Kähler potential for $\rho$ given by [24]

$$K(\rho,\bar{\rho}) = -3\log\left[-i(\rho-\bar{\rho})\right]. \tag{2.101}$$

Let us first discuss the effect of this additional Kähler potential in the absence of non-perturbative effects. Since (for now) $W$ is independent of $\rho$, we see that

$$D_\rho W = (\partial_\rho K)W. \tag{2.102}$$

In particular, we have

$$g^{\rho\bar{\rho}}|D_\rho W|^2 = -\frac{(\rho-\bar{\rho})^2}{3}\left|\frac{3W}{(\rho-\bar{\rho})}\right|^2 = 3|W|^2. \tag{2.103}$$

As a result, the scalar potential simplifies to

$$V = e^K\left(g^{a\bar{b}}D_aW\overline{D_bW} - 3|W|^2\right) = e^K g^{i\bar{\jmath}}D_iW\overline{D_jW}, \tag{2.104}$$

where the indices $a, \bar{b}$ run over the complex structure moduli plus $\rho$, whereas the indices $i, \bar{\jmath}$ only run over the complex structure moduli. Crucially, the final expression is manifestly positive semi-definite and is of no-scale structure [11]. This means that the vacuum structure is independent of the volume modulus $\rho$. Moreover, the condition of a supersymmetric vacuum $D_iW = 0$ implies $V = 0$, hence the only possible vacua are Minkowski. This poses two issues: First, there seems to be no mechanism to stabilize the volume modulus, since it does not appear in the potential. Second, for reasons outlined in [18, p. 13], Minkowski vacua are much harder to study, since the condition $D_\rho W = 0$ additionally imposes $W = 0$, and it is in fact advantageous to include AdS vacua as well. Both these problems can be addressed by considering non-perturbative corrections to the superpotential $W$.

*Non-Perturbative Corrections to W*

Recall that in section 2.1 we stated that the superpotential $W_0$ is given by [15]

$$W_0 = \int_{Y_4} G_4 \wedge \Omega, \tag{2.105}$$

which manifestly only depends on the complex structure moduli, i.e. it does not depend on the Kähler moduli. Due to the nonrenormalization theorem for superpotentials, the expression for $W$ is valid to all orders in perturbation theory [2, p. 474]. However, in [36] it is argued that for the

[15] We now change notation from $W$ to $W_0$ to emphasize that the complete superpotential $W$ is given by $W_0$ plus some correction $\delta W$, as we introduce shortly.

volume modulus $\rho$ there do exist non-perturbative corrections to the superpotential which are of the form

$$\delta W = Ae^{ia\rho}, \tag{2.106}$$

and can be induced due to e.g. D3-brane instantons and gluino condensation. Here $A$ and $a$ are determined by the precise details of these constructions. For simplicity we take $a, A$ to be real. The complete superpotential is then given by

$$W = W_0 + \delta W, \tag{2.107}$$

where $W_0$ is the complex structure moduli dependent superpotential. Crucially, $\partial_\rho W \neq 0$ hence the scalar potential no longer has a no-scale structure. In particular, at a supersymmetric vacuum $D_a W = 0$ one finds

$$V = -3e^K |W|^2, \tag{2.108}$$

which for $W \neq 0$ results in a negative cosmological constant $\Lambda = -3e^K|W|^2$, hence AdS vacua are possible. In short, we see that the inclusion of the Kähler moduli together with non-perturbative corrections to the superpotential results in a very similar situation as simply ignoring the Kähler moduli altogether. This justifies the fact that we only impose $D_i W = 0$ in our derivation of the index density.

To see how the volume modulus can be stabilized, one computes

$$D_\rho W = 0 \iff W = -\frac{2}{3} a A e^{ia\rho} \mathrm{Im}\, \rho, \tag{2.109}$$

where $W$ is evaluated at the minimum. For small negative $W_0$ this has a self-consistent solution $\mathrm{Im}\ \rho \to \infty$, i.e. the large volume limit. It is self-consistent in the sense that in this limit the correction $\delta W$ is exponentially suppressed. Moreover, the scalar potential is given by

$$V = -3e^K |W|^2 = -\frac{a^2 A^2}{6} \frac{e^{-2a \mathrm{Im}\, \rho}}{\mathrm{Im}\, \rho} \tag{2.110}$$

The construction described above was first realized in the KKLT scenario [36], where additionally a number of anti-D3 branes was added to the setup so that the resulting vacuum energy is in fact small and positive. This was one of the first string theoretical constructions which yielded a dS vacuum with a phenomenologically acceptable cosmological constant.

### 2.4.3 Interpretation of the Index Density

*Estimating the Number of Flux Vacua*

As mentioned before, we have chosen to compute the index of supersymmetric vacua as opposed to the actual number, because of computational complexity. Of course, since the index counts the number of supersymmetric vacua with signs, this will merely provide a lower bound on the total number of supersymmetric vacua. In [14, p. 9], the author claims that typically the actual index is equal to the index density times a bounded function greater than one. This is explicitly verified in the one-dimensional moduli space, in the conifold and large complex structure limit, see equations (3.83) and (3.84) of [14, p. 26]. Additionally, in [14] an interesting relation is mentioned between the index and stabilization of Kähler moduli is mentioned. It is stated that the difference between the index and the actual number of vacua in fact measures the number of Kähler stabilized vacua. For our purposes, the conclusion here is that if the index of supersymmetric vacua is finite, then the actual number of such vacua is also expected to be finite.

Continuing on with the assumption that the index density gives a good approximation to the actual number of flux vacua, let us plug in some typical numbers. Considering a simplified model of $Y_4 = X_3 \times \mathbb{T}$, for some Calabi-Yau threefold $X_3$, we have typical numbers of $b = 600$ and $L_* = 2000$, see e.g. [13, p. 106], leading to the infamous result

$$I_{\text{vac}} \approx 10^{500} \times \int \det(R + \omega \cdot \mathbf{1}). \tag{2.111}$$

In other words, a huge number of flux vacua is predicted. Perhaps it is good to emphasize that this is an estimate for a *single* choice of Calabi-Yau background. In general there are still many choices of topologically different Calabi-Yau's. This shows the vast complexity of these higher dimensional spaces. In fact, more exotic F-theory compactifications can lead to even bigger numbers. [16] In [10] the size of the geometrical factor $\int \det(R + \omega \cdot \mathbf{1})$ was additionally estimated in a certain class of type IIB compactifications to not necessarily be of order one. However, in the limit of large fluxes its contribution is still sub-leading in comparison to the pre-factor.

*The String Landscape*

Let us end this section with some more informal remarks on the interpretation of the index density. Regardless of the precise number of vacua that are produced in string theory compactifications, it is now clear that they are numerous. In 2003 this plethora of vacua was described by Leonard Susskind as a *string landscape* [47]. One of the hopes of string theory is to construct a vacuum which has precisely the desired properties to describe our observed physical world. For instance, it should contain a copy of the Standard Model and allow for a positive cosmological constant. However, with these results the search for such a vacuum seems hopeless. Indeed, checking whether a given vacuum has the desired properties is already a huge task, let alone doing it for $10^{500}$ vacua! In fact, it was shown in [16] that obtaining a specific vacuum with, for example, a given cosmological constant is typically NP hard in the context of computational complexity. As such, even if our Universe was described by some particular vacuum of string theory, we might never be able to explicitly find it. As such, one should interpret these results as a change in perspective with regards to model building in string theory. Indeed, in [18] it is discussed how one should rather be interested in the statistical properties of these vacua. Essentially, by analysing some general features of vacua one might be lead to a natural 'selection principle' which prefers certain vacua over others. Let us shortly touch on this by discussing possible extensions of our derivation of the supersymmetric index density.

*Alternative Distributions and Black Hole Attractor Points*

As a first extension, one can study the distribution of cosmological constants by additionally imposing $-3|W|^2 < \Lambda_*$ for some bound $\Lambda_*$. One of the results of [14, p. 24] is that for $|\Lambda_*| \ll M_{\text{pl}}^4$ the distribution is uniform

$$d\mathcal{N}[\Lambda_*] \sim d\Lambda_*. \tag{2.112}$$

By integrating this relation one finds that the cosmological constant can be as small as $M_{\text{pl}}^4/N_{\text{vac}}$. Since we have found that the number of flux vacua is typically very large, this may serve as a reason for the smallness of the observed cosmological constant. This possible explanation of the cosmological constant problem is in fact very similar in spirit to the one proposed in [4]. One can also study the distribution of supersymmetry breaking scales and whether high or low scales are favoured in particular kinds of compactifications, see e.g. [15, 19].

[16] Interestingly, in [48] an elliptic fourfold has been constructed containing $\mathcal{O}(10^{272,000})$ flux vacua, which surpasses the total number of flux vacua in the sum of all other F-theory geometries by a factor $\mathcal{O}(10^{3000})$.

Lastly, we point the interested reader to [14, p. 28], where it is shown that the methods we have used to compute the distribution of flux vacua can also be used to count black hole solutions in the type IIB setting on a Calabi-Yau threefold $Y_3$. The relevant parameter here is the entropy $S$ of the black hole, which is given by

$$S = \pi |\mathcal{Z}|^2, \quad \mathcal{Z} = \int_{Y_3} Q \wedge \Omega, \tag{2.113}$$

here $Q \in H^3(Y_3, \mathbb{Z})$ denotes the charge of a black hole. Note the resemblance between the central charge $\mathcal{Z}$ and the superpotential $W$. Importantly, in the above formula for $S$, the central charge is evaluated at a critical point given by $D_i \mathcal{Z} = 0$, again in analogy with the superpotential. Such critical points are also referred to as *attractor points*. Note that $S$ now plays the role of the tadpole constraint. Indeed, from here one can ask about the distribution of black holes with an entropy $S \leq S_*$ for some $S_*$ and obtain results in the same way as we did for the distribution of supersymmetric flux vacua.

*Finiteness?*

A pressing question is whether the contribution from the geometrical factor $\det(R + \omega \cdot \mathbf{1})$ is at all finite. In other words, do we expect the number of flux vacua to be finite? Clearly this has to do with the tadpole constraint

$$\int_{Y_4} G_4 \wedge G_4 \leq L_*. \tag{2.114}$$

Indeed, *a priori* the LHS is not necessarily positive, hence there may be infinite choices of fluxes satisfying the constraint. In [1, p. 21] an argument is made why this is *not* the case, except at certain limits in the moduli space, which we now adapt to the F-theory setting. As is readily seen from the definition of the superpotential

$$W = \int_{Y_4} G_4 \wedge \Omega \tag{2.115}$$

and the properties of $D_i$, the supersymmetry condition $D_i W = 0$ implies that

$$G_4^{1,3} = 0, \tag{2.116}$$

where $G_4^{p,q}$ denotes the $(p,q)$-part of $G_4$ in the Hodge decomposition of $H^4(Y_4)$. Additionally, it is argued in [32] that the (2,2) part of $G_4$ is primitive. Following the discussion in [32, Appendix I], the result is that $G_4$ is self-dual, i.e.

$$G_4 = \star G_4, \tag{2.117}$$

which implies that

$$\int_{Y_4} G_4 \wedge G_4 = \int_{Y_4} G_4 \wedge \star G_4 \geq 0, \tag{2.118}$$

where equality holds only when $G_4 = 0$. From here one can show that all vacua except for a finite number must lie in a neighbourhood of a so-called D-limit [1]. This is defined as a point in $\mathcal{M}_{\text{cs}}$ where the matrix $D_i \Pi_\alpha$ is reduced in rank. An example of such a limit is the Large Complex Structure (LCS) limit, where

$$\Pi \sim \begin{pmatrix} \tau^3 \\ \tau^2 \\ \tau \\ 1 \end{pmatrix}, \quad \text{as } \operatorname{Im} \tau \to \infty \tag{2.119}$$

In particular, we see that

$$D_\tau \mathbf{\Pi} \sim \begin{pmatrix} 3\tau^2 \\ 2\tau \\ 1 \\ 0 \end{pmatrix}, \quad \text{as } \operatorname{Im}\tau \to \infty \tag{2.120}$$

which indeed lowers in rank at the limit.

In short, we are left to show that even in such limits the index of supersymmetric vacua is still finite. We will do this by obtaining an asymptotic expression for the index density near such singularities and show that it is integrable. This will be the main topic of chapters 3 and 4. To close this chapter, we will rewrite the expression for the index density into a more suitable form.

## 2.5 Rewriting the Index Density

We recall the expression for the index density

$$d\mu = \det(R + \omega \cdot \mathbf{1}). \tag{2.121}$$

where

$$\omega = \frac{i}{2} g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}, \quad R^l{}_i = \frac{i}{2} R^l{}_{i\bar{\jmath}k} dz^i \wedge d\bar{z}^{\bar{\jmath}}. \tag{2.122}$$

The main simplification of $d\mu$ comes from the fact that the Riemann tensor of the Weil-Petersson metric of a Calabi-Yau $D$-fold takes the form [40]

$$R_{i\bar{\jmath}k\bar{l}} = g_{i\bar{\jmath}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{\jmath}} - e^K F_{i\bar{\jmath}k\bar{l}}, \tag{2.123}$$

where

$$F_{i\bar{\jmath}k\bar{l}} := \left( \nabla_i \nabla_k \Omega, \overline{\nabla_j \nabla_l \Omega} \right), \tag{2.124}$$

plays the role of the Yukawa couplings for fourfolds[17]. Here we have made two slight changes in notation. First, the bilinear form $(\cdot, \cdot)$ replaces the role of the intersection form $\eta$, i.e.

$$v, w \in H^4(Y_4): \quad (v, w) := \int_{Y_4} v \wedge w. \tag{2.126}$$

Moreover, we denote the Kähler-Weyl covariant derivative by $\nabla_i$ for easier comparison with the literature we reference in the following chapters. As a shorthand, let us introduce $M := R + \omega \cdot \mathbf{1}$, which is a matrix of two-forms.

The rewriting of $d\mu$ proceeds in three steps. First, we recall that the determinant of a matrix can be written in terms of the Levi-Civita tensor as follows

$$\det M = \epsilon^{k_1 \cdots k_n} \epsilon^{\bar{l}_1 \cdots \bar{l}_n} M_{k_1 \bar{l}_1} \wedge \cdots \wedge M_{k_n \bar{l}_n}. \tag{2.127}$$

Secondly, we note that for a top-form $\alpha$ such as $\det M$, one has

$$\alpha_{i_1 \bar{\jmath}_1 \cdots i_n \bar{\jmath}_n} dz^{i_1} \wedge d\bar{z}^{\bar{\jmath}_1} \cdots dz^{i_n} \wedge d\bar{z}^{\bar{\jmath}_n} \sim \sqrt{g} \epsilon^{i_1 \cdots i_n} \epsilon^{\bar{\jmath}_1 \cdots \bar{\jmath}_n} \alpha_{i_1 \bar{\jmath}_1 \cdots i_n \bar{\jmath}_n} d^n z, \tag{2.128}$$

[17] The name comes from the three-fold setting, where special geometry implies that

$$F_{i\bar{\jmath}k\bar{l}} = e^K F_{ikm} g^{m\bar{n}} \overline{F_{jln}}, \tag{2.125}$$

where the $F_{ijk}$ are historically called the 'Yukawa couplings' (due to their appearance in the compactification of the heterotic string).

where $\sqrt{g}\ d^nz = \sqrt{g}\ dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$ is the volume form on $\mathcal{M}_{\rm cs}$ and $\sim$ means 'up to factors of $n!$'. Combining the above two results yields the following expression for the index density

$$d\mu \sim \sqrt{g}\ d^nz\ \epsilon^{i_1\cdots i_n}\epsilon^{\bar{\jmath}_1\cdots\bar{\jmath}_n}\epsilon^{k_1\cdots k_n}\epsilon^{\bar{l}_1\cdots\bar{l}_n}M_{i_1\bar{\jmath}_1k_1\bar{l}_1}\cdots M_{i_n\bar{\jmath}_nk_n\bar{l}_n}. \tag{2.129}$$

The third step consists of writing the two pairs of epsilon tensors in terms of anti-symmetrized metrics by using the fact that

$$\epsilon^{\mu_1\cdots\mu_p\alpha_1\cdots\alpha_{n-p}}\epsilon_{\mu_1\cdots\mu_p\beta_1\cdots\beta_{n-p}} \sim \delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}. \tag{2.130}$$

From this point onwards, we will not explicitly denote the indices in order to not clutter the notation. Schematically, using the above expression for $p=0$ we can write

$$\epsilon^{\overbrace{\cdots}^{n}}\epsilon^{\overbrace{\cdots}^{n}} \sim \overbrace{g^{[\cdot\cdot}\cdots g^{\cdot\cdot]}}^{n}. \tag{2.131}$$

Hence we have

$$d\mu \sim \sqrt{g}\ d^nz\ \overbrace{g^{\cdot\cdot}\cdots g^{\cdot\cdot}}^{2n}\overbrace{M_{\cdots\cdot}\cdots M_{\cdots\cdot}}^{n} \tag{2.132}$$

Finally, we note that $M$ consists of terms containing $gg$ terms and $e^KF$. Choosing, say, $m$ factors of $e^KF$ in the product of all the $M$'s and letting the remaining factors of $g$ cancel with the preceding factors of $g^{-1}$ yields

$$d\mu \sim \sqrt{g}\ d^nz\ \sum_{m=0}^{n} I_m \tag{2.133}$$

where

$$I_m = e^{mK}\overbrace{g^{\cdot\cdot}\cdots g^{\cdot\cdot}}^{2m}\overbrace{F_{\cdots\cdot}\cdots F_{\cdots\cdot}}^{m} \tag{2.134}$$

The main advantage of this rewriting of the index density is that its dependence on the holomorphic 4-form $\Omega$ is very explicit since the Kähler potential $K$, the metric $g$ and the generalized Yukawa couplings $F_{i\bar{\jmath}k\bar{l}}$ are all directly expressed in terms of it. In the next chapter we will investigate the behaviour of the index density using this formula, together with some remarkable properties of $\Omega$ near specific points in the moduli space.

*A Sidenote on Warping*

As we have stated before, the inclusion of fluxes has an effect on the internal geometry which is described by the warp factor $A$. More precisely, the metric $g$ becomes conformally Calabi-Yau, meaning that

$$g = e^A\tilde{g}, \tag{2.135}$$

with $\tilde{g}$ a Ricci flat metric. In particular, using the fact that the Riemann tensor of $\tilde{g}$ obeys equation (2.123), one finds a similar equation for the Riemann tensor of $g$ of the form

$$e^{-A}R_{i\bar{\jmath}k\bar{l}} = \tilde{g}_{i\bar{\jmath}}\tilde{g}_{k\bar{l}} + \tilde{g}_{i\bar{l}}\tilde{g}_{k\bar{\jmath}} - e^K\tilde{F}_{i\bar{\jmath}k\bar{l}} - \tilde{g}_{i\bar{l}}\bar{\partial}_{\bar{\jmath}}\partial_kA. \tag{2.136}$$

Where $\tilde{F}$ is obtained from $\tilde{g}$. As such we have a correction term

$$g_{i\bar{l}}\bar{\partial}_{\bar{\jmath}}\partial_kA. \tag{2.137}$$

Quite frankly, we are unsure how to deal with this. One might hope that the warp factor varies slowly such that we can ignore this contribution, but this is entirely unclear near singular regions in the moduli space. Nevertheless, we will not consider the effects of warping in the remainder of this work.

# Chapter 3

# Hodge Structures at Singular Points

Much of our discussion so far has revolved around properties of the complex structure moduli space. Indeed, we started with the issue of moduli stabilization and how this can be addressed by introducing fluxes in the internal space. Upon compactification on a Calabi-Yau fourfold $Y_4$, these fluxes induced a superpotential for the complex structure moduli, which were part of the chiral multiplet of an $\mathcal{N} = 1$ supergravity theory with target space $\mathcal{M}_{\text{cs}}$. From there we derived an expression for the index of supersymmetric flux vacua using the properties of the Weil-Petersson metric. It is given by the following expression

$$\text{index density} \sim \sqrt{g}\; d^m z \sum_{k=0}^{m} I_k, \tag{3.1}$$

where

$$I_k = e^{kK} g^{\cdot\cdot} \cdots g^{\cdot\cdot} F_{....} \cdots F_{....}, \tag{3.2}$$

with $2k$ factors of $g^{-1}$ and $k$ factors of $F$. Crucially, all quantities appearing in the index density can be expressed in terms of the holomorphic 4-form $\Omega$ on $Y_4$ as[1]

$$e^{-K} = \left(\Omega, \bar{\Omega}\right), \quad e^{-K} g_{i\bar{j}} = -(\nabla_i \Omega, \overline{\nabla_j \Omega}), \quad F_{i\bar{j}k\bar{l}} = \left(\nabla_i \nabla_k \Omega, \overline{\nabla_j \nabla_l \Omega}\right). \tag{3.4}$$

A priori, it is not clear whether the index density is integrable over the moduli space. Indeed, at singular points where the curvature diverges, this may not be the case. In this chapter, we will further develop the mathematical structure of the moduli space to address this question by studying the behaviour of $\Omega$ near various singular points. More precisely, the relevant object to study is the cohomology group $H^4(Y_4)$ and its Hodge decomposition.

In section 3.1 we will first discuss the singular structure of the moduli space and introduce the concept of a monodromy transformation. It will turn out that the moduli space is such that much of the behaviour of $\Omega$ near singular points is encoded in its transformation under monodromy transformations. This is encapsulated by the *nilpotent orbit theorem* and is described in section 3.2. In section 3.3 we apply the nilpotent orbit theorem to gain insight into the various singularities that can occur and describe the degeneration of $H^4(Y_4)$ using the Deligne splitting. In section 3.5 we use this structure to estimate the leading behaviour of $\Omega$ near one-parameter singularities and thereby

[1] Recall that we introduced the Weil-Petersson metric as

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K, \tag{3.3}$$

which is in fact equal to $-e^K(D_i\Omega, \overline{D_j\Omega})$. Here we prefer the second expression to highlight the similarity between $K$, $g_{i\bar{j}}$ and $F_{i\bar{j}k\bar{l}}$.

obtain concrete expressions for the index density, whose integrability can be tested. Finally, in section 3.6 we provide a discussion of the results and give an outlook on more general singularities, which will be the topic of chapter 4.

## 3.1 Singular Loci and Monodromy

As we have already seen in the example of the complex structure moduli space of the torus, $\mathcal{M}_{\text{cs}}$ is generally not a smooth manifold and admits singularities. It is a fact that these singular points form a singular locus $\Delta$ which can be resolved to $\Delta = \cup \Delta_k$, where locally each $\Delta_k$ is given by the vanishing of one of the moduli, i.e. $z^j = 0$ [35, 50]. It will suffice to think of each $\Delta_k$ as a codimension one hypersurface in $\mathcal{M}_{\text{cs}}$. Of course, one can also consider the vanishing of multiple coordinates $z^{i_1}, \ldots, z^{i_k}$, which will be described by the intersection

$$\Delta_{i_1 \ldots i_k} := \Delta_{i_1} \cap \ldots \cap \Delta_{i_k}. \tag{3.5}$$

Lastly, we denote by

$$\Delta^{\circ}_{i_1 \ldots i_k} := \Delta_{i_1 \ldots i_k} - \bigcup_{j \neq i_1, \ldots, i_k} \Delta_{i_1 \ldots i_k j} \tag{3.6}$$

the points on the divisor $\Delta_{i_1 \ldots i_k}$ which do not lie on any higher intersection locus. See figure 3.1 for a depiction of two intersecting divisors in a local patch. One can consider a slightly different picture, which we will adopt in the coming sections, where instead one uses the coordinates $t^j = \frac{1}{2\pi i} \log z^j$. In these coordinates the singularities are given by Im $t \to \infty$, which brings us to the 'boundary' of $\mathcal{M}_{\text{cs}}$. We have already seen this in the example of the torus, where in the limit Im $\tau \to \infty$ its volume tends to infinity. Topologically, one can view the resulting space as a pinched torus. Generally, one can interpret these singular loci as values for the complex structure moduli which result in a singular $Y_4$. Having described the singularities under consideration more precisely, we can now state the final question we will address in this work

> Q3: Is the AD-density $d\mu$ integrable near a given singular locus $\Delta$ in the complex structure moduli space?

In this chapter we will answer this question with a definite 'yes' in the case of a single divisor $\Delta = \Delta_k$. The more general case is discussed in chapter 4, where the same answer is obtained for particular kinds of singular loci.

### 3.1.1 The Swampland Distance Conjecture

Our interest in the singular points of the moduli space stems from the question of whether the index density is integrable around such regions. However, these singular points are important for many other reasons as well, in particular with regards to the Swampland program and cosmological implications. We will now discuss these reasons in relation to the Swampland Distance Conjecture (SDC), though we must mention that we cannot do it justice in this work. For a review of the recent developments we refer the reader to [45], on which we draw heavily here.

Recall that the complex structure moduli are scalar fields, which we now more generally denote by $\phi$. Let $P$ and $Q$ be two points in the moduli space, and denote by $\Delta\phi$ the distance between them, as measured by the Weil-Petersson metric. Note that the points $P$ and $Q$ each describe a particular effective field theory (EFT), since they correspond to different expectation values of the scalar fields (and therefore e.g. a different cosmological constant). The SDC now asserts that as $\Delta\phi \to \infty$ an

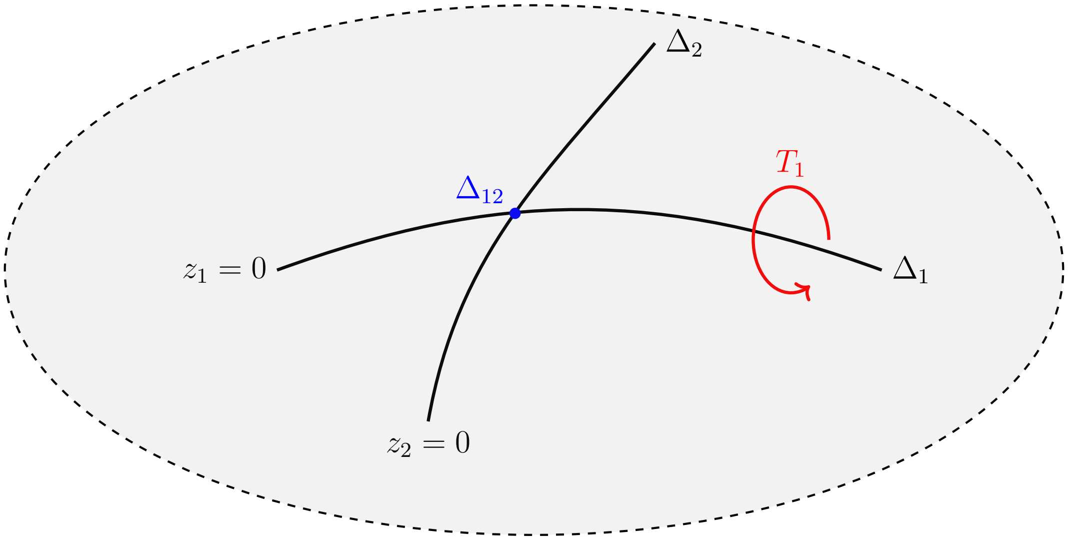



**Figure 3.1:** A local patch in $\mathcal{M}_{\text{cs}}$ containing two intersecting divisors $\Delta_1$ and $\Delta_2$. In red, a monodromy transformation $T_1$ around $\Delta_1$ is depicted.

infinite tower of states becomes light, more precisely the mass scales $m(P)$ and $m(Q)$ at $P$ and $Q$ are related by

$$m(P) \lesssim m(Q) e^{-\lambda \Delta\phi}, \tag{3.7}$$

for some yet undetermined parameter $\lambda$ depending on $P$ and $Q$. Points for which $\Delta\phi \to \infty$ for any path from $P$ to $Q$ are said to lie at infinite distance. Such points necessarily lie at a singular locus we described earlier. Crucially, the limit $\Delta\phi \to \infty$ signals a breakdown of the EFT at $P$, since it is impossible to give a quantum field theoretic description of infinitely many scalar fields that are weakly coupled to gravity. In other words, one requires the full theory of quantum gravity to work in this regime. We have in fact already seen an example of the SDC in action, namely the infinite tower of KK-modes that arose from compactifying a scalar field theory on a circle in section 1.1. Recall that we obtained an infinite tower of states with masses

$$m_n = \frac{n}{R} \tag{3.8}$$

which become massless in the limit $R \to \infty$. With a bit more work one can show that this limit is indeed exponential in the field distance. The SDC also has severe implications for models of cosmological inflation. Indeed, in an EFT valid below some finite cut-off scale $M_*$, it restricts the possible variations in the fields by

$$\lambda \Delta\phi \lesssim \log\left(\frac{M_{\text{pl}}}{M_*}\right). \tag{3.9}$$

In short, the singular points in the moduli space may provide us with new insights into quantum gravity constraints and allowed UV-completions of effective field theories. Note that there may also be singular points which are at finite distance. At such points massless fields do arise, for instance from D3-branes wrapping vanishing three-cycles. This happens, for example, in the conifold. However, there will only be a finite number of them and it is possible to adjust the EFT accordingly. For our purposes it is important to consider both types, since also for finite distance singularities it is known that the Ricci curvature can diverge, hence the integrability of the index density is *a priori* not guaranteed.

### 3.1.2 Monodromy and the Torus Revisited

Before discussing the intricacies of singular loci in general complex structure moduli spaces, let us first return to the familiar example of the torus. We will gain two important insights from this example. First, we will see that at a singular locus, the Hodge structure $H^1(\mathbb{T})$ (or generally $H^D(Y_D)$) breaks down. Secondly, the introduction of a *monodromy transformation* will allow us to write the period vector in a very particular form, which generalizes naturally to a general Calabi-Yau $Y_D$.

Using the same notation as introduced in section 2.3.3, the induced metric on $\mathbb{T}$ is given by:

$$ds^2 = dzd\bar{z} = d\xi_1^2 + |\tau|^2 d\xi_2^2 + 2\text{Re}\tau \, d\xi_1 d\xi_2, \tag{3.10}$$

or

$$g_{ij} = \begin{pmatrix} 1 & \text{Re}\,\tau \\ \text{Re}\,\tau & |\tau|^2 \end{pmatrix}. \tag{3.11}$$

In particular, we have

$$\sqrt{g} = \text{Im}\tau \tag{3.12}$$

and hence the volume of the torus is given by

$$\text{Vol}\,\mathbb{T} = \int_{\mathbb{T}} ds = \int_0^1 d\xi_1 \int_0^1 d\xi_2 \sqrt{g} = \text{Im}\,\tau. \tag{3.13}$$

We see that the large volume limit is exactly given by Im $\tau \to \infty$, corresponding to a singularity. Next consider the (1,0)-form $\Omega$ on $\mathbb{T}$, which is simply given by:

$$\Omega = dz = d\xi_1 + \tau d\xi_2, \quad \bar{\Omega} = d\bar{z} = d\xi_1 + \bar{\tau} d\xi_2. \tag{3.14}$$

In particular, we see that

$$\Omega - \text{Im}\,\tau \frac{\partial \Omega}{\partial \tau} = \bar{\Omega}. \tag{3.15}$$

As expected from Kodaira's formula, we are able to express $\bar{\Omega}$ in terms of $\Omega$ and its derivative with respect to $\tau$. However, we also see that in the limit Im $\tau \to \infty$, this relation breaks down. In fact, this is our first indication that at a singular locus the Hodge structure of $H^D(Y_D)$ breaks down. We will discuss this in detail in the next section. Before that, we turn to our second important observation regarding the period vector. In terms of the canonical basis, the period vector $\mathbf{\Pi}$ is simply given by

$$\mathbf{\Pi} = \begin{pmatrix} 1 \\ \tau \end{pmatrix}. \tag{3.16}$$

We can introduce the coordinate $\omega$ such that

$$\omega = e^{2\pi i \tau}, \tag{3.17}$$

then the singularity we are considering is at $\omega = 0$. We then perform a *monodromy transformation*, essentially encircling the singularity, by mapping

$$\omega \to e^{2\pi i} \omega, \tag{3.18}$$

which corresponds precisely to $\tau \to \tau + 1$. Using the explicit form of $\mathbf{\Pi}$, we see that it also transform as follows

$$\mathbf{\Pi} \to T\mathbf{\Pi}, \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3.19}$$

where $T$ is also called the *monodromy matrix*. Note that $(T - \mathrm{id})^{1+1} = 0$, so $T$ is quasi-unipotent, a property which it will retain in the general setting. Correspondingly, we can define a nilpotent monodromy matrix $N$ associated to this singularity is given by

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.20}$$

satisfying $N^2 = 0$. Finally then, upon introducing the constant vector

$$\mathbf{a}_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.21}$$

we see that $\mathbf{\Pi}$ can be exactly written as follows:

$$\mathbf{\Pi} = e^{\tau N}\mathbf{a}_0. \tag{3.22}$$

In summary, we have found that the period vector can be expressed in terms of a nilpotent matrix $N$ and a constant (i.e. $\tau$-independent) vector $\mathbf{a}_0$ in a very particular way, referred to as a *nilpotent orbit*. As we will discuss in the next section, the result (3.22) is in fact very general and describes (up to corrections) the asymptotic behaviour of the period vector of any Calabi-Yau $D$-fold.

## 3.2 The Nilpotent Orbit Theorem

Our discussion in the following three sections closely follows the works [28, 30]. For a more mathematically oriented review we refer the reader to [7]. For generality, we consider a $D$-dimensional Calabi-Yau manifold $Y_D$. For some examples we will restrict to the fourfold setting, as this is our main case of interest with regards to the distribution of flux vacua. We also refer the reader to Appendix C for an overview of the various definitions regarding (mixed) Hodge structures.

*Polarized Hodge Structures, Filtrations and Monodromy*

The Hodge decomposition[2]

$$H^D = \bigoplus_{p+q=D} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p} \tag{3.23}$$

defines a pure Hodge structure of weight $D$ on the vector space $H^D$. Moreover, it is polarized with respect to the inner product $S$ defined by

$$\alpha, \beta \in H^D: \quad S(\alpha, \beta) := \int_{Y_D} \alpha \wedge \beta. \tag{3.24}$$

That is, $S$ satisfies the following properties w.r.t. the Hodge structure:

1. $S(H^{p,q}, H^{r,s}) = 0, \quad (p,q) \neq (r,s).$
2. $i^{p-q} S(v, \bar{v}) > 0, \quad v \in H^{p,q},\ v \neq 0.$

For example, our particular case of interest $H^4$ enjoys the following decomposition

$$H^4 = H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4} \tag{3.25}$$

[2] We will henceforth write $H^D := H^D(Y_D)$ and $H^{p,q} := H^{p,q}(Y_D)$.

and the Kähler potential can be written as

$$e^{-K} = S(\Omega, \bar{\Omega}) \tag{3.26}$$

From a physical perspective, it is very natural that $H^4$ is relevant to our discussion. Indeed, the three main ingredients that comprise the index density are all related to the above decomposition:

1. Complex structure moduli $z^i$, $\quad i = 1, \dots, \dim H^{3,1}$
2. Fluxes $G_4 \in H^{4,0} \oplus H^{2,2}$
3. $\Omega \in H^{4,0}$.

It is important to realize that the Hodge decomposition of $H^D$ depends on the complex structure of $Y_D$, since it describes the $(p,q)$-decomposition of a differential form into holomorphic and anti-holomorphic parts. This is captured more clearly by introducing the Hodge filtration

$$F^D \subset F^{D-1} \subset \cdots \subset F^1 \subset F^0 = H^D, \tag{3.27}$$

where

$$F^p := \bigoplus_{q \geq p} H^{p,D-q}, \quad 0 \leq p \leq D. \tag{3.28}$$

The crucial result is that the $F^p$ vary holomorphically over the moduli space. As stated before, we see that e.g. for Calabi-Yau fourfolds, $F^4$ is spanned by $\Omega$, $F^3$ is spanned by $\Omega$ and $\nabla_i \Omega$, etc... Indeed, the decomposition of $H^D$ can be recovered from $\Omega$ and its derivatives w.r.t. the complex structure moduli.[3] Combining this result with the earlier observation that $H^4$ contains essentially all quantities we have encountered so far, we see that $\Omega$ is the relevant object to study. We now discuss this in detail.

Recall the description of $\Omega$ in terms of the period vector $\mathbf{\Pi}$ with respect to a basis $\gamma_I$ of $H^D$

$$\Omega = \Pi^\alpha \gamma_\alpha, \quad \alpha = 1. \dots, b_D. \tag{3.29}$$

The essential information on how the period vector behaves near a singular locus $\Delta_j$ is obtained by its transformation under monodromy transformations, i.e. encircling the locus by mapping $z^j \mapsto z^j e^{2\pi i}$. The period vector then transform as

$$\mathbf{\Pi}(\cdots, z^j, \cdots) \mapsto T_j^{-1} \mathbf{\Pi}(\cdots, z^j e^{2\pi i}, \cdots), \tag{3.30}$$

for some monodromy matrices $T_j$. They satisfy the following two properties [46]

1. Quasi-unipotency: $\quad \forall j, \ \exists m_j, n_j \in \mathbb{N}: \quad (T_j^{m_j} - \mathrm{Id})^{n_j+1} = 0.$
2. Commutation: $\quad \forall i, j: \quad [T_i, T_j] = 0.$

By performing a coordinate rescaling $z^j \to (z^j)^{m_j}$, it is easily seen that $T_j$ can be assumed to be unipotent, i.e. we can set $m_j = 1$. Indeed, we will see that all of the important information is contained in the unipotent part of $T_j$. In particular, for general $m_j$ we define

$$N_j := \frac{1}{m_j} \log T_j^{m_j}. \tag{3.31}$$

[3] There is a caveat here when considering $D > 3$. In particular, for our case of interest, namely the fourfold, it is known that only the primitive part of $H^{2,2}$ can be recovered. Recall that in section 2.4.3 we mentioned that $G_4$ is exactly of type (2,2) and is primitive, so this does not pose an issue for us.

Then $N_j$ is nilpotent:

$$N_j^{n_j+1} = 0. \tag{3.32}$$

*The Nilpotent Orbit Theorem*

For convenience, we denote

$$t^j := \frac{1}{2\pi i} \log z^j. \tag{3.33}$$

such that $z^j \to 0$ corresponds to Im $t^j \to \infty$. Using these various monodromy matrices, the nilpotent orbit theorem of Schmid will provide us with a general expression for the period vector which greatly resembles (3.22). The result holds within a patch $\mathcal{E}$ of the moduli space around the singular locus $\Delta$. Let $n_{\mathcal{E}}$ denote the number of intersecting discriminant divisors $\Delta_i$ given by $z^i = 0$. We furthermore denote by $\zeta$ all the coordinates which are not taken to zero. The nilpotent orbit theorem now states that near the point $P$ on $\Delta^\circ_{i_1 \cdots i_{n_{\mathcal{E}}}}$, the period vector is given by [46]

$$\boxed{\mathbf{\Pi}(t,\zeta) = \exp\left[\sum_{j=1}^{n_{\mathcal{E}}} t^j N_j\right] \mathbf{a}_0(\zeta) + \mathcal{O}\left(e^{2\pi i t}\right)} \tag{3.34}$$

where $\mathbf{a}_0$ is a holomorphic function of $\zeta$ and the $\mathcal{O}\left(e^{2\pi i t}\right)$ terms are exponentially suppressed in the limit Im $t^j \to \infty$. We note that this result is not that surprising. Indeed, a loop is described by $t \to t+1$, which by the above formula implies (up to terms of order $e^{2\pi i t}$)

$$\mathbf{\Pi} \to e^N \mathbf{\Pi} = T^{(u)} \mathbf{\Pi}, \tag{3.35}$$

where $T^{(u)}$ denotes the unipotent part of the monodromy matrix $T$. But this is precisely the defining property of $T$! Hence the nilpotent orbit theorem encapsulated the fact that the behaviour of the period vector near a singular point is (up to the vector $\mathbf{a}_0$) completely determined by the (nilpotent) monodromy around the divisor on which this singular point lies.

Taking a step back, we have seen that $\Omega$ and its derivatives, and therefore $\mathbf{\Pi}$ and its derivatives, completely determine the Hodge filtration at non-singular points. Moreover, the nilpotent orbit theorem precisely determines how the construction fails. In fact, this data is encapsulated by the nilpotent monodromy matrices $N_j$ and the holomorphic vector $\mathbf{a}_0$. The goal is now to use this data to construct a finer splitting of the space $H^D$ which is valid on a given singular locus. Since the construction is rather involved, we first present a sketch of the procedure:

1. At a given singular locus $\Delta^\circ_{i_1 \ldots i_k}$ one defines a new filtration $F^p_\infty(\Delta^\circ_{i_1 \ldots i_k})$ in terms of the old filtration $F^p$ and the nilpotent monodromy matrices associated with $\Delta^\circ_{i_1 \ldots i_k}$. However, this filtration will no longer constitute a Hodge filtration for the space $H^D$.

2. To recover the full space $H^D$, one constructs another filtration $W_l$, called the *monodromy weight filtration*, which depends only on the nilpotent monodromy matrices $N_j$. One can then combine the data $(F_\infty, N, W)$ into a *limiting mixed Hodge structure* (LMHS).

3. While the LMHS contains the right information, it will not be very easy to work with. Therefore we combine the $F_\infty$ and $W$ filtrations in a more natural way via the *Deligne splitting* $I^{p,q}$, which gives a finer splitting of $H^D$ in the following sense:

$$H^D = \bigoplus_{0 \le p,q \le D} I^{p,q}. \tag{3.36}$$

Recall that for a standard Hodge decomposition we would restrict the sum to $p + q = D$.

4. Finally, the $I^{p,q}$ spaces still contain some redundant information, in the sense they are partly generated by the $N_j$'s. This gives rise to the *primitive spaces* $P^{p,q}$ which satisfy

$$I^{p,q} = \bigoplus_{i\geq 0} N^i P^{p+i,q+i}, \quad N = \sum_{j=1}^{k} N_j. \tag{3.37}$$

Crucially, the primitive spaces are again *polarized*, but now with respect to the inner product $S(\cdot, N^{p+q-D}\cdot)$. Moreover, they can be combined into *horizontal primitive spaces* as

$$P_l := \bigoplus_{p+q=l} P^{p,q}, \tag{3.38}$$

in terms of which $H^D$ is given by

$$H^D = \bigoplus_{l=0}^{D} \bigoplus_{a=0}^{l} N^a P_{D+l}. \tag{3.39}$$

In short, the above procedure can be summarized as follows: Let $F^p$ be the Hodge filtration of $H^D$ polarized by $S$, away from a singular locus. When moving to a singular locus $\Delta^\circ_{i_1\ldots i_k}$, characterized by nilpotent monodromy matrices $N_{i_1}, \ldots, N_{i_k}$, one obtains a finer splitting of $H^D$ in terms of the primitive spaces, where the $P^{p,q}$ form a Hodge structure of weight $l$ on $P_l$, which is polarized w.r.t. $S(\cdot, N^l\cdot)$.

## 3.3 The Deligne Splitting

*Constructing the Limiting Mixed Hodge Structure*

First, we recall that $\mathbf{\Pi}$ and its derivatives generate $F$ away from the singular loci. The nilpotent orbit theorem gives a precise handle on the divergent behaviour as we approach a singular locus. Consider the singular locus $\Delta^\circ_{i_1\ldots i_k}$, then we define

$$F^p(\Delta^\circ_{i_1\ldots i_k}) := \lim_{\mathrm{Im}\, t^{i_1},\ldots,\mathrm{Im}\, t^{i_k}\to\infty} \exp\left[-\sum_{j=1}^{k} t^j N_j\right] F^p. \tag{3.40}$$

For ease of notation, we will write $F^p_\Delta := F^p(\Delta^\circ_{i_1\ldots i_k})$, but one should keep in mind that the resulting filtration depends on the chosen singular locus. Importantly, one can show that $F^p_\Delta$ is well-defined at the singular locus. However, it no longer constitutes a Hodge filtration, i.e. one cannot (in general) recover the full space $H^D$ from just the $F^p_\Delta$. Intuitively, this happens because we have removed the nilpotent matrices $N_j$ from the nilpotent orbit approximation in (3.40). As such, we will need a second ingredient which combines the $N_j$ with the $F_\Delta$ in an appropriate way. This ingredient is the *monodromy weight filtration*

$$W_{-1} := 0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2D-1} \subset W_{2D} = H^D, \tag{3.41}$$

where the spaces $W_l$ are defined in terms of the nilpotent monodromy matrices $N_j$ by

$$W_l = \bigoplus_{k\geq 1, k\geq l-D+1} \ker N^k \cap \mathrm{im}\, N^{k-l+D-1}, \quad N := N_{i_1} + \cdots + N_{i_k}. \tag{3.42}$$

One can show that this is the unique filtration satisfying the following two properties:

1. $NW_i \subset W_{i-2}$
2. $N^j : \mathrm{Gr}_{D+j} \to \mathrm{Gr}_{D-j}$ is an isomorphism, where

$$\mathrm{Gr}_i := W_i/W_{i-1} \tag{3.43}$$

are the *graded spaces.*

The important result is that in this way the data $(H^D(Y_D), W_j, F^p_\Delta)$ defines a *mixed Hodge structure*, which means that the graded spaces $\mathrm{Gr}_i$ admit a pure Hodge structure of weight $i$ given by

$$\mathrm{Gr}_i = \bigoplus_{p+q=i} \mathcal{H}^{p,q}_i, \quad \mathcal{H}^{p,q}_i = \mathcal{F}^p_i \cap \overline{\mathcal{F}^q_i}, \tag{3.44}$$

where

$$\mathcal{F}^p_i := (F^p \cap W_i)/(F^p \cap W_{i-1}). \tag{3.45}$$

In this manner, we obtain a whole set of Hodge structures of various weights, with $N$ acting as a morphism between them by $N\mathcal{H}^{p,q} \subset \mathcal{H}^{p-1,q-1}$.

*The Deligne Splitting as an Alternative*

While the above construction is certainly a motivation for introducing the monodromy weight filtration, the resulting spaces are difficult to work with. Indeed, the graded spaces are defined as quotients and it will be hard to identify the properties of $\Omega$ at the singular locus from these decompositions. As such, inspired to use the $W_l$ spaces, we introduce the Deligne splitting $I^{p,q}$ by

$$I^{p,q} := F^p_\Delta \cap W_{p+q} \cap \left( \bar{F}^q_\Delta \cap W_{p+q} + \sum_{j\geq 1} \bar{F}^{q-j}_\Delta \cap W_{p+q-j-1} \right), \tag{3.46}$$

which is the unique splitting satisfying

$$F^p_\Delta = \bigoplus_s \bigoplus_{r\geq s} I^{r,s}, \quad W_l = \bigoplus_{p+q\leq l} I^{p,q} \tag{3.47}$$

and

$$\overline{I^{p,q}} = I^{q,p} \bmod \bigoplus_{r<q,s<p} I^{r,s}. \tag{3.48}$$

The first two properties in(3.47) tell us how to recover the spaces $F^p$ and $W_l$ from the $I^{p,q}$'s. To interpret the final property in (3.48), we note that we can recover the graded spaces $\mathrm{Gr}_i$ from the Deligne splitting as well by

$$\mathrm{Gr}_i = \bigoplus_{p+q=i} I^{p,q}, \tag{3.49}$$

in complete analogy to (3.44). In other words, up to the 'mod' factor, the $I^{p,q}$ define a Hodge structure on the graded spaces as well. Importantly, we have traded the difficulty of working with the quotients $\mathrm{Gr}_i$ with the fact that the $I^{p,q}$ spaces satisfy (3.48) instead of the simpler $\overline{I^{p,q}} = I^{q,p}$.[4] In figure 3.2 we have depicted the relations between the various spaces we have introduced in an example of a Deligne diamond. Similarly to the Hodge diamonds, the various vertices correspond to the $I^{p,q}$ spaces, and a dot denotes a non-empty space.

[4] If the Deligne splitting does satisfy $\overline{I^{p,q}} = I^{q,p}$ it is called $\mathbb{R}$-split. In chapter 4 we will see that by choosing an appropriate basis, we can always make the splitting $\mathbb{R}$-split.

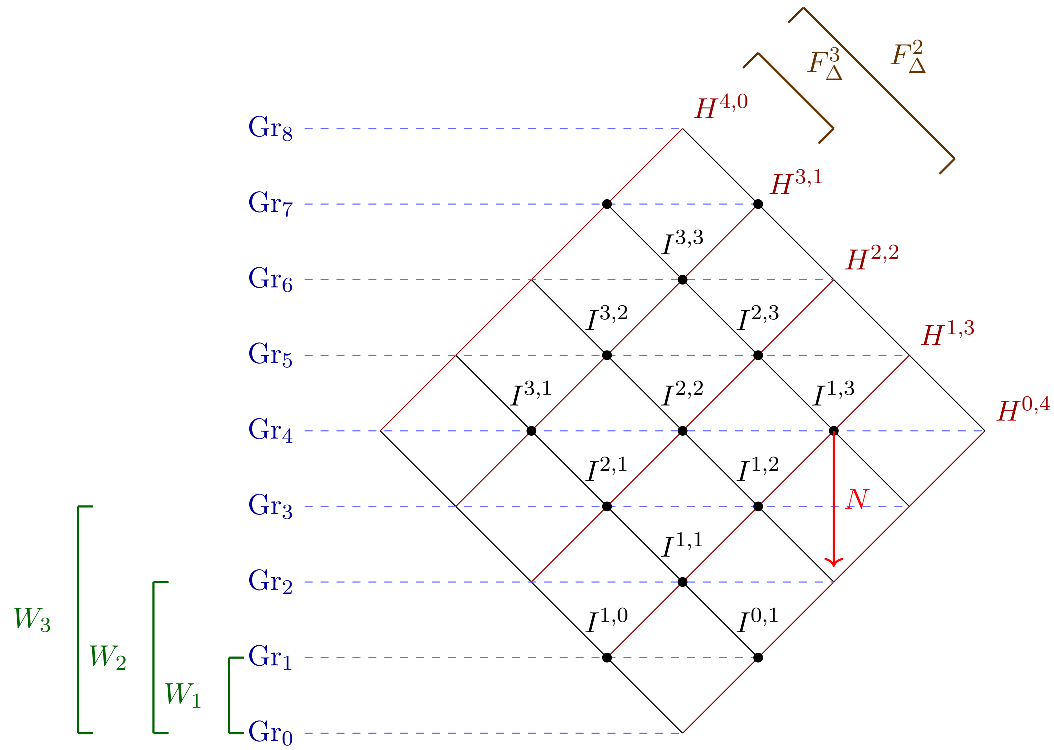



**Figure 3.2:** An example of a type IV Hodge-Deligne diamond which illustrates the properties of the Deligne splitting with respect to the Hodge decomposition $H^{p,q}$, the filtration $F^p$, the graded spaces $\mathrm{Gr}_i$ and the weight filtration $W_l$. The action of $N$ is indicated by the red arrow, from which its action on the various spaces mentioned is easily obtained. The colours match those used in equations (3.47), (3.49) and (3.50).

Continuing on, we note that the action of $N$ is given by

$$NI^{p,q} \subset I^{p-1,q-1} \tag{3.50}$$

This implies that the $I^{p,q}$'s still contain some redundancy, in the sense that part of $I^{p,q}$ can be obtained from $I^{p+1,q+1}$ by the action of $N$. It makes sense, therefore, to define the *primitive spaces* $P^{p,q}$ by

$$P^{p,q} := I^{p,q} \cap \ker N^{p+q-D+1}, \tag{3.51}$$

such that

$$I^{p,q} = \bigoplus_{i\geq 0} N^i P^{p+i,q+i}. \tag{3.52}$$

Indeed, we see that the $I^{p,q}$'s can be recovered completely from the $P^{p,q}$'s and the action of $N$. Finally, we can consider the analogue of (3.49) by defining the *horizontal primitive spaces*

$$P_l := \bigoplus_{p+q=l} P^{p,q}. \tag{3.53}$$

The crucial result is now that the primitive spaces $P^{p,q}$ form a Hodge structure of weight $l$ on each $P_l$, which is moreover polarized w.r.t. $S_l(\cdot,\cdot) := S(\cdot, N^l\cdot)$, i.e.

1. $S_l(P^{p,q}, P^{r,s}) = 0, \quad r+s = D+l = p+q, \quad (p,q) \neq (r,s).$
2. $i^{p-q} S_l(v,\bar{v}) > 0, \quad v \in P^{p,q},\ v \neq 0,\ p+q = D+l.$

It will be these polarization properties which will allow us to estimate the behaviour of the index density near the singularities. We will discuss this in the next section. We end this section by noting that $H^D$ can be recovered from the horizontal primitive spaces and the action of $N$ by

$$H^D = \bigoplus_{l=0}^{D} \bigoplus_{a=0}^{l} N^a P_{D+l}. \tag{3.54}$$

## 3.4 Properties of $a_0$

Recall that the nilpotent orbit theorem characterizes the behaviour of the period vector in terms of the nilpotent monodromy matrices $N_j$ and the vector $\mathbf{a}_0$. In the previous section, we have mostly used the monodromy matrices in order to infer some detailed structures at the singular locus. Now we turn to the properties of $\mathbf{a}_0$ which can be derived from the above construction. First, since $\mathbf{a}_0 \in F^D_\Delta$, we have

$$\mathbf{a}_0 \in I^{D,0} \oplus I^{D,1} \oplus \cdots \oplus I^{D,D}. \tag{3.55}$$

By construction of the Deligne splitting, we have the following relation between the Hodge numbers and the dimensions of the $I^{p,q}$

$$h^{p,D-p} = \sum_{q=0}^{D} i^{p,q}, \quad i^{p,q} = \dim I^{p,q}. \tag{3.56}$$

Crucially, for Calabi-Yau $D$-folds, we have $h^{D,0} = 1$, hence only one of the spaces in (3.55) can be non-trivial. Now let $d$ be an integer such that

$$N^d \mathbf{a}_0 \neq 0, \quad N^{d+1}\mathbf{a}_0 = 0. \tag{3.57}$$

Essentially, $d$ determines the higher power of of $t^i$'s that can appear in the nilpotent orbit expansion and it will play a central role in characterizing the behaviour of the Kähler potential. By the first property in (3.57) and the fact that $NI^{p,q} \subset I^{p-1,q-1}$, we see that

$$\mathbf{a}_0 \in I^{D,d} \oplus \cdots \oplus I^{D,D}. \tag{3.58}$$

On the other hand, since $I^{D,i} = P^{D,i}$ for $i \geq 0$, we can use the polarization property (2) of $P^{D,d+i}$, which states that

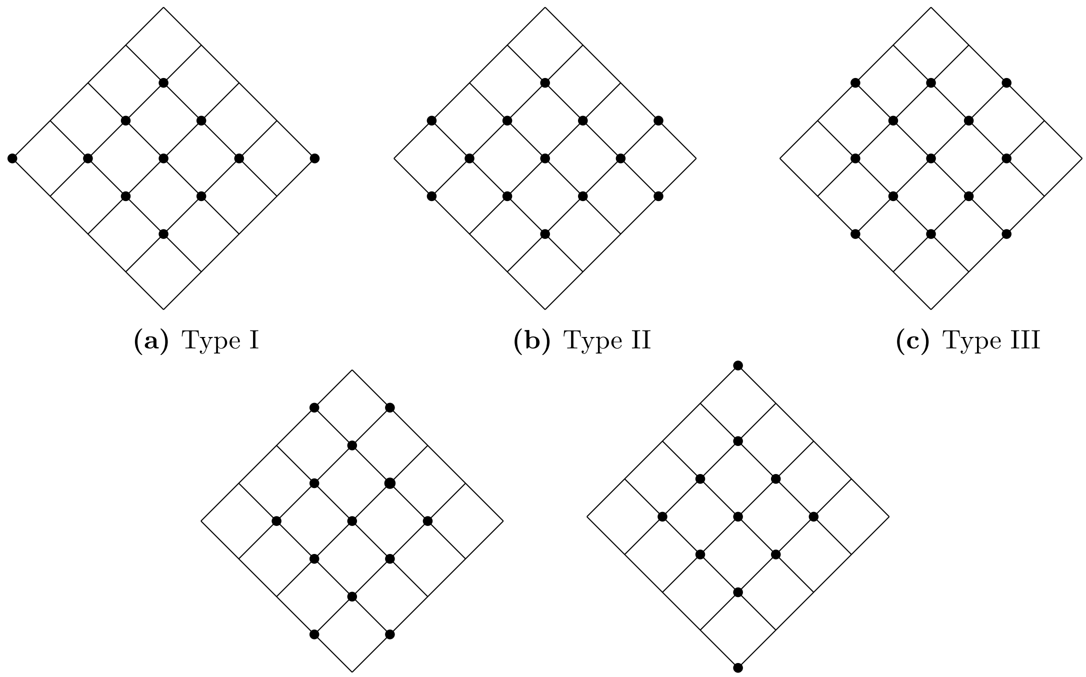
$$0 < i^{D-(d+i)} S_{d+i}(\mathbf{a}_0, \bar{\mathbf{a}}_0) = i^{D-(d+i)} S(\mathbf{a}_0, N^{d+i}\bar{\mathbf{a}}_0). \tag{3.59}$$

But the RHS vanishes for $i > 0$ by the second property in (3.57). We therefore conclude the following:

$$\boxed{d \text{ highest such that } N^d\mathbf{a}_0 \neq 0 \implies d \text{ highest such that } S(\mathbf{a}_0, N^d\bar{\mathbf{a}}_0) \neq 0} \tag{3.60}$$

and moreover

$$i^{D-d} S(\mathbf{a}_0, N^d \bar{\mathbf{a}}_0) > 0. \tag{3.61}$$



(a) Type I (b) Type II (c) Type III

(d) Type IV (e) Type V

**Figure 3.3:** The five types of Deligne diamonds associated to the Deligne splitting of $H^4(Y_4)$ at a singular locus, determined by the conditions $N^d\mathbf{a}_0 \neq 0$ and $N^{d+1}\mathbf{a}_0 = 0$, for $d = 0, \ldots, 4$.

Returning to a more graphical interpretation, we note that the integer $d$ classifies different Deligne diamonds which can occur. Indeed, the integer $d$ labels the row at which $\mathbf{a}_0$ lies, and together with the symmetries of the Deligne diamonds (which are the same as those of the Hodge diamonds) this gives rise to $D-1$ different types. In figure 3.3 we illustrate the five types labelled by Roman numerals I through V for Calabi-Yau fourfolds.

## 3.5 Scaling of the Index Density for Single-Moduli Limits

In this section we will use the established mechanisms of the Deligne splitting and its various properties to discern the scaling of the index density near a given singular locus. We will discuss the one-parameter degeneration in complete generality. Multi-moduli limits will require some more machinery to which we delegate the next chapter. For ease of reference, we collect the relevant formulae for the derivation here.

First, recall the expression for the index density

$$\boxed{d\mu[t] \sim \sqrt{g}\; d^m t \sum_{k=0}^{m} I_k, \quad I_k = e^{kK}\,\overbrace{g^{\cdot\cdot}\cdots g^{\cdot\cdot}}^{2k}\,\overbrace{F_{\cdots\cdot}\cdots F_{\cdots\cdot}}^{k}} \tag{3.62}$$

As stressed earlier, all the quantities in $d\mu$ can be expressed in terms of the period vector $\mathbf{\Pi}$ as follows

$$e^{-K} = (\mathbf{\Pi}, \bar{\mathbf{\Pi}}) =: ||\mathbf{\Pi}||^2, \quad g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K, \quad F_{i\bar{j}k\bar{l}} = (\nabla_i \nabla_k \mathbf{\Pi}, \overline{\nabla_j \nabla_l \mathbf{\Pi}}), \tag{3.63}$$

where $\nabla_i$ denotes the Kähler-Weil covariant derivative. Near a singular locus defined by Im $t \to \infty$, with associated nilpotent monodromy $N$, the period vector is given by

$$\mathbf{\Pi} = \underbrace{e^{tN}\mathbf{a}_0}_{\mathbf{\Pi}_{\text{nil}}} + \mathcal{O}(e^{2\pi i t}). \tag{3.64}$$

There are two cases to distinguish, namely

1. $N\mathbf{a}_0 \neq 0$,
2. $N\mathbf{a}_0 = 0$.

We will discuss these two cases in separate subsections. The case $N\mathbf{a}_0 = 0$ is considerably more complex, as it requires the inclusion of correction terms to restore the $t$-dependence of $\mathbf{\Pi}$. As mentioned in [30], these two cases correspond precisely to singularities which are at infinite or finite distance in the moduli space, respectively. Additionally, we recall the integer $d$ satisfying

$$N^d\mathbf{a}_0 \neq 0, \quad N^{d+1}\mathbf{a}_0 = 0, \tag{3.65}$$

which labels the $D+1$ different types of singularities. In other words, we consider the cases $d > 0$ and $d = 0$ separately. First, we make some remarks that are applicable to both cases. The overall strategy in each case will be very similar and can be summarized as follows:

1. The correct form of the period vector for the specific case at hand is determined. This provides us with the Kähler potential via $e^{-K} = ||\mathbf{\Pi}||^2$.
2. From the Kähler potential the metric components and the covariant derivatives of $\mathbf{\Pi}$ are computed.
3. Finally, it is determined which of the $F_{i\bar{j}k\bar{l}}$ components provide the dominant contribution and their expressions are determined. From here the scaling of the index density follows immediately.

Since we are only interested in the leading order functional behaviour of the index density, we will not be careful with numerical factors but will restore them whenever necessary (this will be relevant in the case $N\mathbf{a}_0 = 0$). Hence throughout the derivation we will make use of the symbol $\sim$, which means 'neglecting sub-leading terms and overall pre-factors'.

Before delving into the specific computations, we make one final remark regarding the computation of $F_{i\bar{j}k\bar{l}}$, or, more precisely, the double covariant derivative $\nabla_i\nabla_k\mathbf{\Pi}$. Here one must be careful, since the result must transform properly under both Kähler-Weil and general coordinate transformations. In other words, one must include the Levi-Civita connection as well, resulting in

$$\nabla_i\nabla_k\mathbf{\Pi} = \nabla_i^K\nabla_k^K\mathbf{\Pi} - \Gamma^m{}_{ik}\nabla_m^K\mathbf{\Pi}, \tag{3.66}$$

where by $\nabla_i^K$ we mean just the Kähler-Weil connection. One checks that this indeed has the right transformation behaviour. Moreover, we will see that in all cases we may approximate $\nabla_i^K \sim \partial_i$, since $\partial_i K$ will give sub-leading contributions. In other words, we will generally find that

$$\nabla_i\nabla_k\mathbf{\Pi} \sim \partial_i\partial_k\mathbf{\Pi} - \Gamma^\mu{}_{ik}\partial_\mu\mathbf{\Pi}. \tag{3.67}$$

### 3.5.1 $N\mathbf{a}_0 \neq 0$

In this case we may approximate the period vector by the nilpotent orbit, i.e. we have

$$\mathbf{\Pi} \sim \mathbf{\Pi}_{\text{nil}} = e^{tN}\mathbf{a}_0, \tag{3.68}$$

neglecting exponentially suppressed terms. Since the bilinear form $(\cdot,\cdot)$ is invariant under monodromy transformations, one sees that it has the following symmetry property with respect to the nilpotent monodromy matrix

$$(N\cdot,\cdot) = -(\cdot, N\cdot). \tag{3.69}$$

As a result, we can rewrite the expression for the Kähler potential as follows

$$e^{-K} = (\mathbf{\Pi}, \bar{\mathbf{\Pi}}) \sim \left(e^{2i(\text{Im } t)N}\mathbf{a}_0, \bar{\mathbf{a}}_0\right) \sim p(\text{Im } t, \zeta), \tag{3.70}$$

where $p$ is a polynomial in Im $t$ of order $d > 0$. It follows that

$$-K \sim \log\left[f(\zeta,\bar{\zeta})(\text{Im } t)^d + g(\zeta,\bar{\zeta})(\text{Im } t)^{d'}\right], \tag{3.71}$$

for some functions $f, g$ depending on the other coordinates, and $0 \leq d' < d$. We then rewrite $K$ into a more convenient form as

$$K \sim \log\left[f(\text{Im } t)^d\left(1 + \frac{g}{f}(\text{Im } t)^{d'-d}\right)\right] \sim d\log \text{Im } t + \log\left[1 + \frac{G(\zeta,\bar{\zeta})}{(\text{Im } t)^x}\right] + F(\zeta,\bar{\zeta}) \tag{3.72}$$

where $F = \log f$, $G = g/f$ and $x = d - d' > 0$. Using the approximation $\log(1+\epsilon) \approx \epsilon$, we arrive at

$$K \sim d\log\ \text{Im } t + \frac{G(\zeta,\bar{\zeta})}{(\text{Im } t)^x} + F(\zeta,\bar{\zeta}) \tag{3.73}$$

Here we once again stress that we are considering the case $d > 0$. Moreover, it is also clear from the expression of $K$ that the precise value of $d = 1, \ldots, D+1$ will not be relevant as it appears as simply a prefactor. We will indeed find that this is the case. By writing $K$ in the above form, one sees that the different metric components $g_{i\bar{\jmath}}$ arise from the three terms separately. Indeed, in terms of the scaling w.r.t. Im $t$ we have

$$g_{t\bar{t}} \sim (\text{Im } t)^{-2}, \quad g_{t\bar{\zeta}} \sim (\text{Im } t)^{-1-x}, \quad g_{\zeta\bar{\zeta}} \sim 1. \tag{3.74}$$

To leading order, the components of the inverse metric are given by

$$g^{t\bar{t}} \sim (\text{Im } t)^2, \quad g^{t\bar{\zeta}} \sim (\text{Im } t)^{1-x}, \quad g^{\zeta\bar{\zeta}} \sim 1 \tag{3.75}$$

One also easily computes

$$\sqrt{g} \sim (\text{Im } t)^{-2}, \quad e^K \sim (\text{Im } t)^{-d} \tag{3.76}$$

Next, we note that

$$\nabla_t\mathbf{\Pi} = \partial_t\mathbf{\Pi} + (\partial_t K)\mathbf{\Pi} \sim N\mathbf{\Pi} + (\text{Im } t)^{-1}\mathbf{\Pi}. \tag{3.77}$$

In particular, in the limit Im $t \to \infty$, we may approximate $\nabla_t \sim \partial_t$. Including the Levi-Civita connection in the double covariant derivative, as discussed in the introduction, then leads to

$$\nabla_t^2\mathbf{\Pi} \sim \partial_t^2\mathbf{\Pi} - \Gamma^m{}_{tt}\partial_m\mathbf{\Pi}. \tag{3.78}$$

One can explicitly compute the Christoffel symbols for $m = t, \zeta$ to find

$$\Gamma^t{}_{tt} \sim (\text{Im } t)^{-1}, \quad \Gamma^\zeta{}_{tt} \sim (\text{Im } t)^{-2-x}. \tag{3.79}$$

In particular, both are again suppressed by at least a factor $(\text{Im } t)^{-1}$, in other words we may further approximate

$$\nabla_t^2 \mathbf{\Pi} \sim \partial_t^2 \mathbf{\Pi} \sim N^2 \mathbf{\Pi}. \tag{3.80}$$

In short, each application of $\nabla_t$ on $\mathbf{\Pi}$ simply brings down a factor of $N$, and the same holds for $\bar{\nabla}_{\bar{t}}$ acting on $\bar{\Pi}$. We now evaluate $F_{i\bar{\jmath}k\bar{l}}$. Suppose $p$ ($\bar{p}$) of the indices are chosen to be $t$ ($\bar{t}$), so $p, \bar{p} = 0, 1, 2$. One readily sees that

$$F_{i\bar{\jmath}k\bar{l}} \sim \left(N^p \mathbf{\Pi}, N^{\bar{p}} \bar{\mathbf{\Pi}}\right) \sim \left(e^{2i(\text{Im } t)\, N} N^p \mathbf{a}_0, N^{\bar{p}} \bar{\mathbf{a}}_0\right) \sim \sum_{q=0}^{d} (\text{Im } t)^q \left(\mathbf{a}_0, N^{p+\bar{p}+q} \bar{\mathbf{a}}_0\right), \tag{3.81}$$

where in the second and third line we again used (3.69). We now recall our result of (3.60), which states that the maximal value of $q$ for which the summand is non-zero is given by

$$p + \bar{p} + q = d. \tag{3.82}$$

This implies that the leading behaviour of $F_{i\bar{\jmath}k\bar{l}}$ is given by

$$F_{i\bar{\jmath}k\bar{l}} \sim (\text{Im } t)^{d-(p+\bar{p})}, \tag{3.83}$$

where we recall that $p + \bar{p}$ is the total number of indices that are chosen to be $t$ or $\bar{t}$. In particular, we have

$$e^K F_{i\bar{\jmath}k\bar{l}} \sim (\text{Im } t)^{-(p+\bar{p})}. \tag{3.84}$$

On the other hand, each choice of index $t$ or $\bar{t}$ yields (at worst) a factor of Im $t$ from the inverse metrics, as can be seen from (3.75). Somewhat curiously, we see that all these factors cancel in $I_k$, for each $k$ separately. The leading behaviour of the index density is therefore simply proportional to the canonical volume form

$$d\mu \sim d^m t \sqrt{g}. \tag{3.85}$$

Concentrating on the part of the measure depending on $t, \bar{t}$, we find

$$d\mu[t] \sim \frac{dt \wedge d\bar{t}}{(\text{Im } t)^2}. \tag{3.86}$$

This is the main result of our calculation.

For the interested reader, we add an additional comment regarding the proportionality factor that is suppressed in the expression above. The result is that each factor of $g^{i\bar{\jmath}} g^{k\bar{l}} e^K F_{i\bar{\jmath}k\bar{l}}$ comes with a pre-factor given by [5]

$$d^{-(p+\bar{p})} \times \frac{d!}{(d-(p+\bar{p}))!} = \prod_{n=0}^{p+\bar{p}-1} \left(1 - \frac{n}{d}\right), \tag{3.87}$$

where we remind the reader that $p$ and $\bar{p}$ denote the number of $t$ and $\bar{t}$ indices that are contracted, respectively. From the second expression we see that these factors are at most of order one. Also, if $p + \bar{p} \neq 4$, there may also be factors of $g^{\zeta\bar{\zeta}}$ involved. The upshot if this discussion is that the prefactor of the index density can be expressed in terms of $d$ and $g^{\zeta\bar{\zeta}}$, although not much can be said about the exact result. We mention this because this will be quite different in the next case.

[5] I thank Erin van der Kamp for bringing my attention to the second expression.

### 3.5.2 $N\mathbf{a}_0 = 0$

Since the derivation in this case is considerably more intricate than in the previous, we first give a short overview of the additional steps that are made throughout this subsection:

1. First, we perform a set of Kähler-Weil transformations to bring the Kähler potential in a more manageable form.

2. Second, we proceed to compute the metric components and the covariant derivatives of $\mathbf{\Pi}$. In contrast to the $N\mathbf{a}_0 \neq 0$ case, we will see that the contribution from the Levi-Civita connection is essential.

3. Finally, again in contrast to the previous case, we will argue that the leading component of $F_{i\bar{j}k\bar{l}}$ is given by $F_{t\bar{t}t\bar{t}}$ and employ the symmetry properties of the index density to argue that it may only occur once.

For the moment we switch to the coordinate $z = e^{2\pi i t}$. Let $q > 0$ be the smallest integer such that $\mathbf{a}_q \neq 0$. The period vector is then of the form

$$\mathbf{\Pi} = \mathbf{a}_0 + z^q \mathbf{\Pi}_q + \mathcal{O}(z^{q+1}), \quad \mathbf{\Pi}_q = e^{\frac{\log z}{2\pi i} N} \mathbf{a}_q. \tag{3.88}$$

Note that are indeed forced to consider higher-order correction terms to $\mathbf{\Pi}$. To compute the Kähler potential, we first note that

$$e^{-K} = ||\mathbf{\Pi}||^2 \sim ||\mathbf{a}_0||^2 + |z|^{2q} \, ||\mathbf{\Pi}_q||^2 + \text{inner products of the form } (\mathbf{a}_i, \bar{\mathbf{a}}_0) \text{ plus c.c,} \tag{3.89}$$

where c.c. stands for complex conjugate. Note that for a given $i$, the inner product $(\mathbf{a}_i, \bar{\mathbf{a}}_0)$ comes with a factor $z^i$. As such, these terms may influence the leading order behaviour of the Kähler potential when $i \leq 2q$. Our first step is to show that we can set $(\mathbf{a}_i, \bar{\mathbf{a}}_0) = 0$ for $q \leq i \leq 2q$ by a finite sequence of Kähler-Weil transformations[6]

$$K(z, \bar{z}) \mapsto K(z, \bar{z}) - f(z) - \bar{f}(\bar{z}), \tag{3.90}$$

where $f$ is any holomorphic function. Since $e^{-K} = (\mathbf{\Pi}, \bar{\mathbf{\Pi}})$, the period vector transforms as

$$\mathbf{\Pi} \mapsto e^f \mathbf{\Pi}. \tag{3.91}$$

We then choose

$$f = f_q = -z^q \frac{(\mathbf{a}_q, \bar{\mathbf{a}}_0)}{(\mathbf{a}_0, \bar{\mathbf{a}}_0)}. \tag{3.92}$$

Using the fact that $e^f = 1 + f + \cdots$, we see that $\mathbf{\Pi}$ transforms as

$$\mathbf{\Pi} \mapsto \mathbf{\Pi} + f\mathbf{\Pi} + \mathcal{O}(z^{2q})\mathbf{\Pi} = \mathbf{a}_0 + z^q e^{tN} \left( \mathbf{a}_q - \frac{(\mathbf{a}_q, \bar{\mathbf{a}}_0)}{(\mathbf{a}_0, \bar{\mathbf{a}}_0)} \mathbf{a}_0 \right) + \mathcal{O}(z^{q+1}). \tag{3.93}$$

In other words, under this transformation

$$\mathbf{a}_q \mapsto \mathbf{a}_q - \frac{(\mathbf{a}_q, \bar{\mathbf{a}}_0)}{(\mathbf{a}_0, \bar{\mathbf{a}}_0)} \mathbf{a}_0, \tag{3.94}$$

which implies that $(\mathbf{a}_q, \bar{\mathbf{a}}_0) \mapsto 0$. Note that due to the higher order $f$-terms in $e^f$, some of the higher-order $\mathbf{a}_{q+j}$, $j > 0$, may also shift. We can now iterate the above process $q$ times (using the

[6] The argument below is due to Damian van de Heisteeg.

shifted $\mathbf{a}_{q+j}$ when needed) by setting $f = f_{q+1}$, $f = f_{q+2}$, etc... until we have obtained $(\mathbf{a}_i, \bar{\mathbf{a}}_0) = 0$ for all $q \leq i \leq 2q$, as desired. Using this result, we have

$$e^{-K} \sim ||\mathbf{a}_0||^2 + |z|^{2q} \; ||\mathbf{\Pi}_q||^2 \tag{3.95}$$

with corrections being suppressed by at least a factor $z$. Let $p$ be the largest integer such that $N^p \mathbf{a}_q \neq 0$. Note that if $p = 0$, then $\mathbf{a}_q$ must be proportional to $\mathbf{a}_0$, in which case the transformation (3.94) sets $\mathbf{a}_q$ to zero. As such we may assume that $p > 0$. For the rest of the derivation we switch back to the coordinate $t = \frac{1}{2\pi i} \log z$. Then the Kähler potential is of the form[7]

$$K \sim 1 + (\text{Im } t)^p e^{2\pi i q (t - \bar{t})}. \tag{3.97}$$

The metric components are given by

$$g_{t\bar{t}} \sim (\text{Im } t)^p e^{2\pi i q(t-\bar{t})}, \quad g_{t\bar{\zeta}} \sim (\text{Im } t)^p e^{2\pi i q(t-\bar{t})}, \quad g_{\zeta\bar{\zeta}} \sim 1. \tag{3.98}$$

and we also note that $\sqrt{g} \sim g_{t\bar{t}}$. Moreover, the inverse components are given by

$$g^{t\bar{t}} \sim g_{t\bar{t}}^{-1}, \quad g^{t\bar{\zeta}} \sim 1, \quad g^{\zeta\bar{\zeta}} \sim 1. \tag{3.99}$$

Finally, we turn to the computation of

$$F_{t\bar{t}t\bar{t}} = (\nabla_t \nabla_t \mathbf{\Pi}, \overline{\nabla_t \nabla_t \mathbf{\Pi}}) = ||\nabla_t^2 \mathbf{\Pi}||^2. \tag{3.100}$$

First, we note that

$$\partial_t K \sim r^p e^{-2qr}, \quad r := 2\pi \text{ Im } t \tag{3.101}$$

is at least of order $\mathcal{O}(e^{-2qr})$. On the other hand, $\partial_t \mathbf{\Pi}$ is of order $\mathcal{O}(e^{-qr})$, which is therefore the leading term in the limit $r \to \infty$. As a result, we may again approximate $\nabla_t = \partial_t$ when acting on the scalar $\mathbf{\Pi}$. Taking the Levi-Civita connection into account, we again find that

$$\nabla_t^2 \mathbf{\Pi} \sim \partial_t^2 \mathbf{\Pi} - \Gamma^\mu{}_{tt} \partial_\mu \mathbf{\Pi}. \tag{3.102}$$

For Kähler geometry, we have

$$\Gamma^\mu{}_{tt} = g^{\mu\nu} \partial_t g_{t\nu}. \tag{3.103}$$

Using the fact that $g^{t\zeta} \sim 1$ one readily sees that the contribution from $\mu = t$, $\nu = \bar{t}$ is leading. In other words, we have

$$\nabla_t^2 \mathbf{\Pi} \sim \partial_t^2 \mathbf{\Pi} - \Gamma^t{}_{tt} \partial_t \mathbf{\Pi}. \tag{3.104}$$

Let us now compute the various quantities in the above expression. Crucially, we will include all factors to obtain precise expressions, because later on a delicate cancellation will take place.[8] First, we have

$$\partial_t \mathbf{\Pi} = (2\pi i q + N) e^{2\pi i q t} \mathbf{\Pi}_q, \tag{3.105}$$

$$\partial_t^2 \mathbf{\Pi} = (2\pi i q + N)(2\pi i q + N) e^{2\pi i q t} \mathbf{\Pi}_q. \tag{3.106}$$

[7] A perhaps even easier way to come to the same result is to expand $e^{-K}$ to see that

$$K \sim ||\mathbf{a}_0||^2 + |z|^{2q} \; ||\mathbf{\Pi}_q||^2 + \left[\text{inner products of the form } (\mathbf{a}_i, \bar{\mathbf{a}}_0) + \mathcal{O}(|z|^{2q} z)\right] + \text{c.c.} \tag{3.96}$$

Then since the terms involving $(\mathbf{a}_i, \bar{\mathbf{a}}_0)$ come with a factor of $z^i$ and are therefore holomorphic, they can be removed directly by a single Kähler-Weil transformation.

[8] The attentive reader might worry about the potential prefactor in $g_{t\bar{t}}$ arising from $||N\mathbf{a}_q||^2$, however this factor will drop out of the expression for $\Gamma^t{}_{tt}$.

Continuing on, we have

$$\Gamma^t{}_{tt} = \partial_t \log(g_{t\bar{t}}) = \frac{p}{2i}(\text{Im } t)^{-1} + 2\pi i q \tag{3.107}$$

It follows that

$$\Gamma^t{}_{tt}\partial_t \mathbf{\Pi} = \left[2\pi i q + \frac{p}{2i}(\text{Im } t)^{-1}\right](2\pi i q + N)e^{2\pi i q t}\mathbf{\Pi}_q \tag{3.108}$$

We see that in the expression for $\nabla_t^2 \mathbf{\Pi}$, the two terms highlighted in red will cancel. This implies that

$$\nabla_t^2 \mathbf{\Pi} \sim \left[N - \frac{p}{2i}(\text{Im } t)^{-1}\right](2\pi i q + N)e^{2\pi i q t}\mathbf{\Pi}_q. \tag{3.109}$$

Noting that $N\mathbf{\Pi}_q \sim t^{-1}\mathbf{\Pi}_q$, we see that the leading behaviour is given by (we now drop unimportant prefactors again)

$$\nabla_t^2 \mathbf{\Pi} \sim pq(\text{Im } t)^{-1}\, e^{2\pi i q t}\mathbf{\Pi}_q. \tag{3.110}$$

Here we remind the reader that $p, q > 0$. This then implies that

$$F_{t\bar{t}t\bar{t}} = ||\nabla_t^2 \mathbf{\Pi}||^2 \sim (\text{Im } t)^{-2}\, e^{2\pi i q(t-\bar{t})}||\mathbf{\Pi}_q||^2 \sim (\text{Im } t)^{-2} g_{t\bar{t}}. \tag{3.111}$$

One can perform similar calculations to obtain the other components of $F_{i\bar{j}k\bar{l}}$. Since the details are not as intricate as the case of $F_{t\bar{t}t\bar{t}}$ we do not give the exact calculation here, but merely state the results:

$$g^{t\bar{t}}g^{t\bar{t}}F_{t\bar{t}t\bar{t}} \sim r^{-2-p}e^{2qr} \tag{3.112}$$

$$\left[g^{t\bar{t}}g^{\zeta\bar{\zeta}} + g^{t\bar{\zeta}}g^{\bar{t}\zeta}\right]F_{t\bar{t}\zeta\bar{\zeta}} \sim 1 + r^p e^{-2qr} \tag{3.113}$$

$$g^{t\bar{\zeta}}g^{t\bar{\zeta}}F_{t\bar{\zeta}t\bar{\zeta}} \sim r^p e^{-2qr} \tag{3.114}$$

$$g^{\zeta\bar{\zeta}}g^{\zeta\bar{\zeta}}F_{\zeta\bar{\zeta}\zeta\bar{\zeta}} \sim 1. \tag{3.115}$$

We see that the dominant term is provided by $F_{t\bar{t}t\bar{t}}$, which is exponentially large. At first glance, we would therefore expect the index density to be exponentially large when $m > 1$. However, we have neglected one aspect of the derivation of the index density. Indeed, in the result (3.62) we have suppressed an additional anti-symmetrization that should be performed. This comes from the original expression $\det(R + \omega)$, which contains both a determinant and also the wedging of two-forms, as discussed in section (2.5). Let us illustrate the consequences of this for a two-dimensional moduli space. For $m = 2$ the term $I_2$ containing two copies of $F_{i\bar{j}k\bar{l}}$ is given by

$$e^{-2K}I_2 = \delta^{[i_1}_{j_1}\delta^{i_2]}_{j_2}\delta^{[k_1}_{l_1}\delta^{k_2]}_{l_2}g^{j_1\bar{j}_1}g^{j_2\bar{j}_2}g^{l_1\bar{l}_1}g^{l_2\bar{l}_2}F_{i_1\bar{j}_1k_1\bar{l}_1}F_{i_2\bar{j}_2k_2\bar{l}_2}. \tag{3.116}$$

Note that this expression is anti-symmetrized w.r.t. $i_1 \leftrightarrow i_2$ and $k_1 \leftrightarrow k_2$. We will now argue that the leading order behaviour is given by

$$I_2 \sim g^{t\bar{t}}g^{t\bar{t}}F_{t\bar{t}t\bar{t}}\left[g^{\bar{\zeta}\zeta}g^{\bar{\zeta}\zeta}F_{\zeta\bar{\zeta}\zeta\bar{\zeta}}\right], \tag{3.117}$$

To this end, we recall that $F_{t\bar{t}t\bar{t}}$ is the dominant component of $F_{i\bar{j}k\bar{l}}$. Without loss of generality, let us therefore assume that $i_1 = k_1 = t$ and $\bar{j}_1 = \bar{l}_1 = \bar{t}$. We make two observations. First, we note that the antisymmetry of $i_1 \leftrightarrow i_2$ and $k_1 \leftrightarrow k_2$ implies that $i_2 = k_2 = \zeta$. Secondly, using the fact that

$$\delta^{[i_1}_{j_1}\delta^{i_2]}_{j_2} = \delta^{i_1}_{[j_1}\delta^{i_2}_{j_2]} \tag{3.118}$$

we see that the expression $g^{j_1\bar{j}_1}g^{j_2\bar{j}_2}$ must be anti-symmetric under $j_1 \leftrightarrow j_2$. This implies that the only term contributing in the contraction is the one where $\bar{j}_1$ is different from $\bar{j}_2$. A similar result holds for the terms containing $l_1, l_2$. Hence we conclude $\bar{j}_2 = \bar{l}_2 = \bar{\zeta}$. We are then left with

$$I_2 \sim \delta^t_{[j_1}\delta^\zeta_{j_2]}\delta^t_{[l_1}\delta^\zeta_{l_2]}g^{j_1\bar{t}}g^{j_2\bar{\zeta}}g^{l_1\bar{t}}g^{l_2\bar{\zeta}}F_{\bar{t}t\bar{t}\bar{t}}F_{\zeta\bar{\zeta}\zeta\bar{\zeta}} \tag{3.119}$$

There are three ways to perform the contraction of the indices, namely

$$g^{t\bar{t}}g^{\zeta\bar{\zeta}}g^{t\bar{t}}g^{\zeta\bar{\zeta}}, \quad \text{or} \quad g^{t\bar{t}}g^{\zeta\bar{\zeta}}g^{\zeta\bar{t}}g^{t\bar{\zeta}}, \quad \text{or} \quad g^{\zeta\bar{t}}g^{t\bar{\zeta}}g^{\zeta\bar{t}}g^{t\bar{\zeta}}. \tag{3.120}$$

However, we can again use the anti-symmetry under $j_1 \leftrightarrow j_2$ and $l_1 \leftrightarrow l_2$ to conclude that the second and third option cannot occur.[9] Alternatively, one can also argue that the first option is in fact the leading behaviour when considering the particular forms of the inverse metric. Hence we find

$$I_2 \sim g^{t\bar{t}}g^{t\bar{t}}F_{\bar{t}t\bar{t}\bar{t}}\left[g^{\bar{\zeta}\zeta}g^{\bar{\zeta}\zeta}F_{\zeta\bar{\zeta}\zeta\bar{\zeta}}\right] \sim g^{t\bar{t}}g^{t\bar{t}}F_{t\bar{t}t\bar{t}} \tag{3.121}$$

as promised. In the case of general $m$, the above analysis can be summarized as follows. The factor $I_m$ containing $m$ copies of $F$ can be written as

$$I_m = \delta^{[i_1}_{j_1}\cdots\delta^{i_m]}_{j_m}\delta^{[k_1}_{l_1}\cdots\delta^{k_m]}_{l_m}g^{j_1\bar{j}_1}\cdots g^{j_m\bar{j}_m}g^{l_1\bar{l}_1}\cdots g^{l_m\bar{l}_m}F_{i_1\bar{j}_1k_1\bar{l}_1}\cdots F_{i_m\bar{j}_mk_m\bar{l}_m}. \tag{3.122}$$

Moreover, the anti-symmetry argument comes down to the following statement: the indices $\alpha_1, \ldots, \alpha_m$ appearing in the $F$'s must all be different, for each fixed $\alpha \in \{i, j, k, l\}$. As a result, the index density becomes

$$d\mu \sim dt \wedge d\bar{t}\ \sqrt{g}g^{t\bar{t}}g^{t\bar{t}}F_{t\bar{t}t\bar{t}}\left[F_{....}\cdots F_{....}g^{..}\cdots g^{..}\right] \tag{3.123}$$

with the term in brackets generically being of order 1. Most importantly, the $F$'s appearing in brackets cannot have $t$ or $\bar{t}$ indices. Using the expression for $F_{t\bar{t}t\bar{t}}$ we obtained earlier, we finally arrive at

$$d\mu \sim \sqrt{g}g^{t\bar{t}}g^{t\bar{t}}F_{t\bar{t}t\bar{t}}\,dt\wedge d\bar{t} \sim g^{t\bar{t}}F_{t\bar{t}t\bar{t}}\,dt\wedge d\bar{t} \sim \frac{dt\wedge d\bar{t}}{(\text{Im}\, t)^2}, \tag{3.124}$$

which is exactly the same result as we found in the case $N\mathbf{a}_0 \neq 0$.

Again, for the interested reader we note that one can perform this analysis a bit more carefully by fully keeping track of the pre-factors that arise in the index density. Here the (surprising) result is that it is simply given by $-4p^2$ together with factors of $g^{\zeta\bar{\zeta}}$ and $F_{\zeta\bar{\zeta}\zeta\bar{\zeta}}$ about which we do not know much, i.e. to leading order

$$d\mu \sim p^2 \times \frac{dt\wedge d\bar{t}}{(t-\bar{t})^2}. \tag{3.125}$$

Here we remind the reader that $p$ is the largest integer such that $N^p\mathbf{a}_q \neq 0$, akin to the integer $d$ for $\mathbf{a}_0$. Note that the pre-factor does not depend on $q$. It would be interesting to investigate this in more detail to understand the qualitative different with the $N\mathbf{a}_0 \neq 0$, but this is beyond the scope of this work.

[9] Here it seems that we are using the fact that we only consider $m = 2$. However for general $m$ it is still true that $g^{t\bar{\zeta}_1} = g^{t\bar{\zeta}_2}$ so this argument generalizes.

## 3.6 Discussion

In this section we will make a couple of observations regarding the results we have derived in the preceding section. We consider the integrability of the index density, its relation to the exact expression for the torus obtained in section 2.3.3 and discuss some aspects of a dual description in terms of a Yang-Mills theory.

Our first observation is that the index density is integrable around $\mathrm{Im}\, t \to \infty$. This is most easily seen by first returning to the coordinate $z = e^{2\pi i t}$, in terms of which the index density is given by

$$d\mu \sim \frac{dz \wedge d\bar{z}}{|z|^2 \log^2 |z|}. \tag{3.126}$$

Passing to a radial coordinate $r = |z|$ with $0 < r < R$, one readily sees that

$$\int_0^R \frac{dr}{r \log^2 r} \sim \frac{1}{\log(1/R)}, \tag{3.127}$$

which is a finite result. Here we stress that this only gives an indication of the index of flux vacua near such a singularity, but as we discussed in chapter 2 this is precisely where the intricacies regarding finiteness lie. In short, we have obtained a definite answer to Q3, namely:

> A3: The AD-density is integrable near any singular loci where a single coordinate $\mathrm{Im}\, t$ in the complex structure moduli space becomes large. More precisely, it takes the universal form
>
> $$d\mu \sim \frac{dt \wedge d\bar{t}}{(\mathrm{Im}\, t)^2}. \tag{3.128}$$

In fact, this answer agrees with the result obtained in [14, p. 17], where a similar computation was performed for the conifold limit. Additionally, the result (3.126) also agrees with the findings of Eguchi and Tachikawa [22], where more general limits were considered in the threefold setting. The upshot is that we have reproduced the results known in the literature for Calabi-Yau threefolds, and generalized them to Calabi-Yau $D$-folds, for any one-parameter limit. However, let us spend a few words on how the results of Eguchi and Tachikawa can be obtained as specific cases of our calculations.

First, when $D = 3$, the conifold is a particular singularity (Type I, or $N\mathbf{a}_0 = 0$) which is characterized by a period vector and monodromy of the form

$$\mathbf{\Pi} = \begin{pmatrix} \frac{m}{2\pi i} z \log z \\ z \end{pmatrix} + \cdots, \quad N = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \tag{3.129}$$

in the limit where $z \to 0$. Here the dots denote analytic functions of $z$. Note that this is readily written in the form of the nilpotent orbit theorem as

$$\mathbf{\Pi} = \mathbf{a}_0 + z e^{\frac{\log z}{2\pi i} N} \mathbf{a}_1, \quad \mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.130}$$

In other words, this is simply the case $q = 1$ in our derivation for the $N\mathbf{a}_0$ case.

Secondly, again when $D = 3$, for a one-dimensional moduli space the asymptotic behaviour of the period vector in the large complex structure limit is given by

$$\mathbf{\Pi} = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}^T \tag{3.131}$$

which corresponds to the case $N\mathbf{a}_0 \neq 0$, specifically $d = 3$. As we discussed in chapter 2, this is an example of a D-limit, where *a priori* the integrability of the index density is not guaranteed. Happily, the upshot of our calculation is that it *is* integrable, at least for one-parameter limits. To make any statements about the large complex structure limits of arbitrary moduli spaces, we need to understand multi-moduli limits, which we discuss in the next chapter.

A second observation is that the result (3.128) is (to leading order and up to a numerical factor) precisely the index density that was obtained in the simple example of flux compactification on the torus, see section 2.3.3. In the case $N\mathbf{a}_0 \neq 0$ this is to be expected, since the metric components obtained in (3.74) agree with the exact metric on the moduli space of the torus. However, it is somewhat unexpected that the same holds in the case $N\mathbf{a}_0 = 0$, since there the metric components are certainly different from that of the torus.

*Gauge/Gravity Duality of the Conifold*

Lastly, we propose an interesting connection with the results of Klebanov and Strassler in [38]. Here a dual description of the gravitational theory of the (complex) three-dimensional conifold is provided in terms of an $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Let us provide some details of this result. For Type IIB string theory on a conifold, a standard result is that the Kähler potential and the superpotential induced by $F_3,\ H_3$ fluxes is given by

$$W = -L\tau z + M\frac{z}{2\pi i}\log z, \quad L = -\log\left[-i|z|^2\log^2|z|\right], \tag{3.132}$$

where $z$ is the complex structure modulus parametrizing the size of the vanishing cycle and $M, -L$ denote the flux quanta of $F_3,\ H_3$. In particular, assuming $|z| \ll 1$ and additionally $L/g_s \gg C_0$, one can solve $D_zW = 0$ to obtain a vacuum given by

$$|z| = e^{-2\pi L/Mg_s}. \tag{3.133}$$

Now the result of [38] is that the conifold has a dual description given by a certain $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theory, where $N = LM$. The string coupling constant $g_s$ is related to the 't Hooft coupling constant of this gauge theory by

$$Mg_s = \lambda = g_{\mathrm{YM}}^2 N. \tag{3.134}$$

In particular, the limit where $L/g_s$ is large corresponds precisely to the large $N$ limit in the dual theory (at fixed $\lambda$). The relation to our results regarding the index density arises by observing that

$$|z| = e^{-b/g_{\mathrm{YM}}^2}, \quad b = \frac{2\pi}{M}, \tag{3.135}$$

which implies that

$$d\mu \sim \frac{d|z|}{|z|\log^2|z|} \sim dg_{\mathrm{YM}}^2. \tag{3.136}$$

Interestingly, the index density is uniform when expressed in terms of $g_{\mathrm{YM}}^2$, which one may interpret as an indication that the dual theory provides a more natural description of the conifold. The interesting fact is that we have obtained exactly the same form of the index density for general one-parameter limits in the moduli space of a Calabi-Yau $D$-fold, as opposed to the specific case of the three-dimensional conifold. This suggests the existence of a dual description of these general limits as well, in the same spirit as above. However, it is beyond the scope of this work to investigate such a duality in detail. It is expected to be challenging, since the exact relation between the singularity of the moduli space and that of the internal Calabi-Yau can be complicated. As such it is not clear at all whether the Calabi-Yau singularity allows for a similar D3-brane construction.

# Chapter 4

# On to Multi-Moduli Limits

In chapter 3 we have provided the general framework used to describe the degeneration of the Hodge structure of $H^D$ at singular loci. Of utmost importance are the nilpotent orbit approximation of the period vector and the properties of the Deligne splitting, most notably the polarization conditions. Using these we were able to obtain the scaling of the index density near singular loci consisting of a single divisor $\Delta_1^\circ$, i.e. where a single parameter is sent to the limit $\text{Im}\ t \to \infty$. The goal of this chapter is to provide more insight into the structure of general singular loci, consisting of higher intersections of divisors $\Delta_{1\cdots k}^\circ$. One of the intricacies that arises when considering multiple-parameter limits is the fact that the results are path-dependent. This is illustrated in figure 4.1, where the complex structure moduli space is depicted in the $t^i$-coordinates, such that the singular loci correspond to the boundary of the moduli space. To be certain that the number of flux vacua is finite, we must check that there are no paths to the boundary that pick up an infinite number of vacua.

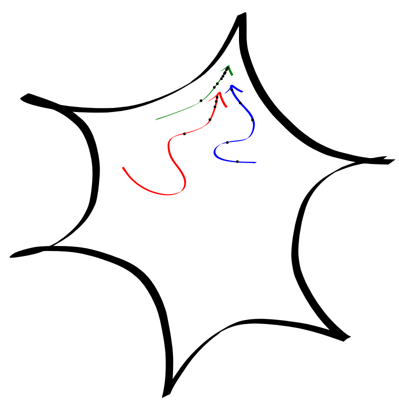


**Figure 4.1:** A depiction of the complex structure moduli space in the $t^i$-coordinates, where the singular loci are seen as the boundary of the space. Three different paths are shown, each passing along some set of flux vacua. The question of interest is whether there exists a path which picks up infinitely many vacua when approaching the boundary.

Clearly, this greatly complicates the question compared to the single-parameter limits. As such, our results rely heavily on the usage of some involved mathematical structures, which is encompassed by the SL(2)-orbit theorem. We discuss its implications in section 4.1. In section 4.2 we turn to the subject of singularity enhancement to gain more insight into the structure of the singular loci $\Delta_{1\cdots k}^\circ$ and in particular their associated Deligne diamonds. Additionally, we motivate and state the growth theorem, which will ultimately provide us with the proper generalization of the scaling of the index density, as we discuss in section 4.3. Note that we will only be able to provide results in a particular case, as will become apparent. Finally, in section 4.4 we relate our results to those found in the mathematics literature and comment on the intricacies that arise when dealing with the most general case and how one might attempt to overcome them.

## 4.1 The SL(2)-Orbit Theorem

Recall that near the intersection of $n_P$ divisors $\Delta_i$ the nilpotent orbit theorem provides an approximation of the period vector by

$$\mathbf{\Pi} = \exp\left[\sum_{i=1}^{n_P} t^i N_i\right] \mathbf{a}_0. \tag{4.1}$$

In chapter 3 we were able to compute the Kähler potential given by

$$e^{-K} = ||\mathbf{\Pi}||^2, \tag{4.2}$$

but only did so in the case of a single divisor, i.e. $n_P = 1$. When turning to the intersection of multiple intersections, a number of complications arise. Most notably, the results highly depend on the specific path taken towards $\Delta^\circ_{1\cdots n_P}$, i.e. which parameters grow the quickest. This is dealt with by introducing a specific growth sector

$$\mathcal{R}_{1\cdots n_P} := \left\{\frac{y^1}{y^2} > \lambda, \dots, \frac{y^{n_P-1}}{y^{n_P}} > \lambda, y^{n_P} > \lambda, x^i < \delta\right\}, \quad t^j = x^j + iy^j \tag{4.3}$$

We stress that our results will hold within this particular sector, as will become apparent when we introduce the SL(2)-orbit. A second complication that arises is the question of how to deal with the various matrices $N_i$. In the one-modulus case, the matrix $N$ had the simple interpretation as a lowering operator, since

$$NI^{p,q} \subseteq I^{p-1,q-1}. \tag{4.4}$$

However, in the multi-modulus case, this property only holds for $N = N_{(n_P)}$, since this is the matrix used to construct the monodromy weight filtration, and in turn the Deligne splitting. As such, it is unclear what exactly the individual matrices $N_i$ do. This is an issue, since whenever we perform a derivative of $\mathbf{\Pi}$ with respect to $t^i$, we will bring down a factor of $N_i$ acting on $\mathbf{a}_0$. To have any hope of estimating the scaling of the index density, which ultimately is expressed in terms of various derivatives of $\mathbf{\Pi}$, we need to deal with this issue. As is often the case, a slight change of perspective will provide us with the answer. The idea is as follows: by 'rotating' the mixed Hodge structure $F_\Delta$ (i.e. performing a change of basis) we will be able to replace the data $(\mathbf{a}_0, N_i)$ with some $(\tilde{\mathbf{a}}_0, N_i^-)$, where the matrices $N_i^-$ are part of $\mathfrak{sl}(2)$-triples whose action on the Deligne splitting and $\tilde{\mathbf{a}}_0$ is more easily understood. Let us discuss this in some detail. We again follow the discussion in [28] and refer the reader to [8], specifically Theorem 4.20, for the mathematical details.

*Uncovering the $\mathfrak{sl}(2)$-algebras*

When we first introduced the Deligne splitting in section 3.3, one of its defining properties was

$$\overline{I^{p,q}} = I^{q,p} \bmod \bigoplus_{r<q,s<p} I^{r,s}. \tag{4.5}$$

We described the complicated 'mod' factor as an alternative to working with the quotient spaces $\mathrm{Gr}_i$. We call a splitting $I^{p,q}$ $\mathbb{R}$-split if it satisfies

$$\overline{I^{p,q}} = I^{q,p}. \tag{4.6}$$

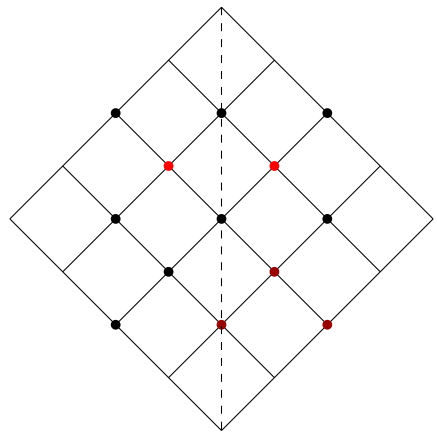


**Figure 4.2:** The property 4.5 is exemplified for a Type III singularity. We have highlighted the spaces $I^{3,2}$ and $\overline{I^{3,2}} = I^{2,3}$ mod $\left(I^{1,1} \oplus I^{0,2} \oplus I^{1,2}\right)$, where the latter is obtained by reflecting $I^{3,2}$ in the vertical axis and modding out the lower spaces.

Of course, the mixed Hodge structure $(F, W)$ does not necessarily induce an $\mathbb{R}$-split Deligne splitting. However, one of the main results of the SL(2)-orbit theorem is that we can always make an $\mathbb{R}$-split Deligne splitting $\tilde{I}^{p,q}$ by performing a rotation of the original mixed Hodge structure:

$$\tilde{F} := e^{\zeta} e^{-i\delta} F, \tag{4.7}$$

for two matrices $\zeta$ and $\delta$. We note that $\zeta$ can be expressed in terms of $\delta$ as described in [28, App. B]. Here $\delta$ can be decomposed as

$$\delta = \sum_{p,q>0} \delta_{-p,-q}, \tag{4.8}$$

where each $\delta_{-p,-q}$ acts on $\tilde{I}^{r,s}$ as

$$\delta_{-p,-q}(\tilde{I}^{r,s}) \subseteq \tilde{I}^{r-p,s-q}. \tag{4.9}$$

To gain some insight into this, we refer the reader to figure 4.2, where the property 4.5 is illustrated. We see that $\overline{I^{p,q}} = I^{q,p}$ up to elements lying in the 'lower spaces'. On the other hand, the action of $\delta$ precisely rearranges these elements in such a way that the resulting $\tilde{I}^{p,q}$ is $\mathbb{R}$-split. We will refer to $(\tilde{F}, W)$ as the SL(2)-splitting of $(F, W)$, for reasons we now describe.

The first advantage of working in the SL(2)-splitting is that one can construct a set of commuting $\mathfrak{sl}(2)$-triples which act nicely on the $\tilde{I}^{p,q}$ spaces. We fix an ordering $(N_1, \ldots, N_{n_\mathcal{E}})$ of the nilpotent monodromy matrices. The result is that we obtain matrices $Y_i, N_i^{\pm}$ satisfying the $\mathfrak{sl}(2)$-algebra

$$[Y_i, N_i^{\pm}] = \pm 2N_i^{\pm}, \quad [N_i^+, N_i^-] = Y_i \tag{4.10}$$

which are moreover pairwise commuting. The action of these matrices on the SL(2)-splitting is given by

$$Y_{(k)} \tilde{I}^{p,q}(\Delta^\circ_{1\cdots k}) = (p + q - D)\tilde{I}^{p,q}(\Delta^\circ_{1\cdots k}) \tag{4.11}$$

and

$$N_k^{\pm} \tilde{I}^{p,q}(\Delta^\circ_{1\cdots k}) \subset \tilde{I}^{p\pm1,q\pm1}(\Delta^\circ_{1\cdots k}). \tag{4.12}$$

In other words, for a given $v \in \tilde{I}^{p,q}(\Delta^\circ_{1\cdots k})$, we see that $Y_{(k)}$ records where $v$ lies in the Deligne diamond of $\Delta^\circ_{1\cdots k}$. Moreover, the operators $N_k^{\pm}$ act as raising/lowering operators, sending $v$ up/down two rows in $\Delta^\circ_{1\cdots k}$ respectively. This is precisely what we set out to achieve at the start of this section.

*The Nilpotent Orbit vs. The SL(2)-Orbit*

The second advantage of working in the SL(2)-splitting is that there is an alternative way to express the asymptotic behaviour of the period vector, via the so-called SL(2)-orbit $\mathbf{\Pi}_{\mathrm{SL}(2)}$. It is related the the nilpotent orbit as follows

$$\mathbf{\Pi}_{\mathrm{nil}} = \exp\left[-\sum_{j=1}^{n_{\mathcal{E}}} t^j N_j\right]\mathbf{a}_0(\zeta) = \exp\left[-\sum_{j=1}^{n_{\mathcal{E}}} x^j N_j\right]\cdot M(y)\cdot \mathbf{\Pi}_{\mathrm{SL}(2)}, \quad t^j = x^j + iy^j \tag{4.13}$$

where

$$\mathbf{\Pi}_{\mathrm{SL}(2)} = \exp\left[-i\sum_{j=1}^{n_{\mathcal{E}}} y^j N_j^-\right]\tilde{\mathbf{a}}_0^{(n\varepsilon)} \tag{4.14}$$

and $M(y)$ is a $y$-dependent matrix which has a power-series expansion in terms of non-positive powers of $y^1/y^2, \ldots, y^{n_P}/y^{n_P-1}$ which is convergent precisely in the growth sector $\mathcal{R}_{1\cdots n_P}$. The upshot of this is as follows: within a chosen growth sector, the SL(2)-orbit approximates the period vector up to polynomially suppressed terms. Compare this with the nilpotent orbit approximation which is valid up to exponentially suppressed terms. Apart from this, the SL(2)-orbit is very similar to the nilpotent orbit, expect it replaces $N_j$ with $N_j^-$ and $\mathbf{a}_0$ with $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$.

To see the usefulness of the SL(2)-orbit, let us give a slightly heuristic argument for computing $||\mathbf{\Pi}||^2$ in this approximation, i.e. we set

$$||\mathbf{\Pi}||^2 \sim ||\mathbf{\Pi}_{\mathrm{SL}(2)}||^2 \sim \left(\exp\left[-2i\sum_{j=1}^{n_{\mathcal{E}}} y^j N_j^-\right]\tilde{\mathbf{a}}_0^{(n\varepsilon)}, \overline{\tilde{\mathbf{a}}_0^{(n\varepsilon)}}\right). \tag{4.15}$$

By expanding the exponential, we see that each factor of $y^j$ in $||\mathbf{\Pi}||^2$ comes with a factor of $N_j^-$ acting on $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$. If we want to find the leading contribution to $||\mathbf{\Pi}||^2$, within the growth sector $R_{1\cdots n_P}$, we therefore want to maximize the number of $N_1^-$'s first, then the number of $N_2^-$'s, etc... To find these numbers, we introduce integers $d_i$ such that

$$(N_{(i)}^-)^{d_i}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0, \quad (N_{(i)}^-)^{d_i+1}\tilde{\mathbf{a}}_0^{(n\varepsilon)} = 0. \tag{4.16}$$

Recall from the discussion in chapter 3 that the integers $d_i$ label the singularity type of $\Delta^\circ_{(i)}$. We then make the following set of observations. First, $d_1$ is the largest integer such that

$$(N_1^-)^{d_1}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0. \tag{4.17}$$

Second, $d_2$ is the largest integer such that

$$(N_1^- + N_2^-)^{d_2}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0. \tag{4.18}$$

Expanding the LHS we see that $d_2$ is also the largest integer such that

$$(N_1^-)^{d_1}(N_2^-)^{d_2-d_1}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0. \tag{4.19}$$

Doing this once more, we see that $d_3$ is the largest integer such that

$$(N_1^-)^{d_1}(N_2^-)^{d_2-d_1}(N_3^-)^{(d_3-d_1)-(d_2-d_1)}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0, \tag{4.20}$$

or

$$(N_1^-)^{d_1}(N_2^-)^{d_2-d_1}(N_3^-)^{d_3-d_2}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0. \tag{4.21}$$

Generally, we find that the $d_i$ are the highest integers such that

$$(N_1^-)^{d_1}(N_2^-)^{d_2-d_1}\cdots(N_{n_P}^-)^{d_{n_P}-d_{n_P-1}}\tilde{\mathbf{a}}_0^{(n\varepsilon)} \neq 0. \tag{4.22}$$

In conclusion, we find that

$$||\mathbf{\Pi}||^2 \sim y_1^{d_1} y_2^{d_2-d_1}\cdots y_{n_P}^{d_{n_P}-d_{n_P-1}}. \tag{4.23}$$

We remark that this is a polynomial of order $d_{n_P}$, as expected (by definition of $d_{n_P}$) since we are considering an intersection of $n_P$ divisors. Moreover, this computation has shed some light on how the singularity structure of $\Delta^\circ_{(n_P)}$ is related to the singularity structures of the lower divisors $\Delta^\circ_{(i)}$, with $1 \leq i \leq n_P$. As far as determining the scaling of the index density, the result (4.23) is already a good starting point. However, to gain more insight into this setting and better understand the multi-parameter degenerations, we will spend some more words on the above relation by discussing the process of singularity enhancement. We will also see that the result (4.23) is in fact a specific case of the *growth theorem*.

## 4.2 Singularity Enhancements and the Growth Theorem

### 4.2.1 Singularity Enhancements

Let us consider a locus $\Delta^\circ_1$ with a given singularity type Type($\Delta^\circ_1$), specified by the integer $d_{(1)} = 0, \ldots, D$. One might imagine moving along $\Delta^\circ_1$ until reaching an intersection $\Delta^\circ_{12}$. At this intersection a second coordinate Im $t^2$ is sent to infinity and we say that the singularity is enhanced to Type($\Delta^\circ_{12}$). Tremendous work has been done in [28, 53] to classify the allowed enhancements for Calabi-Yau threefolds and fourfolds. One of the more intuitive results is that an enhancement can only 'worsen' the singularity, i.e. $d_1 \leq d_2$. In this section we will give some details of such enhancements and in particular focus on its effect on the Deligne diamonds. For simplicity we will stick to an enhancement of the form

$$\text{Type}(\Delta^\circ_1) \to \text{Type}(\Delta^\circ_{12}), \tag{4.24}$$

since the main features will naturally generalize to higher divisors. Recall that upon reaching the locus $\Delta^\circ_1$, the pure Hodge structure of $H^D$ breaks down and we instead require the Deligne splitting to capture the full space (here we illustrate it for fourfolds):

$$(H^{4,0},\ H^{3,1},\ H^{2,2},\ H^{1,3},\ H^{0,4}) \to \begin{matrix} &&&& I^{4,4} &&&& \\ &&& I^{4,3} && I^{3,4} &&& \\ && I^{4,2} && I^{3,3} && I^{4,2} && \\ & I^{4,1} && I^{3,2} && I^{2,3} && I^{1,4} & \\ I^{4,0} && I^{3,1} && I^{2,2} && I^{1,3} && I^{0,4} \\ & I^{3,0} && I^{2,1} && I^{1,2} && I^{0,3} & \\ && I^{2,0} && I^{1,1} && I^{0,2} && \\ &&& I^{1,0} && I^{0,1} &&& \\ &&&& I^{0,0} &&&& \end{matrix} \tag{4.25}$$

or more schematically

$$H^D \mapsto [I^{p,q}]^D(\Delta^\circ_1), \quad 0 \leq p, q \leq D. \tag{4.26}$$

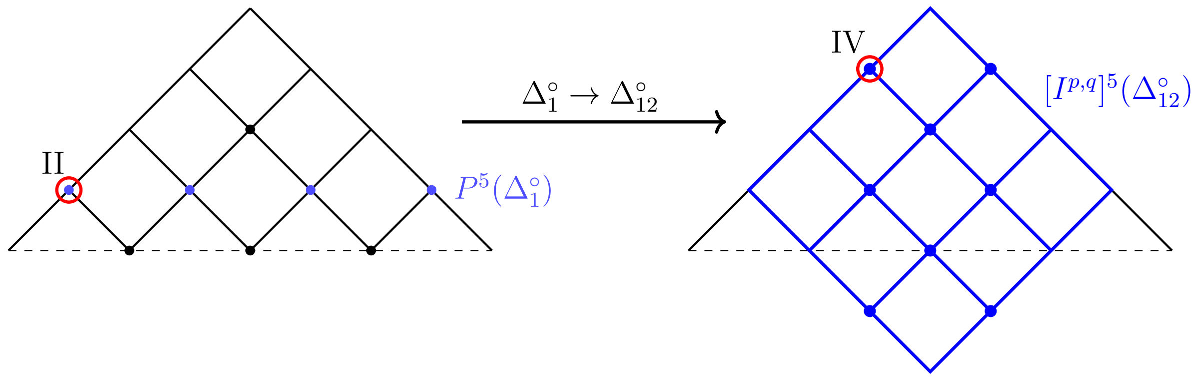



**Figure 4.3:** A Type II → Type IV enhancement is shown at the level of the Deligne diamonds. For convenience only depict the upper half of the diamond. On the LHS we have highlighted the horizontal primitive space $P^5(\Delta_1^\circ)$ which upon enhancement attains a mixed Hodge structure described by the Deligne diamond $[I^{p,q}]^5(\Delta_{12}^\circ)$. Note that we have only displayed the inner part of the latter diamond, since the outer ring only consists of empty vector spaces.

To understand what happens once we move to $\Delta_{12}^\circ$, we recall that from the Deligne splitting we constructed the primitive spaces $P^{p,q}$ which form a pure Hodge structure of weight $p+q=l$ for each $P^l$. The idea of the enhancement is that upon reaching $\Delta_{12}^\circ$, all of $P^l$ spaces split into an even finer Deligne splitting in exactly the same way $H^D$ did upon reaching $\Delta_1^\circ$:

$$P^l(\Delta_1^\circ) \to [I^{p,q}]^l(\Delta_{12}^\circ), \quad 0 \le p, q \le l. \tag{4.27}$$

Pictorially, the primitive part of each horizontal row of the Deligne diamond $I^{p,q}(\Delta_1^\circ)$ forms its own Deligne diamond $[I^{p,q}]^l(\Delta_{12}^\circ)$, as is exemplified in figure 4.3 for a II→ IV enhancement. On the other hand, we could have also analysed the singularity structure of $\Delta_{12}^\circ$ directly by the procedure outlined in section 3.3, leading to yet another Deligne splitting $I^{p,q}(\Delta_{12}^\circ)$. Whether a particular singularity enhancement is allowed is then translated into the question of whether the diamond of $I^{p,q}(\Delta_{12}^\circ)$ can be obtained by summing up the various diamonds of $[I^{p,q}]^l(\Delta_{12}^\circ)$, for $0 \le l \le 2D$, in a particular manner. The details of this procedure are described in [28, 53] for threefolds and fourfolds, respectively. However, for our purposes it will suffice to understand what happens with $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ under an enhancement.

### 4.2.2 The Growth Theorem

In section 3.2 we introduced a bilinear form on $H^D$ given by

$$S(v,w) = \int_{Y_D} v \wedge w, \tag{4.28}$$

and used the notation $||v||^2 = S(v,\bar{v})$. This resembles the *Hodge norm*, defined by

$$||v||_\star^2 := \int_{Y_D} v \wedge \star v, \tag{4.29}$$

where $\star$ is the Hodge star on $Y_D$. The relation between the two is made precise by introducing the Weil operator $C$, which acts as

$$v \in H^{p,q}: \quad Cv = i^{p-q} v. \tag{4.30}$$

Indeed, we can express the Hodge norm in terms of the inner product $S$ and the Weil operator $C$ as

$$||v||_\star^2 = S(Cv, \bar{v}). \tag{4.31}$$

Most importantly, since $\Omega \in H^{D,0}$, we see that the two norms coincide (up to a factor):

$$||\Omega||^2 = i^D ||\Omega||_\star^2. \tag{4.32}$$

In fact, since $\nabla_i \Omega \in H^{D-1,1}$ etcetera, this remains true when acting on $\Omega$ with the covariant derivative $\nabla_i$. When we investigated the scaling of the index density in 3.5, we often expressed our results in terms of the $||\cdot||$ norm[1] of $\mathbf{\Pi}$ or various covariant derivatives thereof. Since for such expressions the norm agrees with the Hodge norm, it may instead be fruitful to investigate the latter in more detail. To this end, we summarize the results of the growth theorem of [7].

We consider the intersection of $n_P$ divisors $\Delta_1, \dots, \Delta_{n_P}$ located at $y^i = \infty$. Let us moreover choose an ordering

$$\text{chosen ordering}: \quad N_1, \dots, N_{n_P}, \tag{4.33}$$

which corresponds to the growth sector $R_{1\cdots n_P}$. Essentially, one can view this as first moving along $\Delta_1^\circ$ to its intersection $\Delta_{12}^\circ$ with $\Delta_2^\circ$, then onto its intersection $\Delta_{123}^\circ$ with $\Delta_3^\circ$, etcetera. Then consider a $D$-form $v$ whose position within the Deligne diamonds of each $\Delta_{1\cdots j}^\circ$ is specified by

$$v \in W_{l_1}(N_{(1)}) \cap \dots \cap W_{l_{n_P}}(N_{(n_P)}), \tag{4.34}$$

where each $l_j$ is the lowest integer such that $v \in W_{l_j}(N_{(j)})$ (recall that $W_j \subseteq W_{j+1}$). Pictorially, each $l_j$ labels the highest row in the Deligne diamond of $\Delta_{1\cdots j}^\circ$ where $v$ has a non-zero part. The growth theorem now states that the leading growth of the Hodge norm of $v$ is given by

$$\boxed{||v||_\star^2 \sim (y^1)^{l_1 - D}(y^2)^{l_2 - l_1} \cdots (y^{n_P})^{l_{n_P} - l_{n_P - 1}}} \tag{4.35}$$

Note the resemblance with (4.23). In fact, we can make the similarity precise by deducing the position of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ within the various Deligne diamonds.

*Application to $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ and a Visualization*

In analogy to the one-modulus case, where the integer $d$ satisfying

$$N^d \mathbf{a}_0 \neq 0, \quad N^{d+1}\mathbf{a}_0 = 0 \tag{4.36}$$

labelled the position of $\mathbf{a}_0$ to be in $I^{D,d}$, see section 3.3, in the multi-moduli case the position of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ is labelled by the integers $d_i$ introduced in section 4.1

$$\tilde{\mathbf{a}}_0^{(n\varepsilon)} \in W_{d_1+D}(N_{(1)}^-) \cap \dots \cap W_{d_{n_P}+D}(N_{(n_P)}^-). \tag{4.37}$$

Indeed, implies that $\tilde{\mathbf{a}}_0^{(n\varepsilon)} \in \tilde{I}^{D,d_i}(\Delta_{1\cdots i}^\circ)$ for $1 \leq i \leq n_P$. Put differently, we have

$$Y_{(i)}\tilde{\mathbf{a}}_0^{(n\varepsilon)} = d_i \tilde{\mathbf{a}}_0^{(n\varepsilon)}, \tag{4.38}$$

[1] Note that stricly speaking $||\cdot||$ is not a norm, since it is not positive-definite. For simplicity we will nevertheless refer to it as a norm.

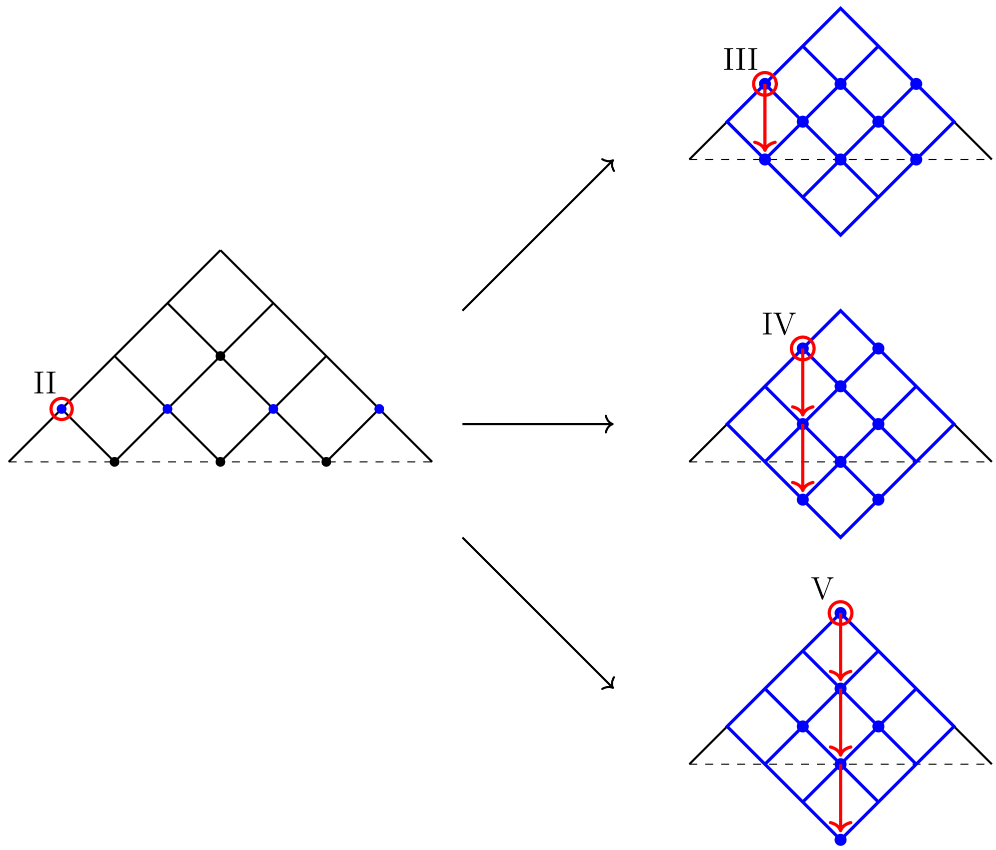



**Figure 4.4:** The various positions of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ across enhancements II→ III, IV, V are shown. On the RHS the action of $N_2^-$ is shown in red. Note that the number of times $N_2^-$ can act non-trivially on $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ is precisely given by one, two and three for the cases III, IV and V, respectively. This corresponds to the integers $d_2 - d_1$.

hence we will also refer to these as the $\mathfrak{sl}(2)$-eigenvalues of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$. It is shown in [7] that $W(N_{(i)}^-) = W(N_i)$, hence we may use the growth theorem to find the growth of the Hodge norm of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, which immediately results in

$$||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||_\star^2 \sim (y^1)^{d_1}(y^2)^{d_2-d_1}\cdots(y^{n_P})^{d_{n_P}-d_{n_P-1}}, \tag{4.39}$$

which is precisely the result we obtained in (4.23). To understand this result from a more graphical point of view, let us make a connection between (4.39) and the singularity enhancements we described earlier. We refer the reader to figure 4.4, where the degeneration of $P^5$, containing $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, to a mixed Hodge structure described by the blue Deligne diamonds is depicted. By tracking the position of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, we see that these are Type II → III, IV, V enhancements. The important observation here is that $N_2^-$ acts as a lowering operator in these latter diamonds. Moreover, depending on the final type, $N_2^-$ can act once, twice or thrice on $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ before reaching a trivial space. This number corresponds precisely to the difference $d_2 - d_1$ for each enhancement, which is the factor appearing in (4.39). In other words, the growth theorem tells us that $||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||^2$ contains a power of $y^{i+1}$ given by the number of times $N_{i+1}^-$ can act on $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ within the mixed Hodge structure of the primitive horizontal space of $\Delta_{1\cdots i}^\circ$ containing $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$.

$\mathbf{\Pi}_{\text{nil}}$ *vs.* $\mathbf{\Pi}_{\text{SL}(2)}$

Having obtained the scaling of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, its connection with the scaling of $\mathbf{\Pi}$ needs some further clarification. The final goal of this section is to show that the growth of $\mathbf{\Pi}$ and $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ agree, and in doing so we will re-derive the result of the growth theorem, yielding further insight. We recall the relation

$$\mathbf{\Pi}_{\text{nil}} = M(y) \cdot \mathbf{\Pi}_{\text{SL}(2)}, \tag{4.40}$$

where we have set each $x^i$ to zero since we are interested in the $y$-dependence. Here $M(y)$ can be written as [8, Thm. 4.20(vii)-(viii)]

$$M(y) = \prod_{r=n}^{1} g_r \left( \frac{y^1}{y^{r+1}}, \ldots, \frac{y^r}{y^{r+1}} \right), \tag{4.41}$$

where each $g_r(y^1, \ldots, y^r)$ is a matrix-valued function which has a power series expansion with constant term one in terms of non-positive powers of $y^1/y^2, y^2/y^3, \ldots, y^r$ in the growth sector $\mathcal{R}_{1\cdots r}$.

Inspired by [7, Thm. 4.8] we consider the matrix

$$e(y) = \left( \frac{y^1}{y^2} \right)^{\frac{1}{2}Y_{(1)}} \cdots \left( \frac{y^{n-1}}{y^n} \right)^{\frac{1}{2}Y_{(n-1)}} (y^n)^{\frac{1}{2}Y_{(n)}}. \tag{4.42}$$

Our first goal is to evaluate $e(y)\mathbf{\Pi}_{\text{nil}}$ using (4.40). We proceed in two steps. First, [8, Thm. 4.20(ix)] implies that the matrix-valued function

$${}^e g_r(y^1, \ldots, y^r) = e(y) \cdot g_r(y^1, \ldots, y^r) \cdot e(y)^{-1} \tag{4.43}$$

has a power series expansion with constant term one in terms of non-positive powers of $(y^1/y^2)^{1/2}, \ldots, (y^r)^{1/2}$ in the growth sector $\mathcal{R}_{1\cdots r}$. Using this relation for each $g_r$ in $M(y)$, we have

$$e(y) \cdot \mathbf{\Pi}_{\text{nil}} = {}^e M(y) \cdot e(y) \cdot \mathbf{\Pi}_{\text{SL}(2)}, \tag{4.44}$$

where ${}^e M(y)$ is defined in the same way as $M(y)$ with each $g_r$ replaced by ${}^e g_r$. To compute the action of $e(y)$ on the SL(2)-orbit, we recall the $\mathfrak{sl}(2)$-eigenvalues of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ defined by

$$Y_{(i)} \tilde{\mathbf{a}}_0^{(n\varepsilon)} = d_i \tilde{\mathbf{a}}_0^{(n\varepsilon)}. \tag{4.45}$$

Since $\mathbf{\Pi}_{\text{SL}(2)}$ contains factors of $N_i^-$ acting on $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, we need to understand how they affect its $\mathfrak{sl}(2)$-eigenvalues. To this end, let $v$ be some $D$-form with

$$v \in W_{l_1}(N_{(1)}^-) \cap \cdots \cap W_{l_{n_P}}(N_{(n_p)}^-). \tag{4.46}$$

For given integers $k, i$ we want to evaluate

$$Y_{(k)} N_i^- v, \quad Y_{(k)} = Y_1 + \cdots + Y_k. \tag{4.47}$$

There are two cases. If $k < i$, then $Y_{(k)}$ commutes with $N_i^-$, hence the action of $N_i^-$ does not change the eigenvalue under $Y_{(k)}$. On the other hand, if $k \geq i$, we use the $\mathfrak{sl}(2)$ algebra to see that

$$\begin{aligned} Y_{(k)} N_i^- v &= Y_i N_i^- v + (Y_1 + \cdots + \hat{Y}_i + \cdots + Y_k) N_i^- v, \quad \text{note: hat means `omit'} \\ &= N_i^- Y_i v - 2 N_i^- v + N_i^- (Y_1 + \cdots + \hat{Y}_i + \cdots + Y_k) v \\ &= N_i^- (Y_{(k)} - 2) v \\ &= (l_k - 2) N_i^- v. \end{aligned}$$

In other words, the action of $N_i^-$ lowers all eigenvalues $l_i, \ldots, l_{n_P}$ by two, i.e.

$$N_i^- v \in W_{l_1}(N_{(1)}^-) \cap \cdots \cap W_{l_{i-1}}(N_{(i-1)}^-) \cap W_{l_i-2}(N_{(i)}^-) \cdots \cap W_{l_{n_P}-2}(N_{(n_p)}^-). \tag{4.48}$$

Returning to the specific case of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$, one readily sees that

$$e(y) \cdot (y^j N_j^-)^k \tilde{\mathbf{a}}_0^{(n\varepsilon)} = \left(\frac{y^1}{y^2}\right)^{\frac{1}{2}d_1} \cdots \left(\frac{y^{n-1}}{y^n}\right)^{\frac{1}{2}d_{n-1}} (y^n)^{\frac{1}{2}d_n} (N_j^-)^k \tilde{\mathbf{a}}_0^{(n\varepsilon)} = ||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||_\star^2 \cdot (N_j^-)^k \tilde{\mathbf{a}}_0. \tag{4.49}$$

Essentially, the factor of $y^j$ appearing in $(y^j N_j^-)^k$ cancels precisely against the one in $e(y)$ after evaluating the action of $Y_{(j)}$. The latter expression is just the result of the growth theorem. We see that $e(y)$ acts nicely on $\mathbf{\Pi}_{\mathrm{SL}(2)}$ by bringing out all the $y$-dependence, resulting in

$$e(y) \cdot \mathbf{\Pi}_{\mathrm{nil}} = {}^e M(y) \cdot e(y) \cdot \exp\left[-i \sum_{j=1}^{n\varepsilon} y^j N_j^-\right] \tilde{\mathbf{a}}_0^{(n\varepsilon)} = ||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||_\star^2 \cdot {}^e M(y) \cdot e^{-iN_{(n)}^-} \tilde{\mathbf{a}}_0^{(n\varepsilon)}. \tag{4.50}$$

Finally, let us compute $||\mathbf{\Pi}_{\mathrm{nil}}||^2$ by employing a trick. We insert unity as $e^{-1}(y)e(y)$ and use the properties of the bilinear form $(\cdot,\cdot)$ to move $e^{-1}(y)$ to the other side as follows[2]

$$||\mathbf{\Pi}_{\mathrm{nil}}||^2 = \big(\mathbf{\Pi}_{\mathrm{nil}}, e^{-1}(y) \cdot e(y) \cdot \bar{\mathbf{\Pi}}_{\mathrm{nil}}\big) = \big(e(y) \cdot \mathbf{\Pi}_{\mathrm{nil}},\ e(y) \cdot \bar{\mathbf{\Pi}}_{\mathrm{nil}}\big) = ||e(y) \cdot \mathbf{\Pi}_{\mathrm{nil}}||^2. \tag{4.51}$$

The final expression above is now easily computed using our result for $e(y) \cdot \mathbf{\Pi}_{\mathrm{nil}}$, where the leading $y$-dependence has been extracted:

$$||e(y) \cdot \mathbf{\Pi}_{\mathrm{nil}}||^2 = ||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||_\star^2 \cdot ||{}^e M(y) \cdot e^{-iN_{(n)}^-} \tilde{\mathbf{a}}_0^{(n\varepsilon)}||^2. \tag{4.52}$$

Neglecting the sub-leading terms in ${}^e M(y)$, we arrive at

$$e^{-K} \sim ||\mathbf{\Pi}_{\mathrm{nil}}||^2 \sim ||\tilde{\mathbf{a}}_0^{(n\varepsilon)}||_\star^2 \tag{4.53}$$

In other words, the leading growths of the nilpotent orbit and $\tilde{\mathbf{a}}_0$ agree. Looking at (4.40) and expanding $M(y)$ one may be tempted to believe that this result is trivial. However, in the form of (4.40) it is not clear that $M(y)$ cannot increase the $\mathfrak{sl}(2)$-eigenvalues of $\tilde{\mathbf{a}}_0^{n\varepsilon}$ (note that $M(y)$ is matrix-valued), leading to a different growth. The result of the above calculation is that this situation cannot occur. Finally, we stress that the first relation in (4.53) holds up to exponentially suppressed corrections, whereas the second relation holds up to polynomially suppressed corrections.

## 4.3 The Scaling of the Index Density for Multi-Moduli Limits

In this section we will use a similar strategy as in 3.5 to obtain the scaling of the index density in the multi-moduli case. A crucial difference is that we will make use of the SL(2)-orbit approximation as we did in the previous section, where we explicitly found that

$$e^{-K} \sim (y^1)^{d_1} (y^2)^{d_2-d_1} \cdots (y^{n_P})^{d_{n_P}-d_{n_P-1}}. \tag{4.54}$$

We now compute the metric components and the covariant derivatives in a similar manner as in chapter 3. starting from this expression. Note that $d_i \leq d_{i+1}$ is always true for the growth sector in

[2]More precisely, $e(y)$ is an isometry of the bilinear form $(\cdot,\cdot)$, see [29, App. B].

question. However, here we will make the stronger assumption that $d_i < d_{i+1}$, so that the leading order behaviour of $K$ depends on all coordinates $y^i$. Indeed, under this assumption we may write

$$K \sim \sum_{i=1}^{n_P} \log y^i. \tag{4.55}$$

Moreover, since the SL(2)-orbit has no $x^j$ dependence, we need only consider the derivatives of $K$ with respect to $y^j$. In particular, the metric components $g_{i\bar{\imath}} := g_{t_i \bar{t}_i}$ are given by

$$g_{i\bar{\imath}} \sim (y^i)^{-2}, \quad g^{i\bar{\imath}} \sim g_{i\bar{\imath}}^{-1} \tag{4.56}$$

and moreover

$$\sqrt{g} \sim (y^1)^{-2} \cdots (y^{n_P})^{-2}. \tag{4.57}$$

Since the Kähler potential does not contain cross-terms in different $y^i$'s, to leading order, the metric components $g_{i\bar{\jmath}}$ with $i \neq j$ will be suppressed w.r.t. $g_{i\bar{\imath}}$. Next, we note that

$$\partial_i \mathbf{\Pi}_{\mathrm{nil}} = \partial_i (M(y) \cdot \mathbf{\Pi}_{\mathrm{SL(2)}}) \sim M(y) \cdot \partial_i \mathbf{\Pi}_{\mathrm{SL(2)}} \sim M(y) \cdot N_i^- \mathbf{\Pi}_{\mathrm{SL(2)}}. \tag{4.58}$$

Here we have used the fact that $\partial_i M(y)$ has no constant term and therefore constitutes a sub-leading contribution. In particular, it follows that

$$\nabla_i \mathbf{\Pi}_{\mathrm{nil}} = \partial_i \mathbf{\Pi}_{\mathrm{nil}} + (\partial_i K) \mathbf{\Pi}_{\mathrm{nil}} \sim M(y) \cdot \left[ N_i^- \mathbf{\Pi}_{\mathrm{SL(2)}} + y_i^{-1} \mathbf{\Pi}_{\mathrm{SL(2)}} \right]. \tag{4.59}$$

In the limit $y^i \to \infty$, we may again approximate $\nabla_i = \partial_i$ when acting on the scalar $\mathbf{\Pi}_{\mathrm{nil}}$. When taking two covariant derivatives, we must again include the Levi-Civita connection, which gives

$$\nabla_i \nabla_j \mathbf{\Pi}_{\mathrm{nil}} \sim \partial_i \partial_j \mathbf{\Pi}_{\mathrm{nil}} - \Gamma^{\mu}{}_{ij} \partial_\mu \mathbf{\Pi}_{\mathrm{nil}}. \tag{4.60}$$

One can then explicitly check that the Levi-Civita term is of order $(y^i)^{-1}$ when $i = j$ and vanishes otherwise. In particular, this means we may approximate

$$\nabla_i \nabla_j \mathbf{\Pi}_{\mathrm{nil}} \sim \partial_i \partial_j \mathbf{\Pi}_{\mathrm{nil}} \sim M(y) \cdot N_i^- N_j^- \mathbf{\Pi}_{\mathrm{SL(2)}}. \tag{4.61}$$

Finally, we turn to the evaluation of

$$F_{i\bar{\jmath}k\bar{l}} = (\nabla_i \nabla_k \mathbf{\Pi}_{\mathrm{nil}}, \overline{\nabla_j \nabla_l \mathbf{\Pi}_{\mathrm{nil}}}). \tag{4.62}$$

First, since the metric components $g^{i\bar{\jmath}}$ for $i \neq j$ are sub-leading, we expect the leading behaviour to be governed by $F_{i\bar{\imath}j\bar{\jmath}}$, which we can alternatively write as[3]

$$F_{i\bar{\imath}j\bar{\jmath}} = ||\nabla_i \nabla_j \mathbf{\Pi}_{\mathrm{nil}}||^2 \sim ||M(y) \cdot N_i^- N_j^- \mathbf{\Pi}_{\mathrm{SL(2)}}||^2. \tag{4.64}$$

The growth of the last term can now be obtained by repeating the trick we used to compute $||\mathbf{\Pi}_{\mathrm{nil}}||^2$, i.e. inserting unity as $e(y)e(y)^{-1}$, resulting in

$$\begin{aligned} ||M(y) \cdot N_i^- N_j^- \mathbf{\Pi}_{\mathrm{SL(2)}}||^2 &\sim ||{}^e M(y) \cdot e(y) N_i^- N_j^- \mathbf{\Pi}_{\mathrm{SL(2)}}||^2 \\ &\sim \left[ \left(\frac{y^1}{y^2}\right)^{d_1} \cdots \left(\frac{y^{n-1}}{y^n}\right)^{d_{n-1}} (y^n)^{d_n} \right] y_i^{-2} y_j^{-2} ||{}^e M(y) \cdot N_i^- N_j^- e^{-iN_{(n)}^-} \tilde{\mathbf{a}}_0||^2 \\ &\sim ||\tilde{\mathbf{a}}_0^{n\varepsilon}||_\star^2 \cdot y_i^{-2} y_j^{-2}, \end{aligned}$$

[3] Note that because we set $\nabla_i \sim \partial_i$, the tensor $F_{i\bar{\jmath}k\bar{l}}$ is symmetric under exchange of $i \leftrightarrow k$ and $\bar{\jmath} \leftrightarrow \bar{l}$. Moreover, using the properties of the bilinear form $(\cdot,\cdot)$, we also have

$$F_{i\bar{\jmath}i\bar{\jmath}} = (\nabla_i^2 \mathbf{\Pi}_{\mathrm{nil}}, \bar{\nabla}_{\bar{\jmath}}^2 \bar{\mathbf{\Pi}}_{\mathrm{nil}}) \sim ((N_i^-)^2 \mathbf{\Pi}_{\mathrm{nil}}, (N_j^-)^2 \bar{\mathbf{\Pi}}_{\mathrm{nil}}) = ||N_i^- N_j^- \mathbf{\Pi}||^2, \tag{4.63}$$

which is the same as $F_{i\bar{\imath}j\bar{\jmath}}$. With this all relevant cases have been considered where in the last line we have inserted the growth of $\tilde{\mathbf{a}}_0^{(n\varepsilon)}$ and neglected subleading order terms in ${}^{e}M(y)$. Note that the presence of $N_i^-$ and $N_j^-$ lowers the scaling of $y_i$ and $y_j$ by two by the same reasoning we used before.

An alternative way to derive this result would be to use the growth theorem directly. Indeed, in the previous section we argued that

$$||\mathbf{\Pi}_{\text{nil}}||^2 \sim ||M(y) \cdot \mathbf{\Pi}_{\text{SL}(2)}||^2 \sim ||\tilde{\mathbf{a}}_0||^2. \tag{4.65}$$

In a similar fashion, we could have immediately used the general result that each application of $N_j^-$ lowers the growth by a factor $y_j^{-2}$, i.e.

$$||N_i^- v||^2 \sim ||v||^2 \cdot (y^i)^{-2}, \tag{4.66}$$

to find

$$||M(y) \cdot N_i^- N_j^- \mathbf{\Pi}_{\text{SL}(2)}||^2 \sim ||\tilde{\mathbf{a}}_0||^2 \cdot y_i^{-2} y_j^{-2}. \tag{4.67}$$

However, this again neglects some non-trivial behaviour that $M(y)$ might cause. The upshot of the more detailed calculation is that $M(y)$ does not induce any additional leading behaviour terms.

Finally then, the index density is given by

$$d\mu \sim d^m t \ \sqrt{g} \sum_{k=0}^{m} I_k, \tag{4.68}$$

where

$$I_k \sim \left(e^K g^{i\bar{\imath}} g^{j\bar{\jmath}} F_{i\bar{\imath}j\bar{\jmath}}\right)^k \sim 1. \tag{4.69}$$

In exactly the same manner as happened in the one-modulus case, we are only left with the volume form

$$d\mu \sim d^m t \ \sqrt{g} \sim \bigwedge_{i=1}^{n_P} \frac{dt^i \wedge d\bar{t}^{\bar{\imath}}}{(\text{Im } t^i)^2}. \tag{4.70}$$

## 4.4 Discussion

From equation 4.70 we see that the result we obtained in chapter 3 generalize naturally to multi-parameter limits in the case where $d_i < d_{i+1}$, resulting in

$$d\mu \sim \bigwedge_{i=1}^{n_P} \frac{dt^i \wedge d\bar{t}^{\bar{\imath}}}{(\text{Im } t^i)^2} = \bigwedge_{i=1}^{n_P} \frac{dz^i \wedge d\bar{z}^{\bar{\imath}}}{|z_i|^2 \log^2 |z^i|}. \tag{4.71}$$

In section 3.6 we commented on the validity of the results by comparison to the works by Denef in [14] and Eguchi and Tachikawa in [22]. To confirm the validity of the multi-moduli result, we compare our findings to the mathematics literature, specifically [40, 42]. Corollary 6.1 of [40] implies that for a general moduli space of a Calabi-Yau manifold, one has

$$\int_{\mathcal{M}} c_{\alpha_1} \wedge \cdots \wedge c_{\alpha_r} \wedge \omega^{\alpha_0} \leq 2^{m-\alpha_0} \int_{\mathcal{M}} \omega_H^m, \quad \sum_{j=0}^{r} r_j = m = \dim \mathcal{M}, \tag{4.72}$$

where $c_{\alpha_i}$ denotes the $\alpha_i$-th Chern class with respect to the Weil-Petersson metric. Moreover, $\omega_H$ denotes the Kähler form of the so-called Hodge metric. Its precise definition is rather involved and explained in e.g. [42]. The important fact is that $\omega_H$ is always bounded by the Kähler form $\omega_P$ of the Poincaré metric, i.e.

$$\omega_H \leq C\omega_P \tag{4.73}$$

for some constant $C$, where

$$\omega_P = \sum_{i=1}^{k} \frac{dz^i \wedge d\bar{z}^{\bar{\imath}}}{|z^i|^2 \log^2 |z^i|} + \sum_{i=k+1}^{m} dz^i \wedge d\bar{z}^{\bar{\imath}}. \tag{4.74}$$

The upshot of this corollary is that the intergral over the index density $\det(R+\omega)$ can be bounded by an integral over the Hodge volume c.f. (4.72), which is in turn bounded by an integral over $m$ wedges of the Poincare Kähler form. It follows that the index of supersymmetric vacua is finite, see also [40, Thm. 6.6]. Crucially, this is a statement involving the entire moduli space and allows for general singularities, which is a stronger result than we have obtained. Indeed, from our findings we can only conclude (in certain cases) that near a singularity the index density is integrable. It may *a priori* still occur that the volume of $\mathcal{M}_{\text{cs}}$ is in fact infinite, leading to an infinite index. In light of the above corollary this is of course not possible, as was also shown earlier in [41] and was also proven by more physical means in [20].

Finally, let us comment on how our results give a slightly different insight than those in the mathematics literature. Note that wedging $\omega_P$ $m$ times yields (to leading order) our result for the index density $d\mu$ expressed in the $z$-coordinates. Therefore, our findings sharpen the behaviour of the index density by showing that it is not only bounded by Poincaré form, but is fact well approximated by the Poincaré form near all singular loci we have considered. As we discussed in some detail in section 3.6, the fact that the index density takes this particular form may be related to an underlying gauge/gravity duality in which its dependence on e.g. the coupling constant is naturally expressed. We believe that the universality of the index density may be a hint at some deeper underlying structure, and our computation of its asymptotic behaviour may serve as a first step in this direction.

*Polarizable Cubes and More General Limits*

Let us once more stress that our results hold for very particular multi-parameter limits, those which are obtained from an enhancement chain which strictly increases, i.e. is of the form

$$\text{Type II} \to \text{Type III} \to \text{Type IV} \to \cdots \tag{4.75}$$

The reason we restrict ourselves to these limits is because of the leading-order behaviour of the Kähler potential, given by

$$e^{-K} \sim (y^1)^{d_1}(y^2)^{d_2-d_1}\cdots(y^{n_P})^{d_{n_P}-d_{n_P-1}}. \tag{4.76}$$

In particular, when e.g. $d_2 = d_1$, meaning that the type does not change in the enhancement chain, the dependence on the $y^2$-coordinate is lost. To properly address this, one must include sub-leading contributions to the Kähler potential whose form is not yet well understood. However, there are still some ways to make progress in this direction, which we will shortly discuss.

One downside of the enhancement chain picture is that is rather restrictive, in the following sense. If one starts with a Type II singularity by taking $y^1$ to be large, and then enhances this to a Type III by taking $y^2$ to be large, one can only make statements about the results in a given growth sector, e.g. where $y^1$ grows quicker than $y^2$. Moreover, we have no way of knowing whether an expression of the form $y^1(y^2)^2$ could appear in $(\mathbf{\Pi}, \bar{\mathbf{\Pi}})$. Instead, one can consider so-called *polarizable cubes*. These are triples of the form

$$\langle \mathrm{II} \mid \mathrm{IV} \mid \mathrm{III} \rangle, \tag{4.77}$$

which indicate that a Type II and a Type III singularity intersect to form a Type IV singularity. Since now both starting singularities are specified, we have much more control over what kind of polynomial expressions in $y^1$ and $y^2$ can occur in $(\mathbf{\Pi}, \bar{\mathbf{\Pi}})$. For the specific example above, the polarization conditions imply that

$$(\mathbf{a}_0, (N_1 + N_2)^3 \bar{\mathbf{a}}_0) \neq 0, \quad (\mathbf{a}_0, N_1 \bar{\mathbf{a}}_0) \neq 0, \quad (\mathbf{a}_0, N_2^2 \bar{\mathbf{a}}_0) \neq 0. \tag{4.78}$$

Combining these three equations, one readily finds that the only cubic term that can occur in $(\mathbf{\Pi}, \bar{\mathbf{\Pi}})$ is given by $y^1(y^2)^2$, whereas $(y^1)^2 y^2$ cannot occur. Still, there are cases where this construction is not conclusive, for example for the triple

$$\langle \mathrm{II} \mid \mathrm{III} \mid \mathrm{III} \rangle. \tag{4.79}$$

Following the same reasoning, one finds that the quadratic term is $(y^1)^2$ is not allowed, however one cannot exclude for instance the term $y^1 y^2$. Nevertheless, it is clear that considering polarizable cubes instead of enhancement chains gives us more control of the kinds of expressions that can appear. One might imagine classifying all possible polarizable cubes and checking for each of them what the behaviour of $(\mathbf{\Pi}, \bar{\mathbf{\Pi}})$ and subsequently apply the analysis of the index density. In general, this is an extremely difficult task and lies beyond the scope of this work. Indeed, checking whether a given cube is polarizable, meaning that the polarization conditions of the different mixed Hodge structures involved must translate properly into each other, is challenging. For Calabi-Yau threefolds, a complete classification of 2-moduli cubes has been obtained in [37], however a general classification for $n$-moduli cubes has yet to be found.

# Conclusion

Throughout this work, we have touched on a number of areas that are relevant in string compactifications. This includes the more physical discussion regarding the statistics of flux vacua, as well as the mathematical story about asymptotic Hodge structures and how they can be used to tackle a variety of problems. We have also provided additional comments regarding the assumptions that are made in the constructions we have considered. In this last section, we will return back to the precise questions we have started out with and provide some steps that can be taken from here.

*The Three Answers*

Following the structure we started in the introduction, let us summarize the answers we have found for the three questions that were posed there. In chapter 1 we performed the dimensional reduction of the eleven-dimensional Ricci scalar and saw the emergence of massless complex structure moduli, scalar fields parametrizing the shape internal geometry. The fact that they are massless constitutes the issue of moduli stabilization (Q1), and its solution is given by:

> A1: The complex structure moduli $z^i$ can be stabilized by extremizing the flux-induced scalar potential $V(z)$, i.e. by setting
>
> $$\frac{\partial V}{\partial z^i} = 0. \tag{4.80}$$
>
> Explicitly, the set of supersymmetric solutions is found by solving
>
> $$D_i W = 0, \quad W = \int_{Y_4} G_4 \wedge \Omega. \tag{4.81}$$

In words, non-zero $G_4$ fluxes in the internal geometry (which were in fact necessitated by the tadpole constraint) contribute an energy to a particular shape of the underlying Calabi-Yau. Minimizing this energy with respect to the moduli then yielded a (typically large) set of string vacua. The analysis of the distribution of such vacua (Q2) was the main content of chapter 2, where we explicitly re-derived the answer that was already given by Ashok and Douglas:

> A2: The distribution of supersymmetric vacua (counted with signs) of Type IIB/F-theory compactifications over the complex structure moduli space is given by the Ashok-Douglas density
>
> $$d\mu = \det(R + \omega \cdot \mathbf{1}). \tag{4.82}$$

We moreover discussed some of the intricacies that arise in this derivation, most notably the quantization of fluxes and the stabilization of the Kähler moduli. Here we found that neglecting the quantization is reasonable in the regime where $L_* \gg b$. Moreover, the Kähler moduli can be stabilized separately by the inclusion of non-perturbative corrections to the superpotential, exactly as in the KKLT construction. Besides obtaining the AD-density, we additionally gave an estimate (or, more precisely, a lower bound) on the total number of flux vacua, which was found to be huge ($10^{500}$ in typical cases, but F-theory easily surpasses this). This raised the question of whether the total number of flux vacua is at all finite, or, more precisely, whether the AD-density is integrable near regions where the curvature diverges (Q3). We devoted the remainder of the text, chapters 3 and 4, to obtain the following answer:

> A3: The AD-density is integrable near any singular loci where a single coordinate $\mathrm{Im}\, t$ in the complex structure moduli space becomes large. More precisely, it takes the universal form
> $$d\mu \sim \frac{dt \wedge d\bar{t}}{(\mathrm{Im}\, t)^2}. \tag{4.83}$$
> Moreover, when multiple coordinates become large, this result generalizes in the natural way when the singularity types in the corresponding enhancement chain strictly increase.

First, we emphasize that the above holds for the complex structure moduli space of any Calabi-Yau $D$-fold. However, the main relevance is for the case $D = 4$ since this applies to the F-theory context and also the case $D = 3$ since this confirms the earlier results by Eguchi and Tachikawa. The other cases are of less physical interest, since they do not arise in string compactifications, but it is nevertheless interesting that the same result holds in such generality. To arrive at A3, we made use of some remarkable properties of the boundary of the complex structure moduli space. Indeed, the nilpotent orbit theorem was crucial in determining the asymptotic form of the period vector in which the AD-density was expressed. Equally crucial are the polarization properties of the mixed Hodge structure at the boundary, which restricted the powers of $\mathrm{Im}\, t$ that could appear in the final expression. Subsequently, to investigate the multi-parameter limits, the usage of the SL(2)-orbit theorem was essential. In particular, the emerging $\mathfrak{sl}(2)$-triplets and their action on the SL(2)-orbit allowed for controlled computations which untangle the various large coordinates, essentially diagonalizing the structure at the boundary. Here we emphasize the fact that we were not able to provide a result in the most general case of arbitrary multi-parameter limits. We actively avoided the case where correction terms to the Kähler potential are relevant. A full understanding will require a thorough treatment of singularity enhancements and the possible corrections to the growth theorem. Adding to this, we discussed a slightly different perspective through the usage of polarizable cubes, although significant work needs to be done to understand their classification in detail for both the threefold and fourfold.

*Further Questions*

As usual, whenever one provides an answer to a question, a number of new questions emerges. Perhaps the most pressing question is *why* the AD-density appears to take such a universal form in these asymptotic limits. Much like how a U(1) symmetry protects the masslessness of the photon, one might imagine some symmetry which emerges at the boundary of the moduli space which dictates this particular form. We have made one proposal how this could arise from a generalized version of the gauge/gravity duality that appears for the conifold, but stress that this is merely speculation at best. However, as these singular loci are currently an active testing ground for the various swampland conjectures, we believe that an investigation of such a duality may yield fruitful insights and add to the holographic nature of the picture.

Another question that arises is whether one can perform a similar calculation for other distributions, such as those for the cosmological constant or the supersymmetry breaking scale. We believe this will be difficult, since very few explicit expressions for such distributions exist. However, one might imagine starting the derivation of such distributions already at the boundary. There one could follow a similar construction as in [29] to decompose the fluxes in a manner adapted to the SL(2)-splitting. In this way the distribution of $\Lambda = -3|W|^2$ may be more conveniently expressed.

Finally, we once more stress that our results ignore the back-reaction of fluxes on the internal geometry, which makes it conformally Calabi-Yau. For such spaces the concept of a moduli space is already quite different and it is unclear whether the results of e.g. the nilpotent orbit theorem still hold. It would be interesting to investigate the effect this has and whether the asymptotic Hodge structure persists.

# Acknowledgements

First of all, I would like to thank Dr. Thomas Grimm for supervising my thesis and providing me with interesting directions to take the project into. Secondly, I would like to thank Damian van de Heisteeg for helping me with the many questions I had as well as taking the time to review my work and checking some details of my calculations. Finally, I want to thank Rigers, Ruben and Paris for the numerous discussions we have had as well as their help with preparing the final presentation.

# Appendix A

# Kähler Manifolds

In this section we will list some basic properties of Kähler manifolds and their associated cohomology groups. We follow the notation and discussion in [43]. In the following, let $M$ be a complex manifold of complex dimension $m$.

**Definition A.1.** A triple $(M, g, J)$ is called a **Kähler manifold** if the following conditions are satisfied:

1. $g$ is a Riemannian metric, i.e. a bilinear, non-degenerate and symmetric map

$$g : TM \times TM \to C^{\infty}(M) \tag{A.1}$$

   which additionally satisfies

$$g(IX, IY) = g(X, Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{A.2}$$

   where $I$ denotes the almost complex structure of $M$. We say that $g$ is a *Hermitian metric* on $M$.

2. $J$ is the *Kähler* form of $g$, i.e. a bilinear map

$$J : TM \times TM \to C^{\infty}(M), \tag{A.3}$$

   which is compatible with $g$ in the following sense:

$$J(X, Y) = g(IX, Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{A.4}$$

   and which is moreover closed, i.e.

$$dJ = 0. \tag{A.5}$$

   In this case, one sometimes calls $g$ a *Kähler metric.*

To be more explicit, let us consider these objects $g$ and $J$ in a local coordinate system $\{y_i, \bar{y}_i\}$, $i = 1, \ldots, m$. A general Riemannian metric $g$ is then of the form:

$$g = g_{a\bar{b}} dy^a \otimes d\bar{y}^b + g_{\bar{a}b} d\bar{y}^{\bar{a}} \otimes dy^b. \tag{A.6}$$

Moreover, the Kähler form $J$ can be written in terms of $g$ as follows:

$$J = \frac{i}{2} g_{a\bar{b}} dy^a \wedge d\bar{y}^{\bar{b}}. \tag{A.7}$$

When the Kähler form is closed, this poses a remarkable simplification, namely that the metric can locally be expressed by a single scalar function as:

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K, \text{ or } \quad J = \frac{i}{2}\partial\bar{\partial}K, \tag{A.8}$$

where $K$ is also called the *Kähler potential*.

**Definition A.2.** An $(r,s)$-form $\omega$ is called **harmonic** if it satisfies the Laplace equation:

$$\Delta\omega = 0, \tag{A.9}$$

where Laplacian is defined by

$$\Delta : \Omega^{(r,s)}(M) \to \Omega^{(r,s)}(M), \quad \Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d, \quad d^\dagger = - \star d \star, \tag{A.10}$$

where $\star$ denotes the Hodge star. Here $\Omega^{r,s}(M)$ denotes the space of all $(r,s)$-forms on $M$.

Note that one can also define the following Laplacian operators:

$$\Delta_\partial = (\partial + \partial^\dagger)^2, \quad \Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^\dagger)^2. \tag{A.11}$$

However, for a Kähler manifold one can show that

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{A.12}$$

Therefore the harmonic forms associated with these operators are the same.

**Definition A.3.** The $(r,s)$**-th $\bar{\partial}$-cohomology group** $H^{r,s}_{\bar{\partial}}(M)$ is defined by

$$H^{r,s}_{\bar{\partial}}(M) := Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M), \tag{A.13}$$

where

$$Z^{r,s}_{\bar{\partial}}(M) = \{\omega \in \Omega^{r,s(M)} : \bar{\partial}\omega = 0\},$$
$$B^{r,s}_{\bar{\partial}}(M) = \{\omega \in \Omega^{r,s}(M) : \exists\alpha \in \Omega^{r,s-1}(M),\ \omega = \bar{\partial}\alpha\}.$$

Throughout the thesis we use the shorter notation $H^{r,s}_{\bar{\partial}}(M) = H^{r,s}(M)$ or simply $H^{r,s}$.

We now state an important theorem which allows us to relate these cohomology groups to harmonic forms.

**Theorem A.4** (Hodge's Decomposition Theorem)**.** *Any $(r,s)$-form $\omega$ can be written in the following form:*

$$\omega = \bar{\partial}\alpha + \bar{\partial}^\dagger\beta + \gamma, \tag{A.14}$$

*for some $(r,s-1)$-form $\alpha$, $(r,s+1)$-form $\beta$ and harmonic form $(r,s)$ form $\gamma$.*

An important consequence of the above theorem is the following

**Corollary A.5.** *Any cohomology class $[\omega]$ in $H^{r,s}_{\bar{\partial}}(M)$ has a unique harmonic representative.*

In other words, given a cohomology class we can uniquely represent it as a harmonic form, and vice versa. This is relevant in the context of string compactifications, where harmonic forms correspond to the internal massless degrees of freedom of the string.

# Appendix B

# Compactification of the Ricci Scalar

We perform the dimensional reduction of the Ricci scalar using the following ansatz for the metric

$$g_{MN}dx^M dx^N = g_{\mu\nu}(x)dx^\mu dx^\nu + h_{a\bar{b}}dy^a d\bar{y}^{\bar{b}} + \bar{z}^{\bar{\imath}}(x)(\bar{b}_{\bar{\imath}})_{ab}dy^a dy^b. \tag{B.1}$$

as introduced in 1.5.

*Computing the Christoffel Symbols*

First, we need to compute the Christoffel symbols:

$$\Gamma^P{}_{MN} = \frac{1}{2}g^{PQ}\left(\partial_M g_{NQ} + \partial_N g_{MQ} - \partial_Q g_{MN}\right). \tag{B.2}$$

To first order in the moduli, the components of the inverse metric on $Y_4$ are given by

$$g^{a\bar{b}} = h^{a\bar{b}}, \quad g^{ab} = -z^i(b_i)^{ab}.$$

We raise the indices using the background metric $h$, which is valid up to first order:

$$(b_i)^{ab} = (b_i)_{\bar{a}\bar{b}}h^{a\bar{a}}h^{b\bar{b}} \tag{B.3}$$

Apart from the Christoffel symbols with purely Greek indices, the non-zero Christoffel symbols are given by

$$\begin{aligned}
\Gamma^\mu{}_{ab} &= -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{ab} = -\frac{1}{2}(\bar{b}_{\bar{\imath}})_{ab}\partial^\mu \bar{z}^{\bar{\imath}} \\
\Gamma^b{}_{\mu a} &= \frac{1}{2}g^{b\bar{c}}\partial_\mu g_{a\bar{c}} \\
&= -\frac{1}{2}z^i(b_i)^{bc}(\bar{b}_{\bar{\jmath}})_{ac}\partial_\mu \bar{z}^{\bar{\jmath}} \\
&= -\frac{1}{2}(b_i \cdot \bar{b}_{\bar{\jmath}})^b{}_a z^i \partial_\mu \bar{z}^{\bar{\jmath}} \\
\Gamma^b{}_{\mu\bar{a}} &= \frac{1}{2}g^{b\bar{c}}\partial_\mu g_{\bar{a}\bar{c}} \\
&= \frac{1}{2}h^{j\bar{c}}(b_i)_{\bar{a}\bar{c}}\partial_\mu z^i \\
&= \frac{1}{2}(b_i)^j{}_{\bar{a}}\partial_\mu z^i
\end{aligned}$$

*On to the Ricci Scalar*

The Riemann tensor is given in terms of the Christoffel symbols as

$$R^P{}_{MNQ} = \partial_M \Gamma^P{}_{NQ} + \Gamma^P{}_{ML}\Gamma^L{}_{NQ} - (M \leftrightarrow N). \tag{B.4}$$

Moreover, we have the following decomposition for the 11-dimensional Ricci scalar:

$$R_{11} = R_3 + g^{\mu\nu}R^a{}_{\mu a\nu} + g^{ab}\left(R^\mu{}_{a\mu b} + R^c{}_{acb} + R^{\bar{c}}{}_{a\bar{c}b}\right) + g^{a\bar{b}}\left(R^\mu{}_{a\mu\bar{b}} + R^c{}_{ac\bar{b}} + R^{\bar{c}}{}_{a\bar{c}\bar{b}}\right) + \text{c.c} \tag{B.5}$$

where again c.c. stands for the complex conjugate of all terms before it, except for $R_3$.

We now make the following important remark. We are only interested in the kinetic terms for the resulting fields, hence we work up to second order in the moduli. This implies that the contribution from the $\Gamma^2$ terms will be very simple. In particular, since the remaining indices will be contracted with the metric, and $g^{ab}$ is proportional to the moduli $z$, the only relevant products will be

$$\begin{aligned}
\Gamma^\mu{}_{\bar{a}M}\Gamma^M{}_{\nu b} &= \Gamma^\mu{}_{\bar{a}\bar{c}}\Gamma^{\bar{k}}{}_{\nu b} = -\frac{1}{4}(b_i \cdot \bar{b}_{\bar{j}})_{\bar{a}b}\partial^\mu z^i \partial_\nu \bar{z}^{\bar{j}} \\
\Gamma^{\bar{c}}{}_{aM}\Gamma^M{}_{\bar{a}\bar{b}} &= \Gamma^{\bar{c}}{}_{a\mu}\Gamma^\mu{}_{\bar{a}\bar{b}} = -\frac{1}{4}(\bar{b}_{\bar{\imath}})^{\bar{c}}{}_a (b_j)_{\bar{a}\bar{b}}\partial_\mu \bar{z}^{\bar{\imath}}\partial^\mu z^j \\
\Gamma^a{}_{\mu M}\Gamma^M{}_{b\nu} &= \Gamma^a{}_{\mu\bar{c}}\Gamma^{\bar{c}}{}_{b\nu} = \frac{1}{4}(b_i \cdot \bar{b}_{\bar{j}})^a{}_b \partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}}.
\end{aligned}$$

Finally we note that none of the quantities depend on the Calabi-Yau coordinates $y^a$, hence the $\partial\Gamma$ terms are often zero. Now let us compute each term in the Ricci decomposition, starting from the easier ones:

$$\begin{aligned}
g^{ab}\left(R^c{}_{acb} + R^{\bar{c}}{}_{a\bar{c}b}\right) &= \mathcal{O}(3) \\
g^{a\bar{b}}R^c{}_{ac\bar{b}} &= 0 \\
g^{a\bar{b}}R^{\bar{c}}{}_{a\bar{c}\bar{b}} &= h^{a\bar{b}}\Gamma^{\bar{c}}{}_{aM}\Gamma^M{}_{\bar{c}\bar{b}} = -\frac{1}{4}h^{a\bar{b}}(\bar{b}_{\bar{\imath}})^{\bar{c}}{}_a (b_j)_{\bar{c}\bar{b}}\partial_\mu \bar{z}^{\bar{\imath}}\partial^\mu z^j = -\frac{1}{4}(\bar{b}_{\bar{\imath}} \cdot b_j)\partial_\mu \bar{z}^{\bar{\imath}}\partial^\mu z^j
\end{aligned}$$

The final three terms are slightly more involved, because some will contains terms involving $\nabla_\mu V^\mu$, for some space-time vector $V^\mu$. Indeed, we have:

$$\begin{aligned}
g^{ab}R^\mu{}_{a\mu b} &= -g^{ab}\left(\partial_\mu \Gamma^\mu{}_{ab} + \Gamma^\nu{}_{\nu\mu}\Gamma^\mu{}_{ab}\right) \\
&= -\frac{1}{2}z^i (b_i \cdot \bar{b}_{\bar{j}})\nabla_\mu \partial^\mu \bar{z}^{\bar{j}} \\
g^{a\bar{b}}R^\mu{}_{a\mu\bar{b}} &= -g^{a\bar{b}}\left(\partial_\mu \Gamma^\mu{}_{a\bar{b}} + \Gamma^\nu{}_{\nu\mu}\Gamma^\mu{}_{a\bar{b}}\right) + h^{a\bar{b}}\left(\Gamma^\mu{}_{ac}\Gamma^c{}_{\mu\bar{b}} + \Gamma^\mu{}_{a\bar{c}}\Gamma^{\bar{c}}{}_{\mu\bar{b}}\right) \\
&= -\frac{1}{4}h^{a\bar{b}}(\bar{b}_{\bar{\imath}} \cdot b_j)_{a\bar{b}}\partial^\mu \bar{z}^{\bar{\imath}}\partial_\mu z^j \\
&= -\frac{1}{4}(b_i \cdot \bar{b}_{\bar{j}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} \\
g^{\mu\nu}R^a{}_{\mu a\nu} &= g^{\mu\nu}\left(\partial_\mu \Gamma^a{}_{a\nu} - \Gamma^a{}_{a\lambda}\Gamma^\lambda{}_{\mu\nu} + \Gamma^a{}_{\mu M}\Gamma^M{}_{a\nu}\right) \\
&= -\frac{1}{2}g^{\mu\nu}(b_i \cdot \bar{b}_{\bar{j}})^a{}_a z^i \nabla_\mu \partial_\nu \bar{z}^{\bar{j}} + \frac{1}{4}g^{\mu\nu}(b_i \cdot \bar{b}_{\bar{j}})^a{}_a \partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} \\
&= -\frac{1}{2}(b_i \cdot \bar{b}_{\bar{j}})z^i \nabla_\mu \partial^\mu \bar{z}^{\bar{j}} + \frac{1}{4}(b_i \cdot \bar{b}_{\bar{j}})\partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}}
\end{aligned}$$

We can deal with the covariant derivatives as follows. Note that all these expressions appear under an integral over $\mathcal{M}_{1,10}$, which contains a factor $\sqrt{-g_3}$. Since the covariant derivatives only have space-time indices, the standard rule for partial integration is adjusted:

$$\sqrt{-g_{11}}\nabla_\mu V^\mu \sim -\nabla_\mu \left(\sqrt{-g_3}\sqrt{g_8}\right) V^\mu = -\sqrt{-g_3}V^\mu \partial_\mu \sqrt{g_8}, \tag{B.6}$$

where $\sim$ means 'up to a total space-time derivative' and the second equality follows from the fact that $\sqrt{g_8}$ is a space-time scalar. Moreover, one readily calculates

$$\partial_\mu \sqrt{g_8} = \frac{1}{2}\sqrt{g_8}\left(g^{ab}\partial_\mu g_{ab} + g^{a\bar{b}}\partial_\mu g_{a\bar{b}}\right) + \text{c.c.}, \tag{B.7}$$

which gives

$$\partial_\mu \sqrt{g_8} = \frac{1}{2}\sqrt{g_8}\left(g^{ab}\partial_\mu g_{ab} + g^{\bar{a}\bar{b}}\partial_\mu g_{\bar{a}\bar{b}}\right) = \mathcal{O}(2) \tag{B.8}$$

In other words, we may freely change each $z^i \nabla_\mu \partial^\mu \bar{z}^{\bar{\jmath}}$ to $-\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}}$ which results in

$$\begin{aligned}
g^{ab} R^\mu{}_{a\mu b} &\sim \frac{1}{2}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \\
g^{a\bar{b}} R^\mu{}_{a\mu\bar{b}} &\sim -\frac{1}{4}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \\
g^{\mu\nu} R^a{}_{\mu a\nu} &\sim \frac{1}{2}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} + \frac{1}{4}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \\
&= \frac{1}{4}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}}
\end{aligned}$$

Combining all these contributions, we obtain the following result for the dimensional reduction of the Ricci scalar

$$\boxed{R_{11} = R_3 + \frac{1}{2}(b_i \cdot \bar{b}_{\bar{\jmath}})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}}} \tag{B.9}$$

# Appendix C

# Hodge Structures

In this section we give an overview of the various Hodge structures that appear in chapters 3 and 4. The aim is to provide a clear and concise list of the important definitions, whereas the motivation and interpretation are left to the main text.

In the following we denote by $V$ a finite-dimensional vector space. Moreover, we denote by $V_{\mathbb{C}} = V \otimes \mathbb{C}$ the complexification of $V$.

**Definition C.1.** A pair $(V, H^{p,q})$ is called a **pure Hodge structure** of weight $w$ if the following conditions are satisfied:

1. $H^{p,q} \subseteq V_{\mathbb{C}}$ is a complex subspace, for all $p, q \in \mathbb{Z}$ satisfying $p + q = w$.
2. $\overline{H^{q,p}} = H^{p,q}$.
3. $V_{\mathbb{C}}$ can be decomposed in terms of the subspaces $H^{p,q}$ as

$$V_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q}. \tag{C.1}$$

An immediate example of a pure Hodge structure which is of relevance for is given by the cohomology groups of a compact Kähler manifold $M$. Indeed, for $n = 0, \ldots, \dim M$, we have the following decomposition:

$$H^n(M) = \bigoplus_{p+q=n} H^{p,q}(M). \tag{C.2}$$

There is an alternative, but equivalent description of a Hodge structure, via a Hodge filtration.

**Definition C.2.** A **Hodge filtration** $F^p$ of weight $n$ of $V_{\mathbb{C}}$ is a collection of complex subspaces $F^p \subseteq V_{\mathbb{C}}$ satisfying the following conditions:

1. The spaces $F^p$ provide a finite decreasing filtration of $V_{\mathbb{C}}$. That is, we have the following sequence

$$0 = F_{w+1} \subseteq F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 = V_{\mathbb{C}}. \tag{C.3}$$

2. For all $p, q \in \mathbb{Z}$ satisfying $p + q = n + 1$, we have

$$F^p \cap \overline{F^q} = 0, \quad F^p \oplus \overline{F^q} = V_{\mathbb{C}}. \tag{C.4}$$

Explicitly, the relation between a pure Hodge structure and a Hodge filtration is given by

$$H^{p,q} = F^p \cap \overline{F^q}, \quad F^p = \bigoplus_{i \geq p} H^{i,w-i}. \tag{C.5}$$

In the following, let $S(\cdot,\cdot)$ be a bilinear form on $V_{\mathbb{C}}$.

**Definition C.3.** A triple $(V, H^{p,q}, S)$ is called a **polarized pure Hodge structure** if $(V, H^{p,q})$ is a pure Hodge structure which has the following additional properties with respect to $S$:

1. $S(H^{p,q}, H^{r,s}) = 0, \quad (p,q) \neq (r,s)$.
2. $i^{p-q} S(v, \bar{v}) > 0, \quad v \in H^{p,q}, \ v \neq 0$.

We say that $H^{p,q}$ is *polarized* with respect to the bilinear form $S$.

Next, we introduce the main ingredient for the construction of a mixed Hodge structure. Let $N$ be a nilpotent matrix.

**Definition C.4.** The **monodromy weight filtration** $W_j$ of weight $w$ of $V_{\mathbb{C}}$ is a collection of complex subspaces $W_j \subseteq V_{\mathbb{C}}$ satisfying the following conditions:

1. The spaces $W_j$ form a finite increasing filtration of $V_{\mathbb{C}}$. That is, we have the following sequence
$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2w} = V_{\mathbb{C}}. \tag{C.6}$$
2. $NW_i \subseteq W_{i-2}$.
3. $N^j : \mathrm{Gr}_{D+j} \to \mathrm{Gr}_{D-j}$ is an isomorphism, where
$$\mathrm{Gr}_i := W_i / W_{i-1} \tag{C.7}$$
are the *graded spaces*.

To emphasize the dependence on $N$, we will often denote the monodromy weight filtration by $W_j(N)$. Additionally, when $V_{\mathbb{C}}$ admits a Hodge filtration $F^p$, we require the following compatibility condition:

$$NF^p \subseteq F^{p-1}. \tag{C.8}$$

The main motivation for the monodromy weight filtration is provided in section 3.3, where it combines with the Hodge structure in an intricate way into a *mixed Hodge structure*.

**Definition C.5.** A triple $(V, W_j(N), F^p)$ is called a **mixed Hodge structure** if the graded spaces $\mathrm{Gr}_j$ each admit a *pure Hodge structure* of weight $j$ given by

$$\mathrm{Gr}_j = \bigoplus_{p+q=j} \mathcal{H}_j^{p,q}, \quad \mathcal{H}_j^{p,q} = \mathcal{F}_j^p \cap \overline{\mathcal{F}_j^q} \tag{C.9}$$

where the induced Hodge filtration $\mathcal{F}_j^p$ of $\mathrm{Gr}_j$ is defined as

$$\mathcal{F}_j^p = (F^p \cap W_j^{\mathbb{C}})/(F^p \cap W_{j-1}^{\mathbb{C}}), \quad W_j^{\mathbb{C}} = W_j \otimes \mathbb{C}. \tag{C.10}$$

Note that the operator $N$ acts as

$$N\mathrm{Gr}_j \subseteq \mathrm{Gr}_{j-2}, \quad N\mathcal{H}^{p,q} \subseteq \mathcal{H}^{p-1,q-1}. \tag{C.11}$$

In other words, it allows us to move down to the lower weight pure Hodge structure.

This concludes our overview of the various Hodge structures that are mentioned throughout the text. In section 3.3 we already give the full details of how to package the information contained in the mixed Hodge structure more elegantly in terms of the Deligne splitting, hence we will not repeat that discussion here.

# Appendix D

# Calculations of $F_{AB\cdots|MN\cdots}$

We first introduce a shorthand notation for the two-point function

$$G_{1\bar{2}} := G(z_1, \bar{z}_2) = e^{\kappa K_{1\bar{2}}}, \quad K_{1\bar{2}} := K(z_1, \bar{z}_2) \tag{D.1}$$

and the constrained two-point function

$$G^0_{1\bar{2}} := G_{z_0}(z_1, \bar{z}_2) = G_{1\bar{2}} - \bar{D}_{\bar{0}\bar{i}} G_{1\bar{0}} \left(\bar{D}_{\bar{0}\bar{i}} D_{0j} G_{0\bar{0}}\right)^{-1} D_{0j} G_{0\bar{2}}. \tag{D.2}$$

Finally, we recall the definition

$$F_{AB\cdots|MN\cdots} = G^{-1}_{1\bar{2}} (D_{1A} D_{1B} \cdots)(D_{2M} D_{2N} \cdots) G^0_{1\bar{2}}\Big|_{z_0=z_1=z_2}. \tag{D.3}$$

## D.1 One Derivative

First, we have

$$\begin{aligned} D_{1k} G_{1\bar{2}}\Big|_{z_0=z_1} &= D_{1k} G_{1\bar{2}} - (D_{1k} \bar{D}_{\bar{0}\bar{i}} G_{1\bar{0}}) \left(\bar{D}_{\bar{0}\bar{i}} D_{0j} G_{0\bar{0}}\right)^{-1} D_{0j} G_{0\bar{2}}\Big|_{z_0=z_1} \\ &= D_k G_{0\bar{2}} - (D_{0k} \bar{D}_{\bar{0}\bar{i}} G_{0\bar{0}}) \left(\bar{D}_{\bar{0}\bar{i}} D_{0j} G_{0\bar{0}}\right)^{-1} D_{0j} G_{0\bar{2}} \\ &= D_{0k} G_{0\bar{2}} - \delta^j{}_k D_{0j} G_{0\bar{2}} \\ &= 0. \end{aligned}$$

A similar expression holds for $D_2 G_{1\bar{2}}$, hence we conclude the following:

$$\boxed{D_1 G_{1\bar{2}}\Big|_{z_0=z_1} = D_2 G_{1\bar{2}}\Big|_{\bar{z}_0=\bar{z}_2} = 0} \tag{D.4}$$

## D.2 Two Derivatives

Since we now have to consider the repeated application of the Kähler-Weil covariant derivative, we should include the Levi-Civita connection as well, i.e.

$$D_i D_j f = D^K_i D^K_j f - \Gamma^k{}_{ij} D_k f, \tag{D.5}$$

where by $D_i^K$ we mean just the Kähler-Weil covariant derivative, without the Levi-Civita connection. However, from the calculation in section D.1 we see that the single derivative term vanishes. Hence we do not need to include the Levi-Civita connection. We then simply compute:

$$\begin{aligned}
G_{0\bar{0}}F_{i|\bar{j}} &= D_{1i}\bar{D}_{\bar{2}\bar{j}}G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= D_{1i}\cdot\kappa\bar{\partial}_{\bar{2}\bar{j}}\left(K_{1\bar{2}}-K_{2\bar{2}}\right)G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= \kappa\partial_{1i}\bar{\partial}_{\bar{2}\bar{j}}K_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= \kappa(\partial_i\bar{\partial}_{\bar{j}}K)G_{0\bar{0}} \\
&= \kappa g_{i\bar{j}}G_{0\bar{0}}.
\end{aligned}$$

Hence our main result for two derivatives is

$$\boxed{F_{i|\bar{j}} = \kappa g_{i\bar{j}}} \tag{D.6}$$

Moreover, using this result we may simplify the expression for $G^0_{1\bar{2}}$ as:

$$G^0_{1\bar{2}} = G_{1\bar{2}} - \kappa^{-1}g^{\bar{i}j}(\bar{D}_{\bar{0}\bar{i}}G_{1\bar{0}})G^{-1}_{0\bar{0}}(D_{0j}G_{0\bar{2}}), \tag{D.7}$$

which we will heavily use in the next section. Finally, using the two-derivative result, we note the following result for the partition function used in chapter 2

$$Z = \pi^n \det(D_i\bar{D}_{\bar{j}}G(z_0,\bar{z}_0)) = \pi^n \det\left(\kappa g_{i\bar{j}}G(z_0,\bar{z}_0)\right) = \kappa^n e^{n\kappa K(z_0,\bar{z}_0)}\det g \tag{D.8}$$

## D.3 Three Derivatives

As argued in [1], since there is no geometrical quantity associated with three (anti)-holomorphic indices, we can immediately conclude that the triple derivative terms will vanish.

## D.4 Four Derivatives

As we have seen in the text, we are interested in the following four expressions:

$$F_{ij|\bar{k}\bar{l}}, \quad F_{i\bar{j}|k\bar{l}}, \quad F_{i\bar{j}|\bar{k}\bar{l}}, \quad F_{\bar{i}\bar{j}|kl}. \tag{D.9}$$

We will dedicate a subsection to each of these terms. We also introduce another shorthand such that

$$G^0_{1\bar{2}} = G_{1\bar{2}} - g^{\bar{m}n}H_{\bar{m}n}, \tag{D.10}$$

where

$$H_{\bar{m}n} = \kappa^{-1}(\bar{D}_{\bar{0}\bar{m}}G_{1\bar{0}})G^{-1}_{0\bar{0}}(D_{0n}G_{0\bar{2}}) = \kappa\bar{\partial}_{\bar{0}\bar{m}}(K_{1\bar{0}}-K_{0\bar{0}})G^{-1}_{0\bar{0}}\partial_{0n}(K_{0\bar{2}}-K_{0\bar{0}}). \tag{D.11}$$

Note that there cannot be terms of order $\kappa^4$ or higher.
Finally, we introduce yet another shorthand

$$\Delta_{0\bar{m}} := \kappa\bar{\partial}_{\bar{0}\bar{m}}\left(K_{1\bar{0}}-K_{0\bar{0}}\right), \quad \Delta_{n0} := \kappa\partial_{0n}\left(K_{0\bar{2}}-K_{0\bar{0}}\right), \quad \Delta_{1i} := \kappa\partial_{1i}(K_{1\bar{2}}-K_{1\bar{1}}), \quad \Delta_{\bar{2}\bar{j}} := \kappa\bar{\partial}_{\bar{2}\bar{j}}(K_{1\bar{2}}-K_{2\bar{2}}). \tag{D.12}$$

This is helpful for the following two reasons. First, the covariant derivative acting on $G_{1\bar{2}}$ is easily expressed in terms of these quantities:

$$D_{1i}G_{1\bar{2}} = \Delta_{1i}G_{1\bar{2}}, \quad \bar{D}_{\bar{1}\bar{\imath}}G_{1\bar{2}} = -\kappa\bar{\partial}_{\bar{1}\bar{\imath}}K_{1\bar{1}} \tag{D.13}$$

and similarly for the others. Secondly, in the limit $z_1 = z_2$, both $\Delta_{1i}$ and $\Delta_{\bar{2}\bar{\jmath}}$ vanish. In particular, the only non-zero terms will be those where a $\bar{\partial}_{\bar{2}\bar{\jmath}}$ acts on $\Delta_{1i}$ and vice-versa. This observation will greatly simplify our calculations. In particular, note that

$$\bar{\partial}_{\bar{2}\bar{\jmath}}\Delta_{1i}\Big|_{z_0=z_1=z_2} = \kappa\bar{\partial}_{\bar{2}\bar{\jmath}}\partial_{1i}K_{1\bar{2}}\Big|_{z_0=z_1=z_2} = \kappa\bar{\partial}_{\bar{\jmath}}\partial_i K = \kappa g_{i\bar{\jmath}}. \tag{D.14}$$

Finally, let us note that, similarly to the discussion for the double derivative case, we should in principle include the Levi-Civita connection as well. However, this will add terms which are proportional to triple derivative terms, which vanish according to our discussion in the previous section. Hence we do not need to include the Levi-Civita connection in our calculations.

### D.4.1 $F_{i\bar{\jmath}|k\bar{l}}$

$$\begin{aligned}
G_{0\bar{0}}F_{i\bar{\jmath}|k\bar{l}}(G_{1\bar{2}}) &= D_{2i}\bar{D}_{\bar{2}\bar{\jmath}}D_{1k}\bar{D}_{\bar{1}\bar{l}}G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= -\kappa D_{2i}\bar{D}_{\bar{2}\bar{\jmath}}D_{1k}(\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}})G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= -\kappa D_{2i}\bar{D}_{\bar{2}\bar{\jmath}}\left[\Delta_{1k}\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}} + \partial_{1k}\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= -\kappa D_{2i}\left[(\bar{\partial}_{\bar{2}\bar{\jmath}}\Delta_{1k})\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}) - \Delta_{\bar{2}\bar{\jmath}}\partial_{1k}\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= -\kappa\left[-\kappa\partial_{2i}K_{2\bar{2}}(\bar{\partial}_{\bar{2}\bar{\jmath}}\Delta_{1k})\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}) - (\partial_{2i}\Delta_{\bar{2}\bar{\jmath}}\partial_{1k}\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= \left[\kappa^3\partial_i K g_{\bar{\jmath}k}\bar{\partial}_{\bar{l}}K + \kappa^2 g_{i\bar{\jmath}}g_{k\bar{l}}\right]G_{0\bar{0}}.
\end{aligned}$$

Hence we find

$$F_{i\bar{\jmath}|c\bar{l}}(G_{1\bar{2}}) = \kappa^2 g_{i\bar{\jmath}}g_{k\bar{l}} + \kappa^3 g_{\bar{\jmath}k}\partial_i K\bar{\partial}_{\bar{l}}K. \tag{D.15}$$

Next, we consider the contributions due to $H_{\bar{m}n}$. We have

$$\begin{aligned}
F_{i\bar{\jmath}|k\bar{l}}(H_{\bar{m}n}) &= \kappa^{-1}D_{2i}\bar{D}_{\bar{2}\bar{\jmath}}D_{1k}\bar{D}_{\bar{1}\bar{l}}\left(\Delta^{(1\bar{0})}_{\bar{0}\bar{m}}G_{1\bar{0}}\cdot\Delta^{(0\bar{2})}_{0n}G_{0\bar{2}}\right)\Big|_{z_0=z_1=z_2} \\
&= \kappa^{-1}\left[D_{2i}\bar{D}_{\bar{2}\bar{\jmath}}\Delta^{(0\bar{2})}_{0n}G_{0\bar{2}}\right]\cdot\left[D_{1k}\bar{D}_{\bar{1}\bar{l}}\Delta^{(1\bar{0})}_{0\bar{m}}G_{1\bar{0}}\right]\Big|_{z_0=z_1=z_2} \\
&= -\left[D_{2i}(\bar{\partial}_{\bar{2}\bar{\jmath}}\Delta^{(0\bar{2})}_{0n})G_{0\bar{2}}\right]\cdot\left[D_{1k}(\bar{\partial}_{\bar{1}\bar{l}}K_{1\bar{1}}\Delta^{(1\bar{0})}_{\bar{0}\bar{m}})G_{1\bar{0}}\right]\Big|_{z_0=z_1=z_2} \\
&= \kappa\left[\partial_{2i}K_{2\bar{2}}(\bar{\partial}_{\bar{2}\bar{\jmath}}\Delta^{(0\bar{2})}_{0n})G_{0\bar{2}}\right]\cdot\left[\partial_{1k}K_{1\bar{1}}\bar{\partial}_{\bar{1}\bar{l}}\Delta^{(1\bar{0})}_{\bar{0}\bar{m}}G_{1\bar{0}}\right]\Big|_{z_0=z_1=z_2} \\
&= \kappa^3 g_{\bar{\jmath}n}g_{k\bar{m}}\partial_i K\bar{\partial}_{\bar{l}}K
\end{aligned}$$

Contracting with $g^{\bar{m}n}$ yields:

$$g^{\bar{m}n}F_{i\bar{\jmath}|k\bar{l}}(H_{\bar{m}n}) = \kappa^3 g_{\bar{\jmath}k}\partial_i K\bar{\partial}_{\bar{l}}K. \tag{D.16}$$

Crucially, we see that this cancels the $\kappa^3$ contribution from $G_{1\bar{2}}$. Therefore the final result is

$$\boxed{F_{i\bar{\jmath}|k\bar{l}} = \kappa^2 g_{i\bar{\jmath}}g_{k\bar{l}}} \tag{D.17}$$

### D.4.2 $F_{ij|\bar{k}\bar{l}}$

Again, we first consider the contribution from $G_{1\bar{2}}$, this is given by

$$\begin{aligned}
G_{0\bar{0}}F_{ij|\bar{k}\bar{l}}(G_{1\bar{2}}) &= D_{1i}D_{1j}\bar{D}_{\bar{2}\bar{k}}\bar{D}_{\bar{2}\bar{l}}G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= D_{1i}D_{1j}\bar{D}_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= D_{1i}D_{1j}\left[\Delta_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}} + \bar{\partial}_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= D_{1i}\left[\Delta_{1j}\bar{\partial}_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}} + \partial_{1j}(\Delta_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}) + \partial_{1j}\bar{\partial}_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= \left[\partial_{1i}\partial_{1j}(\Delta_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}) + \partial_{1i}\partial_{1j}\bar{\partial}_{\bar{2}\bar{k}}\Delta_{\bar{2}\bar{l}}\right]G_{1\bar{2}}\Big|_{z_0=z_1=z_2} \\
&= \left[\kappa^2(g_{i\bar{k}}g_{j\bar{l}} + g_{i\bar{l}}g_{j\bar{k}}) + \kappa\partial_i\partial_j\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}K\right]G_{0\bar{0}}.
\end{aligned}$$

Hence the complete contribution from $G_{1\bar{2}}$ is given by

$$F_{ij|\bar{k}\bar{l}}(G_{1\bar{2}}) = \kappa\partial_i\partial_j\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}K + \kappa^2(g_{i\bar{k}}g_{j\bar{l}} + g_{j\bar{k}}g_{i\bar{l}}) \tag{D.18}$$

Next, we consider the contributions due to $H_{\bar{m}n}$. We have

$$\begin{aligned}
G_{0\bar{0}}^2F_{ij|\bar{k}\bar{l}}(H_{\bar{m}n}) &= \kappa^{-1}D_{1i}D_{1j}\bar{D}_{\bar{2}\bar{k}}\bar{D}_{\bar{2}\bar{l}}\left(\Delta^{(1\bar{0})}_{\bar{0}\bar{m}}G_{1\bar{0}}\cdot\Delta^{(0\bar{2})}_{0n}G_{0\bar{2}}\right)\Big|_{z_0=z_1=z_2} \\
&= \kappa^{-1}\left[D_{1i}D_{1j}\Delta^{(1\bar{0})}_{\bar{0}\bar{m}}G_{1\bar{0}}\right]\cdot\left[\bar{D}_{\bar{2}\bar{k}}\bar{D}_{\bar{2}\bar{l}}\Delta^{(0\bar{2})}_{0n}G_{0\bar{2}}\right]\Big|_{z_0=z_1=z_2} \\
&= \kappa^{-1}\left[(\partial_{1i}\partial_{1j}\Delta^{(1\bar{0})}_{\bar{0}\bar{m}})G_{1\bar{0}}\right]\cdot\left[(\bar{\partial}_{\bar{2}\bar{k}}\bar{\partial}_{\bar{2}\bar{l}}\Delta^{(0\bar{2})}_{0n})G_{0\bar{2}}\right]\Big|_{z_0=z_1=z_2} \\
&= \kappa(\partial_i\partial_j\bar{\partial}_{\bar{m}}K)(\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}\partial_nK)G_{0\bar{0}}^2.
\end{aligned}$$

Hence we find

$$F_{ij|\bar{k}\bar{l}} = \kappa\left(\partial_i\partial_j\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}K - (\partial_i\partial_j\bar{\partial}_{\bar{m}}K)g^{\bar{m}n}(\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}\partial_nK)\right) + \kappa^2(g_{i\bar{k}}g_{j\bar{l}} + g_{j\bar{k}}g_{i\bar{l}}). \tag{D.19}$$

Lastly, we show that the linear term in $\kappa$ is given by the Riemann tensor. Indeed, we have

$$\begin{aligned}
\partial_i\partial_j\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}K - (\partial_i\partial_j\bar{\partial}_{\bar{m}}K)g^{\bar{m}n}(\bar{\partial}_{\bar{k}}\bar{\partial}_{\bar{l}}\partial_nK) &= \bar{\partial}_{\bar{k}}\partial_ig_{j\bar{l}} - g^{\bar{m}n}(\bar{\partial}_{\bar{k}}g_{n\bar{l}})(\partial_ig_{j\bar{m}}) \\
&= g_{n\bar{l}}\bar{\partial}_{\bar{k}}\left(g^{\bar{m}n}\partial_ig_{j\bar{m}}\right) \\
&= -g_{n\bar{l}}R^n{}_{i\bar{k}j} \\
&= -R_{i\bar{k}j\bar{l}},
\end{aligned}$$

hence we conclude that

$$\boxed{F_{ij|\bar{k}\bar{l}} = -\kappa R_{i\bar{k}j\bar{l}} + \kappa^2(g_{i\bar{k}}g_{j\bar{l}} + g_{j\bar{k}}g_{i\bar{l}})} \tag{D.20}$$

# Appendix E

# List of Index Conventions

In the table below we collect the prominent types of indices in this work, what they denote, and the values that they take. We denote anti-holomorphic indices with a bar, e.g. $\bar{a}$.

| **Indices** | **Used for** | **Runs from 1 to ...** |
|---|---|---|
| $M, N, P, Q$ | Tensor indices on the total manifold $\mathcal{M}_{d_c}$ | $d_c$ |
| $\mu, \nu, \rho, \sigma$ | Tensor indices on space-time $\mathcal{M}_{1,d-1}$ | $d$ |
| $a, b, c, d$ | Tensor indices on $Y_D$ | $D$ |
| $I, J, K, L$ | Tensor indices on $\mathcal{M}_K$ | $h^{1,1}(Y_4)$ |
| $i, j, k, l, (m, n)$ | Tensor indices on $\mathcal{M}_{\text{cs}}$ and divisors of $\mathcal{M}_{\text{cs}}$: $\Delta^{\circ}_{i_1 \cdots i_{n_P}}$ | $h^{3,1}(Y_4)$ or dim $\mathcal{M}_{\text{cs}}$ generally |
| | Monodromy matrices $N_j$ and $\mathfrak{sl}(2)$-algebra $\{Y_i, N_i^{\pm}\}$ | number of divisors |
| | Monodromy weight filtration: $W_j$ and $\text{Gr}_i$ | $2\times$ number of divisors |
| | Horizontal primitive spaces: $P_l$ | $2D$ |
| $\alpha, \beta$ | Components of $\mathbf{\Pi}$ and $\mathbf{N}$ | $b_4(Y_4)$ |
| $p, q, r, s$ | Hodge structures: $H^{p,q}$, $F^p$, $I^{p,q}$ and $P^{p,q}$ | $D$ |

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