Institute for Theoretical Physics

Efficient Hilbert Series for Effective Theories

Master's Thesis

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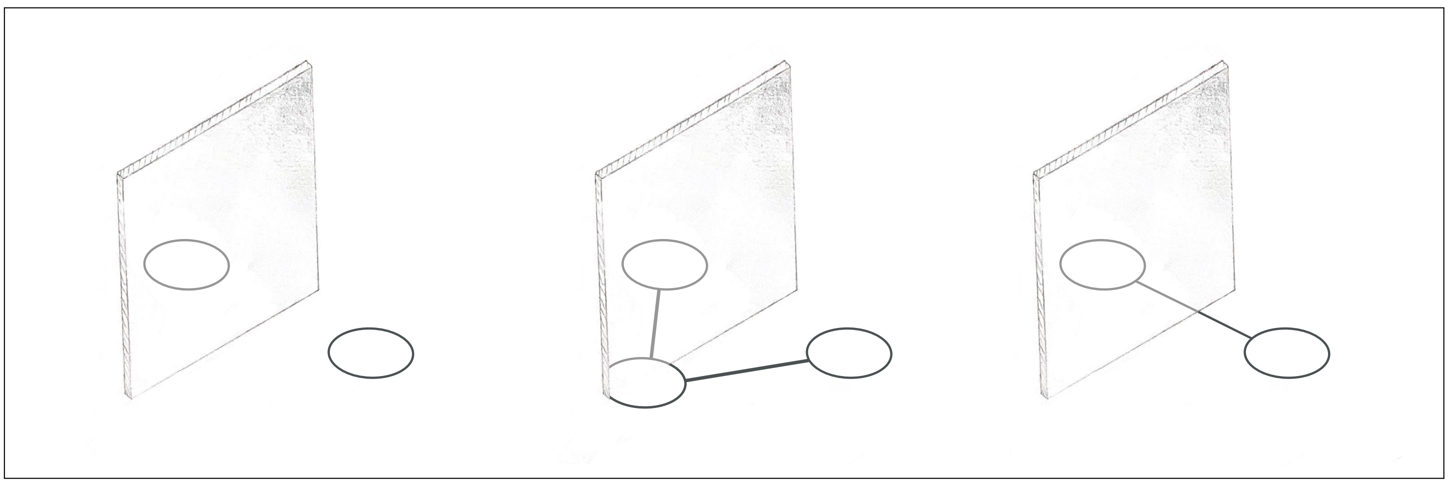

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Abstract

The task of finding operator bases for effective theories can be assisted by using the Hilbert series, which counts the number of independent operators at a given effective order. In this thesis we will introduce the Hilbert series for the Standard Model Effective Field Theory (SMEFT) and some extensions. We present an efficient algorithm for determining the Hilbert series of an effective theory and provide a companion code called Eco (Efficient Counting of Operators) in Form. The implementation can be used to efficiently establish the number of operators at effective orders as high as 20 (or more). While the implementation focusses on SMEFT, we allow for a flexible user input of the light degrees of freedom. We discuss how the Hilbert series technique can be extended to the counting of $\mathcal{CP}$-invariant operators by relating the outer automorphisms of the Lorentz group and the gauge groups to $\mathcal{C}$ and $\mathcal{P}$, respectively. In particular, we show how the outer automorphisms can be classified using the symmetries of the Dynkin diagrams, and how they give rise to an abstract definition of a folding of these diagrams, which can be used in the computation of the Hilbert series.

Contents

1 Introduction

Interesting phenomena have been found at different scales of distance, time and length, that differ by many orders of magnitude. Fortunately most of the time we can describe these phenomena without having to understand everything, which is precisely why physics has been so successful. For example, an engineer is able to design a bridge without knowledge of the Standard Model (SM) of particle physics. The design relies on Newtonian mechanics, which work perfectly well at the scale the engineer is working, i.e. the entire dynamics of the elementary particles are irrelevant, and are not needed to build a reliable bridge on which we can safely cross a river. It is important to note here that the short distance physics still has effects, but these are captured in the parameters that are undetermined by Newtonian physics. This example explains the intuitive idea behind an effective theory: calculate without knowing the exact theory.

The Standard Model (SM) of particle physics has been successfully tested to great precision, and it is the best theory we have at the moment to describe the short distance properties of nature. Nevertheless, it is commonly accepted that the SM is only applicable up to energies not exceeding a certain scale Λ and consequently merely constitutes an effective theory. Because we are not able to probe these energy scales yet with the current particle colliders, we would like a way to capture the imprints of 'new' physics emerging at these energy scales. That is, we want to extend the SM and add the high energy effects perturbatively. This is known as Standard Model Effective Theory. As we are used to describing the fundamental particles using quantum field theories, the effective theory comes in the form of an effective field theory (EFT). EFTs describe the dynamics of the low-energy degrees of freedom (known as the 'light' fields) and the effects of high-energy degrees of freedom (the 'heavy' fields) are included in the form of new effective, higher dimensional, operators. Essentially, these operators and their couplings describe the effective interactions among the light fields mediated by heavy virtual particles. EFTs are most effective when working at energy scales low enough that the heavy particles cannot be produced.

Although the ideas of EFT are simple and maybe even obvious, it is clear that higher dimensional operators will lead to new complications such as non-renormalizable theories. Therefore, implementing them in a mathematically consistent way in an interacting quantum field theory is not so obvious. In this master's thesis we will dive into the set of higher dimensional operators of EFTs and the ultimate goal is to find a recipe to write down a basis, i.e. a minimal set of operators. Roughly, we can state the main research question as

Can we find a general method to construct bases of higher-dimensional operators for effective field theories at any given effective order?

This thesis is organized as follows: we start in Sec. 2 by presenting all the preliminaries of EFTs needed for the rest of this thesis. We carefully explain the concept of the operator basis of the EFT, and in particular how some operators can be related to other operators by integration by parts and through equations of motion relations. This will leads us to the precise statement of the main problem addressed by this thesis at the end of this section.

In Sec. 3, we start by explaining how invariant operators can be constructed for the EFT Lagrangian, after which we work out some of the dimension 6 operators of the Standard Model Effective Field Theory (SMEFT). We will encounter how to deal with integration by parts, equations of motion and Fierz identities to find the minimal set of independent operators. At the end of this section, we summarize the results which are useful for the next section.

In Sec. 4 we introduce the Hilbert series which counts the number of independent operators of a given form. First we will derive the integral formulation of the Hilbert series for operators without derivatives by making use of character orthogonality of irreducible representations. We then discuss that the Lorentz group representations need to be extended to representations of the conformal group in order to include derivatives. We derive a form for the Hilbert series accounting for relations from integration by parts, and subsequently deal with equations of motion by modifying the conformal characters. At the end of this section, we construct the Hilbert series for an EFT with gravity.

With the general form of the Hilbert series derived in the previous section, we first show how the enumeration of operators can be extracted from the Hilbert series at a specific mass dimension. We then discuss one of the main results of this thesis: Eco (Efficient Counting of Operators), our implementation of the Hilbert series in Form. After a discussion of the structure of the algorithm, we provide instructions on how the code can be used, and how it can be applied to different EFTs. We conclude this section by applying our program explicitly to the SMEFT and GRSMEFT up to mass dimension 20, and we reproduce the known results for two-Higgs SMEFT.

In Sec. 6 we discuss the methods we have developed to extend the Hilbert series method for EFTs that are invariant under charge conjugation and parity. We begin with the abstract definition of outer automorphisms of Lie groups and their algebras and show how they can be induced from the symmetries of the Dynkin diagrams. We show explicitly how the outer automorphisms for $SO(3,1)$ and $SU(3)$ are in a one-to-one correspondence with the more familiar parity and charge conjugation transformations, respectively. Furthermore, we will show that the Hilbert series now becomes the average of the Hilbert series as derived in Sec. 4, and a part with modified characters. These modified characters, called the twining characters, can be obtained by folding the Dynkin diagrams of $SO(3,1)$ and $SU(3)$, the only two non-trivial diagrams involved. At the end of this section, we set up the Hilbert series to enumerate $\mathcal{CP}$-invariant operators for the SMEFT and discuss the results up to mass dimension 8.

We conclude in Sec. 7 and give some suggestions for further research.

2 Effective Field Theory

In this section we give a small introduction to effective field theories (EFTs). As it is such a broad topic, we will limit our discussion, to get a general idea of the importance of EFTs and how they are used in practice. We will focus on the elements that will be important for the rest of this thesis. For a much more general and comprehensive overview of EFTs, see e.g. Refs. [1–4].

We will start in Sec. 2.1 by giving some arguments why effective theories are so important for physicists, and in particular what role EFTs play in particle physics. We will introduce the fundamental rules for building an EFT Lagrangian in Sec. 2.2. Just like most other introductions to EFTs, we will finish that section by discussing the classical example of Fermi's theory of the weak interactions. In Sec. 2.4, we derive one of the key concepts on which the rest of this master thesis is built: we prove that the basis of operators for the EFT Lagrangian is redundant by using integration by parts and the equations of motion. We wrap up in Sec. 2.5

2.1 Why Effective Theory?

If we would know the full theory of everything, we can always compute anything in the full theory. However, if we consider more and more fundamental theories, observables become much harder to compute. E.g. computing the hydrogen energy levels in QFT is much harder then in quantum mechanics. Most of the time we can help ourself by using effective theories to make calculations a lot easier. To see to what this schematically boils down to, assume we work with a set of parameters of which some are very large and others are very small compared to the physical quantities (with the same dimension) we are interested in. We can simplify our theory a lot by setting the small parameters equal to zero and letting the large parameters go to infinity. The finite effects that still come from these parameters would be included as small perturbations, which form the effective theory. In this perturbative expansion, we only keep the terms that are relevant up to the required order of accuracy. It is clear that this procedure makes calculations easier, as one is forced to concentrate on the important physics at that scale.

Effective theories in particle physics have proven to be very successful in the last 50 years. For example, QED can be seen as an effective theory of the weak interactions. Results have been obtained much easier with effective theories, because they deal with only one scale at a time. Furthermore, effective theories can be used as probes to identify new physics [5]. In particle physics we work with QFTs, and therefore it is natural that the effective theory will become an effective field theory (EFT). The EFT might have different symmetries, fields (e.g. as in chiral perturbation theory), and calculations will certainly look different than in the original theory, but keeping higher-order terms in the effective expansion, we will be able to correct it to arbitrary precision by including more and more terms. The important parameter in particle physics is distance scale. Expanding the theory in the limit in which higher order terms are small gives us the EFT. However, identifying the correct dynamical degrees of freedom and taking the right perturbations into account can still be a difficult task. We want the simplest framework that captures the essential physics, but in a manner that can be corrected to arbitrary precision. Therefore, the rest of this section is dedicated by explaining this properly.

2.2 The EFT Lagrangian

Just like for any other QFT, the fundamental equation for the EFT is the Lagrangian density. Essentially, we reduce the recipe to construct an EFT Lagrangian to just three ingredients. First of all, we need to determine the relevant degrees of freedom, and thus the field content of the EFT Lagrangian. This is simple in cases where it is obvious what the light particles are: retain the light particles and throw out the heavy ones. However, in many other cases, it can become trickier such as in chiral perturbation theory (χPT) [6, 7].

Second, before we can write down the EFT Lagrangian, we need to determine the symmetries of the theory. This will help us when identifying the allowed interaction terms. To be more precise, symmetries constrain the allowed form of the effective operators for the EFT. In some cases they can also be a guide to get the relevant degrees of freedom, like in χPT.

Third, we need to identify the expansion parameters of the theory and what the leading order description of the EFT is. We will see in a moment that this is closely related to power counting and the mass dimension of the operators, but first we discuss the two approaches to EFTs: top-down and bottom-up.

Operator	Mass dimension	Mass dimension ($d = 4$)
Lagrangian density $\mathcal{L}(x)$	d	4
Integral measure $\mathrm{d}^d x$	$-d$	-4
Scalar ϕ	$\frac{1}{2}(d-2)$	1
Fermion ψ	$\frac{1}{2}(d-1)$	$\frac{3}{2}$
Field Strength $F^{\mu\nu}$	2	2
Derivative D_μ	1	1

Table 1: Mass dimension for various fields and operators in both d and 4 spacetime dimensions.

2.2.1 Top-Down Construction

Assume that a complete fundamental theory is understood, but we find it useful to have a simpler description at low energies $p \ll \Lambda$, where Λ is the energy scale of the UV-theory. In that case, we remove the heavier particles and match onto a low energy EFT. This process is often referred to as 'integrating out' the heavy particles. In order to still capture the effects of the heavy particles, we need to add new (higher dimensional) operators and low energy constants to the EFT Lagrangian. Therefore, the Lagrangian $\mathcal{L}_{\text{high}}$ of the UV-theory is matched onto an infinite sum over these higher dimensional operators:

$$\mathcal{L}_{\text{high}} \to \mathcal{L}_{\text{eff}} = \sum_n \eta^n \mathcal{L}^{(n)}, \tag{2.1}$$

for some small parameter $\eta \ll 1$. It is often convenient to think of η, as being dependent on the energy scales p and Λ, i.e. by explicitly writing $\eta = \frac{p}{\Lambda}$. It means that it is possible to group operators together which capture the effects at order η^n. The desired precision tells us at what n to stop. The two theories $\mathcal{L}_{\text{high}}$ and $\mathcal{L}_{\text{eff}}$ agree in the IR-limit, but differ in the UV-limit. At the scale where they overlap, we can match the coefficients of the low energy theory $\mathcal{L}_{\text{eff}}$ to the known coefficients from $\mathcal{L}_{\text{high}}$. That is, we pick a physical process σ that only has IR-degrees of freedom as external states, and compute σ in the full theory and in the EFT, at a matching scale $\mu \sim \Lambda$. We then compute the couplings in the EFT at this matching scale by setting $\sigma_{\text{eff}} = \sigma_{\text{full}}$. By using renormalization group equations, we can evolve the coefficients of the EFT at matching scale $\mu \sim \Lambda$ down to the lower scale $\mu \sim p$. For a more complete overview of matching, we encourage the reader to have a look at Refs. [1–4].

2.2.2 Bottom-Up Construction

In the bottom up approach of EFTs, the underlying UV-theory is taken to be unknown[1]. In this case, one assumes that the UV-theory exists and that we can write down an EFT in which the heavy degrees of freedom are integrated out. Therefore, one constructs

$$\mathcal{L}_{\text{eff}} = \sum_n \eta^n \mathcal{L}^{(n)}. \tag{2.2}$$

By writing down the most general set of possible operators consistent with all the symmetries we are imposing, we capture the effects of unknown massive degrees of freedom. Unlike in the previous case, the couplings are unknown and cannot be matched with the UV-theory. However, we can fit them to experiment, turning the bottom up approach of EFTs into a valuable probe in the search for new physics out of experimental reach. Again, the desired precision tells us at which n to stop. This means that we need to know how we can group the operators at every order $\eta^n = (\frac{p}{\Lambda})^n$, and this will be the topic of the next section where we will see that it is closely related to the operator dimension.

2.2.3 Power Counting

The traditional definition of a renormalizable theory is that to all orders in perturbation theory, it is possible to absorb the UV-divergences from loop integrals into a finite number of parameters. In this picture, the Lagrangian density $\mathcal{L}$ is heavily constraint. That is, in d spacetime dimensions, the Lagrangian density has mass dimension

$$[\mathcal{L}(x)] = d, \tag{2.3}$$

[1]It can also be the case that the matching is too difficult to carry out, e.g. due to non-perturbative effects.

and as we learned in our field theory class, renormalizable interactions have coefficients with mass dimension ≥ 0. So for $d = 4$, the only invariant operators we can construct out of the fields of Table 1 are[2]

$$\begin{aligned} \delta = 0 : &\quad 1, \\ \delta = 1 : &\quad \phi, \\ \delta = 2 : &\quad \phi^2, \\ \delta = 3 : &\quad \phi^3,\ \bar{\psi}\psi, \\ \delta = 4 : &\quad \phi^4,\ \phi\bar{\psi}\psi,\ D_\mu\phi D^\mu\phi,\ i\bar{\psi}\not{D}\psi,\ F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{2.4}$$

where we have denoted the mass dimension of a certain operator by δ. As mentioned earlier, in EFTs we get higher dimensional operators, which are not renormalizable according to this traditional definition. However, for EFTs we can extend renormalizability by allowing the theory to be renormalizable order by order in its expansion parameters. This definition allows for an infinite number of parameters, but if limited to only a finite number at any given order. We will have more to say about this in a moment. Note that this means that when $\mathcal{L}^{(0)}$ is renormalizable in the traditional sense, we do not know directly about the higher energy scale Λ from just looking at $\mathcal{L}^{(0)}$ only.

Lets put the above statements into more perspective by looking at scalar field theory including at least a ϕ^6 term:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{c_6}{6!}\phi^6 + ... \tag{2.5}$$

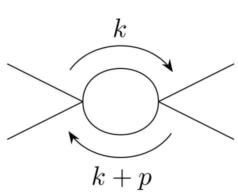
As we are interested in the physics at long distance (small momenta $p \ll \Lambda$), we rescale our coordinates $x^\mu = Sx'^\mu$ by some large constant S, and we redefine our field as

$$\phi(x) = S^{(2-d)/2}\phi'(x'), \tag{2.6}$$

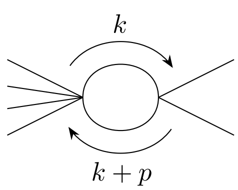
to canonically normalize the kinetic term. By doing so, the Lagrangian becomes

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2S^2\phi'^2 - \frac{\lambda}{4!}S^{4-d}\phi'^4 - \frac{c_6}{6!}S^{6-2d}\phi'^6 + ... \tag{2.7}$$

Set for the moment $d = 4$, and take the limit $S \to \infty$. We notice that the m^2 term becomes more and more dominant, the λ term is unchanged, and the c_6 term contributes less. One calls such terms relevant, marginal and irrelevant respectively. This is obviously related to mass dimension of the operators and we see that relevant operators have mass dimension $\delta < d$, marginal operators have $\delta = d$ and for irrelevant operators we get $\delta > d$. From this discussion it is clear that for finite, but large S, the dimensions of the operators tells us their importance. Therefore, the mass term becomes more important in this limit and the λ term stays equally important. Because the c_6 term becomes less important we expect that it is suppressed by a heavy energy scale. As large S means small momenta $p \ll \Lambda$, we conclude that $c_6 \sim \Lambda^{-2}$. In general, the coupling of an operator of mass dimension $\delta > d$ scales as $\Lambda^{d-\delta}$. Before we can make a general conclusion, we need to have a look at the divergences coming from this theory. Using dimensional regularization ($d = 4 - 2\epsilon$) we compute the following two diagrams



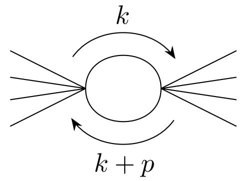
$$\sim \lambda^2 \int \frac{\mathrm{d}^d k}{(k^2 - m^2)((k+p)^2 - m^2)} \sim \lambda^2 \int \frac{\mathrm{d}^d k}{k^4} \sim \frac{\lambda^2}{\epsilon},$$



$$\sim \lambda c_6 \int \frac{\mathrm{d}^d k}{(k^2 - m^2)((k+p)^2 - m^2)} \sim \frac{\lambda}{\Lambda^2}\frac{1}{\epsilon}. \tag{2.8}$$

[2] Note that operators such as $D^2\phi$ vanish upon integration over d^4x, or are related by integration by parts. If one is not convinced yet that these are all operators up to mass dimension 4, we will prove this in later sections when we have a general method for constructing the operator bases of EFTs.

To renormalize these divergences comming from $\frac{1}{\epsilon}$, we need a counter term proportional to $\lambda\phi^4$ and $c_6\phi^6$. As these terms are already present in the EFT lagrangian, we can renormalize our theory. However, we get



$$\sim c_6^2 \int \frac{\mathrm{d}^d k}{(k^2 - m^2)((k+p)^2 - m^2)} \sim \frac{1}{\Lambda^4}\frac{1}{\epsilon}, \tag{2.9}$$

for which we need a ϕ^8 counter term. If we truncate Eq. (2.5), then there is no such ϕ^8 term and the theory is therefore not renormalizable from the traditional point of view. However, because the diagram scales as Λ^{-4}, we see that we only need to add this operator if we work up to order Λ^{-4}, and this is what we mean when calling an EFT renormalizable order by order in $\frac{1}{\Lambda}$.

We can follow the same steps for other fields like complex scalers, fermions, and gauge fields. Accordingly, we can conclude that the EFT Lagrangian has an expansion in powers of the operator dimension:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \sum_{\delta > d} \frac{1}{\Lambda^{\delta - d}} \sum_i c_i^{(\delta)} \mathcal{O}_i^{(\delta)}, \tag{2.10}$$

where the inner sum runs over all possible operators $\mathcal{O}_i^{(\delta)}$ of mass dimension δ (obeying the symmetries of the theory). The dimensionless couplings $c_i^{(\delta)}$ are known as the Wilson-coefficients and are generically taken to be of order 1. For $d = 4$, this boils down to

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(\delta \leq 4)} + \frac{1}{\Lambda} \sum_i c_i^{(5)} \mathcal{O}_i^{(5)} + \frac{1}{\Lambda^2} \sum_i c_i^{(6)} \mathcal{O}_i^{(6)} + \mathcal{O}\Big(\frac{1}{\Lambda^3}\Big). \tag{2.11}$$

Therefore, we see that in a first order EFT, we add all possible operators, obeying the symmetries, at mass dimension 5. For higher order corrections, we also add all operators at mass dimension 6, 7, and so on. In general, for corrections up to $(\frac{p}{\Lambda})^n$, we will need operators up to dimension $d+n$. This will obviously lead to many operators, but we will see in the Sec. 2.4 that the operator basis can be redundant, i.e. different operators yields identical S-matrix elements.

2.3 Fermi's Theory of the Weak Interactions

The classical example of almost every introduction to EFTs is Fermi's theory of the weak interactions, due to its historical significance. Fermi's theory provides a description of β-decay,

$$n \to p + e^- + \bar{\nu}_e, \tag{2.12}$$

proposed by Enrico Fermi in 1934 [8]. Nowadays, we know that the full (UV) theory is the SM, and that the decay is mediated by a W-boson (see Fig. 1a), but at the time Fermi wrote down his theory, this particle was not discovered yet. We will first discuss the full theory, after which we will match this onto the EFT (valid at momenta small compared to the mass M_W of the W-boson). In the SM, the W-boson interacts with the quarks and leptons via the following terms:

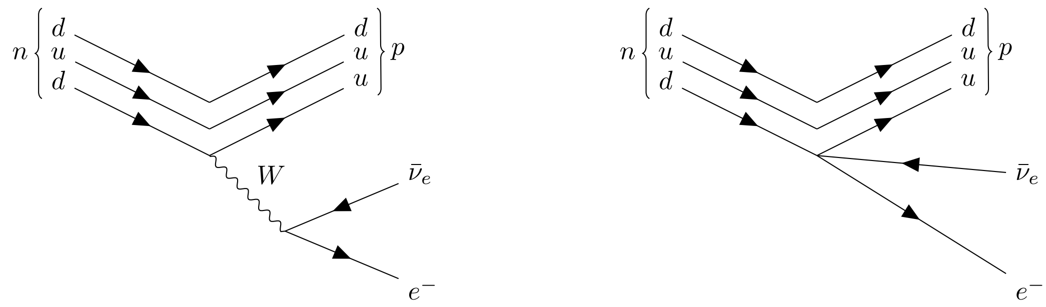
$$\frac{-g}{\sqrt{2}} \eta_{\mu\nu} W^\mu V_{ij} (\bar{u}_i \gamma^\nu P_L d_j) + \frac{-g}{\sqrt{2}} \eta_{\mu\nu} W^\mu (\bar{\nu}_l \gamma^\nu P_L l), \tag{2.13}$$

where $u_i = u, c, t$ are up-type quarks and $d_j = d, s, b$ are down type quarks, $\frac{g}{\sqrt{2}}$ is the coupling constant, and V_{ij} is the CKM mixing matrix. Therefore, the tree level amplitude for β-decay is given by (see Fig. 1a)

$$\mathcal{A} = \left(\frac{-ig}{\sqrt{2}}\right)^2 V_{du} (\bar{u}\gamma^\mu P_L d)(\bar{e}\gamma^\nu P_L \nu_e) \frac{-i\eta_{\mu\nu}}{p^2 - M_W^2} \tag{2.14}$$

For small momentum $p \ll M_W$, we can expand the propagator, such that the amplitude is approximately given by

$$\begin{aligned} \mathcal{A} &= \left(\frac{-ig}{\sqrt{2}}\right)^2 V_{du} (\bar{u}\gamma^\mu P_L d)(\bar{e}\gamma^\nu P_L \nu_e) \frac{i\eta_{\mu\nu}}{M_W^2} \left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + ...\right) \\ &\approx \frac{-ig^2}{2M_W^2} V_{du} (\bar{u}\gamma^\mu P_L d)(\bar{e}\gamma_\mu P_L \nu_e) \left[1 + \mathcal{O}\left(\frac{p^2}{M_W^2}\right)\right]. \end{aligned} \tag{2.15}$$



(a) Tree-level diagram for β-decay.

(b) Effective vertex of β-decay.

Figure 1: Fermi's theory of the weak interactions.

Therefore, the EFT expansion parameter is $\frac{p^2}{M_W^2}$. Effectively, the vertex is now represented by Fig. 1b, with corresponding effective Lagrangian

$$\mathcal{L} = -\frac{g^2}{2M_W^2} V_{du}(\bar{u}\gamma^\mu P_L d)(\bar{e}\gamma_\mu P_L \nu_e) = \frac{-4G_F}{\sqrt{2}} V_{du}(\bar{u}\gamma^\mu P_L d)(\bar{e}\gamma_\mu P_L \nu_e), \tag{2.16}$$

where we wrote the coupling constant in terms of Fermi's constant

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}. \tag{2.17}$$

The EFT Lagrangian Eq. (2.16) is part of the low energy limit of the SM. The EFT no longer has the dynamical W-boson, and although the heavy particle is integrated out, its effect has now been included via the dimension six four-fermion operator of the EFT Lagrangian.

2.4 Operator Redundancies

Although we terminate the sum in the EFT Lagrangian at some point, it still leaves a large number of terms to be considered. The operator basis can be reduced as operators can be related by integration by parts (IBP) and through the equations of motion (EOM).

1. **IBP:** Two operators are equivalent if they are related by a total derivative. That is, we have the equivalence relation

$$\mathcal{O}_a \sim \mathcal{O}_b \quad \text{if} \quad \mathcal{O}_a = \mathcal{O}_b + \mathrm{d}\mathcal{O}_c \tag{2.18}$$

   where we have used that

$$\int \mathrm{d}^4x \, \mathrm{d}\mathcal{O} = 0 \tag{2.19}$$

   Relations like this are called integration by parts (IBP) relations.

2. **EOM:** Operators related by the classical EOM lead to the same physical effect. We have the following equivalence relation

$$\mathcal{O}_a \sim \mathcal{O}_b \quad \text{if} \quad \mathcal{O}_a = \mathcal{O}_b + \mathcal{O}_c \frac{\delta S_{\text{eff}}}{\delta \phi} \tag{2.20}$$

   We may choose such a set to minimize the number of higher dimensional terms in which derivatives are inserted.

We are used to the fact that operators can be related by integration by parts, however that they are also related through the classical EOM is far from trivial and we will prove this in the following.

2.4.1 Field Redefinitions

We can get experimentally observable quantities by computing S-matrix elements. The S-matrix depends on particle states, the physical spectrum of the theory. However, in QFT we work with fields (in particular we compute correlation functions), and these quantum fields and particles are not the same. The Lehmann-Symanzik-Zimmermann (LSZ) reduction formula can be used to relate the two. Before we give the LSZ reduction formula, recall that the momentum Green's functions for a scalar field $\phi(x)$ are defined by

$$G(q_1,...,q_m;p_1,...,p_n)=\prod_{i=1}^{m}\int \mathrm{d}^4y_i\ e^{iq_i\cdot y_i}\prod_{j=1}^{n}\int \mathrm{d}^4x_j\ e^{-ip_j\cdot x_j}\langle 0|T\{\phi(y_1)...\phi(y_m)\phi(x_1)...\phi(x_n)\}|0\rangle. \tag{2.21}$$

where p_i are the incoming momenta and q_i the outgoing. The ϕ propagator $D(p)$ is a special case of Eq. (2.21):

$$D(p)=\int \mathrm{d}^4x\ e^{ip\cdot x}\langle 0|T\{\phi(x)\phi(0)\}|0\rangle. \tag{2.22}$$

As long as

$$\langle p|\phi(x)|0\rangle\neq 0, \tag{2.23}$$

the field $\phi(x)$ can produce a single particle state $|p\rangle$ from the vacuum, with invariant mass m. In that case, the propagator $D(p)$ has a pole at $p^2=-m^2$

$$D(p)=\frac{iZ}{p^2+m^2}+... \tag{2.24}$$

where the ... denote non-pole terms, and Z is the normalization factor. We can now compute the S-matrix from the Green's function by picking out the poles for each particle [9]:

$$\lim_{q_i^2\to -m^2}\lim_{p_j^2\to -m^2}\prod_{i=1}^{m}(q_i^2+m^2)\prod_{j=1}^{n}(p_j^2+m^2)G(q_1,...,q_m;p_1,...,p_n)=\prod_{i=1}^{m}(i\sqrt{Z})\prod_{j=1}^{n}(i\sqrt{Z})\langle f|i\rangle, \tag{2.25}$$

where $|i\rangle=|p_1,...,p_n\rangle_{\text{in}}$ and $|f\rangle|q_1,...,q_m\rangle_{\text{out}}$ are some initial and final particle states, respectively. This is the LSZ reduction formula for a scalar field, and the only complication for fermions and gauge fields is that one has to contract with spinors and polarization vectors.

We will now show that field redefinitions do not change the S-matrix, i.e. do not change observable quantities. In order to do so, we turn to the path integral formulation of quantum field theory. From the functional integral

$$Z[J]=\int D\phi\ e^{i\int \mathcal{L}[\phi]+J\phi}, \tag{2.26}$$

we can compute the correlation functions as follows:

$$\langle 0|T\{\phi(x_1)...\phi(x_n)\}|0\rangle=\frac{1}{Z[J]}\frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)}Z[J]\bigg|_{J=0}. \tag{2.27}$$

Consider the field redefinition

$$\phi(x)=F[\phi'(x)]=\phi'+..., \tag{2.28}$$

where the dots can involve integer powers of ϕ and a finite number of derivatives. Under this field redefinition, the Lagrangian is also redefined by

$$\mathcal{L}[\phi]=\mathcal{L}[F[\phi'(x)]]\equiv\mathcal{L}'[\phi'] \tag{2.29}$$

From the redefined field ϕ' and Lagrangian $\mathcal{L}'$, we can compute correlation functions using the redefined functional integral

$$Z'[J]=\int D\phi' e^{i\int \mathcal{L}'[\phi']+J\phi'}=\int D\phi\, e^{i\int \mathcal{L}'[\phi]+J\phi}, \tag{2.30}$$

where in the last step, we could drop the prime since ϕ' is a dummy variable in the path integral. On the other hand, we can also compute the change of the original functional integral under the change of variables Eq. (2.28),

$$Z[J]=\int D\phi'\left|\frac{\delta F}{\delta\phi'}\right|e^{i\int \mathcal{L}'[\phi']+JF[\phi']}=\int D\phi\, e^{i\int \mathcal{L}'[\phi]+JF[\phi]}, \tag{2.31}$$

where we dropped the primes again on ϕ', and used that the Jacobian $\left|\frac{\delta F}{\delta\phi'}\right|$ is unity in the absence of an anomaly [10] (we compute the Jacobian explicitly in Sec. 2.4.2). From Eqs. (2.30) and (2.31) we see that we have two ways to compute correlation functions. Both use the redefined Lagrangian $\mathcal{L}'[\phi]$, but in the former the field ϕ is used and in the latter $F[\phi]$. As discussed in Eq. (2.23), the S-matrix does not care about the choice of field, as long as

$$\langle p|\phi(x)|0\rangle\neq 0,\quad\text{and}\quad\langle p|F[\phi(x)]|0\rangle\neq 0 \tag{2.32}$$

holds. Therefore, we can conclude that both the Lagrangian $\mathcal{L}[\phi]$ and $\mathcal{L}'[\phi]$ can be used to compute the same observable quantities.

2.4.2 A Special Case: Equations of Motion

Above, we showed that field redefinitions do not change observable quantities. We will now show that a special case of field redefinitions can be used to remove operators which are related through EOM-relations from the EFT Lagrangian. We will show this for a real scalar field following Refs. [1, 11, 12] . For a proof for the other fields, see in particular [12]. To be more precise what we mean by removing operator redundancies by EOM, lets work out an example before we go over the formal proof.

Suppose we have the following EFT Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \frac{c_1}{\Lambda^2}\phi^3\partial^2\phi + \frac{c_2}{\Lambda^2}\phi^6 + \mathcal{O}\Big(\frac{1}{\Lambda^4}\Big), \tag{2.33}$$

for a real scalar field ϕ. In the fourth term $\phi^3\partial^2\phi$, we recognise (a part of) the equation of motion $\partial^2\phi = ...$, and the idea now is that we can remove operators like this performing a suitable field redefinition. Remember that $\partial^2\phi$ in the EOM arises from the kinetic term in Eq. (2.33), so we can use this term to remove the fourth term. Therefore, it is not hard to see that the following field redefinition will do the job:

$$\phi \to \phi + \frac{c_1}{\Lambda^2}\phi^3. \tag{2.34}$$

Maintaining the EFT power counting, we get for the kinetic term

$$\begin{aligned}\frac{1}{2}\partial_\mu\phi\partial^\mu\phi &\to \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\Big(\frac{c_1}{\Lambda^2}\phi^3\Big)\partial^\mu\phi + \mathcal{O}\Big(\frac{1}{\Lambda^4}\Big)\\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{c_1}{\Lambda^2}\phi^3\partial^2\phi + \mathcal{O}\Big(\frac{1}{\Lambda^4}\Big),\end{aligned} \tag{2.35}$$

where we have dropped the total derivative in the second line. Indeed, the term we want to eliminate drops outs and we get for the redefined EFT Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \Big[\frac{1}{4!}\lambda + \frac{c_1}{\Lambda^2}m^2\Big]\phi^4 + \Big[\frac{c_2}{\Lambda^2} - \frac{c_1\lambda}{3!\Lambda^2}\Big]\phi^6 + \mathcal{O}\Big(\frac{1}{\Lambda^4}\Big) \tag{2.36}$$

The two Lagrangians Eqs. (2.33) and (2.36) give the same S-matrix, but the latter is easier to do computations with as we have eliminated a term. Note that we redefined the coefficients of the ϕ^4 and ϕ^6 terms, but this is not a problem as we fit the parameters to experiment.

With this example in mind, we can go back to the general proof. Like before, we write the EFT Lagrangian as

$$\mathcal{L}_{\text{eff}} = \sum_{n=0}^{\infty} \eta^n \mathcal{L}^{(n)} \tag{2.37}$$

From now on, we work up to order $\eta = \Lambda^{-1}$, but it is straightforward to show that what follows generalizes to arbitrary order in η^n. In general, we want to remove operators of the form

$$\eta\theta[\varphi_l] = \eta F[\varphi_l]E[X], \tag{2.38}$$

where $F[\varphi_l]$ is an arbitrary function of any of the fields appearing in the theory, denoted by φ_l, and their derivatives. Here $E[X]$ is the term that is proportional to the EOM, which reduces to

$$E[X] = \begin{cases} E[\phi] = D^2\phi & \text{for scalars,}\\ E[\psi] = \not{D}\psi & \text{for fermions,}\\ E[F] = D^\mu F^a_{\mu\nu} & \text{for field strength tensors,}\end{cases} \tag{2.39}$$

with a some group index. Note that in general $F[\varphi_l]$ can carry Lorentz, spinor and other group indices. We want to show that an operator of this form does not give a contribution to the S-matrix, and therefore does not alter experimental observables. For convenience, we write the functional integral as

$$\begin{aligned}Z[J_i] &= \int \prod_l D\varphi_l \, \exp\Big[i\int \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \sum_i J_i\varphi_i + \mathcal{O}(\eta^2)\Big]\\ &= \int \prod_l D\varphi_l \, \exp\Big[i\int \mathcal{L}^{(0)} + \eta\big(\mathcal{L}^{(1)} - \theta[\varphi_l]\big) + \eta\theta[\varphi_l] + \sum_i J_i\varphi_i + \mathcal{O}(\eta^2)\Big],\end{aligned} \tag{2.40}$$

to make it clear that we want to remove θ from the EFT Lagrangian. For now we restrict to a real scalar field, but in the end we point out the differences for complex scalar fields, fermions and field strength tensors. In Sec. 2.4.1, we showed that we have the freedom to make a change of variables. Exploiting this fact, we can make the following change of variables

$$\phi \to \phi + \eta F[\varphi_l]. \tag{2.41}$$

With this, the Lagrangian transforms in a similar way as in Eqs. (2.35) and (2.36):

$$\begin{aligned} \mathcal{L}^{(0)} + \eta\mathcal{L}^{(1)} &= \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{2}m^2\phi^2 + ... + \eta\mathcal{L}^{(1)} \\ &\to \mathcal{L}^{(0)} + D_\mu\Big(\eta F[\varphi_l]\Big)D^\mu\phi + \eta(-m^2F[\varphi_l] + ...) + \eta\mathcal{L}^{(1)} + \mathcal{O}(\eta^2) \\ &= \mathcal{L}^{(0)} - \eta F[\varphi_l]D^2\phi + \eta\big(-m^2\phi F[\varphi_l] + ... + \eta\mathcal{L}^{(1)}\big) + \mathcal{O}(\eta^2) \\ &= \mathcal{L}^{'(0)} - \eta\theta[\varphi_l] + \eta\mathcal{L}^{'(1)} + \mathcal{O}(\eta^2), \end{aligned} \tag{2.42}$$

where we neglected the total derivative again. Note that the field redefinition of Eq. (2.41) induces other terms, captured by the dots. However, we are not bothered by this as the EFT Lagrangian consists of all possible operators, so it boils down to a redefinition of the coefficients in $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$. We can compute the Jacobian using ghost fields

$$\left|\frac{\delta\phi(x)}{\delta\phi'(y)}\right| = \det\left[\delta(x-y) + \eta\frac{\delta F[\phi'(x)]}{\delta\phi'(y)}\right] = \int Dc D\bar{c}\exp\left[i\int\bar{c}\Big(1+\eta\frac{\delta F}{\delta\phi}\Big)c\right] \tag{2.43}$$

Here we should recall that the EFT is valid only up to energies of order $\Lambda = \frac{1}{\eta}$. By rescaling the ghost field, $c \to \frac{c}{\sqrt{\eta}}$, the ghost Lagrangian becomes

$$\mathcal{L}_{Ghost} = \eta\bar{c}c + \bar{c}\frac{\delta F}{\delta\phi}c. \tag{2.44}$$

So even if the ghost field propagates, their mass is of the order of the cutoff scale, meaning that just like the other massive degrees of freedom, we can integrate this out. This leads again only to a shift of the coefficients of the EFT Lagrangian. Therefore, we conclude that one can set the Jacobian equal to unity. As a final remark on this, note that the linear term in Eq. (2.41) is necessary for this argument to hold. Inserting Eqs. (2.42) and (2.43) in the functional integral of Eq. (2.40) we get

$$\begin{aligned} Z[J_i] &= \int\prod_l D\varphi_l\left|\frac{\delta\phi}{\delta\phi'}\right|\exp\left[i\int\mathcal{L}^{'(0)} - \eta\theta[\varphi_l] + \eta\big(\mathcal{L}^{'(1)} - \theta[\varphi_l]\big) + \eta\theta[\varphi_l] + \sum_i J_i\varphi_i + \eta J_\phi F[\varphi_l] + \mathcal{O}(\eta^2)\right] \\ &= \int\prod_l D\varphi_l\ \exp\left[i\int\mathcal{L}^{'(0)} + \eta\big(\mathcal{L}^{'(1)} - \theta[\varphi_l]\big) + \sum_i J_i\varphi_i + \eta J_\phi F[\varphi_l] + \mathcal{O}(\eta^2)\right]. \end{aligned} \tag{2.45}$$

After differentiating $Z[J_i]$ multiple times with respect to J_ϕ, an arbitrary correlation function becomes

$$\langle 0|T\{(\phi(x_1) + \eta F(x_1))...(\phi(x_n) + \eta F(x_n))\}|0\rangle. \tag{2.46}$$

In order to obtain the S-matrix, the LSZ reduction formula, given in Eq. (2.25), tells us that we must multiply the Green's functions (in momentum space), by (p^2+m^2) for each external leg carrying momentum p and then set p^2 to $-m$. So, the only part that survives is

$$\langle 0|T\{\phi(x_1)...\phi(x_n)\}|0\rangle, \tag{2.47}$$

because it has a pole that goes like $(p^2+m^2)^{-1}$ and any insertion of $F(x_n)$ is eliminated as it corresponds to terms with fewer than n poles. Therefore, Eq. (2.45) gives the same S-matrix as the generating functional

$$Z'[J_i] = \int\prod_l D\varphi_l\ \exp\left[i\int\mathcal{L}^{'(0)} + \eta\big(\mathcal{L}^{'(1)} - \theta[\varphi_l]\big) + \sum_i J_i\varphi_i + \mathcal{O}(\eta^2)\right], \tag{2.48}$$

and we conclude that we have removed the operator θ (Eq. (2.38)) for a real scalar field from the EFT Lagrangian. The preceding derivation can also be done for complex scalars, fermions, and field strength tensors. In that case, we can replace the change of variables from Eq. (2.41) by

$$\begin{cases} \phi^\dagger \to \phi^\dagger + \eta F[\varphi_l] & \text{for complex scalars,} \\ \bar{\psi} \to \bar{\psi} + \eta F[\varphi_l] & \text{for fermions,} \\ A^a_\mu \to A^a_\mu + \eta F[\varphi_l]^a_\mu & \text{for field strength tensors.} \end{cases} \tag{2.49}$$

and in the last case, we use the gauge fixing procedure to take care of the Faddeev-Popov ghost term in the Jacobian [12].

2.5 The Main Problem

Let's repeat the research question of this master's thesis:

Can we find a general method to construct bases of higher dimensional operators for effective field theories at any given order?

With everything that we have seen in this section, we can place this research question into more context. In order to use EFTs, we now know that a very important step is to write down the EFT Lagrangian. We saw that the EFT Lagrangian becomes an expansion in all possible operators of higher mass dimension, that are compatible with the symmetries of the EFT. Finding all these operators can be difficult, especially because the operator bases can be redundant by IBP-relations and through EOM. Mistakes are easily made and a general method would be therefore very helpful. Not only is this of phenomenological utility, but application of this general method to certain models might lead to new interesting theoretical questions and developments. In our route to finding this general method, our focus in this thesis is mainly on the bottom up approach of EFTs, but many of the techniques we will discuss can be used for the top down approach as well.

3 Standard Model Effective Field Theory

The Standard Model (SM) of particle physics is the theory describing the electromagnetic, weak and strong interactions, and it includes all known elementary particles. In Table 2, the field content of the SM is summarized together with the representation under the Lorentz group and the SM gauge group $SU(3) \times SU(2)_L \times U(1)$. We stick to the same conventions outlined in Sec. 2 of [13]. An overview of all notations in this thesis, including these, can be found in App. A. The renormalizable SM Lagrangian for one fermion generation before spontaneous symmetry breaking is given by

$$\mathcal{L}_{\rm SM}^{(\delta\leq 4)} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^I_{\mu\nu}W^{I\mu\nu} - \frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) + m^2\varphi^\dagger\varphi - \frac{1}{2}\lambda(\varphi^\dagger\varphi)^2 \\ + \; i\big(\bar{l}\not{D}l + \bar{e}\not{D}e + \bar{q}\not{D}q + \bar{u}\not{D}u + \bar{d}\not{D}d\big) - \big(\bar{l}\Gamma_e e\varphi + \bar{q}\Gamma_u u\widetilde{\varphi} + \bar{q}\Gamma_d d\varphi + \text{h.c.}\big), \quad (3.1)$$

where $\Gamma_{e,u,d}$ are the Yukawa couplings. For the sign convention of the covariant derivative D_μ, see Eq. (A.9). Although the SM has been successfully tested to great precision, it is commonly accepted that it should in fact be an effective theory of some yet unknown UV-theory with energy scale $\Lambda \gg m_t, m_W$, the masses of the heaviest degrees of freedom of the SM. Assuming that there is no unknown particle at the scale of the SM, we can take the field content of the SM as our light degrees of freedom to construct an EFT, called the Standard Model Effective Field Theory (SMEFT). From our discussion in Sec. 2.2, we know that the SMEFT Lagrangian becomes an expansion in powers of the operator dimension:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm SM}^{(\delta\leq 4)} + \frac{1}{\Lambda}\sum_i c_i^{(5)}\mathcal{O}_i^{(5)} + \frac{1}{\Lambda^2}\sum_i c_i^{(6)}\mathcal{O}_i^{(6)} + \mathcal{O}\Big(\frac{1}{\Lambda^3}\Big). \quad (3.2)$$

In this section, we will explore the operator basis of the Standard Model Effective Field Theory (SMEFT). The original analysis of the operator basis at dimension 6 was by Buchmuller and Wyler [14] in 1986. Although the presence of several EOM-vanishing combinations was already pointed out in the literature, it took up to 2010 before an updated list was published by Grzadkowski et al. [13]. Attempts were made to classify the operators at dimension 7 [15] and 8 [16] as well. However, due to the somewhat complicated structure of the field content of the SM and the fact that operators can be related by EOM-relations and IBP, mistakes are easily made. It became clear that a general method was needed and this will be the topic of Sec. 4.

In this section, we will repeat part of the analysis done by Grzadkowski et al. [13] for the operator basis at dimension 5 and 6. As will become clear, writing down operators by hand is a difficult and tedious task in which mistakes are easily made. However, we can learn a lot that will be very convenient for finding the general method later on, such as taking tensor products of the representations of the fields. We explain how we can construct invariant operators for the EFT Lagrangian in Sec. 3.1, after which we apply these techniques in Secs. 3.2 and 3.3 to construct some of the operators for the SMEFT at mass dimension 6. We discuss briefly how the other operators can be constructed in Sec. 3.4. We conclude in Sec. 3.5.

Operator class	Field		Lorentz	$SU(3)$	$SU(2)_L$	$U(1)$	Mass dimension
φ	Higgs	φ	$(0,0)$	**1**	**2**	$\frac{1}{2}$	1
X	B boson	$B_{\mu\nu}$	$(1,0)\oplus(0,1)$	**1**	**1**	0	2
	W boson	$W^I_{\mu\nu}$	$(1,0)\oplus(0,1)$	**1**	**3**	0	2
	Gluon	$G^A_{\mu\nu}$	$(1,0)\oplus(0,1)$	**8**	**1**	0	2
ψ	Leptons (×3 generations)	l	$(\frac{1}{2},0)$	**1**	**2**	$-\frac{1}{2}$	$\frac{3}{2}$
		e	$(0,\frac{1}{2})$	**1**	**1**	-1	$\frac{3}{2}$
	Quarks (×3 generations)	q	$(\frac{1}{2},0)$	**3**	**2**	$\frac{1}{6}$	$\frac{3}{2}$
		u	$(0,\frac{1}{2})$	**3**	**1**	$\frac{2}{3}$	$\frac{3}{2}$
		d	$(0,\frac{1}{2})$	**3**	**1**	$-\frac{1}{3}$	$\frac{3}{2}$

Table 2: Field content of the SM together with the representation under the gauge group $SU(3) \times SU(2)_L \times U(1)$.

3.1 Invariants

The symmetry groups of the SM are the Lorentz group[3] and the gauge group $SU(3) \times SU(2)_L \times U(1)$, and it is reasonable to assume that also the SMEFT obeys these symmetries. In order to impose these symmetries at the level of the (EFT) Lagrangian, we can only add invariant operators. To be more precise, the operators of the Lagrangian should transform as trivial representations under the symmetry groups. For the symmetry groups at hand, these are

Lorentz	$SU(3)$	$SU(2)_L$	$U(1)$
$(0,0)$	$\mathbf{1}$	$\mathbf{1}$	0

where we denoted the $SU(3)$ and $SU(2)$ representations by their dimension. In general if we denote the representation by its dimension, we will do this in bold. As summarized in Table 2, the building blocks for the operators do not transform trivially, and it is our task to find out how we can make combinations of the fields that will do. In order to do so, we can turn to representation theory.

As we can only add scalar quantities to the Lagrangian, we need to contract all the indices of the fields. From a more mathematically point of view, this is the same as writing down a tensor to which we feed the fields. Therefore, constructing tensor product representations will do the job. In general, these tensor products transform as reducible representations. If we are able to project out the trivial representation, we get an operator (although it may be redundant through IBP and using the EOM-relations). To show this with an example, lets assume that we have two spin-$\frac{1}{2}$ representations of $SU(2)$ and denote by ϕ and $\widetilde{\phi}$ the corresponding fields. Every physics student should be familiar with the following tensor product

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1, \tag{3.3}$$

in which the tensor products decomposes into the irreducible representations of spin-0 and spin-1. We know that the spin-0 component is the anti-symmetric combination of ϕ and $\widetilde{\phi}$, and thus the invariant operator is given by:

$$\phi^1\widetilde{\phi}^2 - \phi^2\widetilde{\phi}^1 = \epsilon_{ij}\phi^i\widetilde{\phi}^j, \tag{3.4}$$

because applying a group element $U \in SU(2)$ on this operator yields

$$\epsilon_{ij}\phi^i\widetilde{\phi}^j \to \epsilon_{ij}U^i_m\phi^m U^j_n\widetilde{\phi}^n = \det(U)\epsilon_{mn}\phi^m\widetilde{\phi}^n. \tag{3.5}$$

Using that $\det(U) = 1$ for a group element of $SU(2)$, we conclude that this operator indeed is invariant.

In general we can find operators by computing the tensor products for all representations of the fields of Table 2. Counting the number of trivial representations in the tensor product decomposition will help us in finding all independent operators. For $SU(2)$ we have the general rule that the tensor product of spin-j and spin-k decomposes as

$$j \otimes k = \bigoplus_{|j-k|\le \ell \le j+k} \ell. \tag{3.6}$$

The following tensor products for $SU(3)$ will be useful:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \quad \mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}, \quad \mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{1} \oplus [...] \tag{3.7}$$

For the tensor products of the Lorentz representations we use Eq. (C.11). Note that an operator is invariant under $U(1)$ if the total hypercharge vanishes. When computing tensor product decompositions, we will often denote by [...] all non-trivial representations. Lets put these ideas into practice to construct some of the operators of the SMEFT at dimension 6.

[3]To be more precise: the part of the Lorentz group that is connected to the identity (see App. C). We will have much more to say about this in later sections.

3.2 Bosonic operator classification

The operators for dimension 6 are presented in Tables 3 and 4 at the end of this section. For now we are only interested in the first two columns. In this subsection, we construct the purely bosonic operators of classes $\varphi^6, \mathcal{D}^2\varphi^4, X^3$, and $X^2\varphi^2$ shown in Table 3. With our explanations one should be able to construct the other operators as well using [13]. Before we get started, note that from φ and $\varphi^\dagger$ we can construct a singlet (spin-0) and triplet (spin-1) representation of $SU(2)$ (see Eq. (3.3)):

$$\varphi^\dagger\varphi, \quad \text{and} \quad \varphi^\dagger\tau^I\varphi \tag{3.8}$$

respectively. To show this, let $U \in SU(2)$, then $\varphi^\dagger\varphi \to \varphi^\dagger U^\dagger U\varphi = \varphi^\dagger\varphi$ where we used that $U^\dagger U = \mathbb{1}$. Similar, we get $\varphi^\dagger\tau^I\varphi \to \varphi^\dagger U^\dagger\tau^I U\varphi$. This transforms as a triplet, because the Pauli matrices transform in the adjoint representation (the triplet representation of $SU(2)$ is equivalent to the adjoint representation). Furthermore, we can form one singlet out of two $SU(2)$ triplets using δ_{IJ}. Similarly, for two $SU(3)$ octets we can use δ_{AB} to project out the trivial representation.

φ^6 and $\mathcal{D}^2\varphi^4$

As the Higgs field already transforms trivially under the Lorentz group and $SU(3)$, we do not have to worry about these groups. For the total hypercharge to vanish, we must have an equal number of Higgs fields and its complex conjugate. For the $SU(2)$ structure, we have to compute the tensor product of 6 spin-$\frac{1}{2}$ representations, which we write for convenience as

$$\begin{aligned}[0\oplus 1]\otimes[0\oplus 1]\otimes[0\oplus 1] &= [0\otimes 0\otimes 0]\oplus[0\otimes 1\otimes 1]\oplus[1\otimes 0\otimes 1]\oplus[1\otimes 1\otimes 0]\oplus[1\otimes 1\otimes 1]\oplus[...]\\ &= 0\oplus 0\oplus 0\oplus 0\oplus 0\oplus[...],\end{aligned} \tag{3.9}$$

where we used Eq. (3.3). Although we get 5 times the trivial representation, it will turn out that these are not all independent. Using Eqs. (3.8) and (3.9) it is clear that one operator follows from 3 singlets, so $(\varphi^\dagger\varphi)^3$ is an invariant operator. Three times an operator is constructed out of one singlet and two triplets. However, they are all the same, because

$$(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^I\varphi) = (\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi) = (\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\varphi). \tag{3.10}$$

Furthermore, we can use the relation[4]

$$\tau^I_{jk}\tau^I_{mn} = 2\delta_{jn}\delta_{mk} - \delta_{jk}\delta_{mn} \tag{3.11}$$

to show that

$$\begin{aligned}(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^I\varphi) &= (\varphi^\dagger\varphi)\varphi^\dagger_j\varphi_k\varphi^\dagger_m\varphi_n(2\delta_{jn}\delta_{mk} - \delta_{jk}\delta_{mn})\\ &= 2(\varphi^\dagger\varphi)\varphi^\dagger_j\varphi_k\varphi^\dagger_k\varphi_j - (\varphi^\dagger\varphi)\varphi^\dagger_j\varphi_j\varphi^\dagger_m\varphi_m\\ &= 2(\varphi^\dagger\varphi)^3 - (\varphi^\dagger\varphi)^3.\end{aligned} \tag{3.12}$$

So this is not an independent operator. We can construct one more operator out of 3 triplets:

$$\epsilon^{IJK}(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^J\varphi)(\varphi^\dagger\tau^K\varphi), \tag{3.13}$$

but this operator vanishes because the triplets are identical. Therefore, $(\varphi^\dagger\varphi)^3$ is the only pure Higgs operator at mass dimension 6.

We need at least two derivatives D_μ in order to get a trivial Lorentz representation:

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right) = (0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1). \tag{3.14}$$

The Lorentz singlet is obtained by contraction of the indices. This means that the derivatives should act on two different fields, as we can use EOM-relations to remove operator with an insertion of $D^2\varphi$ or $D^2\varphi^\dagger$. Again, we do not have to worry about the $SU(3)$ structure and we must have an equal number of Higgs fields and its complex conjugate for the total hypercharge to vanish. This leaves us with the following $SU(2)$ structure

$$[0\oplus 1]\otimes[0\oplus 1] = 0\oplus[1\otimes 1]\oplus 1\oplus 1 = 0\oplus 0\oplus[...], \tag{3.15}$$

[4] It is not hard to show this relation using the explicit form of the Pauli matrices (see Eq. (A.2))

so there should be two operators of this form. The derivatives can act on two conjugated or unconjugated fields

$$[(D^\mu\varphi)^\dagger\varphi][(D_\mu\varphi)^\dagger\varphi], \quad [\varphi^\dagger D^\mu\varphi][\varphi^\dagger D_\mu\varphi], \tag{3.16}$$

but we can use IBP and induce the EOM to show that these should be equivalent to

$$[(D^\mu\varphi)^\dagger\varphi][\varphi^\dagger(D_\mu\varphi)] = (\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi), \quad \text{and} \quad [\varphi^\dagger\varphi][(D_\mu\varphi)^\dagger(D_\mu\varphi)] \tag{3.17}$$

We can also relate the second one to the standard basis of Table 3, because

$$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi) = (\varphi^\dagger\varphi)D^\mu[(D_\mu\varphi)^\dagger\varphi + \varphi^\dagger(D_\mu\varphi)] = 2[\varphi^\dagger\varphi][(D_\mu\varphi)^\dagger(D_\mu\varphi)] + (\varphi^\dagger\varphi)[(D^2\varphi)^\dagger\varphi + \varphi^\dagger(D^2\varphi)], \tag{3.18}$$

and we can drop the terms containing $D^2\varphi$ and $D^2\varphi^\dagger$ through EOM relations. From Eq. (3.15) we see that we could also form a singlet out of two triplets, however we can use Eq. (3.11) again to show that we can relate such operators to the ones of above.

***X*³**

Lets start with the Lorentz structure:

$$(1,0)\otimes(1,0)\otimes(1,0) = [(0,0)\oplus(1,0)\oplus(2,0)]\otimes(1,0) = (0,0)\oplus[...] \tag{3.19}$$

Likewise for the tensor product of three times $(0,1)$ we find also one trivial representation. Therefore, we should be able to write two independent operators with 3 field strength tensors. We need three different fields for non-vanishing contraction of the Lorentz indices:

$$X_\mu^{\ \nu}Y_\nu^{\ \rho}Z_\rho^{\ \mu}, \quad \widetilde{X}_\mu^{\ \nu}Y_\nu^{\ \rho}Z_\rho^{\ \mu}. \tag{3.20}$$

Otherwise the anti-symmetry of the indices forces the operator to be zero. Instead of writing the $\epsilon_{\mu\nu\rho\sigma}$ explicitely, we use the dual tensor $\widetilde{X}_{\mu\nu}$. Because of the $SU(2)$ and $SU(3)$ structure of W and G, we cannot have an operator of the form BWG. We can form a gauge singlet out of two $SU(2)$ triplets or $SU(3)$ octets. However, they both vanish:

$$B_\mu^{\ \nu}W_\nu^{I\rho}\widetilde{W}_\rho^{I\mu} = 0 \quad \text{and} \quad B_\mu^{\ \nu}G_\nu^{A\rho}\widetilde{G}_\rho^{A\mu} = 0, \tag{3.21}$$

because $X_{\nu\rho}\widetilde{X}^\rho_{\ \mu}$ is symmetric in the indices μ and ν, while $X^{\mu\nu}$ is anti-symmetric in these indices. Therefore, the only other option is to construct gauge singlets out of three triplets or octets. We can do this using the structure constants ϵ^{IJK} and f^{ABC}:

$$\epsilon^{IJK}W_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}, \qquad \epsilon^{IJK}\widetilde{W}_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}, \qquad f^{ABC}G_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}, \qquad f^{ABC}\widetilde{G}_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}. \tag{3.22}$$

***X*²*φ*²**

We can form a $SU(2)$ singlet or triplet out of the two Higgs fields, and they must be of the form of Eq. (3.8) again because of the hypercharge constraints. We can form two Lorentz invariants out of the field strengths because

$$[(1,0)\oplus(0,1)]\otimes[(1,0)\oplus(0,1)] = (0,0)\oplus(0,0)\oplus[...], \tag{3.23}$$

and they are given by $X_{\mu\nu}Y^{\mu\nu}$ and $\widetilde{X}_{\mu\nu}Y^{\mu\nu}$ (X and Y can be the same in this case). Therefore, using δ_{IJ} and δ_{AB} to project out the trivial representations of $SU(2)$ and $SU(3)$ respectively, we can form with $\varphi^\dagger\varphi$ the following operators:

$$\varphi^\dagger\varphi\, B_{\mu\nu}B^{\mu\nu}, \quad \varphi^\dagger\varphi\, \widetilde{B}_{\mu\nu}B^{\mu\nu}, \quad \varphi^\dagger\varphi\, W^I_{\mu\nu}W^{I\mu\nu}, \quad \varphi^\dagger\varphi\, \widetilde{W}^I_{\mu\nu}W^{I\mu\nu}, \quad \varphi^\dagger\varphi\, G^A_{\mu\nu}G^{A\mu\nu}, \quad \varphi^\dagger\varphi\, \widetilde{G}^A_{\mu\nu}G^{A\mu\nu}. \tag{3.24}$$

If we use $\varphi^\dagger\tau^I\varphi$, we can combine this with the $SU(2)$ triplet $W^I_{\mu\nu}$ to get

$$\varphi^\dagger\tau^I\varphi\, W^I_{\mu\nu}B^{\mu\nu}, \quad \text{and} \quad \varphi^\dagger\tau^I\varphi\, \widetilde{W}^I_{\mu\nu}B^{\mu\nu}. \tag{3.25}$$

In principle, three $SU(2)$ triplets can combine to spin an overall singlet, but

$$\epsilon^{IJK}\varphi^\dagger\tau^I\varphi\, W^J_{\mu\nu}W^{K\mu\nu}, \quad \text{and} \quad \epsilon^{IJK}\varphi^\dagger\tau^I\varphi\, \widetilde{W}^J_{\mu\nu}W^{K\mu\nu} \tag{3.26}$$

both vanish.

3.3 Fermionic operator classification

In this subsection we will derive some of the four-fermion operators of Table 4. Because the fermions are chiral spinors, we can order these operators according to their chiral structure. In particular, we will discuss the $(\bar{L}L)(\bar{L}L)$, $(\bar{R}R)(\bar{R}R)$, and $(\bar{L}L)(\bar{R}R)$ classes.

$\boldsymbol{(\bar{L}L)(\bar{L}L)}$
Imposing hypercharge constrains, we see that we can only allow for the combinations $(\bar{l}l)^2$, $(\bar{q}q)^2$, and $\bar{l}l\bar{q}q$. Because $\bar{L}$ transforms as a right-handed spinor representation we get

$$\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right). \tag{3.27}$$

For the $SU(2)$ structure we can use the result for four Higgs fields of Eq. (3.15). However, q now also has $SU(3)$ structure. Because $\bar{q}$ transforms in the anti-fundamental representation, we can make use Eq. (3.7). Therefore, we can construct the two-fermion currents

$$(\bar{l}\gamma_\mu l),\quad (\bar{q}\gamma_\mu q),\quad (\bar{l}\gamma_\mu\tau^I l),\quad (\bar{q}\gamma_\mu\tau^I q),\quad (\bar{q}\gamma_\mu T^A q),\quad \text{and}\quad (\bar{q}\gamma_\mu T^A\tau^I q). \tag{3.28}$$

Two of such currents can combine to a Lorentz singlet by contracting the Lorentz index (see Eq. (3.14)). Using what we have seen earlier to get gauge invariants out of two $SU(2)$ triplets or $SU(3)$ octets, we get the following operators

$$(\bar{l}\gamma_\mu l)(\bar{l}\gamma^\mu l),\quad (\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q),\quad (\bar{q}\gamma_\mu\tau^I q)(\bar{q}\gamma^\mu\tau^I q),\quad (\bar{l}\gamma_\mu l)(\bar{q}\gamma^\mu q),\quad (\bar{l}\gamma_\mu\tau^I l)(\bar{q}\gamma^\mu\tau^I q), \tag{3.29}$$

$$(\bar{q}\gamma_\mu T^A q)(\bar{q}\gamma_\mu T^A q)\quad\text{and}\quad (\bar{q}\gamma_\mu T^A\tau^I q)(\bar{q}\gamma_\mu T^A\tau^I q) \tag{3.30}$$

However, these last two operators are not independent and we can relate them to the other operators. Using the following relation for the generators of $SU(3)$

$$T^A_{\alpha\beta}T^A_{\gamma\delta}=\frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}-\frac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta}, \tag{3.31}$$

we can write

$$\begin{aligned}(\bar{q}\gamma_\mu T^A q)(\bar{q}\gamma_\mu T^A q)&=(\bar{q}_{\alpha j}\gamma_\mu q_{\beta j})(\bar{q}_{\gamma k}\gamma_\mu q_{\delta k})\Big(\frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}-\frac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta}\Big)\\&=\frac{1}{2}(\bar{q}_{\alpha j}\gamma_\mu q_{\beta j})(\bar{q}_{\beta k}\gamma_\mu q_{\alpha k})-\frac{1}{6}(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)\end{aligned} \tag{3.32}$$

In order to rewrite the first part, we can use the following Fierz identity (see App. C.3)

$$(\bar{\psi}_L\gamma^\mu\psi_L)(\bar{\chi}_L\gamma_\mu\chi_L)=(\bar{\psi}_L\gamma^\mu\chi_L)(\bar{\chi}_L\gamma_\mu\psi_L). \tag{3.33}$$

With this, we get

$$(\bar{q}_{\alpha j}\gamma_\mu q_{\beta j})(\bar{q}_{\beta k}\gamma_\mu q_{\alpha k})=(\bar{q}_{\alpha j}\gamma_\mu q_{\alpha k})(\bar{q}_{\beta m}\gamma_\mu q_{\beta n})\delta_{jn}\delta_{km}. \tag{3.34}$$

Rewriting Eq. (3.11) as $\delta_{jn}\delta_{mk}=\frac{1}{2}\delta_{jk}\delta_{mn}+\frac{1}{2}\tau^I_{jk}\tau^I_{mn}$, we obtain

$$\begin{aligned}(\bar{q}_{\alpha j}\gamma_\mu q_{\beta j})(\bar{q}_{\beta k}\gamma_\mu q_{\alpha k})&=(\bar{q}_{\alpha j}\gamma_\mu q_{\alpha k})(\bar{q}_{\beta m}\gamma_\mu q_{\beta n})\frac{1}{2}\big(\delta_{jk}\delta_{mn}+\tau^I_{jk}\tau^I_{mn}\big)\\&=\frac{1}{2}(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)+\frac{1}{2}(\bar{q}\gamma_\mu\tau^I q)(\bar{q}\gamma^\mu\tau^I q)\end{aligned} \tag{3.35}$$

So, we see finally how the operator is a linear combinations of the other operators of Eq. (3.29):

$$(\bar{q}\gamma_\mu T^A q)(\bar{q}\gamma_\mu T^A q)=\frac{1}{4}(\bar{q}\gamma_\mu\tau^I q)(\bar{q}\gamma^\mu\tau^I q)-\frac{1}{12}(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q). \tag{3.36}$$

With a similar calculation we obtain

$$(\bar{q}\gamma_\mu T^A\tau^I q)(\bar{q}\gamma_\mu T^A\tau^I q)=\frac{3}{4}(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)-\frac{5}{12}(\bar{q}\gamma_\mu\tau^I q)(\bar{q}\gamma^\mu\tau^I q), \tag{3.37}$$

thereby completing our derivation of all independent operators of class $\boldsymbol{(\bar{L}L)(\bar{L}L)}$.

$(\bar{R}R)(\bar{R}R)$

Again, hypercharge constraints force us to look only at operators with $\bar{u}u$, $\bar{d}d$, and $\bar{e}e$. As $\bar{R}$ transforms as a left handed spinor, and we do not have to worry about $SU(2)$ structure, we can construct the following two-fermion currents

$$(\bar{u}\gamma_\mu u), \quad (\bar{d}\gamma_\mu d), \quad (\bar{e}\gamma_\mu e), \quad (\bar{u}\gamma_\mu T^A u), \quad \text{and} \quad (\bar{d}\gamma_\mu T^A d). \tag{3.38}$$

With these currents, we can construct the following operators

$$\begin{gathered}(\bar{u}\gamma_\mu u)(\bar{u}\gamma^\mu u), \quad (\bar{d}\gamma_\mu d)(\bar{d}\gamma^\mu d), \quad (\bar{e}\gamma_\mu e)(\bar{e}\gamma^\mu e), \quad (\bar{u}\gamma_\mu u)(\bar{d}\gamma^\mu d), \quad (\bar{u}\gamma_\mu u)(\bar{e}\gamma^\mu e), \quad (\bar{d}\gamma_\mu d)(\bar{e}\gamma^\mu e), \\ (\bar{u}\gamma_\mu T^A u)(\bar{d}\gamma^\mu T^A d), \quad (\bar{d}\gamma_\mu T^A d)(\bar{d}\gamma^\mu T^A d), \quad \text{and} \quad (\bar{u}\gamma_\mu T^A u)(\bar{u}\gamma^\mu T^A u). \end{gathered} \tag{3.39}$$

However, we can use Eqs. (3.31) and (3.33) to show that the last two operators can be written as a linear combination of the others:

$$\begin{aligned}(\bar{u}\gamma_\mu T^A u)(\bar{u}\gamma^\mu T^A u) &= (\bar{u}_\alpha\gamma_\mu u_\beta)(\bar{u}_\gamma\gamma^\mu u_\delta)\Big(\frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta}\Big) \\ &= \frac{1}{2}(\bar{u}_\alpha\gamma_\mu u_\beta)(\bar{u}_\beta\gamma^\mu u_\alpha) - \frac{1}{6}(\bar{u}\gamma_\mu u)(\bar{u}\gamma^\mu u) \\ &= \frac{1}{2}(\bar{u}_\alpha\gamma_\mu u_\alpha)(\bar{u}_\beta\gamma^\mu u_\beta) - \frac{1}{6}(\bar{u}\gamma_\mu u)(\bar{u}\gamma^\mu u) \\ &= \frac{1}{3}(\bar{u}\gamma_\mu u)(\bar{u}\gamma^\mu u),\end{aligned} \tag{3.40}$$

and similarly

$$(\bar{d}\gamma_\mu T^A d)(\bar{d}\gamma^\mu T^A d) = \frac{1}{3}(\bar{d}\gamma_\mu d)(\bar{d}\gamma^\mu d). \tag{3.41}$$

$(\bar{L}L)(\bar{R}R)$

For this class, we simply pick one current out of Eq. (3.28) and one out of Eq. (3.38). Contracting the Lorentz indices gives the following operator

$$\begin{gathered}(\bar{l}\gamma_\mu l)(\bar{u}\gamma^\mu u), \quad (\bar{l}\gamma_\mu l)(\bar{d}\gamma^\mu d), \quad (\bar{l}\gamma_\mu l)(\bar{e}\gamma^\mu e), \quad (\bar{q}\gamma_\mu q)(\bar{u}\gamma^\mu u), \quad (\bar{q}\gamma_\mu q)(\bar{d}\gamma^\mu d), \quad (\bar{q}\gamma_\mu q)(\bar{e}\gamma^\mu e) \\ (\bar{q}\gamma_\mu T^A q)(\bar{u}\gamma^\mu T^A u), \quad \text{and} \quad (\bar{q}\gamma_\mu T^A q)(\bar{d}\gamma^\mu T^A d).\end{gathered} \tag{3.42}$$

One might suspect that the last two operators can be related to the others as well, but this is not the case as the two octets in these operators include different fields.

3.4 Remaining classes

For the pure bosonic operators, one would expect to see terms of the form $\varphi^2\mathcal{D}^4, X^2\mathcal{D}^2, X\mathcal{D}^2\varphi^2$. However, these classes are absent in Table 3 and this follows because we can use IBP and EOM relations to remove such operators. For example, we can use IBP to move all derivatives D^μ on one Higgs field:

$$\varphi^\dagger D^\mu D^\nu D^\rho D^\sigma \varphi. \tag{3.43}$$

We can construct invariants using the epsilon tensor $\epsilon_{\mu\nu\rho\sigma}$, but this leads to an appearance of $[D_\mu, D_\nu] \sim X_{\mu\nu}$ which reduces to the $X\varphi^4$ class. The other way is to contract with two metrics $\eta_{\delta\lambda}\eta_{\alpha\beta}$, which gives[5] $D^2\varphi$ and can therefore be removed through EOM relations. With similar arguments one can show that $X^2\mathcal{D}^2, X\mathcal{D}^2\varphi^2$ can be removed from the SMEFT operator basis.

We did not construct all four-fermion operators of Table 4. For the remaining operators one should start by looking at the constraints by hypercharge to reduce the possible terms a lot. With what we did above, a straightforward calculation will give the other terms.

There are also mixed terms with fermions and bosons as the building blocks for the operator. Those of the form $\psi^2 X\varphi$ and $\psi^2\varphi^3$ will be straightforward to check. For those with a derivative one should expect the following classes

$$\psi^2\varphi^2\mathcal{D}, \quad \psi^2 X\mathcal{D}, \quad \psi^2\mathcal{D}^3 \quad \text{and} \quad \psi^2\varphi\mathcal{D}^2. \tag{3.44}$$

[5] And possible also a term with one or two $X_{\mu\nu}$'s by interchanging the derivatives, which is in one of the $X^2\varphi^2$ or $X\mathcal{D}^2\varphi^2$ classes.

However only the class $\psi^2\varphi^2\mathcal{D}$ appears in Table 3, as all other operators can be related by IBP en EOM relations. To get a flavour of what it boils down to, lets look at an operator of the form $\psi^2\mathcal{D}^3$. The three covariant derivatives are contracted with a current $\bar{\psi}\gamma_\mu\psi$ such as in Eqs. (3.28) and (3.38). We can use IBP to move all operators on ψ to get

$$\bar{\psi}D_\mu D^\mu\gamma_\nu D^\nu\psi = \bar{\psi}D^2 \not{D}\psi, \tag{3.45}$$

which is proportional to an EOM operator and we can therefore remove this operator from the EFT Lagrangian.

3.5 Summary

In the previous subsections we discussed how one can construct some of the operators for the basis at mass dimension 6 of the SMEFT. We saw that not only relations by IBP and EOM make the derivation difficult, but also the Fierz identities of Eqs. (3.11), (3.31) and (3.33) make it hard to find out if some operators can be related. It is clear that this will become harder at higher mass dimensions. Of great help for now was that we could count the number of operators of a given form by working out the tensor products. However, it still happened quite often that we overcounted, with the overcounting of 4 for φ^6 being an extreme case. The counting will obviously not become easier at higher mass dimension[6]. So a method that can do the counting for us will already be of much help, as knowing the number of operators of a given form has proven to be an excellent start for constructing operators. Furthermore, a general counting method can be of great use for exploring the operator bases of other EFTs as well.

[6] Try to work out the tensor product of 12 times the **8** representation of $SU(3)$.

	φ^6 and $\varphi^4\mathcal{D}^2$	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
Q_φ	$(\varphi^\dagger\varphi)^3$	$(\varphi^\dagger\varphi)^3$	φ^6	φ^6	✓
$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$2\mathcal{D}^2(\varphi^\dagger\varphi)^2$	$2\mathcal{D}^2\varphi^4$	$2\mathcal{D}^2\varphi^4$	✓
$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^\star\left(\varphi^\dagger D_\mu\varphi\right)$				✓
	X^3				
Q_G	$f^{ABC}G_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$	$2G^3$	0	G^3	✓
$Q_{\widetilde{G}}$	$f^{ABC}\widetilde{G}_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$				-
Q_W	$\epsilon^{IJK}W_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}$	$2W^3$	0	W^3	✓
$Q_{\widetilde{W}}$	$\epsilon^{IJK}\widetilde{W}_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}$				-
	$X^2\varphi^2$				
$Q_{\varphi G}$	$\varphi^\dagger\varphi\, G_{\mu\nu}^A G^{A\mu\nu}$	$2\varphi^\dagger\varphi G^2$	0	φ^2G^2	✓
$Q_{\varphi\widetilde{G}}$	$\varphi^\dagger\varphi\, \widetilde{G}_{\mu\nu}^A G^{A\mu\nu}$				-
$Q_{\varphi W}$	$\varphi^\dagger\varphi\, W_{\mu\nu}^I W^{I\mu\nu}$	$2\varphi^\dagger\varphi W^2$	0	φ^2W^2	✓
$Q_{\varphi\widetilde{W}}$	$\varphi^\dagger\varphi\, \widetilde{W}_{\mu\nu}^I W^{I\mu\nu}$				-
$Q_{\varphi B}$	$\varphi^\dagger\varphi\, B_{\mu\nu}B^{\mu\nu}$	$2\varphi^\dagger\varphi B^2$	0	φ^2B^2	✓
$Q_{\varphi\widetilde{B}}$	$\varphi^\dagger\varphi\, \widetilde{B}_{\mu\nu}B^{\mu\nu}$				-
$Q_{\varphi WB}$	$\varphi^\dagger\tau^I\varphi\, W_{\mu\nu}^I B^{\mu\nu}$	$2\varphi^\dagger\varphi WB$	0	φ^2WB	✓
$Q_{\varphi\widetilde{W}B}$	$\varphi^\dagger\tau^I\varphi\, \widetilde{W}_{\mu\nu}^I B^{\mu\nu}$				-
	$\psi^2X\varphi$				
Q_{eW}	$(\bar{l}_p\sigma^{\mu\nu}e_r)\tau^I\varphi W_{\mu\nu}^I +$ h.c.	$l^\dagger e\varphi W + le^\dagger\varphi^\dagger W$	0	$le\varphi W$	✓
Q_{eB}	$(\bar{l}_p\sigma^{\mu\nu}e_r)\varphi B_{\mu\nu} +$ h.c.	$l^\dagger e\varphi B + le^\dagger\varphi B$	0	$le\varphi B$	✓
Q_{uG}	$(\bar{q}_p\sigma^{\mu\nu}T^A u_r)\widetilde{\varphi}\, G_{\mu\nu}^A +$ h.c.	$q^\dagger u\varphi^\dagger G + qu^\dagger\varphi G$	0	$qu\varphi G$	✓
Q_{uW}	$(\bar{q}_p\sigma^{\mu\nu}u_r)\tau^I\widetilde{\varphi}\, W_{\mu\nu}^I +$ h.c.	$q^\dagger u\varphi^\dagger W + qu^\dagger\varphi W$	0	$qu\varphi W$	✓
Q_{uB}	$(\bar{q}_p\sigma^{\mu\nu}u_r)\widetilde{\varphi}\, B_{\mu\nu} +$ h.c.	$q^\dagger u\varphi^\dagger B + qu^\dagger\varphi B$	0	$qu\varphi B$	✓
Q_{dG}	$(\bar{q}_p\sigma^{\mu\nu}T^A d_r)\varphi\, G_{\mu\nu}^A +$ h.c.	$q^\dagger d\varphi G + qd^\dagger\varphi^\dagger G$	0	$qd\varphi G$	✓
Q_{dW}	$(\bar{q}_p\sigma^{\mu\nu}d_r)\tau^I\varphi\, W_{\mu\nu}^I +$ h.c.	$q^\dagger d\varphi W + qd^\dagger\varphi^\dagger W$	0	$qd\varphi W$	✓
Q_{dB}	$(\bar{q}_p\sigma^{\mu\nu}d_r)\varphi\, B_{\mu\nu} +$ h.c.	$q^\dagger d\varphi B + qd^\dagger\varphi^\dagger B$	0	$qd\varphi B$	✓
	$\psi^2\varphi^3$				
$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar{l}_p e_r\varphi) +$ h.c.	$\varphi^\dagger\varphi^2 l^\dagger e + \varphi(\varphi^\dagger)^2 le^\dagger$	0	$\varphi^3 le$	✓
$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p u_r\widetilde{\varphi}) +$ h.c.	$(\varphi^\dagger)^2\varphi q^\dagger u + \varphi^2\varphi^\dagger qu^\dagger$	0	$\varphi^3 qu$	✓
$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p d_r\varphi) +$ h.c.	$\varphi^\dagger\varphi^2 q^\dagger d + \varphi(\varphi^\dagger)^2 qd^\dagger$	0	$\varphi^3 qd$	✓
	$\psi^2\varphi^2\mathcal{D}$				
$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r)$	$2\mathcal{D}\varphi^\dagger\varphi l^\dagger l$	$2\mathcal{D}\varphi^2 l^2$	$2\mathcal{D}\varphi^2 l^2$	✓
$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu^I\varphi)(\bar{l}_p\tau^I\gamma^\mu l_r)$				✓
$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{q}_p\gamma^\mu q_r)$	$2\mathcal{D}\varphi^\dagger\varphi q^\dagger q$	$2\mathcal{D}\varphi^2 q^2$	$2\mathcal{D}\varphi^2 q^2$	✓
$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu^I\varphi)(\bar{q}_p\tau^I\gamma^\mu q_r)$				✓
$Q_{\varphi e}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{e}_p\gamma^\mu e_r)$	$\mathcal{D}\varphi^\dagger\varphi e^\dagger e$	$\mathcal{D}\varphi^2 e^2$	$\mathcal{D}\varphi^2 e^2$	✓
$Q_{\varphi u}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{u}_p\gamma^\mu u_r)$	$\mathcal{D}\varphi^\dagger\varphi u^\dagger u$	$\mathcal{D}\varphi^2 u^2$	$\mathcal{D}\varphi^2 u^2$	✓
$Q_{\varphi d}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{d}_p\gamma^\mu d_r)$	$\mathcal{D}\varphi^\dagger\varphi d^\dagger d$	$\mathcal{D}\varphi^2 d^2$	$\mathcal{D}\varphi^2 d^2$	✓
$Q_{\varphi ud}$	$i(\widetilde{\varphi}^\dagger D_\mu\varphi)(\bar{u}_p\gamma^\mu d_r) +$ h.c.	$\mathcal{D}\varphi^2 u^\dagger d + \mathcal{D}(\varphi^\dagger)^2 ud^\dagger$	0	$\mathcal{D}\varphi^2 ud$	✓

Table 3: All dimension-six operators which are not four-fermionic. First two columns are taken from Ref. [13]. In the third column, the results for the Hilbert series method are shown (see Secs. 4 and 5). Column 4, 5 and 6 show the results of the Hilbert series when enumerating $\mathcal{CP}$ invariant operators (see Sec. 6).

	$(\bar{L}L)(\bar{L}L)$	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
Q_{ll}	$(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t)$	$\bar{l}^2 l^2$	l^4	l^4	✓
$Q_{qq}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{q}_s\gamma^\mu q_t)$	$2\bar{q}^2 q^2$	$2q^4$	$2q^4$	✓
$Q_{qq}^{(3)}$	$(\bar{q}_p\gamma_\mu \tau^I q_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$				✓
$Q_{lq}^{(1)}$	$(\bar{l}_p\gamma_\mu l_r)(\bar{q}_s\gamma^\mu q_t)$	$2\bar{q}q\bar{l}l$	$2q^2 l^2$	$2q^2 l^2$	✓
$Q_{lq}^{(3)}$	$(\bar{l}_p\gamma_\mu \tau^I l_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$				✓
	$(\bar{R}R)(\bar{R}R)$				
Q_{ee}	$(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma^\mu e_t)$	$\bar{e}^2 e^2$	e^4	e^4	✓
Q_{uu}	$(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu u_t)$	$\bar{u}^2 u^2$	u^4	u^4	✓
Q_{dd}	$(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu d_t)$	$\bar{d}^2 d^2$	d^4	d^4	✓
Q_{eu}	$(\bar{e}_p\gamma_\mu e_r)(\bar{u}_s\gamma^\mu u_t)$	$\bar{e}e\bar{u}u$	e^2u^2	e^2u^2	✓
Q_{ed}	$(\bar{e}_p\gamma_\mu e_r)(\bar{d}_s\gamma^\mu d_t)$	$\bar{e}e\bar{d}d$	e^2d^2	e^2d^2	✓
$Q_{ud}^{(1)}$	$(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu d_t)$	$2\bar{u}u\bar{d}d$	$2u^2d^2$	$2u^2d^2$	✓
$Q_{ud}^{(8)}$	$(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t)$				✓
	$(\bar{L}L)(\bar{R}R)$				
Q_{le}	$(\bar{l}_p\gamma_\mu l_r)(\bar{e}_s\gamma^\mu e_t)$	$\bar{l}l\bar{e}e$	l^2e^2	l^2e^2	✓
Q_{lu}	$(\bar{l}_p\gamma_\mu l_r)(\bar{u}_s\gamma^\mu u_t)$	$\bar{l}l\bar{u}u$	l^2u^2	l^2u^2	✓
Q_{ld}	$(\bar{l}_p\gamma_\mu l_r)(\bar{d}_s\gamma^\mu d_t)$	$\bar{l}l\bar{d}d$	l^2d^2	l^2d^2	✓
Q_{qe}	$(\bar{q}_p\gamma_\mu q_r)(\bar{e}_s\gamma^\mu e_t)$	$\bar{q}q\bar{e}e$	q^2e^2	q^2e^2	✓
$Q_{qu}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{u}_s\gamma^\mu u_t)$	$2\bar{q}q\bar{u}u$	$2q^2u^2$	$2q^2u^2$	✓
$Q_{qu}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{u}_s\gamma^\mu T^A u_t)$				✓
$Q_{qd}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{d}_s\gamma^\mu d_t)$	$2\bar{q}q\bar{d}d$	$2q^2d^2$	$2q^2d^2$	✓
$Q_{qd}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{d}_s\gamma^\mu T^A d_t)$				✓
	$(\bar{L}R)(\bar{R}L)$ and $(\bar{L}R)(\bar{L}R)$				
Q_{ledq}	$(\bar{l}_p^j e_r)(\bar{d}_s q_t^j)$+ h.c.	$\bar{l}e\bar{d}q + l\bar{e}d\bar{q}$	0	$ledq$	✓
$Q_{quqd}^{(1)}$	$(\bar{q}_p^j u_r)\epsilon_{jk}(\bar{q}_s^k d_t)$+ h.c.	$2\bar{q}^2ud + 2q^2\bar{u}\bar{d}$	0	$2q^2ud$	✓
$Q_{quqd}^{(8)}$	$(\bar{q}_p^j T^A u_r)\epsilon_{jk}(\bar{q}_s^k T^A d_t)$+ h.c.				✓
$Q_{lequ}^{(1)}$	$(\bar{l}_p^j e_r)\epsilon_{jk}(\bar{q}_s^k u_+$ h.c.	$2\bar{l}e\bar{q}u + 2l\bar{e}q\bar{u}$	0	$2lequ$	✓
$Q_{lequ}^{(3)}$	$(\bar{l}_p^j \sigma_{\mu\nu} e_r)\epsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$+ h.c.				✓
	B-violating				
Q_{duq}	$\epsilon^{\alpha\beta\gamma}\epsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$ + h.c.	$duql + \bar{d}\bar{u}\bar{q}\bar{l}$	0	$duql$	✓
Q_{qqu}	$\epsilon^{\alpha\beta\gamma}\epsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$ + h.c.	$q^2ue + \bar{q}^2\bar{u}\bar{e}$	0	q^2ue	✓
Q_{qqq}	$\epsilon^{\alpha\beta\gamma}\epsilon_{jn}\epsilon_{km}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$ + h.c.	$q^3l + \bar{q}^3\bar{l}$	0	q^3l	✓
Q_{duu}	$\epsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$ + h.c.	$u^2de + \bar{u}^2\bar{d}\bar{e}$	0	u^2de	✓

Total:	63	84	30	57	57

Table 4: All dimension-six four-fermion operators. First two columns are taken from [13]. In the third column, the results for the Hilbert series method are shown (see Secs. 4 and 5). Column 4, 5 and 6 show the results of the Hilbert series when enumerating $\mathcal{CP}$ invariant operators (see Sec. 6).

4 Hilbert Series

From the previous section, it became clear that constructing the minimal operator basis for the SMEFT is cumbersome, and this problem will also occur for other EFTs. A general method that can do the counting of operators of a given form can be therefore very helpful. To be more precise: in general the building blocks for the EFT Lagrangian are a set of N fields ϕ_i transforming in a representation of some symmetry group G, and the (covariant) derivative. We are interested in the number $c_{k,n_1,...,n_N}$ of invariant operators one can construct from n_i fields ϕ_i and k derivatives. Therefore, we define the Hilbert series (HS) as [17]

$$H(\mathcal{D}, \{\phi_i\}) = \sum_{k=0}^{\infty} \sum_{n_1=0}^{\infty} ... \sum_{n_N=0}^{\infty} c_{k,n_1,...,n_N} \mathcal{D}^k \phi_1^{n_1} ... \phi_N^{n_N} . \tag{4.1}$$

Here ϕ_i and $\mathcal{D}$ are simply used to count the number of occurrences of the field and the derivative in the operator and should not be confused with the fields itself. The direct computation of the HS is the topic of this section. We follow the approach of [18, 19], but restrict our discussion to EFTs in four space-time dimensions. The methods and results covered in this section are summarized in [20].

We start by deriving the HS for operators without derivatives in Sec. 4.1. In order to include derivatives we need to extend the Lorentz representations to representations of the conformal group, which we discuss in Sec. 4.2. In Sec. 4.3 we derive the HS accounting for relations from integration by parts (IBP). We subsequently deal with equations of motion (EOM) in Sec. 4.4. We conclude this section by constructing the HS for an EFT with gravity in Sec. 4.5.

4.1 Hilbert Series for operators without derivatives

We start with the construction of the HS for operators without derivatives. This means we do not need to think about EOM and IBP redundancies yet. For simplicity, consider just a single field ϕ transforming under the representation R of some symmetry group G. The HS therefore takes the following form:

$$H(0, \phi) = \sum_{n=0}^{\infty} c_{0,n} \phi^n \tag{4.2}$$

As became clear from the previous section, an operator made out of n fields ϕ transforms under the tensor product

$$\underbrace{R \otimes ... \otimes R}_{\text{n-times}} . \tag{4.3}$$

Furthermore, we decomposed such tensor products into the irreducible components by hand, and we tried to compute the coefficients $c_{0,n}$ by counting the number of trivial representations. Although this procedure turned out to give an overcounting from time to time, lets stick with this naive way of computing $c_{0,n}$ in the hope we learn enough to correct for the overcounting later. We can generalize this counting procedure up to any order by using characters, the trace of the representation matrices. Although characters do not tell us much about the group structure, they encode some valuable information about the representations. This is best shown with the following properties of characters [21]:

$$\chi_{R_1 \otimes R_2} = \chi_{R_1} \cdot \chi_{R_2} \qquad \text{and} \qquad \chi_{R_1 \oplus R_2} = \chi_{R_1} + \chi_{R_2} , \tag{4.4}$$

where $R_{1,2}$ are representations of G with characters χ_{R_1} and χ_{R_2} respectively. Furthermore, if R_1 and R_2 are irreducible, then they obey the following character orthogonality:

$$\int_G \mathrm{d}\mu \, \chi^*_{R_i}(g)) \chi_{R_2}(g) = \delta_{R_1,R_2} , \tag{4.5}$$

where $\mathrm{d}\mu$ is the Haar measure on G. See App. B for some formulas for the characters and Haar measures of simple, semi-simple Lie groups. With these properties, we can simply express the character for the tensor representation of Eq. (4.3) as $\chi_{R\otimes ... \otimes R} = (\chi_R)^n$. Using Eq. (4.4), the character decomposes into a sum of the characters of the irreducible components of Eq. (4.3). Exploiting the orthogonality of characters, we can project out the number of trivial representations

$$c_{0,n} = \int_G \mathrm{d}\mu \, 1 \cdot \big(\chi_R(g)\big)^n , \tag{4.6}$$

where we used that the trivial representation has character equal to 1[7]. Inserting this into Eq. (4.2), we get

$$H(0,\phi) = \int_G \mathrm{d}\mu \sum_{n=0}^{\infty} \phi^n \big(\chi_R(g)\big)^n = \int_G \mathrm{d}\mu \, \frac{1}{1-\phi\chi_R(g)}. \tag{4.7}$$

However, this is not what we want in the context of EFTs, because if we have n copies of the field ϕ, we can always (anti-)commute these n identical copies of the field in the case of bosons (fermions). Therefore, we are only interested in the HS where ϕ^n transforms in the symmetric (anti-symmetric) tensor product $\mathrm{sym}^n(R)$ ($\wedge^n(R)$), and this corrects for the overcounting of operators in Eq. (4.7). By projecting out the trivial represenation again, the HS for bosonic operators takes the following form

$$H(0,\phi) = \int_G \mathrm{d}\mu \, 1 \cdot \sum_{n=0}^{\infty} \phi^n \chi_{\mathrm{sym}^n(R)}(g) = \int_G \mathrm{d}\mu \, \exp\left[\sum_{n=1}^{\infty} \frac{\phi^n}{n}\chi_R(g^n)\right] \equiv \int_G \mathrm{d}\mu \, \mathrm{PE}\left[\phi\chi_R(g)\right], \tag{4.8}$$

where we made use of the plethystic exponential. For a derivation of this exponential, see App. B.1.1. For Fermionic operators we get a slightly different form

$$H(0,\phi) = \int_G \mathrm{d}\mu \, 1 \cdot \sum_{n=0}^{\infty} \phi^n \chi_{\wedge^n(R)}(g) = \int_G \mathrm{d}\mu \, \exp\left[-\sum_{n=1}^{\infty} \frac{(-\phi)^n}{n}\chi_R(g^n)\right] \equiv \int_G \mathrm{d}\mu \, \mathrm{PEF}\left[\phi\chi_R(g)\right], \tag{4.9}$$

where the Pletystic exponential for anti-symmetric tensor products was used (see App. B.1.2).

For N bosonic fields ϕ_i we need to deal with tensor products of the form

$$\mathrm{sym}^{n_1}(R_1) \otimes ... \otimes \mathrm{sym}^{n_N}(R_N), \tag{4.10}$$

which has character equal to

$$\chi_{\mathrm{sym}^{n_1}(R_1)} \cdots \chi_{\mathrm{sym}^{n_N}(R_N)}. \tag{4.11}$$

The HS for multiple bosonic fields therefore becomes

$$\begin{aligned} H(0,\{\phi_i\}) &= \int_G \mathrm{d}\mu \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \phi_1^{n_1} \ldots \phi_N^{n_N} \chi_{\mathrm{sym}^{n_1}(R_1)} \cdots \chi_{\mathrm{sym}^{n_N}(R_N)} \\ &= \int_G \mathrm{d}\mu \sum_{n_1=0}^{\infty} \phi_1^{n_1} \chi_{\mathrm{sym}^{n_1}(R_1)} \cdots \sum_{n_N=0}^{\infty} \phi_N^{n_N} \chi_{\mathrm{sym}^{n_N}(R_N)} \\ &= \int_G \mathrm{d}\mu \prod_{i=1}^{N} \mathrm{PE}\left[\phi_i \chi_{R_i}(g)\right], \end{aligned} \tag{4.12}$$

and we get the fermionic case by replacing PE with PEF. This generalizes to EFTs with both fermions and bosons:

$$H(0,\{\phi_i\}) = \int_G \mathrm{d}\mu \prod_{i=1}^{N} \begin{cases} \mathrm{PE}\left[\phi_i \chi_{R_i}(g)\right] & \phi_i \text{ is boson,} \\ \mathrm{PEF}\left[\phi_i \chi_{R_i}(g)\right] & \phi_i \text{ is fermion.} \end{cases} \tag{4.13}$$

As a final remark, note that we have projected out the number of trivial representations. However, we can project out every representation R' we like by multiplying above integrands by $\chi^*_R(g)$.

4.2 Conformal Representations

The construction of the HS for operators without derivatives was relatively easy as the only redundancy in operators was due to the bosonic or fermionic nature of the fields. When we include the (covariant) derivative, it gets harder as operator redundancies arise from IBP and EOM relations. Henning et al. [19] pointed out that these difficulties can be tackled by working with representations of the conformal group $SO(d,2)$, and they were the first to compute the HS for operators that cannot be related by IBP and EOM relations. However, their construction is in d space-time

[7] We can project out any irreducible representation $\widetilde{R}$ of G by replacing 1 with $\chi^*_{\widetilde{R}}(g)$.

dimension and can therefore become hard to grasp. In this section we try to strip off the abstractness and focus on EFTs in 4 space-time dimensions.

Luckily we do not need to know everything about representations of the conformal group in order to work with them. This is mainly due to the fact that the Lorentz group $SO(3,1)$ is a subgroup of the conformal group $SO(4,2)$. In fact, we can construct the conformal representations out of the representations of the Lorentz group and we will rely on this result in this section. This is very useful as we expect that most physicists are more familiar with representations of the Lorentz group than of the conformal group. In App. D we show this connection explicitly by constructing the algebra of $\mathfrak{so}(4,2)$ by extending the algebra of the Lorentz group. We encourage the interested reader to have a look.

Lets start by considering a single field ϕ_ℓ which transforms under a representation $\ell = (\ell_1, \ell_2)$ of the Lorentz group. Of course, ϕ_ℓ can transform under some other symmetry group G as well, but we will focus on the Lorentz group due to its interplay with derivatives. As discussed in App. D.3, we can extend this to a representation (Δ, ℓ) of $SO(4,2)$, where Δ is an extra label that is needed because the rank[8] of $\mathfrak{so}(4,2)$ is 3. As Δ is the eigenvalue of the dilaton operator, we can physically associate this with the scaling dimension of ϕ_ℓ. The conformal representation $R_{\Delta,\ell}$ now acts on objects like

$$\begin{pmatrix} \phi_\ell \\ D_{\mu_1}\phi_\ell \\ D_{\mu_1}D_{\mu_2}\phi_\ell \\ D_{\mu_1}D_{\mu_2}D_{\mu_3}\phi_\ell \\ \vdots \end{pmatrix}, \tag{4.14}$$

and the highest weight ϕ_ℓ is called the primary operator. The (covariant) derivative acts as a lowering operator, so IBP identities can be taken into account by projecting out the highest weight state of $R_{\Delta,\ell}$.

From our discussion of the previous section it is clear that we need the character of the conformal representation. Commutators of derivatives yield field strength (and Weyl) tensors when including gauge groups (gravity), so we may treat multiple derivatives as if they are transforming under $\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})$ under the Lorentz group. Therefore, the derivation of the character in App. D.4 still holds and we get

$$\chi_{\Delta,\ell}(q,x) = q^\Delta \chi_\ell(x) \sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x) \equiv q^\Delta \chi_\ell(x_1,x_2) P(q,x). \tag{4.15}$$

Here $\chi_\ell(x)$ is the Lorentz character of ℓ, $x = (x_1, x_2)$ parametrizes the Lorentz group, and q denotes the scaling dimension[9]. Furthermore, we defined the momentum generating function

$$P(q,x) \equiv \sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x) = \mathrm{PE}\left[q\chi_{(\frac{1}{2},\frac{1}{2})}(x)\right], \tag{4.16}$$

which we can write as a plethystic exponential. It is important to note here that in $\chi_{\Delta,\ell}(q,x)$ we can relate q directly with the counting of (covariant) derivatives by dividing out the scaling dimension of ϕ_ℓ. The characters are orthogonal when we define the measure of $SO(4,2)$ as follows (see App. D.4)

$$\int \mathrm{d}\mu_{SO(4,2)}\, \chi^*_{\Delta,\ell}\chi_{\Delta',\ell'} = \int \mathrm{d}\mu_L \oint \frac{dq}{2\pi i q} \frac{1}{|P(q,x)|^2} \chi^*_{\Delta,\ell}\chi_{\Delta',\ell'} = \delta_{\ell,\ell'}\delta_{\Delta,\Delta'}, \tag{4.17}$$

where $\mathrm{d}\mu_L$ is the measure of the Lorentz group.

[8]To be precise, the Cartan algebra now consist of $J^3_{L,R}$ the two $SU(2)$ Cartan matrices from the Lorentz group, and the dilaton operator D.

[9]Characters are class functions, meaning they are conjugation invariant. Therefore, we can compute the conformal characters in terms of the $U(1)$-variables $q = e^{i\theta D}$, $x_1 = e^{i\theta_L J^3_L}$ and $x_2 = e^{i\theta_R J^3_R}$

4.3 Integration by Parts

Now that we have constructed (the orthonormal) characters for conformal representations, we can construct the HS for the operators with derivatives. First, we write down the generating function $Z[\phi, q, x]$ for a bosonic field ϕ with spin ℓ and scaling dimension Δ:

$$Z[\phi, q, x] = 1 + \phi\chi_{\Delta,\ell}(q, x) + \phi^2\chi_{\mathrm{sym}^2(\Delta,\ell)}(q, x) + ... = 1 + \sum_{n=1}^{\infty} \phi^n \chi_{\mathrm{sym}^n(\Delta,\ell)}(q, x) = \mathrm{PE}\left[\phi\chi_{\Delta,\ell}(q, x)\right]. \tag{4.18}$$

We can decompose the tensor products into the direct sum of irreducible representations $R_{\Delta',\ell'}$:

$$\chi_{\mathrm{sym}^n(\Delta,\ell)}(q, x) = \sum_{\Delta',\ell'} b^{(n)}_{\Delta',\ell'} \chi_{\Delta',\ell'}(q, x), \tag{4.19}$$

with $b^{(n)}_{(\Delta',\ell')}$ the multiplicities of the (symmetric) tensor product decomposition. Inserting this in Eq. (4.18) and swapping the two sums, we get

$$Z[\phi, q, x] = 1 + \sum_{\Delta',\ell'} \sum_{n=1}^{\infty} \phi^n b^{(n)}_{\Delta',\ell'} \chi_{\Delta',\ell'}(q, x) = 1 + \sum_{\Delta',\ell'} c_{\Delta',\ell'}(\phi)\chi_{\Delta',\ell'}(q, x), \tag{4.20}$$

were we defined $c_{\Delta',\ell'}(\phi) = \sum_{n=1}^{\infty} \phi^n b^{(n)}_{\Delta',\ell'}$. As explained before, we are not only interested to project out a particular representation, but we want are interested in the highest weights to take IBP identities into account. Furthermore, in order for these highest weights to be proper operators for the EFT Lagrangian, we need them to be Lorentz invariant. Therefore, we are only interested in projecting out the coefficients $c_{\Delta',(0,0)}(\phi)$, which give precisely the number of invariant operators of dimension Δ', weighted by the number of fields ϕ. As the derivatives in such an operator are the only other objects carrying mass dimension we can count the number of derivatives $\mathcal{D}$ from this mass dimension and the number of fields. Therefore, we can write the HS as

$$H(\mathcal{D}, \phi) = 1 + \sum_{\Delta'} c_{\Delta',(0,0)}\Big(\frac{\phi}{\mathcal{D}^\Delta}\Big)\mathcal{D}^{\Delta'} = 1 + \sum_{n=0}^{\infty} c_{(\Delta+n),(0,0)}\Big(\frac{\phi}{\mathcal{D}^\Delta}\Big)\mathcal{D}^{\Delta+n}. \tag{4.21}$$

Looking at Eq. (4.20), we see that we can compute the coefficients by exploiting the orthogonality of the characters:

$$\begin{aligned} c_{\Delta',(0,0)}\Big(\frac{\phi}{\mathcal{D}^\Delta}\Big) &= \int \mathrm{d}\mu_{SO(4,2)}\, \chi^*_{\Delta',(0,0)} \Big(Z\Big[\frac{\phi}{\mathcal{D}^\Delta}, q, x\Big] - 1\Big) \\ &= \int \mathrm{d}\mu_L \oint \frac{\mathrm{d}q}{2\pi i q} \frac{1}{|P(q, x)|^2} q^{-\Delta'} P^*(q, x) \Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi_{\Delta,\ell}(q, x)\Big] - 1\Big), \end{aligned} \tag{4.22}$$

where we used that $q \in U(1)$. Plugging this back into the HS gives

$$\begin{aligned} H(\mathcal{D}, \phi) &= 1 + \int \mathrm{d}\mu_L \oint \frac{\mathrm{d}q}{2\pi i q} \sum_{n=0}^{\infty} \Big(\frac{\mathcal{D}}{q}\Big)^{\Delta+n} \frac{1}{P(q, x)} \Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi_{\Delta,\ell}(q, x)\Big] - 1\Big) \\ &= 1 + \int \mathrm{d}\mu_L \oint \frac{\mathrm{d}q}{2\pi i} \Big(\frac{\mathcal{D}}{q}\Big)^{\Delta} \frac{1}{q - \mathcal{D}} \frac{1}{P(q, x)} \Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi_{\Delta,\ell}(q, x)\Big] - 1\Big) \end{aligned} \tag{4.23}$$

where we used the geometric series to compute the sum over n. Before we perform the integral over q, we have to distinguish between two cases. First of all, if $\Delta > 1$ the integral picks up the residue at the single pole $q = \mathcal{D}$. If $\Delta = 1$, we also get a contribution from the pole $q = 0$ ($\frac{1}{P}$ and PE are regular in $q = 0$), however this contribution vanishes as $\chi_{1,\ell}(0, x) = 0$, and $\mathrm{PE}[0] = 1$. Therefore, the only contribution is from $q = \mathcal{D}$ and we obtain

$$H(\mathcal{D}, \phi) = 1 + \int \mathrm{d}\mu_L \frac{1}{P(\mathcal{D}, x)} \Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi_{\Delta,\ell}(\mathcal{D}, x)\Big] - 1\Big). \tag{4.24}$$

A straightforward calculations leads to the HS for N fields ϕ_i with conformal representation R_{Δ_i,ℓ_i}, which are now also transforming under some representation R_i of a gauge symmetry group G. The generating function of Eq. (4.18) becomes

$$Z[\{\phi_i\}, q, x, g] = \prod_{i=1}^{N} \begin{cases} \mathrm{PE}\left[\phi_i\chi_{\Delta_i,\ell_i}(q, x)\chi_{R_i}(g)\right] & \phi_i \text{ is boson,} \\ \mathrm{PEF}\left[\phi_i\chi_{\Delta_i,\ell_i}(q, x)\chi_{R_i}(g)\right] & \phi_i \text{ is fermion,} \end{cases} \tag{4.25}$$

with $g \in G$. We can follow the same steps that led from Eq. (4.18) to Eq. (4.24), but now we also need to integrate over G to project out gauge singlets. We therefore obtain:

$$H(\mathcal{D}, \{\phi_i\}) = 1 + \int_G \mathrm{d}\mu \int \mathrm{d}\mu_L \, \frac{1}{P(\mathcal{D}, x)} \Big(Z\Big[\Big\{ \frac{\phi_i}{\mathcal{D}^{\Delta_i}} \Big\}, \mathcal{D}, x, g\Big] - 1 \Big) \tag{4.26}$$

with $\mathrm{d}\mu$ the Haar measure of G.

As a final remark, note that $\frac{1}{P}$ accounts for the IBP relations. Looking back at Eq. (4.21) we see that from $c_{\Delta',(0,0)}$ we can only extract the number of derivatives and fields that make up the highest weight. Accounting for all descendants in the conformal representation can be done by multiplying with the momentum generating function $P(\mathcal{D}, x)$. The integral over the Lorentz group will project out the invariant operators. Looking at Eqs. (4.24) and (4.26), we see that in this case the $\frac{1}{P}$ drops out. This means that we have a method to compute both the HS for operators related by IBP and for operators which are not redundant by IBP relations.

4.4 Equations of Motion

EOM redundancies can be removed by modifying the characters for the conformal representation in the previous section. As discussed in [19], the proper way of doing this is to look at so-called short representations. We provide some background reading on short representations in App. D.2, where we show that it is permitted to remove the EOM redundancies from the conformal representations as long as we choose our scaling dimension Δ right.

4.4.1 Scalar Field

We start with the EOM for a Lorentz scalar ϕ ($\Delta = 1, \ell = (0,0)$). Redundancies through EOM allows us to drop operators with an insertion of $D^2\phi$. The contracted indices correspond to taking a trace, which we therefore need to remove from Eq. (4.14). For example, we can split the row with two derivatives in Eq. (4.14) into the trace and a traceless part

$$D_{\mu_1} D_{\mu_2} \phi = \underbrace{(D_{\mu_1} D_{\mu_2} - \tfrac{1}{4}\eta_{\mu_1\mu_2} D^2)\phi}_{\text{Traceless}} + \tfrac{1}{4}\underbrace{\eta_{\mu_1\mu_2} D^2 \phi}_{\text{Trace}}, \tag{4.27}$$

we only want the first term on the right-hand side. We can generalize this to a row with n derivatives by contracting two indices and leaving all other indices symmetric. It is hard to write this generalization as explicit as we did in Eq. (4.27), but in terms of characters under the Lorentz group this boils down to the following:

$$\chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x) = \widetilde{\chi}_n(x) + \chi_{\mathrm{sym}^{n-1}(\frac{1}{2},\frac{1}{2})}(x) \cdot 1, \tag{4.28}$$

with 1 the trivial character of $D^2\phi$ and $\widetilde{\chi}$ the character of the traceless part that we seek[10]. Implementing this modified Lorentz character changes the character of the conformal representation to

$$\begin{aligned}\widetilde{\chi}_{[1,(0,0)]}(q, x) &= \sum_{n=0}^{\infty} q^{1+n} \widetilde{\chi}_n(x) = q \left(\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x) - \sum_{n=2}^{\infty} q^n \chi_{\mathrm{sym}^{n-2}(\frac{1}{2},\frac{1}{2})}(x) \right) \\ &= q\left(1 - q^2\right) \sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x) = q\left(1 - q^2\right) P(q, x).\end{aligned} \tag{4.29}$$

4.4.2 Chiral Fermionic Field

We can apply this reasoning to a left handed fermionic field ψ_L with spin $\ell = (\frac{1}{2}, 0)$ under the Lorentz Group and scaling dimension $\Delta = \frac{3}{2}$. The EOM for this fermionic field allows us to remove operators with an insertion of $\not{D}\psi_L = \gamma^\mu D_\mu \psi_L$. It follows from the tensor product of the representation $(\frac{1}{2}, \frac{1}{2})$ for the derivative combined with that of ψ_L:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, 0\right) = \left(0, \frac{1}{2}\right) \oplus \left(1, \frac{1}{2}\right) \tag{4.30}$$

that $\not{D}\psi_L$ transforms as $(0, \frac{1}{2})$ under the Lorentz group. Therefore, the character of a line with n derivatives in Eq. (4.14) can be decomposed as follows:

$$\chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(x)\chi_{(\frac{1}{2},0)}(x) = \widetilde{\chi}(x) + \chi_{\mathrm{sym}^{n-1}(\frac{1}{2},\frac{1}{2})}(x)\chi_{(0,\frac{1}{2})}(x). \tag{4.31}$$

[10] This corresonds with case 2. of the short representations as discussed in App. D.2

where $\tilde{\chi}$ is again the character we need in order to remove the operators related through EOM[11]. With this we get the modified conformal character

$$\begin{aligned}\tilde{\chi}_{[\frac{3}{2},(\frac{1}{2},0)]}(q,x_1,x_2) &= q^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} q^n \chi_{\text{sym}^n(\frac{1}{2},\frac{1}{2})}\chi_{(\frac{1}{2},0)} - \sum_{n=1}^{\infty} q^n \chi_{\text{sym}^{n-1}(\frac{1}{2},\frac{1}{2})}\chi_{(0,\frac{1}{2})}\right)\\ &= q^{\frac{3}{2}}\left(\chi_{(\frac{1}{2},0)}(x) - q\chi_{(0,\frac{1}{2})}(x)\right)P(q,x). \end{aligned} \tag{4.32}$$

One gets the (conformal) character for right handed fermions by interchanging $\chi_{(\frac{1}{2},0)}$ and $\chi_{(0,\frac{1}{2})}$ in the above lines.

4.4.3 Gauge Fields

A gauge fields $F_{\mu\nu}$ transforms as $\ell = (1,0) \oplus (0,1)$ under the Lorentz group and the scaling dimension is equal to $\Delta = 2$. It is easiest to work with the chiral components $F_L^{\mu\nu} = F^{\mu\nu} + \widetilde{F}^{\mu\nu}$ and $F_R^{\mu\nu} = F^{\mu\nu} - \widetilde{F}^{\mu\nu}$, which transform as $(1,0)$ and $(0,1)$, respectively. Here $\widetilde{F}^{\mu\nu}$ is the dual field strength tensor. A derivative acting on a $(1,0)$ then decomposes as

$$\left(\frac{1}{2},\frac{1}{2}\right) \otimes (1,0) = \left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{3}{2},\frac{1}{2}\right). \tag{4.33}$$

and we identify the first term on the RHS with the EOM $D_\mu F^{\mu\nu} = J^\nu$. However, removing this from Eq. (4.14) is too much as the current is conserved[12]: $D_\mu J^\mu = 0$. This corresponds to the trace again as can also be seen from

$$\left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{1}{2},\frac{1}{2}\right) = (0,0) \oplus (1,1) \oplus (1,0) \oplus (0,1). \tag{4.34}$$

So, the Lorentz character of a line with n derivatives in Eq. (4.14) becomes

$$\chi_{\text{sym}^n(\frac{1}{2},\frac{1}{2})}(x)\chi_{(1,0)}(x) = \widetilde{\chi}(x) + \left(\chi_{\text{sym}^{n-1}(\frac{1}{2},\frac{1}{2})}(x)\chi_{(\frac{1}{2},\frac{1}{2})}(x) - \chi_{\text{sym}^{n-2}(\frac{1}{2},\frac{1}{2})}(x)\cdot 1\right), \tag{4.35}$$

and $\widetilde{\chi}$ is again what we seek. The modified character for the conformal representation is therefore given by

$$\begin{aligned}\tilde{\chi}_{[2,(1,0)]}(q,x) &= q^2\left(\chi_{(1,0)}(x)\sum_{n=0}^{\infty} q^n \chi_{\text{sym}^n}(x) - \chi_{(\frac{1}{2},\frac{1}{2})}(x)\sum_{n=1}^{\infty} q^n\chi_{\text{sym}^{n-1}}(x) + \sum_{n=2}^{\infty} q^n \chi_{\text{sym}^{n-2}}(x)\right)\\ &= q^2\left(\chi_{(1,0)} - q\chi_{(\frac{1}{2},\frac{1}{2})} + q^2\right)P(q,x). \end{aligned} \tag{4.36}$$

A similar derivation holds for the $(0,1)$ component $F_R^{\mu\nu}$ of the field strength. Combining these, we find that the character of the conformal representation for the field strength after removing EOM is given by

$$\tilde{\chi}_{[2,(1,0)\oplus(0,1)]}(q,x) = q^2\left(\chi_{(1,0)\oplus(0,1)}(x) - 2q\chi_{(\frac{1}{2},\frac{1}{2})}(x) + 2q^2\right)P(q,x). \tag{4.37}$$

4.5 Gravity

We have seen how we construct the HS for the field content of the SMEFT. We will now discuss how to include gravity, providing a brief summary of the approach in Ref. [22]. Quantizing the action of general relativity (GR)

$$S = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}R \tag{4.38}$$

yields a non-renormalizable theory. This can of course be used as an effective theory, for which we must include operators of higher mass dimension in the Riemann tensor $R_{\mu\nu\rho\sigma}$, whose reducible representation under the Lorentz group is $(2,0)\oplus(0,2)\oplus(1,1)\oplus(0,0)$. To see which EOM redundancies we need to remove, we look at the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{4.39}$$

with $R_{\mu\nu}$ and R the Ricci tensor and scalar, respectively. In vacuum the energy momentum tensor $T_{\mu\nu}$ is zero and the Einstein equations reduce to $R_{\mu\nu} = 0$. As explained in Ref. [22], we can perform field redefinitions in the metric

[11]See case 3 of the short representations in App. D.2

[12]If the states we remove in Eq. (4.14) build a short representation themselves, we should remove only the short representation. See also case 3 of App. D.2

tensor to remove any occurrence of the Ricci scalar and tensor from the operator basis. Therefore, the building block in the EFT of gravity is the Weyl tensor $C_{\mu\nu\rho\sigma}$, which is the Riemann tensor sans its Ricci tensor/scalar traces. The representation of $R_{\mu\nu}$ under the Lorentz group is $(1,1) \oplus (0,0)$, meaning that the Weyl tensor has to transform as $(2,0) \oplus (0,2)$. Besides satisfying the Einstein equations, we have to take the contracted Bianchi identity

$$\nabla^{\mu} C_{\mu\nu\rho\sigma} = \nabla_{[\rho} R_{\sigma]\nu} + \frac{1}{6} g_{\nu[\rho} \nabla_{\sigma]} R \tag{4.40}$$

into account, which can be regarded as an additonal equation of motion. To see which expressions we must subtract in the character for $C_{\mu\nu\rho\sigma}$, we again work with the self-dual and anti-self-dual of the Weyl tensor, which yields the irreducible subspaces $(2,0)$ and $(0,2)$. As in the gauge field case, commutators of derivatives yield building blocks that are already included, and the Bianchi identity implies that $\nabla^2 C_{\mu\nu\rho\sigma}$ is not an independent quantity, so we again consider symmetrized and traceless products of derivatives. The tensor product of a derivative acting on $(2,0)$ decomposes as

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes (2,0) = \left(\frac{5}{2}, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \tag{4.41}$$

and we identify $(\frac{3}{2}, \frac{1}{2})$ with the contracted Bianchi identity. Similar to the gauge field case, subtracting the full $(\frac{3}{2}, \frac{1}{2})$ removes too much, as $\nabla^{\mu}\nabla^{\nu} C_{\mu\nu\rho\sigma} \equiv 0$ by anti-symmetry. This vanishing object is an anti-symmetric rank-2 tensor, and so transforms as $(1,0)$. The corresponding conformal representation is fixed by $\Delta = 3$, and $\ell = (2,0)$[13], and the character of the conformal representation is given by

$$\tilde{\chi}_{[3,(2,0)]} = q^3 \big(\chi_{(2,0)} - q\chi_{(\frac{3}{2},\frac{1}{2})} + q^2 \chi_{(1,0)}\big) P(q,x), \tag{4.42}$$

and we find a similar expression for the $(0,2)$ component. Combining these results yields the EOM-removed character for the conformal representation of the Weyl tensor as

$$\widetilde{\chi}_{[3,(2,0)\oplus(0,2)]} = q^3 \Big(\chi_{(2,0)\oplus(0,2)}(x) - q\chi_{(\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{3}{2})}(x) + q^2 \chi_{(1,0)\oplus(0,1)}(x)\Big) P(q,x). \tag{4.43}$$

[13] In order to satisfy the unitary bound in Eq. (D.17) to get the corresponding short conformal representation, the Weyl tensor should have scaling dimension $\Delta = 3$. Formally, the Weyl tensor can therefore not be identified with the conformal representation as it has mass dimension 2. However, the aim is to enumerate operators and we can therefore assign a conformal scaling dimension of $\Delta = 3$, but when we expand the Hilbert series around the mass dimension of the operators, we can choose a grading scheme in which the Weyl tensor has mass dimension 2.

5 Hilbert Series in Practice

In the previous section we discussed how the HS can be computed by integrating over the generating function for the characters of the symmetric and anti-symmetric tensor products. In this section we will break this infinite series down and extract the information from the HS that will be useful in practice. We start in Sec. 5.1 with the application of the HS to the SMEFT, i.e. we show how the enumeration of operators can be extracted from the HS at a given mass dimension. In Sec. 5.2 we explain that it is straightforward to implement the HS in a computer code that is able to work with algebraic expressions. We then arrive at one of the main results of this thesis: Eco (Efficient Counting of Operators), which is our implementation of the HS in Form. We will discuss the structure of the algorithm, plus some of the methods that are key to turning it into an efficient implementation in Sec. 5.3. In Sec. 5.4 we provide instructions on how the code can be used, and how it can be applied to different EFTs. The possibility to add local and/or global $U(1)$ symmetries is discussed in Sec. 5.4.1. In Sec. 5.5 we illustrate the use of our program by applying it to SMEFT, SMEFT for a Higgs-doublet model, and GRSMEFT, reproducing known results and obtaining new ones at higher dimensions. Most of the methods and results covered in this section closely follow the content of Ref. [20].

5.1 Hilbert Series for SMEFT

In order to apply the HS technique to the SMEFT, we recall Eq. (4.26), the integral form of the HS derived in the previous section. The field content of the SM was summarized in Table 2. Because the SM describes both the fundamental particles and their anti-particles, we need to add the anti-particles as independent building blocks to the HS. Therefore, the HS becomes a function of the following variables (for one fermion generation):

$$H_{SM}(\mathcal{D},\{\phi_a\}) = H(\mathcal{D},\varphi,\varphi^\dagger,B,W,G,l,l^\dagger,e,e^\dagger,q,q^\dagger,u,u^\dagger,d,d^\dagger). \tag{5.1}$$

For a field ϕ_a, we can write the character that enters the argument of the plethystic exponentials as

$$\chi_a(\mathcal{D},x_1,x_2,x,y,z_1,z_2) = \widetilde{\chi}_{[\Delta_a,\ell_a]}(\mathcal{D},x_1,x_2)\chi_a^{U(1)}(x)\chi_a^{SU(2)}(y)\chi_a^{SU(3)}(z_1,z_2), \tag{5.2}$$

where we use the $SU(2)$ variables x_1, x_2 to parametrize the Lorentz group, x for $U(1)$, y for $SU(2)$, and z_1, z_2 for $SU(3)$. The different characters for the conformal representations and the representations of the gauge groups are given in Tables 5 and 18 respectively. For example the character of $q^\dagger$ is given by

$$\begin{aligned}\chi_{q^\dagger} &= \widetilde{\chi}_{[\frac{3}{2},(0,\frac{1}{2})]}(\mathcal{D},x_1,x_2)\chi_{-\frac{1}{6}}^{U(1)}(x)\chi_{\mathbf{2}}^{SU(2)}(y)\chi_{\bar{\mathbf{3}}}^{SU(3)}(z_1,z_2)\\ &= \mathcal{D}^{\frac{3}{2}}\Big(x_2+\frac{1}{x_2}-\mathcal{D}\Big(x_1+\frac{1}{x_1}\Big)\Big)P(\mathcal{D},x_1,x_2)\;x^{-\frac{1}{6}}\Big(y+\frac{1}{y}\Big)\Big(z_2+\frac{z_1}{z_2}+\frac{1}{z_1}\Big),\end{aligned} \tag{5.3}$$

and the plethystic exponential becomes

$$\text{PEF}\Big[\frac{q^\dagger}{\mathcal{D}^{\frac{3}{2}}}\chi_{q^\dagger}\Big] = \exp\Big[-\sum_{n=1}^{\infty}\frac{1}{n}\frac{(-q^\dagger)^n}{\mathcal{D}^{\frac{3}{2}n}}\widetilde{\chi}_{[\frac{3}{2},(0,\frac{1}{2})]}(\mathcal{D}^n,x_1^n,x_2^n)\chi_{-\frac{1}{6}}^{U(1)}(x^n)\chi_{\mathbf{2}}^{SU(2)}(y^n)\chi_{\bar{\mathbf{3}}}^{SU(3)}(z_1^n,z_2^n)\Big]. \tag{5.4}$$

The integral measure appearing in the HS (Eq. (4.26)) becomes

$$\int \mathrm{d}\mu_L \int \mathrm{d}\mu_{\text{gauge}} = \int \mathrm{d}\mu_L \int \mathrm{d}\mu_{U(1)} \int \mathrm{d}\mu_{SU(2)} \int \mathrm{d}\mu_{SU(3)}\,, \tag{5.5}$$

where the Haar measures of the Lorentz and gauge groups are given in Table 18.

To obtain the counting of the SMEFT operators at a given mass dimension n, we rescale the fields and the derivative according to their mass dimension:

$$\begin{aligned}\mathcal{D} &\to \epsilon\mathcal{D}\\ \phi_a &\to \epsilon^{\delta_a}\phi_a,\end{aligned} \tag{5.6}$$

with δ_a the mass dimension of field ϕ_a. Note that the mass dimension δ_a of the fields is the same as the scaling dimension Δ_a appearing in the conformal representations, except for gravity, which we will discuss in a moment.

Representation	Character	FORM procedure
$[1,(0,0)]$	$\mathcal{D}\big(1-\mathcal{D}^2\big)P(\mathcal{D},x_1,x_2)$	`addScalar()`
$[\frac{3}{2},(\frac{1}{2},0)]$	$\mathcal{D}^{\frac{3}{2}}\Big(\chi_{(\frac{1}{2},0)}(x_1,x_2)-\mathcal{D}\chi_{(0,\frac{1}{2})}(x_1,x_2)\Big)P(\mathcal{D},x_1,x_2)$	`addLHFermion()`
$[\frac{3}{2},(0,\frac{1}{2})]$	$\mathcal{D}^{\frac{3}{2}}\Big(\chi_{(0,\frac{1}{2})}(x_1,x_2)-\mathcal{D}\chi_{(\frac{1}{2},0)}(x_1,x_2)\Big)P(\mathcal{D},x_1,x_2)$	`addRHFermion()`
$[\frac{3}{2},(\frac{1}{2},0)\oplus(0,\frac{1}{2})]$	$\mathcal{D}^{\frac{3}{2}}\Big(\chi_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(x_1,x_2)-\mathcal{D}\chi_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(x_1,x_2)\Big)P(\mathcal{D},x_1,x_2)$	`addDiracFermion()`
$[2,(1,0)\oplus(0,1)]$	$\mathcal{D}^2\Big(\chi_{(1,0)\oplus(0,1)}(x_1,x_2)-2\mathcal{D}\chi_{(\frac{1}{2},\frac{1}{2})}(x_1,x_2)+2\mathcal{D}^2\Big)P(\mathcal{D},x_1,x_2)$	`addFieldStrength()`
$[3,(2,0)\oplus(0,2)]$	$\mathcal{D}^3\Big(\chi_{(2,0)\oplus(0,2)}(x_1,x_2)-\mathcal{D}\chi_{(\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{3}{2})}(x_1,x_2)$ $+\mathcal{D}^2\chi_{(1,0)\oplus(0,1)}(x_1,x_2)\Big)P(\mathcal{D},x_1,x_2)$	`addGravity()`

Table 5: Some conformal representation and their characters. For the explicit form of the Lorentz characters $\chi_\ell(x_1,x_2)$ see Table 18. The momentum generating function $P(\mathcal{D},x)$ is given in Eq. (4.16). In the last column the corresponding name of the FORM procedure is given.

With this rescaling, we can expand the plethystic exponentials around ϵ and retain the coefficient in front of ϵ^n. For example, for mass dimension 5 we get

$$\begin{aligned} H_{SM}\Big|_{\mathcal{O}(\epsilon^5)} \\ &= \frac{1}{(2\pi\mathrm{i})^6}\oint\frac{\mathrm{d}x_1}{x_1}(1-x_1^2)\oint\frac{\mathrm{d}x_2}{x_2}(1-x_2^2)\oint\frac{\mathrm{d}x}{x}\oint\frac{\mathrm{d}y}{y}(1-y^2)\oint\frac{\mathrm{d}z_2}{z_2}\oint\frac{\mathrm{d}z_2}{z_2}(1-z_1z_2)\Big(1-\frac{z_1^2}{z_2}\Big)\Big(1-\frac{z_2^2}{z_1}\Big) \\ &\epsilon^5\Big[\varphi^2l^2\frac{(1+y^2+y^4)(x_1^2+y^4x_1^2+y^2(1+x_1^2)^2)}{y^2x_1^2}+(\varphi^\dagger)^2(l^\dagger)^2\frac{(1+y^2+y^4)(x_2^2+y^4x_2^2+y^2(1+x_2^2)^2)}{y^2x_2^2}+...\Big] \\ &= \varphi^2l^2+(\varphi^\dagger)^2(l^\dagger)^2, \end{aligned} \tag{5.7}$$

where the +... denote terms that do not contribute to the HS at mass dimension 5, i.e. these terms become zero after the integrals are performed. Furthermore, we have set $\epsilon=1$ in the last line. Although the expansion of the plethystic exponentials is a bit cumbersome due to the large amount of terms, performing the integrals is relatively easy because it boils down to picking the residues at 0 in all variables x_1,x_2,x,y,z_1,z_2. We know that the only dimension 5 operator of the SMEFT is given by [13]

$$\epsilon_{jk}\epsilon_{mn}\varphi^j\varphi^m(l_p^k)^TCl_r^n. \tag{5.8}$$

However, the HS in Eq. (5.7), tells us now that there are two operators, one of the form φ^2l^2 and the other the form $(\varphi^\dagger)^2(l^\dagger)^2$. It is clear that we can relate the former to the dimension 5 operator of Eq. (5.8). The second operator is implicit there, as it corresponds to the Hermitian conjugate that is also added to the dimension 5 SMEFT Lagrangian. Therefore, the HS counts all operators including the Hermitian conjugates, which is not a strange result, as we have started with the conjugate particles as separate building blocks for the HS. The results for the HS at mass dimension 6 are given in the third column of Tables 3 and 4[14]. We find 84 operators instead of the more familiar 59, but this is due to the fact that we enumerated the baryon violating, and hermitian conjugate operators, as well.

5.1.1 GRSMEFT

In Sec. 4.5 we explained how we can describe general relativity using an EFT. The relevant degree of freedom in the EFT is the Weyl tensor, and assuming that it transforms trivially under the SM gauge group we can extend the SMEFT to include gravity, called GRSMEFT. It is important to mention here that the conformal scaling dimension of the Weyl tensor is $\Delta=3$. However, the Weyl tensor has mass dimension $\delta=2$ and we should therefore be careful when we rescale the fields according to their mass dimension.

[14]This column is called H^+ for now, a notation that will become clear in Sec. 6

5.2 Implementation

From the discussion in Sec. 5.1, we see that enumerating the operators in a minimal basis at a desired mass dimension is straightforward by using the HS. It amounts to inserting the explicit form of the characters for all the different fields, expanding the plethystic exponentials to that mass dimension, and integrating over the Lorentz group and the gauge groups. In principle, integrating can be a hard problem, but from the form of the characters and the Haar measure in Table 18 we see that this amounts to the simple task of taking residues. Therefore, it is straightforward to implement the HS into computer code, as taking residues can be easily done with a programming language that is able to handle algebraic expression. However, if this is implemented without any refinement, it results in an inefficient algorithm that generates a huge number of terms, many of which will be zero in the end. Henning et al. [23] were the first to give an implementation for SMEFT, and their Mathematica code takes around 7 minutes to count the minimal basis of mass dimension 6. Going to dimension 8 already takes more than two hours, and the runtime quickly diverges. Their code can be optimized such that dimension 15 can be done in about two hours, but after that the limits of Mathematica are quickly reached, i.e. pushing the implementation to order 16 or 17 (or looking at extensions of the SMEFT) will produce intermediate expressions that become too large for Mathematica to process efficiently, and it even crashes.

Therefore, we should turn to programming languages better suited for calculations of this type. One of these languages is Form [24, 25], which is aimed at performing large symbolic calculations in theoretical physics very efficiently. We have implemented the HS into Form, and the code, called ECO (Efficient Counting of Operators), can be found in Ref. [20]. In the next section, we discuss the algorithm and the Form features that will be useful for implementing an efficient algorithm.

5.3 Structure of the algorithm

The trick to getting an efficient implementation is keeping the number of terms in the expansion as small as possible, and figuring out terms that will be zero before carrying out the whole expansion. To illustrate how to keep the number of terms small, we discuss the example of the Hilbert series for multiple left-handed fermions ψ_i ($\Delta = \frac{3}{2}$ and $\ell = (\frac{1}{2}, 0)$), all charged differently under some gauge group G. Denote by $\chi_i(g)$ the character of the representation of ψ_i under G, then the HS becomes

$$\begin{aligned} H(\epsilon\mathcal{D}, \{\epsilon^{\frac{3}{2}}\psi_i\}) &= 1 + \int \mathrm{d}\mu_G \int \mathrm{d}\mu_L \, \frac{1}{P(\mathcal{D}, x)} \Big(\prod_i \mathrm{PEF}\Big[\frac{\epsilon^{\frac{3}{2}}\psi_i}{(\epsilon\mathcal{D})^{\frac{3}{2}}} \tilde{\chi}_{[\frac{3}{2},(\frac{1}{2},0)]}(\epsilon\mathcal{D}, x)\chi_i(g) \Big] - 1 \Big) \\ &= 1 + \int \mathrm{d}\mu_G \int \mathrm{d}\mu_L \, \frac{1}{P(\mathcal{D}, x)} \Big(\exp\Big[-\sum_{n=1}^{\infty} \frac{1}{n} \frac{(-1)^n}{\mathcal{D}^{\frac{3}{2}n}} \tilde{\chi}_{[\frac{3}{2},(\frac{1}{2},0)]}((\epsilon\mathcal{D})^n, x^n) \sum_i \psi_i^n \chi_i(g^n) \Big] - 1 \Big), \end{aligned} \tag{5.9}$$

where we rescaled the labels in terms of the mass dimension according to Eq. (5.6). Because the only dependence on the mass dimension in the argument of the plethystic exponential is in $\chi_{\frac{3}{2},(\frac{1}{2},0)}(\epsilon\mathcal{D}, x)$, we see that $\sum_i \psi_i^n \chi_i(g^n)$ can be treated as one element during the expansion in mass dimension, and the only important feature that we have to keep track of is the power n.

To efficiently expand the plethystic exponential, which involves many terms, we make great use of the `Brackets+` and `id, once` features of Form. With the `Brackets+` statement we can order the expression with all terms of equal mass dimension in a single bracket. The `id, once` statement tries to match terms one by one with another expression, giving Form the possibility to sort the whole expression after each insertion i.e. combining same terms. The interested reader can have a look at the comments in the code to see how we used these Form statements for an efficient expansion. Furthermore, we can postpone inserting the explicit form of $\chi_{(\frac{1}{2},0)}$ and $\chi_{(0,\frac{1}{2})}$ until after the expansion in mass dimension (once again, we only have to keep track of the power of the variables of these characters), when we integrate over the Lorentz group, thus reducing the number of terms in intermediate expressions.

From the character, which is the product of the Lorentz characters with the character of the gauge group (see Eq. (5.2)), we note that after the expansion we can first perform the integral over the Lorentz group and still treat $\sum_i \psi_i^n \chi_i(g^n)$ just as one term. The integral over the Lorentz group will set many terms equal to zero, so it is not until we perform the integral over the gauge group, that we have to expand the sum $\sum_i \psi_i^n \chi_i(g^n)$. Of course, when the gauge group is the product of other groups, the character is a product of characters meaning we can use this trick again; only inserting explicit expressions for these characters when we perform the integral over the

Procedure	Description
`add‘Field’(‘symbol’,‘SU3’,‘SU2’,‘Q’)`	Adds the field with its `symbol` (`symbol` needs to be declared first in FORM) to the Hilbert series. For an efficient algorithm to only count the operators replace symbol by 1, that is use `#call add‘Field’(1,‘SU3’,‘SU2’,‘Q’)`. See Table 5 for an overview of the different fields that can be used.
`HilbertSeries(‘symbol’)`	Computes the HS at mass dimension `massDim` with `symbol` the symbol for the derivative (`symbol` needs to be declared first in FORM). Needs to be called after all particles are added with `add‘Field’`
`counting`	Counts the number of operators in the output of `HilbertSeries`. Needs to be called after `HilbertSeries`.

Table 6: Overview of all procedures

corresponding group. Inserting the characters of the gauge groups can again be done efficiently by making use of `id, once`.

Therefore, the algorithm has to perform one expansion for the plethystic exponential of a scalar field, fermion, field strength or gravity tensor, and for every field type a commuting function[15] is used that can be replaced later by the field content (i.e. the symbol to count the occurrence of a field and its character under the gauge group). Furthermore, to account for IBP relations, we have to expand the prefactor $\frac{1}{P(\mathcal{D},x)}$. From Eq. (4.16), we see that this is a plethystic exponential itself,

$$\frac{1}{P(\mathcal{D},x)} = \mathrm{PE}[-\mathcal{D}\chi_{(\frac{1}{2},\frac{1}{2})}(x)], \tag{5.10}$$

and in order to expand this we can follow the recipe of subsequent substitutions, as described above.
For the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$, our implementation takes on the following form:

- Read in which fields are present, and store their representation under the symmetry groups.
- Expand just one plethystic exponential for every type of field (scalar, fermion, field strength or gravity), and multiply these to get the Hilbert series.
- Insert expressions for Lorentz characters, and perform the integral over the Lorentz group (from residues).
- Insert expressions for the gauge group characters and perform the integral, one at a time. First for $SU(2)$, then $U(1)$, and finally $SU(3)$.

Another trick we employ is to express characters in terms of other characters. For example, $2 \otimes 2 = 3 \oplus 1$, so $\chi_3 = \chi_2^2 - 1$. This results in a faster algorithm as we can now use the power of FORM to combine terms, and only need to substitute fewer characters. For $SU(2)$, and similarly the Lorentz group, we therefore only require the character for spin $\frac{1}{2}$.

5.4 How to use ECO

The user can specify the input in the file `main.frm`, and execute ECO by calling `form main` or `tform -wn main`, where n denotes the number of cores. The structure of this file is as follows: the first part contains settings, such as the desired mass dimension, after which the different fields of the model are specified. The final part of the file performs the calculation described above, giving the Hilbert Series as the output. A summary of all procedures and their action can be found in Table 6. All the declarations the program needs and the procedures that can be used to add fields are in the files `declare.h`, `addField.h`, and `HilbertSeries.h`, respectively. These files are included as header files in the `main` file.

We start by discussing the settings, of which the most important one is the desired mass dimension, which is specified using the variable `massDim`. One can choose whether or not EOM and IBP relations should be used to

[15] If all objects in a power are commuting, FORM makes use of binomial expansions, making the expansion a lot faster.

reduce the basis by setting `EOM` and `IBP` to `1` or `0`, respectively. As an additional feature (useful e.g. in SMEFT), the number of Fermion generations can be defined with **`numFermGen`** (by default this is 1). For example, if we want to generate a basis at mass dimension 6, subtract both EOM and IBP relations and with one fermion generation, the `Settings` section in the main file includes

```
#define massDim "6"
#define EOM "1"
#define IBP "1"
#define numFermGen "1"
  ⋮
```

Next we specify the fields by calling the procedure

```
#call add‘Field’(‘symbol’,‘SU3’,‘SU2’,‘Q’)
```

for every field separately. Here **`‘Field’`** refers to the transformation under the Lorentz group, e.g. scalars or left-handed fermions, and a complete list of all supported particles is given in Table 5. The argument `Symbol` of this procedure encodes the symbol used to denote the field, which needs to be declared before calling the procedure.[16] The next three parameters are the representations under $SU(3) \times SU(2) \times U(1)$. For $SU(3)$ and $SU(2)$ the input of the representation is equal to the dimension of the representation. For the representation under SU(3) the possibilities are the singlet (`1`), 3 (`3`), $\bar{3}$ (`3B`) and 8 (`8`) representations, and for $SU(2)$ one can choose between the singlet (`1`), doublet (`2`) and triplet (`3`) representations. The charge under $U(1)$ needs to be an integer `Q`, which we achieve by rescaling the fractional SM charges with a factor of 6 (similarly we rescaled the mass dimensions in our internal code such that they are integers). Charges for additional $U(1)$ symmetries can be added at the end of the string (not shown), as discussed in Sec. 5.4.1.

An example in which the Higgs field and the left-handed quark doublet of the SM are added looks as follows

```
  ⋮
Symbol h,hd,q,qd;

#call addScalar(h,1,2,3)
#call addScalar(hd,1,2,-3)

#call addLHFermion(q,3,2,1)
#call addRHFermion(qd,3B,2,-1)
  ⋮
```

Note that the conjugate particles need to be added as independent building blocks.

When all fields are declared, the HS is computed by calling the `HilbertSeries` procedure, which takes the symbol used for momentum as its argument. To count operators one can set this argument to 1, and when set to 0 only operators without derivatives are produced. This procedure carries out the calculation discussed before and therefore takes up the bulk of the run time. The output of this procedure is a `Local` expression `Hilbert` that gives the basis as a polynomial of the symbols of the fields declared above and the user-specified symbol for the derivative. For the above example this yields

```
  ⋮
Symbol p;
#call HilbertSeries(p)
Print;
.sort

Hilbert = 2*h^2*hd^2*p^2 + 2*q^2*qd^2 + 2*h*hd*q*qd*p + h^3*hd^3;

#call counting
```

[16] All protected symbols can be found in the `declare.h` file, the most important of which are `x,y,y1,y2,z1,z2`.

```
Number of operators at mass dimension 6 is 7.

.end
0.03 sec out of 0.03 sec
```

where we also display the result of the FORM program in blue. The number of operators can be counted and printed by calling the `counting` procedure. This procedure only prints the total number of operators that are in the expression `Hilbert`, without overwriting it. The output of ECO tells us that there should be two operators of the form $h^2(h^\dagger)^2p^2$, and we relate this to the operators that we explicitly constructed in Sec. 3, i.e. $(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$ and $(\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi)$. Likewise

$$\texttt{2*q\^{}2*qd\^{}2} \leftrightarrow \begin{cases}(\bar{q}_p\gamma_\mu q_r)(\bar{q}_s\gamma^\mu q_t)\\(\bar{q}_p\gamma_\mu\tau^I q_r)(\bar{q}_s\gamma^\mu\tau^I q_t)\end{cases} \quad \texttt{2*h*hd*q*qd*p} \leftrightarrow \begin{cases}(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{q}_p\gamma^\mu q_r)\\(\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{q}_p\tau^I\gamma^\mu q_r)\end{cases} \quad \texttt{h\^{}3*hd\^{}3} \leftrightarrow (\varphi^\dagger\varphi)^3 \quad (5.11)$$

5.4.1 Additional U(1) symmetries and generations

In addition to the gauge groups of the SM, we have included the possibility to add one (or more) $U(1)$ symmetry group(s). Such a $U(1)$ can be either an extra gauge symmetry, e.g. for a Z' model, or a global symmetry such as baryon number. From the point of view of enumerating operators, the only difference between a gauge and global symmetry is that in the former case one needs to include the corresponding field strength in the list of particles. The charges under these additional $U(1)$ symmetries can be added as extra arguments when listing particles. E.g. for baryon number,

```
   ⋮
#call addLHFermion(q,3,2,1,1)
#call addRHFermion(qd,3B,2,-1,-1)
   ⋮
```

If for a field no charge corresponding to an additional $U(1)$ is provided, we assume that it has charge 0. We remind the reader that these charges must be integers, which is why we multiplied baryon number by a factor of 3.

To get general dependence on the number of fermion generations, we can declare a symbol for this (at the beginning of the `main` file)

```
Symbol Nf;
#define numFermGen "Nf"
   ⋮
#call HilbertSeries(p)
Print;
.sort
```

```
Hilbert = 2*h^2*hd^2*p^2 + h^3*hd^3 + Nf^2*q^2*qd^2 + 2*Nf^2*h*hd*Q*Qd*p + Nf^4*q^2*qd^2;
```

To organize this expression in powers of `Nf` one can use the `Brackets` command. Details on its use and other tips can be found in the ECO package.

5.5 Results

With ECO it is now straightforward to reproduce known results for (extensions of) the SMEFT. The results for SMEFT up to mass dimension 15 were already given in [23] and with the FORM code we extended this up to dimension 20 in a reasonably short amount of time, see Table 7. In Fig. 2 the number of operators is shown for one and three fermion generations. Although there is some offset in the growth of the odd and even mass dimensions, it is clear that the number of operators follows exponential growth at least to this order.

As an illustration of including an additional $U(1)$ symmetry, we counted how many of the operators in the SMEFT conserve baryon minus lepton number $(B-L)$. The results are denoted in parentheses after the number of SMEFT operators in Table 7. None of the operators of odd dimension do, while all of the operators of even dimension do up to and including dimension 8 (10) for three (one) generations of fermions. An example of an operator of dimension 10 that violates $B-L$ for two or more generations is $h^4 l_1^2 l_2^2$. This operator must vanish for $l_1 = l_2$ due to the antisymmetry of fermion fields. We reproduced the results of Ref. [22] for the GRSMEFT at dimension 8 and extended it also to dimension 20, see Table 7. We did not plot the growth for these numbers as the differences are not visible compared to the results for the SMEFT shown in Fig. 2.

We have accomplished an enormous speed up with the FORM code. For example, computing the HS for the dimension 15 operators in the SMEFT can be obtained in a minute on a single CPU core (in a laptop with a 2.6 GHz Intel Core i7 processor). Even the enumeration for dimension 20 can be done within one hour. Calculating the HS for the GRSMEFT is about a factor two slower than for the SMEFT, while obtaining the $B-L$ conserving operators is about 10% faster.

The Hilbert series approach was also applied to the Two Higgs Doublet Model (2HDM) in Ref. [26]. In this extension of the SM, an identical second Higgs doublet is added. We reproduced the 228 operators found in [26] at mass dimension 6. As most of the operators in the SM have a coupling to the Higgs field, it is not surprising that we find a sizeable number of additional operators for the 2HDM. This has of course implications for the run time of the program. At mass dimension 15 we find a number of 22020182 (16181746764) operators with one (three) fermion generation(s) respectively. Producing the full operator basis results in a run time which is a factor 2 slower compared to the SMEFT. When counting the total number of operators only, more terms can of course be combined, giving a run time which is just a few percent slower.

Because we allow for flexible user input in ECO and due to its speed, new problems and questions can be encountered, e.g. running ECO with some fields of the SMEFT turned off gives the possibility to see which fields and gauge group representations drive the exponential growth we observed in Fig. 2. Furthermore, new fields are easily added to the SMEFT and can therefore be valuable in the building of new models. And ECO can even help in constructing the contraction of the indices explicitly for the operators, despite the fact that the HS do not encode any information about this. As an example of this, we can run ECO for the quark doublet (and its conjugate) for dimension 6, but do not charge the fields under $SU(3)$ and $SU(2)$ yet, i.e. we first run ECO with

```
  ⋮
#call addLHFermion(q,1,1,1)
#call addRHFermion(qd,1,1,-1)
#call HilbertSeries(p)
Print;
.sort

Hilbert = q^2*qd^2;
```

We find just one operator, which we expect to be explicitly written as $(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)$. But what happens when we change the $SU(2)$ representation to the actual doublet? In that case we also get just one independent operator because

```
#call addLHFermion(q,1,2,1)
#call addRHFermion(qd,1,2,-1)
  ⋮
Hilbert = q^2*qd^2;
```

Although we would expect an operator of the form $(\bar{q}_p\gamma_\mu\tau^I q_r)(\bar{q}_s\gamma^\mu\tau^I q_t)$, we can relate this to $(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)$[17] by using the Fierz relation of $SU(2)$, i.e. Eq. (3.11). Finally, we also turn on the correct $SU(3)$ representation, which gives

```
#call addLHFermion(q,3,2,1)
#call addRHFermion(qd,3B,2,-1)
  ⋮
Hilbert = 2q^2*qd^2;
```

[17] Note that now there are also $SU(2)$ indices contracted which we did not write down explicitly.

	SMEFT		GRSMEFT	
Dim.	One generation	Three generations	One generation	Three generations
5	2 (0)	12 (0)	2	12
6	84 (84)	3045 (3045)	94	3055
7	30 (0)	1542 (0)	30	1542
8	993 (993)	44807 (44807)	1096	45816
9	560 (0)	90456 (0)	580	91284
10	15456 (15456)	2092441 (2091965)	17797	2160964
11	11962 (0)	3472266 (0)	12936	3567228
12	261485 (261421)	75577476 (75497816)	314650	79514441
13	257378 (0)	175373592 (0)	291702	182542620
14	4614554 (4612082)	2795173575 (2788483269)	5812440	2995340275
15	5474170 (0)	7557369962 (0)	6518462	8023911776
16	83106786 (83018832)	104832630678 (104309538256)	109518595	114544709924
17	114382724 (0)	320370940524 (0)	143038374	346787656718
18	1509048322 (1506287470)	3877200543051 (3844527891431)	2077921838	4318404186688
19	2343463290 (0)	13044941495798 (0)	3073825028	14409316160246
20	27410087742(27331077766)	141535779949640(139703579253606)	3939163204	160724199619554

Table 7: Number of operators in the SMEFT and GRSMEFT of a given dimension with 1 or 3 generations. For the SMEFT, we also counted the operators that conserve baryon minus lepton number, and the results are given in parentheses.

The reason that we now find an extra operator is because we cannot use the Fierz identity for $SU(2)$, as we must also contract the $SU(3)$ colour indices, so $(\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q)$ and$(\bar{q}_p\gamma_\mu\tau^I q_r)(\bar{q}_s\gamma^\mu\tau^I q_t)$ are independent operators. However, the HS tells us that operators of the form $(\bar{q}\gamma_\mu T^A q)(\bar{q}\gamma_\mu T^A q)$ and $(\bar{q}\gamma_\mu T^A\tau^I q)(\bar{q}\gamma_\mu T^A\tau^I q)$ are not independent and we expect that we can relate them by using the $SU(3)$ Fierz identity (see Eq. (3.31)), which was what we did explicitly in Sec. 3.3.

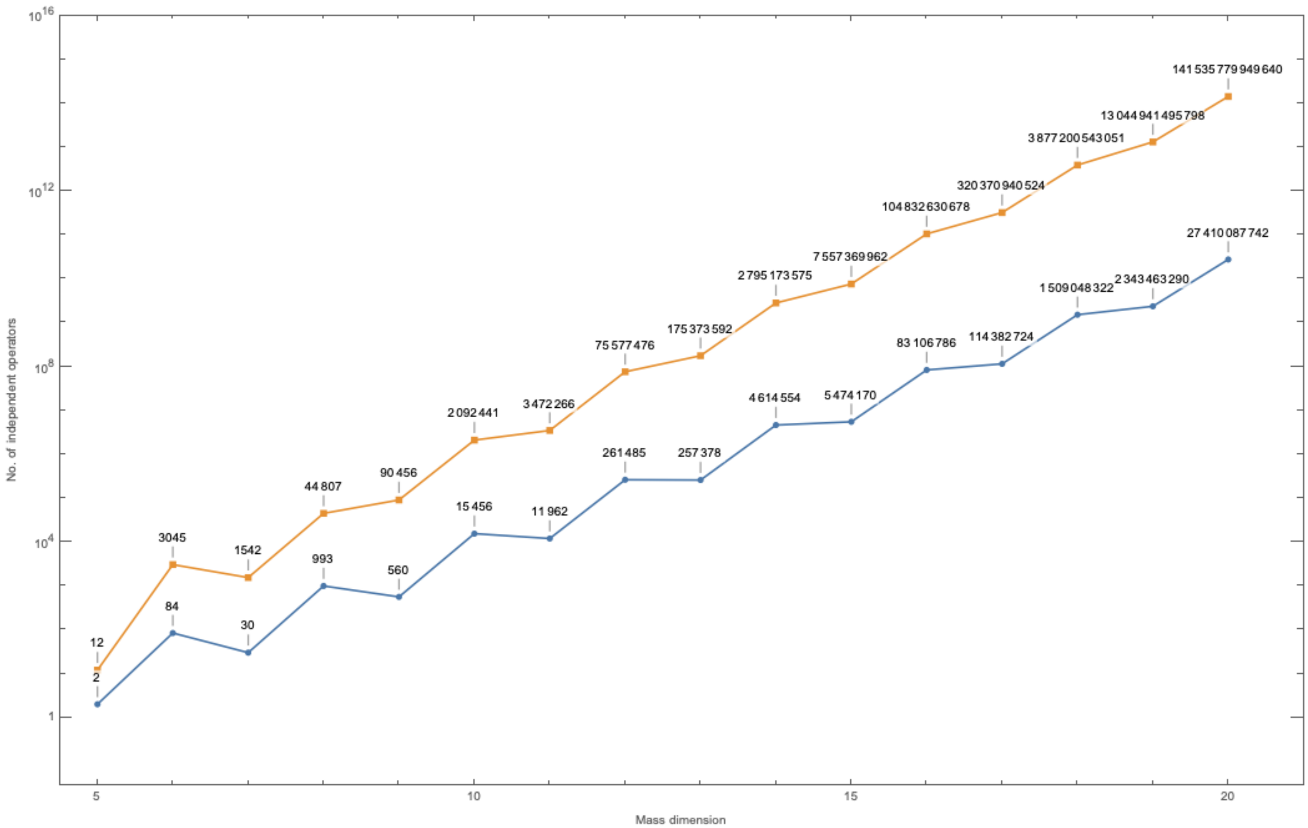


Figure 2: Growth of the number of operators in the SMEFT up to mass dimension 20. Note the logarithmic scale on the vertical axis. The points joined by the lower line are for one fermion generation and the upper line denotes the number of operators for three generations.

6 Discrete Symmetries

In this section we will discuss two discrete symmetries that are particularly relevant for the SM, viz. charge conjugation and parity. In order to include parity and charge conjugation in the Hilbert series, we need to have an idea at which level these symmetries act. If we look back at the previous sections we see that we demanded that the EFT is invariant under $SO(3,1)_o$, the proper orthochronous Lorentz subgroup. As this is the part of the Lorentz group that is connected to the identity, the representations can be directly identified from the representations of the Lie algebra $\mathfrak{so}(3,1)$ (see App. C). However, in order to include parity as a proper symmetry of the SMEFT, we need to extend our symmetry group to the component $O(3,1)_o$ of the Lorentz group that includes the parity operator which also means that the fields should transform under representations of $O(3,1)_o$. We can segment the 'new' symmetry group into two components

$$O(3,1)_o = \{O_+, O_-\}, \quad \text{with} \quad O_+ \equiv SO(3,1)_o, \quad \text{and} \quad O_- \equiv O(3,1)_o \setminus SO(3,1)_o = SO(3,1)_o\mathcal{P}, \tag{6.1}$$

where in the last equality we used that we can write every element $g_- \in O_-$ as $g_- = g_+\mathcal{P}$ for some $g_+ \in SO(3,1)_o$. One of the key features in the computation of the Hilbert series was that we could enumerate operators by using the characters of those representations. Because we would like that the representations of $O(3,1)_o$ reduce to the 'old' representations on $SO(3,1)_o$, we demand that the new characters $\chi(g)$ coincide with the characters of the $SO(3,1)_o$ representations if $g = g_+$. This means that the task we are facing is building the representations for $O(3,1)_o$ by extending the representations of $SO(3,1)_o$ and computing the characters for elements g_-.

In this section we show that this problem can be solved at the level of representation theory by understanding the concept of outer automorphisms of Lie groups and Lie algebras. Furthermore, the general methods that we will come across can also be applied to treat charge conjugation at the level of the Hilbert series. We start by defining outer automorphisms for Lie groups and their algebras in Sec. 6.1. We derive that this leads to classifying all automorphisms on the Dynkin diagrams, after which we will reverse the discussion in Sec. 6.2 and see how the symmetries of Dynkin diagrams lead to all outer automorphisms. These outer automorphisms then yield a way to define representations of $O(3,1)_o$ out of the representations of $SO(3,1)_o$. Additionally, we will find out that we can define a folded Lie algebra, called the orbit Lie algebra. This in turn gives an easy way to compute the so called twining character. It will turn out that this twining character will effectively be the object that describes the character for elements g_-. We apply this approach to the example of parity in Sec. 6.3 and see how everything comes together by computing the Hilbert series with parity as a symmetry. We then apply the same methods in Sec. 6.4 to the representations of $SU(3)$ in order to describe charge conjugation. Finally, we will combine everything to set up the Hilbert series for $\mathcal{CP}$-invariant operators in Sec. 6.5, and we will discuss some results in Sec. 6.5.1.

6.1 Outer automorphism

In order to define the outer automorphisms, we first need to define inner automorphisms. Note that both can be defined for finite and continuous groups, but in this section we limit our discussion to the case of continuous Lie groups G and their Lie algebras $\mathfrak{g}$, as they are most relevant for us.

Automorphism: An automorphism of G is a bijective group homomorphism $a : G \to G$, which means that every automorphism is both invertible, and conserves the group multiplication law, i.e.

$$a(gh) = a(g)a(h), \quad \forall g, h \in G. \tag{6.2}$$

Likewise for the Lie algebra $\mathfrak{g}$, we can define an automorphism as the linear mapping $a : \mathfrak{g} \to \mathfrak{g}$ that conserves the structure of the Lie algebra, i.e.

$$a([x,y]) = [a(x), a(y)], \quad \forall x, y \in \mathfrak{g}. \tag{6.3}$$

We will denote the group of automorphisms of Lie groups and Lie algebras by $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\mathfrak{g})$ respectively.

Inner automorphism: For each group element $g \in G$, the adjoint map

$$\mathrm{Ad}_g(h) = ghg^{-1} \tag{6.4}$$

defines an automorphism of G. Similarly for the Lie algebra, the map

$$a_x(y) = e^{\mathrm{ad}_x}(y) = e^x y e^{-x}, \quad \text{with} \quad \mathrm{ad}_x : y \mapsto [x,y] \tag{6.5}$$

forms an automorphism $\forall x \in \mathfrak{g}$ (note that ad_x itself is an automorphism). Notice that e^x is an element of the component of G that is connected to the identity. All automorphisms represented like this form a subgroup of $\text{Aut}(G)$ ($\text{Aut}(\mathfrak{g})$), called the inner automorphism group $\text{Inn}(G)$ ($\text{Inn}(\mathfrak{g})$).

Outer automorphism: Roughly, the outer automorphisms are defined as all automorphisms which are not inner. So for Lie groups (algebras), it is not possible to represent these by the adjoint map of Eq. (6.4) (Eq. (6.5)) for some element of G ($\mathfrak{g}$). More formally, we can define the outer automorphisms as follows:

$$\text{Out}(G) \equiv \text{Aut}(G)/\,\text{Inn}(G). \tag{6.6}$$

A similar definition holds for the outer automorphisms of $\mathfrak{g}$. Note that from this definition, it follows that outer automorphisms are determined up to inner automorphisms.

We know that characters are unique for different irreducible representations and that they are invariant under conjugation (they are class functions). From the definition of inner automorphisms, it follows therefore that $\text{Inn}(G)$ leaves the characters alone. However, the action of $\text{Out}(G)$ can induce permutations among the conjugacy classes and this means the outer automorphisms can switch between inequivalent representations. In other words, an outer automorphism maps representation matrices to other representation matrices, but the latter do not necessarily correspond to the same representation, i.e.

$$R(a(g)) = AR'(g)A^{-1}, \quad \forall g \in G, \tag{6.7}$$

where A is some unitary matrix and R and R' are two (possibly inequivalent) representations. In fact, Eq. (6.7) holds for all irreducible representations R of G, if and only if $a : g \mapsto a(g)$ is an automorphism. Note that only for inner automorphisms $R = R'$ is automatically implied. To prove this statement, we first assume that a is an automorphism and we show that Eq. (6.7) holds. Therefore, define $R(a(g)) \equiv \Gamma(g)$ and from

$$\Gamma(gh) = R(a(gh)) = R(a(g)a(h)) = R(a(g))R(a(h)) = \Gamma(g)\Gamma(h) \tag{6.8}$$

it follows that $\Gamma(g)$ is a representation itself. Therefore, we can find an equivalent represenation R' that is related to $\Gamma(g)$ via a basis transformation. To be more precise, we can find a matrix A such that $\Gamma(g) = AR'(g)A^{-1}$. To show the reverse direction, we assume that Eq. (6.7) holds and show that a is an automorphism. Therefore, we need to show that a is bijective and a homomorphism. In order to show that a is injective, we need to show that

$$a(g) = a(h) \implies g = h, \quad \forall g, h \in G. \tag{6.9}$$

Therefore, assume that $a(g) = a(h)$ and apply R on both sides. Then $R(a(g)) = R(a(h))$ and by using our assumption Eq. (6.7), we find

$$R'(g) = R'(h). \tag{6.10}$$

Because Eq. (6.7) holds for all irreducible representation of G, it will hold for at least one invertible representation R' such that Eq. (6.10) can be inverted. Applying the inversion map to both sides shows that $g = h$ and therefore that a is injective. In order to show injectivity, we need to prove that

$$\forall g \in G,\ \exists h \in G \quad \text{such that} \quad a(h) = g. \tag{6.11}$$

Now require that $a(h) = g$ and use Eq. (6.7) again for an invertible representation R', then

$$h = R^{-1}(A^{-1}R(g)A) \tag{6.12}$$

and we therefore constructed the element h explicitly. In order to show the homomorphism property, we compute

$$R(a(gh)) = AR'(gh)A^{-1} = AR'(g)A^{-1}AR'(h)A^{-1} = R(a(g))R(a(h)). \tag{6.13}$$

Taking for R an invertible representation we can invert Eq. (6.13), which completes our proof that a is a bijective group homomorphism $G \to G$.

The above theorem can now be used to find an explicit matrix representation A of an outer automorphism a by solving Eq. (6.7), and we will see in the next subsection that we can do this by looking at induced outer automorphisms on the Lie algebra. Eq. (6.7) has a straightforward translation to the Lie algebra $\mathfrak{g}$:

$$R(a(x)) = AR'(x)A^{-1}, \quad \forall x \in \mathfrak{g}, \tag{6.14}$$

where R and R' are two, not necessarily equivalent, representations of $\mathfrak{g}$, and A some unitary matrix. Because the adjoint representation is unique in its dimension we note that all possible automorphism for Lie groups can be classified by looking at the non-trivial linear mappings of the adjoint representation to itself. The adjoint representations plays an important role in representation theory because the weights of this representation are the roots of the Lie algebra. Therefore, all possible automorphisms of a Lie algebra can be classified from the symmetries of its root system. Furthermore, as the Weyl symmetries of the root system leave conjugacy classes invariant, we see that these symmetries correspond with the inner automorphisms. The outer automorphism are captured by non-trivial symmetries of the Dynkin diagram, as these corresponds to the ordering of simple roots. The formal construction of the outer automorphisms therefore becomes:

$$\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\,\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}(\rho)/W = S_{Dyn}, \tag{6.15}$$

with ρ the root system, W the Weyl group and S_{Dyn} the symmetries of the Dynkin diagram. Therefore, the task of finding all outer automorphisms is reduced to finding all symmetries of the Dynkin diagrams, or equivalently of the symmetries of the Cartan matrix. This will be the topic of the next subsection.

6.2 Dynkin Diagrams and Induced Outer Automorphisms

In this subsection we follow the content of Ref. [27]. Every simple or semi-simple Lie algebra $\mathfrak{g}$ can be obtained from Dynkin diagrams. These diagrams are in a one-to-one correspondence with the Cartan matrix, which is an $n \times n$-matrix $A = (a_{ij})_{i,j\in I}$, with $I = \{1, 2, ..., n\}$. This matrix has the following properties:

$$(i)\ a_{ij} \leq 0 \ \text{ if } \ i \neq j, \qquad (ii)\ \frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}, \qquad (iii)\ \text{ if } \ a_{ij} = 0, \ \text{ then } \ a_{ji} = 0. \tag{6.16}$$

We can construct the Lie algebra from the Cartan matrix by choosing an Abelian Lie algebra $\mathfrak{h}$, consisting of n elements $H_i, i \in I$ and n roots $\alpha^{(j)}$ such that $\alpha_i^{(j)} = a_{ij}$. The Lie algebra $\mathfrak{g}$ is then generated by H_i and $E_i^{\pm}, i \in I$, obeying the relations

$$[E_i^+, E_j^-] = \delta_{ij} H_j, \qquad [H_i, E_j^+] = \alpha_i^{(j)} E_j^+, \qquad [H_i, E_j^-] = -\alpha_i^{(j)} E_j^-, \tag{6.17}$$

where H_i are known as the Cartan operators and $E^{\pm}$ as the raising and lowering operators. A symmetry of the Dynkin diagram is now determined by an automorphism $\widetilde{\omega} : I \to I$ that leaves the Cartan matrix invariant, i.e.

$$a_{\widetilde{\omega}(i),\widetilde{\omega}(j)} = a_{ij}. \tag{6.18}$$

This automorphism $\widetilde{\omega}$ can be used to define a folding of the Dynking diagram, but before we do this, we have a look at how this induces an (outer) automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ on the Lie algebra. The action of ω on the Lie algebra is defined by its action on the generators H_i and $E_i^{\pm}$ as

$$\omega(E_i^+) = E^+_{\widetilde{\omega}(i)}, \quad \omega(E_i^-) = E^-_{\widetilde{\omega}(i)}, \quad \omega(H_i) = H_{\widetilde{\omega}(i)}. \tag{6.19}$$

For a representation (R, V) of the Lie algebra, where V is the vector space on which R acts, this means that

$$(R^{(\omega)}, V), \quad \text{with} \quad R^{(\omega)}(x) \equiv R(\omega(x))), \quad x \in \mathfrak{g}, \tag{6.20}$$

is also a representation. We know that for (semi-)simple Lie algebras, we can classify (R, V) and $(R^{(\omega)}, V)$ in terms of some highest weight λ and $\lambda^{(\omega)}$ respectively. We therefore write (R, V_λ) and $(R^{(\omega)}, V_{\lambda^{(\omega)}})$ from now on. In the case that $\lambda = \lambda^{(\omega)}$, we call λ a symmetric highest weight. This gives rise to a map $\tau_\omega : V_\lambda \to V_{\lambda^{(\omega)}}$ of the highest weight vector spaces, which intertwines the action of $\mathfrak{g}$

$$\tau_\omega(R(x) \cdot v) = R^{(\omega)}(x) {\cdot} \tau_\omega(v), \quad \forall v \in V_\lambda, \tag{6.21}$$

and we call τ_ω the intertwining map. This means that τ_ω maps the highest weight vector v_λ of V_λ to the highest weight vector $v_{\lambda^{(\omega)}}$ of $V_{\lambda^{(\omega)}}$. As Eq. (6.21) holds $\forall v \in V_\lambda$, we can strip off v to get

$$\tau_\omega R(x) = R^{(\omega)}(x) {\cdot} \tau_\omega \tag{6.22}$$

Notice that we can relate τ_ω and $R^{(\omega)}$ to A and R' in Eq. (6.14), respectively. Now let N be the order of $\widetilde{\omega}$, that is the smallest integer such that $\widetilde{\omega}^N = \mathbb{1}$. Then N is also the order of ω, i.e. $\omega^N x = \mathbb{1}$ and we can apply Eq. (6.22) N times:

$$R(x) = R(\omega^N(x)) \equiv R^{(\omega^N)} = \tau_\omega R^{(\omega^{N-1})}(x) \tau_\omega^{-1} = (\tau_\omega)^2 R^{(\omega^{N-2})}(x)(\tau_\omega)^{-2} = ... = (\tau_\omega)^N R(x)(\tau_\omega)^{-N}, \tag{6.23}$$

which we can solve for $(\tau_\omega)^N = \pm\mathbb{1}$, meaning that we have some freedom in choosing the phase of τ_ω.

6.2.1 Orbit Lie algebra

We can use $\widetilde{\omega}$ to construct a folded Dynkin diagram, whose induced Lie algebra is called the orbit Lie algebra. However, with the notation of above it is easier to define the folding for the Cartan matrix[18]. In order to construct this folding, recall that N is the order of $\widetilde{\omega}$, then denote by

$$N_i = \left|\{i, \widetilde{\omega}i, \widetilde{\omega}^2 i, ..., \widetilde{\omega}^{N-1} i\}\right| \tag{6.24}$$

the length of the $\widetilde{\omega}$-orbits. The folded Cartan matrix becomes now a matrix $\widetilde{A}$ with index set $\widetilde{I}$. It is convenient to choose the latter to contain the smallest representatives of the orbits:

$$\widetilde{I} = \{i \in I | i \leq \widetilde{\omega}^n i, \text{ for } 1 \leq n < N\} \tag{6.25}$$

For each $i \in I$ we define the integer

$$s_i = 1 - \sum_{l=1}^{N_i - 1} a_{\widetilde{\omega}^l(i),i}. \tag{6.26}$$

With this, the Cartan matrix $\widetilde{A} = (\widetilde{a}_{ij})_{i,j\in\widetilde{I}}$ of the orbit Lie algebra $\widetilde{\mathfrak{g}}$ is defined by

$$\widetilde{a}_{ij} = s_i \frac{N_i}{N} \sum_{l=0}^{N_i - 1} a_{\widetilde{\omega}^l(i),j} \tag{6.27}$$

One can show that this folded Cartan matrix satisfies all properties of Eq. (6.16), so it is a Cartan matrix itself (for a proof, see Ref. [27]). Therefore, the folded Cartan matrix induces a new Lie algebra, which we call the orbit Lie algebra $\widetilde{\mathfrak{g}}$[19].

The map τ_ω can be used to define the twining character on the Cartan subalgebra:

$$\chi_\lambda^{(\omega)}(h) = \text{tr}_{V_\lambda}\left[\tau_\omega e^{iR_\lambda(h)}\right], \quad h \in \mathfrak{h}. \tag{6.28}$$

It is clear that for the trivial automorphism $\omega = \mathbb{1}$, the character reduces to the ordinary character of (R, V_λ). An important result is that

The twining character of Eq. (6.28) *is in a one to one correspondence with the character of the orbit Lie algebra.*

For a proof of this statement, we refer the reader to Ref. [27].

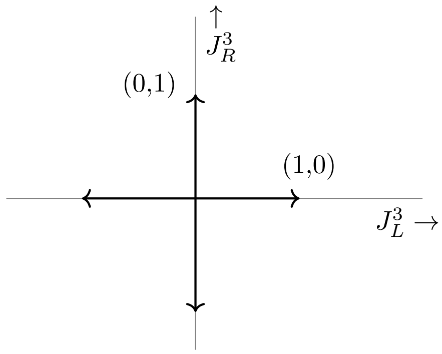
This is a good point to pause for a moment, and summarize what we have done so far. We mentioned at the beginning of this section that we want to extend the representations of the fields such that parity and charge conjugation become honest symmetries of the EFT. We discussed how outer automorphism can induce permutations among representations and in particular the map τ_ω can intertwine the action of the representations. Assuming that the order of ω is $N = 2$ and R is an irreducible representation[20], we can then extend R to an irreducible representation containing τ_ω. We have to distinquish between two cases: in the case that λ is a symmetric highest weight, R extended with τ_ω is irreducible. In the case that λ is not symmetric, $R \oplus R^{(\omega)}$ becomes an irreducible representation as τ_ω intertwines the action between R and $R^{(\omega)}$. We can also extend the characters of these representations to elements of the form $\tau_\omega R(g)$ by using Eq. (6.28).

Until now, the discussion has been rather abstract, but we will show in the next sections explicitly that τ_ω can describe the matrix representation of parity and charge conjugation. In particular, we can make use of Eq. (6.28) to compute the characters.

[18] In the end, the folded Dynkin diagram and the folded Cartan matrix are in a one-to-one correspondence.
[19] It is important to note that $\widetilde{\mathfrak{g}}$ is not a subalgebra of $\mathfrak{g}$.
[20] Generalization to arbitrary N and reducible representations is straightforward.

6.3 Parity

In App. C we show that the Lorentz group has two positive orthogonal roots $(1,0)$ and $(0,1)$. The root system, Cartan matrix, and Dynkin diagram therefore take the following form:



$$, \qquad A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A_1 \cup A_1 = \tag{6.29}$$

With this, it is not hard to see that the (non-trivial) automorphism of the Cartan matrix is given by $\widetilde{\omega} : 1 \leftrightarrow 2$. We can construct the induced outer automorphism $\omega : \mathfrak{so}(3,1) \rightarrow \mathfrak{so}(3,1)$, by constructing the explicit action on the generators of the Lorentz algebra, i.e. $H_1 = J_L^3$, $H_2 = J_R^3$, $E_1^\pm = J_L^\pm$ and $E_2^\pm = J_R^\pm$. We get for example:

$$\omega(J_3^L) = \omega(H_1) = H_{\widetilde{\omega}(1)} = H_2 = J_3^R, \tag{6.30}$$

and a similar derivation for the other generators yields

$$\omega(J_3^R) = J_3^L, \quad \omega(J_\pm^L) = J_\pm^R, \quad \text{and} \quad \omega(J_\pm^R) = J_\pm^L. \tag{6.31}$$

We see in particular that the induced outer automorphism ω interchanges the left and right generators, and we conclude that ω indeed is in a one-to-one correspondence with the (more) familiar parity transformation.

In order to see how ω can induce other representations, we look at an arbitrary eigenstate $|\ell_1, \ell_2\rangle$ of some representation R, i.e.

$$R(J_L^3)|\ell_1, \ell_2\rangle = \ell_1 |\ell_1, \ell_2\rangle, \quad \text{and} \quad R(J_R^3)|\ell_1, \ell_2\rangle = \ell_2 |\ell_1, \ell_2\rangle \tag{6.32}$$

The induced representation $R^{(\omega)}$ then has eigenstates $|\ell_2, \ell_1\rangle$ because

$$R^{(\omega)}(J_L^3)|\ell_1, \ell_2\rangle \equiv R(\omega(J_L^3))|\ell_1, \ell_2\rangle = R(J_R^3)|\ell_1, \ell_2\rangle = \ell_2 |\ell_1, \ell_2\rangle, \text{ and similarly } R^{(\omega)}(J_R^3)|\ell_1, \ell_2\rangle = \ell_1 |\ell_1, \ell_2\rangle. \tag{6.33}$$

Therefore, the intertwining map Eq. (6.21) should act on the eigenstates according to

$$\tau_\omega |\ell_1, \ell_2\rangle = |\ell_2, \ell_1\rangle. \tag{6.34}$$

From this action, we see that the intertwining map τ_ω is actually the matrix representation of parity, and to make this analogy explicit we will write $\mathcal{P} = \tau_\omega$ in what follows. This means that we can extend the following representations of $SO(3,1)_o$ to representations that transform irreducibly under the component $O(3,1)_o$ that includes parity:

$$(\ell, \ell), \quad \text{as well as} \quad (\ell_1, \ell_2) \oplus (\ell_2, \ell_1) \quad \text{with} \quad \ell_1 \neq \ell_2. \tag{6.35}$$

In general, the representations of $O(3,1)_o$ are tensor products and direct sums of these.

Before we compute the twining characters for both cases, we construct the folding of the Cartan matrix (or Dynkin diagram). As there is just one $\widetilde{\omega}$-orbit, we see that $N = N_i = 2$ and we get $\widetilde{I} = \{1\}$. So Eq. (6.26) in this case is given by

$$s_1 = 1 - a_{\widetilde{\omega}(1),1} = 1 - a_{2,1} = 1, \tag{6.36}$$

and the folded Cartan matrix becomes

$$\widetilde{A} = s_1 \frac{N_1}{N}(a_{1,1} + a_{2,1}) = 2. \tag{6.37}$$

We recognize the Cartan matrix of $\mathfrak{su}(2)$, and with the results of the previous subsection this means that the twining characters are in a one-to-one correspondence with the characters of $SU(2)$.

6.3.1 Twining characters

We now compute the twining character of Eq. (6.28) and make the analogy with the character of the orbit Lie algebra explicit. Assume that we have a representation R that is of the form of Eq. (6.35). The twining character then becomes

$$\chi^{(\omega)}(h) \equiv \mathrm{tr}\big[\tau_\omega e^{i\theta_1 R(J_L^3)+i\theta_2 R(J_R^3)}\big] = \sum_\mu \langle\mu|\mathcal{P}e^{i\theta_1 R(J_L^3)+i\theta_2 R(J_R^3)}|\mu\rangle, \quad h \in \mathfrak{so}(3,1), \tag{6.38}$$

where $h = \theta_1 J_L^3 + \theta_2 J_R^3$, and the sum runs over all weight vectors $|\mu\rangle$. From the explicit action of $\mathcal{P}$ (see Eq. (6.34)), we see that two weights get permuted under $\mathcal{P}$. Because the weights are orthogonal, i.e. $\langle\mu|\mu'\rangle = \delta_{\mu\mu'}$, we see that Eq. (6.38) only picks up terms in which the weights are invariant under $\mathcal{P}$, i.e. $\mathcal{P}|\mu\rangle = |\mu\rangle$. For a representation of the form (ℓ,ℓ), we can easily pick out the weights that are invariant. Denote by $|\ell\rangle$ the highest weight of this representation, and because this state is invariant under parity, we have $\mathcal{P}|\ell\rangle = |\ell\rangle$. Starting from the highest weight, we can always construct the other eigenstates by lowering with J_L^- and J_R^-

$$|\mu\rangle = (J_L^-)^{m_L}(J_R^-)^{m_R}|\ell\rangle. \tag{6.39}$$

Then $\mathcal{P}|\mu\rangle = |\mu\rangle$ if

$$\mathcal{P}(J_L^-)^{m_L}(J_R^-)^{m_R}\mathcal{P}^{-1} \tag{6.40}$$

is invariant. Because $\mathcal{P}$ interchanges left and right, and as $(J_-^{L,R})$ commute, the only states $|\mu\rangle$ invariant under $\mathcal{P}$, are those with $m = m_L = m_R$:

$$|\mu\rangle = (J_-^L J_-^R)^m |l\rangle, \tag{6.41}$$

which are the states with an equal number of left and right lowering operators. Effectively, this means that there is just one lowering operator, from which it follows that the character obtained in this way is indeed the character of an $SU(2)$ representation. In order words, an (ℓ,ℓ) representation behaves like a spin ℓ representation of $SU(2)$. In the case that R is the direct sum $(\ell_1,\ell_2)\oplus(\ell_2,\ell_1)$ with $\ell_1 \neq \ell_2$, none of the weight states $|\mu\rangle$ is invariant under $\mathcal{P}$ and therefore we see that the twining characters vanishes.

We can now discuss the general case where the representation might decompose into a direct sum of multiple irreducible representations under $O(3,1)_o$ (see Eq. (6.35)). In general, we can write an element $g_- = g_+\mathcal{P}$, with $g_+ = \exp(i\theta_1 R(J_L^3) + i\theta_2 R(J_R^3))$ as

$$g_- = \left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & (x_1x_2)^{m_i} & & & & \\ & & \ddots & & & \\ \hline & & & \ddots & & \\ & & & & \begin{pmatrix} x_1^{m_j}x_2^{n_j} & 0 \\ 0 & x_1^{n_j}x_2^{m_j} \end{pmatrix} & \\ & & & & & \ddots \end{array}\right)\left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ \hline & & & \ddots & & \\ & & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & & & \ddots \end{array}\right), \tag{6.42}$$

where we defined the $U(1)$ variables $x_i = e^{i\theta_i}$. Furthermore, we went to a basis in which g_+ is diagonal. Note that the following matrix can be diagonalised

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{pmatrix} \tag{6.43}$$

where $\pm\sqrt{ab}$ are the eigenvalues the matrix. Using this fact, we find that we can diagonalise Eq. (6.42):

$$g_- = \left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & (x_1x_2)^{m_i} & & & & \\ & & \ddots & & & \\ \hline & & & \ddots & & \\ & & & & \begin{pmatrix} \sqrt{(x_1x_2)^{(m_j+n_j)}} & 0 \\ 0 & -\sqrt{(x_1x_2)^{(m_j+n_j)}} \end{pmatrix} & \\ & & & & & \ddots \end{array}\right). \tag{6.44}$$

We see that the variables of $SO(3,1)_o$ only group together in the form $(x_1x_2)^n$. Using this, we can make the translation to the variable α of the characters for the orbit $SU(2)$ algebra, i.e.

$$x_1x_2 \mapsto \alpha. \tag{6.45}$$

Therefore, we find that the twining character is given by

$$\chi^{(\mathcal{P})}(g_-) = \sum_{n=1}^{dim(R)} \alpha^{e(n)}, \tag{6.46}$$

where $e(n)$ is some appropriate exponent.

This was rather abstract, so let's work out the explicit example of the $\ell = (\frac{1}{2}, \frac{1}{2})$ representation. We get that

$$g_- = \begin{pmatrix} x_1x_2 & & & \\ & \frac{x_1}{x_2} & & \\ & & \frac{x_2}{x_1} & \\ & & & \frac{1}{x_1x_2} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} x_1x_2 & & & \\ & 1 & & \\ & & -1 & \\ & & & \frac{1}{x_1x_2} \end{pmatrix} \mapsto \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & -1 & \\ & & & \frac{1}{\alpha} \end{pmatrix}, \tag{6.47}$$

where in the first step we diagonalised the matrix and in the second step we translated the variables x_1, x_2 of the matrix to the variables α of the orbit Lie algebra $SU(2)$. From the explicit form of Eq. (6.47) we see immediately that the twining character of $\ell = (\frac{1}{2}, \frac{1}{2})$ corresponds with the character of spin $\frac{1}{2}$:

$$\chi^{(\mathcal{P})}_{(\frac{1}{2},\frac{1}{2})}(g_-) = \alpha + \frac{1}{\alpha}. \tag{6.48}$$

Diagonalising the matrix seems unnecessary, as it does not alter the trace, i.e. the character. However, we will see in the next section that the diagonal matrix will be a good form to work with in the Hilbert series.

Before we continue, let's recall what we have done so far and discuss how it will help us in computing the Hilbert series. At the beginning of this section, we pointed out that in order to generate operators that are invariant under parity, we need to extend our representations to $O(3,1)_o$ representations. In order to do so, we showed that there is one non-trivial outer automorphism of $SO(3,1)_o$ which we related to the object known as parity. Furthermore, we showed that only the $SO(3,1)_o$ representations of the form (ℓ, ℓ) and $(\ell_1, \ell_2) \oplus (\ell_2, \ell_1)$ can be extended to irreducible representations of $O(3,1)_o$. General representations then become direct sums and tensor products of these. We saw that the splitting of Eq. (6.1) is indeed correct for elements $g_- = g_+\mathcal{P}$ with $g_+ = \exp(\theta_1 J_L^3 + \theta_2 J_R^3)$ and we were able to extend the characters to these elements in Eq. (6.46). Before we can plug these characters into the HS we need to know how if these characters still obey an orthogonality relation.

6.3.2 Character orthogonality

Because we extended the symmetry group from $SO(3,1)_o$ to $O(3,1)_o$, integrals over $SO(3,1)_o$ should now become integrals over $O(3,1)_o$ which we can write as

$$\int_{O(3,1)_o} = \int_{SO(3,1)} + \int_{O_-}. \tag{6.49}$$

As the characters can be related to the characters of $SU(2)$ on O_-, we relate the integral over O_- with the integral over $SU(2)$ as well. Therefore, when integrating over characters, we choose our measure as follows

$$\int \mathrm{d}\mu_{O(3,1)_o} = \frac{1}{2}\Big[\int \mathrm{d}\mu_{SO(3,1)} + \int \mathrm{d}\mu_{SU(2)}\Big]. \tag{6.50}$$

In order to show that this leads to the correct character orthogonality for the irreducible representations of Eq. (6.35), we have to distinguish between three cases. First of all, assume we have two representations (ℓ, ℓ) and (ℓ', ℓ'), then

$$\int \mathrm{d}\mu_{O(3,1)_o}\, \chi_{(\ell,\ell)} \cdot \chi_{(\ell',\ell')} = \frac{1}{2}\Big[\int \mathrm{d}\mu_{SO(3,1)}\, \chi_{(\ell,\ell)}(x)\chi_{(\ell',\ell')}(x) + \int \mathrm{d}\mu_{SU(2)}\, \chi_{\ell}(\alpha)\chi_{\ell'}(\alpha)\Big] = \delta_{\ell\ell'}, \tag{6.51}$$

where both integrals contributed a kronecker delta. In the case that we replace (ℓ, ℓ) by $(\ell_1, \ell_2) \oplus (\ell_2, \ell_1)$ the twining character vanishes and therefore the integral over $SU(2)$ does not contribute. However, the integral over $SO(3,1)_o$ contributes twice $\delta_{\ell'\ell_1}\delta_{\ell'\ell_2}$ because of the direct sum. As the factor of two is canceled by the factor of $\frac{1}{2}$ in the measure we conclude that this case gives also the character orthogonality we want. A similar argument also shows that we get the correct character orthogonality when we replace also (ℓ', ℓ') by $(\ell'_1, \ell'_2) \oplus (\ell'_2, \ell'_1)$.

6.3.3 Plethystic exponential and Parity

The next thing to do is to compute the generating function of the characters of the (anti-)symmetric tensor products of a representation ℓ with corresponding spin $\widetilde{\ell}$ representation on the orbit Lie algebra

$$\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\ell)}(g) = \frac{1}{\det_\ell(1-qg)} = \exp\Big[\sum_{k=1}^{\infty}\frac{q^k}{k}\mathrm{tr}(g^n)\Big],$$
$$\sum_{n=0}^{\infty} q^n \chi_{\wedge^n(\ell)}(g) = \det_\ell(1+qg) = \exp\Big[-\sum_{k=1}^{\infty}\frac{(-q)^k}{k}\mathrm{tr}(g^k)\Big]. \tag{6.52}$$

For elements g_+ this reduces to the plethystic exponential $\mathrm{PE}[q\chi_\ell(x_1,x_2)]$ ($\mathrm{PEF}[q\chi_\ell(x_1,x_2)]$), as we show explicitly in App. B.1. However, for elements $g_- = g_+\mathcal{P}$ this does not reduce to the plethystic exponential $\mathrm{PE}[q\chi_{\widetilde{\ell}}(\alpha)]$ ($\mathrm{PE}[q\chi_{\widetilde{\ell}}(\alpha)]$), as $\mathrm{tr}[g^n] \neq \chi_\ell(g^n)$. To see this, we notice that from the diagonal form of Eq. (6.44) that

$$\mathrm{tr}[g_-^n] = \begin{cases} ... + \alpha^{m_i\cdot 2k} + ... + 2\alpha^{(m_j+n_j)k} + ... & \text{if } n=2k, \\ ... + \alpha^{m_i(2k+1)} + ... & \text{if } n = 2k+1. \end{cases} \tag{6.53}$$

where we made the translation to the $SU(2)$ variable α. We see that for $n = 2k$ the minus signs do not cancel, which directly causes the trouble. However, we can still express $\mathrm{tr}[g_-^n]$ in terms of the character $\chi_\ell(x_1,x_2)$ and the character of the Orbit Lie algebra $\chi_\ell^{(\mathcal{P})}(\alpha)$, which gives the important formula

$$\mathrm{tr}[g_-^n] = \begin{cases} \chi_\ell(\alpha^k,\alpha^k) & \text{if } n=2k, \\ \chi_\ell^{(\mathcal{P})}(\alpha^{2k+1}) & \text{if } n = 2k+1. \end{cases} \tag{6.54}$$

With this, the symmetric tensor product yields

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\ell)}(g_-) &= \exp\Big[\sum_{k=0}^{\infty}\frac{q^{2k+1}}{2k+1}\chi_\ell^{(\mathcal{P})}(\alpha^{2k+1}) + \sum_{k=1}^{\infty}\frac{q^{2k}}{2k}\chi_\ell(\alpha^k,\alpha^k)\Big], \\ &= \exp\Big[\sum_{k=1}^{\infty}\frac{q^k}{k}\chi_\ell^{(\mathcal{P})}(\alpha^k) + \sum_{k=1}^{\infty}\frac{q^{2k}}{k}\frac{1}{2}\Big(\chi_\ell(\alpha^k,\alpha^k) - \chi_\ell^{(\mathcal{P})}(\alpha^{2k})\Big)\Big], \\ &\equiv \mathrm{PE}\left[q\chi_\ell^{(\mathcal{P})}(\alpha)\right]\mathrm{PE}\Big[\frac{1}{2}q^2\Big(\chi_\ell(\alpha,\alpha) - \chi_\ell^{(\mathcal{P})}(\alpha^2)\Big)\Big].\end{aligned} \tag{6.55}$$

And likewise for the anti-symmetric tensor products we get

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi_{\wedge^n(\ell)}(g_-) &= \exp\Big[-\sum_{k=1}^{\infty}\frac{(-q)^k}{k}\chi_\ell^{(\mathcal{P})}(\alpha^k) - \sum_{k=1}^{\infty}\frac{q^{2k}}{k}\frac{1}{2}\Big(\chi_\ell(\alpha^k,\alpha^k) - \chi_\ell^{(\mathcal{P})}(\alpha^{2k})\Big)\Big], \\ &\equiv \mathrm{PEF}\left[q\chi_\ell^{(\mathcal{P})}(\alpha)\right]\mathrm{PE}\Big[-\frac{1}{2}q^2\Big(\chi_\ell(\alpha,\alpha) - \chi_\ell^{(\mathcal{P})}(\alpha^2)\Big)\Big].\end{aligned} \tag{6.56}$$

To see these formulas in practice, we look again at the example of Eq. (6.47), where now

$$\mathrm{tr}[g_-^n] = \begin{cases} \alpha^{2k} + 2 + \frac{1}{\alpha^{2k}} & \text{if } n=2k, \\ \alpha^{2k+1} + \frac{1}{\alpha^{2k+1}} & \text{if } n = 2k+1. \end{cases} \tag{6.57}$$

From this, we can compute the momentum generating function $P(q,g)$ that was so important in dealing with derivatives for the Hilbert series. We get

$$\begin{aligned}P(q,g_-) \equiv \sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(g_-) &= \exp\Big[\sum_{k=1}^{\infty}\frac{q^k}{k}\Big(\alpha^k + \frac{1}{\alpha^k}\Big) + \sum_{k=1}^{\infty}\frac{q^{2k}}{2k}\Big(\alpha^{2k} + 2 + \frac{1}{\alpha^{2k}} - \alpha^{2k} - \frac{1}{\alpha^{2k}}\Big)\Big], \\ &= \exp\Big[-\log\Big(1-q\alpha\Big) - \log\Big(1-\frac{q}{\alpha}\Big) - \log\Big(1-q^2\Big)\Big] = \frac{1}{(1-q^2)(1-q\alpha)(1-\frac{q}{\alpha})} \\ &= \frac{1}{1-q^2}P^{(2)}(q,\alpha),\end{aligned} \tag{6.58}$$

where we defined

$$P^{(2)}(q,\alpha) \equiv \frac{1}{\big(1-q\alpha\big)\big(1-\frac{q}{\alpha}\big)}. \tag{6.59}$$

To make the distinction between the form of the momentum generating function found evaluated on $O_+ = SO(3,1)_o$, we denote that one defined in Eq. (4.16) by $P^{(4)}(q,x)$.

6.3.4 Hilbert Series and Parity

Recall Sec. 4.2, where we extended the representations of $SO(3,1)$ to representations of the conformal group $SO(4,2)$. In order to include parity, we should now extend the representations of $O(3,1)_o$ that we found above, to representations of $O(4,2)$. We get that an irreducible conformal representation with parity as a symmetry is completely determined by (Δ,ℓ), where ℓ is either equal to (l,l) or $(l_1,l_2)\oplus(l_2,l_1)$. The effective characters of these representations then become

$$q^\Delta \chi_\ell \sum_{n=0}^{\infty} \chi_{\mathrm{sym}^n(\frac{1}{2},\frac{1}{2})}(g) = \begin{cases} q^\Delta \chi_\ell(x_1,x_2) P^{(4)}(q,x_1,x_2) \equiv \chi_{\Delta,\ell}(q,x_1,x_2) & \text{if } g = g_+, \\ \frac{q^\Delta}{1-q^2}\chi_\ell^{(\mathcal{P})}(\alpha) P^{(2)}(q,\alpha) \equiv \chi^{(\mathcal{P})}_{\Delta,\ell}(q,\alpha) & \text{if } g = g_-. \end{cases} \tag{6.60}$$

Taking the measure now equal to

$$\oint \frac{dq}{2\pi i q} \int \mathrm{d}\mu_{O(3,1)_o} \frac{1}{|P|^2} = \frac{1}{2}\oint \frac{dq}{2\pi i q}\left[\int \mathrm{d}\mu_{SO(3,1)} \frac{1}{P^*(q,g_+)P(q,g_+)} + \int \mathrm{d}\mu_{SU(2)} \frac{1}{P^*(q,g_-)P(q,g_-)}\right] \tag{6.61}$$

gives the correct normalization of the characters. Recalling the explicit form for $P(q,g_\pm)$ from Eq. (4.16) and Eq. (6.58) the HS now becomes

$$H(\mathcal{D},\phi) = \frac{1}{2}(H^+(\mathcal{D},\phi) + H^-(\mathcal{D},\phi)), \tag{6.62}$$

where we defined

$$H^+(\mathcal{D},\phi) \equiv 1 + \int \mathrm{d}\mu_{SO(3,1)} \frac{1}{P^{(4)}(\mathcal{D},x_1,x_2)}\Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi_{\Delta,\ell}(\mathcal{D},x_1,x_2)\Big] - 1\Big), \tag{6.63}$$

$$H^-(\mathcal{D},\phi) \equiv 1 + \int \mathrm{d}\mu_{SU(2)} \frac{1-\mathcal{D}^2}{P^{(2)}(\mathcal{D},\alpha)}\Big(\mathrm{PE}\Big[\frac{\phi}{\mathcal{D}^\Delta}\chi^{(\mathcal{P})}_{\Delta,\ell}(\mathcal{D},\alpha)\Big]\,\mathrm{PE}\Big[\frac{\phi^2}{2\mathcal{D}^{2\Delta}}\Big(\chi_{\Delta,\ell}(\mathcal{D}^2,\alpha,\alpha) - \chi^{(\mathcal{P})}_{\Delta,\ell}(\mathcal{D}^2,\alpha^2)\Big)\Big] - 1\Big).$$

For a fermionic field, the PE should be replaced by PEF. Of course H_+ reduces to the form of the Hilbert series which was derived in Eq. (4.24). In order to see how everything comes together, we work out the above formulas for a scalar field, field strength tensor, and a vector field. Of course, we can look at the action of parity on fermions, but this will not be useful for our discussion of the SMEFT as the left and right handed components of the Fermions in the SM are in different representations of the gauge groups. We therefore discuss fermions once we also know how to treat charge conjugation.

Scalar Field
We start with the example of a scalar field ϕ. Because the representation is fixed by $\ell = (0,0)$ and $\Delta = 1$, it follows that ϕ is an irreducible representation under $O(3,1)_o$. From the discussion of above, it follows that the conformal character is given by

$$\widetilde{\chi}(q,x) = q^\Delta\big(1-q^2\big)P(q,x) = \begin{cases} q\big(1-q^2\big)P^{(4)}(q,x_1,x_2) \equiv \widetilde{\chi}_{1,(0,0)}(q,x_1,x_2) & \text{on } SO(3,1)_o, \\ qP^{(2)}(q,\alpha) \equiv \widetilde{\chi}^{(\mathcal{P})}_{1,(0,0)}(q,\alpha) & \text{on } O_-, \end{cases} \tag{6.64}$$

where we subtracted the EOM. Note that the factor $1-q^2$ drops out on O_-. As H^+ was already derived in Sec. 4, we focus on the derivation of H^-. Therefore, we first compute (see Eqs. (6.55) and (6.63))

$$\begin{aligned} \frac{1}{2\mathcal{D}^2}\Big(\widetilde{\chi}_{1,(0,0)}(\mathcal{D}^2,\alpha,\alpha) - \widetilde{\chi}^{(\mathcal{P})}_{1,(0,0)}(\mathcal{D}^2,\alpha^2)\Big) &= \frac{1}{2}\left(\frac{1-\mathcal{D}^4}{(1-\mathcal{D}^2\alpha^2)(1-\mathcal{D}^2)^2(1-\mathcal{D}^2/\alpha^2)} - \frac{1}{(1-\mathcal{D}^2\alpha^2)(1-\mathcal{D}^2/\alpha^2)}\right) \\ &= \frac{1+\mathcal{D}^2-(1-\mathcal{D}^2)}{2(1-\mathcal{D}^2\alpha^2)(1-\mathcal{D}^2)(1-\mathcal{D}/\alpha^2)} \\ &= \frac{\mathcal{D}^2}{1-\mathcal{D}^2}P^{(2)}(\mathcal{D}^2,\alpha^2), \end{aligned} \tag{6.65}$$

which gives for the H^-:

$$H^-(\mathcal{D},\phi) = \int \mathrm{d}\mu_{SU(2)}\ \mathrm{PE}\Big[\pm\phi P^{(2)}(\mathcal{D},\alpha) + (\pm\phi)^2\frac{\mathcal{D}^2}{1-\mathcal{D}^2}P^{(2)}(\mathcal{D}^2,\alpha^2)\Big]. \tag{6.66}$$

Here, we also allow for intrinsic parity, i.e. a pseudoscalar ϕ. We can treat this case by letting H^+ remain the same, but we evaluate the Hilbert series on O_- as $H^-(\mathcal{D},-\phi)$. This means that operators with an odd number of ϕ's found on O_- drop out of the Hilbert series as they come with a minus sign on H^-. The results are shown in Table 8 and we see that we get the expected results. That is, without intrinsic parity all operators are parity invariant. With intrinsic parity, the operators with an odd number of ϕ insertions drop out.

Field Strength

The field strength tensor $F^{\mu\nu}$ transforms under the representation $(1,0)\oplus(0,1)$, which is now an irreducible representation of $O(3,1)$. We get for the conformal character

$$\widetilde{\chi}_{[2,(1,0)\oplus(0,1)]}(q,x) = \begin{cases} q^2\Big(\chi_{(1,0)\oplus(0,1)}(x) - 2q\chi_{(\frac{1}{2},\frac{1}{2})}(x) + 2q^2\Big)P^{(4)}(q,x) & \text{on } SO(3,1), \\ 0 & \text{on } O_-, \end{cases} \tag{6.67}$$

where the character on O_- completely vanishes. It is now straightforward to show that H^- is given by

$$\begin{aligned} H^-(\phi,p) &= \int \mathrm{d}\mu_{SU(2)}\ \mathrm{PE}\Big[\frac{F^2}{2\mathcal{D}^4}\widetilde{\chi}_{[2,(1,0)\oplus(0,1)]}(\mathcal{D}^2,\alpha,\alpha)\Big] \\ &= \int \mathrm{d}\mu_{SU(2)}\ \mathrm{PE}\Big[F^2\frac{2(\alpha^2+1+1/\alpha^2-\mathcal{D}^2)}{1-\mathcal{D}^2}P^{(2)}(\mathcal{D}^2,\alpha^2)\Big]. \end{aligned} \tag{6.68}$$

In Table 9 we show some results with one field strength tensor.

Vector field

In the case of a vector field we are interested in the case that there are four different vector fields because we know that one such operator contracts with the epsilon tensor which should be the only parity odd operator. Therefore, we do not include derivatives, but just look at the character of the $\ell = (\frac{1}{2},\frac{1}{2})$ representation. The character then becomes (see Eq. (6.48) again)

$$\chi_{(\frac{1}{2},\frac{1}{2})} = \begin{cases} \big(x_1+\frac{1}{x_1}\big)\big(x_2+\frac{1}{x_2}\big) \equiv \chi_{(\frac{1}{2},\frac{1}{2})}(\alpha,\alpha) & \text{on } SO(3,1), \\ \big(\alpha+\frac{1}{\alpha}\big) \equiv \chi^{(\mathcal{P})}_{(\frac{1}{2},\frac{1}{2})}(\alpha) & \text{on } O_-, \end{cases} \tag{6.69}$$

From which we obtain

$$\frac{1}{2}\Big(\chi_{(\frac{1}{2},\frac{1}{2})}(\alpha,\alpha) - \chi^{(\mathcal{P})}_{(\frac{1}{2},\frac{1}{2})}(\alpha^2)\Big) = \frac{1}{2}\Big(\alpha^2+2+\frac{1}{\alpha^2}-\alpha^2-\frac{1}{\alpha^2}\Big) = 1. \tag{6.70}$$

Therefore, H^- for four different vector fields A_i, $i=1,2,3,4$, is given by

$$H^-(0,A_i) = \int \mathrm{d}\mu_{SU(2)} \prod_{i=1}^{4} \mathrm{PE}\Big[A_i\chi^{(\mathcal{P})}_{(\frac{1}{2},\frac{1}{2})}(\alpha) + A_i^2\Big] \tag{6.71}$$

In Table 10 we show the results of this. Unlike the previous two results, we only show the number of operators that are both generated by H^+ and H^-, plus the explicit term that has a different pre-factor for H^+ and H^-. So, we conclude that indeed only one operator is parity odd.

Mass dim.	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$
5	ϕ^5	$\pm\phi^5$	ϕ^5, (0)
6	ϕ^6	ϕ^6	ϕ^6
7	ϕ^7	$\pm\phi^7$	ϕ^7, (0)
8	$\phi^8 + \phi^4\mathcal{D}^4$	$\phi^8 + \phi^4\mathcal{D}^4$	$\phi^8 + \phi^4\mathcal{D}^4$
9	$\phi^9 + \phi^5\mathcal{D}^4$	$\pm\phi^9 \pm \phi^5\mathcal{D}^4$	$\phi^9 + \phi^5\mathcal{D}^4$, (0)
10	$\phi^{10} + \phi^6\mathcal{D}^4 + \phi^4\mathcal{D}^6$	$\phi^{10} + \phi^6\mathcal{D}^4 + \phi^4\mathcal{D}^6$	$\phi^{10} + \phi^6\mathcal{D}^4 + \phi^4\mathcal{D}^6$
11	$\phi^{11} + \phi^7\mathcal{D}^4 + \phi^5\mathcal{D}^6$	$\pm\phi^{11} \pm \phi^7\mathcal{D}^4 \pm \phi^5\mathcal{D}^6$	$\phi^{11} + \phi^7\mathcal{D}^4 + \phi^5\mathcal{D}^6$, (0)
12	$\phi^{12} + \phi^8\mathcal{D}^4 + 2\phi^6\mathcal{D}^6 + \phi^4\mathcal{D}^8$	$\phi^{12} + \phi^8\mathcal{D}^4 + 2\phi^6\mathcal{D}^6 + \phi^4\mathcal{D}^8$	$\phi^{12} + \phi^8\mathcal{D}^4 + 2\phi^6\mathcal{D}^6 + \phi^4\mathcal{D}^8$
13	$\phi^{13} + \phi^9\mathcal{D}^4 + 2\phi^7\mathcal{D}^6 + 2\phi^5\mathcal{D}^8$	$\pm\phi^{13} \pm \phi^9\mathcal{D}^4 \pm 2\phi^7\mathcal{D}^6 \pm 2\phi^5\mathcal{D}^8$	$\phi^{13} + \phi^9\mathcal{D}^4 + 2\phi^7\mathcal{D}^6 + 2\phi^5\mathcal{D}^8$, (0)

Table 8: Enumerating parity invariant operators for a real scalar field. In the third column, the minus (plus) signs correspond to a scalar field with(out) intrinsic parity. In the last column, the terms inside brackets are the results in the case of intrinsic parity.

Mass dim.	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$
6	0	0	0
8	$3F^4$	F^4	$2F^4$
10	$5F^4\mathcal{D}^2$	$F^4\mathcal{D}^2$	$3F^4\mathcal{D}^2$
12	$4F^6 + 4F^4\mathcal{D}^4$	$2F^4\mathcal{D}^4$	$2F^6 + 3F^4\mathcal{D}^4$
14	$13F^6\mathcal{D}^2 + 4F^5\mathcal{D}^4 + 6F^4\mathcal{D}^6$	$F^6\mathcal{D}^2 + 2F^4\mathcal{D}^6$	$7F^6\mathcal{D}^2 + 2F^5\mathcal{D}^4 + 4F^4\mathcal{D}^6$

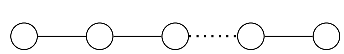
Table 9: Enumerating parity invariant operators with one field strength tensor $F^{\mu\nu}$.

Mass dim.	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$
4	$52 + 4A_1A_2A_3A_4$	$52 + 2A_1A_2A_3A_4$	$52 + 3A_1A_2A_3A_4$

Table 10: Parity invariant operators for four different vector fields A_i, $i = 1, 2, 3, 4$. The number corresponds to the operators both generated by H^+ and H^-. The term that is explicitly written down is the term that has a different pre-factor for H^+ and H^-.

6.4 Charge Conjugation

In order to describe charge conjugation, we look at all the outer automorphisms of the gauge groups $SU(2)$ and $SU(3)$. The dynkin diagram for $SU(N)$ is given by A_{N-1};



which means that in the case of $SU(2)$ there is just one simple root, and it follows that the only symmetry on this Dynkin diagram is the trivial symmetry. Therefore, we conclude that $SU(2)$ has no interesting outer automorphisms and therefore does not transform under charge conjugation. That the outer automorphisms of $SU(N)$ indeed correspond to the object we know as charge conjugation follows if we construct the outer automorphism for $SU(3)$ explicitly, which is what we will do next. We first derive the root system of $\mathfrak{su}(3)$ after which we will be able to construct the induced outer automorphism that follows from the symmetry of the Cartan matrix. We will then point out the correspondence with charge conjugation after which we will be able to compute the twining characters.

6.4.1 Root System and Induced Outer Automorphism of SU(3)

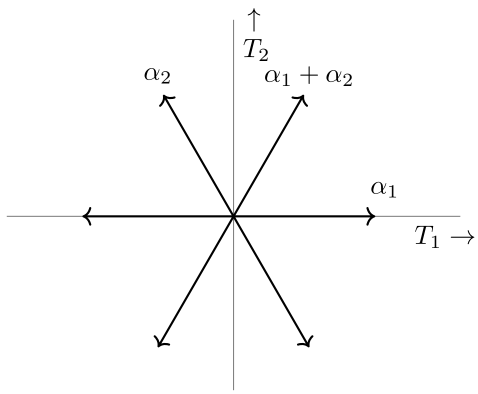
The standard basis for the Lie algebra $\mathfrak{su}(3)$ is defined in terms of the Gell-Mann matrices as $T_i = \frac{1}{2}\lambda_i$, with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{6.72}$$

The straightforward choice for the Cartan subalgebra are the generators T_3 and T_8, from which it follows that the eigenstates (roots) of the adjoint representation are given by

$$E_{\pm(1,0)} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2), \qquad E_{\pm(1/2,\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_4 \pm iT_5), \quad \text{and} \quad E_{(\mp 1/2, \pm\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_6 \pm iT_7). \tag{6.73}$$

For the positive roots we may choose $\alpha_1 = (1,0)$, $\alpha_2 = (-1/2, \sqrt{3}/2)$ and $\alpha_3 = (1/2, \sqrt{3}/2) = \alpha_1 + \alpha_2$, and therefore the root systems of $\mathfrak{su}(3)$ becomes



We can now also compute the Cartan matrix and Dynkin diagram

$$A_{ij} = \frac{2\alpha^i \cdot \alpha^j}{(\alpha^j)^2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad A_2 = \bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigcirc \tag{6.74}$$

The automorphism of the Cartan matrix is again given by $\widetilde{\omega} : 1 \leftrightarrow 2$. In order to define the action of the induced outer automorphism on the generators, we change the basis such that the generators satisfy Eq. (6.17). The two raising and lowering operators corresponding to the simple roots become

$$E_1^{\pm} = E_{\pm(1,0)}, \quad E_2^{\pm} = E_{\pm(-1/2,\sqrt{3}/2)}, \tag{6.75}$$

and we rotate the generators of the Cartan sub-algebra as follows:

$$H_1 = 2T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = -T_3 + \sqrt{3}T_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.76}$$

One can easily check that Eq. (6.17) is indeed satisfied.

To see that the induced outer automorphism corresponds to what we know as charge conjugation, we compute the action of the induced outer automorphism ω on the fundamental representation **3**. As the explicit matrix form of Eq. (6.76) corresponds to the Cartan matrices of the fundamental representation, we immediately see that the vectors $v_1 = (1,0,0)^T, v_2 = (0,1,0)^T$ and $v_3 = (0,0,1)^T$ are the eigenvectors of this representation with weights $(1,0), (-1,1)$ and $(0,-1)$, respectively. From

$$R^{(\omega)}(H_1) \cdot v_1 \equiv R_{\mathbf{3}}(\omega(H_1)) \cdot v_1 = R_{\mathbf{3}}(H_2) \cdot v_1 = 0, \tag{6.77}$$
$$R^{(\omega)}(H_2) \cdot v_2 \equiv R_{\mathbf{3}}(\omega(H_2)) \cdot v_1 = R_{\mathbf{3}}(H_1) \cdot v_1 = v_1. \tag{6.78}$$

we see that v_1 is also an eigenvector of the representation $R^{(\omega)}$, but with weight $(0,1)$ instead. Similar we get that v_2 and v_3 are eigenvectors with weights $(1,-1)$ and $(-1,0)$ respectively. We recognize the weight vectors of the anti-fundamental representation $\bar{\mathbf{3}}$, from which we conclude that the induced representation $R^{(\omega)}$ is the $\bar{\mathbf{3}}$ representation of $SU(3)$. The induced outer automorphism is therefore indeed the object we know as charge conjugation. To make the distinction between the eigenvalues of both representations we denote by $|1,0\rangle$, $|-1,1\rangle$, and $|0,-1\rangle$ the eigenvectors of the $\mathbf{3}$ representation and the eigenvectors of $\bar{\mathbf{3}}$ are denoted by $|0,1\rangle$, $|1,-1\rangle$, and $|-1,0\rangle$, respectively. From this, it immediately follows that the intertwining map of Eq. (6.21) is completely determined by

$$\tau_\omega : |m,n\rangle \mapsto |n,m\rangle. \tag{6.79}$$

Similar to what we did for parity, it is now reasonable to define $\mathcal{C} \equiv \tau_\omega$ as the matrix representation of charge conjugation. However, unlike in the case of parity, this will give some difficulties as states of the same weight can be degenerate, meaning that $\mathcal{C}$ can still induce permutations between eigenstates with the same weight. We will say more about this when we discuss the twining characters and finish this section by folding the Dynkin diagram.

As $\widetilde{\omega}$ is the same as for parity, we have again $N = N_i = 2$ and $\widetilde{I} = \{1\}$. However, we get

$$s_1 = 1 - a_{2,1} = 2 \tag{6.80}$$

which leads to

$$\widetilde{A} = s_1 \frac{N_1}{N}(a_{11} + a_{21}) = 2. \tag{6.81}$$

Therefore, the folded Cartan matrix is again that of an $\mathfrak{su}(2)$ algebra.

6.4.2 Twining Character of SU(3)

As the induced outer automorphism and folded Cartan matrix are the same as in the case of parity, the derivation of the twining character for an $SU(3)$ representations R, i.e.

$$\chi^{(\mathcal{C})}(h) \equiv \mathrm{tr}\left[\tau_\omega e^{i\theta_1 R(H_1) + i\theta_2 R(H_2)}\right] = \sum_\mu \langle\mu|\mathcal{C} e^{i\theta_1 R(H_1)+i\theta_2 R(H_2)}|\mu\rangle, \quad h = \theta_1 H_1 + i\theta_2 H_2 \in \mathfrak{su}(3), \tag{6.82}$$

is similar to the example of parity discussed in Sec. 6.3.1. The only complication arises because the lowering operators of $\mathfrak{su}(3)$ do not commute, so we cannot make the same conclusion as we did in Eq. (6.41). To show this with an example, let's look at the adjoint representation which has highest weight state $|\lambda\rangle = |1,1\rangle$. Then both states $E_1^- E_2^- |1,1\rangle$ and $E_2^- E_1^- |1,1\rangle$ have weight $(0,0)$, but they are not the same state, i.e.

$$E_1^- E_2^- |1,1\rangle \neq E_2^- E_1^- |1,1\rangle. \tag{6.83}$$

This means that we do not get just an effective lowering of the form $(E_1^- E_2^-)$, because this operator is different compared to $(E_2^- E_1^-)$, and $\mathcal{C}$ still permutes the states $E_1^- E_2^- |1,1\rangle$ and $E_2^- E_1^- |1,1\rangle$.

That said, we can still compute the twining characters case by case if we carefully construct the intertwining map $\mathcal{C}$ explicitly and check which states are invariant under $\mathcal{C}$, such that we know what terms in Eq. (6.82) do not vanish. We will discuss the HS for two examples that are most useful to us: the adjoint representation $\mathbf{8}$ and the direct sum of the fundamental and anti-fundamental representations $\mathbf{3} \oplus \bar{\mathbf{3}}$. In particular we will work out the plethystic exponentials[21], for which we will need to compute $\mathrm{tr}[g_-^n]$ (see Eq. (6.52)), where we now have that $g_- = g_+\mathcal{C}$, with $g_+ = \exp(i\theta_1 R(H_1) + i\theta_2 R(H_2))$ and H_1 and H_2 given in Eq. (6.76).

Fundamental and Anti-Fundamental representation

Let's start with the representation $\mathbf{3} \oplus \bar{\mathbf{3}}$. An element $g_- = g_+\mathcal{C}$ can be written in terms of the variables $z_i =$

[21] The derivation of the plethystic exponentials is completely analogous to what we did in Sec. 6.3.3

$\exp(i\theta_i R(H_i))$, so we get

$$g_- = \begin{pmatrix} z_1 &&&&& \\ & z_2 &&&& \\ && \frac{z_2}{z_1} &&& \\ &&& \frac{z_1}{z_2} && \\ &&&& \frac{1}{z_2} & \\ &&&&& \frac{1}{z_1} \end{pmatrix} \begin{pmatrix} 0 & 1 &&&& \\ 1 & 0 &&&& \\ && 0 & 1 && \\ && 1 & 0 && \\ &&&& 0 & 1 \\ &&&& 1 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} \sqrt{z_1 z_2} &&&&& \\ & -\sqrt{z_1 z_2} &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& \sqrt{\frac{1}{z_1 z_2}} & \\ &&&&& -\sqrt{\frac{1}{z_1 z_2}} \end{pmatrix}, \tag{6.84}$$

where we diagonalized the matrix. We see that the variables z_1 and z_2 always group together in the form $(z_1 z_2)^n$ in the diagonalized matrix, and we make the translation to the variable z of the folded $SU(2)$ Lie algebra

$$z_1 z_2 \mapsto z, \tag{6.85}$$

and it follows that

$$\mathrm{tr}[g_-^n] = \begin{cases} 2z^k + 2 + \frac{2}{z^k} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k+1. \end{cases} \tag{6.86}$$

Because the trace completely vanishes in the case that n is odd, the plethystic exponentials become

$$\sum_{n=0}^{\infty} A^n \chi_{\mathrm{sym}^n(\mathbf{8})}(g_-) = \exp\left[\sum_{k=1}^{\infty} \frac{A^{2k}}{2k}\Big(2z^k + 2 + \frac{2}{z^k}\Big)\right], \tag{6.87}$$

$$\sum_{n=0}^{\infty} A^n \chi_{\wedge^n(\mathbf{8})}(g_-) = \exp\left[-\sum_{k=1}^{\infty} \frac{A^{2k}}{2k}\Big(2z^k + 2 + \frac{2}{z^k}\Big)\right]. \tag{6.88}$$

The results for the symmetric and anti-symmetric case can be found in Table 11 and Table 12 respectively, and we see that the symmetric case gives the correct answer. However, in the anti-symmetric case the operators of the form A^2 and A^6 drop out and seem therefore to be odd under charge conjugation. Now recall Eq. (6.23), where we showed that we have some freedom in choosing the phase of the intertwining map $\mathcal{C}$. Therefore, we replace $\mathcal{C}$ of Eq. (6.84) by

$$\mathcal{C} = \begin{pmatrix} 0 & i &&&& \\ i & 0 &&&& \\ && 0 & i && \\ && i & 0 && \\ &&&& 0 & i \\ &&&& i & 0 \end{pmatrix}, \tag{6.89}$$

from which we obtain

$$\mathrm{tr}[g_-^n] = \begin{cases} 2(-1)^k\big[z^k + 1 + \frac{1}{z^k}\big] & \text{if } n = 2k, \\ 0 & \text{if } n = 2k+1. \end{cases} \tag{6.90}$$

With this choice of phase for the matrix representation of $\mathcal{C}$, we get the correct answer $H^- = A^2 + A^4 + A^6$.

Adjoint representation

We can follow the same strategy for the adjoint representation, also denoted by $(1,1)$, of $SU(3)$. Using the diagonal form of g_+, we get

$$g_- = \begin{pmatrix} z_1 z_2 &&&&&&& \\ & \frac{z_1^2}{z_2} &&&&&& \\ && \frac{z_2^2}{z_1} &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& \frac{z_1}{z_2^2} && \\ &&&&&& \frac{z_2}{z_1^2} & \\ &&&&&&& \frac{1}{z_1 z_2} \end{pmatrix} \begin{pmatrix} 1 &&&&&&& \\ & 0 & 1 &&&&& \\ & 1 & 0 &&&&& \\ &&& 0 & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 0 & 1 & \\ &&&&& 1 & 0 & \\ &&&&&&& 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} z_1 z_2 &&&&&&& \\ & \sqrt{z_1 z_2} &&&&&& \\ && -\sqrt{z_1 z_2} &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& \sqrt{\frac{1}{z_1 z_2}} && \\ &&&&&& -\sqrt{\frac{1}{z_1 z_2}} & \\ &&&&&&& \frac{1}{z_1 z_2} \end{pmatrix}, \tag{6.91}$$

where we again diagonalized the matrix. Note that (as discussed above) there are two degenerate states with the same weight, and $\mathcal{C}$ permutes these two states. Making the replacement $z_1 z_2 \to z$, we see that the twining character of the adjoint representation is the spin-$\frac{1}{2}$ representation of $SU(2)$

$$\chi_{\mathbf{8}}^{(\mathcal{C})}(g_-) = z + \frac{1}{z}. \tag{6.92}$$

In order to work with the plethystic exponentials, we compute

$$\mathrm{tr}[g_-^n] = \begin{cases} z^{2k} + 2z^k + 2 + \frac{2}{z^k} + \frac{1}{z^{2k}} & \text{if } n = 2k, \\ z^{2k+1} + \frac{1}{z^{2k+1}} & \text{if } n = 2k+1. \end{cases} \tag{6.93}$$

Using this for the plethystic exponential, we get for the symmetric and anti-symmetric tensor products:

$$\begin{aligned} \sum_{n=0}^{\infty} A^n \chi_{\mathrm{sym}^n(\mathbf{8})}(g_-) &= \exp\left[\sum_{k=0}^{\infty} \frac{A^{2k+1}}{2k+1}\Big(z^{2k+1} + \frac{1}{z^{2k+1}}\Big) + \sum_{k=1}^{\infty} \frac{A^{2k}}{2k}\Big(z^{2k} + 2z^k + 2 + \frac{2}{z^k} + \frac{1}{z^{2k}}\Big)\right], \\ &= \exp\left[\sum_{k=1}^{\infty} \frac{A^k}{k}\Big(z^k + \frac{1}{z^k}\Big) + \sum_{k=1}^{\infty} \frac{A^{2k}}{2k}\Big(2z^k + 2 + \frac{2}{z^k}\Big)\right], \end{aligned} \tag{6.94}$$

$$\begin{aligned} \sum_{n=0}^{\infty} A^n \chi_{\wedge^n(\mathbf{8})}(g_-) &= \exp\left[-\sum_{k=0}^{\infty} \frac{(-A)^{2k+1}}{2k+1}\Big(z^{2k+1} + \frac{1}{z^{2k+1}}\Big) - \sum_{k=1}^{\infty} \frac{(-A)^{2k}}{2k}\Big(z^{2k} + 2z^k + 2 + \frac{2}{z^k} + \frac{1}{z^{2k}}\Big)\right], \\ &= \exp\left[-\sum_{k=1}^{\infty} \frac{(-A)^k}{k}\Big(z^k + \frac{1}{z^k}\Big) - \sum_{k=1}^{\infty} \frac{A^{2k}}{2k}\Big(2z^k + 2 + \frac{2}{z^k}\Big)\right]. \end{aligned} \tag{6.95}$$

The results for this are summarized in Table 13 and Table 14, and we see that all operators in the symmetric case seem to be $\mathcal{C}$-invariant. However, in the anti-symmetric case the operators of the form A^3 and A^8 drop out and seem therefore to be odd under $\mathcal{C}$. We can use Eq. (6.23) again to choose another phase for the representation matrix of $\mathcal{C}$, but everything we tried does not give the correct answer. We leave this as an open question for now, to be answered in further research.

A^n	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}	A^{11}	A^{12}	A^{13}	A^{14}	A^{15}
H^+	$AA^\dagger$	0	$(AA^\dagger)^2$	0	$(AA^\dagger)^3$	0	$(AA^\dagger)^4$	0	$(AA^\dagger)^5$	0	$(AA^\dagger)^6$	0	$(AA^\dagger)^7$	0
H^-	A^2	0	A^4	0	A^6	0	A^8	0	A^{10}	0	A^{12}	0	A^{14}	0
$\mathcal{C}$-inv.	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 11: Number of symmetric invariants formed out of the $\mathbf{3} \oplus \bar{\mathbf{3}}$ representation of $SU(3)$.

A^n	A^2	A^3	A^4	A^5	A^6
H^+	$AA^\dagger$	$A^3 + (A^\dagger)^3$	$(AA^\dagger)^2$	0	$(AA^\dagger)^3$
H^-	$-A^2$	0	A^4	0	$-A^6$
$\mathcal{C}$-inv.	-	✓	✓	✓	-

Table 12: Number of anti-symmetric invariants formed out of the $\mathbf{3} \oplus \bar{\mathbf{3}}$ representation of $SU(3)$. Note that there are no operators higher than A^6, as the vector space over which we anti-symmetrize has dimension 6.

A^n	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}	A^{11}	A^{12}	A^{13}	A^{14}	A^{15}	A^{16}	A^{17}	A^{18}	A^{19}
H^+	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4	3
H^-	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4	3
$\mathcal{C}$-inv.	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 13: Number of symmetric invariants formed out of the adjoint representation of $SU(3)$.

A^n	A^2	A^3	A^4	A^5	A^6	A^7	A^8
H^+	0	1	0	1	0	0	1
H^-	0	-1	0	1	0	0	-1
$\mathcal{C}$-inv.	✓	-	✓	✓	✓	✓	-

Table 14: Number of anti-symmetric invariants formed out of the adjoint representation of $SU(3)$. Note that there are no operators higher than A^8, as the vector space over which we anti-symmetrize has dimension 8.

6.5 CP

Finally, we arrive at the point where we can combine the above formulas to enumerate the $\mathcal{CP}$-invariant operators of the SMEFT. Before we can do this, we need to know what representations R can be extended such that $\mathcal{CP}$ acts as an honest symmetry on these representations. Because $\mathcal{P}$ only acts on the Lorentz part, and $\mathcal{C}$ only on the gauge groups, it follows that R is the tensor product of an $O(3,1)_o$ representation (recall Eq. (6.35) for the irreducible $O(3,1)_o$ representations) and a representation that transforms well under the gauge group extended with the outer automorphism $\mathcal{C}$, e.g. the trivial representation or $\mathbf{3} \oplus \bar{\mathbf{3}}$ and $\mathbf{8}$ of $SU(3)$. For the SM these are all $SU(3)$ representations that we need to know, but what about $SU(2)$ and $U(1)$? At the beginning of Sec. 6.4 we pointed out that $SU(2)$ has no interesting outer automorphisms, so this means that all $SU(2)$ representations already transform well under charge conjugation. For $U(1)$ charge conjugation acts by sending the charge Q of a field to $-Q$. We therefore conclude that the extended representations for the SM are the direct sum of the representation of the field and the representation of the conjugate field.

The Hilbert series still takes the form of Eq. (6.62), but the generating functions for the characters of the symmetric and anti-symmetric tensor products change slightly because we include the gauge group $SU(3) \times SU(2) \times U(1)$ on which $\mathcal{C}$ acts. Just like we did for parity and charge conjugation separately, we can compute the plethystic exponentials of Eq. (6.52) by deriving $\text{tr}[g_-^n]$, with $g_- = g_+\mathcal{CP}$, where we will parametrize g_+ in terms of the usual variables x_1, x_2, x, y, z_1, z_2. Therefore, we compute for all fields of the SM the explicit matrix form of an element $g_- = g_+\mathcal{CP}$ which we then sub-sequentially diagonalize. Just like in the previous cases, there is some freedom in choosing the phase for the matrix $\mathcal{CP}$, and we will fix this phase by demanding that the kinetic terms of the SM Lagrangian of mass dimension 4 are $\mathcal{CP}$ invariant (see Eq. (3.1) and Table 15).

Higgs field

Because the Higgs field φ is a scalar, it transforms already trivially under $\mathcal{P}$, but still transforms under $\mathcal{C}$. That is, $\mathcal{CP}$ acts on φ by mapping it to $\varphi^\dagger$. Therefore, the representation of the Higgs field becomes the direct sum of the representations of φ and $\varphi^\dagger$, i.e.

$$\Big[(0,0)\otimes\mathbf{1}\otimes\mathbf{2}\otimes Y_{1/2}\Big]\oplus\Big[(0,0)\otimes\mathbf{1}\otimes\mathbf{2}\otimes Y_{-1/2}\Big]. \tag{6.96}$$

Without derivatives, the matrix $g_- = g_+\mathcal{CP}$ becomes

$$g_- = \begin{pmatrix} x^{\frac{1}{2}}y &&& \\ & x^{-\frac{1}{2}}y && \\ && x^{\frac{1}{2}}\frac{1}{y} & \\ &&& x^{-\frac{1}{2}}\frac{1}{y}\end{pmatrix}\begin{pmatrix} 0 & i && \\ i & 0 && \\ && 0 & i \\ && i & 0\end{pmatrix} \mapsto \begin{pmatrix} iy &&& \\ & -iy && \\ && \frac{i}{y} & \\ &&& -\frac{i}{y}\end{pmatrix}, \tag{6.97}$$

with x the $U(1)$ variable and y the $SU(2)$ variable. We have included a complex phase in the matrix representation of $\mathcal{CP}$, which is necessary because otherwise the kinetic term $\mathcal{D}^2\phi^2$ will not be counted as a $\mathcal{CP}$-invariant operator. That is, we would find $H^- = -\mathcal{D}^2\phi^2$ without this phase which gives the incorrect answer $\frac{1}{2}(H^+ + H^-) = 0$, and we have more to say about this in Sec. 6.5.1. Notice that we do not have to worry about $U(1)$ charges as they drop out in the diagonalization process, and this will be true for all other fields as well. We can include the derivatives by extending the Lorentz part of the representation to the conformal representation of $O(4,2)$, just like we did in Sec. 6.3.4. Using Eq. (6.54) then yields

$$\mathrm{tr}[g_-^n] = \begin{cases} 2(-1)^k P^{(4)}(q^{2k},\alpha^k,\alpha^k)\big(y^{2k}+\frac{1}{y^{2k}}\big) & \text{if } n=2k, \\ 0 & \text{if } n=2k+1. \end{cases} \tag{6.98}$$

Up and down quark

For the up quark u, the extended representation with $\mathcal{CP}$ included becomes the direct sum of the 'old' representations of u and its antiparticle $\bar{u}$. Therefore we get

$$\Big[\Big(0,\frac{1}{2}\Big)\otimes\mathbf{3}\otimes\mathbf{1}\otimes Y_{2/3}\Big]\oplus\Big[\Big(\frac{1}{2},0\Big)\otimes\bar{\mathbf{3}}\otimes\mathbf{1}\otimes Y_{-2/3}\Big]. \tag{6.99}$$

Without derivatives, we can write an element g_- as

$$g_- = \begin{pmatrix} x_2z_1 &&&&&& \\ & x_1z_2 &&&&& \\ && x_2\frac{z_2}{z_1} &&&& \\ &&& x_1\frac{z_1}{z_2} &&& \\ &&&& x_2\frac{1}{z_2} && \\ &&&&& x_1\frac{1}{z_1} & \\ &&&&&& x_i\leftrightarrow\frac{1}{x_i}\end{pmatrix}\begin{pmatrix} 0 & 1 &&&&& \\ 1 & 0 &&&&& \\ && 0 & 1 &&& \\ && 1 & 0 &&& \\ &&&& 0 & 1 & \\ &&&& 1 & 0 & \\ &&&&&& \ddots\end{pmatrix}, \tag{6.100}$$

$$\to \begin{pmatrix} \sqrt{x_1x_2z_1z_2} &&&&&& \\ & -\sqrt{x_1x_2z_1z_2} &&&&& \\ && \sqrt{x_1x_2} &&&& \\ &&& -\sqrt{x_1x_2} &&& \\ &&&& \sqrt{\frac{x_1x_2}{z_1z_2}} && \\ &&&&& -\sqrt{\frac{x_1x_2}{z_1z_2}} & \\ &&&&&& x_i\leftrightarrow\frac{1}{x_i}\end{pmatrix} \tag{6.101}$$

where we have neglected the $U(1)$ charges because they drop out in the diagonalization process. We can restore the derivatives by using Eq. (6.54):

$$\mathrm{tr}[g_-^n] = \begin{cases} 2P^{(4)}(q^{2k},\alpha^k,\alpha^k)\Big[\big(\alpha^k+\frac{1}{\alpha^k}\big)-q^{2k}\big(\alpha^k+\frac{1}{\alpha^k}\big)\Big]\Big(z^k+1+\frac{1}{z^k}\Big) & \text{if } n=2k, \\ 0 & \text{if } n=2k+1. \end{cases} \tag{6.102}$$

Likewise, the down quark d is in the representation

$$\left[\left(0,\frac{1}{2}\right)\otimes\mathbf{3}\otimes\mathbf{1}\otimes Y_{2/3}\right]\oplus\left[\left(\frac{1}{2},0\right)\otimes\bar{\mathbf{3}}\otimes\mathbf{1}\otimes Y_{-2/3}\right], \tag{6.103}$$

but this leads to the same equation as Eq. (6.102), because the only difference is the charge under $U(1)$, which drops out in this formula.

Electron
The extended representation with $\mathcal{CP}$ included becomes for the (right handed) electron e:

$$\left[\left(0,\frac{1}{2}\right)\otimes\mathbf{1}\otimes\mathbf{1}\otimes Y_{-1}\right]\oplus\left[\left(\frac{1}{2},0\right)\otimes\mathbf{1}\otimes\mathbf{1}\otimes Y_{1}\right]. \tag{6.104}$$

From now on, we will not write down the explicit matrices as we did in the last two examples as they can become quite large and they are straightforward to compute. For the electron we therefore get

$$\mathrm{tr}[g_-^n]=\begin{cases}2P^{(4)}(q^{2k},\alpha^k,\alpha^k)\Big[\big(\alpha^k+\frac{1}{\alpha^k}\big)-q^{2k}\big(\alpha^k+\frac{1}{\alpha^k}\big)\Big] & \text{if } n=2k,\\ 0 & \text{if } n=2k+1.\end{cases} \tag{6.105}$$

Lepton and quark doublet
The representations for the lepton l and quark doublet q become

$$\begin{gathered}\left[\left(\frac{1}{2},0\right)\otimes\mathbf{1}\otimes\mathbf{2}\otimes Y_{-1/2}\right]\oplus\left[\left(0,\frac{1}{2}\right)\otimes\mathbf{1}\otimes\mathbf{2}\otimes Y_{1/2}\right],\\ \left[\left(\frac{1}{2},0\right)\otimes\mathbf{3}\otimes\mathbf{2}\otimes Y_{1/6}\right]\oplus\left[\left(0,\frac{1}{2}\right)\otimes\bar{\mathbf{3}}\otimes\mathbf{2}\otimes Y_{-1/6}\right],\end{gathered} \tag{6.106}$$

respectively. With little effort we then compute for the lepton doublet

$$\mathrm{tr}[g_-^n]=\begin{cases}2(-1)^kP^{(4)}(q^{2k},\alpha^k,\alpha^k)\Big[\big(\alpha^k+\frac{1}{\alpha^k}\big)-q^{2k}\big(\alpha^k+\frac{1}{\alpha^k}\big)\Big]\Big(y^{2k}+\frac{1}{y^{2k}}\Big) & \text{if } n=2k,\\ 0 & \text{if } n=2k+1,\end{cases} \tag{6.107}$$

and for the quark doublet

$$\mathrm{tr}[g_-^n]=\begin{cases}2(-1)^kP^{(4)}(q^{2k},\alpha^k,\alpha^k)\Big[\big(\alpha^k+\frac{1}{\alpha^k}\big)-q^{2k}\big(\alpha^k+\frac{1}{\alpha^k}\big)\Big]\Big(y^{2k}+\frac{1}{y^{2k}}\Big)\Big(z^k+1+\frac{1}{y^k}\Big) & \text{if } n=2k,\\ 0 & \text{if } n=2k+1.\end{cases} \tag{6.108}$$

Notice that in both cases we had to include a non-trivial phase such that the kinetic terms are $\mathcal{CP}$ invariant, i.e. otherwise we would find $H^-=-e^2\mathcal{D}-q^2\mathcal{D}$ which would kill the contribution from H^+ .

Field strength tensors
We begin this case by noting that the field strength tensors already transform well under $\mathcal{CP}$ (they are their own anti-particle), e.g. for $B^{\mu\nu}$ we have

$$[(1,0)\oplus(0,1)]\otimes\mathbf{1}\otimes\mathbf{1}\otimes Y_0. \tag{6.109}$$

This is of course the same case as we already discussed when we treated parity only. So recalling Eqs. (6.54) and (6.67), we get

$$\begin{aligned}\mathrm{tr}[g_-^n]&=\begin{cases}\widetilde{\chi}_{[2,(1,0)\oplus(0,1)]}(q^{2k},\alpha^k,\alpha^k) & \text{if } n=2k,\\ 0 & \text{if } n=2k+1,\end{cases}\\ &=\begin{cases}P^{(4)}(q^{2k},\alpha^k,\alpha^k)\Big[2\big(\alpha^{2k}+1+\frac{1}{\alpha^{2k}}\big)-2q^{2k}\big(\alpha^{2k}+2+\frac{1}{\alpha^{2k}}\big)+2q^{4k}\big)\Big] & \text{if } n=2k,\\ 0 & \text{if } n=2k+1.\end{cases}\end{aligned} \tag{6.110}$$

Likewise, the representations for $G^{\mu\nu}$ and $W^{\mu\nu}$ yield

$$[(1,0)\oplus(0,1)]\otimes\mathbf{8}\otimes\mathbf{1}\otimes Y_0,\qquad [(1,0)\oplus(0,1)]\otimes\mathbf{1}\otimes\mathbf{3}\otimes Y_0, \tag{6.111}$$

respectively. We can 'tensor' the result of Eqs. (6.93) and (6.110) together to get

$$\mathrm{tr}[g_-^n] = \begin{cases} \widetilde{\chi}_{[2,(1,0)\oplus(0,1)]}(q^{2k},\alpha^k,\alpha^k)\Big(z^{2k}+2z^k+2+\frac{2}{z^k}+\frac{1}{z^{2k}}\Big) & \text{if } n=2k, \\ 0 & \text{if } n=2k+1, \end{cases} \tag{6.112}$$

for $G^{\mu\nu}$, and for $W^{\mu\nu}$ we use that all $SU(2)$ representations transform well under $\mathcal{C}$:

$$\mathrm{tr}[g_-^n] = \begin{cases} \widetilde{\chi}_{[2,(1,0)\oplus(0,1)]}(q^{2k},\alpha^k,\alpha^k)\Big(y^{4k}+1+\frac{1}{y^{4k}}\Big) & \text{if } n=2k, \\ 0 & \text{if } n=2k+1. \end{cases} \tag{6.113}$$

6.5.1 Results

With the formulas we found above, we can now enumerate the $\mathcal{CP}$ invariant operators for the SMEFT. As discussed, we had some freedom in choosing the phase for the matrix representations of $\mathcal{CP}$ and we fixed it to produce the correct $\mathcal{CP}$ invariant operators of mass dimension 4. These operators are summarized in the first two columns of Table 15, and in column 3 and 4 we give the results for H^+ and H^-, where in the latter we show in red the operators for which we had to choose a non-trivial phase for the $\mathcal{CP}$ matrix. Note that for the enumeration in H^-, we can only use one symbol per field in the Hilbert series as the fields transform now in an irreducible representation in which the particle and anti-particle get intertwined. Therefore, we decided make the convention to give the field the name without the dagger in the Hilbert series. To get from the results of H^+ and H^- to the $\mathcal{CP}$-invariant operators, we drop the daggers in H^+. These results are shown in the fifth column. In the last column we add a checkmark for all operators that are $\mathcal{CP}$-invariant.

We have shown the results for the dimension 6 operators earlier in Tables 3 and 4. We also worked out the result for dimension 5 and 7 and they are shown in Tables 16, and 19, respectively. In Table 20 to Table 22, we have worked out the bosonic operators of mass dimension 8. A summary of all operators classes of mass dimension 8 is given in Table 17. For the explicit contraction of the operators at mass dimension 8, we have used [28]. Note that in all tables, the results for which we need the complex phase in the matrix representation of $\mathcal{CP}$ are coloured red in the tables.

We can make three important observations from the results of Table 15 that turn out to be true in general. First of all, if an operator is $\mathcal{CP}$ invariant, then it is enumerated on both H^+ and H^- such that when we average we get the correct answer. As an example, see the pure Higgs operators at mass dimension 4, 6 and 8. Second, the field strength tensors often come in pairs where one of the operators is contracted with the epsilon tensor. Because the epsilon tensor is $\mathcal{CP}$ odd, this operator should drop out. In cases like this, we find 2 operators on H^+ and none on H^- (see e.g. the second class of Table 15). Averaging then gives the correct answer that there is just one $\mathcal{CP}$ invariant operator. Third, it might happen that we have an operator plus its hermitian conjugate (see e.g. the third class in Table 15). In that case, there are two operators enumerated on H^+ and zero on H^-. Because $\mathcal{CP}$ sends the operator to its complex conjugate we see that there is effectively one operator that is $\mathcal{CP}$ invariant[22], which we also find when we average over H^+ and H^-.

For mass dimension 4, 5, 6 and 7 we enumerate the correct operators that are $\mathcal{CP}$ invariant and the explicit output is given in the corresponding tables. We conclude that there are 13 $\mathcal{CP}$ invariant operators at mass dimension 4, 1 at dimension 5, 57 at dimension 6, and 15 at dimension 7. We also checked the operator basis of dimension 8, by using the complete set of operators published in Ref. [28]. All bosonic operators are reproduced correctly, and the results are explicitly shown in the tables in App. E. For an overview of the enumeration in operator classes see Table 17, and we see that 563 out of the 993 operators are $\mathcal{CP}$ invariant. For the following classes with fermions: $\psi^2X^2\varphi, \psi^2X\varphi^3, \psi^2\varphi^5, \psi^2X\varphi^2\mathcal{D}, \psi^2X\varphi\mathcal{D}^2, \psi^4X$, and $\psi^4\varphi\mathcal{D}$ we do not find any operators on H^- because it is either the case that a field strength tensor appears together with a dual field strength tensor, or that an operator appears with its complex conjugate. Therefore, the only classes which can cause problems are $\psi^2\varphi^2\mathcal{D}^3$, $\psi^2\varphi^4\mathcal{D}$, $\psi^2X^2\mathcal{D}$, $\psi^4\varphi^2$, and $\varphi^4\mathcal{D}^2$, but they are fine as some differences between H^+ and H^- arise due to the fact that some of the operators in these classes appear together with a dual field strength tensor, or their hermitian conjugate.

We can therefore conclude that the methods we have developed in this section to enumerate the $\mathcal{CP}$-invariant operators of the SMEFT works fine up to at least mass dimension 8. However, before we can be fully conclusive

[22] As long as the Wilson-coefficients are real.

about the results for higher dimensional operators, we need to have better understanding why we needed to make the non-trivial choice of phase in the representation matrix of $\mathcal{CP}$ for the Higgs scalar, lepton doublet and quark doublet (see Eqs. (6.98), (6.107), and (6.108)).

φ^4 and $\varphi^2\mathcal{D}^2$		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
Q_{φ^4}	$(\varphi^\dagger\varphi)^2$	$(\varphi^\dagger)^2\varphi^2$	φ^4	φ^4	✓
$Q_{\varphi^2\mathcal{D}^2}$	$(D_\mu\varphi)^\dagger(D^\mu\varphi)$	$\varphi^\dagger\varphi\mathcal{D}^2$	$\varphi^2 D^2$	$\varphi^2\mathcal{D}^2$	✓
X^2					
$Q_{G^2}^{(1)}$	$G_{\mu\nu}^A G^{A\mu\nu}$	$2G^2$	0	G^2	✓
$Q_{G^2}^{(2)}$	$\widetilde{G}_{\mu\nu}^A G^{A\mu\nu}$				-
$Q_{W^2}^{(1)}$	$W_{\mu\nu}^I W^{I\mu\nu}$	$2W^2$	0	W^2	✓
$Q_{W^2}^{(2)}$	$\widetilde{W}_{\mu\nu}^I W^{I\mu\nu}$				-
$Q_{B^2}^{(1)}$	$B_{\mu\nu}B^{\mu\nu}$	$2B^2$	0	B^2	✓
$Q_{B^2}^{(2)}$	$\widetilde{B}_{\mu\nu}B^{\mu\nu}$				-
$\psi^2\mathcal{D}$ and $\psi^2\varphi$					
Q_{l^2D}	$\bar{l}\not{D}l$	$l^\dagger l\mathcal{D}$	$l^2\mathcal{D}$	$l^2\mathcal{D}$	✓
Q_{e^2D}	$\bar{e}\not{D}e$	$e^\dagger e\mathcal{D}$	$e^2\mathcal{D}$	$e^2\mathcal{D}$	✓
Q_{q^2D}	$\bar{q}\not{D}q$	$q^\dagger q\mathcal{D}$	$q^2\mathcal{D}$	$q^2\mathcal{D}$	✓
Q_{u^2D}	$\bar{u}\not{D}u$	$u^\dagger u\mathcal{D}$	$u^2\mathcal{D}$	$u^2\mathcal{D}$	✓
Q_{d^2D}	$\bar{d}\not{D}d$	$d^\dagger d\mathcal{D}$	$d^2\mathcal{D}$	$d^2\mathcal{D}$	✓
$Q_{le\varphi}$	$\bar{l}e\varphi$+ h.c.	$l^\dagger e\varphi + le^\dagger\varphi^\dagger$	0	$le\varphi$	✓
$Q_{qu\varphi}$	$\bar{q}u\widetilde{\varphi}$+ h.c.	$q^\dagger u\varphi^\dagger + qu^\dagger\varphi$	0	$qu\varphi$	✓
$Q_{qd\varphi}$	$\bar{q}d\varphi$+ h.c.	$q^\dagger d\varphi + qd^\dagger\varphi^\dagger$	0	$qd\varphi$	✓
Total:	16	19	-	13	13

Table 15: All dimension 4 operators of the SMEFT. Note that we have included the operators with two field strength tensors and an epsilon tensor, as it is one of the operators that fall out of the Hilbert series.

		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
$Q_{\nu\nu}$	$\epsilon_{jk}\epsilon_{mn}\varphi^j\varphi^m(l_p^k)^T C l_r^n$+ h.c.	$\varphi^2 l^2 + (\varphi^\dagger)^2(l^\dagger)^2$	0	$\varphi^2 l^2$	✓

Table 16: All dimension 5 operators of the SMEFT.

Class	H^+	H^-	$\frac{1}{2}(H^+ + H^-)$
X^4	43	9	26
φ^8	1	1	1
$\varphi^6\mathcal{D}^2$	2	2	2
$\varphi^4\mathcal{D}^4$	3	3	3
$X^3\varphi^2$	6	0	3
$X^2\varphi^4$	10	0	5
$X^2\varphi^2\mathcal{D}^2$	18	4	11
$X\varphi^4\mathcal{D}^2$	6	4	3
$\psi^2 X^2\varphi$	96	0	48
$\psi^2 X\varphi^3$	22	0	11
$\psi^2\varphi^2\mathcal{D}^3$	16	14	15
$\psi^2\varphi^5$	6	0	3
$\psi^2\varphi^4\mathcal{D}$	13	7	10
$\psi^2 X^2\mathcal{D}$	57	23	40
$\psi^2 X\varphi^2\mathcal{D}$	92	0	46
$\psi^2 X\varphi\mathcal{D}^2$	48	0	24
$\psi^2\varphi^3\mathcal{D}^2$	36	0	18
$\psi^4\varphi^2$	87	31	59
$\psi^4 X$	200	0	100
$\psi^4\varphi\mathcal{D}$	166	0	83
$\psi^4\mathcal{D}^2$	65	39	52
Total	993	133	563

Table 17: Summary of all dimension-8 operators of the SMEFT

7 Conclusion

The initial goal of this masters thesis was to find a general method that can assist in constructing a minimal operator basis for effective field theories at any effective order. Before we were able to do so, we took some time to properly define what the operator basis is, and in particular what we actual mean by *minimal* in this sense. We showed that the operator basis consists of all independent invariants under the symmetry groups and that operators can be related by integration by parts and through usage of the classical equations of motion. The minimal operator basis then is the subset of independent operators that cannot be related by these relations. We found out that finding the minimal operator basis can be assisted by the Hilbert series which counts the number of independent operators of a particular form. Using some beautiful results of group theory, we managed to derive general formulas to compute the HS for EFTs in 4 dimensional space-time with SM like field content, i.e. scalars, field strength tensors and fermions, and we extended this with the Weyl tensor: the building block for an EFT with gravity.

We explicitly computed the HS for the SMEFT and some extension, i.e. GRSMEFT and 2HDM. By doing so, we have implemented an efficient algorithm of the HS called Eco (Efficient Counting of Operators), in Form, and the code can be found in [20]. Eco can be used to enumerate operators for EFTs in 4 dimensional space time with the above mentioned field content. The user has the freedom to include as many of these fields as desired, together with their representations under the SM gauge group $SU(3) \times SU(2) \times U(1)$, as well as additional $U(1)$ global or gauge symmetries. The user flexibility, together with the speed-up due to how we structured the calculation, makes Eco a valuable addition to the model-building toolkit.

We have shown that the HS technique can be extended to EFTs that are symmetric under parity and charge conjugation, by relating these to the outer automorphisms of the symmetry groups. With the outer automorphisms, we were able to extend the representations of the Lorentz group and $SU(3)$, and we explicitly computed the extended representation matrices. Furthermore, we showed that all outer automorphisms can be classified using the symmetries of the Dynkin diagrams, and this gave us a way to fold these diagrams, which we used to explicitly compute the HS. We managed to implement this for the SMEFT, and we obtained the correct enumeration of $\mathcal{CP}$ invariant operators up to and including mass dimension 8. Because we had to manually insert a phase for the matrix representation of $\mathcal{CP}$, we cannot yet be fully conclusive that we will obtain the correct answer for higher mass dimensions. Therefore, some open questions remain, but it also opens new interesting questions about the nature of orbit Lie algebras, which we might address in the future.

Although we were able to produce the number of operators of the SMEFT up to mass dimension 20 with Eco, something that was out of practical reach before, we have yet been far away from the limits of Eco. Pushing the program further might result in new interesting questions. For instance, what drives the exponential growth of the number of operators of the SMEFT, observed in Fig. 2? In order to answer this question, Eco can be an excellent starting point as the user flexibility and Eco's speed allows for easy changes in the field content of the SMEFT or their representations under the gauge group. Furthermore, it will certainly be interesting to extend the methods of the HS technique to other EFTs like chiral perturbation theory or soft collinear effective theory, and implement these new degrees of freedom in Eco.

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A Notations and Conventions

Throughout this thesis, we use the following notations and conventions. The signature of the Minkowski metric tensor $\eta_{\mu\nu}$ is taken to be $(-+++)$. We work in natural units in which

$$\hbar = c = 1. \tag{A.1}$$

We will denote by $j = 1, 2$, and $I = 1, 2, 3$ the $SU(2)$ doublet and triplet indices, by $\alpha = 1, 2, 3$, and $A = 1, ..., 8$ the $SU(3)$ colour and octet indices, and by $p = 1, 2, 3$ the generation indices. The Pauli matrices τ^I, the generators of $SU(2)$ are given by

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.2}$$

The projection matrices for Lorentz spinors are

$$P_L = \tfrac{1}{2}(\mathbb{1} - \gamma_5), \quad \text{and} \quad P_R = \tfrac{1}{2}(\mathbb{1} + \gamma_5) \tag{A.3}$$

where γ_5 is given by

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{A.4}$$

The epsilon tensors in 2, 3, and 4 dimensions are defined with

$$\epsilon_{12} = +1, \qquad \epsilon_{123} = +1, \quad \text{and} \quad \epsilon_{0123} = +1, \tag{A.5}$$

respectively. We denote by φ^j the Higgs scalar, and the complex conjugate will always occur either as

$$\varphi^\dagger, \quad \text{or} \quad \widetilde{\varphi}^j = \epsilon_{jk}(\varphi^k)^*. \tag{A.6}$$

We write

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ W^I_{\mu\nu} &= \partial_\mu W^I_\nu - \partial_\nu W^I_\mu - g\epsilon^{IJK} W^J_\mu W^K_\nu, \\ G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu, \end{aligned} \tag{A.7}$$

for the field strength tensors of the $U(1)$, $SU(2)$, and $SU(3)$ gauge groups respectively. Here g_s is the strong coupling constant, and g and g' the $SU(2)$ and $U(1)$ coupling constants respectively. For the quark content, we write $q^{\alpha j}_{Lp}$ for the left handed quark doublet, u^α_{Rp} for the right handed up type quarks and d^α_{Rp} for the right handed down type quarks. The leptons are denoted by l^j_{Lp} and e_{Rp} for the lepton doublet and singlet respectively. We drop the index structure if it is clear from the context. The dual tensor of a field strength tensor $X_{\mu\nu}$ is given by

$$\widetilde{X}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} X^{\rho\sigma}, \tag{A.8}$$

where X stands for B, W^I, or G^A. Our sign convention for the covariant derivative is

$$(D_\mu q)^{\alpha j} = \left[\delta_{\alpha\beta}\delta_{jk}(\partial_\mu + ig' Y_q B_\mu) + ig\delta_{\alpha\beta} S^I_{jk} W^I_\mu + ig_s \delta_{jk} T^A_{\alpha\beta} G^A_\mu\right] q^{\beta k}, \tag{A.9}$$

where $T^A = \frac{1}{2}\lambda^A$ and $S^I = \frac{1}{2}\tau^I$ are the usual $SU(3)$ and $S(U(2)$ generators, with λ^A the Gell-Mann matrices[23]. We will denote the mass dimension of a operator by δ and if not specified, the number of space-time dimensions by d.

[23] They are explicitly given in Eq. (6.72).

B Characters

Let G be a (Lie) group and R a representation of this group. Then the character of this representation is equal to the trace of the matrix constituting this representation, i.e.

$$\chi(g) = \mathrm{tr}[R(g)]. \tag{B.1}$$

From the cyclic property of the trace we get that characters of conjugate group elements are equal, and such functions are called class functions. A torus T in a compact Lie group G is defined as a compact connected abelian Lie subgroup of G. We can define a maximal torus as one which is maximal among such subgroups, that is T is maximal if for any torus T' containing T we have $T = T'$. Note that this does not imply that a maximal torus is unique. We can make use of the Torus Theorem which states that every element of G is conjugate to an element of a maximal torus T [29]. For simple, semi-simple Lie groups, we can choose the maximal torus in terms of the Cartan matrices H_k, i.e. T consists of all elements of the form $e^{i\theta_k H_k}$. This means that we can calculate the characters in terms of the Cartan matrices' parameters:

$$\chi(g) = \chi(x_k) = \mathrm{tr}\left[e^{i\sum_k \theta_k H_k}\right], \tag{B.2}$$

with $x_k = e^{i\theta_k}$. Note that this implies that the x_k are on the unit circle. Table 18 contains explicit expressions of some of the characters for representations of the Lorentz group and the SM gauge group $SU(3) \times SU(2) \times U(1)$. Because we often integrate over these characters, we showed the Haar measure in the fourth column. Note that this measure only holds for class functions like the characters.

Group	Representation	Character	Haar measure
Lorentz	$(0,0)$	1	$\frac{1}{(2\pi i)^2}\oint_{\lvert y_1\rvert=1}\frac{dy_1}{y_1}(1-y_1^2) \times \oint_{\lvert y_2\rvert=1}\frac{dy_2}{y_2}(1-y_2^2)$
	$(\frac{1}{2},0)$	$y_1+\frac{1}{y_1}$	
	$(0,\frac{1}{2})$	$y_2+\frac{1}{y_2}$	
	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$y_1+\frac{1}{y_1}+y_2+\frac{1}{y_2}$	
	$(\frac{1}{2},\frac{1}{2})$	$(y_1+\frac{1}{y_1})(y_2+\frac{1}{y_2})$	
	$(1,0)\oplus(0,1)$	$y_1^2+1+\frac{1}{y_1^2}+(y_1\leftrightarrow y_2)$	
	$(2,0)\oplus(0,2)$	$y_1^4+y_1^2+1+\frac{1}{y_1^2}+\frac{1}{y_1^4}+(y_1\leftrightarrow y_2)$	
	$(\frac{3}{2},\frac{1}{2})$	$(y_1^3+y_1+\frac{1}{y_1}+\frac{1}{y_1^3})(y_2+\frac{1}{y_2})$	
	$(\frac{1}{2},\frac{3}{2})$	$(y_1+\frac{1}{y_1})(y_2^3+y_1+\frac{1}{y_2}+\frac{1}{y_2^3})$	
$U(1)$	charge Q	x^Q	$\frac{1}{2\pi i}\oint_{\lvert x\rvert=1}\frac{dx}{x}$
$SU(2)$	singlet	1	$\frac{1}{2\pi i}\oint_{\lvert y\rvert=1}\frac{dy}{y}(1-y^2)$
	fundamental/doublet	$y+\frac{1}{y}$	
	triplet/adjoint	$y^2+1+\frac{1}{y^2}$	
$SU(3)$	singlet	1	$\frac{1}{(2\pi i)^2}\oint_{\lvert z_1\rvert=1}\frac{dz_2}{z_2}\oint_{\lvert z_2\rvert=1}\frac{dz_2}{z_2} \times(1-z_1z_2)(1-\frac{z_1^2}{z_2})(1-\frac{z_2^2}{z_1})$
	fundamental/3	$z_1+\frac{z_2}{z_1}+\frac{1}{z_2}$	
	antifundamental/$\bar{3}$	$z_2+\frac{z_1}{z_2}+\frac{1}{z_1}$	
	adjoint	$z_1z_2+\frac{z_2^2}{z_1}+\frac{z_1^2}{z_2}+2+\frac{z_2}{z_1^2}+\frac{z_1}{z_2^2}+\frac{1}{z_1z_2}$	

Table 18: Explicit formulas for some of the characters of the Lorentz group and the SM gauge groups $SU(3) \times SU(2) \times U(1)$ representations and the Haar measures (see also Eq. (B.2)).

B.1 Plethystic Exponential

Consider a d-dimensional representation R of some group G. We assume that R can be diagonalized, i.e. we can find a set of eigenvectors $\{e_i\}_{i=1,\ldots,d}$ with eigenvalues $R(g)e_i = \lambda_i e_i$, which is also a basis for the representation. If we know the eigenvalues λ_i, the character of R is just the sum over these eigenvalues $\chi_R(g) = \sum_{i=1}^{n} \lambda_i$. Using this basis for R we know that a basis of the n-th symmetric tensor product $\mathrm{sym}^n(R)$ is given by

$$\left\{\frac{1}{n!}\sum_{\sigma\in S_n} e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(n)}}\middle| 1\leq i_1\leq\cdots\leq i_n\leq n\right\}, \tag{B.3}$$

where S_n is the symmetric group. For the n-th anti-symmetric tensor product $\wedge^n(R)$ we get the basis

$$\left\{\frac{1}{n!}\sum_{\sigma\in S_n} \epsilon(\sigma) e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(n)}}\middle| 1\leq i_1\leq\cdots\leq i_n\leq n\right\}, \tag{B.4}$$

with $\epsilon(\sigma)$ the sign of the permutation. As an explicit example of this, lets take $d = 3$ and $n = 2$. Then the basis for the symmetric representation is

$$\left\{e_1\otimes e_2, e_2\otimes e_2, e_3\otimes e_3, \frac{1}{2}(e_1\otimes e_2+e_2\otimes e_1), \frac{1}{2}(e_1\otimes e_3+e_3\otimes e_1), \frac{1}{2}(e_2\otimes e_3+e_3\otimes e_2)\right\}, \tag{B.5}$$

and for the anti-symmetric representation

$$\left\{\frac{1}{2}(e_1\otimes e_2-e_2\otimes e_1), \frac{1}{2}(e_1\otimes e_3-e_3\otimes e_1), \frac{1}{2}(e_2\otimes e_3-e_3\otimes e_2)\right\}. \tag{B.6}$$

B.1.1 Bosonic Plethystic Exponential

We want to compute the sum over all the characters of all symmetric tensor products of R

$$\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(R)}(g), \tag{B.7}$$

where the character is weighted by q. Using the eigenbasis of R, we easily compute for the example of Eq. (B.5) that the character $\chi_{\mathrm{sym}^2(R)}(g)$ is given by

$$\chi_{\mathrm{sym}^2(R)}(g) = \mathrm{tr}[\mathrm{sym}^2(R)] = \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3, \tag{B.8}$$

which are all symmetric polynomials in λ_i of order 2. In general we get for arbitrary d and n that the character is given by symmetric polynomials of order n in λ_i:

$$\chi_{\mathrm{sym}^n(R)}(g) = \mathrm{tr}[\mathrm{sym}^n(R)] = \sum_{i_1+\cdots+i_d=n} \lambda_1^{i_1}\ldots\lambda_d^{i_d}. \tag{B.9}$$

Plugging this into Eq. (B.7), we can express this as

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(R)}(g) &= \sum_{n=0}^{\infty} q^n \sum_{i_1+\cdots+i_d=n} \lambda_d^{i_1}\ldots\lambda_d^{i_d} = \sum_{i_1=0}^{\infty}(q\lambda_1)^{i_1}\cdots\sum_{i_d=0}^{\infty}(q\lambda_d)^{i_d}\\ &= \prod_{i=1}^{d}\frac{1}{1-q\lambda_i} = \frac{1}{\det\left(\mathbb{1}-qR(g)\right)}.\end{aligned} \tag{B.10}$$

Using $\log(\det(A)) = \mathrm{tr}(\log(A))$ and $\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$ we get

$$\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(R)}(g) = \exp\left[\mathrm{tr}\sum_{k=1}^{\infty}\frac{q^k}{k}R(g)^k\right] = \exp\left[\sum_{k=1}^{\infty}\frac{q^k}{k}\mathrm{tr}[R(g^k)]\right]. \tag{B.11}$$

Note that when we go to the eigenbasis of $R(g)$ we can write this equation in terms of the characters

$$\sum_{n=0}^{\infty} q^n \chi_{\mathrm{sym}^n(R)}(g) = \exp\left[\sum_{k=1}^{\infty}\frac{q^k}{k}\sum_{i=1}^{d}\lambda_i^k\right] = \exp\left[\sum_{k=1}^{\infty}\frac{q^k}{k}\chi_R(\lambda_i^k)\right] \equiv \mathrm{PE}[q\chi_R(\lambda_i)] \tag{B.12}$$

where in the last step we defined the Pletystic exponential

$$\mathrm{PE}[f(x)] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)\right], \tag{B.13}$$

for some arbitrary function $f(x)$. So, we see that we can express the sum over the symmetric tensor products of R in terms of only the character of R.

B.1.2 Fermionic Plethystic Exponential

In analogy with above, we derive the sum over the characters of all anti-symmetric tensor products of a representation R

$$\sum_{n=0}^{\infty} q^n \chi_{\wedge^n(R)}(g), \tag{B.14}$$

For $d = 3$ and $n = 2$, we can use Eq. (B.6) to compute the character

$$\chi_{\wedge^2(R)}(g) = \mathrm{tr}[\wedge^2(R)] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3. \tag{B.15}$$

where we used the eigenbasis of R. For arbitrary d and n this becomes

$$\chi_{\wedge^n(R)}(g) = \mathrm{tr}[\wedge^n(R)] = \sum_{1\leq i_1\leq\cdots\leq i_n\leq n} \lambda_{i_1}\ldots\lambda_{i_n}. \tag{B.16}$$

Plugging this back into Eq. (B.14) we can do a similar computation as for the symmetric case:

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi_{\wedge^n(R)}(g) &= \sum_{n=0}^{\infty} q^n \sum_{1\leq i_1\leq\cdots\leq i_d\leq n} \lambda_{i_1}\ldots\lambda_{i_d} = \prod_{i=1}^{d}(1+q\lambda_i) = \det(\mathbb{1}+qR(g)) \\ &= \exp\left[-\sum_{k=1}^{\infty}\frac{(-q)^k}{k}\mathrm{tr}[R(g)^k]\right] = \exp\left[-\sum_{k=1}^{\infty}\frac{(-q)^k}{k}\chi_R(\lambda_i^k)\right]\end{aligned} \tag{B.17}$$

where we went to the eigenbasis of R again. Defining the anti-symmetric Pletystic exponential as

$$\mathrm{PEF}[f(x)] \equiv \exp\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}f(x^k)\right], \tag{B.18}$$

for some function $f(x)$, we finally obtain

$$\sum_{n=0}^{\infty} q^n \chi_{\wedge^n(R)}(g) = \mathrm{PEF}[q\chi_R(\lambda_i)]. \tag{B.19}$$

C Lorentz Group

In this appendix we present some background information on the Lorentz group $O(3,1)$ and it representations. We denote by $\Lambda^\mu_{\ \nu}$ an element of $O(3,1)$. This matrix obeys the following property:

$$\eta_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma} = \eta_{\rho\sigma}, \tag{C.1}$$

with $\eta_{\mu\nu}$ the Minkowski metric. One can show that either $\det\Lambda = +1$ or $\det\Lambda = -1$, and we call such transformations proper and improper respectively. Notice that the product of two proper Lorentz transformations is proper, meaning that they form a subgroup of $O(3,1)$. From Eq. (C.1), we note that $(\Lambda^0_{\ 0})^2 - \Lambda^i_{\ 0}\Lambda^i_{\ 0} = 1$. Therefore either $\Lambda^0_{\ 0} \geq +1$, which are called orthochronous Lorentz transformations, or $\Lambda^0_{\ 0} \leq -1$. It is straightforward to show that the product of two orthochronous transformations is also orthochronous, so this is also a subgroup of $O(3,1)$. We denote by $SO(3,1)_o$ the proper orthochronous subgroup and when it is clear from the context that we mean the orthochronous transformations, we drop the subscript.

C.1 Lorentz Algebra

For an infinitesimal Lorentz transformation, we can write

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + i\delta\omega^\mu_{\ \nu}. \tag{C.2}$$

Eq. (C.1) can be used to show that

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}. \tag{C.3}$$

Thus, there are six independent infinitesimal Lorentz transformations and we can divide them into three rotations L_i and three boosts K_j. Such an infinitesimal transformation is both proper and orthochronous and they form the Lie algebra $\mathfrak{so}(3,1)$ of the Lorentz group. A common choice of basis for the generators of the algebra is

$$L_1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad L_2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad L_3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$K_1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{C.4}$$

and their algebra[24] is given by

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [L_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_k] = -i\epsilon_{ijk}J_k. \tag{C.5}$$

In order to classify the algebra, we make the following change of basis

$$J^i_L = \frac{1}{2}(L_i - iK_i), \qquad J^i_R = \frac{1}{2}(L_i + iK_i). \tag{C.6}$$

Notice that (J^i_L) and J^i_R are conjugate to each other. These generators obey the Lie algebra

$$[J^i_L, J^j_R] = 0, \qquad [J^i_L, J^j_L] = i\epsilon_{i,j,k}J^k_L, \qquad [J^i_R, J^k_R] = i\epsilon_{i,j,k}J^k_R, \tag{C.7}$$

thus the splitting of the Lie algebra in terms of two $\mathfrak{su}(2)$ sub-algebras. Because of the first commutator in Eq. (C.7), a natural choice for the Cartan subalgebra is the algebra spanned by $\{J^L_3, J^R_3\}$, and we can define the raising and lowering operators by

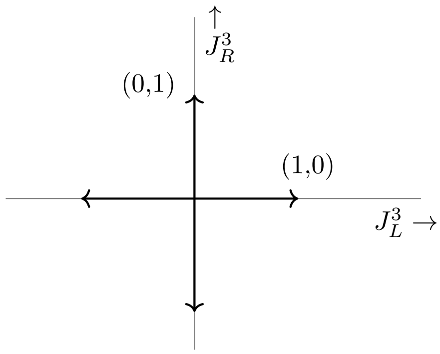
$$J^\pm_L = \frac{1}{\sqrt{2}}(J^1_L \mp iJ^2_L), \qquad J^\pm_R = \frac{1}{\sqrt{2}}(J^1_R \mp iJ^2_R), \tag{C.8}$$

such that

$$[J^3_L, J^\pm_L] = \pm J^\pm_L, \qquad [J^3_R, J^\pm_R] = \pm J^\pm_R, \tag{C.9}$$

[24] This is the same algebra as that of $\mathfrak{sl}(2,\mathbb{C})$ (viewed as a real Lie algebra), the algebra of $SL(2,\mathbb{C})$ which is the double covering group of $SO(3,1)$.

with all other commutators vanishing. We now read off that the roots (weights of the adjoint representation) are given by $\alpha_1 = (1,0)$ and $\alpha_2 = (0,1)$. Therefore, the root diagram, Cartan matrix and Dynkin diagram are given by



$$, \qquad A_{ij} = \frac{2\alpha^i \cdot \alpha^j}{(\alpha^j)^2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A_1 \cup A_1 = \tag{C.10}$$

C.2 Representations

We know that a d-dimensional irreducible representation of $SU(2)$ is specified by an integer of half-integer j, where $2j+1 = d$. Eq. (C.7) shows that the Lie algebra of the Lorentz group is that of two $SU(2)$ algebras. As these two sub-algebras are commuting, we therefore conclude that an irreducible representation of the Lorentz group is specified by two integers or half integers ℓ_1 and ℓ_2, so we label these representations by $\ell = (\ell_1, \ell_2)$[25]. Note that the dimension of such a representation is equal to $(2\ell_1 + 1)(2\ell_2 + 1)$.

The general decomposition of the tensor products of two irreducible representations (j_1, j_2) and (k_1, k_2) is given by [30]:

$$(j_1, j_2) \otimes (k_1, k_2) = \bigoplus_{\ell_1, \ell_2} (\ell_1, \ell_2), \tag{C.11}$$

where $|j_1 - k_1| \leq \ell_1 \leq j_1 + k_1$ and $|j_2 - k_2| \leq \ell_2 \leq j_2 + k_2$.

C.3 Fierz Identities

In EFTs, one often encounters products of four spinors (here spinor refers to Weyl or Dirac spinors) and often we can relate such operators using Fierz identities [31]. Lets denote spinors by ψ_1, ψ_2, etc. First note that we can write $\psi_2 \bar{\psi}_1$ as a 4×4-matrix and it is convenient to express this matrix in a basis constructed from the gamma matrices:

$$\psi_2 \bar{\psi}_1 = a\mathbb{1} + b^\mu \gamma_\mu + c^{\mu\nu} \sigma_{\mu\nu} + d^\mu \gamma_\mu \gamma_5 + e\gamma_5. \tag{C.12}$$

We can determine the coefficients $a, ..., e$ by taking traces. For example, taking the trace of Eq. (C.12) gives

$$\text{tr}[\psi_2 \bar{\psi}_1] = a \text{tr}[\mathbb{1}] = 4a, \tag{C.13}$$

where we used that the trace of any odd number of γ-matrices is zero. Also the trace of $\sigma_{\mu\nu}$ is zero as it is a commutator. Using the cyclic property of the trace, we conclude

$$a = \frac{1}{4} \text{tr}[\bar{\psi}_2 \psi_1] = \frac{1}{4} \bar{\psi}_1 \psi_2 \tag{C.14}$$

Likewise, we can multiply Eq. (C.12) by γ^ρ before taking the trace to get

$$b^\mu = \frac{1}{4} \bar{\psi}_1 \gamma^\mu \psi_2. \tag{C.15}$$

Repeating this process yields

$$\psi_2 \bar{\psi}_1 = \frac{1}{4}(\bar{\psi}_1 \psi_2)\mathbb{1} + \frac{1}{4}(\bar{\psi}_1 \gamma^\mu \psi_2)\gamma_\mu + \frac{1}{8}(\bar{\psi}_1 \sigma^{\mu\nu} \psi_2)\sigma_{\mu\nu} - \frac{1}{4}(\bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2)\gamma_\mu \gamma_5 + \frac{1}{4}(\bar{\psi}_1 \gamma_5 \psi_2)\gamma_5. \tag{C.16}$$

[25] In fact, this is not completely true. We get representations of the double covering group $SL(2, \mathbb{C})$ and they are honest $SO(3,1)_o$ representations if $\ell_1 + \ell_2$ is an integer [30]

We can find all Fierz identies from this equation. For example multiply both sides from the left with $\bar{\psi}_3$ and with ψ_4 from the right. We get

$$(\bar{\psi}_3\psi_2)(\bar{\psi}_1\psi_4) = \frac{1}{4}(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) + \frac{1}{4}(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) + \frac{1}{8}(\bar{\psi}_1\sigma^{\mu\nu}\psi_2)(\bar{\psi}_3\sigma_{\mu\nu}\psi_4) - \frac{1}{4}(\bar{\psi}_1\gamma^\mu\gamma_5\psi_2)(\bar{\psi}_3\gamma_\mu\gamma_5\psi_4) + \frac{1}{4}(\bar{\psi}_1\gamma_5\psi_2)(\bar{\psi}_3\gamma_5\psi_4). \tag{C.17}$$

Similar multiply with $\bar{\psi}_3\gamma^\lambda$ from the left and with $\gamma_\lambda\psi_4$ from the right to get $(\bar{\psi}_3\gamma^\lambda\psi_2)(\bar{\psi}_1\gamma_\lambda\psi_4)$. The RHS can then be simplified using some identities for the gamma matrices. In particular, the RHS simplifies a lot in the case of Weyl spinors. All Fierz identities can be obtained in this way. Some Fierz identities worth remembering are

$$\begin{aligned}(\bar{\psi}_{1L}\gamma^\mu\psi_{2L})(\bar{\psi}_{3R}\gamma_\mu\psi_{4R}) &= -2(\bar{\psi}_{1L}\psi_{4R})(\bar{\psi}_{3R}\psi_{2L}),\\ (\bar{\psi}_{1L}\gamma^\mu\psi_{2L})(\bar{\psi}_{3L}\gamma_\mu\psi_{4L}) &= (\bar{\psi}_{1L}\gamma^\mu\psi_{4L})(\bar{\psi}_{3L}\gamma_\mu\psi_{2L}),\\ (\bar{\psi}_{1R}\gamma^\mu\psi_{2R})(\bar{\psi}_{3R}\gamma_\mu\psi_{4R}) &= (\bar{\psi}_{1R}\gamma^\mu\psi_{4R})(\bar{\psi}_{3R}\gamma_\mu\psi_{2R}).\end{aligned} \tag{C.18}$$

D Conformal Group

In order to deal with IBP and EOM redundancies in the HS, we will need representations (in particular the characters of these representations) of the conformal group. In this appendix, we follow the discussion of [32] to construct the representations in four dimensional space-time. For a review of conformal representations in arbitrary space-time dimension, see [33].

D.1 Conformal Algebra

Let us start by showing that we can extend the Lorentz group $SO(3,1)_o$ to get the conformal group $SO(4,2)$. In general, $O(p,q)$ is the group of real matrices Λ such that

$$\Lambda^T \eta \Lambda = \eta, \tag{D.1}$$

with

$$\eta = \text{diag}(\underbrace{+1,...,+1}_{p}, \underbrace{-1,...,-1}_{q}). \tag{D.2}$$

The generators form the algebra of $\mathfrak{so}(p,q)$, and they are given by a set of Hermitian operators $M_{\mu\nu} = -M_{\nu\mu}$ obeying the following algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho}\big). \tag{D.3}$$

We will show in a moment how this relates to the generators $J^i_{L,R}$ of Eq. (C.7) that we used in App. C.1 in the case of $\mathfrak{so}(3,1)$. The conformal group of space-time is an extension of the Poincaré group, the Lie group of space-time isometries:

$$\mathbb{R}^{p,q} \rtimes O(p,q), \tag{D.4}$$

which includes the orthogonal transformations of Eq. (D.1) and the translations. The algebra of the Poincaré group is therefore an extension of the algebra of Eq. (D.3):

$$\begin{aligned}
[P_\mu, P_\nu] &= 0,\\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu),\\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho}),
\end{aligned} \tag{D.5}$$

with the momentum operator P_μ as the generator of the translations. We can now extend the Poincaré algebra by adding the Hermitian operators D and K_μ with the extra commutators

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu,\\
[D, K_\mu] &= iK_\mu,\\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D + M_{\mu\nu}),\\
[M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu),
\end{aligned} \tag{D.6}$$

with all other commutators involving D and K_μ vanishing. The operators D and K_μ are called the generators of dilatations and special conformal transformations respectively. We can define the generators $J_{mn} = -J_{nm}$, with $m = -1, 0, 1, ..., p$, and $n = -1, 0, 1, ..., q$, by

$$J_{\mu\nu} \equiv M_{\mu\nu}, \quad J_{-1\mu} \equiv \frac{1}{2}(P_\mu - K_\mu), \quad J_{0\mu} \equiv \frac{1}{2}(P_\mu + K_\mu), \quad \text{and} \quad J_{-1,0} \equiv D, \tag{D.7}$$

and it is straightforward to show that these form the algebra $\mathfrak{so}(p+1, q+1)$ as they satisfy the commutation relations of Eq. (D.3) [34]. Going back to four space-time dimensions, we can now conclude that $SO(3,1)_o$ can be extended to the conformal group $SO(4,2)$, showing that $SO(3,1)$ is indeed a subgroup.

Instead of working with Hermitian generators, it will be more convenient to redefine some of the generators as follows: $D' = iD$ such that $D'^\dagger = -D'$, $P'_\mu = iP_\mu$, and K'_μ such that $K'^\dagger_\mu = P'_\mu$. Dropping primes in what follows, we get the following non-zero commutators

$$\begin{aligned}
[D, P_\mu] &= P_\mu, \\
[D, K_\mu] &= -K_\mu, \\
[P_\mu, K_\nu] &= 2(\eta_{\mu\nu} D + iM_{\mu\nu}), \\
[M_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho}).
\end{aligned} \tag{D.8}$$

We can extract the two sets of $SU(2)$ generators for the Lorentz group from $M_{\mu\nu}$ by defining

$$\begin{aligned}
J_L^3 &= \frac{1}{2}(M_{23} + M_{01} + i(M_{02} - M_{13})), & J_L^3 &= \frac{1}{2}(M_{23} - M_{01} - i(M_{02} + M_{13})), \\
J_L^+ &= \frac{1}{2}(M_{12} + M_{03}), & J_R^+ &= \frac{1}{2}(M_{12} - M_{03}), \\
J_R^- &= J_R^{+\dagger}, & J_L^- &= J_L^{+\dagger}.
\end{aligned} \tag{D.9}$$

It is straightforward to show that these satisfy the commutation relations of Eq. (C.8). We denote by $\mathfrak{h}_0$, $\mathfrak{h}_+$, and $\mathfrak{h}_-$ the subspaces generated by $H_0 = \{D, J_L^3, J_R^3\}$, $H_+ = \{P_\mu, J_L^+, J_R^+\}$, and $H_- = \{K_\mu, J_L^-, J_R^-\}$ respectively. With this, it follows from Eq. (D.8) that

$$[\mathfrak{h}_0, \mathfrak{h}_0] \in \mathfrak{h}_0, \qquad [\mathfrak{h}_0, \mathfrak{h}_+] \in \mathfrak{h}_+, \qquad [\mathfrak{h}_0, \mathfrak{h}_-] \in \mathfrak{h}_-, \qquad [\mathfrak{h}_+, \mathfrak{h}_-] \in \mathfrak{h}_0, \tag{D.10}$$

so the elements of $\mathfrak{h}_+$ ($\mathfrak{h}_-$) act as raising (lowering) operators on the commuting Cartan sub-algebra $\mathfrak{h}_0$. As D commutes with $M_{\mu\nu}$ and therefore also with $J_{L,R}^\pm$, we get that the weights of $J_{L,R}^\pm$ under $\mathfrak{h}_0$ are given by

	D	J_L^3	J_R^3
$J_L^\pm$	0	± 1	0
$J_R^\pm$	0	0	± 1

(D.11)

where we used Eq. (C.9). As P_μ has one Lorentz index, we know that we can take a linear combination that has the weights of the fundamental Lorentz representation under $J_{L,R}^3$. In fact, the following combinations achieve this:

$$P_w = P_1 + iP_2, \qquad P_{\bar{w}} = P_1 - iP_2, \qquad P_z = P_3 + iP_4, \qquad P_{\bar{z}} = P_3 - iP_4. \tag{D.12}$$

Defining similar linear combinations of K_μ, and using $[D, P_\mu] = P_\mu$ and $[D, K_\mu] = -K_\mu$, we get the weights under $\mathfrak{h}_0$:

	D	J_L^3	J_R^3
P_w	1	$\frac{1}{2}$	$\frac{1}{2}$
$P_{\bar{w}}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$
P_z	1	$-\frac{1}{2}$	$\frac{1}{2}$
$P_{\bar{z}}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

	D	J_L^3	J_R^3
K_w	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
$K_{\bar{w}}$	-1	$\frac{1}{2}$	$\frac{1}{2}$
K_z	-1	$\frac{1}{2}$	$-\frac{1}{2}$
$K_{\bar{z}}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$

(D.13)

D.2 Representations

We found that the Cartan subalgebra of the Conformal group is spanned by $H_0 = \{D, J_L^3, J_R^3\}$. As J_L^3 and J_R^3 span the algebra of the Lorentz group, we see that we can label weights of a conformal representation by a set of two $SU(2)$ quantum numbers $\ell = (\ell_1, \ell_2)$, and an extra label Δ, which is the eigenvalue under D. As Δ is the eigenvalue under the dilaton D, we can physically associate this with the scaling dimension of a field. From Eq. (D.11) we read of that the Lorentz generators $J_{L,R}^3$ do not change the weight Δ under D. However, we see from Eq. (D.13) that P_μ (K_μ) raise (lower) the scaling dimension by 1. Of course, this is to be expected as the derivative (the generator of the translations) has scaling dimension 1. As the weights of P_μ under J_L^3 and J_R^3 are that of the fundamental Lorentz representation we see that P_μ shifts weights as follows:

$$P_\mu : (\Delta, \ell) \to \left(\Delta + 1, \ell \otimes \left(\frac{1}{2}, \frac{1}{2}\right)\right) \tag{D.14}$$

As K_μ has precisely opposite weights of P_μ, we see that K_μ reverses the action of P_μ on the weights. Therefore, a conformal representations can be written as a direct sum of representations R_{ℓ_i} of the Lorentz group

$$R_{SO(4,2)} = \bigoplus_i R_{\ell_i}, \tag{D.15}$$

where all states inside R_{ℓ_i} have the same scaling dimension Δ_i. The operators P_μ and K_μ move us through all the R_{ℓ_i}, and we can up and down using $J^{\pm}_{L,R}$ inside a single R_{ℓ_i}. We get an infinite sum as the momentum operator P_μ increases the dimension of the R_{ℓ_i}, meaning that we can act with an infinite number of P_μ. However, K_μ decreases the dimension of the R_{ℓ_i} and this implies that there must be some term R_ℓ that has lowest dimension. The states of this term, denoted by $|\Delta, \ell\rangle$, all have the same scaling dimension Δ and are annihilated by the K_μ operators. We can build the whole representation out of the $|\Delta, \ell\rangle$ by acting with the momentum operators P^μ on these states

$$|\Delta, \ell\rangle^* = \Big\{ \sum_{n_w, n_{\bar{w}}, n_z, n_{\bar{z}}} P_w^{n_w} P_{\bar{w}}^{n_{\bar{w}}} P_z^{n_z} P_{\bar{z}}^{n_{\bar{z}}} \times |\Delta, \ell\rangle \Big\}, \tag{D.16}$$

where we have denoted the (infinite) spectrum of states by $|\Delta, \ell\rangle^*$. It is therefore natural to say that the P_μ acts as a lowering operator on conformal representations (although they are elements of $\mathfrak{h}_+$), and we call $|\Delta, \ell\rangle$ the highest weight state.

D.2.1 Unitary representations

It can be shown that the conformal representation is unitary if one of the following conditions holds for the highest weight state $|\Delta, \ell_1, \ell_2\rangle$ [35]:

$$\begin{aligned} &(i) \quad \Delta \geq \ell_1 + \ell_2 + 2 \qquad \ell_1 \neq 0, \ell_1 \neq 0, \\ &(ii) \quad \Delta \geq \ell_1 + \ell_2 + 1 \qquad \ell_1 \ell_2 = 0. \end{aligned} \tag{D.17}$$

In the case where equality does not hold, we call the representation long and all the states of Eq. (D.16) are non-zero. However, if equality holds in one of the conditions, then we speak of a short representation and some of the states listed in Eq. (D.16) vanish. To be more precise, we can find a non-zero state $|\lambda\rangle$ on which we can apply a linear combination of lowering operators $a_\mu P^\mu$ such that $a_\mu P^\mu |\lambda\rangle = 0$. This implies that then also all the descendants of $a_\mu P^\mu |\lambda\rangle$ do not occur in the representation. One can show that in 4 dimensional space-time, a short representation is one of the following cases [32]:

1. In the case $\ell_1 \neq 0, \ell_2 \neq 0, \Delta = \ell_1 + \ell_2 + 2$, the state $|\Delta + 1, \ell_1 - \frac{1}{2}, \ell_2 - \frac{1}{2}\rangle$ and its descendants also vanish in the representation.
2. In the case $\ell_1 = \ell_2 = 0, \Delta = 1$, the state $|3, 0, 0\rangle$ and its descendants are not found in the representation.
3. In the case $\ell_1 > 0, \ell_2 = 0, \Delta = \ell_1 + 1$, the state $|\Delta + 1, \ell_1 - \frac{1}{2}, \frac{1}{2}\rangle$ and its descendants are absent. Note that when we delete the descendant states $|\Delta + 1, \ell_1 - \frac{1}{2}, \frac{1}{2}\rangle^*$ from Eq. (D.16) we need to treat them as a short representation, that is we must only delete the states that do not occur in in the short representation $(\Delta + 1, \ell_1 - \frac{1}{2}, \frac{1}{2})$. The case $\ell_1 = 0, \ell_2 > 0, \Delta = \ell_2 + 1$ can be treated similar.

D.3 EFTs and Conformal Representations

We can now start thinking about building a conformal representation for the fields of the EFTs. Lets focus on a single field ϕ for now, transforming in some representation $\ell = (\ell_1, \ell_2)$ of the Lorentz group[26]. Furthermore, we assign a mass dimension Δ to the field. From the above discussion, it is clear that we can build a conformal representation out of the Lorentz representation by acting with the momentum operator on the states of ℓ. As the derivative is the momentum operator, we get that the conformal representation $R_{\Delta,\ell}$ acts on objects like:

$$R_{\Delta,\ell} \sim \begin{pmatrix} \phi \\ \partial_{\mu_1}\phi \\ \partial_{\mu_1}\partial_{\mu_2}\phi \\ \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi \\ \vdots \end{pmatrix}, \tag{D.18}$$

[26] And some gauge symmetry group G, but we focus on the Lorentz group now.

where the highest weight ϕ is called the primary state. Using that the derivatives commute, we can simply read of the weights under $H_0 = \{D, J_L^3, J_R^3\}$ from

$$\begin{array}{c|c|c} & \text{Scaling dim.} & \text{Spin} \\ \hline \phi & \Delta & \ell \\ \partial_{\mu_1}\phi & \Delta+1 & (\frac{1}{2},\frac{1}{2})\otimes\ell \\ \partial_{\mu_1}\partial_{\mu_2}\phi & \Delta+2 & \text{sym}^2(\frac{1}{2},\frac{1}{2})\otimes\ell \\ \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi & \Delta+2 & \text{sym}^3(\frac{1}{2},\frac{1}{2})\otimes\ell \\ \vdots & \vdots & \vdots \end{array} \tag{D.19}$$

The conformal representations are very useful for the HS technique because the operators related through EOM relations correspond with the states that do not appear in the short representations. Furthermore, we can take care of IBP identities by noting that only the highest weight in Eq. (D.18) is not a total derivative.

D.4 Characters

The characters for a representation $R_{\Delta,\ell}$, with $\ell = (\ell_1, \ell_2)$ are given by:

$$\begin{aligned} \chi_{\Delta,\ell} &= \text{tr}_{|\Delta,\ell\rangle^*}\left[e^{i\theta D + i\theta_L J_L^3 + i\theta_R J_R^3}\right] \\ &= \sum_{n_w,n_{\bar{w}},n_z,n_{\bar{z}}} \sum_{|m_{1,2}|\le l_{1,2}} \langle \Delta, m_1, m_2 | e^{i\theta D + i\theta_L J_L^3 + i\theta_R J_R^3} P_w^{n_w} P_{\bar{w}}^{n_{\bar{w}}} P_z^{n_z} P_{\bar{z}}^{n_{\bar{z}}} |\Delta, m_1, m_2\rangle, \end{aligned} \tag{D.20}$$

which we can easily compute using Eq. (D.19). Defining the $U(1)$-variables $q = e^{i\theta D}$, $x_1 = e^{i\theta_L J_L^3}$ and $x_2 = e^{i\theta_R J_R^3}$, we get

$$\chi_{\Delta,\ell}(q,x) = \sum_{n=0}^{\infty} q^{\Delta+n}\chi_{\text{sym}^n(\frac{1}{2},\frac{1}{2})\otimes\ell}(x) = q^\Delta \chi_\ell(x) \sum_{n=0}^{\infty} q^n \chi_{\text{sym}^n(\frac{1}{2},\frac{1}{2})}(x) \equiv q^\Delta \chi_\ell(x_1,x_2) P(q,x), \tag{D.21}$$

where we notated $x = (x_1, x_2)$. Note that the characters of the short representations are modified, which we discuss in Sec. 4.4.

D.4.1 Character Orthogonality

In order for these characters to be orthogonal, we choose the Haar measure to be

$$\int d\mu_{SO(4,2)} = \int d\mu_L \oint \frac{dq}{2\pi i q} \frac{1}{P^*(q,x)P(q,x)}. \tag{D.22}$$

with $d\mu_L$ the Haar measure of the Lorentz group. With this, we get the correct normalized orthogonality of the characters

$$\begin{aligned} \int d\mu_{SO(4,2)}\, \chi^*_{\Delta,\ell}\chi_{\Delta',\ell'} &= \int d\mu_L \oint \frac{dq}{2\pi i q} \frac{1}{P^*(q,x)P(q,x)} \left(q^\Delta \chi_\ell(x) P(q,x)\right)^* q^{\Delta'} \chi_{\ell'}(x) P(q,x) \\ &= \int d\mu_L\, \chi^*_\ell(x)\chi_{\ell'}(x) \oint \frac{dq}{2\pi i q} \frac{q^\Delta}{q^{\Delta'}} \\ &= \delta_{\ell,\ell'} \oint \frac{dq}{2\pi i} q^{\Delta-\Delta'-1} \\ &= \delta_{\ell,\ell'}\delta_{\Delta,\Delta'}, \end{aligned} \tag{D.23}$$

were we used the orthonormality of the characters of representations of the Lorentz group and that $q \in U(1)$.

E SMEFT Operators

In all tables in this appendix, the first two columns show the explicit operator contraction. The results of the Hilbert series for H^+ and H^- are shown in column 3 and 4. The enumeration of $\mathcal{CP}$-invariant operators is given in column 5, and in column 6 we show a checkmark for every $\mathcal{CP}$-invariant operator.

$\psi^2 X \varphi^2$		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
$Q_{l^2 W \varphi^2}$	$\epsilon_{mn}(\tau^I \epsilon)_{jk}(l_p^m C i \sigma^{\mu\nu} l_r^j)\varphi^n \varphi^k W^I_{\mu\nu}$+ h.c.	$l^2\varphi^2 W + (l^\dagger)^2(\varphi^\dagger)^2 W$	0	$l^2\varphi^2 W$	✓
$\psi^2\varphi^4$					
$Q_{l^2\varphi^4}$	$\epsilon_{mn}\epsilon_{jk}(l_p^m C l_r^j)\varphi^n\varphi^k(\varphi^\dagger\varphi)$+ h.c.	$l^2\varphi^3\varphi^\dagger + (l^\dagger)^2(\varphi^\dagger)^3\varphi$	0	$l^2\varphi^4$	✓
$\psi^4\varphi$					
$Q_{l^3 e\varphi}$	$\epsilon_{jk}\epsilon_{mn}(\bar{e}_p l_r^j)(l_s^k C l_t^m)\varphi^n$+ h.c.	$e^\dagger l^3\varphi + e(l^\dagger)^3\varphi^\dagger$	0	$el^3\varphi$	✓
$Q_{leud\varphi}$	$\epsilon_{jk}(\bar{d}_p l_r^j)(u_s C e_t)\varphi^k$+ h.c.	$d^\dagger lue\varphi + dl^\dagger u^\dagger e^\dagger\varphi^\dagger$	0	$dlue\varphi$	✓
$Q^{(1)}_{l^2 qd\varphi}$	$\epsilon_{jk}\epsilon_{mn}(\bar{d}_p l_r^j)(q_s^k C l_t^m)\varphi^n$+ h.c.	$2d^\dagger l^2 q\varphi + 2d(l^\dagger)^2 q^\dagger\varphi^\dagger$	0	$2dl^2q\varphi$	✓
$Q^{(2)}_{l^2 qd\varphi}$	$\epsilon_{jm}\epsilon_{kn}(\bar{d}_p l_r^j)(q_s^k C l_t^m)\varphi^n$+ h.c.				✓
$Q_{l^2 qu\varphi}$	$\epsilon_{jk}(\bar{q}_p^m u_r)(l_{sm} C l_t^j)\varphi^k$+ h.c.	$q^\dagger ul^2\varphi + qu^\dagger(l^\dagger)^2\varphi^\dagger$	0	$qul^2\varphi$	✓
$Q_{lud^2\varphi}$	$\epsilon_{\alpha\beta\gamma}(\bar{l}_p d_r^\alpha)(u_s^\beta C d_t^\gamma)\tilde{\varphi}$+ h.c.	$l^\dagger d^2 u\varphi^\dagger + l(d^\dagger)^2 u^\dagger\varphi$	0	$ld^2u\varphi$	✓
$Q_{lq^2 d\varphi}$	$\epsilon_{\alpha\beta\gamma}\epsilon_{jk}(\bar{l}_p^m d_r^\alpha)(q_{sm}^\beta C q_t^{j\gamma})\tilde{\varphi}^k$+ h.c.	$l^\dagger dq^2\varphi^\dagger + ld^\dagger(q^\dagger)^2\varphi$	0	$ldq^2\varphi$	✓
$\psi^2\varphi^3 D$					
$Q_{le\varphi^3 D}$	$\epsilon_{mn}\epsilon_{jk}(l_p^m C\gamma^\mu e_r)\varphi^n\varphi^j iD_\mu\varphi^k$+ h.c.	$le\varphi^3\mathcal{D} + l^\dagger e^\dagger(\varphi^\dagger)^3\mathcal{D}$	0	$le\varphi^3\mathcal{D}$	✓
$\psi^4 D$					
$Q_{l^2 udD}$	$\epsilon_{jk}(\bar{d}_p\gamma^\mu u_r)(l_s^j CiD_\mu l_t^k)$+ h.c.	$d^\dagger ul^2\mathcal{D} + du^\dagger(l^\dagger)^2\mathcal{D}$	0	$dul^2\mathcal{D}$	✓
$\psi^2\varphi^2 D^2$					
$Q^{(1)}_{l^2\varphi^2 D^2}$	$\epsilon_{jk}\epsilon_{mn}(l_p^j CD^\mu l_r^k)\varphi^m(D_\mu\varphi^n)$+ h.c.	$2l^2\varphi^2\mathcal{D}^2 + 2(l^\dagger)^2(\varphi^\dagger)^2\mathcal{D}^2$	0	$2l^2\varphi^2\mathcal{D}^2$	✓
$Q^{(2)}_{l^2\varphi^2 D^2}$	$\epsilon_{jm}\epsilon_{kn}(l_p^j CD^\mu l_r^k)\varphi^m(D_\mu\varphi^n)$+ h.c.				✓
$\psi^4\mathcal{D}$					
$Q_{lqd^2 D}$	$\epsilon_{\alpha\beta\gamma}(\bar{l}_p\gamma^\mu q_r^\alpha)(d_s^\beta CiD_\mu d_t^\gamma)$+ h.c.	$l^\dagger qd^2\mathcal{D} + lq^\dagger(d^\dagger)^2\mathcal{D}$	0	$lqd^2\mathcal{D}$	✓
$Q_{ed^3 D}$	$\epsilon_{\alpha\beta\gamma}(\bar{e}_p\gamma^\mu d_r^\alpha)(d_s^\beta CiD_\mu d_t^\gamma)$+ h.c.	$e^\dagger d^3\mathcal{D} + e(d^\dagger)^3\mathcal{D}$	0	$ed^3\mathcal{D}$	✓

Total:	15	30	0	15	15

Table 19: SMEFT operators at mass dimension 7. The first two columns are taken from [28].

X^4, X^3X' and $X^2X'^2$		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
$Q_{G^4}^{(1)}$	$(G_{\mu\nu}^A G^{A\mu\nu})(G_{\rho\sigma}^B G^{B\rho\sigma})$	$9G^4$	$3G^4$	$6G^4$	✓
$Q_{G^4}^{(2)}$	$(G_{\mu\nu}^A \widetilde{G}^{A\mu\nu})(G_{\rho\sigma}^B \widetilde{G}^{B\rho\sigma})$				✓
$Q_{G^4}^{(3)}$	$(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^A G^{B\rho\sigma})$				✓
$Q_{G^4}^{(4)}$	$(G_{\mu\nu}^A \widetilde{G}^{B\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{B\rho\sigma})$				✓
$Q_{G^4}^{(5)}$	$(G_{\mu\nu}^A G^{A\mu\nu})(G_{\rho\sigma}^B \widetilde{G}^{B\rho\sigma})$				-
$Q_{G^4}^{(6)}$	$(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{B\rho\sigma})$				-
$Q_{G^4}^{(7)}$	$d^{ABE}d^{CDE}(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^C G^{D\rho\sigma})$				✓
$Q_{G^4}^{(8)}$	$d^{ABE}d^{CDE}(G_{\mu\nu}^A \widetilde{G}^{B\mu\nu})(G_{\rho\sigma}^C \widetilde{G}^{D\rho\sigma})$				✓
$Q_{G^4}^{(9)}$	$d^{ABE}d^{CDE}(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^C \widetilde{G}^{D\rho\sigma})$				-
$Q_{W^4}^{(1)}$	$(W_{\mu\nu}^I W^{I\mu\nu})(W_{\rho\sigma}^J W^{J\rho\sigma})$	$6W^4$	$2W^4$	$4W^4$	✓
$Q_{W^4}^{(2)}$	$(W_{\mu\nu}^I \widetilde{W}^{I\mu\nu})(W_{\rho\sigma}^J \widetilde{W}^{J\rho\sigma})$				✓
$Q_{W^4}^{(3)}$	$(W_{\mu\nu}^I W^{J\mu\nu})(W_{\rho\sigma}^I W^{J\rho\sigma})$				✓
$Q_{W^4}^{(4)}$	$(W_{\mu\nu}^I \widetilde{W}^{J\mu\nu})(W_{\rho\sigma}^I \widetilde{W}^{J\rho\sigma})$				✓
$Q_{W^4}^{(5)}$	$(W_{\mu\nu}^I W^{I\mu\nu})(W_{\rho\sigma}^J \widetilde{W}^{J\rho\sigma})$				-
$Q_{W^4}^{(6)}$	$(W_{\mu\nu}^I W^{J\mu\nu})(W_{\rho\sigma}^I \widetilde{W}^{J\rho\sigma})$				-
$Q_{B^4}^{(1)}$	$(B_{\mu\nu} B^{\mu\nu})(B_{\rho\sigma} B^{\rho\sigma})$	$3B^4$	B^4	$2B^4$	✓
$Q_{B^4}^{(2)}$	$(B_{\mu\nu} \widetilde{B}^{\mu\nu})(B_{\rho\sigma} \widetilde{B}^{\rho\sigma})$				✓
$Q_{B^4}^{(3)}$	$(B_{\mu\nu} B^{\mu\nu})(B_{\rho\sigma} \widetilde{B}^{\rho\sigma})$				-
$Q_{G^3B}^{(1)}$	$d^{ABC}(B_{\mu\nu} G^{A\mu\nu})(G_{\rho\sigma}^B G^{C\rho\sigma})$	$4G^3B$	0	$2G^3B$	✓
$Q_{G^3B}^{(2)}$	$d^{ABC}(B_{\mu\nu} \widetilde{G}^{A\mu\nu})(G_{\rho\sigma}^B \widetilde{G}^{C\rho\sigma})$				✓
$Q_{G^3B}^{(3)}$	$d^{ABC}(B_{\mu\nu} \widetilde{G}^{A\mu\nu})(G_{\rho\sigma}^B G^{C\rho\sigma})$				-
$Q_{G^3B}^{(4)}$	$d^{ABC}(B_{\mu\nu} G^{A\mu\nu})(G_{\rho\sigma}^B \widetilde{G}^{C\rho\sigma})$				-
$Q_{G^2W^2}^{(1)}$	$(W_{\mu\nu}^I W^{I\mu\nu})(G_{\rho\sigma}^A G^{A\rho\sigma})$	$7G^2W^2$	G^2W^2	$4G^2W^2$	✓
$Q_{G^2W^2}^{(2)}$	$(W_{\mu\nu}^I \widetilde{W}^{I\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2W^2}^{(3)}$	$(W_{\mu\nu}^I G^{A\mu\nu})(W_{\rho\sigma}^I G^{A\rho\sigma})$				✓
$Q_{G^2W^2}^{(4)}$	$(W_{\mu\nu}^I \widetilde{G}^{A\mu\nu})(W_{\rho\sigma}^I \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2W^2}^{(5)}$	$(W_{\mu\nu}^I \widetilde{W}^{I\mu\nu})(G_{\rho\sigma}^A G^{A\rho\sigma})$				✓
$Q_{G^2W^2}^{(6)}$	$(W_{\mu\nu}^I W^{I\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2W^2}^{(7)}$	$(W_{\mu\nu}^I G^{A\mu\nu})(W_{\rho\sigma}^I \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2B^2}^{(1)}$	$(B_{\mu\nu} B^{\mu\nu})(G_{\rho\sigma}^A G^{A\rho\sigma})$	$7G^2B^2$	G^2B^2	$4G^2B^2$	✓
$Q_{G^2B^2}^{(2)}$	$(B_{\mu\nu} \widetilde{B}^{\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2B^2}^{(3)}$	$(B_{\mu\nu} G^{A\mu\nu})(B_{\rho\sigma} G^{A\rho\sigma})$				✓
$Q_{G^2B^2}^{(4)}$	$(B_{\mu\nu} \widetilde{G}^{A\mu\nu})(B_{\rho\sigma} \widetilde{G}^{A\rho\sigma})$				✓
$Q_{G^2B^2}^{(5)}$	$(B_{\mu\nu} \widetilde{B}^{\mu\nu})(G_{\rho\sigma}^A G^{A\rho\sigma})$				-
$Q_{G^2B^2}^{(6)}$	$(B_{\mu\nu} B^{\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{A\rho\sigma})$				-
$Q_{G^2B^2}^{(7)}$	$(B_{\mu\nu} G^{A\mu\nu})(B_{\rho\sigma} \widetilde{G}^{A\rho\sigma})$				-
$Q_{W^2B^2}^{(1)}$	$(B_{\mu\nu} B^{\mu\nu})(W_{\rho\sigma}^I W^{I\rho\sigma})$	$7W^2B^2$	W^2B^2	$4W^2B^2$	✓
$Q_{W^2B^2}^{(2)}$	$(B_{\mu\nu} \widetilde{B}^{\mu\nu})(W_{\rho\sigma}^I \widetilde{W}^{I\rho\sigma})$				✓
$Q_{W^2B^2}^{(3)}$	$(B_{\mu\nu} W^{I\mu\nu})(B_{\rho\sigma} W^{I\rho\sigma})$				✓
$Q_{W^2B^2}^{(4)}$	$(B_{\mu\nu} \widetilde{W}^{I\mu\nu})(B_{\rho\sigma} \widetilde{W}^{I\rho\sigma})$				✓
$Q_{W^2B^2}^{(5)}$	$(B_{\mu\nu} \widetilde{B}^{\mu\nu})(W_{\rho\sigma}^I W^{I\rho\sigma})$				-
$Q_{W^2B^2}^{(6)}$	$(B_{\mu\nu} B^{\mu\nu})(W_{\rho\sigma}^I \widetilde{W}^{I\rho\sigma})$				-
$Q_{W^2B^2}^{(7)}$	$(B_{\mu\nu} W^{I\mu\nu})(B_{\rho\sigma} \widetilde{W}^{I\rho\sigma})$				-

Table 20: SMEFT operators at mass dimension 8. The first two columns are taken from [28].

φ^8, $\varphi^6\mathcal{D}^2$ and $\varphi^4\mathcal{D}^4$		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
Q_{φ^8}	$(\varphi^\dagger\varphi)^4$	$(\varphi^\dagger\varphi)^4$	φ^8	φ^8	✓
$Q_{\varphi^6}^{(1)}$	$(\varphi^\dagger\varphi)^2(D_\mu\varphi^\dagger D^\mu\varphi)$	$2(\varphi^\dagger\varphi)^3\mathcal{D}^2$	$2\varphi^6\mathcal{D}^2$	$2\varphi^6\mathcal{D}^2$	✓
$Q_{\varphi^6}^{(2)}$	$(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi)(D_\mu\varphi^\dagger\tau^I D^\mu\varphi)$				✓
$Q_{\varphi^4}^{(1)}$	$(D_\mu\varphi^\dagger D_\nu\varphi)(D^\nu\varphi^\dagger D^\mu\varphi)$	$3(\varphi^\dagger\varphi)^2\mathcal{D}^4$	$3\varphi^4\mathcal{D}^4$	$3\varphi^4\mathcal{D}^4$	✓
$Q_{\varphi^4}^{(2)}$	$(D_\mu\varphi^\dagger D_\nu\varphi)(D^\mu\varphi^\dagger D^\nu\varphi)$				✓
$Q_{\varphi^4}^{(3)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)(D^\nu\varphi^\dagger D_\nu\varphi)$				✓
$X^3\varphi^2$					
$Q_{G^3\varphi^2}^{(1)}$	$f^{ABC}(\varphi^\dagger\varphi)G_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$	$2\varphi^\dagger\varphi G^3$	0	$\varphi^\dagger\varphi G^3$	✓
$Q_{G^3\varphi^2}^{(2)}$	$f^{ABC}(\varphi^\dagger\varphi)G_\mu^{A\nu}G_\nu^{B\rho}\widetilde{G}_\rho^{C\mu}$				-
$Q_{W^3\varphi^2}^{(1)}$	$\epsilon^{IJK}(\varphi^\dagger\varphi)W_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}$	$2\varphi^\dagger\varphi W^3$	0	$\varphi^\dagger\varphi W^3$	✓
$Q_{W^3\varphi^2}^{(2)}$	$\epsilon^{IJK}(\varphi^\dagger\varphi)W_\mu^{I\nu}W_\nu^{J\rho}\widetilde{W}_\rho^{K\mu}$				-
$Q_{W^2B\varphi^2}^{(1)}$	$\epsilon^{IJK}(\varphi^\dagger\tau^I\varphi)B_\mu^{\nu}W_\nu^{J\rho}W_\rho^{K\mu}$	$2\varphi^\dagger\varphi W^2B$	0	$\varphi^\dagger\varphi W^2B$	✓
$Q_{W^2B\varphi^2}^{(2)}$	$\epsilon^{IJK}(\varphi^\dagger\tau^I\varphi)(\widetilde{B}^{\mu\nu}W_{\nu\rho}^J W_\mu^{K\rho} + B^{\mu\nu}W_{\nu\rho}^J\widetilde{W}_\mu^{K\rho})$				-
$X^2\varphi^4$					
$Q_{G^2\varphi^4}^{(1)}$	$(\varphi^\dagger\varphi)^2 G_{\mu\nu}^A G^{A\mu\nu}$	$2(\varphi^\dagger\varphi)^2G^2$	0	$(\varphi^\dagger\varphi)^2G^2$	✓
$Q_{G^2\varphi^4}^{(2)}$	$(\varphi^\dagger\varphi)^2 \widetilde{G}_{\mu\nu}^A G^{A\mu\nu}$				✓
$Q_{W^2\varphi^4}^{(1)}$	$(\varphi^\dagger\varphi)^2 W_{\mu\nu}^I W^{I\mu\nu}$	$4(\varphi^\dagger\varphi)^2W^2$	0	$2(\varphi^\dagger\varphi)^2W^2$	✓
$Q_{W^2\varphi^4}^{(2)}$	$(\varphi^\dagger\varphi)^2 \widetilde{W}_{\mu\nu}^I W^{I\mu\nu}$				-
$Q_{W^2\varphi^4}^{(3)}$	$(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^J\varphi)W_{\mu\nu}^I W^{J\mu\nu}$				✓
$Q_{W^2\varphi^4}^{(4)}$	$(\varphi^\dagger\tau^I\varphi)(\varphi^\dagger\tau^J\varphi)\widetilde{W}_{\mu\nu}^I W^{J\mu\nu}$				-
$Q_{WB\varphi^4}^{(1)}$	$(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi)W_{\mu\nu}^I B^{\mu\nu}$	$2(\varphi^\dagger\varphi)^2WB$	0	$(\varphi^\dagger\varphi)^2WB$	✓
$Q_{WB\varphi^4}^{(2)}$	$(\varphi^\dagger\varphi)(\varphi^\dagger\tau^I\varphi)\widetilde{W}_{\mu\nu}^I B^{\mu\nu}$				✓
$Q_{B^2\varphi^4}^{(1)}$	$(\varphi^\dagger\varphi)^2 B_{\mu\nu}B^{\mu\nu}$	$2(\varphi^\dagger\varphi)^2B^2$	0	$(\varphi^\dagger\varphi)^2B^2$	✓
$Q_{B^2\varphi^4}^{(2)}$	$(\varphi^\dagger\varphi)^2 \widetilde{B}_{\mu\nu}B^{\mu\nu}$				-
$X^2\varphi^2\mathcal{D}^2$					
$Q_{G^2\varphi^2D^2}^{(1)}$	$(D^\mu\varphi^\dagger D^\nu\varphi)G_{\mu\rho}^A G_\nu^{A\rho}$	$3\varphi^\dagger\varphi G^2\mathcal{D}^2$	$\varphi^2G^2\mathcal{D}^2$	$2\varphi^2G^2\mathcal{D}^2$	✓
$Q_{G^2\varphi^2D^2}^{(2)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)G_{\nu\rho}^A G^{A\nu\rho}$				✓
$Q_{G^2\varphi^2D^2}^{(3)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)G_{\nu\rho}^A \widetilde{G}^{A\nu\rho}$				-
$Q_{W^2\varphi^2D^2}^{(1)}$	$(D^\mu\varphi^\dagger D^\nu\varphi)W_{\mu\rho}^I W_\nu^{I\rho}$	$6\varphi^\dagger\varphi W^2\mathcal{D}^2$	$2\varphi^2W^2\mathcal{D}^2$	$4\varphi^2W^2\mathcal{D}^2$	✓
$Q_{W^2\varphi^2D^2}^{(2)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)W_{\nu\rho}^I W^{I\nu\rho}$				✓
$Q_{W^2\varphi^2D^2}^{(3)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)W_{\nu\rho}^I \widetilde{W}^{I\nu\rho}$				-
$Q_{W^2\varphi^2D^2}^{(4)}$	$\epsilon^{IJK}(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)W_{\mu\rho}^J W_\nu^{K\rho}$				✓
$Q_{W^2\varphi^2D^2}^{(5)}$	$\epsilon^{IJK}(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(W_{\mu\rho}^J\widetilde{W}_\nu^{K\rho} - \widetilde{W}_{\mu\rho}^J W_\nu^{K\rho})$				✓
$Q_{W^2\varphi^2D^2}^{(6)}$	$\epsilon^{IJK}(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(W_{\mu\rho}^J\widetilde{W}_\nu^{K\rho} + \widetilde{W}_{\mu\rho}^J W_\nu^{K\rho})$				-
$Q_{WB\varphi^2D^2}^{(1)}$	$(D^\mu\varphi^\dagger\tau^I D_\mu\varphi)B_{\nu\rho}W^{I\nu\rho}$	$6\varphi^\dagger\varphi WB\mathcal{D}^2$	0	$3\varphi^2WB\mathcal{D}^2$	✓
$Q_{WB\varphi^2D^2}^{(2)}$	$(D^\mu\varphi^\dagger\tau^I D_\mu\varphi)B_{\nu\rho}\widetilde{W}^{I\nu\rho}$				-
$Q_{WB\varphi^2D^2}^{(3)}$	$(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(B_{\mu\rho}W_\nu^{I\rho} - B_{\mu\rho}W_\nu^{I\rho})$				-
$Q_{WB\varphi^2D^2}^{(4)}$	$(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(B_{\mu\rho}W_\nu^{I\rho} + B_{\mu\rho}W_\nu^{I\rho})$				✓
$Q_{WB\varphi^2D^2}^{(5)}$	$(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(B_{[\mu}^\rho\widetilde{W}_{\nu]\rho}^I - \widetilde{B}_{[\mu}^\rho W_{\nu]\rho}^I)$				✓
$Q_{WB\varphi^2D^2}^{(6)}$	$(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)(B_{(\mu}^\rho\widetilde{W}_{\nu)\rho}^I + \widetilde{B}_{(\mu}^\rho W_{\nu)\rho}^I)$				-
$Q_{B^2\varphi^2D^2}^{(1)}$	$(D^\mu\varphi^\dagger D^\nu\varphi)B_{\mu\rho}B_\nu^\rho$	$3\varphi^\dagger\varphi B^2\mathcal{D}^2$	$\varphi^2B^2\mathcal{D}^2$	$2\varphi^2B^2\mathcal{D}^2$	✓
$Q_{B^2\varphi^2D^2}^{(2)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)B_{\nu\rho}B^{\nu\rho}$				✓
$Q_{B^2\varphi^2D^2}^{(3)}$	$(D^\mu\varphi^\dagger D_\mu\varphi)B_{\nu\rho}\widetilde{B}^{\nu\rho}$				-

Table 21: SMEFT operators at mass dimension 8. The first two columns are taken from [28].

$X\varphi^4\mathcal{D}^2$		H^+	H^-	$\frac{1}{2}(H^+ + H^-)$	$\mathcal{CP}$-inv.
$Q^{(1)}_{W\varphi^4D^2}$	$(\varphi^\dagger\varphi)(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)W^I_{\mu\nu}$	$4(\varphi^\dagger\varphi)^2 W\mathcal{D}^2$	0	$2\varphi^4 W\mathcal{D}^2$	✓
$Q^{(2)}_{W\varphi^4D^2}$	$(\varphi^\dagger\varphi)(D^\mu\varphi^\dagger\tau^I D^\nu\varphi)\widetilde{W}^I_{\mu\nu}$				-
$Q^{(3)}_{W\varphi^4D^2}$	$\epsilon^{IJK}(\varphi^\dagger\tau^I\varphi)(D^\mu\varphi^\dagger\tau^J D^\nu\varphi)W^K_{\mu\nu}$				✓
$Q^{(4)}_{W\varphi^4D^2}$	$\epsilon^{IJK}(\varphi^\dagger\tau^I\varphi)(D^\mu\varphi^\dagger\tau^J D^\nu\varphi)\widetilde{W}^K_{\mu\nu}$				-
$Q^{(1)}_{B\varphi^4D^2}$	$(\varphi^\dagger\varphi)(D^\mu\varphi^\dagger D^\nu\varphi)B_{\mu\nu}$	$2(\varphi^\dagger\varphi)^2 B\mathcal{D}^2$	0	$\varphi^4 B\mathcal{D}^2$	✓
$Q^{(2)}_{B\varphi^4D^2}$	$(\varphi^\dagger\varphi)(D^\mu\varphi^\dagger D^\nu\varphi)\widetilde{B}_{\mu\nu}$				-

Table 22: SMEFT operators at mass dimension 8. The first two columns are taken from [28].