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Master's Thesis

Ward-Takahashi identities for a scalar field on de Sitter space with gauged Weyl symmetry

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Abstract

This thesis explores a recently developed approach to rescaling symmetry of quantum field theories in an extension of general relativity. Unfortunately it has not been named yet, but the model developed by Lucat and Prokopec is best described as gauging Weyl transformations by a vector field interpreted as space-time torsion. After introducing Riemann-Cartan geometry as extension of general gelativity with torsion, the appropriate gauge connection is constructed. A massless, non-minimally coupled scalar field on de Sitter space is selected to test for the conformal anomaly in the framework of the new theory. Integrating out the scalar fluctuations yields an effective field theory description, that is determined to one loop in the scalar field and up to second order in external graviton and torsion perturbations. By renormalizing the corresponding vertex functions using dimensional regularization, the Ward-Takahashi identities are shown to remain consistent to arbitrary linear perturbations. The results are discussed in the context of the trace anomaly.

Contents

1 Introduction

As every physics student learns at some point in their career, physical experiments can only measure dimensionless quantities which are assigned a physical unit by comparing the experimental outcome to a reference scale. Any set of physical units is as good as any other and performing an experiment using a different scale cannot change the result. Consequently, one may start to wonder if a physical theory can be free of any inherent scale at all. Unsurprisingly, this is indeed the case and the concept of rescaling invariance has led to many applications in modern physics, most notably the development of conformal field theories, or CFTs for short.

A universe described by a theory without intrinsic scale could be compared to a fractal like structure, that is, observed as identical regardless how far one zooms in or out. This stands in clear contradiction to reality, where one can distinguish between cosmic scales, human scales and (sub-)atomic scales. As the important physics is different in each regime (general relativity, classical mechanics and quantum mechanics, respectively), it seems obvious that the symmetry is not realized today. Accordingly, current physical models contain some elementary scales, induced by dimensionfull coupling constants. One of them is the mass of the Higgs particle $m_h \approx 125$GeV in the standard model of particles, which generates various other mass scales by Yukawa coupling to different fields. For instance, the fermionic interaction term

$$\mathcal{L}_{\text{int}} = g\bar{\psi}h\psi \tag{1.1}$$

is interpreted as a mass term for the fermion ψ in a Higgs field condensate $\langle h \rangle \neq 0$. Hence, $m_\psi = g\langle h \rangle$ acts as an apparent mass, where g is the Yukawa coupling constant. In this way, all fields in the standard model obtain their respective mass; for the Higgs field, however, there is currently no such explanation and it is thought of as fundamental.

Additionally, when gravity is taken into account, one can combine Newton's constant G_N, Planck constant $\hbar$ and speed of light c to obtain the Planck length

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-35}\,\text{m}\,, \tag{1.2}$$

which is thought of as a fundamental length scale of physics as it is built from the constants of nature. For instance, l_P is used as a typical length-scale in quantum loop gravity and string theory or, alternatively, could be thought of as a lattice spacing on a discretized space-time. On distances shorter than the Planck length, space is expected to look and behave vastly different as the current understanding of the laws of physics does not extent into this regime. A pressing question arises immediately: why is this symmetry investigated so thoroughly in research and also in this thesis, if it is apparently not present in nature? The approach pursued here is that scaling symmetry ought to be realized by the fundamental theory, valid on scales below l_P or at energies higher than the Planck mass. In such a theory, no dimensionfull parameter can be present and all physical scales observed today arise spontaneously, for instance via the condensation of a field by radiative corrections as in the Coleman-Weinberg mechanism [3]. In order to motivate this choice, it seems helpful to first get the different notions of the underlying symmetry principle straight.

Mathematically speaking, a rescaling of dimensionfull parameters is mediated by transformations of the type

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu \to \Omega^2 ds^2 \,, \tag{1.3}$$

where ds^2 is the line element and $g_{\mu\nu}$ denotes the metric tensor, used to determine lengths. For constant $\Omega \in \mathbb{R}$, it is a question of choice if the metric tensor or the coordinates x^μ are changed. The resulting transformation is known as dilation which stretches or contracts space uniformly, in the sense that an experimenter using the metric $\Omega^2 g_{\mu\nu}$ would measure each length by a factor Ω larger compared to the experimenter using the metric $g_{\mu\nu}$. In the more general case of $\Omega(x)$ being a smooth function of space-time, the transformation (1.3) is known as a Weyl transformation, which, by definition, can only act on fields and not on coordinates. In this case, (1.3) should more precisely be written as $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, $x^\mu \to x^\mu$, and a theory is said to be Weyl invariant, if the corresponding action does not change under it. A distinct, yet similar notion is that of conformal transformations. These *are* coordinate transformations $x \to \tilde{x}(x)$ that may result in a change of the metric tensor up to a Weyl factor (1.3). As conformal symmetry transformations are closed under composition, which means that two successive transformations can be combined into a single one, they form a group, known as the conformal group. Although the notions are not exactly the same, following common practice, both 'conformal invariance' and 'Weyl invariance' shall be used interchangeably in this thesis for theories that exhibit local rescaling symmetry.

Theoretical and experimental evidence for Weyl symmetry in theories of nature is plentiful. For instance, the standard model of particles is in fact almost conformal invariant [4]. Except for the Higgs mass (and the neutrino's, which could like the electron's be Yukawa generated), all coupling constants are dimensionless in four dimensions, which means that they cannot induce a physical scale. Furthermore, all other terms in the standard model Lagrangian are also classically conformal. In other words, the standard model presents itself as the first example of a fundamental theory almost obeying the symmetry principle, which can be made exact if one considers the Higgs mass to be dynamically generated, e.g. by condensation of a conformal scalar field.

Another piece of evidence is given by the density fluctuations of the early universe which are considered the seeds for structure formation of the universe, that eventually developed to galaxies, clusters of galaxies and the large scale structure visible today. These perturbations are described by their statistical properties such as the power spectrum $P(k)$ that determines the strength of the fluctuations as a function of momentum k. Early studies on galaxy formation [5] implied an almost flat power spectrum, later confirmed by cosmic microwave background measurements. That is to say, the fluctuations are nearly independent of the scale (or time) of their creation with only small corrections up to a few percent. While inflation itself provides an explanation for this observation, it follows quite naturally in a theory with Weyl symmetry, as there exists no dimensionfull parameter that could potentially spoil the symmetry.

Other applications of the principle of scaling invariance include the solution of the gauge hierarchy problem [4, 6] and the study of quantum field theories at renormalization group fixed points [7]. At such points, the running of the coupling constants as induced by renormalization ceases and the corresponding physical systems typically become self-similar. In other words, they look identical on all scales and thus conformal symmetry is realized. This fact has been crucial for understanding the ultraviolet completion of QCD, for example. In the high energy regime, quarks and gluons behave essentially as free particles, a phenomenon known as asymptotic freedom. From a theoretical perspective, the theory runs into a renormalization group fixed point with vanishing coupling constants.

Therefore, the idea that the fundamental theory of nature lacks any dimensionfull parameter and thus exhibits Weyl invariance is certainly appealing. However, the symmetry is assumed to be realized only in the high energy regime, say, at the Planck scale or during inflation of the early universe.

In [2], the authors have pushed the idea of conformal symmetry even further. They argue, that just like a coordinate transformation is nothing but a change in the observers point of view, the same interpretation can be applied to conformal transformations. In this picture, a change of scale is considered nothing more but a change of reference frame. It also explains the necessity for Weyl transformations over dilations; just as a diffeomorphism can be considered as a local Lorentz transformation with $\Lambda^{\mu}{}_{\nu}(x) = \frac{\partial \tilde{x}^{\mu}(x)}{\partial x^{\nu}}$, rescalings ought to be local as well. After all, every experimenter is free to choose their own scale locally for observations in their region of space-time.

The problem with this construction, however, is gravity, as general relativity ceases to be Weyl invariant. While there already exist conformal models of gravity, for instance based on the Weyl tensor, they are plagued by many theoretical obstacles, such as being both classical and quantum unstable and containing ghost fields [8]. The approach pursued in [2] is different, as gravity is made invariant by introducing a gauge vector field T_{α} that compensates for the transformation (1.3). The authors further show that T_{α} admits a natural interpretation as torsion, a geometrical property of space-time absent in Einstein's theory of relativity. These observations are fantastic for many reasons. Not only does this construction yield a consistent and completely conformally invariant theory, that reduces for vanishing torsion to general relativity, an extremely successful theory describing the physics of the universe. It furthermore explains why the symmetry is not observed today. Current measurements are in accordance with vanishing torsion [9], which is interpreted as the symmetry broken phase. On the other hand, in the early universe, when densities are high, torsion is known to be more significant and can thus restore Weyl invariance. This new model has been thoroughly investigated in [1] and indeed appears very promising.

However, there exist more arguments in favor of including these additional degrees of freedom in the form of space-time torsion. Apart from the purely mathematical reasons, this construction can, as inflation once did, provide very natural explanations for many problems in modern theoretical cosmology such as the singularity problem and dark energy. A broader overview of these benefits will be given in chapter 2 where torsion is properly introduced, but the application most significant for this thesis is directly related to its role as symmetry restoring gauge field, namely the resolution of a problem known as the conformal anomaly.

To get a grasp on this anomaly, notice how, in accordance with Noether's theorem, Weyl symmetry has direct consequences for the observables of the theory. To wit, consider the change of an action under the infinitesimal version of (1.3):

$$g_{\mu\nu} \to (1+2\omega)g_{\mu\nu} \implies S \to S + \int d^D x\sqrt{-g}\left[\frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\omega(x)g^{\mu\nu}\right] \tag{1.4}$$

Upon recalling the definition for the energy momentum tensor $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$, demanding conformal invariance leads to the condition

$$g^{\mu\nu}T_{\mu\nu} = 0\,. \tag{1.5}$$

In other words, theories that exhibit Weyl symmetry have a traceless stress-energy tensor. Although being true on a classical level, it was found that upon quantization $T_{\mu\nu}$ may develop a trace [10], a phenomenon known as the conformal (or trace-) anomaly. Its explicit form depends on the matter content of the theory, but for a scalar field it is given by [11]

$$\langle \hat{T}^{\mu}{}_{\mu}\rangle = g^{\mu\nu}\langle \hat{T}_{\mu\nu}\rangle = \frac{1}{16\pi^2}\left(\frac{1}{120}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - \frac{1}{360}E + \frac{1}{180}\Box R\right)\,, \tag{1.6}$$

where $C_{\alpha\beta\gamma\delta}$ denotes the Weyl tensor, E the Gauss-Bonnet term[1], R the Ricci scalar and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the d'Alembertian operator.

The anomaly has a long and sometimes controversial history and has thus received much attention in research; see also [12] for an overview. For instance, it might be related to the

[1] See (6.57) for a definition.

Hawking effect, which states that due to spontaneous particle-antiparticle pair creations near the horizon, black holes radiate and therefore evaporate over time. In 2 dimensions, the anomalous contributions to $T_{\mu\nu}$ have been proven to completely determine the strength of the Hawking radiation, or vice versa [13]. Moreover, the anomaly has also applications in low energy physics. As an example, it induces corrections to the partition function of a 2 dimensional statistical system at a conformally invariant fixed point at finite temperature [14]. The physical significance, but also the historical development of the anomaly will be discussed in more detail in section 6.4, where a proof of the identity (1.6) is presented.

In quantum field theories with gauge symmetries, Noether's theorem is replaced by Ward-Takahashi identities, which constrain the quantum correlators of the fields. The crucial observation is that the introduction of a torsion vector as a compensating field for Weyl transformations leads to a correction to the identity $T^{\mu}{}_{\mu} = 0$ of the form

$$g^{\mu\nu}\langle\hat{T}_{\mu\nu}\rangle - \nabla_{\alpha}\langle\hat{D}^{\alpha}\rangle = 0\,, \tag{1.7}$$

where the vector D^{α} is known as the dilation current. The proposition is that there will be no anomaly in the setup with gauged conformal symmetry, that is, no anomalous terms are to be written on the right-hand side of the Ward identity (1.7). Lucat has proven the absence of $\Box R$ in his PhD thesis [1]. The purpose of the present thesis is to continue this investigation for the other terms on the right-hand side of (1.6). Upon a positive outcome, Weyl symmetry is found to be respected in the quantum theory, contrary to the statement of the anomaly.

To set up the calculations for proving this proposition, a scalar field on de Sitter space is selected to test for the anomaly. This choice is motivated by theoretical, but mostly practical arguments. As the universe can be very accurately described by de Sitter during inflation and also approaches this space-time asymptotically in the future, the results will have direct applications to our universe. On the other hand, it also facilitates the explicit calculations as the de Sitter space has non-vanishing curvature which certainly helps to detect curvature built terms, as the ones in the anomaly (1.6). Moreover, because of its cosmological significance, much research about quantization in this space-time has been carried out and can be invoked here.

Straightforward quantization of classical theories typically leads to infinities, that have to dealt with somehow, a process known as regularization and renormalization. As Weyl invariance forbids the introduction of a length scale, regularization schemes such as cut-off are ruled out. Instead, dimensional regularization is the natural choice as it preserves Lorentz and Gauge invariance and breaks rescaling symmetry only weakly by logarithmic corrections. To summarize quickly how it works, calculations are done in an arbitrary, analytically continued number of space-time dimensions D and only at the end the limit $D \to 4$ is taken. The aforementioned divergences show in poles $1/(D-4)$ (regularization), which are removed by addition of so-called counterterms, whose purpose is to produce the same infinities but with opposite sign (renormalization).

This thesis is roughly structured as follows. First, in chapter 2 the concept of space-time torsion as an extension of general relativity is introduced. Building on that, chapter 3 outlines the construction of gauging conformal symmetry by vectorial torsion as introduced in [2]. The de Sitter space is defined in chapter 4, and will be used as a classical background upon which the field theory will be quantized, starting with the scalar field in chapter 5 . The subsequent chapter 6 contains the main calculations of this thesis. There, the one-loop effective action is derived in a perturbative setup and the Ward identities are proven. It continues with a proof of the trace anomaly as it is frequently found in the literature and ends with a discussion about the relation of the anomaly with the results presented here and in particular, why no anomaly is found when Weyl symmetry is gauged. The concluding part 7 rounds everything up. Throughout the entire thesis, conformal symmetry will be used as a guiding principle. The reader should be acquainted with general relativity and quantum field theory; in particular, familiarity with dimensional regularization might be helpful.

Notation and Conventions

In this thesis, the speed of light c, Newton's constant G_N and Planck constant $\hbar$ are set to unity, i.e. $c = G = \hbar = 1$.
The Einstein sum convention will always be used unless stated explicitly. Greek indices, such as $\alpha, \beta, ..., \mu, \nu, ...$ take the values $0,\ 1,\ ..,\ D-1$, where D is the space-time dimension. The metric tensor is denoted by $g_{\mu\nu}$ with signature $(-,+,+,+)$. If $g_{\mu\nu}$ is a metric, then g equals its determinant.
$\mathring{\nabla}$ stands for differentiation using the Levi-Civita connection. The covariant derivative in Einstein-Cartan theory uses the standard ∇ and the conformally gauged derivative is denoted by $\overline{\nabla}$. The latter two are introduced in chapters 2 and 3, respectively.
The Riemann tensor is defined as

$$R^{\lambda}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\sigma\nu} - \partial_\nu \Gamma^{\lambda}_{\sigma\mu} + \Gamma^{\lambda}_{\kappa\mu}\Gamma^{\kappa}_{\sigma\nu} - \Gamma^{\lambda}_{\kappa\nu}\Gamma^{\kappa}_{\sigma\mu} \,. \tag{1.8}$$

The Ricci tensor is $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$.
In order to simplify long expressions the $\pm$ symbol is used in connection with the $\Gamma(x)$ function and its logarithmic derivative $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$:

$$\begin{aligned} \Gamma(x \pm y) &\equiv \Gamma(x+y) \times \Gamma(x-y) \\ \psi(x \pm y) &\equiv \psi(x+y) + \psi(x-y) \end{aligned} \tag{1.9}$$

A Remark Regarding Non-Local Covariant Differentiation

This part concerns a serious issue with the covariant derivative the author had to struggle for some time to resolve. It is rather technical and can be skipped without affecting the comprehensibility of the rest of the thesis. Although not crucial at the end of the day, it seemed too important not to be mentioned.

It is well known that partial derivatives are manifestly local objects as they do not act on functions at different space-time points,

$$\partial^y_\alpha A^{\nu_1..\nu_m}_{\mu_1..\mu_n}(x) = 0. \tag{1.10}$$

Here $A^{\nu_1..\nu_m}_{\mu_1..\mu_n}(x)$ denotes an arbitrary tensor field and $\partial^y_\alpha \equiv \frac{\partial}{\partial y^\alpha}$. It seems natural to demand the same property from covariant differentiation, that is

$$\nabla^y_\alpha A^{\nu_1..\nu_m}_{\mu_1..\mu_n}(x) = 0. \tag{1.11}$$

This, however, is *not* true in general. Let ∇ be the Levi-Civita connection for now. A straightforward calculation shows

$$\begin{aligned} g_{\mu\nu}(y)\nabla^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} &= \nabla^y_\alpha \left(g_{\mu\nu}(y)\frac{\delta^4(x-y)}{\sqrt{-g}}\right) = \nabla^y_\alpha \left(g_{\mu\nu}(x)\frac{\delta^4(x-y)}{\sqrt{-g}}\right) \\ &= g_{\mu\nu}(x)\nabla^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} + \frac{\delta^4(x-y)}{\sqrt{-g}}\nabla^y_\alpha g_{\mu\nu}(x), \end{aligned} \tag{1.12}$$

where metric compatibility and the Leibniz rule were used. On the other hand

$$\begin{aligned} g_{\mu\nu}(y)\nabla^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} &= g_{\mu\nu}(y)\partial^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} = \partial^y_\alpha \left(g_{\mu\nu}(y)\frac{\delta^4(x-y)}{\sqrt{-g}}\right) - \frac{\delta^4(x-y)}{\sqrt{-g}}\partial^y_\alpha g_{\mu\nu}(y) \\ &= \partial^y_\alpha \left(g_{\mu\nu}(x)\frac{\delta^4(x-y)}{\sqrt{-g}}\right) - \frac{\delta^4(x-y)}{\sqrt{-g}}\partial^y_\alpha g_{\mu\nu}(y) \\ &= g_{\mu\nu}(x)\partial^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} - \frac{\delta^4(x-y)}{\sqrt{-g}}\partial^y_\alpha g_{\mu\nu}(y) \\ &= g_{\mu\nu}(x)\nabla^y_\alpha \frac{\delta^4(x-y)}{\sqrt{-g}} - \frac{\delta^4(x-y)}{\sqrt{-g}}\partial^y_\alpha g_{\mu\nu}(y), \end{aligned} \tag{1.13}$$

where once more the product rule and the fact that $\frac{\delta^4(x-y)}{\sqrt{-g}}$ is a scalar in the sense that $\partial\frac{\delta^4(x-y)}{\sqrt{-g}} = \nabla\frac{\delta^4(x-y)}{\sqrt{-g}}$ were used. Comparison yields

$$-\frac{\delta^4(x-y)}{\sqrt{-g}}\partial^y_\alpha g_{\mu\nu}(y) = \frac{\delta^4(x-y)}{\sqrt{-g}}\nabla^y_\alpha g_{\mu\nu}(x)$$

and thus in particular

$$\nabla^y_\alpha g_{\mu\nu}(x) \neq 0$$

for a general metric $g_{\mu\nu}$. With a bit more effort one can show that the correct expression is

$$\nabla^y_\alpha g_{\mu\nu}(x) = -\Gamma^\lambda_{\alpha\mu}(y) g_{\nu\lambda}(x) - \Gamma^\lambda_{\alpha\nu}(y) g_{\mu\lambda}(x). \tag{1.14}$$

It seems worth noting that the right-hand side is exactly what one would compute naively using the standard rules for covariant differentiation since $\partial^y_\alpha g_{\mu\nu}(x) = 0$.
So what is the point of all of this? As done in [15], for instance, there appears to be a common understanding in the physics community that the covariant derivative is local in the sense of equation (1.11). However, as shown above, this is not the case. Yet, having a local generalization of partial differentiation is clearly advantageous. One way to resolve the problem is to use a different covariant derivative than the mathematically unambiguously defined one and enforce the locality condition (1.11) to hold. This gives rise to a well-defined derivative operator which is commonly used in the literature and will also be used for the rest of this thesis. The only subtlety is that one has to be very careful which indices belong to which space-time points, as transferring tensors to different points will in general give wrong results in combination with this 'physical' covariant derivative (even when the points become the same eventually, that is, for $\lim_{x\to y}$). The issue of translating indices to different space-time points requires careful treatment involving bi-local tensors (bitensors) and is postponed to chapter4; see in particular the discussion around equation (4.19).

2 Riemann-Cartan Geometry

General relativity (GR) is far from being a complete description of gravitational physics, despite its tremendous success and large variety of applications [16], such as explaining the expansion of the universe, gravitational wave physics and deviations from Newton's law of gravity in the range of the solar system. There is, as of today, no quantum mechanical formulation of the theory and its unification with the standard model of particles remains largely unknown. A lot of recent research concerns this fundamental issue and gave birth to some branches of modern physics, most famously string theory. Other attempts to cure the shortcomings of general relativity have led to a variety of so-called extended theories of gravity. As GR is in great observational concordance with observations, there is little room for such extensions. One approach, nearly as old as GR itself, is to equip space-time with torsion resulting in a manifold structure known as Riemann Cartan geometry. In the way that curvature is typically interpreted as a 'bending' of the space which causes parallel transport to change the direction of vectors, torsion is frequently associated with a 'twist' in space-time. It can result in a shift or rescaling of vectors after parallel translation.

In the axiomatic derivation of GR, torsion is required to vanish. An early attempt to include torsion into the theory was via the coupling to spin, in a similar way as GR predicts coupling of the space-time metric to matter. The resulting theory is known as Einstein-Cartan-Sciama-Kibble theory [17, 18] and has been intensively investigated. It was found [19] that the interpretation of torsion as a spin coupling contains the risk of inconsistent predictions and in fact, torsion allows for much more applications when it is regarded as a free and independent field.

The modern, revived interest in torsion is caused by new ideas to resolve current problems of cosmology. To name only two of them, torsion might be able to solve the initial singularity problem [20] (or see also the explanation after equation (3.4)) and there exists an argument that it could also avert Black Hole singularities. Namely, by investigating collapsing uniformly distributed thermal matter, one finds in place of a singularity a bounce at finite size, most likely due to the violation of the null-energy condition and consecutive negative contributions to the energy momentum tensor from fermions coupled to skew symmetric torsion. Additionally, it can explain the current accelerated expansion of the universe [21]. Theoretical evidence, or at least arguments in favor of torsion are therefore abundant. It is expected to be of significance only in the early, dense universe when all symmetries were restored. Current observations can only give upper bounds and indeed, torsion is found to be very weak today [9]. Hence, the correction to GR is very small and Einstein's theory remains a valid approximation for all its successful applications. Unfortunately, though, this means that only future experiments can reach a final verdict about the presence and nature of space-time torsion.

The following discusses the geometric and algebraic properties of torsion in differential geometry on an introductory level, mainly following the construction given in the extended review on gravity with torsion [11]. For more on its geometry see also [22].

A key notion in differential geometry is the covariant derivative. Its non-trivial nature stems from the fact that partial derivatives are not (that is, do not transform as) tensors on general manifolds. This is cured by adding another, necessarily also non-tensorial quantity resulting in the covariant derivative. For instance, its action on a vector field is given by

$$\nabla_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda_{\nu\mu} V^\nu \,. \tag{2.1}$$

The connection $\Gamma^\lambda_{\mu\nu}$ appearing on the right-hand side is not unique, as adding any tensor $C^\lambda_{\mu\nu}$ to it results in another, equally valid definition of the covariant derivative. In general relativity this choice is restricted by the additional requirements (i) symmetry, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$ and (ii) metric compatibility $\nabla_\mu g_{\alpha\beta} = 0$, where indices in parenthesis denote symmetrization,

$$\Gamma^\lambda_{(\mu\nu)} \coloneqq \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu}\right) \,. \tag{2.2}$$

The unique derivative satisfying (i) & (ii) is known as the Levi-Civita derivative and will be regarded as a reference point for other possible connections. Here it is denoted with a circle on top,

$$\mathring{\nabla}_\mu V^\lambda = \partial_\mu V^\lambda + \mathring{\Gamma}^\lambda_{\mu\nu} V^\nu \,, \tag{2.3}$$

and the connection coefficients can be computed from its defining properties to equal

$$\mathring{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \,. \tag{2.4}$$

The failure of covariant derivatives to commute is encoded in the Riemann curvature tensor

$$[\mathring{\nabla}_\mu, \mathring{\nabla}_\nu] V^\lambda = \mathring{R}^\lambda{}_{\sigma\mu\nu} V^\sigma \,, \tag{2.5}$$

or, inserting the connection

$$\mathring{R}^\lambda{}_{\sigma\mu\nu} = \partial_\mu \mathring{\Gamma}^\lambda_{\sigma\nu} - \partial_\nu \mathring{\Gamma}^\lambda_{\sigma\mu} + \mathring{\Gamma}^\lambda_{\kappa\mu}\mathring{\Gamma}^\kappa_{\sigma\nu} - \mathring{\Gamma}^\lambda_{\kappa\nu}\mathring{\Gamma}^\kappa_{\sigma\mu} \,. \tag{2.6}$$

Physically, equation (2.5) describes the effect of parallel transport of the vector V^σ along an infinitesimal rectangle spanned by the x^μ and x^ν coordinate lines. Thus, a non-vanishing right-hand side indicates the presence of curvature, while spaces with zero Riemann tensor are called flat. Note that curvature can only change the direction of the vector V^σ, but neither its length nor position.

Relaxing one (or both) of the conditions (i), (ii), that led to the Levi-Civita connection, results in alternative geometries. Violating requirement (i) gives torsion, while violating (ii)

gives non-metricity. Frequently, these features are summarized in the statements that torsion induces displacement and non-metricity changes the length of vectors under parallel transport[2]. While there is not so much to say about the latter, as it generally receives the least attention, the following observation shows that it is potentially of interest on its own. Namely, define condition (iii) no curvature, that is $R^{\lambda}{}_{\sigma\mu\nu} = 0$. The geometrical trinity of gravity [23] states that one can satisfy (at most) two of the three requirements (i), (ii), (iii) and the resulting theory will be classically equivalent to general relativity, although they are expected to differ on the quantum level. Nonetheless, metric compatibility shall be retained here.
The torsion tensor is defined as the antisymmetric part of the connection,

$$T^{\sigma}{}_{\mu\nu} := \Gamma^{\sigma}_{[\mu\nu]} \,, \tag{2.7}$$

where the square brackets denote normalized antisymmetrization,

$$A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right) . \tag{2.8}$$

By construction, it is skewsymmetric in the last two indices. The inversion of the above formula is

$$\Gamma^{\sigma}_{\mu\nu} = \mathring{\Gamma}^{\sigma}_{\mu\nu} + K^{\sigma}{}_{\mu\nu} \,, \tag{2.9}$$

where the tensor

$$K^{\sigma}{}_{\mu\nu} = T^{\sigma}{}_{\mu\nu} + T_{\mu\nu}{}^{\sigma} + T_{\nu\mu}{}^{\sigma} \tag{2.10}$$

is known as contorsion tensor, which is antisymmetric in the first two indices. In the presence of non-vanishing torsion the commutator of covariant derivatives acquires an additional term,

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\sigma} = T^{\lambda}{}_{\mu\nu}\nabla_{\lambda}V^{\sigma} + R^{\sigma}{}_{\lambda\mu\nu}V^{\lambda} \,. \tag{2.11}$$

Using this result, one can examine the effect of torsion on vectors during parallel transport. To this end, it is helpful to decompose $T^{\sigma}{}_{\mu\nu}$ into 3 Lorentz invariant components:

- The trace $T_{\nu} = \frac{2}{D-1}T^{\lambda}{}_{\lambda\nu}$
- The axial vector (or totally antisymmetric part) $S^{\rho} = \epsilon^{\mu\nu\sigma\rho}T_{\mu\nu\sigma}$
- The tensor $q^{\sigma}{}_{\mu\nu}$ that satisfies $q^{\lambda}{}_{\lambda\nu} = 0$ and $\epsilon^{\mu\nu\sigma\rho}q_{\mu\nu\sigma} = 0$

Each of these can be further categorized depending on whether the contributing components are time-, light- or space-like. It turns out that the different theoretical applications of torsion

[2] As shown below, this is not quite correct. Also torsion can affect the length of vectors in parallel translation.

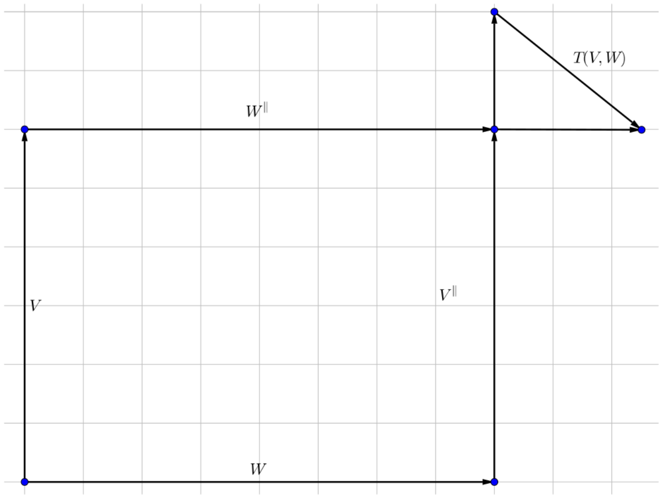

Figure 2.1: Effect of the torsion trace on the parallel transport of V along W and vice versa. The failure to yield a closed rectangle is directly proportional to torsion. Note that the parallel translated vectors lie in the plane spanned by their parent vectors, which is no longer the case if more components of torsion are included. This graphics is taken from[2].

mentioned before are mediated by different parts in this decomposition of $T^{\sigma}{}_{\mu\nu}$. Thus, in general, the torsion used in one theory can be completely different from the torsion used in other theories. An exhaustive geometrical classification and their different applications can, for instance, be found in [24]. To summarize the results, the totally antisymmetric part induces a rotation orthogonal to the trajectory during parallel transport and is therefore linked to parity and chiral transformation. Likewise, the mixed component $q^{\sigma}{}_{\mu\nu}$ can induce rotation and shear of vectors and the torsion trace can yield a rescaling of the length of vectors as shown for instance in figure 2.1. Because Weyl transformation are essentially a change of length scale, it will be the torsion trace that gives rise to a non-trivial interplay with conformal symmetry. Thus, in the rest of the thesis, a purely vectorial torsion is assumed and S^{ρ} and $q^{\sigma}{}_{\mu\nu}$ are set to zero.

For a purely vectorial torsion, the trace can be inverted to obtain the torsion tensor, contorsion tensor and connection:

$$T^{\sigma}{}_{\mu\nu} = \delta^{\sigma}{}_{[\mu}T_{\nu]} \tag{2.12}$$

$$K^{\sigma}{}_{\mu\nu} = g_{\mu\nu}T^{\sigma} - \delta^{\sigma}{}_{\nu}T_{\mu} \tag{2.13}$$

$$\Gamma^{\sigma}{}_{\mu\nu} = \mathring{\Gamma}^{\sigma}_{\mu\nu} + g_{\mu\nu}T^{\sigma} - \delta^{\sigma}{}_{\nu}T_{\mu} \tag{2.14}$$

Given the connection coefficients (2.14), the Riemann curvature tensor is computed in the same way as in general relativity:

$$
\begin{aligned}
R^{\lambda}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\lambda}_{\sigma\nu} - \partial_\nu \Gamma^{\lambda}_{\sigma\mu} + \Gamma^{\lambda}_{\kappa\mu}\Gamma^{\kappa}_{\sigma\nu} - \Gamma^{\lambda}_{\kappa\nu}\Gamma^{\kappa}_{\sigma\mu} \\
&= \mathring{R}^{\lambda}{}_{\sigma\mu\nu} - 2g_{\sigma[\mu}\mathring{\nabla}_{\nu]}T^{\lambda} + 2\delta^{\lambda}{}_{[\mu}\mathring{\nabla}_{\nu]}T_\sigma - 2\delta^{\lambda}{}_{[\mu}g_{\nu]\sigma}T_\kappa T^\kappa - 2g_{\sigma[\mu}T_{\nu]}T^\lambda + 2\delta^{\lambda}{}_{[\mu}T_{\nu]}T_\sigma
\end{aligned}
$$

Contractions yield the Ricci tensor and Ricci scalar, respectively:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \mathring{R}_{\mu\nu} + g_{\mu\nu}(\mathring{\nabla}_\lambda T^\lambda - (D-2)T_\lambda T^\lambda) + (D-2)(\mathring{\nabla}_\nu T_\mu + T_\mu T_\nu) \tag{2.15}$$

$$R = g^{\mu\nu}R_{\mu\nu} = \mathring{R} + 2(D-1)\mathring{\nabla}_\mu T^\mu - (D-1)(D-2)T_\mu T^\mu \tag{2.16}$$

Note that the Ricci tensor is not symmetric due to the term $\mathring{\nabla}_\nu T_\mu$. This is contrary to GR, where both sides of the Einstein equations

$$\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = 8\pi T_{\mu\nu}$$

are manifestly symmetric in μ and ν. In other words, the presence of torsion allows for a non-symmetric stress-energy tensor.

The Weyl tensor is defined as the traceless part of the Riemann curvature tensor, but having the same symmetries. It is invariant under a Weyl transformation and is thus naturally of interest in theories with conformal symmetry. Taking into account the correct index symmetries, one finds that it is given by

$$
\begin{aligned}
C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{2}{D-2}\big(R_{\sigma[\nu}g_{\mu]\rho} - R_{\rho[\nu}g_{\mu]\sigma}\big) + \frac{2}{(D-1)(D-2)}g_{\rho[\mu}g_{\nu]\sigma}R \\
&= \mathring{R}_{\rho\sigma\mu\nu} - 2g_{\sigma[\mu}\mathring{\nabla}_{\nu]}T_\rho + 2g_{\rho[\mu}\mathring{\nabla}_{\nu]}T_\sigma - 2g_{\rho[\mu}g_{\nu]\sigma}T_\kappa T^\kappa - 2g_{\sigma[\mu}T_{\nu]}T_\rho + 2g_{\rho[\mu}T_{\nu]}T_\sigma \\
&\quad - \frac{2}{D-2}g_{\rho[\mu}\mathring{R}_{\nu]\sigma} - \frac{2}{D-2}g_{\rho[\mu}g_{\nu]\sigma}\mathring{\nabla}_\kappa T^\kappa + 2g_{\rho[\mu}g_{\nu]\sigma}T_\kappa T^\kappa - 2g_{\rho[\mu}\mathring{\nabla}_{\nu]}T_\sigma - 2g_{\rho[\mu}T_{\nu]}T_\sigma \\
&\quad + \frac{2}{D-2}g_{\sigma[\mu}\mathring{R}_{\nu]\rho} + \frac{2}{D-2}g_{\sigma[\mu}g_{\nu]\rho}\mathring{\nabla}_\kappa T^\kappa - 2g_{\sigma[\mu}g_{\nu]\rho}T_\kappa T^\kappa + 2g_{\sigma[\mu}\mathring{\nabla}_{\nu]}T_\rho + 2g_{\sigma[\mu}T_{\nu]}T_\rho \\
&\quad + \frac{2}{(D-1)(D-2)}g_{\rho[\mu}g_{\nu]\sigma}\mathring{R} + \frac{4}{D-2}g_{\rho[\mu}g_{\nu]\sigma}\mathring{\nabla}_\kappa T^\kappa - 2g_{\rho[\mu}g_{\nu]\sigma}T_\kappa T^\kappa \\
&= \mathring{R}_{\rho\sigma\mu\nu} - \frac{2}{D-2}\big(\mathring{R}_{\sigma[\nu}g_{\mu]\rho} - \mathring{R}_{\rho[\nu}g_{\mu]\sigma}\big) + \frac{2}{(D-1)(D-2)}g_{\rho[\mu}g_{\nu]\sigma}\mathring{R} = \mathring{C}_{\rho\sigma\mu\nu}
\end{aligned} \tag{2.17}
$$

That is, the Weyl tensor does not depend on T_μ. One could interpret this as another hint about the connection of Weyl symmetry and the torsion trace.

In the perturbative picture employed in this thesis, the geometrical quantities metric and torsion will be treated as external fields. Thus, there is no need to worry about quantization of either of them. However, to get a full picture one should include the effects of propagating torsion, which is still an active field of research and new tests for the detection of its imprints are proposed regularly [25], [26].

3 Weyl Invariant Geometry with Torsion

In this chapter, the techniques developed in the preceding section are refined to create the framework of the Weyl invariant theory, which will be used in the rest of the thesis. Recall the definition of a Weyl rescaling (1.3):

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \tag{3.1}$$

Equivalently, one can say that the metric is a field of conformal weight $w = 2$. The first proposal to introduce a fundamental field compensating for this transformation was by Hermann Weyl himself using the Weyl vector W_α defined by

$$\nabla_\alpha g_{\mu\nu} = W_\alpha g_{\mu\nu} \, . \tag{3.2}$$

In the language of chapter 2 this means that metric compatibility of the covariant derivative is violated. Despite looking very promising when comparing (3.2) with (3.1), this idea was not further pursued after a criticism from Einstein: Since the metric is used to measure distance, and therefore proper time, it cannot change under parallel transport, as equation (3.2) suggests. If it would, then different observers reuniting after traveling along distinct paths could not agree on the time lapse passed, since this is now a path-dependent quantity. In other words, the absolute notion of time, which is crucial to define the causal structure of space-time, would become meaningless.

Since, in chapter 2, the trace of the torsion tensor was shown to change the length of vectors as well, another natural candidate to absorb transformation (3.1) into a geometric redefinition is the usage of torsion. This can yield a self-consistent, conformally invariant theory which shall be outlined below, following the derivation in [2].

To obtain an intuition about the required transformation for the torsion trace, it is sufficient to consider the change of the Ricci scalar of general relativity under the infinitesimal version of (3.1):

$$g_{\mu\nu} \to (1 + 2\omega(x)) g_{\mu\nu} \quad \Longrightarrow \quad \mathring{R} \to \mathring{R} - 2(D-1)\mathring{\Box}\omega - (D-1)(D-2)(\mathring{\nabla}\omega)^2 \tag{3.3}$$

A naive comparison with the expression for the Ricci scalar in Riemann-Cartan geometry (2.16) shows that the transformation cancels to linear order in ω for the choice $T_\mu \to T_\mu + \partial_\mu \omega$. In fact, on a closer inspection, one finds that this cancellation is exact to all orders when properly taking into account the transformation of the covariant derivative in (2.16). While being far from a proof, of course, this transformation is arguably the most natural choice, as it has been shown to leave invariant the Riemann tensor (and its contractions), Geodesic equation, Raychaudhuri equation and was already considered in the literature before [27].

Because of its significance, this main result shall be rephrased once more: Under the combined conformal transformations

$$
\begin{aligned}
g_{\mu\nu} &\to e^{2\omega(x)} g_{\mu\nu}, \\
T_\mu &\to T_\mu + \partial_\mu \omega(x)\,,
\end{aligned}
\tag{3.4}
$$

the Riemann tensor and therefore the left-hand side of the Einstein equations are invariant. From a physical perspective, this means that the scale transformations can be pushed forward onto the same level as diffeomorphisms, as they also leave the geometry unaltered, but correspond to a change of frame. The same interpretation can now be applied to conformal transformations, describing nothing more than the different observers. Moreover, this also explains how the introduction of torsion can solve the singularity problem, which is defined as geodesic incompleteness of space-time. Upon performing the transformation (3.1), a finite range of proper time can be stretched to infinite length making the singularity unreachable. This statement is further supported by the investigation of nearby geodesics, governed by the Jacobi equation. Torsion eventually slows down approaching geodesics preventing them from running into a singularity [2].

While the geometric side of the Einstein equations has been found invariant under (3.4), the right-hand side is in general not. Explicitly, a rescaling implies

$$
\begin{aligned}
\sqrt{-g} &\to e^{D\,\omega(x)}\sqrt{-g}\,, \\
T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} &\to e^{-(D-2)\omega(x)} T_{\mu\nu}\,,
\end{aligned}
\tag{3.5}
$$

where the matter action S_m is assumed to be conformally invariant. This issue is most easily resolved by replacing the coupling constant, G_N, on the matter side by a scalar field (dilaton),

$$
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\alpha^2}{\Phi^2(x)} T_{\mu\nu}. \tag{3.6}
$$

Here α^2 is a dimensionless coupling constant and the scalar field is required to have a conformal weight of $w_\Phi = -(D-2)/2$. Under these assumptions, the altered Einstein equations (3.6) are conformal in any space-time dimension D. They follow from the action

$$
S = \frac{1}{\alpha^2} \int d^D x \sqrt{-g}\,\Phi^2 R + S_m\,. \tag{3.7}
$$

Of course, this means that Newton's constant G_N is no longer regarded as a fundamental constant of nature, but rather generated as a condensate of some scalar field in a similar way as the Higgs field condensate generates the mass of particles in the standard model.

A final remark concerning the transformation of T_μ is in order. While (3.4) resembles the gauge transformation of an abelian $U(1)$ symmetry, these two concepts are in fact very distinct. Most notably, the gauge group $U(1)$ is compact (that is, the gauge parameter is periodic $\alpha \sim \alpha + 2\pi$), which causes the electromagnetic charge to be quantized. On the other hand, there is no such restriction for Weyl transformations as the scale can be changed by any real number. There exist more observable differences between the two cases, which are for instance discussed in [26], but they are of no importance for considerations in this thesis.

The search for conformal invariant theories requires the introduction of an appropriate gauge connection. Consider the derivative of some tensor field of type $\binom{p}{q}$ as it might appear in the action,

$$\nabla_\lambda A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} \,, \tag{3.8}$$

which transforms under a conformal rescaling as

$$\nabla_\lambda A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} \to e^{(p-q)\omega} e^{-\omega} \nabla_\lambda A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} + A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} (p-q) e^{(p-q)\omega} e^{-\omega} \nabla_\lambda \omega \,. \tag{3.9}$$

To get rid of the second piece one thus defines

$$\overline{\nabla}_\lambda A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} := (\nabla_\lambda + w_g T_\lambda) A^{\mu_1 \cdots \mu_p}_{\nu_1 \ldots \nu_q} \tag{3.10}$$

such that the transformation of the torsion trace cancels the additional term on the right-hand side of (3.9). The integer $w_g = q - p$ is known as the geometrical conformal weight. Unfortunately this is not yet the end of the construction of the Weyl invariant derivative. Physical fields can, due to their units, have an internal conformal weight as well. An an example, consider the kinetic term of a scalar field

$$\int d^D x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \,.$$

Rescaling invariance of the action requires that ϕ transforms as

$$\phi(x) \to e^{-\frac{D-2}{2}\omega} \phi(x). \tag{3.11}$$

The conformally invariant derivative acting on the scalar would thus be

$$\overline{\nabla}_\mu \phi = (\nabla_\mu - w\, T_\mu)\phi = \left(\partial_\mu + \frac{D-2}{2} T_\mu \right) \phi \,, \tag{3.12}$$

where the real number w will be called the scaling dimension of the field. The conformal covariant derivative $\overline{\nabla}$ is obtained by combining the two possible extra contributions due to the geometrical weight and physical unit. It acts on a physical field Ψ as

$$\overline{\nabla}_\mu \Psi = \nabla_\mu \Psi + (w_g - w) T_\mu \Psi \,, \tag{3.13}$$

with w_g its geometrical weight, that is $w_g = q - p$ for a tensor field of type $\binom{p}{q}$. Notice that $\overline{\nabla}$ maintains metric compatibility, because the geometric weight of $g_{\mu\nu}$ equals its scaling dimension of $w = 2$:

$$\overline{\nabla}_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} = 0 \tag{3.14}$$

As a second instructive example, consider a vector field A_ν. Its scaling dimension w can be found from the kinetic term $g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$ to be $w = -(D-4)/2$ and the geometrical weight is $w_g = 1$, because it has one lower index. Thus, application of equation (3.13) implies

$$\begin{aligned} \overline{\nabla}_\mu A_\nu &= \nabla_\mu A_\nu + \frac{D-2}{2} T_\mu A_\nu \\ &= \mathring{\nabla}_\mu A_\nu + T_\nu A_\mu - g_{\mu\nu} T_\lambda A^\lambda + \frac{D-2}{2} T_\mu A_\nu \,, \end{aligned} \tag{3.15}$$

which also shows how to step-by-step calculate the conformal covariant derivative.

Having established the correct way to use the torsion trace as a gauge boson for conformal transformations, the non-minimally coupled scalar field action of general relativity,

$$S_\phi = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi \mathring{R} \, \phi^2 \, . \right] \tag{3.16}$$

is refined according to the minimal coupling procedure $\mathring{\nabla} \to \overline{\nabla}$, $\mathring{R} \to R$ to yield

$$\begin{aligned} S_\phi &= \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \phi \overline{\nabla}_\nu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \phi^2 \right] \\ &= \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi + \frac{D-2}{2} T_\mu \phi \right) \left(\partial_\nu \phi + \frac{D-2}{2} T_\nu \phi \right) - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \phi^2 \right] \, . \end{aligned} \tag{3.17}$$

Observe that the non-minimal coupling term $R\phi^2$ is automatically conformal, as $R \to e^{-2\omega} R$ under (3.4) due to the inverse metric in $R = g^{\mu\nu} R_{\mu\nu}$ and the Ricci tensor being invariant. The conformal weight of this term is thus equal to $-2 - 2(\frac{D-2}{2}) = -D$ which compensates the $+D$ scaling dimension of $\sqrt{-g}$. Only the mass term can spoil the symmetry. This seems natural, as a mass would introduce a (length-) scale into the theory which cannot be present when Weyl invariance is demanded. Thus, when necessary, m^2 will be set to 0.

4 The de Sitter Space-Time

This chapter defines the background geometry which is set up in the framework of general relativity, so that there will be no torsion. For a review about the history of cosmology, see [28].

About a century ago, Edwin Hubble discovered in 1929 that the universe is expanding. By that time, Friedmann and Lemaître had already found their solution

$$ds^2 = -dt^2 + a^2(t)dx^2 \tag{4.1}$$

of the Einstein equations, which can also predict an expanding space via an increasing scale factor $a(t)$. Under the fundamental assumptions of homogeneity and isotropy on large scales ($\gtrapprox 100$ Mpc[3]), which are both confirmed by experiments, this solution is found by modeling the universe as an ideal fluid with spatially constant energy density. Further generalizations of (4.1) include spherical and hyperbolic spatial geometry and these solutions are commonly known as Friedmann–Lemaître–Robertson–Walker (FLRW) space-times.

Over the decades, more observations led to the development of the standard model of cosmology, according to which the universe went through a stage of accelerated expansion ($a(t) \propto e^{Ht}$) shortly after the big bang, also known as inflation. This stage was followed by the eras of radiation ($a(t) \propto t^{1/2}$) and matter ($a(t) \propto t^{2/3}$) domination and current observations indicate that the universe is again in a state of accelerated expansion ($a(t) \propto e^{\Lambda t}$). To our current knowledge, despite their similar behavior, the two stages of accelerated expansion are not related. For instance, most popular inflationary models are driven by a scalar field condensate known as the inflaton, while all observations are consistent with the late time acceleration being driven by the cosmological constant Λ, even though more general models have been proposed as the explanation.

A space-time (4.1) with $a(t) = e^{Ht}$ is known as de Sitter space (dS), and the early universe is a very good approximation. Indeed, an exact de Sitter solution would cause eternal inflation in obvious conflict with observation and hence this era is called quasi de Sitter stage. Nonetheless, because it is such a close approximation, understanding elementary physics in this space-time is crucial to obtaining a deeper understanding of the nature of inflation. Moreover, observations suggest that de Sitter geometry is the one asymptotically approached by our universe. The remainder of this chapter is therefore devoted to a thorough introduction to this space-time.

The geometrical discussion follows the lines of [29] with some additional remarks from [30], [31]. A way to visualize the D dimensional de Sitter space dS$_D$ that makes its symmetries manifest is via the embedding into the $(D+1)$ dimensional Minkowski space. In this picture,

[3] $1parsec \equiv 1$ pc $\approx 3.1 \times 10^{16}$ m ≈ 3.3 ly

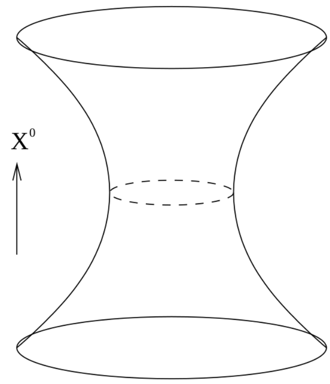

Figure 4.1: Embedding of the 2 dimensional de Sitter dS_2 space into 1+2 dimensional flat Minkowski space. Horizontal/spatial sections correspond to circles S_1. This graphic is included from[29].

the de Sitter space of radius $1/H$ is given as the hypersurface parameterized by

$$-X_0^2 + X_1^2 + \ldots + X_D^2 = \frac{1}{H^2}\,. \tag{4.2}$$

An example of this embedding for the 2 dimensional case can be seen in figure 4.1 which also reflects the hyperbolic nature of equation (4.2). The embedding gives an easy way to find the isometries of de Sitter space, which is analogous to finding the symmetry group $SO(3)$ in the more familiar context of the embedding of the sphere S_2 in $\mathbb{R}^3$. The symmetries of the ambient Minkowski space that also leave the embedding (4.2) invariant are the usual rotations of the D-sphere and boosts. For each spatial direction there is one linearly independent boost, so that

$$\#isometries\ of\ \mathsf{dS}_D = \frac{D(D-1)}{2} + D = \frac{D(D+1)}{2},$$

having thus established that de Sitter is a maximally symmetric space as it contains the maximal number of Killing vector fields. There are countless coordinate systems on de Sitter, each constructed to highlight certain features of the space. Two of them are of primary interest here. First, introduce the so-called planar or flat-slicing coordinates (t, x^i) via the transformations

$$\begin{aligned} X^0 &= \frac{1}{H}\sinh Ht + \frac{H}{2}x_i x^i e^{Ht}\,, \\ X^i &= x^i e^{Ht}\,, \qquad\qquad i = 1, \cdots, D-1, \\ X^D &= \frac{1}{H}\cosh Ht - \frac{H}{2}x_i x^i e^{Ht}\,. \end{aligned} \tag{4.3}$$

These coordinates cover only half of the space as it can intuitively be seen by the constraint $X^0 + X^D = \frac{1}{H}e^{Ht} > 0$. However, every point of de Sitter can still be reached if one includes the antipodal transformation

$$X^a(t,x) \to \hat{X}^a(t,x) \equiv -X^a(t,x) \tag{4.4}$$

that mirrors all coordinate values about the origin of the ambient Minkowski space. The antipodal transformation will return below when examining the causal structure. The induced metric in this coordinate system reads

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j, \tag{4.5}$$

which confirms the previous result that de Sitter space is equivalent to a spatially flat FLRW space-time with scale factor

$$a(t) = e^{Ht}.$$

By performing yet another transformation of the time coordinate given by $d\eta = \frac{dt}{a}$, it is possible to write the line element in conformally flat form

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2) \tag{4.6}$$

with scale factor

$$a(\eta) = -\frac{1}{H\eta}.$$

The parameter η is called conformal time and has range $\eta \in (-\infty, 0)$ for an expanding universe or $\eta \in (0, \infty)$ for a contracting one. Unless stated differently conformal time shall be used in the remainder of this thesis.

Having established the preferred coordinate system one can readily examine differential geometry on de Sitter space. Recall the metric from equation (4.6):

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \tag{4.7}$$

Either direct calculation or using the formulas for maximally symmetric space-times gives the Riemann tensor:

$$\mathring{R}_{\mu\nu\rho\sigma} = 2H^2 g_{\mu[\rho}g_{\sigma]\nu} \tag{4.8}$$

Also recall the notation $\mathring{A}$ used for quantities that are computed with the Levi-Civita connection

$$\mathring{\Gamma}^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) = Ha\left(\delta^0{}_\mu\delta^\sigma{}_\nu + \delta^0{}_\nu\delta^\sigma{}_\mu - \eta^{\sigma 0}\eta_{\mu\nu}\right), \tag{4.9}$$

where the last evaluation holds for conformal coordinates. Contractions yield the Ricci tensor and scalar, respectively:

$$\begin{aligned} \mathring{R}_{\mu\nu} &= \mathring{R}^{\lambda}{}_{\mu\lambda\nu} = H^2(D-1)g_{\mu\nu} \\ \mathring{R} &= H^2 D(D-1) \end{aligned} \tag{4.10}$$

It is worth noting that the de Sitter space-time is also a solution to the vacuum Einstein equations with cosmological constant,

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{4.11}$$

Insertion of the Ricci tensor yields the relation between Hubble constant and cosmological constant,

$$H^2 = \frac{2\Lambda}{(D-1)(D-2)}\,.$$

Considering de Sitter as cosmological fluid with an equation of state $\rho = -p$ (inflation) or as vacuum solution at the price of a cosmological constant (current acceleration) is just a physical re-interpretation. Mathematically they are equivalent and describe the same space-time.

To further explore the causal structure of de Sitter, it is helpful to construct its Penrose diagram. Recall the 3 defining properties of a Penrose (or conformal) diagram: (i) it covers the entire space-time, (ii) light rays traverse under $\pm 45^\circ$ and (iii) the diagram has finite size. Although metric (4.6) is already in conformal form, that is, null geodesics are given by $x = x_0 \pm \eta$ and would thus show as lines with slope $\pm 45^\circ$ in a (η, x) diagram, neither of the other conditions (i), (iii) are satisfied. Both coordinate ranges are infinite, and as remarked above, the coordinates do not cover the full space. Instead, it is easier to start directly from the embedding (4.2) and perform the transformations

$$\begin{aligned} X^0 &= \pm\sqrt{\frac{1}{\cos^2 T} - 1}\,, \\ X^i &= w^i \frac{1}{\cos T}\,, \end{aligned} \tag{4.12}$$

where the domain of T is $(-\pi/2, \pi/2)$ and the $+$ should be chosen for $T \in (0, \pi/2)$ and $-$ for $T \in (-\pi/2, 0)$. The w^i with $i = 1, \dots, D$ are an arbitrary, but redundant coordinate system of the $(D-1)$-sphere and typically parameterized by $(D-1)$ angles $\theta_1, \dots, \theta_{D-1}$. Hence, all coordinates have finite range and the Penrose diagram will be drawn with axes (θ, η) belonging to the metric

$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\theta^2 + d\Omega^2_{D-2}), \tag{4.13}$$

where $d\Omega^2_{D-2}$ denotes the metric of the $(D-2)$-sphere. Figure 4.2 shows the causal structure

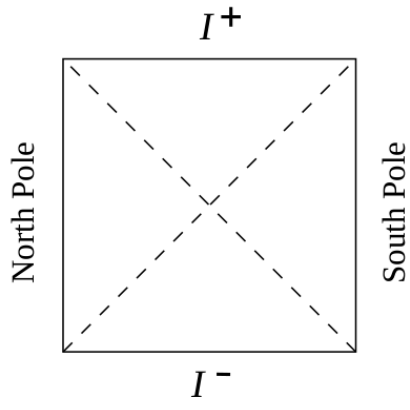

Figure 4.2: Conformal diagram for de Sitter space. Horizontal axis: θ with $\theta = 0$ at the north pole and $\theta = \pi$ at the south pole. Each interior point corresponds to a S_{D-2} while the poles are timelike lines. A horizontal section is thus an entire S_{D-1}. Also indicated are past- and future timelike infinity I^+, I^- as well as the past and future horizon (dashed lines) for an observer at the north pole (or south pole, respectively). This graphic is taken from[29].

of de Sitter space. Perhaps the most prominent feature is the existence of horizons. An observer at the north pole $\theta = 0$ will never be able to observe events that occur at space-time points in the upper right corner of the diagram, because there is not enough (conformal) time left for the events to reach him. This future horizon is sometimes called curvature radius of de Sitter space. Similarly, events at the north pole can only influence the upper left half of the diagram. The intersection of the causal past (bottom left) and causal future (top left) of an observer at the north pole is therefore called the (northern) causal diamond or causal patch. It is mapped to the same region in the southern hemisphere by the antipodal transformation (4.4). Both causal patches are completely causally disconnected and for simplicity one typically restricts to one of them when considering a physical application, so that all points can be in causal contact.

For doing physics, in particular for comprehending how events at point x can be observed at point x' one needs a profound understanding of distance in de Sitter space. Once more, an analogy with a lower dimensional and more familiar case can provide helpful intuition. Consider two points x and x' on a sphere of radius R. The unique $SO(3)$ invariant distance between them is given by the angle θ swept out by a great circle, the geodesic on the sphere. Thus, $l(x, x') = R\theta$ is called the geodesic distance between the points. However, if one imagines the sphere to be embedded into $\mathbb{R}^3$ via $\delta_{ij} X_i X_j = R^2$, a natural and useful notion of separation appears in form of the quantity Z defined by $R^2 Z \equiv \delta_{ij} X_i X'_j = R^2 \cos\theta$. The same construction will now be applied to de Sitter space. Let $l(x, x')$ be the unique $SO(1, D-1)$ invariant geodesic distance between two points x, x' in dS$_D$. It can be related to the invariant distance defined by

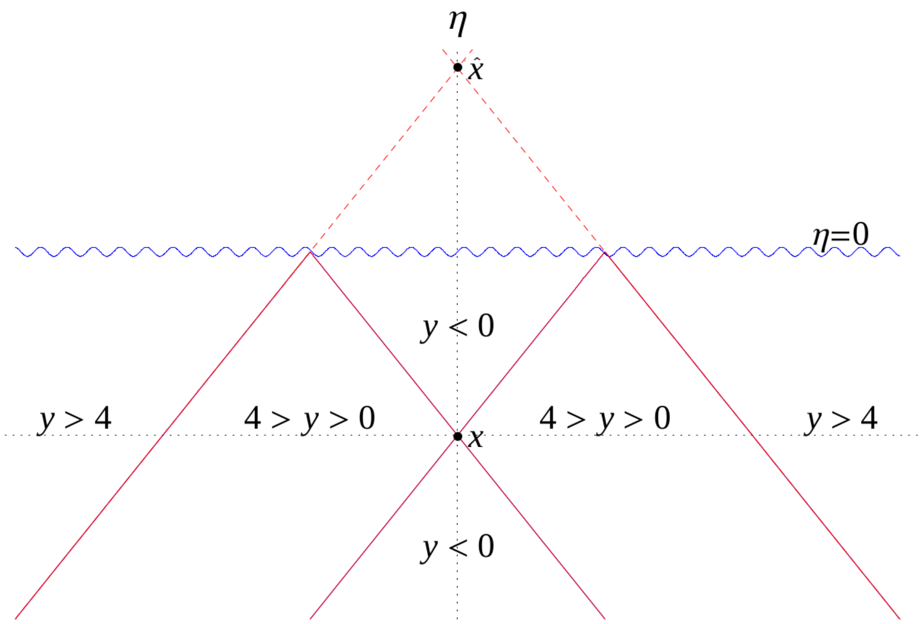

Figure 4.3: Causal structure of de Sitter space using the distance function y. The wavy line at $\eta = 0$ corresponds to future infinity. The antipodal point of x is denoted by $\hat{x}$. Graphics adapted from[32]

the embedding (4.2) via

$$Z = H^2 \eta_{ij} X^i X'^j = \cos Hl, \qquad i, j = 0, 1, \ldots, D. \tag{4.14}$$

The function Z is frequently used in the literature as de Sitter invariant distance. A more convenient choice for calculations is the function $y(x, x')$ defined by $1 - \frac{y}{2} = Z$, or in terms of the geodesic distance

$$y(x, x') = 4 \sin^2 \left(\frac{Hl}{2} \right) . \tag{4.15}$$

It vanishes for null separation which makes the advantage of the variable y as a more intuitive generalization for distance from special relativity even more apparent. In the conformal coordinates (4.6) it takes the form

$$y(x, x') = H^2 a a' \Delta x^2 = \frac{-(\eta - \eta')^2 + \Delta \vec{x}^2}{\eta \eta'} . \tag{4.16}$$

The causal structure of de Sitter space using the y coordinate is shown in figure 4.3. The geodesic distance l and hence y vanish for points x' on the lightcone of x, while the lightcone of the antipodal point $\hat{x}$ is given by $y(x,x') = 4$. Notice, however, that the antipodal point is not covered by the conformal coordinate system because η is strictly negative there.

Another interesting feature of the distance function y is contained in its derivatives. By using equation (4.16) one can verify

$$\mathring{\nabla}_\mu \mathring{\nabla}_\nu\, y = (2-y)H^2 g_{\mu\nu}\,, \tag{4.17}$$

so that the metric can be recovered as a coincidence limit:

$$\lim_{x'\to x} \frac{1}{2H^2}\mathring{\nabla}_\mu \mathring{\nabla}_\nu y = g_{\mu\nu}(x) \tag{4.18}$$

This observation motivates the definition

$$\tilde{g}_{\mu\rho'}(x,x') \coloneqq -\frac{1}{2H^2}\mathring{\nabla}_\mu \mathring{\nabla}'_\rho y \tag{4.19}$$

as a bilocal generalization of the metric tensor[4], where the additional minus accounts for the differentiation at x' ($\nabla' \equiv \nabla^{x'}$) instead of x. From a differential geometric point of view, $\tilde{g}_{\mu\rho'}$ behaves as a covector at both x and x' and hence it is sometimes called a bivector. Its purpose is to translate other (co-)vectors between the two points in a consistent manner, for instance

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu \frac{\delta^D(x-x')}{\sqrt{-g}} = -\partial'_\rho \frac{\delta^D(x-x')}{\sqrt{-g}}\,, \tag{4.20}$$

as an explicit calculation confirms. The notation with a prime on the index ρ in $\tilde{g}_{\mu\rho'}$ indicates that the corresponding derivative is calculated in the tangent space of x'. It is indispensable to keep track of this information, because the covariant derivative used in this thesis does not act on indices at different points. The bilocal metric also reduces to the metric tensor at coincidence,

$$\lim_{x'\to x} \tilde{g}_{\mu\rho'} = g_{\mu\rho}\,, \tag{4.21}$$

though it is not recommended to write it this way as derivatives acting on it will likely give wrong results. In particular, (4.21) does **not**[5] imply

$$\mathring{\nabla}_\alpha \left(\tilde{g}_{\mu\rho'}\delta^D(x-x')\right) = \mathring{\nabla}_\alpha \left(g_{\mu\rho}\delta^D(x-x')\right)\,, \tag{4.22}$$

since the covariant derivative is sensitive to changing the space-time point of indices it is acting on. The bitensor $\tilde{g}_{\mu\rho'}$ facilitates the computation of many bilocal quantities and will appear frequently in the remainder of this thesis. Appendix A contains all necessary identities for the distance function y including $\tilde{g}_{\mu\rho'}$.

---

[4]Another, more commonly used choice in the literature is the so called parallel propagator $\bar{g}_{\mu\rho'} \coloneqq 2\left(1-\frac{y}{4}\right)\nabla_\mu\nabla'_\rho \ln\left(1-\frac{y}{4}\right)$. Together with the unit vectors $n_\mu \coloneqq \frac{1}{H\sqrt{\frac{y}{4}}\sqrt{1-\frac{y}{4}}}\nabla_\mu \frac{y}{4}$ (and similarly at x') it forms a basis for maximally symmetric bilocal tensors (bitensors), as each such tensor can be expanded in terms of the metric, parallel propagator, unit vectors and derivatives thereof. While having advantages on their own, using (4.19) as bilocal metric is more convenient for the calculations in this thesis. See also [33, 15].

[5]See also the comment at the end of the introduction. Relation (4.22) would perfectly hold if ∇ was the proper covariant derivative instead of the 'physical' one which is used in this thesis.

5 Non-Minimally Coupled Scalar Field

The theory under consideration will be a massive scalar field coupled to gravity

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \phi \, \overline{\nabla}_\nu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \, \phi^2 \,, \tag{5.1}$$

where ξ is the non-minimal coupling parameter. Note that the ϕ here should not be considered the inflaton field, but rather a so-called *spectator* during the de Sitter-like stages of the universe. This means that ϕ is assumed not to contribute significantly to the energy density and thus also not to change the cosmic evolution. Its sole purpose is to test the quantum behavior of the geometric quantities. In other words, the theory can be quantized around the vanishing expectation value of the scalar field without impact. In this picture, the quadratic action decouples fluctuations of torsion and the graviton from the scalar sector, which implies that they can be considered separately. In particular, quantization of the scalar field can be performed on de Sitter space with vanishing torsion, which will be assumed for the rest of this chapter.

The equation of motion follows by varying the action $S = \int d^D x \sqrt{-g} \mathcal{L}$ with respect to ϕ,

$$(\mathring{\Box} - m^2 + \xi \mathring{R}) \phi = 0 \,, \tag{5.2}$$

which is of course the familiar Klein-Gordon equation with an effective mass

$$m^2_{\mathsf{eff}} \coloneqq m^2 - \xi \mathring{R} \,.$$

The stress-energy tensor of the theory is given by

$$T_{\mu\nu} \coloneqq \frac{-2}{\sqrt{-g}} \frac{\delta S_\phi}{g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi + g_{\mu\nu} \mathcal{L} - \xi (\mathring{R}_{\mu\nu} + g_{\mu\nu} \mathring{\Box} - \mathring{\nabla}_\mu \mathring{\nabla}_\nu) \phi^2 \,, \tag{5.3}$$

where it is useful to recall the variation of the Ricci scalar $\delta \mathring{R} = (\mathring{R}_{\mu\nu} + g_{\mu\nu} \mathring{\Box} - \mathring{\nabla}_\mu \mathring{\nabla}_\nu) \delta g^{\mu\nu}$. $T_{\mu\nu}$ becomes traceless for $\xi = -\frac{(D-2)}{4(D-1)}$, or $\xi = -\frac{1}{6}$ in 4 dimensions, and thus this value will be referred to as conformal coupling.

After establishing these basic results, the next step is the quantization of the scalar degree of freedom, that is, promoting the field ϕ to an operator $\hat{\phi}$. However, quantization on curved backgrounds is known to be a delicate matter [34, 35]. For instancen, the definitions of a vacuum state and excited, n-particle states become typically ambiguous in the presence of curvature. For de Sitter space it is known that there exists a one parameter family of vacuum states, so called α vacua. As there is no global timelike Killing vector field, also the definition of energy becomes a frame dependent quantity. Luckily, most of these problems are of minor interest for the work in this thesis. On the contrary, it will be sufficient for now to know the propagator of the scalar field, that shall be determined now.

Starting point is the Wightman function [29]

$$G(x,x') \coloneqq \langle\Omega|\hat{\phi}(x)\hat{\phi}(x')|\Omega\rangle \tag{5.4}$$

where the state $|\Omega\rangle$ is chosen to be the Bunch-Davies vacuum, which is defined by reducing to the flat space limit in the far past $\eta \to -\infty$. $G(x,x')$ satisfies the free field equation

$$(\mathring{\Box} - m^2)\, G(x,x') = 0\,, \tag{5.5}$$

which can easily be solved under the reasonable assumption of de Sitter invariance, so that $G(x,x') = G(y(x,x'))$. Using for instance the expression (4.16) for the de Sitter invariant distance $y(x,x')$ one finds the action of the d'Alembertian on a function of y to be given by

$$\frac{\mathring{\Box}}{H^2}F(y) = (4y-y^2)\frac{d^2F(y)}{dy^2} + D(2-y)\frac{dF(y)}{dy}\,. \tag{5.6}$$

The homogeneous equation of motion thus reads

$$\left[(4y-y^2)\frac{d^2}{dy^2} + D(2-y)\frac{d}{dy} - \frac{m_{\text{eff}}^2}{H^2}\right] G(y) = 0\,, \tag{5.7}$$

which is known as Euler's hypergeometric differential equation. Its solution is the hypergeometric function

$$G(y) = c \times {}_2F_1\left(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4}\right)\,, \tag{5.8}$$

where $\nu^2 = (\frac{D-1}{2})^2 - \frac{m_{\text{eff}}^2}{H^2}$ and a so far arbitrary constant c. Physical requirements will both yield the value for c and also modify the right-hand side slightly. The hypergeometric function can be expanded as an infinite series

$${}_2F_1\left(a,b;c;z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \tag{5.9}$$

where $(d)_n$ denotes the Pochhammer symbol which has the two useful representations

$$(d)_n = \frac{\Gamma(d+n)}{\Gamma(d)} = d\,(d+1)\cdots(d+n-1)\,. \tag{5.10}$$

Thus, expression (5.8) is singular for $y=4$, or, on the lightcone of the antipodal point $\hat{x}$. The Hadamard form, in which all singularities of the propagator are demanded to be on the lightcone of x, can be obtained by using one of the Gauss identities for ${}_2F_1$:

$$\begin{aligned} G(x,x') = {} & c\frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma(\frac{1}{2}\pm\nu)}\,{}_2F_1\left(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; \frac{y}{4}\right) \\ & + c\frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(\frac{D-1}{2}\pm\nu)}\left(\frac{y}{4}\right)^{1-D/2}{}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; 2-\frac{D}{2}; \frac{y}{4}\right) \end{aligned} \tag{5.11}$$

Here and in the following the $\pm$ symbol is used to abbreviate $\Gamma(x \pm y) \equiv \Gamma(x+y) \times \Gamma(x-y)$. Since linear ODEs obey the superposition principle, one could theoretically add the original solution (5.8) with another constant $\tilde{c}$. This would lead to no observable consequences because x and its antipodal point $\hat{x}$ do neither share a common history nor a common future. Still, for the sake of simplicity and fulfilling the Hadamard criterion, this second linearly independent solution shall be discarded.[6]

The physical expression that can be used in the path integral, however, requires usage of the time ordered two-point function

$$i\Delta(x,x') \coloneqq \langle \Omega | \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\} | \Omega \rangle \,, \tag{5.12}$$

where the symbol $\mathcal{T}$ stands for time ordering

$$\mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\} = \theta(\eta - \eta')\hat{\phi}(x)\hat{\phi}(x') - \theta(\eta' - \eta)\hat{\phi}(x')\hat{\phi}(x) \,.$$

Due to the Heaviside step-function, the equation of motion (5.5) acquires a dirac-delta $\delta^D(x-x')$ source term. The Chernikov-Tagirov propagator [36] is then defined for the particular source

$$(\mathring{\Box} - m^2) i\Delta(x,x') = \frac{i\delta^D(x-x')}{\sqrt{-g}} \,, \tag{5.13}$$

and can be obtained from the Wightman function 5.11 with little effort. In fact, a pole prescription [37]

$$y \to y_{++} = H^2 a a' \left(-(|\eta - \eta'| - i\epsilon)^2 + \Delta\vec{x}^2 \right) \tag{5.14}$$

sources the δ-function on the right-hand side of (5.13) correctly, as will readily be proven. For completeness, other possible prescriptions are

$$\begin{aligned}
y_{+-} &= H^2 a a' \left(-(\eta - \eta' + i\epsilon)^2 + \Delta\vec{x}^2 \right) \,, \\
y_{-+} &= H^2 a a' \left(-(\eta - \eta' - i\epsilon)^2 + \Delta\vec{x}^2 \right) \,, \\
y_{--} &= H^2 a a' \left(-(|\eta - \eta'| + i\epsilon)^2 + \Delta\vec{x}^2 \right) \,,
\end{aligned}$$

and yield the Wightman propagators $(+-, -+)$ and the anti-time ordered propagator $(--)$, respectively. Unless stated differently, all instances of the distance y in the remainder of the thesis are meant to include the pole prescription $(++)$ implicitly. The value of the constant c

[6]The one parameter family of propagators obtained by including this linearly independent solution corresponds to the one parameter family of α vacua mentioned earlier. Choosing the Hadamard state, i.e neglecting the undesired solution is in fact equivalent to choosing the Bunch Davies vacuum in (5.4).

follows directly from normalization. First, notice that due to the pole prescription the action of the d'Alembertian changes according to

$$\frac{\mathring{\square}}{H^2}F(y) = (4y - y^2)\frac{d^2F(y)}{dy^2} + D(2-y)\frac{dF(y)}{dy} - 4i\epsilon\delta(\eta - \eta')$$
$$- 2i\epsilon Ha(\eta')\mathrm{sgn}(\eta - \eta')\left[2y\frac{d^2F(y)}{dy^2} + D\frac{dF(y)}{dy}\right] . \tag{5.15}$$

When $F(y)$ is a non-singular function of y, that is, its expansion does not contain the power $y^{1-D/2}$, the right-hand side is regular and one can take the $\epsilon \to 0$ limit so that the d'Alembertian reduces to (5.6). However, when $F(y) = y^{1-D/2}$ one gets

$$\frac{\mathring{\square}}{H^2}y^{1-D/2} = \frac{D(D-2)}{4}y^{1-D/2} - 2(D-2)\frac{i\epsilon\delta(\eta - \eta')}{y^{D/2}} .$$

The last term gives rise to a D-dimensional delta function in the limit $\epsilon \to 0$ [38],

$$\lim_{\epsilon \to 0} 2(D-2)\frac{i\epsilon\delta(\eta - \eta')}{y^{D/2}} = \frac{4\pi^{D/2}}{\Gamma\left(\frac{D}{2} - 1\right)}\frac{i\delta^D(x - x')}{\sqrt{-g}} , \tag{5.16}$$

which can, for instance, be obtained by multiplying both sides with a test function $f(x)$ and evaluation in an integral over x. The identity

$$\frac{\mathring{\square}}{H^2}\left(\frac{y}{4}\right)^{1-D/2} = \frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2} - 1\right)H^D}\frac{i\delta^D(x - x')}{\sqrt{-g}} + \frac{D(D-2)}{4}\left(\frac{y}{4}\right)^{1-D/2} , \tag{5.17}$$

where the implicit pole prescription is crucial, will be of great importance in regularizing the theory. All other terms in the expansion of the hypergeometric functions in (5.11) are regular so that the constant can be inferred to be $c = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D-1}{2} \pm \nu)}{\Gamma(\frac{D}{2})}$. Thus, the final result eventually reads:

$$i\Delta(x, x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\Bigg\{\frac{\Gamma(\frac{D-1}{2} \pm \nu)\Gamma\left(1 - \frac{D}{2}\right)}{\Gamma(\frac{1}{2} \pm \nu)}\,{}_2F_1\left(\frac{D-1}{2} + \nu, \frac{D-1}{2} - \nu; \frac{D}{2}; \frac{y}{4}\right)$$
$$+ \Gamma\left(\tfrac{D}{2} - 1\right)\left(\frac{y}{4}\right)^{1-D/2}{}_2F_1\left(\frac{1}{2} + \nu, \frac{1}{2} - \nu; 2 - \frac{D}{2}; \frac{y}{4}\right)\Bigg\} \tag{5.18}$$

Recall once more the three requirements that led to this unique result: (i) De Sitter invariance, (ii) the vacuum state is the Bunch Davies vacuum and (iii) Hadamard form, that is all singularities lie on the lightcone. This scalar propagator will be heavily used in the following chapter to obtain the explicit form of the effective field theory.

6 Effective Field Theory

This chapter constitutes the main part of the thesis. It will be investigated to what extent Weyl symmetry of the scalar theory defined by (6.3) survives at the quantum level. Following the core reference [1], the first step is to prove how the inclusion of torsion will lead to a modification of the condition that the stress-energy tensor ought to be traceless for a conformal theory:

$$\overline{\nabla}_\alpha \langle \hat{D}^\alpha \rangle - \langle \hat{T}^\mu{}_\mu \rangle = 0 \tag{6.1}$$

The vector D^α is known as the dilation current, which acts as a source for the torsion trace. As it stems from a gauge symmetry, identity (6.1) is called a Ward identity and should be interpreted as follows. While the energy momentum tensor might acquire a trace, there exists a dilation current that compensates these contributions in a way that can make Weyl transformations a symmetry of the quantum theory.

This stands in contrast with the torsionless case, where no vector D^α can be defined. There, it has been shown that the energy momentum tensor develops a non-vanishing trace at the quantum level, known as the conformal anomaly. Its explicit form for a scalar field can, for instance, be found in [11] and is also derived in section 6.4 below:

$$\langle \hat{T}^\mu{}_\mu \rangle = \frac{1}{16\pi^2}\left(\frac{1}{120} C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - \frac{1}{360}\mathring{E} + \frac{1}{180}\mathring{\Box}\mathring{R}\right) \tag{6.2}$$

Here E stands for the Gauss-Bonnet term as defined in equation (6.57). The claim is that there is no anomaly when Weyl symmetry is gauged by torsion. In other words, no additional terms are found on the right-hand side of (6.1). Lucat has shown the absence of the $\mathring{\Box}\mathring{R}$ term by an explicit calculation of the left-hand side in his PhD thesis [1]. Because he only calculated the two-point functions of an effective scalar field theory in Minkowski space, there was no way to test for the Weyl tensor or Gauss-Bonnet term which contribute, at lowest order on a flat background, to the three-point functions of the theory. This thesis is meant to close this gap. Because de Sitter space has non-vanishing curvature, the Gauss-Bonnet term is finite and calculation of the two-point functions is sufficient to determine its presence (or absence) in the Ward identity. Interestingly, although de Sitter is conformally flat and hence $C_{\alpha\beta\gamma\delta} = 0$, the analysis will also yield information about the Weyl tensor.

The scalar field theory used to verify this proposition was defined in (3.17), which gives, after inserting the Ricci scalar from (2.16) and integrating by parts,

$$\begin{aligned} S_\phi = \int d^D x\sqrt{-g}\Big[&-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(\frac{D-2}{2} + 2\xi(D-1)\right)T^\mu\phi\partial_\mu\phi \\ &- \frac{D-2}{4}\left(\frac{D-2}{2} + 2\xi(D-1)\right)T_\lambda T^\lambda\phi^2 + \frac{1}{2}\xi\mathring{R}\phi^2\Big]. \end{aligned} \tag{6.3}$$

Note that the dependence on T_μ vanishes completely for the conformal coupling $\xi = -\frac{D-2}{4(D-1)}$. The re-appearance of this value is a direct consequence of the construction as torsion implementing conformal symmetry. Additionally, m^2 has been set to 0 as a non-vanishing mass would introduce a scale into the theory and thereby breaking the symmetry.

The perturbative quantization of the theory defined by (6.3) is also known as the background field method. The fields are split according to

$$\begin{aligned} \phi(x) &= \phi_{\mathsf{cl}}(x) + \delta\phi(x) \\ g_{\mu\nu}(x) &= g^{\mathsf{dS}}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(x) \\ T_\alpha(x) &= \delta T_\alpha(x)\,, \end{aligned} \tag{6.4}$$

where $g^{\mathsf{dS}}_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu}$ is the de Sitter metric and it has already been employed that the background geometry has no torsion. Because ϕ has merely the purpose of a spectator field, the classical solution is in accordance with the considerations in chapter 5 chosen to be $\phi_{\mathsf{cl}} = 0$. To simplify notation, the fluctuations $\delta\phi$, which then have the same two-point function (5.8) as the original scalar field, will be called ϕ again. As their dynamics is of no interest here, they are integrated out by defining the following vacuum to vacuum transition amplitude

$$\langle in|out\rangle = \int \overline{\mathcal{D}}\phi e^{iS_\phi[\phi,g_{\mu\nu},T_\alpha]} \tag{6.5}$$

with scalar field action (6.3). As it was pointed out in [39], the path measure

$$\overline{\mathcal{D}}\phi = \prod_x d\phi(x)\left(\frac{\sqrt{-g(x)}}{\|n(x)\|^2}\right)^{1/2}$$

is in fact conformally invariant, despite the fact that ϕ transforms non-trivially. The metric dependent result of the integration of canonical momenta is typically omitted in the notation and absorbed into the measure $\mathcal{D}$, but it is crucial for the symmetry. The vector n^μ is of no importance here, but necessary to set a direction of time to obtain a Hamiltionian description, required for the definition of the path integral.

A comment about the amplitude (6.5) is in order. The in-out formalism with asymptotic boundary states at infinity is not well-defined on de Sitter space because fields are interacting through all of space. Instead, this amplitude is understood in the general boundary formalism [40], where the quantum fluctuations are confined to a finite, but arbitrarily large volume of space and vanish outside. Hence, integration in (6.5) is only over this volume and yields a well-defined path integral, still general enough to draw conclusive statements. For instance, the domain of the fluctuations can be chosen to contain all of inflation in an early universe application.

Performing an infinitesimal Weyl transformation

$$
\begin{aligned}
g_{\mu\nu}(x) &\to (1+2\omega(x))g_{\mu\nu}(x) \\
T_\alpha(x) &\to T_\alpha(x) + \partial_\alpha\omega(x) \\
\phi(x) &\to \left(1 - \tfrac{D-2}{2}\omega(x)\right)\phi(x)
\end{aligned}
$$

on the $\langle in|out\rangle$ amplitude leads to

$$
\int \overline{\mathcal{D}}\phi e^{iS_\phi} \to \int \overline{\mathcal{D}}\phi e^{iS_\phi} \times \Bigg\{1 + i\int d^Dx\sqrt{-g}\Bigg[-\frac{D-2}{2\sqrt{-g}}\frac{\delta S_\phi}{\delta\phi(x)}\omega(x)\phi(x) \\
- \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}\omega(x)g^{\mu\nu} - \mathring{\nabla}_\alpha\left(\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta T_\alpha(x)}\right)\omega(x)\Bigg]\Bigg\}. \tag{6.6}
$$

The first term in the square brackets vanishes due to the Ehrenfest theorem [39], or the fact that for any operator $\hat{\mathcal{O}}$ in the theory

$$
\left\langle \frac{\delta S_\phi}{\delta\phi}\hat{\mathcal{O}}\right\rangle = 0\,.
$$

Demanding Weyl invariance means that (6.6) has to equal $\int \overline{\mathcal{D}}\phi e^{iS_\phi}$ for any choice of $\omega(x)$. This implies

$$
\int \overline{\mathcal{D}}\phi e^{iS_\phi}(T^\mu{}_\mu - \mathring{\nabla}_\alpha D^\alpha) = 0\,, \tag{6.7}
$$

where the stress-energy tensor and dilation current are defined by

$$
T_{\mu\nu}(x) = \frac{-2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)} \tag{6.8}
$$

$$
D^\alpha(x) = \frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta T_\alpha(x)}\,. \tag{6.9}
$$

Dividing (6.7) by $\langle in|out\rangle$ leads to the Ward-Takahashi identity

$$
\langle \hat{T}^\mu{}_\mu\rangle - \mathring{\nabla}_\alpha\langle\hat{D}^\alpha\rangle = 0\,, \tag{6.10}
$$

and thus proves (6.1), because $\mathring{\nabla}_\alpha D^\alpha = \overline{\nabla}_\alpha D^\alpha$ as a weight analysis shows. It will sometimes be referred to as the fundamental Ward identity to distinguish it from the perturbative identites satisfied by the two-point vertex functions in the effective field theory formulation.

The only issue that might still spoil conformal symmetry is whether the path integral is well-defined. This could indeed become a problem, because amplitudes such as (6.5) or (6.7) are in general divergent and thus, equation (6.10) expresses formally $'\infty - \infty = 0'$. However, as it shall be proven in due time, the infinities can be consistently renormalized by conformal counterterms and the finite, physical results obey the Ward identity without anomalous terms.

The effective action $\Gamma[g_{\mu\nu}, T_\alpha]$ for metric and torsion is defined by

$$i\Gamma[g_{\mu\nu}, T_\alpha] = \ln(\langle in|out\rangle) = \ln\left(\int \overline{\mathcal{D}}\phi e^{iS_\phi}\right) , \tag{6.11}$$

such that it generates the quantum expectation values

$$\langle \mathbb{T}^*\{\hat{T}_{\mu\nu}\}\rangle = \frac{-2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g^{\mu\nu}(x)} = \frac{-2}{\sqrt{-g}}\left\langle \mathbb{T}^*\left\{\frac{\delta S_\phi}{\delta g^{\mu\nu}}\right\}\right\rangle_{\mathcal{C}} , \tag{6.12}$$

$$\langle \mathbb{T}^*\{\hat{D}^\alpha\}\rangle = \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta T_\alpha(x)} = \frac{1}{\sqrt{-g}}\left\langle \mathbb{T}^*\left\{\frac{\delta S_\phi}{\delta T_\alpha}\right\}\right\rangle_{\mathcal{C}} . \tag{6.13}$$

$\langle \mathbb{T}^*(\cdot)\rangle := \frac{\int \mathcal{D}\phi(\cdot)e^{iS}}{\int \mathcal{D}\phi e^{iS}}$ denotes $\mathbb{T}^*$ ordering, that is the time ordered product of operators where all derivatives are to be evaluated outside the expectation value, while the subscript $\mathcal{C}$ indicates that only connected Feynman diagrams contribute. The vertex functions are defined by expansion of the functional Γ according to the number of external legs. Up to second order this reads

$$\begin{aligned}\Gamma = &\int d^Dx\sqrt{-g}\left({}_g\Gamma_{\mu\nu}(x)\,\delta g^{\mu\nu}(x) + {}_T\Gamma^\alpha(x)\,\delta T_\alpha(x)\right)\\ &+ \int d^Dx\,d^Dx'\sqrt{-g(x)}\sqrt{-g(x')}\Big(\tfrac{1}{2}\,{}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')\,\delta g^{\mu\nu}(x)\,\delta g^{\rho\sigma}(x')\\ &\qquad + {}_{gT}\Gamma^\alpha_{\mu\nu}(x,x')\,\delta g^{\mu\nu}(x)\,\delta T_\alpha(x') + \tfrac{1}{2}\,{}_{TT}\Gamma^{\alpha\beta}(x,x')\,\delta T_\alpha(x)\,\delta T_\beta(x')\Big) + \dots ,\end{aligned} \tag{6.14}$$

where the dots stand for terms with 3 or more external fields, for instance ${}_{TTT}\Gamma^{\alpha\beta\gamma}\delta T_\alpha\delta T_\beta\delta T_\gamma$. The external graviton perturbation $\delta g^{\mu\nu}$ and torsion trace perturbation δT_α are understood as small deviations from their de Sitter values $g^{\mathsf{dS}}_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu}$ and $T_\alpha = 0$. Equation (6.14) implies the following expansions for the dilation current and energy momentum tensor, respectively:

$$\langle \hat{D}^\alpha(x)\rangle = {}_T\Gamma^\alpha(x) + \int d^Dx'\sqrt{-g'}\left({}_{gT}\Gamma^\alpha_{\mu\nu}(x',x)\,\delta g^{\mu\nu}(x') + {}_{TT}\Gamma^{\alpha\beta}(x,x')\,\delta T_\beta(x')\right) , \tag{6.15}$$

$$\langle \hat{T}_{\mu\nu}(x)\rangle = -2\;{}_g\Gamma_{\mu\nu}(x) - 2\int d^Dx'\sqrt{-g'}\left({}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')\,\delta g^{\rho\sigma}(x') + {}_{gT}\Gamma^\alpha_{\mu\nu}(x,x')\,\delta T_\alpha(x')\right) , \tag{6.16}$$

up to higher order terms, of course, and obvious notation $\sqrt{-g'} \equiv \sqrt{-g(x')}$. Insertion of (6.15), (6.16) into the fundamental Ward identity (6.1) yields two derived Ward identities which have to be satisfied by the vertex functions separately. One of them is obtained by acting with $\frac{\delta}{\delta T_\beta(x')}\Big|_{g_{\mu\nu}=g^{\mathsf{dS}}_{\mu\nu},T_\alpha=0}$ on (6.1),

$$\mathring{\nabla}_\alpha\,{}_{TT}\Gamma^{\alpha\beta}(x,x') + 2g^{\mu\nu}(x)\,{}_{gT}\Gamma^\beta_{\mu\nu}(x,x') = 0 . \tag{6.17}$$

Now act with $\frac{\delta}{\delta g^{\rho\sigma}(x')}\Big|_{g_{\mu\nu}=g^{\mathsf{dS}}_{\mu\nu},T_\alpha=0}$ on (6.1):

$$\mathring{\nabla}_\alpha\, {}_{gT}\Gamma^\alpha_{\rho\sigma}(x',x) + 2\frac{\delta^D(x-x')}{\sqrt{-g}}\, {}_g\Gamma_{\rho\sigma}(x) + 2g^{\mu\nu}(x)\, {}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x') = 0\,. \tag{6.18}$$

Relations (6.17), (6.18) will be referred to as the first and second Ward identity, respectively. The remainder of this chapter is devoted to a proof of their validity for the theory defined by (6.3) and their relation to the commonly known trace anomaly. To this end, an explicit expression for the vertex functions will be derived using the scalar propagator obtained in chapter 5. The divergences contained in the vertex functions are extracted using dimensional regularization, that is, they appear as simple poles $1/(D-4)$, where D is the number of space-time dimensions. This last statement holds true because for a scalar field without self-interaction (e.g. a term $\propto \lambda\phi^4$ in the Lagrangian), the effective action is fully determined by its one-loop expression [34], while higher order loops could also cause higher order divergences such as $1/(D-4)^2$. After removing the divergent parts by addition of suitable counterterms in section 6.2, the renormalized vertices are shown to obey the identities (6.17) and (6.18). Subsequently, a typical derivation of the anomaly shall be outlined. The discussion in section 6.5 gives the authors interpretation of his results, in particular in relation to the anomaly, as well as further insights and open questions.

6.1 One-Loop Vertex Functions

Here, all terms on the right-hand side of (6.14) are computed in increasing order of difficulty. Consider the two one-point functions first:

$$\begin{aligned}
{}_g\Gamma_{\mu\nu}(x) &= \frac{1}{\sqrt{-g}}\left\langle \mathbb{T}^*\left\{\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)}\right\}\Big|_{g=g^{\mathsf{dS}},T=0}\right\rangle_{\mathcal{C}}\\
{}_T\Gamma^\alpha(x) &= \frac{1}{\sqrt{-g}}\left\langle \mathbb{T}^*\left\{\frac{\delta S_\phi}{\delta T_\alpha(x)}\right\}\Big|_{g=g^{\mathsf{dS}},T=0}\right\rangle_{\mathcal{C}}
\end{aligned}$$

By using the conformal action (6.3) one finds

$$\begin{aligned}
{}_g\Gamma_{\mu\nu}(x) &= -\tfrac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\langle\mathbb{T}^*\{\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}\}\rangle\\
&\quad - \tfrac{1}{2}\Big(\tfrac{1}{2}g_{\mu\nu}\xi\mathring{R} - \xi\mathring{R}_{\mu\nu} - \xi g_{\mu\nu}\mathring{\Box} + \xi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\Big)\langle\mathbb{T}^*\{\hat{\phi}^2(x)\}\rangle\,,\\
{}_T\Gamma^\alpha(x) &= -\left(\tfrac{D-2}{2} + 2\xi(D-1)\right)g^{\alpha\beta}\langle\mathbb{T}^*\{\hat{\phi}\partial_\beta\hat{\phi}\}\rangle\,.
\end{aligned} \tag{6.19}$$

The diagrammatic representation of these vertices is shown in figure 6.1. From the defining equation

$$\langle\mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle = i\Delta(x,x')$$

Figure 6.1: One-loop Feynman diagrams constituting the vertex functions in (6.19) with the external lines attached at point x. Solid lines correspond to the scalar propagator. Because of their shape, these kind of diagrams are typically called *tadpole* diagrams.

for the propagator, the contractions can be computed by the point splitting technique

$$\begin{aligned}\langle \mathbb{T}^*\{\hat{\phi}^2(x)\}\rangle &= \lim_{x'\to x}\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle = i\Delta(x,x)\,,\\ \left\langle \mathbb{T}^*\{\hat{\phi}\partial_\beta\hat{\phi}\}\right\rangle &= \partial_\beta\Big(\lim_{x'\to x}\left\langle \mathcal{T}\{\hat{\phi}(x')\hat{\phi}(x)\}\right\rangle\Big) = \partial_\beta(i\Delta(x,x))\,,\\ \langle \mathbb{T}^*\{\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}\}\rangle &= \lim_{x'\to x}\partial_\alpha\partial'_\beta i\Delta(x,x')\,,\end{aligned} \tag{6.20}$$

because $\mathbb{T}^*$-ordering reduces to usual time ordering once all derivatives are pulled out of the expectation value. The Chernikov-Tagirov propagator was derived in chapter 5,

$$\begin{aligned}i\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\Bigg\{&\Gamma\left(\tfrac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-D/2} + \sum_{n=0}^{\infty}\Bigg[\frac{\Gamma(\frac{D-1}{2}\pm\nu)}{\Gamma(\frac{1}{2}\pm\nu)}\frac{(\frac{D-1}{2}+\nu)_n(\frac{D-1}{2}-\nu)_n}{(D/2)_n n!}\times\\ &\Gamma\left(1-\tfrac{D}{2}\right)\left(\frac{y}{4}\right)^n + \Gamma\left(\tfrac{D}{2}-1\right)\frac{\frac{1}{4}-\nu^2}{2-D/2}\frac{(\frac{3}{2}+\nu)_n(\frac{3}{2}-\nu)_n}{(3-D/2)_n(n+1)!}\left(\frac{y}{4}\right)^{n+2-D/2}\Bigg]\Bigg\}\,,\end{aligned} \tag{6.21}$$

where the series representation (5.9) for the hypergeometric function has been applied. Also recall the $\pm$ notation $\Gamma(x\pm y) = \Gamma(x+y)\times\Gamma(x-y)$ used in context with the gamma function. To obtain the coincidence limit $x'\to x$, it is important to note that any D-dependent power of y vanishes for $y\to 0$, when evaluated at the complex continued dimension D. According to the rules of dimensional regularization [41], the identity $0^n = 0$, which is valid for positive n, is analytically continued to all $n\in\mathbb{C}$. Since D-dependent powers of y at coincidence vanish wherever they are defined, analyticity ensures that they vanish everywhere in the complex plane. In particular $\left(\frac{y}{4}\right)^{1-D/2}\to 0$ because the limit $y\to 0$ is taken before setting D to 4. This means that only the constant term ($\propto y^0$) in (6.21) survives the limit

$$\lim_{x'\to x} i\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D-1}{2}\pm\nu)\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}\pm\nu)}\,, \tag{6.22}$$

so that one can immediately conclude that the one-point function ${}_T\Gamma^\alpha(x)$ vanishes because

$$\langle \mathbb{T}^*\{\hat{\phi}\partial_\beta\hat{\phi}\}\rangle = \partial_\beta i\Delta(x,x) = \partial_\beta\, const = 0\,. \tag{6.23}$$

The last contraction in (6.20) also receives a contribution from the linear term of the propagator. Two derivatives acting on y give (see also appendix A)

$$\partial_\alpha \partial'_\beta \frac{y}{4} = H^2 a a' \left(\delta^0{}_\alpha \delta^0{}_\beta \frac{y}{4} - \frac{1}{2} H a \delta^0{}_\alpha \Delta x_\beta + \frac{1}{2} H a' \delta^0{}_\beta \Delta x_\alpha - \frac{1}{2} \eta_{\alpha\beta} \right) ,$$

which implies

$$\langle \mathbb{T}^* \{ \partial_\alpha \hat{\phi} \partial_\beta \hat{\phi} \} \rangle = \frac{\xi \mathring{R}}{D} g_{\alpha\beta} i\Delta(x,x) \,, \tag{6.24}$$

where the definition $\nu^2 = \frac{(D-1)^2}{4} + \frac{\xi \mathring{R}}{H^2}$ has been used. Applying these results to (6.19) shows that also ${}_g\Gamma_{\mu\nu}$ vanishes. In other words, no tadpole contributes to the effective action.

Next, the function ${}_{TT}\Gamma^{\alpha\beta}(x,x')$ will be derived. Out of the three vertices with two external lines, it is the easiest to compute and will thus be used to illustrate how to regularize expressions involving the Chernikov-Tagirov propagator (6.21) within the framework of dimensional regularization. Most of the manipulation rules involving the de Sitter invariant distance y are explained in and adapted from [37].

By making use of the expansion (6.15) one finds

$$\begin{aligned} {}_{TT}\Gamma^{\alpha\beta}(x,x') &= \frac{1}{\sqrt{-g(x')}} \left. \frac{\delta \langle \mathbb{T}^* \{ \hat{D}^\alpha(x) \} \rangle}{\delta T_\beta(x')} \right|_{g=g^{\mathrm{dS}}, T=0} \\ &= \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}} \left\langle \mathbb{T}^* \left\{ i \frac{\delta S_\phi}{\delta T_\alpha(x)} \frac{\delta S_\phi}{\delta T_\beta(x')} + \frac{\delta^2 S_\phi}{\delta T_\alpha(x) \delta T_\beta(x')} \right\} \right|_{g=g^{\mathrm{dS}}, T=0} \right\rangle_{\mathcal{C}} , \end{aligned} \tag{6.25}$$

where the last step follows from the defining equation (6.13). Figure 6.2 shows the Feynman diagrams contributing to this vertex. Insertion of the action yields

$$\begin{aligned} {}_{TT}\Gamma^{\alpha\beta}(x,x') &= i \left(\frac{D-2}{2} + 2\xi(D-1) \right)^2 \left\langle \mathbb{T}^* \left\{ \hat{\phi} \, (\mathring{\nabla}^\alpha \hat{\phi}) \, \hat{\phi}' \, (\mathring{\nabla}'^\beta \hat{\phi}') \right\} \right\rangle_{\mathcal{C}} \\ &\quad - \frac{D-2}{2} \left(\frac{D-2}{2} + 2\xi(D-1) \right) g^{\alpha\beta}(x) \frac{\delta^D(x-x')}{\sqrt{-g}} \left\langle \mathbb{T}^* \{ \hat{\phi}^2(x) \} \right\rangle_{\mathcal{C}} , \end{aligned} \tag{6.26}$$

with obvious short-hand notation $\phi \equiv \phi(x)$, $\phi' \equiv \phi(x')$ and $\nabla' \equiv \nabla_{x'}$. Here, and in what follows, derivatives are always meant to act at point x if not indicated differently. Computing the first term involves Wicks theorem [42], which states that the vacuum expectation value of a product of four operators can be decomposed according to

$$\langle \mathcal{T}(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \hat{\mathcal{O}}_4) \rangle = \langle \mathcal{T}(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2) \rangle \langle \mathcal{T}(\hat{\mathcal{O}}_3 \hat{\mathcal{O}}_4) \rangle + \langle \mathcal{T}(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_3) \rangle \langle \mathcal{T}(\hat{\mathcal{O}}_2 \hat{\mathcal{O}}_4) \rangle + \langle \mathcal{T}(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_4) \rangle \langle \mathcal{T}(\hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3) \rangle .$$

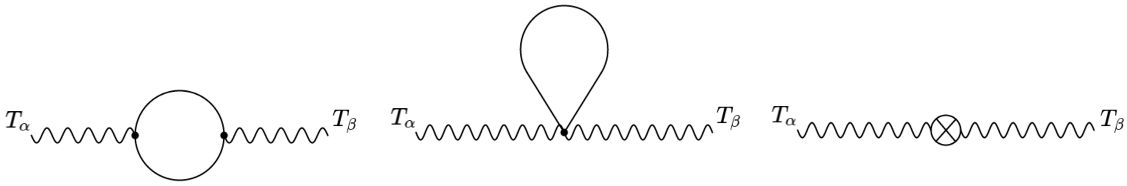

Figure 6.2: One-loop Feynman diagrams with two external torsion fields contributing to the TT vertex. The left diagram corresponds to the first term in (6.25), while the middle diagram is given by the second. The third graph represents the counterterm contribution that is necessary to cancel the divergences and will be explicitly computed in section 6.2. The external lines are attached to points x and x', respectively, and hence the second and third diagram are referred to as local (that is, $\propto \delta^D(x-x')$), while the left diagram gives also non-local terms.

Notice that this is only the leading order of a loop expansion but sufficient here. Moreover, only two terms of the right-hand side contribute because $\langle \hat{\phi}\, \mathring{\nabla}^\alpha \hat{\phi} \rangle \langle \hat{\phi}'\, \mathring{\nabla}'^\beta \hat{\phi}' \rangle = 0$ according to equation (6.23). By making use of the identities (6.20) and

$$\left\langle \mathbb{T}^* \left\{ \hat{\phi}(x') \mathring{\nabla}^\alpha \hat{\phi}(x) \right\} \right\rangle = \mathring{\nabla}^\alpha \left\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\} \right\rangle = \mathring{\nabla}^\alpha \, i\Delta(x,x') \tag{6.27}$$

one arrives at

$$\begin{aligned} {}_{TT}\Gamma^{\alpha\beta}(x,x') = {} & \frac{i}{2}\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \mathring{\nabla}^\alpha \mathring{\nabla}'^\beta (i\Delta(x,x'))^2 \\ & - \frac{D-2}{2}\left(\frac{D-2}{2} + 2\xi(D-1)\right) g^{\alpha\beta}(x) \frac{\delta^D(x-x')}{\sqrt{-g}} i\Delta(x,x)\,. \end{aligned} \tag{6.28}$$

The first line corresponds to the first diagram in figure 6.2, while the second line gives the local one in the middle. In order to proceed, an expression for the square of the scalar propagator $i\Delta(x,x')$ is required. Notice that the sum on the right-hand side of (6.21) is in fact finite in $D=4$, despite seemingly being divergent due to the poles in $(D-4)$. To prove this claim, it suffices to expand all terms to first order in $(D-4)$,

$$\frac{\Gamma(\frac{D-3}{2} \pm \nu)}{\Gamma(\frac{1}{2} \pm \nu)} = 1 + \frac{D-4}{2}\left[\psi\left(\frac{1}{2} + \nu_4\right) + \psi\left(\frac{1}{2} - \nu_4\right)\right] + \mathcal{O}(D-4)^2 \tag{6.29}$$

$$\Gamma\left(1 - \tfrac{D}{2}\right) = \frac{2}{D-4} + (-1 - \psi(1)) + \mathcal{O}(D-4) \tag{6.30}$$

$$\Gamma\left(\tfrac{D}{2} - 1\right) = 1 + \frac{D-4}{2}\psi(1) + \mathcal{O}(D-4)^2 \tag{6.31}$$

$$\left(\frac{D-3}{2}\right)^2 - \nu^2 = \left(\frac{1}{4} - \nu^2\right) + \frac{D-4}{2} + \mathcal{O}(D-4)^2\,, \tag{6.32}$$

where ν_4 stands for $\nu|_{D=4} = \sqrt{\frac{9}{4}+\frac{\xi\mathring{R}}{H^2}}$, and $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma function with $\psi(1) = -\gamma_E$ the Euler-Mascheroni constant. Upon defining the function $f_0(y)$ as the $D \to 4$ limit of the sum appearing in (6.21), one thus finds

$$f_0(y) = \left(\frac{1}{4}-\nu_4^2\right)\sum_{n=0}^{\infty}\frac{(\frac{3}{2}+\nu_4)_n(\frac{3}{2}-\nu_4)_n}{(2)_n}\frac{\left(\frac{y}{4}\right)^n}{n!}\times \\ \left[\ln\frac{y}{4}+\psi\left(\frac{3}{2}+n\pm\nu_4\right)-\psi\left(1+n\right)-\psi\left(2+n\right)\right]+\mathcal{O}(D-4)\,, \quad (6.33)$$

with $\pm$ notation in the digamma function $\psi(x\pm y)\equiv\psi(x+y)+\psi(x-y)$. In other words, the infinite terms cancel to each order and can be 'resummed' to give the finite function $f_0(y)$. With this definition the square now becomes

$$(i\Delta(x,x'))^2 = \frac{H^{2D-4}}{(4\pi)^D}\left\{\Gamma^2\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{2-D}+\frac{8}{y}f_0(y)+f_0^2(y)\right\}. \quad (6.34)$$

Although the first term has a finite prefactor in $D=4$, it is not integrable as it scales $\sim y^{-2}$, which means that it diverges more than logarithmically at the lower boundary when integrated over y. To make it integrable, one has to increase the power of y by extracting derivatives [37]. This is, in general, done as follows. First, observe that the action of the d'Alembertian (5.6) implies

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{\mathring{\Box}}{H^2}\left(\frac{y}{4}\right)^{3-D}-\frac{4}{D-4}\left(\frac{y}{4}\right)^{3-D}. \quad (6.35)$$

In order to localize the divergence of the $\propto y^{3-D}$ term on a delta function $\delta^D(x-x')$, subtract a zero in the form of the fundamental identity (5.17) with coefficient $\frac{2}{(D-3)(D-4)}$,

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{\mathring{\Box}}{H^2}\left[\left(\frac{y}{4}\right)^{3-D}-\left(\frac{y}{4}\right)^{1-D/2}\right] \\ -\frac{4}{D-4}\left[\left(\frac{y}{4}\right)^{3-D}-\frac{D(D-2)}{8(D-3)}\left(\frac{y}{4}\right)^{1-D/2}\right] \\ +\frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}}\,, \quad (6.36)$$

where one should recall that the distance $y(x,x')$ contains the pole prescription (5.14). The first two terms on the right-hand side are finite in $D=4$ as expansion shows

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}}-4\frac{\mathring{\Box}}{H^2}\frac{\ln(\frac{y}{4})}{y}+8\frac{\ln(\frac{y}{4})}{y}-\frac{4}{y} \quad (6.37)$$

plus terms proportional to $(D-4)$. As this combination appears very frequently, it makes sense to define another auxiliary function by

$$g(y) = -4\frac{\mathring{\Box}}{H^2}\frac{\ln(\frac{y}{4})}{y} + 8\frac{\ln(\frac{y}{4})}{y} - \frac{4}{y}\,, \tag{6.38}$$

such that

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}} + g(y)$$

up to terms that vanish when $D \to 4$. In appendix A.1 it is shown how to apply the same construction also to other powers of y. Replacing this term in (6.34) makes everything integrable and yields the desired split into divergent and finite terms:

$$\begin{aligned}(i\Delta(x,x'))^2 &= \frac{H^{D-4}}{(4\pi)^{D/2}}\frac{2\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)}\frac{i\delta^D(x-x')}{\sqrt{-g}} + \frac{H^4}{(4\pi)^4}\left\{g(y) + \frac{8}{y}f_0(y) + f_0^2(y)\right\} \\ &\equiv \frac{H^{D-4}}{(4\pi)^{D/2}}\frac{2\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)}\frac{i\delta^D(x-x')}{\sqrt{-g}} + (i\Delta(x,x'))^2_{\mathsf{fin}}\end{aligned} \tag{6.39}$$

The local part of the TT vertex additionally involves the propagator at coincidence (6.22),

$$\begin{aligned}&\lim_{x'\to x} i\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D-1}{2}\pm\nu)\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}\pm\nu)} \\ &= \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{2\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)}\left(\frac{1}{4}-\nu^2\right) + \frac{H^{D-2}}{(4\pi)^{D/2}}\left[\left(\frac{1}{4}-\nu_4^2\right)\left[1-2\psi(1)+\psi\left(\tfrac{1}{2}\pm\nu_4\right)\right]+1\right],\end{aligned} \tag{6.40}$$

which gives another infinite contribution. To get to the last line, expansions (6.29)-(6.32) were used once more such that the divergence is presented in the same way as above and the resulting finite terms are evaluated in $D=4$. Upon inserting (6.39), (6.40) into the formula for the TT vertex and using $\mathring{R} = D(D-1)H^2$ for de Sitter space one obtains the regularized result

$$\begin{aligned}&{}_{TT}\Gamma^{\alpha\beta}(x,x') = \\ &= \left(\frac{D-2}{2}+2\xi(D-1)\right)^2\frac{H^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D}{2}-1\right)}{(D-3)(D-4)}\left(-\mathring{\nabla}^\alpha\mathring{\nabla}'^\beta + DH^2g^{\alpha\beta}\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \\ &+ \frac{1}{16\pi^2}H^2g^{\alpha\beta}\left\{-(1+6\xi)+2(1+6\xi)^2\left[2-2\psi(1)+\psi(1/2\pm\nu_4)\right]\right\}\frac{\delta^4(x-x')}{\sqrt{-g}} \\ &+ \frac{i}{2}(1+6\xi)^2\,\mathring{\nabla}^\alpha\mathring{\nabla}'^\beta(i\Delta(x,x'))^2_{\mathsf{fin}}\,.\end{aligned} \tag{6.41}$$

The last two lines are finite and hence written in four dimensions, while the counterterm that is required to cancel the divergence in the first line will be computed in the following section.

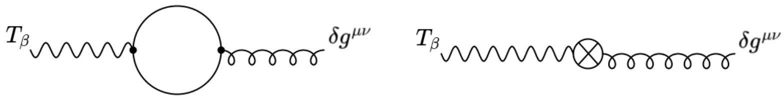

Figure 6.3: Diagrams contributing to ${}_{gT}\Gamma^{\beta}_{\mu\nu}$. Again, the external legs are attached to points x and x', respectively.

Next, the gT vertex function ${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')$ will be calculated. Following the same procedure as before, one finds that it is given by

$$
\begin{aligned}
&{}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x') \\
&= \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\left\langle \mathbb{T}^* \left\{ i\frac{\delta S_\phi}{\delta T_\beta(x')}\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)} + \frac{\delta^2 S_\phi}{\delta T_\beta(x')\delta g^{\mu\nu}(x)}\right\}\Big|_{g=g^{\mathsf{dS}},T=0}\right\rangle_{\mathcal{C}} \\
&= \frac{i}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)\left(\delta^\rho{}_\mu\delta^\sigma{}_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\right)\left\langle \mathbb{T}^*\left\{\left(\hat\phi'\,\mathring{\nabla}'^{\beta}\hat\phi'\right)\left(\partial_\rho\hat\phi\,\partial_\sigma\hat\phi\right)\right\}\right\rangle_{\mathcal{C}} \\
&\quad - \frac{i}{2}\xi\left(\frac{D-2}{2}+2\xi(D-1)\right)\left(\mathring{R}_{\mu\nu}-\tfrac{1}{2}\mathring{R}g_{\mu\nu}+g_{\mu\nu}\mathring{\Box}-\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\left\langle \mathbb{T}^*\left\{\left(\hat\phi'\mathring{\nabla}'^{\beta}\hat\phi'\right)\hat\phi^2\right\}\right\rangle_{\mathcal{C}} \\
&\quad - \frac{1}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)\frac{\delta^D(x-x')}{\sqrt{-g}}\left\langle \mathbb{T}^*\left\{\delta^\beta{}_\mu\,\hat\phi\,\partial_\nu\hat\phi + \delta^\beta{}_\nu\,\hat\phi\,\partial_\mu\hat\phi - g_{\mu\nu}\,\hat\phi\,\partial^\beta\hat\phi\right\}\right\rangle_{\mathcal{C}} .
\end{aligned}
\tag{6.42}
$$

The terms in the last line vanish according to (6.23), which means that no local diagram contributes as shown in figure 6.3. Wick contraction of the remaining pieces yields

$$
\begin{aligned}
&{}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x') \\
&= \frac{i}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)\mathring{\nabla}'^{\beta}\left[\partial_\mu i\Delta(x,x')\partial_\nu i\Delta(x,x') - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho i\Delta(x,x')\partial_\sigma i\Delta(x,x')\right] \\
&\quad - \frac{i}{2}\xi\left(\frac{D-2}{2}+2\xi(D-1)\right)\left(\mathring{R}_{\mu\nu}-\tfrac{1}{2}\mathring{R}g_{\mu\nu}+g_{\mu\nu}\mathring{\Box}-\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\mathring{\nabla}'^{\beta}(i\Delta(x,x'))^2 .
\end{aligned}
\tag{6.43}
$$

In the second line one can simply use the formula for the square of the scalar propagator obtained in (6.39). The first term is more subtle, though. Due to the additional derivatives acting on the propagator, not only is the square of the $y^{1-D/2}$ term not integrable, but also its product with the power $y^{2-D/2}$. To wit,

$$
\begin{aligned}
\partial_\mu i\Delta(x,x')\partial_\nu i\Delta(x,x') = \frac{H^{2D-4}}{(4\pi)^D}\Bigg[&\Gamma^2\left(\frac{D}{2}-1\right)\partial_\mu\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\left(\frac{y}{4}\right)^{1-D/2} \\
&+ 2\Gamma^2\left(\frac{D}{2}-1\right)\frac{\frac{1}{4}-\nu^2}{2-D/2}\partial_{(\mu}\left(\frac{y}{4}\right)^{1-D/2}\partial_{\nu)}\left(\frac{y}{4}\right)^{2-D/2} \\
&+ 2\Gamma\left(\frac{D}{2}-1\right)\partial_{(\mu}\left(\frac{y}{4}\right)^{1-D/2}\partial_{\nu)}f_1(y) + \partial_\mu f_0(y)\,\partial_\nu f_0(y)\Bigg] ,
\end{aligned}
\tag{6.44}
$$

where $f_1(y)$ is defined by the same series (6.33) as $f_0(y)$ but with the sum starting at $n = 1$. The appearance of the additional infinity can be understood from a simple argument. Previously, when showing that $f_0(y)$ contains no divergence, the $1/(D-4)$ in the coefficient of $y^{2-D/2}$ was balanced against the constant term $\propto y^0$. Here, however, the derivative acting on the scalar propagator removes this constant and hence the infinity needs to be considered separately. The first two lines in (6.44) are simplified by using the identities (A.18) and (A.19) from appendix A, where it is shown how to combine them into a single power of $\left(\frac{y}{4}\right)$. They contain the power y^{1-D} which requires the extraction of one more d'Alembertian, also calculated in (A.36). The main strategy remains the same as before, lowering the degree of divergence until the result becomes integrable, that is, of power $\propto y^{-1}$ or higher. Keeping track of the additional arising poles in $1/(D-4)$ this eventually results in

$$
\begin{aligned}
&\partial_\mu i\Delta(x,x')\partial_\nu i\Delta(x,x') = \\
&= \frac{H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)} \Bigg[\frac{D-2}{2(D-1)} \mathring{\nabla}_\mu \mathring{\nabla}_\nu + \frac{1}{2(D-1)} g_{\mu\nu} \mathring{\Box} - \frac{D-2}{2} H^2 g_{\mu\nu} \\
&\qquad\qquad - \left(\frac{1}{4} - \nu^2\right) H^2 g_{\mu\nu} \Bigg] \frac{i\delta^D(x-x')}{\sqrt{-g}} \\
&+ \frac{H^4}{(4\pi)^4} \Bigg\{ \frac{1}{12} g_{\mu\nu} \mathring{\Box}\, g(y) + \frac{1}{6} \mathring{\nabla}_\mu \mathring{\nabla}_\nu\, g(y) - \frac{1}{2} H^2 g_{\mu\nu}\, g(y) - \frac{1}{2} H^2 g_{\mu\nu} \left(\frac{1}{4} - \nu_4^2\right) g(y) \\
&\quad - \left(\frac{1}{4} - \nu_4^2\right) \mathring{\nabla}_\mu \mathring{\nabla}_\nu \frac{4}{y} + H^2 g_{\mu\nu} \left(\frac{1}{4} - \nu_4^2\right) \frac{4}{y} + 2\partial_{(\mu} \frac{4}{y}\, \partial_{\nu)} f_1(y) + \partial_\mu f_0(y)\, \partial_\nu f_0(y) \Bigg\},
\end{aligned}
$$

where, as before, D has been set to 4 in the finite terms. For the regularized gT vertex function one then finds

$$
\begin{aligned}
&{}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x') = \\
&= \left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \frac{H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-1)(D-3)(D-4)} \times \\
&\qquad\qquad \left[g_{\mu\nu} \mathring{\Box} - \mathring{\nabla}_\mu \mathring{\nabla}_\nu + \mathring{R}_{\mu\nu} \right] \mathring{\nabla}'^{\beta} \frac{\delta^D(x-x')}{\sqrt{-g}} \\
&+ \frac{H^2}{16\pi^2} (1+6\xi)^2\, g_{\mu\nu} \mathring{\nabla}'^{\beta} \frac{\delta^4(x-x')}{\sqrt{-g}} \\
&+ \frac{i}{4} \frac{H^4}{(4\pi)^4} \mathring{\nabla}'^{\beta} \Bigg\{ (1+6\xi)^2 \left[-\frac{1}{3} H^2 g_{\mu\nu} \frac{\mathring{\Box}}{H^2}\, g(y) + \frac{1}{3} \mathring{\nabla}_\mu \mathring{\nabla}_\nu\, g(y) - H^2 g_{\mu\nu}\, g(y) + 4 \mathring{\nabla}_\mu \mathring{\nabla}_\nu \frac{4}{y} \right] \\
&\quad + 2\,(1+6\xi) \left(\delta^\rho{}_{(\mu} \delta^\sigma{}_{\nu)} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \right) \left(\partial_\rho \frac{8}{y}\, \partial_\sigma f_1(y) + \partial_\rho f_0(y)\, \partial_\sigma f_0(y) \right) \\
&\quad + 2\xi\,(1+6\xi) \left(\mathring{R}_{\mu\nu} - g_{\mu\nu} \mathring{\Box} + \mathring{\nabla}_\mu \mathring{\nabla}_\nu \right) \left(\frac{8}{y} f_0(y) + f_0^2(y) \right) \Bigg\},
\end{aligned}
\tag{6.45}
$$

which makes the separation into infinite and finite terms manifest. Again, renormalization of the divergent terms is postponed to the next section.

Only the calculation of the gg-vertex is left. As for the other vertices, the starting point is a functional derivative of (6.12):

$$
\begin{aligned}
{}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x') &= -\frac{1}{2\sqrt{-g(x')}}\,\frac{\delta}{\delta g^{\rho\sigma}(x')}\bigg|_{g=g^{\mathrm{dS}},T=0}\langle\mathbb{T}^*\{\hat{T}_{\mu\nu}(x)\}\rangle \\
&= \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\left\langle\mathbb{T}^*\left\{i\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)}\frac{\delta S_\phi}{\delta g^{\rho\sigma}(x')}+\frac{\delta^2 S_\phi}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\right\}\bigg|_{g=g^{\mathrm{dS}},T=0}\right\rangle_{\mathcal{C}}
\end{aligned}
\tag{6.46}
$$

The diagrammatic representation is the same as for the TT-vertex in figure 6.2 but with the two external lines being replaced with gravitons. By definition, the vertex is manifestly symmetric in $\mu\nu$ and $\rho\sigma$ and also under the bose symmetry $(\mu\nu,x)\leftrightarrow(\rho\sigma,x')$. To simplify notation and make the expressions more readable, symmetrization of the index pairs $(\mu\nu)$ and $(\rho\sigma)$ will be omitted. Nonetheless it is important to keep in mind that this symmetrization is implicit in all the following formulas.

This time, it makes sense to consider each term separately, starting with the local contribution, that is, the second term in (6.46). Its calculation requires the knowledge of the second order variation of the Ricci scalar, which is given in the appendix, more precisely, in identity (D.5). This result can be used, because $\langle\mathcal{T}\{\hat{\phi}^2\}\rangle$ yields a constant, which means that it is not necessary to keep terms that are proportional to a derivative of ϕ^2 that naturally arise by partial integration. Explicitly, one finds

$$
\begin{aligned}
&\left\langle\mathbb{T}^*\left\{\frac{\delta^2 S_\phi}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\right\}\bigg|_{g=g^{\mathrm{dS}},T=0}\right\rangle_{\mathcal{C}} = \\
&= \left(-\frac{1}{8}g_{\mu\nu}g'_{\rho\sigma}-\frac{1}{4}\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\right)\frac{\delta^D(x-x')}{\sqrt{-g}}g^{\alpha\beta}\langle\mathbb{T}^*\{\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}\}\rangle \\
&\quad+\frac{1}{4}g_{\mu\nu}\langle\mathbb{T}^*\{\partial'_\rho\hat{\phi}'\partial'_\sigma\hat{\phi}'\}\rangle+\frac{1}{4}g'_{\rho\sigma}\langle\mathbb{T}^*\{\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}\}\rangle \\
&\quad+\frac{1}{4}\xi\langle\mathcal{T}\{\hat{\phi}^2\}\rangle\bigg(-H^2 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\square}}{H^2}+(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu)+H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\square}}{H^2} \\
&\qquad+2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma+\frac{(D-1)(D-4)}{2}H^2 g_{\mu\nu}g'_{\rho\sigma}+(D-1)(D-2)H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\bigg)\frac{\delta^D(x-x')}{\sqrt{-g}} \\
&\quad+terms\propto\mathring{\nabla}\langle\mathcal{T}\{\hat{\phi}^2\}\rangle \\
&= \frac{1}{4}\xi i\Delta(x,x)\bigg[-H^2 g_{\mu\nu}g_{\rho\sigma}\frac{\mathring{\square}}{H^2}+(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu)+\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\mathring{\square}+2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma- \\
&\qquad-2(D-1)H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\bigg]\frac{\delta^D(x-x')}{\sqrt{-g}},
\end{aligned}
\tag{6.47}
$$

with $\tilde{g}_{\mu\rho'}$ being the bilocal generalization of the metric tensor as defined in (4.19). To obtain the last expression, identity (6.24) was used once more.

The first term in (6.46) is known as TT-correlator in the literature, because it is proportional to $\langle \hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x')\rangle$ with $T_{\mu\nu}$ the stress-energy tensor. This name will not be used here, however, to avoid the ambiguity with T standing for torsion. Inserting the metric derivatives and Wick contracting gives:

$$
\begin{aligned}
&\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\Big\langle \mathbb{T}^* \left\{ i\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)}\frac{\delta S_\phi}{\delta g^{\rho\sigma}(x')}\right\}\Big|_{g=g^{\mathrm{dS}},T=0}\Big\rangle_{\mathcal{C}} \\
&= \frac{i}{4}\Big\langle \mathbb{T}^*\Big\{ \Big[\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\partial_\alpha\phi\partial_\beta\phi + \left(\tfrac{1}{2}g_{\mu\nu}\xi\mathring{R} - \xi\mathring{R}_{\mu\nu} - \xi g_{\mu\nu}\mathring{\Box} + \xi\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\hat{\phi}^2\Big] \times \\
&\qquad \Big[\left(\delta^\gamma_\rho\delta^\delta_\sigma - \tfrac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\right)\partial'_\gamma\phi'\partial'_\delta\phi' + \left(\tfrac{1}{2}g'_{\rho\sigma}\xi\mathring{R} - \xi\mathring{R}'_{\rho\sigma} - \xi g'_{\rho\sigma}\mathring{\Box}' + \xi\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\right)\hat{\phi}'^2\Big]\Big\}\Big\rangle_{\mathcal{C}} \\
&= \frac{i}{4}\Big\{ 2\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\left(\delta^\gamma_\rho\delta^\delta_\sigma - \tfrac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\right)\partial_\alpha\partial'_\gamma i\Delta(x,x')\partial_\beta\partial'_\delta i\Delta(x,x') \\
&\quad + 2\xi\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\left(\tfrac{1}{2}g'_{\rho\sigma}\mathring{R} - \mathring{R}'_{\rho\sigma} - g'_{\rho\sigma}\mathring{\Box}' + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\right)\partial_\alpha i\Delta(x,x')\partial_\beta i\Delta(x,x') \\
&\quad + 2\xi\left(\delta^\gamma_\rho\delta^\delta_\sigma - \tfrac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\right)\left(\tfrac{1}{2}g_{\mu\nu}\mathring{R} - \mathring{R}_{\mu\nu} - g_{\mu\nu}\mathring{\Box} + \mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\partial'_\gamma i\Delta(x,x')\partial'_\delta i\Delta(x,x') \\
&\quad + 2\xi^2\left(\tfrac{1}{2}g_{\mu\nu}\mathring{R} - \mathring{R}_{\mu\nu} - g_{\mu\nu}\mathring{\Box} + \mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\left(\tfrac{1}{2}g'_{\rho\sigma}\mathring{R} - \mathring{R}'_{\rho\sigma} - g'_{\rho\sigma}\mathring{\Box}' + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\right)(i\Delta(x,x'))^2\Big\}
\end{aligned}
\tag{6.48}
$$

Regularization requires knowledge of the term $\partial_\alpha\partial'_\gamma i\Delta(x,x')\partial_\beta\partial'_\delta i\Delta(x,x')$. This turns out to have a vast variety of infinities whose computation is very lengthy and thus, it is presented in appendix B in detail. The basic strategy remains the same as for the similar expression with only two derivatives acting on the propagators. By usage of the bitensor $\tilde{g}_{\mu\rho'}$ one can extract the derivatives to write individual terms of the product as a single power of y and then raise the power to split off the divergence and make it integrable. The remaining terms in (6.48) are known, and putting all together yields:

$$
\begin{aligned}
&{}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x') = \\
&\left(\tfrac{D-2}{2} + 2\xi(D-1)\right)^2 \tfrac{H^D}{4(4\pi)^{D/2}}\tfrac{\Gamma\left(\frac{D}{2}-1\right)}{(D-1)(D-2)(D-3)(D-4)}\Big\{ -\tfrac{D-2}{D-1}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma \\
&\quad + \tfrac{D-2}{D-1}H^2\left(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\tfrac{\mathring{\Box}}{H^2} - \tfrac{D-2}{D-1}H^4 g_{\mu\nu}g'_{\rho\sigma}\tfrac{\mathring{\Box}}{H^2}\tfrac{\mathring{\Box}}{H^2} \\
&\quad - 2H^2(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu) + 2(D-1)H^4 g_{\mu\nu}g'_{\rho\sigma} \\
&\quad - DH^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\tfrac{\mathring{\Box}}{H^2} - 2DH^2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma\Big\}\frac{\delta^D(x-x')}{\sqrt{-g}} \\
&+ \tfrac{H^D}{4(4\pi)^{D/2}}\tfrac{\Gamma\left(\frac{D}{2}-1\right)}{(D+1)(D-1)(D-3)(D-4)}\Big\{ -\tfrac{D-2}{2(D-1)}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma\mathring{\nabla}'_\rho
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{1}{2(D-1)} H^2 \left(g_{\mu\nu} \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + g'_{\rho\sigma} \mathring{\nabla}_\mu \mathring{\nabla}_\nu \right) \tfrac{\mathring{\Box}}{H^2} + H^2 (g_{\mu\nu} \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + g'_{\rho\sigma} \mathring{\nabla}_\mu \mathring{\nabla}_\nu) \\
&+ \tfrac{1}{2(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} \tfrac{\mathring{\Box}}{H^2} \tfrac{\mathring{\Box}}{H^2} + \tfrac{1}{2} H^4 g_{\mu\nu} g'_{\rho\sigma} \tfrac{\mathring{\Box}}{H^2} - (D-1) H^4 g_{\mu\nu} g'_{\rho\sigma} - \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma \tfrac{\mathring{\Box}}{H^2} \\
&+ 2H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma - \tfrac{1}{2} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \tfrac{\mathring{\Box}}{H^2} \tfrac{\mathring{\Box}}{H^2} + \tfrac{D-2}{2} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \tfrac{\mathring{\Box}}{H^2} \Big\} \frac{\delta^D(x-x')}{\sqrt{-g}} \\
&+ \tfrac{1}{4(4\pi)^2} \Big\{ \left[-\tfrac{2}{15} - \tfrac{1}{6} (1+6\xi)^2 - 2\xi (1+6\xi) \left(-\tfrac{1}{2} - 2\psi(1) + \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] \times \\
&\quad \times H^2 (g_{\mu\nu} \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + g'_{\rho\sigma} \mathring{\nabla}_\mu \mathring{\nabla}_\nu) \\
&\quad + \left[\tfrac{1}{3} - \tfrac{1}{3}(1+6\xi) - \tfrac{5}{6}(1+6\xi)^2 - (2\xi - \tfrac{1}{2})(1+6\xi) \left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\mathring{\Box}}{H^2} \\
&\quad + \left[-\tfrac{11}{10} - \tfrac{4}{3}(1+6\xi) + \tfrac{5}{2}(1+6\xi)^2 + 12\xi(1+6\xi) \left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] H^4 g_{\mu\nu} g'_{\rho\sigma} \\
&\quad + \left[-\tfrac{4}{15} + \tfrac{5}{3}(1+6\xi) - \tfrac{4}{3}(1+6\xi)^2 + 4\xi(1+6\xi) \left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] H^2 \tilde{g}_{\mu\rho'} \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma \\
&\quad + \left[-\tfrac{1}{6} + \tfrac{5}{6}(1+6\xi) - \tfrac{2}{3}(1+6\xi)^2 + 2\xi(1+6\xi) \left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \frac{\mathring{\Box}}{H^2} \\
&\quad + \left[1 - 5(1+6\xi) + (1+6\xi)^2 - 12\xi(1+6\xi) \left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\right] H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \Big\} \frac{\delta^4(x-x')}{\sqrt{-g}} \\
&+ \frac{iH^4}{4(4\pi)^4} \times \Big\{ \mathring{\nabla}_\mu \mathring{\nabla}_\nu \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma \Big[\left(\tfrac{1}{90} + \tfrac{1}{18}(1+6\xi)^2 \right) g(y) + \left(-\tfrac{2}{3}(1+6\xi) + \tfrac{4}{3}(1+6\xi)^2 \right) \tfrac{4}{y} \\
&\qquad + \left(\tfrac{8}{3}(1+6\xi) - 4(1+6\xi)^2 \right) \ln \tfrac{y}{4} + 2\xi^2 \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + H^2 \left(g_{\mu\nu} \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + g'_{\rho\sigma} \mathring{\nabla}_\mu \mathring{\nabla}_\nu \right) \frac{\mathring{\Box}}{H^2} \Big[\left(\tfrac{1}{180} - \tfrac{1}{18}(1+6\xi)^2 \right) g(y) \\
&\qquad + \left(-\tfrac{1}{3}(1+6\xi) - \tfrac{2}{3}(1+6\xi)^2 \right) \tfrac{4}{y} - 2\xi^2 \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + H^2 \left(g_{\mu\nu} \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + g'_{\rho\sigma} \mathring{\nabla}_\mu \mathring{\nabla}_\nu \right) \Big[\left(-\tfrac{1}{30} + \tfrac{1}{3}(1+6\xi) - \tfrac{1}{6}(1+6\xi)^2 \right) g(y) \\
&\qquad + 6\xi^2 \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\mathring{\Box}}{H^2} \frac{\mathring{\Box}}{H^2} \Big[\left(-\tfrac{1}{180} + \tfrac{1}{18}(1+6\xi)^2 \right) g(y) + \left(\tfrac{1}{8} + 2\xi^2 \right) \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\mathring{\Box}}{H^2} \Big[\left(-\tfrac{1}{60} - \tfrac{5}{6}(1+6\xi) + \tfrac{1}{3}(1+6\xi)^2 \right) g(y) + 2(1+6\xi) \tfrac{4}{y} \\
&\qquad + (6\xi - 12\xi^2) \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + H^4 g_{\mu\nu} g'_{\rho\sigma} \Big[\left(\tfrac{1}{10} + 2(1+6\xi) - \tfrac{5}{2}(1+6\xi)^2 \right) g(y) + \left(-6(1+6\xi) + 6(1+6\xi)^2 \right) \frac{4}{y} \\
&\qquad + 90\xi^2 \left(\tfrac{8}{y} f_0(y) + f_0^2(y) \right) \Big] \\
&\quad + \tfrac{1}{30} H^2 \tilde{g}_{\mu\rho'} \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma \tfrac{\mathring{\Box}}{H^2} g(y)
\end{aligned}
$$

$$
\begin{aligned}
&+ H^2 \tilde{g}_{\mu\rho'} \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma \bigg[\left(-\tfrac{1}{15} + \tfrac{2}{3}(1+6\xi)\right) g(y) - 96\xi(1+6\xi) \frac{\ln(\frac{y}{4})}{y} \\
&\qquad + \Big[-2 + \tfrac{14}{3}(1+6\xi) + \tfrac{14}{3}(1+6\xi)^2 - 24\xi(1+6\xi)\left(2\psi(1) - \psi\left(\tfrac{1}{2} \pm \nu_4\right)\right)\Big] \frac{4}{y} \bigg] \\
&+ \tfrac{1}{60} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \tfrac{\mathring{\Box}}{H^2} \tfrac{\mathring{\Box}}{H^2} g(y) + \left(-\tfrac{1}{30} + \tfrac{1}{3}(1+6\xi)\right) H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \tfrac{\mathring{\Box}}{H^2} g(y) \\
&+ \left(-2(1+6\xi) + 2(1+6\xi)^2\right) H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} g(y) \\
&+ 4 \mathring{\nabla}_\mu \mathring{\nabla}'_\rho \frac{4}{y} \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma f_2(y) - 8(1+6\xi) \mathring{\nabla}_\mu \mathring{\nabla}'_\rho \ln\frac{y}{4} \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma f_1(y) + 2 \mathring{\nabla}_\mu \mathring{\nabla}'_\rho f_1(y) \mathring{\nabla}_\nu \mathring{\nabla}'_\sigma f_1(y) \\
&+ \left(-\tfrac{1}{2} H^2 g_{\mu\nu} \tfrac{\mathring{\Box}}{H^2} - 12\xi H^2 g_{\mu\nu}\right) \left(\partial'_\rho \tfrac{8}{y} \partial'_\sigma f_1(y) + \partial'_\rho f_0(y)\, \partial'_\sigma f_0(y)\right) \\
&+ \left(-\tfrac{1}{2} H^2 g'_{\rho\sigma} \tfrac{\mathring{\Box}}{H^2} - 12\xi H^2 g'_{\rho\sigma}\right) \left(\partial_\mu \tfrac{8}{y} \partial_\nu f_1(y) + \partial_\mu f_0(y)\, \partial_\nu f_0(y)\right) \\
&+ 2\xi(-g_{\mu\nu}\mathring{\Box} + \mathring{\nabla}_\mu \mathring{\nabla}_\nu + 3H^2 g_{\mu\nu}) \left(\delta^\gamma{}_\rho \delta^\delta{}_\sigma - \tfrac{1}{2} g'_{\rho\sigma} g'^{\gamma\delta}\right) \left(\partial'_\gamma \tfrac{8}{y} \partial'_\delta f_1(y) + \partial'_\gamma f_0(y)\, \partial'_\delta f_0(y)\right) \\
&+ 2\xi(-g'_{\rho\sigma}\mathring{\Box}' + \mathring{\nabla}'_\rho \mathring{\nabla}'_\sigma + 3H^2 g'_{\rho\sigma}) \left(\delta^\alpha{}_\mu \delta^\beta{}_\nu - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta}\right) \left(\partial_\alpha \tfrac{8}{y} \partial_\beta f_1(y) + \partial_\alpha f_0(y)\, \partial_\beta f_0(y)\right) \bigg\}
\end{aligned}
\tag{6.49}
$$

In the same spirit as for $f_1(y)$, the function $f_2(y)$ appearing here is defined by (6.33) with the series starting at $n = 2$. In other words, the two lowest order contributions of the propagator (6.21) have to be kept separated before one can set $D = 4$ in the tail of the sum. The divergences are given by the first two blocks of terms and organized whether they are proportional to the conformal factor $\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2$ or not. The next task is to determine the counterterms that will renormalize them.

6.2 Renormalization

Renormalizing a quantum field theory means to add terms to the Lagrangian, so called counterterms, that give the same divergences as the vertex functions but with opposite sign so that their combination is finite. To be more specific, consider the Lagrangian (5.1),

$$
\mathcal{L}_0 = -\frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \phi_0 \, \overline{\nabla}_\nu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{2} \xi_0 R\, \phi_0^2 \,, \tag{6.50}
$$

where the subscript 0 indicates bare quantities, which possibly contain divergences. They differ from the physical observed ones by

$$
\phi_0 = \sqrt{Z_\phi}\, \phi = \sqrt{1 + \delta Z_\phi}\, \phi \tag{6.51}
$$

$$
m_0^2 = m^2 + \delta m^2 \tag{6.52}
$$

$$
\xi_0 = \xi + \delta\xi, \tag{6.53}
$$

where the δ-terms are the counterterm corrections, of order $\mathcal{O}(\hbar)$ for one-loop. If renormalization is done correctly, physical results computed this way are finite.

However, not all divergences can be absorbed by redefinitions such as (6.51)-(6.53), in which case new, previously not present interactions have to be added to the Lagrangian. This is more than certainly true in the effective theory obtained by integrating out ϕ, where one needs to find new four dimensional operators. Of course, the structure of the terms one has to add cannot be arbitrary. Apart from obvious conditions, for example obeying causality, only a finite number of such terms is allowed. If a theory needs an infinite number of different terms to remove the divergences, it is said to be not renormalizable, which typically occurs if one of the coupling constants of the theory has a negative mass dimension.[7]

For the theory at hand, the most general counterterm action is [11]

$$S_{\text{ct}} = \int d^D x \sqrt{-g} \left[a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + a_4 \overline{\Box} R + a_5 \mathcal{T}_{\mu\nu} \mathcal{T}^{\mu\nu} \right] , \qquad (6.54)$$

where the curvature tensors include torsion implicitly, $\overline{\Box} = g^{\mu\nu} \overline{\nabla}_\mu \overline{\nabla}_\nu$ is the conformal d'Alembertian and $\mathcal{T}_{\mu\nu} = \partial_\mu \mathcal{T}_\nu - \partial_\nu \mathcal{T}_\mu$ the field strength of the torsion trace. One now has to find the correct coefficients $a_i,\ i = 1, \dots 5$ such that the addition of S_{ct} to the effective action renders the theory finite. Is is clear that all a_i will be proportional to [8]

$$a_i \propto \frac{\mu^{D-4}}{D-4} , \qquad (6.55)$$

where μ is an arbitrary mass scale necessary in dimensional regularization to give the action the correct units in D dimensions. The role of the denominator is to possibly remove the poles in $1/(D-4)$. In general relativity, demanding conformal symmetry places further constrictions on the coefficients a_1, a_2, a_3, [44]. Only the two linearly independent combinations

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \tfrac{4}{D-2} R_{\mu\nu} R^{\mu\nu} - \tfrac{2}{(D-1)(D-2)} R^2 \qquad (6.56)$$

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4\, R_{\mu\nu} R^{\mu\nu} + R^2 \qquad (6.57)$$

are allowed, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor (2.17), which is conformal in any D and, in particular, independent of torsion. Some authors, such as Duff in [44], use the four dimensional version of the Weyl tensor and therefore obtain results which are conformal in $D = 4$ only. This ambiguity

[7] For instance, the scalar field theory with interaction $\mathcal{L}_{\text{int}} = \lambda \phi^6$ is not renormalizable, because of the mass dimension $[\lambda] = -2$ in four space-time dimensions. For more on renormalization of quantum fields see also [43].

[8] Technically, in the spirit of (6.51), the a_i appearing here should be written as $a_{i,0} = a_i + \delta a_i$, where δa_i denotes the divergence and a_i is finite. However, as none of the counterterms terms are present in the tree level effective action, both a_i and δa_i are of order $\mathcal{O}(\hbar)$. As finite terms can be arbitrarily distributed between a_i and δa_i without physical consequences, the symbol a_i shall be used as the coefficient for the counterterm, which is determined only up to finite ($\mathcal{O}(D-4)^0$) corrections.

in choosing the counterterm will return in the trace anomaly and is further discussed in section 6.5. The symbol E is used to denote the term known as the Gauss-Bonnet integrand which is a topological contribution in $D = 4$, [45].

Due to the enhanced symmetry induced by torsion, more general combinations are allowed in the theory under consideration. As found in chapter 3, the Riemann tensor is invariant under a Weyl rescaling, making each R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ a valid counterterm on their own. More precisely, the conformal weight of each term is $w = -4$, which implies that the action is invariant in $D = 4$ as the square root of the metric determinant has a weight of $+D$. The same holds true for $\overline{\Box}R$ and the kinetic term $\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$, both having a weight of -4.

Insertion of the Ricci scalar, Ricci tensor, Riemann tensor results in:

$$\begin{aligned}
S_{\text{ct1}} = \int d^D x \sqrt{-g}\, a_1 \Big\{ & \mathring{R}^2 + 4\mathring{R}(D-1)\,\mathring{\nabla}_\lambda T^\lambda - 2\mathring{R}(D-1)(D-2)T_\lambda T^\lambda \\
& + 4(D-1)^2(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3) \Big\} \qquad (6.58)
\end{aligned}$$

$$\begin{aligned}
S_{\text{ct2}} = \int d^D x \sqrt{-g}\, a_2 \Big\{ & \mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + 2\mathring{R}\,\mathring{\nabla}_\lambda T^\lambda - 2(D-2)\mathring{R}\,T_\lambda T^\lambda + 2(D-2)\mathring{R}^{\mu\nu}\mathring{\nabla}_\nu T_\mu + \\
& 2(D-2)\mathring{R}^{\mu\nu}T_\mu T_\nu + (3D-4)(\mathring{\nabla}_\lambda T^\lambda)^2 - (D-2)^2(T_\mu \mathring{\Box} T^\mu) + \mathcal{O}(T_\alpha^3) \Big\} \qquad (6.59)
\end{aligned}$$

$$\begin{aligned}
S_{\text{ct3}} = \int d^D x \sqrt{-g}\, a_3 \Big\{ & \mathring{R}_{\mu\nu\rho\sigma}\mathring{R}^{\mu\nu\rho\sigma} + 8\mathring{R}^{\mu\nu}(\mathring{\nabla}_\nu T_\mu + T_\mu T_\nu) - 4\mathring{R}\,T_\lambda T^\lambda + 4(\mathring{\nabla}_\lambda T^\lambda)^2 \\
& - 4(D-2)(T_\mu \mathring{\Box} T^\mu) + \mathcal{O}(T_\alpha^3) \Big\} \qquad (6.60)
\end{aligned}$$

$$\begin{aligned}
S_{\text{ct4}} = \int d^D x \sqrt{-g}\, a_4 \Big\{ & \mathring{\Box}\mathring{R} + 2(D-1)\mathring{\Box}(\mathring{\nabla}_\lambda T^\lambda) + (D-4)\mathring{R}(\mathring{\nabla}_\lambda T^\lambda) - 2(D-4)\mathring{R}\,T_\lambda T^\lambda \\
& + 2(D-1)(D-4)(\mathring{\nabla}_\lambda T^\lambda)^2 - (D-1)(D-2)\mathring{\Box}(T_\lambda T^\lambda) + \mathcal{O}(T_\alpha^3) \Big\} \qquad (6.61)
\end{aligned}$$

$$S_{\text{ct5}} = \int d^D x \sqrt{-g}\, 2a_5 \big[- T_\mu \mathring{\Box} T^\mu + \mathring{R}^{\mu\nu}T_\mu T_\nu - (\mathring{\nabla}_\lambda T^\lambda)^2 \big] \qquad (6.62)$$

Expansion up to second order in T_μ is sufficient here, because at most two functional derivatives with respect to torsion will be taken. In order to get to the fourth result, the definition for the conformal covariant derivative (3.13) was used:

$$\begin{aligned}
\overline{\Box}R &= g^{\mu\nu}\overline{\nabla}_\mu \overline{\nabla}_\nu R \\
&= g^{\mu\nu}(\nabla_\mu + 3T_\mu)(\nabla_\nu + 2T_\nu)R \\
&= g^{\mu\nu}\Big[(\mathring{\nabla}_\mu + 3T_\mu)(\mathring{\nabla}_\nu + 2T_\nu)R - (g_{\mu\nu}T^\sigma - \delta^\sigma{}_\mu T_\nu)(\mathring{\nabla}_\sigma + 2T_\sigma)R\Big] \\
&= \mathring{\Box}R + 2(\mathring{\nabla}_\lambda T^\lambda)R - (D-6)T^\lambda\,\mathring{\nabla}_\lambda R - 2(D-4)R\,T_\lambda T^\lambda \qquad (6.63)
\end{aligned}$$

Equation (6.61) now follows by inserting the Ricci scalar and integration by parts. Obviously, each term in this action is either proportional to $(D-4)$ or gives a boundary contribution and hence it cannot be used for renormalization of the theory. One can safely set $a_4 = 0$.

A careful inspection immediately results in another simplification. The term $T_\mu \mathring{\Box} T^\mu$ present in S_{ct2}, S_{ct3} and S_{ct5} will induce a contribution to the TT-vertex function

$$ {}_{TT}\Gamma^{\alpha\beta}(x,x') \sim \frac{\delta^2}{\delta T_\alpha(x)\delta T_\beta(x')} \int d^D x \sqrt{-g}\, T_\mu \mathring{\Box} T^\mu \sim \tilde{g}^{\alpha\beta'}\, \mathring{\Box} \frac{\delta^D(x-x')}{\sqrt{-g}} \,, $$

which is not present as divergence in (6.41). It is therefore necessary to remove it from the counterterm action which is done by forming the linear combinations

$$ \begin{aligned} S_{\text{ct2,5}} &\equiv S_{\text{ct2}} + S_{\text{ct5}}|_{a_5 = -a_2(D-2)^2/2} = \\ &= \int d^D x \sqrt{-g}\, a_2 \Big\{ \mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + 2\mathring{R}\big(\mathring{\nabla}_\lambda T^\lambda - (D-2)T_\lambda T^\lambda\big) + 2(D-2)\mathring{R}^{\mu\nu}\mathring{\nabla}_\nu T_\mu \\ &\qquad - (D-2)(D-4)\mathring{R}^{\mu\nu}T_\mu T_\nu + D(D-1)(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3)\Big\} \,, \end{aligned} \tag{6.64} $$

$$ \begin{aligned} S_{\text{ct3,5}} &\equiv S_{\text{ct3}} + S_{\text{ct5}}|_{a_5 = -2(D-2)a_3} = \\ &= \int d^D x \sqrt{-g}\, a_3 \Big\{ \mathring{R}_{\mu\nu\rho\sigma}\mathring{R}^{\mu\nu\rho\sigma} + 8\mathring{R}^{\mu\nu}\nabla_\nu T_\mu - 4(D-4)\mathring{R}^{\mu\nu}T_\mu T_\nu - 4\mathring{R}\, T_\lambda T^\lambda \\ &\qquad + 4(D-1)(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3)\Big\} \,, \end{aligned} \tag{6.65} $$

so that only three independent coefficients a_1, a_2 and a_3 are left. Just as in the theory without torsion, conformal symmetry will be the guiding principle to determine their relative values. Because the counterterm actions are by construction conformal in $D=4$ only, they can violate the Ward identity (6.1) by terms $\propto (D-4)$. This is indeed the case and demanding cancellation of these violations for general D will lead to two linear combinations, very much like the restriction to the Weyl tensor and Gauss-Bonnet terms in (6.56), (6.57). Nonetheless, one should not confuse these two, admittedly, similar results. The fact that only (6.56), (6.57) are allowed for a conformal theory in general relativity is a consequence of the transformation of the curvature tensors under a Weyl rescaling and, in particular, the Gauss-Bonnet term is conformal in $D=4$ only. In the theory presented here, on the other hand, the curvature tensors are invariant and one thus seeks for *a priori* different combinations obeying the Ward identities in any number of space-time dimensions D.

The vertex functions for the individual counterterms and their failure to be conformal in general D are computed in the appendices C and D. By inspection of the tensor structures in the divergences of the regularized gg vertex (6.49) *and* demanding conformal invariance in arbitrary space-time dimension, one arrives at the following results: The first block of divergences is renormalized by a term proportional to

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + \frac{4D-6}{D(D-1)}R^2 + 2(D-2)(D-3)\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} =$$
$$= E - \frac{(D-2)(D-3)}{D(D-1)}R^2 + 2(D-2)(D-3)\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}\,, \tag{6.66}$$

where in the second line the definition of the Gauss-Bonnet term (6.57) has been used. It seems reasonable to write it this way, because E yields a mere boundary contribution in $D = 4$ and its presence is only required to make the term conformal. With this choice, also the gT- and TT-vertices can be rendered finite. The second block of divergences in (6.49) is renormalized by the square of the Weyl tensor,

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{4}{D-2}R_{\mu\nu}R^{\mu\nu} - \frac{2}{(D-1)(D-2)}R^2\,, \tag{6.67}$$

which is independent of torsion and will thus not change the other vertices. The final counterterm action reads

$$S_{ct} = \int d^Dx\sqrt{-g}\left\{b_1\Big[R^2 - \frac{D(D-1)}{(D-2)(D-3)}E - 2D(D-1)\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}\Big] + b_2 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right\} \tag{6.68}$$

where the coefficients are [9]

$$b_1 = \frac{\mu^{D-4}}{(4\pi)^{D/2}}\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \frac{\Gamma\left(\frac{D}{2}-1\right)}{8(D-1)^2(D-3)^2(D-4)} \tag{6.69}$$
$$b_2 = \frac{\mu^{D-4}}{(4\pi)^{D/2}}\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{16(D+1)(D-1)(D-3)^3(D-4)}\,. \tag{6.70}$$

The final vertex corrections induced by these choices are also given in appendix C. In order to add them to the original theory, one has to expand $H^D = H^4\mu^{D-4}\left(1 + (D-4)\ln\frac{H}{\mu}\right)$ plus higher order terms, which will potentially induce a scale dependence of H on μ. This is also called running of the coupling constants and is typically used to explore the infrared ($\mu \to 0$) or ultraviolet ($\mu \to \infty$) behavior of a quantum theory. Perhaps the most famous example is asymptotic freedom of QCD [46], which refers to the phenomenon that the elementary particles of QCD, are essentially free particles at very high energies. It is predicted by renormalization as the induced running coupling constants vanish at high energies.

[9] The same comment as in footnote 8 applies.

Adding the counterterm vertex (C.20) to the regulated result (6.41) yields:

$$
{}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ren}} = \frac{1}{32\pi^2}(1+6\xi)^2 \ln\tfrac{H^2}{\mu^2}\left(-\mathring{\nabla}^{\alpha}\mathring{\nabla}'^{\beta} + 4H^2 g^{\alpha\beta}\right)\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{1}{16\pi^2}H^2 g^{\alpha\beta}\Big\{-(1+6\xi) + 2(1+6\xi)^2\left[2 - 2\psi(1) + \psi\left(\tfrac{1}{2}\pm\nu_4\right)\right]\Big\}\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{i}{2}(1+6\xi)^2\,\mathring{\nabla}^{\alpha}\mathring{\nabla}'^{\beta}(i\Delta(x,x'))^2_{\mathsf{fin}} \tag{6.71}
$$

Renormalizing (6.45) by means of (C.21) results in:

$$
{}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ren}} = \frac{1}{192\pi^2}(1+6\xi)^2\ln\tfrac{H^2}{\mu^2}\left(g_{\mu\nu}\mathring{\Box} - \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu} + \mathring{R}_{\mu\nu}\right)\mathring{\nabla}'^{\beta}\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{1}{16\pi^2}(1+6\xi)^2 H^2 g_{\mu\nu}\mathring{\nabla}'^{\beta}\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{i}{4}\frac{H^4}{(4\pi)^4}\mathring{\nabla}'^{\beta}\Big\{(1+6\xi)^2\left[-\frac{1}{3}H^2 g_{\mu\nu}\frac{\mathring{\Box}}{H^2}g(y) + \frac{1}{3}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}g(y) - H^2 g_{\mu\nu}g(y) + 4\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\frac{4}{y}\right]
+ 2(1+6\xi)\left(\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\right)\left(\partial_{\rho}\frac{8}{y}\,\partial_{\sigma}f_1(y) + \partial_{\rho}f_0(y)\,\partial_{\sigma}f_0(y)\right)
+ 2\xi(1+6\xi)\left(\mathring{R}_{\mu\nu} - g_{\mu\nu}\mathring{\Box} + \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\right)\left(\frac{8}{y}f_0(y) + f_0^2(y)\right)\Big\} \tag{6.72}
$$

At last, the renormalized gg vertex becomes (symmetrization in $(\mu\nu)$ and $(\rho\sigma)$ is again implicit):

$$
{}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\mathsf{ren}} =
$$

$$
= \frac{1}{1152\pi^2}(1+6\xi)^2\ln\tfrac{H^2}{\mu^2}\Big\{-\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + H^2\left(g_{\mu\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + g'_{\rho\sigma}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\right)\frac{\mathring{\Box}}{H^2}
- 3H^2(g_{\mu\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + g'_{\rho\sigma}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}) - H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2} + 9H^4 g_{\mu\nu}g'_{\rho\sigma}
- 6H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2} - 12H^2\tilde{g}_{\mu\rho'}\mathring{\nabla}_{\nu}\mathring{\nabla}'_{\sigma}\Big\}\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{1}{720}\frac{1}{(4\pi)^2}\ln\tfrac{H^2}{\mu^2}\Big\{-2\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\mathring{\nabla}'_{\sigma}\mathring{\nabla}'_{\rho} - H^2\left(g_{\mu\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + g'_{\rho\sigma}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\right)\frac{\mathring{\Box}}{H^2} + H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2}
+ 6H^2(g_{\mu\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + g'_{\rho\sigma}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}) + 3H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2} - 18H^4 g_{\mu\nu}g'_{\rho\sigma} - 6\tilde{g}_{\mu\rho'}\nabla_{\nu}\nabla'_{\sigma}\frac{\mathring{\Box}}{H^2}
+ 12H^2\tilde{g}_{\mu\rho'}\nabla_{\nu}\nabla'_{\sigma} - 3\frac{1}{2}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2} + 6H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2}\Big\}\frac{\delta^4(x-x')}{\sqrt{-g}}
+ \frac{1}{64\pi^2}\Big\{\left[-\tfrac{2}{15} - \tfrac{1}{6}(1+6\xi)^2 - 2\xi(1+6\xi)\left(-\tfrac{1}{2} - 2\psi(1) + \psi\left(\tfrac{1}{2}\pm\nu_4\right)\right)\right]\times
\times H^2(g_{\mu\nu}\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + g'_{\rho\sigma}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu})
+ \left[\tfrac{5}{15} + \tfrac{1}{3}(1+6\xi) - \tfrac{2}{3}(1+6\xi)^2 - (2\xi - \tfrac{1}{2})(1+6\xi)\left(2\psi(1) - \psi\left(\tfrac{1}{2}\pm\nu_4\right)\right)\right]H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}
$$

$$+\Big[-\tfrac{11}{10}-\tfrac{4}{3}(1+6\xi)+\tfrac{5}{2}(1+6\xi)^2+12\xi(1+6\xi)\big(2\psi(1)-\psi\big(\tfrac{1}{2}\pm\nu_4\big)\big)\Big]H^4 g_{\mu\nu}g'_{\rho\sigma}$$

$$+\Big[-\tfrac{4}{15}+\tfrac{5}{3}(1+6\xi)-\tfrac{4}{3}(1+6\xi)^2+4\xi(1+6\xi)\big(2\psi(1)-\psi\big(\tfrac{1}{2}\pm\nu_4\big)\big)\Big]H^2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma$$

$$+\Big[-\tfrac{1}{6}+\tfrac{5}{6}(1+6\xi)-\tfrac{2}{3}(1+6\xi)^2+2\xi(1+6\xi)\big(2\psi(1)-\psi\big(\tfrac{1}{2}\pm\nu_4\big)\big)\Big]H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2}$$

$$+\Big[1-5(1+6\xi)+(1+6\xi)^2-12\xi(1+6\xi)\big(2\psi(1)+\psi\big(\tfrac{1}{2}\pm\nu_4\big)\big)\Big]H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Big\}\frac{\delta^4(x-x')}{\sqrt{-g}}$$

$$+\frac{i}{4}\frac{H^4}{(4\pi)^4}\Big\{\mathring{\nabla}_\mu\mathring{\nabla}_\nu\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\Big[\Big(\tfrac{1}{90}+\tfrac{1}{18}(1+6\xi)^2\Big)g(y)+\Big(-\tfrac{2}{3}(1+6\xi)+\tfrac{4}{3}(1+6\xi)^2\Big)\tfrac{4}{y}$$

$$+\Big(\tfrac{8}{3}(1+6\xi)-4(1+6\xi)^2\Big)\ln(\tfrac{y}{4})+2\xi^2\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+H^2\Big(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\Big)\frac{\mathring{\Box}}{H^2}\Big[\Big(\tfrac{1}{180}-\tfrac{1}{18}(1+6\xi)^2\Big)g(y)$$

$$+\Big(-\tfrac{1}{3}(1+6\xi)-\tfrac{2}{3}(1+6\xi)^2\Big)\tfrac{4}{y}-2\xi^2\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+H^2\Big(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\Big)\Big[\Big(-\tfrac{1}{30}+\tfrac{1}{3}(1+6\xi)-\tfrac{1}{6}(1+6\xi)^2\Big)g(y)+6\xi^2\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2}\Big[\Big(-\tfrac{1}{180}+\tfrac{1}{18}(1+6\xi)^2\Big)g(y)+\big(\tfrac{1}{8}+2\xi^2\big)\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\Big[\Big(-\tfrac{1}{60}-\tfrac{5}{6}(1+6\xi)+\tfrac{1}{3}(1+6\xi)^2\Big)g(y)+(1+6\xi)\tfrac{8}{y}$$

$$+(6\xi-12\xi^2)\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+H^4 g_{\mu\nu}g'_{\rho\sigma}\Big[\Big(\tfrac{1}{10}+2(1+6\xi)-\tfrac{5}{2}(1+6\xi)^2\Big)g(y)+\Big(-6(1+6\xi)+6(1+6\xi)^2\Big)\frac{4}{y}$$

$$+90\xi^2\Big(\tfrac{8}{y}f_0(y)+f_0^2(y)\Big)\Big]$$

$$+\tfrac{1}{30}H^2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma\tfrac{\mathring{\Box}}{H^2}g(y)$$

$$+H^2\tilde{g}_{\mu\rho'}\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma\Bigg[\big(-\tfrac{1}{15}+\tfrac{2}{3}(1+6\xi)\big)g(y)-96\xi(1+6\xi)\frac{\ln(\frac{y}{4})}{y}$$

$$+\Big[-2+\tfrac{14}{3}(1+6\xi)+\tfrac{14}{3}(1+6\xi)^2-24\xi(1+6\xi)\big(2\psi(1)-\psi\big(\tfrac{1}{2}\pm\nu_4\big)\big)\Big]\frac{4}{y}\Bigg]$$

$$+\tfrac{1}{60}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\tfrac{\mathring{\Box}}{H^2}\tfrac{\mathring{\Box}}{H^2}g(y)+\big(-\tfrac{1}{30}+\tfrac{1}{3}(1+6\xi)\big)H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\tfrac{\mathring{\Box}}{H^2}g(y)$$

$$+\Big(-2(1+6\xi)+2(1+6\xi)^2\Big)H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y)$$

$$+4\mathring{\nabla}_\mu\mathring{\nabla}'_\rho\frac{4}{y}\,\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma f_2(y)-8(1+6\xi)\,\mathring{\nabla}_\mu\mathring{\nabla}'_\rho\ln\frac{y}{4}\,\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma f_1(y)+2\mathring{\nabla}_\mu\mathring{\nabla}'_\rho f_1(y)\,\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma f_1(y)$$

$$+\Big(-\tfrac{1}{2}H^2 g_{\mu\nu}\tfrac{\mathring{\Box}}{H^2}-12\xi H^2 g_{\mu\nu}\Big)\Big(\partial'_\rho\tfrac{8}{y}\,\partial'_\sigma f_1(y)+\partial'_\rho f_0(y)\,\partial'_\sigma f_0(y)\Big)$$

$$+\Big(-\tfrac{1}{2}H^2 g'_{\rho\sigma}\tfrac{\mathring{\Box}}{H^2}-12\xi H^2 g'_{\rho\sigma}\Big)\Big(\partial_\mu\tfrac{8}{y}\,\partial_\nu f_1(y)+\partial_\mu f_0(y)\,\partial_\nu f_0(y)\Big)$$

$$+2\xi(-g_{\mu\nu}\mathring{\Box}+\mathring{\nabla}_\mu\mathring{\nabla}_\nu+3H^2 g_{\mu\nu})\Big(\delta^\gamma{}_\rho\delta^\delta{}_\sigma-\tfrac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\Big)\Big(\partial'_\gamma\tfrac{8}{y}\,\partial'_\delta f_1(y)+\partial'_\gamma f_0(y)\,\partial'_\delta f_0(y)\Big)$$

$$+ 2\xi(-g'_{\rho\sigma}\mathring{\Box}' + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + 3H^2 g'_{\rho\sigma})\left(\delta^\alpha{}_\mu\delta^\beta{}_\nu - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\left(\partial_\alpha\tfrac{8}{y}\,\partial_\beta f_1(y) + \partial_\alpha f_0(y)\,\partial_\beta f_0(y)\right)\Bigg\} \tag{6.73}$$

It remains to be shown that these finite vertices are still conformal, that is, obey the Ward identities.

6.3 Ward Identities

Recall the two derived Ward identities (6.17) and (6.18) that ought to be satisfied by the one-loop vertex functions:

$$\mathring{\nabla}_\alpha\, {}_{TT}\Gamma^{\alpha\beta}(x,x') + 2g^{\mu\nu}(x)\, {}_{gT}\Gamma^\beta_{\mu\nu}(x,x') = 0$$
$$\mathring{\nabla}_\alpha\, {}_{gT}\Gamma^\alpha_{\rho\sigma}(x',x) + 2\frac{\delta^D(x-x')}{\sqrt{-g}}\, {}_g\Gamma_{\rho\sigma}(x) + 2g^{\mu\nu}(x)\, {}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x') = 0$$

The trace of the energy momentum tensor descends to a trace of the gT- and gg-vertex functions. For the first Ward identity, it can be computed from (6.72) by means of the identities given in appendix A.2:

$$\begin{aligned} g^{\mu\nu}\, {}_{gT}\Gamma^\beta_{\mu\nu}(x,x')_{\mathsf{ren}} = &\\ = \frac{\ln\frac{H^2}{\mu^2}}{64\pi^2}&(1+6\xi)^2\left(4H^2 + \mathring{\Box}\right)\mathring{\nabla}'^\beta\frac{\delta^4(x-x')}{\sqrt{-g}} \\ + \frac{H^2}{32\pi^2}&\left\{-(1+6\xi) + 2(1+6\xi)^2\left[2 - 2\psi(1) + \psi\left(\tfrac{1}{2}\pm\nu_4\right)\right]\right\}\mathring{\nabla}'^\beta\frac{\delta^4(x-x')}{\sqrt{-g}} \\ - \frac{i}{4}&(1+6\xi)^2\,\mathring{\nabla}'^\beta\mathring{\Box}(i\Delta(x,x'))^2_{\mathsf{fin}} \end{aligned} \tag{6.74}$$

On the other hand, the covariant divergence of (6.71) becomes (using $\partial^x_\alpha(g^{\alpha\beta}\delta^D(x-x')) = -\partial^\beta_{x'}\delta^D(x-x')$)

$$\begin{aligned} \mathring{\nabla}_\alpha\, {}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ren}} = &\\ = \frac{\ln\frac{H^2}{\mu^2}}{32\pi^2}&(1+6\xi)^2\left(-4H^2 - \mathring{\Box}\right)\mathring{\nabla}'^\beta\frac{\delta^4(x-x')}{\sqrt{-g}} \\ - \frac{H^2}{16\pi^2}&\left\{-(1+6\xi) + 2(1+6\xi)^2\left[2 - 2\psi(1) + \psi\left(\tfrac{1}{2}\pm\nu_4\right)\right]\right\}\mathring{\nabla}'^\beta\frac{\delta^4(x-x')}{\sqrt{-g}} \\ + \frac{i}{2}&(1+6\xi)^2\,\mathring{\nabla}'^\beta\mathring{\Box}(i\Delta(x,x'))^2_{\mathsf{fin}}\,, \end{aligned} \tag{6.75}$$

which is indeed twice the negative of (6.74). Thus, the first Ward identity is satisfied. Similarly, the trace appearing in the second Ward identity is computed by using the bitensor traces given in appendix A.3:

$$
\begin{aligned}
&g^{\mu\nu}{}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\text{ren}} = \\
&\quad = \frac{1}{384\pi^2}\,(1+6\xi)^2 \ln\frac{H^2}{\mu^2}\left[-g'_{\rho\sigma}\mathring{\Box}\mathring{\Box} + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box} - 3H^2 g'_{\rho\sigma}\mathring{\Box}\right]\frac{\delta^4(x-x')}{\sqrt{-g}} \\
&\quad - \frac{1}{32\pi^2}\,(1+6\xi)^2\, H^2 g'_{\rho\sigma}\mathring{\Box}\frac{\delta^4(x-x')}{\sqrt{-g}} \\
&\quad + \frac{i}{4}\frac{H^4}{(4\pi)^4}\Bigg\{ (1+6\xi)^2\left[\frac{1}{6}g'_{\rho\sigma}\mathring{\Box}\mathring{\Box}g(y) - \frac{1}{6}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box}g(y) + \frac{1}{2}H^2 g'_{\rho\sigma}\mathring{\Box}g(y) - 2\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box}\frac{4}{y}\right] \\
&\quad - (1+6\xi)\left(\delta^\gamma{}_\rho\delta^\delta{}_\sigma - \frac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\right)\mathring{\Box}\left(\partial'_\gamma\frac{8}{y}\,\partial'_\delta f_1(y) + \partial'_\gamma f_0(y)\,\partial'_\delta f_0(y)\right) \\
&\quad - \xi\,(1+6\xi)\left(3H^2 g'_{\rho\sigma} - g'_{\rho\sigma}\mathring{\Box} + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\right)\mathring{\Box}\left(\frac{8}{y}f_0(y) + f_0^2(y)\right)\Bigg\}
\end{aligned}
\tag{6.76}
$$

Because neither the original theory nor the counterterms induce a one-point function $\propto {}_g\Gamma_{\mu\nu}$, the middle term of the second Ward identity can be ignored. The covariant divergence of the renormalized gT vertex (6.72) on the other hand is

$$
\begin{aligned}
&\mathring{\nabla}_\alpha\,{}_{gT}\Gamma^\alpha_{\rho\sigma}(x',x)_{\text{ren}} = \\
&\quad = \frac{1}{192\pi^2}\,(1+6\xi)^2 \ln\frac{H^2}{\mu^2}\left[g'_{\rho\sigma}\mathring{\Box}\mathring{\Box} - \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box} + 3H^2 g'_{\rho\sigma}\mathring{\Box}\right]\frac{\delta^4(x-x')}{\sqrt{-g}} \\
&\quad + \frac{1}{16\pi^2}\,(1+6\xi)^2\, H^2 g'_{\rho\sigma}\mathring{\Box}\frac{\delta^4(x-x')}{\sqrt{-g}} \\
&\quad + \frac{i}{4}\frac{H^4}{(4\pi)^4}\Bigg\{ (1+6\xi)^2\left[-\frac{1}{3}g'_{\rho\sigma}\mathring{\Box}\mathring{\Box}g(y) + \frac{1}{3}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box}g(y) - H^2 g'_{\rho\sigma}\mathring{\Box}g(y) + 4\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\mathring{\Box}\frac{4}{y}\right] \\
&\quad + 2\,(1+6\xi)\left(\delta^\gamma{}_\rho\delta^\delta{}_\sigma - \frac{1}{2}g'_{\rho\sigma}g'^{\gamma\delta}\right)\mathring{\Box}\left(\partial'_\gamma\frac{8}{y}\,\partial'_\delta f_1(y) + \partial'_\gamma f_0(y)\,\partial'_\delta f_0(y)\right) \\
&\quad + 2\xi\,(1+6\xi)\left(3H^2 g'_{\rho\sigma} - g'_{\rho\sigma}\mathring{\Box} + \mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma\right)\mathring{\Box}\left(\frac{8}{y}f_0(y) + f_0^2(y)\right)\Bigg\},
\end{aligned}
\tag{6.77}
$$

which is also twice the negative of (6.76). That is, both Ward identities are satisfied by the one-loop vertices and no anomalous terms are found.

6.4 Interlude: A Derivation of the Trace Anomaly

This part is a sketch of the calculation of the trace anomaly for a scalar field following chapters 3 and 6 of Birrell and Davies' book [34]. The aim is to be as understandable as possible and if in doubt, one should go back to the original source. Many authors refer to either their or very similar derivations when citing the anomaly. While being arguably the most technical chapter of the thesis, it will be of great use in understanding the nature of the anomaly, as well as why the results (6.74), (6.76) deviate from it even in their local pieces. In this section, the spacetime is general and does not necessarily have to be de Sitter. Furthermore, Birrell and Davies work in the realm of general relativity, and thus there will be no torsion. These simplifications are sufficient for the present purpose of highlighting the main steps that lead to the anomaly. The same analysis for non-vanishing torsion is performed in [1] and the derived anomaly has qualitatively the same form as (6.97).

It turns out that the anomaly is a consequence of renormalization and one thus seeks for an expression of the divergences of the effective action. In general, these divergences are caused by the ultraviolet (or short distance) behavior of the scalars Green's function. This implies that a local construction will be sufficient and a preferred choice of local reference frame are Riemann normal coordinates (RNC). By definition, in a RNC system of x around the point x', geodesics are locally straight lines, that is, the geodesic equation reduces to

$$\frac{d^2x^\mu}{d\tau^2} = 0\,. \tag{6.78}$$

Equivalently, after introducing $z^\alpha(x,x') = x^\alpha - x'^\alpha$ as normal coordinates, one may expand the metric tensor around $\eta_{\mu\nu}$,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \tfrac{1}{3}\mathring{R}_{\mu\alpha\nu\beta}\, z^\alpha z^\beta - \tfrac{1}{6}\mathring{\nabla}_\gamma \mathring{R}_{\mu\alpha\nu\beta}\, z^\alpha z^\beta z^\gamma + \dots \tag{6.79}$$

such that $g_{\mu\nu}(x') = \eta_{\mu\nu}$ and the Christoffel symbols vanish at x' because there is no linear term in z. The expansion of a function into the coordinate z will therefore also yield an expansion in the curvature tensors. To calculate the anomaly, it turns out to be more convenient to work with a slightly modified scalar propagator

$$\mathcal{G}_F(x,x') := \sqrt[4]{-g}\,G_F(x,x')\,, \tag{6.80}$$

where G_F is the Feynman propagator of the massive scalar field with equation of motion

$$[\mathring{\Box} - m^2 + \xi\mathring{R}]\phi = 0\,.$$

The inclusion of the mass helps to avoid infrared singularities that can occur when $\mathring{R}$ vanishes. The Fourier transformation of the Green's function using RNC is defined by

$$\mathcal{G}_F(x,x') = \int \frac{d^Dk}{(2\pi)^D} e^{-ikz}\mathcal{G}_F(k)\,, \tag{6.81}$$

with $kz = \eta^{\mu\nu}k_\mu z_\nu$ owing the locally flat coordinate system. Next, Birell and Davies use the asymptotic momentum space expansion of this propagator [47], which follows from the equation of motion, to conclude

$$\mathcal{G}_F(x,x') = \int \frac{d^Dk}{(2\pi)^D} e^{-ikz}\left[a_0(x,x') + a_1(x,x')\frac{\partial}{\partial m^2} + a_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\right]\frac{1}{k^2+m^2}\,, \tag{6.82}$$

plus higher derivative terms. The coefficients $a_j(x,x')$ are up to fourth order in metric derivatives given by

$$a_0(x,x') = 1 \tag{6.83}$$

$$a_1(x,x') = \tfrac{1}{6}\,(1+6\xi)\,\mathring{R} - \tfrac{1}{12}\,(1+6\xi)\,\mathring{\nabla}_\alpha \mathring{R}\, z^\alpha - \tfrac{1}{3} a_{\alpha\beta}\, z^\alpha z^\beta \tag{6.84}$$

$$a_2(x,x') = \tfrac{1}{12}\,(1+6\xi)^2\,\mathring{R}^2 + \tfrac{1}{3} a^\lambda{}_\lambda \tag{6.85}$$

with

$$a_{\alpha\beta} = \tfrac{1}{12}\,(1+6\xi)\,\mathring{\nabla}_\alpha \mathring{\nabla}_\beta \mathring{R} + \tfrac{1}{120}\mathring{\nabla}_\alpha \mathring{\nabla}_\beta \mathring{R} - \tfrac{1}{40}\mathring{\Box}\mathring{R}_{\alpha\beta} - \tfrac{1}{30}\mathring{R}_\alpha{}^\lambda \mathring{R}_{\lambda\beta} + \tfrac{1}{60}\mathring{R}_{\kappa\alpha\lambda\beta}\mathring{R}^{\kappa\lambda} + \tfrac{1}{60}\mathring{R}^{\lambda\mu\kappa}{}_\alpha \mathring{R}_{\lambda\mu\kappa\beta}\,. \tag{6.86}$$

All geometric quantities on the right-hand sides are to be evaluated at the anchor point x'. Also notice how (6.82) reduces (up to a pole prescription) to the familiar result

$$G_F(k) = \mathcal{G}_F(k) = \frac{1}{k^2+m^2} \tag{6.87}$$

in the special relativistic limit $g_{\mu\nu} \to \eta_{\mu\nu}$. By using the integral formula of the flat space propagator

$$\frac{1}{k^2+m^2+i\epsilon} = -i\int_0^\infty ds\, e^{is(k^2+m^2+i\epsilon)} \tag{6.88}$$

in (6.82), and performing the k integration, one arrives at

$$\mathcal{G}_F(x,x') = \frac{-i}{(4\pi)^{D/2}}\int_0^\infty ds \frac{1}{(is)^{D/2}} e^{-ism^2+isz^2/4}\, F(x,x';is)\,, \tag{6.89}$$

which is known as Schwinger-DeWitt proper time representation. While this derivation only holds in Riemann normal coordinates, the general expression is only slightly more complicated, but not required here. Equation (6.82) implies that $F(x,x';is)$ admits the expansion

$$F(x,x';is) = a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + \ldots$$

To obtain the implications for the one-loop effective action Γ one needs the standard result from quantum field theory (see Birell, Davies (6.25))

$$\Gamma = \frac{-i}{2}\int d^D x\sqrt{-g}\, \ln(-G_F(x,x))\,. \tag{6.90}$$

Finally, the logarithm can be recast as m^2 integration so that at last the general effective Lagrangian takes the form

$$L_{\text{eff}} = \frac{i}{2}\lim_{x'\to x}\int_{m^2}^\infty dm^2\, G_F(x,x')\,,$$

where $G_F(x,x')$ is up to a factor of the metric determinant given by (6.89). The divergent terms are determined by inserting the Schwinger-DeWitt expansion and performing the m^2 integration, which leads to

$$\begin{aligned} L_{\text{eff}} &= \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\sum_{j=0}^{\infty} a_j(x)\int_0^{\infty}(is)^{j-1-D/2}\,e^{-ism^2}\,ids \\ &= \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\sum_{j=0}^{\infty} a_j(x)(m^2)^{D/2-j}\Gamma(j-D/2)\,, \end{aligned} \tag{6.91}$$

and $a_j(x) \equiv a_j(x,x)$. In $D=4$ dimensions, $j=0,1,2$ constitute the divergent terms which follow from the poles in the Γ function at non-positive integers. However, by conformal invariance m^2 has to be set to 0 at last (ignoring the already mentioned infrared divergences) and hence only $a_2(x)$ will survive. The divergent effective action that ought to be renormalized is thus

$$\Gamma_{\text{div}} = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\Gamma(2-D/2)\int d^Dx\sqrt{-g}\,a_2(x)\,, \tag{6.92}$$

with

$$a_2(x) = \frac{1}{180}\mathring{R}_{\alpha\beta\gamma\delta}\mathring{R}^{\alpha\beta\gamma\delta} - \frac{1}{180}\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} - \frac{1}{30}(1+5\xi)\,\mathring{\Box}\mathring{R} + \frac{1}{72}(1+6\xi)^2\,\mathring{R}^2\,. \tag{6.93}$$

Birrell and Davies drop the $\mathring{\Box}\mathring{R}$ and $\mathring{R}^2$ terms because the former is a boundary contribution and the latter vanishes for conformal coupling[10]. The other two summands can be rewritten according to

$$\frac{1}{180}\mathring{R}_{\alpha\beta\gamma\delta}\mathring{R}^{\alpha\beta\gamma\delta} - \frac{1}{180}\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} = \frac{1}{120}\,{}_4\mathring{C}_{\alpha\beta\gamma\delta}\,{}_4\mathring{C}^{\alpha\beta\gamma\delta} - \frac{1}{360}\mathring{E}\,,$$

where ${}_4\mathring{C}_{\alpha\beta\gamma\delta}$ denotes the Weyl tensor in four dimensions and $\mathring{E} = \mathring{R}_{\alpha\beta\gamma\delta}\mathring{R}^{\alpha\beta\gamma\delta} - 4\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} + \mathring{R}^2$ is the Gauss-Bonnet integrand. From here, the anomaly arises as follows: By means of renormalization, a counterterm action

$$\Gamma_{ct} = \frac{\mu^{D-4}}{D-4}\frac{1}{(4\pi)^{D/2}}\int d^Dx\sqrt{-g}\left[\frac{1}{120}\,{}_4\mathring{C}_{\alpha\beta\gamma\delta}\,{}_4\mathring{C}^{\alpha\beta\gamma\delta} - \frac{1}{360}\mathring{E}\right] \tag{6.94}$$

is added to the effective action that cancels the divergences. Upon using the relations

$$\frac{-2g^{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}\,{}_4\mathring{C}_{\alpha\beta\gamma\delta}\,{}_4\mathring{C}^{\alpha\beta\gamma\delta} = (D-4)\left({}_4\mathring{C}_{\alpha\beta\gamma\delta}\,{}_4\mathring{C}^{\alpha\beta\gamma\delta} + \tfrac{2}{3}\mathring{\Box}\mathring{R}\right) \tag{6.95}$$

$$\frac{-2g^{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}\mathring{E} = (D-4)\mathring{E}\,, \tag{6.96}$$

[10]More precisely, $(1+6\xi)^2 \propto (D-4)^2$ so that the $1/(D-4)$ of the divergence cancels and the result still vanishes when $D \to 4$.

one finds that the renormalized stress-energy tensor has acquired the anomalous trace

$$
\begin{aligned}
\langle \hat{T}^{\mu}{}_{\mu} \rangle_{\text{ren}} &= \frac{-2g^{\mu\nu}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\Gamma + \Gamma_{ct}) = \frac{-2g^{\mu\nu}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \Gamma_{ct} \\
&= \frac{1}{16\pi^2} \left(\frac{1}{120} {}_4\mathring{C}_{\alpha\beta\gamma\delta} \, {}_4\mathring{C}^{\alpha\beta\gamma\delta} - \frac{1}{360} \mathring{E} + \frac{1}{180} \mathring{\Box} \mathring{R} \right) ,
\end{aligned} \tag{6.97}
$$

as the finite combination of $(D-4)$ on the right-hand side of (6.95), (6.96) with the $1/(D-4)$ prefactor of the counterterm action. In the first line it was used that the original theory was assumed to be conformal, so that Γ does not contribute to the trace. The result (6.97) is commonly known as the *conformal anomaly*.

Historically, the presence of the $\mathring{\Box}\mathring{R}$ term was found first, originally by Capper and Duff [10]. Subsequently, further investigation by Deser, Duff, Isham [48] showed also the potential presence of the other terms and over the years, the coefficients were determined for various different field content (see also [12] for an extended overview). A geometrical analysis [49] revealed that the role of the Weyl tensor and the Gauss-Bonnet term in the anomaly are quite distinct. The former requires the introduction of a renormalization scale μ which breaks conformal symmetry, while the latter is scale independent and in general necessary for consistency. For instance, its coefficient can also be determined by demanding diffeomorphism invariance. The authors also show that there is always one term of the second type, namely the Euler density, which equals the Gauss-Bonnet term in four dimensions, while the number of terms of the first type increase drastically with dimension, and only in $D = 4$ there happens to be only one, namely the square of the Weyl tensor. The distinction into scale dependent and scale independent terms is important because the former may induce a running of the coupling constants of the theory.

Despite being discovered first, the $\mathring{\Box}\mathring{R}$ term in the anomaly is not universal. In fact, it would not even arise in the given derivation if one decides to renormalize with the D dimensional Weyl tensor, as it was done in section 6.2. In this case, the identity (6.95) is replaced by

$$
\frac{-2g^{\mu\nu}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^D x \sqrt{-g} \, {}_D C_{\alpha\beta\gamma\delta} \, {}_D C^{\alpha\beta\gamma\delta} = (D-4) \, {}_D C_{\alpha\beta\gamma\delta} \, {}_D C^{\alpha\beta\gamma\delta} , \tag{6.98}
$$

and thus no $\mathring{\Box}\mathring{R}$ term on the right-hand side of (6.97). Additionally, its coefficient can be changed by a finite and hence ambiguous counterterm. For instance, the inclusion of $\mathring{R}^2$ leads to

$$
\frac{-2g^{\mu\nu}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^D x \sqrt{-g} \mathring{R}^2 = -4(D-1) \mathring{\Box} \mathring{R} + \mathcal{O}(D-4) \tag{6.99}
$$

as modification of the trace anomaly. On the other hand, there is no local action[11] that can reproduce the other terms of the anomaly which are thus considered unambiguous: Indeed, all renormalization schemes predict the same terms with the same coefficients.

[11] There is, however, a non-local one known as the Riegert action [50].

6.5 Discussion

The main conclusion of section 6.3 came, in principle, as no surprise. By starting with a conformal action and applying a conformal renormalization scheme, the Ward identities were found to be satisfied identically. The non-trivial part of this statement lies in the renormalization. It was not obvious that the required counterterms could be chosen to obey the Ward identities themselves. In general relativity, this is obviously not possible as the renormalization spoils conformal symmetry, which ultimately leads to the anomaly. In the approach presented here, on the other hand, the stress-energy tensor may develop a non-vanishing trace which the dilation current always compensates for and it is therefore an elementary part of the Ward identity.

If Weyl symmetry is not gauged by a vector field and therefore no dilation current is present, the trace of the energy momentum tensor after quantization is interpreted as an anomaly, because the symmetry predicts vanishing of the trace at classical level. Accordingly, in the language of this thesis, the anomaly should be handed down from $\langle \hat{T}^{\mu}{}_{\mu} \rangle$ to local terms in the traces of the gT and gg-vertex functions, explicitly given by (6.74) and (6.76), respectively. The precise relation could, for instance, be found by functional variation of the anomaly (6.97), although it is clear that the calculation can be spared. A prominent feature of the anomaly is its presence even for a conformally coupled scalar. Since the traced vertices vanish for $\xi = -\frac{1}{6}$, none of the anomalous terms on the right-hand side of (6.97) can be observed.

For the square of the Weyl tensor and the $\Box R$ term this poses no problem. The prefactor of the latter in the anomaly is not unique anyway and it would not show up there when renormalizaton is performed as in section 6.2, as remarked above. Additionally, an explicit calculation [1] already revealed that it does not arise in the theory with gauged Weyl symmetry. $C^2_{\mu\nu\rho\sigma}$, on the other hand, contributes to lowest order to the three-point function because de Sitter is a conformally flat space. Hence it does also not show itself in the gg-vertex.

This leaves only the Gauss-Bonnet term in the anomaly unexplained. It is finite on de Sitter and one would thus expect to find its imprints in the one-loop vertices, which unfortunately cannot be confirmed in the calculations done here. Going back to the derivation of the anomaly, one realizes that it arises in (6.97) because the divergences of the general one-loop effective action are renormalized by a counterterm

$$\propto \frac{\mu^{D-4}}{D-4} \left(\frac{1}{120} C^2_{\mu\nu\rho\sigma} - \frac{1}{360} E \right) .$$

On the other hand, for the theory under investigation, the corresponding counterterm (that was determined by the divergences of the vertex (6.49)) is given by (6.68), (6.70)

$$\frac{\mu^{D-4}}{D-4}\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{16(D+1)(D-1)(D-3)^3}C^2_{\mu\nu\rho\sigma} \sim \frac{\mu^{D-4}}{D-4}\frac{1}{120}C^2_{\mu\nu\rho\sigma} + \mathcal{O}(D-4)^0\,,$$

which is the term even present for a conformally coupled scalar field. That is, there is agreement in the coefficient of the $C^2_{\mu\nu\rho\sigma}$ term which implicitly confirms it in the anomaly, though there is no explicit contribution.

The Gauss-Bonnet term is not added here, because it does not renormalize any vertex. By using the expressions given in appendices C, D one finds that all its potential corrections are proportional to $(D-4)$, and hence, it is impossible to obtain it as a counterterm by only considering the divergences. In fact, its presence in the counterterm action (6.94), which eventually induces the anomaly, is also not entirely obvious. A term proportional to $\Box R$ is suppressed at this point when going from (6.93) to (6.94) because it is a boundary term. But the same can be said about the Gauss-Bonnet integrand, which gives rise to a boundary contribution in $D = 4$. Birrell and Davies use four dimensional counterterms which means it could be dropped by the same reason, changing the anomaly accordingly.

Moreover, there exists another argument in favor of not including Gauss-Bonnet as a counterterm on the same level as the Weyl tensor. The explicit calculations in section 6.2 show that it is not conformal in D dimensions, that is, its induced vertices do not satisfy the Ward identities. On the contrary, it was added to the R^2 counterterm to make their combination conformal, while being only a boundary term itself.

Other authors prove the Gauss-Bonnet term by different means. In [51], the unique combination $\frac{1}{120}C^2_{\mu\nu\rho\sigma} - \frac{1}{360}\mathring{E}$ appears as a consequence of diffeomorphism invariance as the resulting stress-energy tensor would fail to be conserved, unless both terms are added. Unfortunately, this violation is proportional to a derivative of the curvature scalars, which are constant on de Sitter space. Also this way of detecting E is ruled out in the present context. Nonetheless, all instances of $T_{\mu\nu}$ and its functional derivatives, the gT- and gg-vertices, appearing in this thesis have been verified to be conserved.

Lucat has presented a different interpretation of the Gauss-Bonnet term in his PhD thesis [1]. It is known that in Riemannian geometry in four dimensions one can write

$$\mathring{E} = \mathring{\nabla}_\mu V^\mu$$

for some vector field V^μ, making its role as boundary term manifest. If one replaces the connection by its conformal cousin (3.13), a weight analysis shows

$$\mathring{E} = \mathring{\nabla}_\mu V^\mu \to E = \overline{\nabla}_\mu V^\mu = \mathring{\nabla}_\mu V^\mu - (D-4)T_\mu V^\mu\,. \tag{6.100}$$

He concludes that the contribution to the dilation current $\frac{\delta}{\delta T^\alpha}\int d^Dx\sqrt{-g}E$ is the negative of the one to the energy momentum tensor given by (6.96). If this was the case, then Gauss-Bonnet is indeed conformal and could be added to the counterterm action for free. It would obey the Ward identity itself and because it does not contribute to renormalization, its coefficient could be chosen arbitrarily. In particular, one was free to add

$$\Gamma_{\mathsf{GB}} = -\frac{1}{360}\frac{\mu^{D-4}}{D-4}\int d^Dx\sqrt{-g}\left[\mathring{E} + (D-4)T_\mu V^\mu\right] \tag{6.101}$$

to the counterterm action (6.68), such that this action reproduces the same anomaly as (6.94) in the case without torsion. More precisely, inclusion of the conformal Gauss-Bonnet counterterm (6.101) gives to lowest order in torsion the following corrections:

$$\begin{aligned}
T^{\mathsf{GB}}_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\left.\frac{\delta\Gamma_{\mathsf{GB}}}{\delta g^{\mu\nu}}\right|_{g=g^{\mathsf{dS}},T_\alpha=0} \\
&= \frac{1}{180\sqrt{-g}}\frac{\mu^{D-4}}{D-4}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}\left[\mathring{R}_{\alpha\beta\gamma\delta}\mathring{R}^{\alpha\beta\gamma\delta} - 4\,\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} + \mathring{R}^2\right] \\
&= \frac{1}{90}\frac{\mu^{D-4}}{D-4}\left(\delta^\rho{}_\mu\delta^\sigma{}_\nu - \tfrac{1}{4}g_{\mu\nu}g^{\rho\sigma}\right)\left[\mathring{R}_{\alpha\beta\gamma\rho}\mathring{R}^{\alpha\beta\gamma}{}_\sigma - 4\,\mathring{R}_{\alpha\rho}\mathring{R}^\alpha{}_\sigma + \mathring{R}\mathring{R}_{\rho\sigma}\right]
\end{aligned} \tag{6.102}$$

$$D^\alpha_{\mathsf{GB}} = \frac{1}{\sqrt{-g}}\left.\frac{\delta\Gamma_{\mathsf{GB}}}{\delta T_\alpha}\right|_{g=g^{\mathsf{dS}},T_\alpha=0} = -\frac{\mu^{D-4}}{360}V^\alpha \tag{6.103}$$

Upon the observation $g^{\mu\nu}T^{\mathsf{GB}}_{\mu\nu} = \frac{1}{90}\frac{\mu^{D-4}}{D-4}(1-\frac{D}{4})\mathring{E} = -\frac{\mu^{D-4}}{360}\mathring{\nabla}_\mu V^\mu$, these two terms satisfy the Ward identity (6.1) trivially. As for the other terms appearing in the trace of $T_{\mu\nu}$, it is compensated for by the contribution to the dilation current, here induced by the second term in (6.101). The linear combination $R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ found in section 6.2 should in this case not be called E, but rather something that resembles Gauss-Bonnet, but is not conformal.

Literature on the Gauss-Bonnet term is scarce and not always coherent. In [45], the authors give a (non-covariant) expression for the vector field V^μ that satisfies $\mathring{E} = \mathring{\nabla}_\mu V^\mu$ in Riemannian geometry, but neither with their result generalized to theories with torsion nor with the explicit results obtained in this thesis was it possible to recover a form such as (6.100). It might be due to the fact that the Gauss-Bonnet term is inherently only defined in four dimensions and inclusion of torsion is not as straightforward as hoped. In any case, the question about its presence in the counterterm action and hence in the anomaly, as well as the question if it is itself conformal or not cannot be fully answered by this thesis.

As a concluding remark, the observation that E does not induce any divergent contributions even when accompanied by a $1/(D-4)$ prefactor has led to the development of an extension of general relativity, where this term is added to the Einstein-Hilbert action,

$$S = \int d^Dx\sqrt{-g}\left[\frac{M_P^2}{2}\mathring{R} + \frac{\alpha}{D-4}\mathring{E}\right], \tag{6.104}$$

with M_P denoting the Planck mass. The resulting theory is known as $4D$-Einstein-Gauss-Bonnet gravity [52] and has received much attention in recent research, thereby causing some controversy (see e.g. [53] and references therein).

On the positive side, the derivation of the trace anomaly yields additional information about the non-conformally coupled scalar. According to equation (6.93), the divergent part for $\xi \neq -\frac{1}{6}$ also includes $\frac{-1}{(4\pi)^{D/2}}\frac{1}{72}\left(1+6\xi\right)^2 \mathring{R}^2$. It is renormalized by a counterterm with opposite sign, and inspection of (6.68), (6.69) shows that this is essentially done:

$$\frac{1}{(4\pi)^{D/2}}\left(\frac{D-2}{2}+2\xi(D-1)\right)^2 \frac{\Gamma\left(\frac{D}{2}-1\right)}{8(D-1)^2(D-3)^2}R^2 \sim \frac{(1+6\xi)^2}{(4\pi)^{D/2}}\frac{1}{72}R^2+\mathcal{O}(D-4) \tag{6.105}$$

The counterterm chosen in this thesis only differs by a boundary (E) and a torsion dependent ($\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$) term, showing that this part of the theory is in agreement with the literature. One can go even a step further. The difference between the general divergence induced by (6.93) and the particular counterterm (6.105) is of order $\mathcal{O}(D-4)$ and hence gives a finite contribution when multiplying $1/(D-4)$. This in turn induces a finite correction according to equation (6.99), which descends down to the vertex functions. Indeed, a careful analysis shows that this is the origin of the first line in the trace (6.76) of the renormalized gg-vertex, which cancels in the Ward identity against a similar expression originating from the gT-vertex.

7 Conclusion

Nature presents many motivations to study rescaling invariant systems. By using Weyl invariance as a guiding principle, vectorial space-time torsion appears natural in the form of a $U(1)$-like gauge field for Weyl transformations in the sense that it changes as $T_\alpha \to T_\alpha + \partial_\alpha \omega$. This has two important consequences. Contrary to general relativity, the curvature tensors are invariant under Weyl transformations and can therefore be used to obtain a conformal theory of gravity that reduces to GR for vanishing torsion. Moreover, the matter sector also benefits from this construction, as T_α is used to build a conformally invariant gauge connection (3.13). In this way, another application is added to the so-far tremendously successful interplay between physics and geometry.

In general relativity, Weyl symmetry gets broken in the quantum theory by renormalization which results in the famous anomaly in the trace of the stress-energy tensor. To investigate the corresponding situation in this new setup, a massless, non-minimally coupled scalar field is quantized on a de Sitter background, while the geometric quantities graviton and torsion are kept as external perturbations. Some long, intricate calculations lead to the one-loop expressions (6.41), (6.45), (6.49) for the vertices with two external legs. Remarkably, though, their divergences can be removed by a fully conformal counterterm action (6.68), so that the renormalized results still satisfy the Ward-Takahashi identities. In other words, Weyl symmetry is promoted to a true gauge-like symmetry, also respected by the quantum theory.

The presented results agree with the literature in the case of vanishing torsion. Only the role of the Gauss-Bonnet term in the divergence, and hence in the anomaly, cannot be fully determined, as it avoids being detected by the means of the calculations in this thesis. Despite a prefactor $\propto 1/(D-4)$, it in fact does not yield any divergent contributions, because every vertex induced by E is itself proportional to $(D-4)$. Moreover, because the Gauss-Bonnet term has a constant value on de Sitter space, it vanishes whenever it appears under a derivative. A recalculation for a more general background might be able to shed some light on this issue. However, whatever the outcome of such an analysis will be, it does not change the fact that conformal symmetry remains non-anomalous for this theory. This is especially true as the Gauss-Bonnet term (6.100) might be conformal itself.

Although the universe is clearly not scaling invariant, it seems like a reasonable assumption that the symmetry is spontaneously broken. In this picture, a new degree of freedom emerges that ought to dynamically create the Planck scale. It has been shown [1] that a scalar field can take this role, resulting in a non-trivial interplay between the scalar-gravity sector and torsion in the high energy regime. Because the coupling of these new fields to the standard model of particles is suppressed by the Planck scale $\propto \frac{k^2}{M_p^2}$, where k is the four momentum of the scattering process, they are undetectable by current accelerators as the LHC. In the inflationary universe, on the other hand, such energy scales are available and the degrees of freedom could be excited and propagate at large distances. If existent, the next generation space-based gravitational waves observatories, for instance, LISA, are therefore expected to detect their effects [1].

This thesis is all but the first step in establishing the ground for this new theory with gauged Weyl symmetry. To obtain more conclusive statements, the Ward-Takahashi identities should be fulfilled not only to linear perturbations, but to quadratic order. Furthermore, as trace anomalies also exist for different matter content such as fermions and vector fields, their respective absence has to be shown separately. However, given the combined evidence in [1] and this thesis, the conclusion that the symmetry is realized in their quantum theories seems very close.

On top of these generalizations, there remain an equal number of related open questions, which the aforementioned Gauss-Bonnet term is only one of. Particular interest also lies in the way the new scalar degree of freedom mixes with the gravitational ones, both at classical and quantum level. We are just at the beginning of understanding Weyl symmetry in nature, which certainly deserves more investigation.

A De Sitter Identities

Throughout: Torsion vanishes, that is, $\mathring{\nabla} = \nabla$, $\mathring{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu}$, etc.
The aim of this appendix is to provide an extension (and sometimes repetition) of the identities given in appendix A of [37] that facilitate calculations of derivatives on the de Sitter space.
Recall the invariant distance

$$\frac{y}{4} = \frac{y(x,x')}{4} = \frac{H^2 a a' \Delta x^2}{4} = \frac{(x-x')^{\mu}(x-x')_{\mu}}{4\eta\eta'} \tag{A.1}$$

from chapter 4 with $a' \equiv a(\eta') = -\frac{1}{H\eta'}$. For derivatives acting on y there are the following relations:

$$\partial_{\mu}\frac{y}{4} = Ha\delta^{0}{}_{\mu}\frac{y}{4} + \frac{1}{2}H^2 a a' \Delta x_{\mu} \tag{A.2}$$

$$\partial'_{\rho}\frac{y}{4} = Ha'\delta^{0}{}_{\rho}\frac{y}{4} - \frac{1}{2}H^2 a a' \Delta x_{\rho} \tag{A.3}$$

$$\partial_{\mu}\partial'_{\rho}\frac{y}{4} = H^2 a a'\left(\delta^{0}{}_{\mu}\delta^{0}{}_{\rho}\frac{y}{4} - \frac{1}{2}Ha\delta^{0}{}_{\mu}\Delta x_{\rho} + \frac{1}{2}Ha'\delta^{0}{}_{\rho}\Delta x_{\mu} - \frac{1}{2}\eta_{\mu\rho}\right) \tag{A.4}$$

$$\partial_{\mu}\partial_{\nu}\frac{y}{4} = H^2 a^2\left(2\delta^{0}{}_{\mu}\delta^{0}{}_{\nu}\frac{y}{4} + Ha'\delta^{0}{}_{(\mu}\Delta x_{\nu)}\right) + \frac{1}{2}H^2 a a'\eta_{\mu\nu} \tag{A.5}$$

$$\nabla_{\mu}\nabla_{\nu}\frac{y}{4} = \left(\frac{1}{2} - \frac{y}{4}\right)H^2 g_{\mu\nu} \tag{A.6}$$

$$\frac{\Box}{H^2}\frac{y}{4} = D\left(\frac{1}{2} - \frac{y}{4}\right) \tag{A.7}$$

$$\nabla_{\mu}\frac{y}{4}\nabla^{\mu}\frac{y}{4} = H^2\left(\frac{y}{4} - \left(\frac{y}{4}\right)^2\right) \tag{A.8}$$

$$\nabla_{\mu}\frac{y}{4}\nabla^{\mu}\nabla'_{\rho}\frac{y}{4} = H^2\left(\frac{1}{2} - \frac{y}{4}\right)\nabla'_{\rho}\frac{y}{4} \tag{A.9}$$

$$\frac{\Box}{H^2}\nabla_{\mu}\frac{y}{4} = -\nabla_{\mu}\frac{y}{4} \tag{A.10}$$

$$\frac{\Box}{H^2}\nabla'_{\rho}\frac{y}{4} = -D\nabla'_{\rho}\frac{y}{4} \tag{A.11}$$

$$\frac{\Box}{H^2}(\nabla'_{\rho}\frac{y}{4}\nabla'_{\sigma}\frac{y}{4}) = -2(D+1)\nabla'_{\rho}\frac{y}{4}\nabla'_{\sigma}\frac{y}{4} + \frac{1}{2}H^2 g'_{\rho\sigma} \tag{A.12}$$

In $D = 4$ one has:

$$\frac{\Box}{H^2}\frac{y}{4} = -4\frac{y}{4} + 2 \tag{A.13}$$

$$\frac{\Box}{H^2}\ln\frac{y}{4} = \frac{4}{y} - 3 \tag{A.14}$$

$$\frac{\Box}{H^2}\left(\frac{y}{4}\ln\frac{y}{4}\right) = -4\frac{y}{4}\ln\frac{y}{4} + 2\ln\frac{y}{4} - 5\frac{y}{4} + 3 \tag{A.15}$$

$$\frac{\Box}{H^2}\ln^2\frac{y}{4} = 8\frac{\ln(\frac{y}{4})}{y} - 6\ln\frac{y}{4} + 2\frac{4}{y} - 2 \tag{A.16}$$

By using (A.6) one can extract derivatives of mixed powers of y:

$$\partial_\mu \left(\frac{y}{4}\right)^{-\alpha} \partial_\nu \left(\frac{y}{4}\right)^{-\beta} = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} \nabla_\mu \nabla_\nu \left(\frac{y}{4}\right)^{-(\alpha+\beta)} + \frac{\alpha\beta}{\alpha+\beta+1} H^2 g_{\mu\nu} \left(\frac{1}{2} - \frac{y}{4}\right) \times \left(\frac{y}{4}\right)^{-(\alpha+\beta+1)} \tag{A.17}$$

The relevant cases are $(\alpha, \beta) = (D/2 - 1.D/2 - 1)$ and $(\alpha, \beta) = (D/2 - 1.D/2 - 2)$.

$$\partial_\mu \left(\frac{y}{4}\right)^{1-D/2} \partial_\nu \left(\frac{y}{4}\right)^{1-D/2} = \frac{D-2}{4(D-1)} \nabla_\mu \nabla_\nu \left(\frac{y}{4}\right)^{2-D} + \frac{(D-2)^2}{4(D-1)} H^2 g_{\mu\nu} \left(\frac{1}{2} - \frac{y}{4}\right) \times \left(\frac{y}{4}\right)^{1-D} \tag{A.18}$$

$$\partial_\mu \left(\frac{y}{4}\right)^{1-D/2} \partial_\nu \left(\frac{y}{4}\right)^{2-D/2} = \frac{D-4}{4(D-3)} \nabla_\mu \nabla_\nu \left(\frac{y}{4}\right)^{3-D} + \frac{(D-2)(D-4)}{4(D-3)} H^2 g_{\mu\nu} \left(\frac{1}{2} - \frac{y}{4}\right) \times \left(\frac{y}{4}\right)^{2-D} \tag{A.19}$$

These are supplemented by two limiting cases $(\alpha + \beta) \to 0$ in $D = 4$:

$$\partial_\mu \frac{4}{y} \partial_\nu \frac{y}{4} = \nabla_\mu \nabla_\nu \ln \frac{y}{4} - \frac{1}{2} H^2 g_{\mu\nu} \frac{4}{y} + H^2 g_{\mu\nu} \tag{A.20}$$

$$\partial_\mu \frac{4}{y} \partial_\nu \left(\frac{y}{4} \ln \frac{y}{4}\right) = \frac{1}{2} \nabla_\mu \nabla_\nu \ln^2 \frac{y}{4} + 2 \nabla_\mu \nabla_\nu \ln \frac{y}{4} - 2H^2 g_{\mu\nu} \frac{\ln(\frac{y}{4})}{y} + H^2 g_{\mu\nu} \ln \frac{y}{4} - H^2 g_{\mu\nu} \frac{4}{y} + 2H^2 g_{\mu\nu} \tag{A.21}$$

Also recall the bilocal generalization for the metric tensor from (4.19),

$$H^2 \tilde{g}_{\mu\rho'} = -2 \nabla_\mu \nabla'_\rho \frac{y}{4} \, . \tag{A.22}$$

This definition immediately implies:

$$\nabla_\nu \tilde{g}_{\mu\rho'} = 2 g_{\mu\nu} \nabla'_\rho \frac{y}{4} \tag{A.23}$$

$$\nabla'_\sigma \tilde{g}_{\mu\rho'} = 2 g'_{\rho\sigma} \nabla_\mu \frac{y}{4} \tag{A.24}$$

$$\nabla_\nu \nabla'_\sigma \tilde{g}_{\mu\rho'} = (1 - y/2) g_{\mu\nu} g'_{\rho\sigma} \tag{A.25}$$

$$\frac{\square}{H^2} \tilde{g}_{\mu\rho'} = -\tilde{g}_{\mu\rho'} \tag{A.26}$$

A.1 Extracting d'Alembertians

In order to lower the degree of divergence one can raise the power of y-dependent terms by extracting derivatives. The action of the d'Alembertian on a non-singular function of y was given in (5.6),

$$\frac{\square}{H^2} F(y) = (4y - y^2) F''(y) + D(2 - y) F'(y) \, , \tag{A.27}$$

where a prime denotes a derivative with respect to y. A non-singular function contains no term proportional to $y^{1-D/2}$ when expanded in powers of y. Equation (A.27) can be used to prove

$$\left(\frac{y}{4}\right)^{-\alpha} = -\frac{2}{(\alpha-1)(D-2\alpha)}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-\alpha} + \frac{2(D-\alpha)}{D-2\alpha}\left(\frac{y}{4}\right)^{1-\alpha} \qquad (\alpha \neq D/2) \tag{A.28}$$

$$\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-D/2} = \frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}} + \frac{D(D-2)}{4}\left(\frac{y}{4}\right)^{1-D/2}, \tag{A.29}$$

where the pole prescription (5.14) is necessary for the second equality to hold. Application of identity (A.28) to $\alpha = D, D-1, D-2$ yields

$$\left(\frac{y}{4}\right)^{-D} = \frac{2}{D(D-1)}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-D} \tag{A.30}$$

$$\left(\frac{y}{4}\right)^{1-D} = \frac{2}{(D-2)^2}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{2-D} - \frac{2}{D-2}\left(\frac{y}{4}\right)^{2-D} \tag{A.31}$$

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{3-D} - \frac{4}{D-4}\left(\frac{y}{4}\right)^{3-D} \tag{A.32}$$

In order to localize the divergence of the $\propto y^{3-D}$ term subtract the fundamental identity (A.29) of (A.32) with prefactor $\frac{2}{(D-3)(D-4)}$:

$$\begin{aligned}\left(\frac{y}{4}\right)^{2-D} &= \frac{2}{(D-3)(D-4)}\frac{\Box}{H^2}\left[\left(\frac{y}{4}\right)^{3-D} - \left(\frac{y}{4}\right)^{1-D/2}\right] \\ &- \frac{4}{D-4}\left[\left(\frac{y}{4}\right)^{3-D} - \frac{D(D-2)}{8(D-3)}\left(\frac{y}{4}\right)^{1-D/2}\right] \\ &+ \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}}\end{aligned} \tag{A.33}$$

The first two terms on the right-hand side are finite in $D=4$ and expansion shows

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}} + g(y) + \mathcal{O}(D-4). \tag{A.34}$$

Upon recalling the definition $g(y) = \left(-4\frac{\Box}{H^2}\frac{\ln(\frac{y}{4})}{y} + 8\frac{\ln(\frac{y}{4})}{y} - \frac{4}{y}\right)$ from (6.38) one can thus recast the identities (A.30)-(A.32) in $D=4$ space-time dimensions:

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)\Gamma(\frac{D}{2}-1)}\frac{(4\pi)^{D/2}}{H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}} + g(y) \tag{A.35}$$

$$\begin{aligned}\left(\frac{y}{4}\right)^{1-D} &= \frac{4}{(D-2)^2(D-3)(D-4)\Gamma(\frac{D}{2}-1)}\frac{(4\pi)^{D/2}}{H^D}\frac{\Box}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}} - \\ &- \frac{4}{(D-2)(D-3)(D-4)\Gamma(\frac{D}{2}-1)}\frac{(4\pi)^{D/2}}{H^D}\frac{i\delta^D(x-x')}{\sqrt{-g}} + \frac{1}{2}\frac{\Box}{H^2}g(y) - g(y)\end{aligned} \tag{A.36}$$

$$\left(\frac{y}{4}\right)^{-D} = \frac{8}{D(D-1)(D-2)^2(D-3)(D-4)\Gamma(\frac{D}{2}-1)}\frac{(4\pi)^{D/2}}{H^D}\frac{\square}{H^2}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}} -$$
$$- \frac{8}{D(D-1)(D-2)(D-3)(D-4)\Gamma(\frac{D}{2}-1)}\frac{(4\pi)^{D/2}}{H^D}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}} +$$
$$+ \frac{1}{12}\frac{\square}{H^2}\frac{\square}{H^2}g(y) - \frac{1}{6}\frac{\square}{H^2}g(y) \tag{A.37}$$

A.2 Chernikov-Tagirov Propagator

This section provides relations for the various finite parts of the Chernikov-Tagirov propagator introduced in chapter 5. Recall the definitions

$$f_{0,1,2} = \sum_{n=0,1,2}^{\infty}\left[\frac{\Gamma(\frac{D-1}{2}\pm\nu)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma(\frac{1}{2}\pm\nu)}\frac{(\frac{D-1}{2}+\nu)_n(\frac{D-1}{2}-\nu)_n}{(D/2)_n n!}\left(\frac{y}{4}\right)^n + \right.$$
$$\left. + \Gamma\left(\frac{D}{2}-1\right)\frac{\frac{1}{4}-\nu^2}{2-D/2}\frac{(\frac{3}{2}+\nu)_n(\frac{3}{2}-\nu)_n}{(3-D/2)_n(n+1)!}\left(\frac{y}{4}\right)^{n+2-D/2}\right]$$
$$= \left(\frac{1}{4}-\nu_4^2\right)\sum_{n=0,1,2}^{\infty}\frac{(\frac{3}{2}+\nu_4)_n(\frac{3}{2}-\nu_4)_n}{(2)_n}\frac{\left(\frac{y}{4}\right)^n}{n!}\left[\ln\frac{y}{4} + \psi\left(\frac{3}{2}+n\pm\nu_4\right) - \psi\left(1+n\right)\right.$$
$$\left. - \psi\left(2+n\right)\right], \tag{A.38}$$

$$g(y) = -4\frac{\square}{H^2}\frac{\ln(\frac{y}{4})}{y} + 8\frac{\ln(\frac{y}{4})}{y} - \frac{4}{y}, \tag{A.39}$$

for which the propagator and its square become (after setting $D = 4$ wherever possible)

$$i\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\left[\Gamma\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-D/2} + f_0(y)\right], \tag{A.40}$$

$$(i\Delta(x,x'))^2 = \frac{H^{D-4}}{(4\pi)^{D/2}}\frac{2\,\Gamma\left(\frac{D}{2}-1\right)}{(D-3)(D-4)}\frac{i\delta^D(x-x')}{\sqrt{-g}} + \frac{H^{2D-4}}{(4\pi)^D}\left[g(y) + 2\frac{4}{y}f_0(y) + f_0^2(y)\right]$$
$$= \frac{H^{D-4}}{(4\pi)^{D/2}}\frac{2\,\Gamma\left(\frac{D}{2}-1\right)}{(D-3)(D-4)}\frac{i\delta^D(x-x')}{\sqrt{-g}} + (i\Delta(x,x'))^2_{\mathsf{fin}}\,. \tag{A.41}$$

The equation of motion (5.13) for the propagator implies in arbitrary dimension D:

$$\square f_0(y) = (-\xi\mathring{R} + m^2)f_0(y) + H^2\,\Gamma\left(\tfrac{D}{2}-1\right)\left(\frac{1}{4}-\nu_D^2\right)\left(\frac{y}{4}\right)^{1-D/2} \tag{A.42}$$

$$\square f_1(y) = (-\xi\mathring{R} + m^2)f_0(y) + \frac{D+2}{2}H^2\,\Gamma\left(\tfrac{D}{2}-1\right)\left(\frac{1}{4}-\nu_D^2\right)\left(\frac{y}{4}\right)^{2-D/2} \tag{A.43}$$

When $D = 4$ one furthermore has:

$$f_0(y) = f_1(y) + \left(\frac{1}{4} - \nu_4^2\right)\left[-1 - 2\psi(1) + \psi\left(\frac{1}{2} \pm \nu_4\right) + \ln\left(\frac{y}{4}\right)\right] + 1 \tag{A.44}$$

$$f_1(y) = f_2(y) + \frac{1}{2}\left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\left[(\psi\left(\frac{1}{2} \pm \nu_4\right) - \frac{5}{2} - 2\psi(1))\frac{y}{4} + \frac{y}{4}\ln\frac{y}{4}\right]$$
$$+ \frac{3}{2}\left(\frac{1}{4} - \nu_4^2\right)\frac{y}{4} + \frac{1}{2}\left(\frac{9}{4} - \nu_4^2\right)\frac{y}{4} \tag{A.45}$$

$$\frac{\Box}{H^2} f_0(y) = \left(\frac{9}{4} - \nu_4^2\right) f_0(y) + \left(\frac{1}{4} - \nu_4^2\right)\frac{4}{y} \tag{A.46}$$

$$\frac{\Box}{H^2} f_1(y) = \left(\frac{9}{4} - \nu_4^2\right) f_0(y) + 3\left(\frac{1}{4} - \nu_4^2\right)$$
$$= \left(\frac{9}{4} - \nu_4^2\right) f_1(y) + \left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\left[-1 - 2\psi(1) + \psi\left(\frac{1}{2} \pm \nu_4\right)\right]$$
$$+ \left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\ln\frac{y}{4} + \left(\frac{9}{4} - \nu_4^2\right) + 3\left(\frac{1}{4} - \nu_4^2\right) \tag{A.47}$$

$$\frac{\Box}{H^2} f_2(y) = \left(\frac{9}{4} - \nu_4^2\right) f_1(y) + 2\left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\left[(\psi\left(\frac{1}{2} \pm \nu_4\right) - \frac{5}{4} - 2\psi(1))\frac{y}{4} + \frac{y}{4}\ln\frac{y}{4}\right]$$
$$+ 6\left(\frac{1}{4} - \nu_4^2\right)\frac{y}{4} + 2\left(\frac{9}{4} - \nu_4^2\right)\frac{y}{4} \tag{A.48}$$

In the calculation of the Ward-identities one has to trace some combinations of derivatives of these functions. Traces with two derivatives ($D = 4$):

$$\frac{g^{\mu\nu}}{H^2}\partial_\mu \frac{4}{y}\,\partial_\nu f_1(y) = -\frac{1}{2}\left(\frac{1}{4} - \nu_4^2\right)\left[-1 - 2\psi(1) + \psi\left(\frac{1}{2} \pm \nu_4\right)\right]\frac{(4\pi)^2}{H^4}\frac{i\delta^4(x - x')}{\sqrt{-g}}$$
$$+ \frac{1}{2}\frac{\Box}{H^2}\left(\frac{4}{y} f_0(y)\right) - \frac{4}{y} f_0(y) - \frac{1}{2}\left(\frac{9}{4} - \nu_4^2\right)\frac{4}{y} f_0(y) - \frac{1}{2}\frac{\Box}{H^2}\frac{4}{y}$$
$$+ \frac{1}{2}\left(\frac{1}{4} - \nu_4^2\right) g(y) - \left(\frac{1}{4} - \nu_4^2\right)\frac{4}{y} + \frac{4}{y} \tag{A.49}$$

$$\frac{g^{\mu\nu}}{H^2}\partial_\mu f_0(y)\,\partial_\nu f_0(y) = \frac{1}{2}\frac{\Box}{H^2} f_0^2(y) - \left(\frac{1}{4} - \nu_4^2\right)\frac{4}{y} f_0(y) - \left(\frac{9}{4} - \nu_4^2\right) f_0^2(y) \tag{A.50}$$

Traces with four derivatives ($D = 4$):

$$\frac{g^{\mu\nu}}{H^2}\partial_\mu \partial'_\rho \frac{4}{y}\,\partial_\nu \partial'_\sigma f_2(y) =$$
$$= \frac{1}{2}\frac{\Box}{H^2}\left(\partial'_\rho \frac{4}{y}\partial'_\sigma f_1(y)\right) - \partial'_\rho \frac{4}{y}\partial'_\sigma f_1(y) - \frac{1}{2}\left(\frac{9}{4} - \nu_4^2\right)\partial'_\rho \frac{4}{y}\partial'_\sigma f_1(y)$$
$$- \left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\left[\nabla'_\rho \nabla'_\sigma \frac{\ln(\frac{y}{4})}{y} + \frac{1}{4}\nabla'_\rho \nabla'_\sigma \ln^2\frac{y}{4}\right]$$
$$+ \left[\left(\frac{1}{4} - \nu_4^2\right)\left(\frac{9}{4} - \nu_4^2\right)\left(-\frac{1}{2} - 2\psi(1) + \psi\left(\frac{1}{2} \pm \nu_4\right)\right) + \left(\frac{9}{4} - \nu_4^2\right) + 3\left(\frac{1}{4} - \nu_4^2\right)\right]\times$$

$$\times\left[-\frac{1}{4}\nabla'_\rho\nabla'_\sigma\frac{4}{y}-\frac{1}{2}\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{1}{8}H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y}+\frac{1}{4}H^2g'_{\rho\sigma}\frac{4}{y}-\frac{1}{2}H^2g'_{\rho\sigma}\right]$$
$$+\left(\frac{1}{4}-\nu_4^2\right)\left(\frac{9}{4}-\nu_4^2\right)\left[-\frac{1}{4}\nabla'_\rho\nabla'_\sigma\frac{4}{y}-\frac{1}{2}\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{5}{8}H^2g'_{\rho\sigma}\frac{4}{y}-\frac{5}{4}H^2g'_{\rho\sigma}\right]$$
$$+\left(\frac{1}{4}-\nu_4^2\right)\left(\frac{9}{4}-\nu_4^2\right)\left[\frac{1}{4}H^2g'_{\rho\sigma}\frac{\Box}{H^2}\left(2\frac{\ln(\frac{y}{4})}{y}-\ln\frac{y}{4}\right)+\frac{1}{2}H^2g'_{\rho\sigma}\left(2\frac{\ln(\frac{y}{4})}{y}-\ln\frac{y}{4}\right)\right] \tag{A.51}$$

$$\frac{g^{\mu\nu}}{H^2}\left[2\left(\frac{1}{4}-\nu_4^2\right)\partial_\mu\partial'_\rho\ln\frac{y}{4}\,\partial_\nu\partial'_\sigma f_1(y)+\partial_\mu\partial'_\rho f_1(y)\,\partial_\nu\partial'_\sigma f_1(y)\right]=$$
$$=\frac{1}{2}\frac{\Box}{H^2}\left[\partial'_\rho f_0(y)\,\partial'_\sigma f_0(y)\right]-\left(\frac{9}{4}-\nu_4^2\right)\partial'_\rho f_0(y)\,\partial'_\sigma f_0(y)+\frac{1}{2}\left(\frac{1}{4}-\nu_4^2\right)^2\frac{\Box}{H^2}\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}$$
$$-\frac{1}{4}\left(\frac{1}{4}-\nu_4^2\right)^2H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y}-\left(\frac{1}{4}-\nu_4^2\right)\partial'_\rho\frac{4}{y}\,\partial'_\sigma f_1(y) \tag{A.52}$$

A.3 Traces involving Bitensors

This section provides some $g^{\mu\nu}$ traces of terms containing bitensors that appear in the gg-vertex. Traces with $\tilde{g}_{\mu\rho'}$:

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\frac{\delta^D(x-x')}{\sqrt{-g}}=-\nabla'_\rho\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{A.53}$$
$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\partial'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}}=-\nabla'_\rho\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}}+DH^2g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{A.54}$$
$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\partial'_\sigma\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}=-\nabla'_\rho\nabla'_\sigma\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}+(D+2)\nabla'_\rho\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}}+$$
$$+(D+2)H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}-D^2H^2g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{A.55}$$
$$g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}=H^2g'_{\rho\sigma}-4\nabla'_\rho\frac{y}{4}\nabla'_\sigma\frac{y}{4} \tag{A.56}$$
$$g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}=H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}-2H^2g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{A.57}$$
$$g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\Box}{H^2}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}=H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}-4H^2g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\delta^D(x-x')}{\sqrt{-g}}-$$
$$-8\nabla'_\rho\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}}+4(D+1)H^2g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{A.58}$$
$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\frac{y}{4}=-(1-y/2)\partial'_\rho\frac{y}{4} \tag{A.59}$$

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\left(\frac{y}{4}\right)^{-\alpha} = -\nabla'_\rho\left(\frac{y}{4}\right)^{-\alpha} + 2\frac{\alpha}{\alpha-1}\nabla'_\rho\left(\frac{y}{4}\right)^{-(\alpha-1)} \tag{A.60}$$

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\left(\frac{y}{4}\right)^{-1} = -\nabla'_\rho\left(\frac{y}{4}\right)^{-1} - 2\nabla'_\rho\ln\frac{y}{4} \tag{A.61}$$

$$\begin{aligned} g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{-\alpha} &= -\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{-\alpha} + 2\frac{\alpha}{\alpha-1}\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{-(\alpha-1)} + \\ &\quad + 2\alpha H^2 g'_{\rho\sigma}\left(\left(\frac{y}{4}\right)^{-\alpha} - \left(\frac{y}{4}\right)^{-(\alpha-1)}\right) \end{aligned} \tag{A.62}$$

In $D = 4$ there are the two relevant cases:

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\partial'_\sigma\frac{4}{y} = -\nabla'_\rho\nabla'_\sigma\frac{4}{y} - 2\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4} + 2H^2 g'_{\rho\sigma}\frac{4}{y} - 2H^2 g'_{\rho\sigma} \tag{A.63}$$

$$\begin{aligned} g^{\mu\nu}\tilde{g}_{\mu\rho'}\partial_\nu\partial'_\sigma\frac{\ln(\frac{y}{4})}{y} &= -\nabla'_\rho\nabla'_\sigma\frac{\ln(\frac{y}{4})}{y} + \frac{1}{2}\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4} - \frac{1}{4}\nabla'_\rho\nabla'_\sigma\ln^2\left(\frac{y}{4}\right) \\ &\quad + \frac{1}{2}H^2 g'_{\rho\sigma} - \frac{1}{2}H^2 g'_{\rho\sigma}\frac{4}{y} + 2H^2 g'_{\rho\sigma}\frac{\ln(\frac{y}{4})}{y} - \frac{1}{2}H^2 g'_{\rho\sigma}\ln\frac{y}{4} \end{aligned} \tag{A.64}$$

For the function $g(y) = \left(-4\frac{\Box}{H^2}\frac{\ln(\frac{y}{4})}{y} + 8\frac{\ln(\frac{y}{4})}{y} - \frac{4}{y}\right)$ it furthermore holds ($D = 4$):

$$g^{\mu\nu}\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\rho g(y) = -\nabla'_\rho\nabla'_\sigma g(y) + 4\nabla'_\rho\nabla'_\sigma\frac{4}{y} + 4H^2 g'_{\rho\sigma}g(y) + 2H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y} - 8H^2 g'_{\rho\sigma}\frac{4}{y} \tag{A.65}$$

$$\begin{aligned} g^{\mu\nu}\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\rho\frac{\Box}{H^2}g(y) &= -\nabla'_\rho\nabla'_\sigma\frac{\Box}{H^2}g(y) + 6\nabla'_\rho\nabla'_\sigma g(y) + 8\nabla'_\rho\nabla'_\sigma\frac{\Box}{H^2}\frac{4}{y} - 8\nabla'_\rho\nabla'_\sigma\frac{4}{y} \\ &\quad + 6H^2 g'_{\rho\sigma}\frac{\Box}{H^2}g(y) - 16H^2 g'_{\rho\sigma}g(y) + 2H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\Box}{H^2}\frac{4}{y} \\ &\quad - 24H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y} + 32H^2 g'_{\rho\sigma}\frac{4}{y} \end{aligned} \tag{A.66}$$

$$g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y) = 4\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4} + H^2 g'_{\rho\sigma}g(y) - 2H^2 g'_{\rho\sigma}\frac{4}{y} + 4H^2 g'_{\rho\sigma} \tag{A.67}$$

$$\begin{aligned} g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\Box}{H^2}g(y) &= -4\nabla'_\rho\nabla'_\sigma\frac{4}{y} + 8\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4} + H^2 g'_{\rho\sigma}\frac{\Box}{H^2}g(y) - 2H^2 g'_{\rho\sigma}g(y) \\ &\quad - 2H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y} + 4H^2 g'_{\rho\sigma}\frac{4}{y} + 8H^2 g'_{\rho\sigma} \end{aligned} \tag{A.68}$$

$$\begin{aligned} g^{\mu\nu}H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\Box}{H^2}\frac{\Box}{H^2}g(y) &= -8\nabla'_\rho\nabla'_\sigma g(y) - 12\nabla'_\rho\nabla'_\sigma\frac{\Box}{H^2}\frac{4}{y} + 16\nabla'_\rho\nabla'_\sigma\frac{4}{y} + 16\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4} \\ &\quad + H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\Box}{H^2}g(y) - 4H^2 g'_{\rho\sigma}\frac{\Box}{H^2}g(y) + 20H^2 g'_{\rho\sigma}g(y) \\ &\quad - 2H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{\Box}{H^2}\frac{4}{y} + 40H^2 g'_{\rho\sigma}\frac{\Box}{H^2}\frac{4}{y} - 72H^2 g'_{\rho\sigma}\frac{4}{y} + 16H^2 g'_{\rho\sigma} \end{aligned} \tag{A.69}$$

B An Expression for $\partial_\mu \partial'_\rho i\Delta(x,x') \partial_\nu \partial'_\sigma i\Delta(x,x')$

Throughout: Torsion vanishes, that is, $\mathring{\nabla} = \nabla$, $\mathring{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu}$, etc.
Throughout: $(\mu\nu)$ and $(\rho\sigma)$ are understood to be symmetrized, which will be omitted in the notation for clarity, i.e $\partial_\mu\partial'_\rho i\Delta(x,x')\partial_\nu\partial'_\sigma i\Delta(x,x') \equiv \partial_{\mu)}\partial'_{(\rho} i\Delta(x,x')\partial'_{\sigma)}\partial_{(\nu} i\Delta(x,x')$, etc.
The calculation of $\partial_\mu\partial'_\rho i\Delta(x,x')\partial_\nu\partial'_\sigma i\Delta(x,x')$ essentially means to find an an expression for $\partial_\mu\partial'_\rho \left(\frac{y}{4}\right)^{-\alpha} \partial_\nu\partial'_\sigma \left(\frac{y}{4}\right)^{-\beta}$. Using the results of appendix A one can compute:

$$\partial_\mu\partial'_\rho \left(\frac{y}{4}\right)^{-\alpha} \partial_\nu\partial'_\sigma \left(\frac{y}{4}\right)^{-\beta} = \alpha(\alpha+1)\beta(\beta+1)\left(\frac{y}{4}\right)^{-(\alpha+\beta+4)} \partial_\mu\frac{y}{4}\partial_\nu\frac{y}{4}\partial'_\rho\frac{y}{4}\partial_\sigma\frac{y}{4} + \\ + \frac{\alpha\beta}{2(\alpha+\beta+1)} H^2 \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma \left(\frac{y}{4}\right)^{-(\alpha+\beta+1)} \tag{B.1}$$

On the other hand:

$$\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma \left(\frac{y}{4}\right)^{-\gamma} = \gamma(\gamma+1)(\gamma+2)(\gamma+3)\left(\frac{y}{4}\right)^{-(\gamma+4)} \partial_\mu\frac{y}{4}\partial_\nu\frac{y}{4}\partial'_\rho\frac{y}{4}\partial_\sigma\frac{y}{4} \\ - \gamma\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\gamma+1)} - \left(\frac{y}{4}\right)^{-\gamma}\right) \\ + \gamma^2 H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\gamma+1)} - \left(\frac{y}{4}\right)^{-\gamma}\right) - \frac{\gamma(\gamma+1)}{2} H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\gamma+2)} - \left(\frac{y}{4}\right)^{-(\gamma+1)}\right) \\ - 2\gamma H^2 \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{-(\gamma+1)} - \frac{\gamma(\gamma+1)}{2} H^4 \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{-(\gamma+2)} \tag{B.2}$$

Combination with $\gamma = \alpha + \beta$ yields:

$$\partial_\mu\partial'_\rho \left(\frac{y}{4}\right)^{-\alpha} \partial_\nu\partial'_\sigma \left(\frac{y}{4}\right)^{-\beta} = \\ \frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}\Bigg\{\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{-(\alpha+\beta)} \\ + (\alpha+\beta)\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\alpha+\beta+1)} - \left(\frac{y}{4}\right)^{-(\alpha+\beta)}\right) \\ - (\alpha+\beta)^2 H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\alpha+\beta+1)} - \left(\frac{y}{4}\right)^{-(\alpha+\beta)}\right)\Bigg\} \\ + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2(\alpha+\beta+2)(\alpha+\beta+3)} H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-(\alpha+\beta+2)} - \left(\frac{y}{4}\right)^{-(\alpha+\beta+1)}\right) \\ + \left[-2\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)} + \frac{\alpha\beta}{2(\alpha+\beta+1)}\right] H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{-(\alpha+\beta+1)} \\ + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2(\alpha+\beta+2)(\alpha+\beta+3)} H^4 \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{-(\alpha+\beta+2)} \tag{B.3}$$

The relevant cases are $(\alpha,\beta) = (D/2-1, D/2-1).(D/2-1, D/2-2), (D/2-1), (D/2-3), (D/2-1, 1), (D/2-2, D/2-2)$
Non-integrable powers of y are replaced according to equations (A.35)-(A.37).

B.1 $\alpha = D/2-1$, $\beta = D/2-1$

The prefactor of this term is $\Gamma^2\left(\frac{D}{2}-1\right)$. Application of (B.3) yields:

$$\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{1-D/2} = \frac{D(D-2)}{16(D+1)(D-1)}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{2-D} +$$
$$+\frac{D(D-2)^2}{16(D+1)(D-1)}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{1-D}-\left(\frac{y}{4}\right)^{2-D}\right)-$$
$$-\frac{D(D-2)^3}{16(D+1)(D-1)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{1-D}-\left(\frac{y}{4}\right)^{2-D}\right)+$$
$$+\frac{D(D-2)^2}{32(D+1)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{-D}-\left(\frac{y}{4}\right)^{1-D}\right)+$$
$$+\frac{(D-2)^2}{8(D+1)(D-1)}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{1-D}+\frac{D(D-2)^2}{32(D+1)}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{-D} \quad \text{(B.4)}$$

Thus with prefactor it takes the following form in $D=4$:

$$\Gamma^2\left(\frac{D}{2}-1\right)\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{1-D/2} =$$
$$=\frac{D(D-2)\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{D\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{D(D-2)\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\square}{H^2}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{((D+1)(D-2)+D(D-1))\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{D(D-1)(D-2)\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{\Gamma\left(\frac{D}{2}-1\right)}{2(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{2(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{\Gamma\left(\frac{D}{2}-1\right)}{4(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\square}{H^2}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{4(D+1)(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\square}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{1}{30}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma g(y)$$

$$+\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left[\frac{1}{60}\frac{\Box}{H^2}g(y) - \frac{1}{10}g(y)\right]$$
$$+ H^4 g_{\mu\nu}g'_{\rho\sigma}\left[\frac{1}{240}\frac{\Box}{H^2}\frac{\Box}{H^2}g(y) - \frac{11}{120}\frac{\Box}{H^2}g(y) + \frac{3}{10}g(y)\right]$$
$$+ H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left[\frac{1}{60}\frac{\Box}{H^2}g(y) - \frac{1}{30}g(y)\right]$$
$$+ H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left[\frac{1}{120}\frac{\Box}{H^2}\frac{\Box}{H^2}g(y) - \frac{1}{60}\frac{\Box}{H^2}g(y)\right] \tag{B.5}$$

B.2 $\alpha = D/2-1$, $\beta = D/2-2$

The prefactor of this term is $-\frac{2}{D-4}\Gamma^2\left(\frac{D}{2}-1\right)\left(\frac{1}{4}-\nu^2\right)$. Application of (B.3) yields:

$$\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{2-D/2} = \frac{(D-2)(D-4)}{16(D-1)(D-3)}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{3-D} +$$
$$+\frac{(D-2)(D-4)}{16(D-1)}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{2-D} - \left(\frac{y}{4}\right)^{3-D}\right) -$$
$$-\frac{(D-2)(D-3)(D-4)}{16(D-1)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{2-D} - \left(\frac{y}{4}\right)^{3-D}\right) +$$
$$+\frac{(D-2)^2(D-4)}{32(D-1)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{1-D} - \left(\frac{y}{4}\right)^{2-D}\right) +$$
$$+\frac{(D-4)}{8(D-1)}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{2-D} + \frac{(D-2)^2(D-4)}{32(D-1)}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{1-D} \tag{B.6}$$

Thus with prefactor and additional symmetry factor of 2 it takes the following form in $D=4$:

$$-2\frac{2}{D-4}\Gamma^2\left(\frac{D}{2}-1\right)\left(\frac{1}{4}-\nu^2\right)\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{2-D/2} = \left(\frac{1}{4}-\nu^2\right)\times\Bigg\{$$
$$-\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{4(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{\Gamma\left(\frac{D}{2}-1\right)}{4(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\Box}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{(D-2)^2\Gamma\left(\frac{D}{2}-1\right)}{2(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{\Gamma\left(\frac{D}{2}-1\right)}{(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$-\frac{\Gamma\left(\frac{D}{2}-1\right)}{2(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\Box}{H^2}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{2(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{i\delta^D(x-x')}{\sqrt{-g}}\Bigg\}+$$

$$
+\left(\frac{1}{4}-\nu_4^2\right)\times\Big\{-\frac{1}{6}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\frac{4}{y}+\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left[-\frac{1}{12}g(y)+\frac{1}{6}\frac{4}{y}\right]
$$

$$
+H^4 g_{\mu\nu}g'_{\rho\sigma}\left[-\frac{1}{24}\frac{\Box}{H^2}g(y)+\frac{1}{3}g(y)-\frac{1}{6}\frac{4}{y}\right]
$$

$$
-\frac{1}{6}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma g(y)+H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left[-\frac{1}{12}\frac{\Box}{H^2}g(y)+\frac{1}{6}g(y)\right]\Big\} \tag{B.7}
$$

B.3 $\alpha = D/2-1$, $\beta = D/2-3$ and $\alpha = D/2-1$, $\beta = 1$

Consider $(\alpha,\beta)=(D/2-1,D/2-3)$ first.
The prefactor of this term is $\frac{2}{(D-4)(D-6)}\Gamma^2\left(\frac{D}{2}-1\right)\left(\frac{1}{4}-\nu^2\right)\left(\frac{9}{4}-\nu^2\right)$. Application of (B.3) yields:

$$
\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{3-D/2}=\frac{D(D-6)}{16(D-1)(D-3)}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{4-D}+
$$

$$
+\frac{D(D-4)(D-6)}{16(D-1)(D-3)}\left[H^2 g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2 g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{3-D}-\left(\frac{y}{4}\right)^{4-D}\right)-
$$

$$
-\frac{D(D-4)^2(D-6)}{16(D-1)(D-3)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{3-D}-\left(\frac{y}{4}\right)^{4-D}\right)+
$$

$$
+\frac{D(D-4)(D-6)}{32(D-1)}H^4 g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{2-D}-\left(\frac{y}{4}\right)^{3-D}\right)+
$$

$$
+\left[-\frac{D(D-4)(D-6)}{8(D-1)(D-3)}+\frac{(D-2)(D-6)}{8(D-3)}\right]H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{3-D}
$$

$$
+\frac{D(D-4)(D-6)}{32(D-1)}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{2-D} \tag{B.8}
$$

Consider now $(\alpha,\beta)=(D/2-1,-1)$.
The prefactor of this term is $\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)\frac{2}{D}\left(\frac{(D-1)^2}{4}-\nu^2\right)\frac{\Gamma(\frac{D-1}{2}\pm\nu)}{\Gamma(\frac{1}{2}\pm\nu)}$. Because $\beta+1=0$, formula (B.3) is particularly simple.

$$
\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)=-\frac{1}{2}\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{1-D/2} \tag{B.9}
$$

Thus the combined contribution of $(\alpha,\beta)=(D/2-1,D/2-3)$ and $(\alpha,\beta)=(D/2-1,-1)$ together witch a symmetry factor of 2 reads in $D=4$:

$$
2\Big[\frac{2}{(D-4)(D-6)}\Gamma^2\left(\frac{D}{2}-1\right)\left(\frac{1}{4}-\nu^2\right)\left(\frac{9}{4}-\nu^2\right)\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{3-D/2}+
$$

$$
+\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)\frac{2}{D}\left(\frac{(D-1)^2}{4}-\nu^2\right)\frac{\Gamma(\frac{D-1}{2}\pm\nu)}{\Gamma(\frac{1}{2}\pm\nu)}\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{1-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)\Big]=
$$

$$
=\left(\frac{1}{4}-\nu^2\right)\left(\frac{9}{4}-\nu^2\right)\times\Big\{\frac{D\Gamma\left(\frac{D}{2}-1\right)}{8(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{i\delta^D(x-x')}{\sqrt{-g}}
$$

$$+\frac{D\Gamma\left(\frac{D}{2}-1\right)}{4(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{i\delta^D(x-x')}{\sqrt{-g}}\Big\}$$
$$+\left[-\frac{1}{2}\left(\frac{9}{4}-\nu_4^2\right)-\frac{3}{2}\left(\frac{1}{4}-\nu_4^2\right)\right]H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}$$
$$+\left(\frac{1}{4}-\nu_4^2\right)\left(\frac{9}{4}-\nu_4^2\right)\times\Big\{-\frac{1}{3}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{1}{6}\left[H^2g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{4}{y}$$
$$+\frac{1}{12}H^4g_{\mu\nu}g'_{\rho\sigma}g(y)-\frac{1}{6}H^4g_{\mu\nu}g'_{\rho\sigma}\frac{4}{y}-2H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\ln(\frac{y}{4})}{y}$$
$$+\left[\psi(1)-\frac{5}{12}-\frac{1}{2}\left(\psi(\frac{1}{2}+\nu_4)+\psi(\frac{1}{2}-\nu_4)\right)\right]H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}+\frac{1}{6}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y)\Big\} \quad \text{(B.10)}$$

B.4 $\alpha = D/2-2$, $\beta = D/2-2$

The prefactor of this term is $\left(-\frac{2}{D-4}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{1}{4}-\nu^2\right)\right)^2$. Application of (B.3) yields:

$$\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{2-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{2-D/2}=\frac{(D-2)(D-4)}{16(D-1)(D-3)}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\left(\frac{y}{4}\right)^{4-D}+$$
$$+\frac{(D-2)(D-4)^2}{16(D-1)(D-3)}\left[H^2g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\left(\frac{1}{2}\left(\frac{y}{4}\right)^{3-D}-\left(\frac{y}{4}\right)^{4-D}\right)-$$
$$-\frac{(D-2)(D-4)^3}{16(D-1)(D-3)}H^4g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{3-D}-\left(\frac{y}{4}\right)^{4-D}\right)+$$
$$+\frac{(D-2)(D-4)^2}{32(D-1)}H^4g_{\mu\nu}g'_{\rho\sigma}\left(\frac{1}{2}\left(\frac{y}{4}\right)^{2-D}-\left(\frac{y}{4}\right)^{3-D}\right)+$$
$$+\frac{(D-4)^2}{8(D-1)(D-3)}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\left(\frac{y}{4}\right)^{3-D}$$
$$+\frac{(D-2)(D-4)^2}{32(D-1)}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\left(\frac{y}{4}\right)^{2-D} \quad \text{(B.11)}$$

Thus with prefactor it takes the following form in $D=4$:

$$\left(\frac{2}{D-4}\Gamma(\frac{D}{2}-1)\left(\frac{1}{4}-\nu^2\right)\right)^2\partial_\mu\partial'_\rho\left(\frac{y}{4}\right)^{2-D/2}\partial_\nu\partial'_\sigma\left(\frac{y}{4}\right)^{2-D/2}=$$
$$\left(\frac{1}{4}-\nu^2\right)^2\times\Big\{\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{8(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4g_{\mu\nu}g'_{\rho\sigma}\frac{i\delta^D(x-x')}{\sqrt{-g}}$$
$$+\frac{(D-2)\Gamma\left(\frac{D}{2}-1\right)}{4(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{H^D}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{i\delta^D(x-x')}{\sqrt{-g}}\Big\}$$
$$+\left(\frac{1}{4}-\nu_4^2\right)^2\times\Big\{-\frac{1}{6}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{1}{12}\left[H^2g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{4}{y}$$
$$+\frac{1}{24}H^4g_{\mu\nu}g'_{\rho\sigma}g(y)-\frac{1}{12}H^4g_{\mu\nu}g'_{\rho\sigma}\frac{4}{y}+\frac{1}{6}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}+\frac{1}{12}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y)\Big\} \quad \text{(B.12)}$$

B.5 Full Expression

Combining yields:

$$
\begin{aligned}
&\partial_\mu \partial'_\rho i\Delta(x-x')\partial_\nu \partial'_\sigma i\Delta(x-x') = \frac{H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D}{2}-1\right)}{(D-3)(D-4)} \Bigg\{ \\
&\quad \frac{D(D-2)}{8(D+1)(D-1)} \nabla_\mu \nabla_\nu \nabla'_\rho \nabla'_\sigma + \frac{D}{8(D+1)(D-1)} \left[H^2 g_{\mu\nu} \nabla'_\rho \nabla'_\sigma + H^2 g'_{\rho\sigma} \nabla_\mu \nabla_\nu\right] \frac{\square}{H^2} \\
&\quad - \frac{D(D-2)}{8(D+1)} \left[H^2 g_{\mu\nu} \nabla'_\rho \nabla'_\sigma + H^2 g'_{\rho\sigma} \nabla_\mu \nabla_\nu\right] + \frac{1}{8(D+1)(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\square}{H^2} \frac{\square}{H^2} \\
&\quad - \frac{((D+1)(D-2)+D(D-1))}{8(D+1)(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\square}{H^2} + \frac{D(D-1)(D-2)}{8(D+1)} H^4 g_{\mu\nu} g'_{\rho\sigma} \\
&\quad + \frac{1}{2(D+1)(D-1)} H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma \frac{\square}{H^2} - \frac{(D-2)}{2(D+1)(D-1)} H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma \\
&\quad + \frac{1}{4(D+1)(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \frac{\square}{H^2} \frac{\square}{H^2} - \frac{(D-2)}{4(D+1)(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \frac{\square}{H^2} \\
&\quad + \left(\frac{1}{4}-\nu^2\right) \times \Bigg[-\frac{(D-2)}{4(D-1)} \left[H^2 g_{\mu\nu} \nabla'_\rho \nabla'_\sigma + H^2 g'_{\rho\sigma} \nabla_\mu \nabla_\nu\right] - \frac{1}{4(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} \frac{\square}{H^2} \\
&\quad + \frac{(D-2)^2}{2(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} - \frac{1}{(D-1)} H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma - \frac{1}{2(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \frac{\square}{H^2} \\
&\quad + \frac{(D-2)}{2(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \Bigg] \\
&\quad + \left(\frac{1}{4}-\nu^2\right)\left(\frac{9}{4}-\nu^2\right) \times \left[\frac{D}{8(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} + \frac{D}{4(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'}\right] \\
&\quad + \left(\frac{1}{4}-\nu^2\right)^2 \times \left[\frac{(D-2)}{8(D-1)} H^4 g_{\mu\nu} g'_{\rho\sigma} + \frac{(D-2)}{4(D-1)} H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'}\right] \Bigg\} \frac{i\delta^D(x-x')}{\sqrt{-g}} \\
&+ \frac{H^4}{(4\pi)^4} \Bigg\{ \frac{1}{30} \nabla_\mu \nabla_\nu \nabla'_\rho \nabla'_\sigma g(y) + \left[H^2 g_{\mu\nu} \nabla'_\rho \nabla'_\sigma + H^2 g'_{\rho\sigma} \nabla_\mu \nabla_\nu\right] \left(\frac{1}{60} \frac{\square}{H^2} g(y) - \frac{1}{10} g(y)\right) \\
&\quad + H^4 g_{\mu\nu} g'_{\rho\sigma} \left(\frac{1}{240} \frac{\square}{H^2} \frac{\square}{H^2} g(y) - \frac{11}{120} \frac{\square}{H^2} g(y) + \frac{3}{10} g(y)\right) \\
&\quad + H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma \left(\frac{1}{60} \frac{\square}{H^2} g(y) - \frac{1}{30} g(y)\right) + H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \left(\frac{1}{120} \frac{\square}{H^2} \frac{\square}{H^2} g(y) - \frac{1}{60} \frac{\square}{H^2} g(y)\right) \\
&\quad + \left(\frac{1}{4}-\nu_4^2\right) \times \Bigg[-\frac{1}{6} \nabla_\mu \nabla_\nu \nabla'_\rho \nabla'_\sigma \frac{4}{y} + \left(H^2 g_{\mu\nu} \nabla'_\rho \nabla'_\sigma + H^2 g'_{\rho\sigma} \nabla_\mu \nabla_\nu\right) \left(-\frac{1}{12} g(y) + \frac{1}{6} \frac{4}{y}\right) \\
&\quad + H^4 g_{\mu\nu} g'_{\rho\sigma} \left(-\frac{1}{24} \frac{\square}{H^2} g(y) + \frac{1}{3} g(y) - \frac{1}{6} \frac{4}{y}\right) \\
&\quad - \frac{1}{6} H^2 \tilde{g}_{\mu\rho'} \nabla_\nu \nabla'_\sigma g(y) + H^4 \tilde{g}_{\mu\rho'} \tilde{g}_{\nu\sigma'} \left(-\frac{1}{12} \frac{\square}{H^2} g(y) + \frac{1}{6} g(y)\right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{1}{2}\left(\frac{9}{4}-\nu_4^2\right)-\frac{3}{2}\left(\frac{1}{4}-\nu_4^2\right)\right]H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}\\
&+\left(\frac{1}{4}-\nu_4^2\right)\left(\frac{9}{4}-\nu_4^2\right)\times\Bigg[-\frac{1}{3}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{1}{6}\left[H^2g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{4}{y}\\
&\quad+\frac{1}{12}H^4g_{\mu\nu}g'_{\rho\sigma}g(y)-\frac{1}{6}H^4g_{\mu\nu}g'_{\rho\sigma}\frac{4}{y}-2H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\ln(\frac{y}{4})}{y}\\
&\quad+\left(\psi(1)-\frac{5}{12}-\frac{1}{2}\psi\left(\frac{1}{2}\pm\nu_4\right)\right)H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}+\frac{1}{6}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y)\Bigg]\\
&+\left(\frac{1}{4}-\nu_4^2\right)^2\times\Bigg[-\frac{1}{6}\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma\ln\frac{y}{4}+\frac{1}{12}\left[H^2g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+H^2g'_{\rho\sigma}\nabla_\mu\nabla_\nu\right]\frac{4}{y}\\
&\quad+\frac{1}{24}H^4g_{\mu\nu}g'_{\rho\sigma}g(y)-\frac{1}{12}H^4g_{\mu\nu}g'_{\rho\sigma}\frac{4}{y}+\frac{1}{6}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{4}{y}+\frac{1}{12}H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}g(y)\Bigg]\\
&+2\partial_\mu\partial'_\rho\,\frac{4}{y}\,\partial_\nu\partial'_\sigma\,f_2(y)+2\left(\frac{1}{4}-\nu_4^2\right)\partial_\mu\partial'_\rho\,\ln\frac{y}{4}\,\partial_\nu\partial'_\sigma\,f_1(y)+\partial_\mu\partial'_\rho\,f_1(y)\,\partial_\nu\partial'_\sigma\,f_1(y)\Bigg\}
\end{aligned}
\tag{B.13}
$$

C Counterterm Vertices

Here the vertices for the countererm actions of section 6.2 are computed. Recall the 'improved' results (6.58),(6.64),(6.65),

$$\begin{aligned} S_{\mathsf{ct1}} = \int d^D x\sqrt{-g}\, a_1\Big\{ & \mathring{R}^2 + 4\mathring{R}(D-1)\,\mathring{\nabla}_\lambda T^\lambda - 2\mathring{R}(D-1)(D-2)g^{\mu\nu}T_\mu T_\nu \\ & + 4(D-1)^2(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3)\Big\} \end{aligned} \tag{C.1}$$

$$\begin{aligned} S_{\mathsf{ct2,5}} = \int d^D x\sqrt{-g}\, a_2\Big\{ & \mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + 2\mathring{R}\big(\mathring{\nabla}_\lambda T^\lambda - (D-2)T_\lambda T^\lambda\big) + 2(D-2)\mathring{R}^{\mu\nu}\mathring{\nabla}_\nu T_\mu \\ & - (D-2)(D-4)\mathring{R}^{\mu\nu}T_\mu T_\nu + D(D-1)(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3)\Big\} \end{aligned} \tag{C.2}$$

$$\begin{aligned} S_{\mathsf{ct3,5}} = \int d^D x\sqrt{-g}\, a_3\Big\{ & \mathring{R}_{\mu\nu\rho\sigma}\mathring{R}^{\mu\nu\rho\sigma} + 8\mathring{R}^{\mu\nu}\nabla_\nu T_\mu - 4(D-4)\mathring{R}^{\mu\nu}T_\mu T_\nu - 4\,\mathring{R}T_\lambda T^\lambda \\ & + 4(D-1)(\mathring{\nabla}_\lambda T^\lambda)^2 + \mathcal{O}(T_\alpha^3)\Big\}\,, \end{aligned} \tag{C.3}$$

with the $\mathring{B}$ notation for quantities computed using the Levi-Civita connection.
The relevant vertices are computed by

$$T^{\mathsf{cti}}_{\mu\nu}(x) = \frac{-2}{\sqrt{-g}}\,\frac{\delta S_{\mathsf{cti}}}{\delta g^{\mu\nu}(x)}\bigg|_{g=g^{\mathsf{dS}},T=0} \tag{C.4}$$

$${}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{cti}} = \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\,\frac{\delta^2 S_{\mathsf{cti}}}{\delta T_\alpha(x)\delta T_\beta(x')}\bigg|_{g=g^{\mathsf{dS}},T=0} \tag{C.5}$$

$${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{cti}} = \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\,\frac{\delta^2 S_{\mathsf{cti}}}{\delta g^{\mu\nu}(x)\delta T_\beta(x')}\bigg|_{g=g^{\mathsf{dS}},T=0} \tag{C.6}$$

$${}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\mathsf{cti}} = \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\,\frac{\delta^2 S_{\mathsf{cti}}}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\bigg|_{g=g^{\mathsf{dS}},T=0} \tag{C.7}$$

with $i \in \{1,(2,5),(3,5)\}$. As for the vertices of the original theory, the second order metric variation in (C.7) is the most intricate calculation, which is done in the appendix D.
Upon using (C.1) one finds:

$$\begin{aligned} T^{\mathsf{ct1}}_{\mu\nu} &= a_1 g_{\mu\nu}\mathring{R}^2\frac{D-4}{D} \\ {}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct1}} &= 4a_1(D-1)\left(-\mathring{R}(D-2)g^{\alpha\beta} + 2(D-1)\mathring{\nabla}^\alpha\mathring{\nabla}'^\beta\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \end{aligned} \tag{C.8}$$

$${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct1}} = -4a_1(D-1)\left(\mathring{R}_{\mu\nu} + g_{\mu\nu}\mathring{\Box} - \mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\mathring{\nabla}'^\beta\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.9}$$

As in section 6.1, derivatives with prime act at x', while unprimed derivatives are to be evaluated at x.

It violates the Ward identities (6.17),(6.18) according to:

$$\mathring{\nabla}_{\alpha\ TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct1}} + 2g^{\mu\nu}(x)\ {}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct1}} = 4a_1 H^2 D(D-1)^2(D-4)\mathring{\nabla}'^{\beta}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.10}$$

$$\begin{aligned}&\mathring{\nabla}_{\beta\ gT}\Gamma^{\beta}_{\rho\sigma}(x',x)_{\mathsf{ct1}} - \frac{\delta^D(x-x')}{\sqrt{-g}}T^{\mathsf{ct1}}_{\rho\sigma} + 2g^{\mu\nu}\ {}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\mathsf{ct1}} =\\ &= 2a_1 D(D-1)(D-4)\left(-H^2 g'_{\rho\sigma}\mathring{\Box} + H^2\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + \tfrac{1}{4}(D-1)(D-4)H^4 g'_{\rho\sigma}\right)\frac{\delta^D(x-x')}{\sqrt{-g}}\end{aligned} \tag{C.11}$$

Counterterm action 2:

$$T^{\mathsf{ct2,5}}_{\mu\nu} = a_2 g_{\mu\nu}\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta}(D-4)$$

$${}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct2,5}} = 2a_2(D-1)\left(-(D-2)(3D-4)H^2 g^{\alpha\beta} + D\mathring{\nabla}^{\alpha}\mathring{\nabla}'^{\beta}\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.12}$$

$${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct2,5}} = -Da_2\left(\mathring{R}_{\mu\nu} + g_{\mu\nu}\mathring{\Box} - \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\right)\mathring{\nabla}'^{\beta}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.13}$$

It violates the Ward identities (6.17),(6.18) according to:

$$\mathring{\nabla}_{\alpha\ TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct2,5}} + 2g^{\mu\nu}(x)\ {}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct2,5}} = 4a_2 H^2(D-1)^2(D-4)\mathring{\nabla}'^{\beta}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.14}$$

$$\begin{aligned}&\mathring{\nabla}_{\beta\ gT}\Gamma^{\beta}_{\rho\sigma}(x',x)_{\mathsf{ct2,5}} - \frac{\delta^D(x-x')}{\sqrt{-g}}T^{\mathsf{ct2,5}}_{\rho\sigma} + 2g^{\mu\nu}\ {}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\mathsf{ct2,5}} =\\ &= 2a_2(D-1)(D-4)\left(-H^2 g'_{\rho\sigma}\mathring{\Box} + H^2\mathring{\nabla}'_{\rho}\mathring{\nabla}'_{\sigma} + \tfrac{1}{4}(D-1)(D-4)H^4 g'_{\rho\sigma}\right)\frac{\delta^D(x-x')}{\sqrt{-g}}\end{aligned} \tag{C.15}$$

Counterterm action 3:

$$T^{\mathsf{ct3,5}}_{\mu\nu} = a_3 g_{\mu\nu}\mathring{R}_{\alpha\beta\gamma\delta}\mathring{R}^{\alpha\beta\gamma\delta}\frac{D-4}{D}$$

$${}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct3,5}} = 8a_3(D-1)\left(-2(D-2)H^2 g^{\alpha\beta} + \mathring{\nabla}^{\alpha}\mathring{\nabla}'^{\beta}\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.16}$$

$${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct3,5}} = -4a_3\left(\mathring{R}_{\mu\nu} + g_{\mu\nu}\mathring{\Box} - \mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\right)\mathring{\nabla}'^{\beta}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.17}$$

It violates the Ward identities (6.17),(6.18) according to:

$$\mathring{\nabla}_\alpha \, {}_{TT}\Gamma^{\alpha\beta}(x,x')_{\mathsf{ct3,5}} + 2g^{\mu\nu}(x)\, {}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{\mathsf{ct3,5}} = 8a_3 H^2(D-1)(D-4)\mathring{\nabla}'^{\beta}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.18}$$

$$\mathring{\nabla}_\beta \, {}_{gT}\Gamma^{\beta}_{\rho\sigma}(x',x)_{\mathsf{ct3,5}} - \frac{\delta^D(x-x')}{\sqrt{-g}} T^{\mathsf{ct3,5}}_{\rho\sigma} + 2g^{\mu\nu}\, {}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{\mathsf{ct3,5}} =$$
$$= a_3 4(D-4)\left(-H^2 g'_{\rho\sigma}\mathring{\Box} + H^2\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + \tfrac{1}{4}(D-1)(D-4)H^4 g'_{\rho\sigma}\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.19}$$

Final counterterm vertices

After determining the correct linear combinations and coefficients in (6.68) and (6.69),(6.70), respectively, the theory is eventually renormalized with the following corrections: (R^2 denoting contributions from term (6.66))

$${}_{TT}\Gamma^{\alpha\beta}(x,x')_{R^2} = \left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \frac{\mu^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D}{2}-1\right)}{(D-3)(D-4)}\times$$
$$\times\left(-DH^2 g^{\alpha\beta} + \mathring{\nabla}^\alpha\mathring{\nabla}'^\beta\right)\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.20}$$

$${}_{gT}\Gamma^{\beta}_{\mu\nu}(x,x')_{R^2} = -\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \frac{\mu^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D}{2}-1\right)}{2(D-1)(D-3)(D-4)}\times$$
$$\times\left(\mathring{R}_{\mu\nu} + g_{\mu\nu}\mathring{\Box} - \mathring{\nabla}_{(\mu}\mathring{\nabla}_{\nu)}\right)\mathring{\nabla}'^\beta\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.21}$$

The way to read the gg vertices is that all terms in the curly brackets act on the δ function outside of it. Also recall the bitensor $\tilde{g}_{\mu\rho'}$ from (4.19).

$${}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{R^2} = \frac{1}{4}\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \frac{\mu^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D}{2}-1\right)}{(D-1)(D-2)(D-3)(D-4)}\times$$
$$\times\Bigg\{\frac{D-2}{D-1}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma - \frac{D-2}{D-1}H^2\left(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\frac{\mathring{\Box}}{H^2}$$
$$+ 2H^2(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma + g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu) + \frac{D-2}{D-1}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2}$$
$$+ (D-4)H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2} - 2(D-1)H^4 g_{\mu\nu}g'_{\rho\sigma}$$
$$+ DH^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2} + 2DH^2\tilde{g}_{\mu)(\rho'}\mathring{\nabla}_{\sigma')}\mathring{\nabla}_{(\nu}\Bigg\}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{C.22}$$

$${}_{gg}\Gamma_{\mu\nu\rho\sigma}(x,x')_{C^2} = \frac{\mu^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D}{2}-1\right)}{8(D+1)(D-1)(D-3)(D-4)}\times\Bigg\{\frac{D-2}{D-1}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\mathring{\nabla}'_\sigma\mathring{\nabla}'_\rho$$

$$
\begin{aligned}
&+\frac{1}{D-1}H^2\left(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu\right)\frac{\mathring{\Box}}{H^2}-\frac{1}{D-1}H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2}\\
&-2H^2(g_{\mu\nu}\mathring{\nabla}'_\rho\mathring{\nabla}'_\sigma+g'_{\rho\sigma}\mathring{\nabla}_\mu\mathring{\nabla}_\nu)-H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\mathring{\Box}}{H^2}+2(D-1)H^4 g_{\mu\nu}g'_{\rho\sigma}\\
&+2\tilde{g}_{\mu)(\rho'}\mathring{\nabla}_{\sigma')}\mathring{\nabla}_{(\nu}\frac{\mathring{\Box}}{H^2}-4H^2\tilde{g}_{\mu)(\rho'}\mathring{\nabla}_{\sigma')}\mathring{\nabla}_{(\nu}+H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2}\frac{\mathring{\Box}}{H^2}\\
&-(D-2)H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\mathring{\Box}}{H^2}\bigg\}\frac{\delta^D(x-x')}{\sqrt{-g}} \qquad \text{(C.23)}
\end{aligned}
$$

Neither C^2 nor the R^2 term give rise to a tadpole-like contribution $T_{\mu\nu} \propto \frac{\delta S}{\delta g^{\mu\nu}}$.

D Second Order Metric Variations of Curvature Scalars

Throughout: Torsion vanishes, that is, $\mathring{\nabla} = \nabla$, $\mathring{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu}$, etc.
Throughout: $(\mu\nu)$ and $(\rho\sigma)$ are understood to be symmetrized, which will be omitted in the notation for clarity.
In the second order variation of the counterterm action (6.54) with respect to $g^{\mu\nu}$ appear 4 tensor structures that require the usage of the bilocal metric $\tilde{g}_{\mu\rho'}$ together with the relations (A.23)-(A.26). The metric variations on the left-hand side are meant to only act on $\delta g^{\mu\nu}$ on the right-hand side only, while all other metric tensors on the right side are kept fixed.

$$\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}\nabla_\beta \delta g^{\alpha\beta}g_{\alpha\gamma}\nabla_\delta \delta g^{\gamma\delta} =$$
$$= \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}} - H^2 g_{\mu\nu}g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{D.1}$$

$$\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}\Box\delta g^{\alpha\beta}g_{\alpha\gamma}g_{\beta\delta}\delta g^{\gamma\delta} =$$
$$= \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\frac{\delta^D(x-x')}{\sqrt{-g}} + 2H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{D.2}$$

$$\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}\Box\nabla_\beta\delta g^{\alpha\beta}g_{\alpha\gamma}\nabla_\delta\delta g^{\gamma\delta} =$$
$$= \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\Box\frac{\delta^D(x-x')}{\sqrt{-g}} + H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}} - H^2 g_{\mu\nu}g'_{\rho\sigma}\Box\frac{\delta^D(x-x')}{\sqrt{-g}}$$
$$- 2H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + g'_{\rho\sigma}\nabla_\mu\nabla_\nu)\frac{\delta^D(x-x')}{\sqrt{-g}} + (D-1)H^4 g_{\mu\nu}g'_{\rho\sigma}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{D.3}$$

$$\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}}\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}\Box\delta g^{\alpha\beta}g_{\alpha\gamma}g_{\beta\delta}\Box\delta g^{\gamma\delta} =$$
$$= \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\Box\frac{\delta^D(x-x')}{\sqrt{-g}} + 4H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\frac{\delta^D(x-x')}{\sqrt{-g}}$$
$$- 8H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\frac{\delta^D(x-x')}{\sqrt{-g}} + 4H^4(g_{\mu\nu}g'_{\rho\sigma} + \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'})\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{D.4}$$

With the four basic results at hand the second order variations of the curvature scalars follow after a short calculation: (All results are valid up to terms $\propto \nabla R_{\mu\nu\rho\sigma}$, which vanish on de Sitter)

$$\frac{\delta^2}{\delta g^{\mu\nu}(x)g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}R =$$
$$= \sqrt{-g(x)}\sqrt{-g(x')}\Big\{ -\frac{1}{2}g_{\mu\nu}g'_{\rho\sigma}\Box + \frac{1}{2}(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + g'_{\rho\sigma}\nabla_\mu\nabla_\nu) + \frac{(D-1)(D-4)}{4}H^2 g_{\mu\nu}g'_{\rho\sigma}$$
$$+ \frac{1}{2}\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box + \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma + \frac{(D-1)(D-2)}{2}\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Big\}\frac{\delta^D(x-x')}{\sqrt{-g}} \tag{D.5}$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta g^{\mu\nu}(x) g^{\rho\sigma}(x')} \int d^D x \sqrt{-g} R^2 = \\
&= \sqrt{-g(x)}\sqrt{-g(x')}\Big\{ 2 g_{\mu\nu} g'_{\rho\sigma} \Box\Box - 2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma\Box + g'_{\rho\sigma}\nabla_\mu\nabla_\nu\Box) + 2\nabla_\mu\nabla_\nu\nabla'_\rho\nabla'_\sigma \\
&\quad - (D-1)(D-4)H^2 g_{\mu\nu} g'_{\rho\sigma}\Box + (D-1)(D-2)H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + g'_{\rho\sigma}\nabla_\mu\nabla_\nu) \\
&\quad + \left[\frac{D^2(D-1)^2}{4} - 2(D-1)^3\right] H^4 g_{\mu\nu} g'_{\rho\sigma} \\
&\quad + D(D-1)H^2 \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box + 2D(D-1)H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma \\
&\quad + \left[\frac{D^2(D-1)^2}{2} - 2D(D-1)^2\right] H^4 \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Big\}\frac{\delta^D(x-x')}{\sqrt{-g}}
\end{aligned} \tag{D.6}
$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta g^{\mu\nu}(x) g^{\rho\sigma}(x')} \int d^D x \sqrt{-g} R_{\alpha\beta}R^{\alpha\beta} = \\
&= \sqrt{-g(x)}\sqrt{-g(x')}\Big\{ \frac{1}{2} g_{\mu\nu} g'_{\rho\sigma} \Box\Box' - \frac{1}{2}(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma\Box + g'_{\rho\sigma}\nabla_\mu\nabla_\nu\Box') + \nabla_\mu\nabla_\nu\nabla'_\sigma\nabla'_\rho \\
&\quad - \frac{D-3}{2}H^2 g_{\mu\nu} g'_{\rho\sigma}\Box + (D-3)H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + g'_{\rho\sigma}\nabla_\mu\nabla_\nu) \\
&\quad + \left[\frac{D(D-1)^2}{4} - 2(D-1)(D-2)\right] H^4 g_{\mu\nu} g'_{\rho\sigma} \\
&\quad + \frac{1}{2}\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\Box' + \tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\Box + (D-1)H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box + 3(D-2)H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma \\
&\quad + \left[\frac{D(D-1)^2}{2} - 2(D-1)^2\right] H^4 \tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Big\}\frac{\delta^D(x-x')}{\sqrt{-g}}
\end{aligned} \tag{D.7}
$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta g^{\mu\nu}(x) g^{\rho\sigma}(x')} \int d^D x \sqrt{-g} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \\
&= \sqrt{-g(x)}\sqrt{-g(x')}\Big\{ 2\nabla_\mu\nabla_\nu\nabla'_\sigma\nabla'_\rho - 2H^2 g_{\mu\nu} g'_{\rho\sigma}\Box - 2H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma + g'_{\rho\sigma}\nabla_\mu\nabla_\nu) \\
&\quad + \frac{D(D-1)}{2}H^4 g_{\mu\nu} g'_{\rho\sigma} + 2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\Box' + 4\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\Box - 2(D-4)H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box \\
&\quad + (D-1)(D-4)H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Big\}\frac{\delta^D(x-x')}{\sqrt{-g}}
\end{aligned} \tag{D.8}
$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta g^{\mu\nu}(x) g^{\rho\sigma}(x')} \int d^D x \sqrt{-g} C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \\
&= \sqrt{-g(x)}\sqrt{-g(x')}\Big\{ -2\frac{D-3}{(D-1)(D-2)} g_{\mu\nu} g'_{\rho\sigma}\Box\Box' \\
&\quad + 2\frac{D-3}{(D-1)(D-2)}(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma\Box + g'_{\rho\sigma}\nabla_\mu\nabla_\nu\Box') + 2\frac{D-3}{D-1}\nabla_\mu\nabla_\nu\nabla'_\sigma\nabla'_\rho
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{D-3}{D-2}H^2 g_{\mu\nu}g'_{\rho\sigma}\Box-4\frac{D-3}{D-2}H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+g'_{\rho\sigma}\nabla_\mu\nabla_\nu)+4\frac{(D-1)(D-3)}{D-2}H^4 g_{\mu\nu}g'_{\rho\sigma}\\
&+2\frac{D-3}{D-2}\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\Box'+4\frac{D-3}{D-2}\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\Box-2(D-3)H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\\
&-8\frac{D-3}{D-2}H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma\Bigg\}\frac{\delta^D(x-x')}{\sqrt{-g}}
\end{aligned}
\tag{D.9}
$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta g^{\mu\nu}(x)g^{\rho\sigma}(x')}\int d^Dx\sqrt{-g}E=\\
&\quad=\sqrt{-g(x)}\sqrt{-g(x')}(D-3)(D-4)\Bigg\{-H^2 g_{\mu\nu}g'_{\rho\sigma}\Box+H^2(g_{\mu\nu}\nabla'_\rho\nabla'_\sigma+g'_{\rho\sigma}\nabla_\mu\nabla_\nu)\\
&\qquad+\tfrac{1}{4}(D-1)(D-6)H^4 g_{\mu\nu}g'_{\rho\sigma}+H^2\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Box\\
&\qquad+2H^2\tilde{g}_{\mu\rho'}\nabla_\nu\nabla'_\sigma+\tfrac{1}{2}(D-1)(D-2)H^4\tilde{g}_{\mu\rho'}\tilde{g}_{\nu\sigma'}\Bigg\}\frac{\delta^D(x-x')}{\sqrt{-g}}
\end{aligned}
\tag{D.10}
$$

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