Utrecht University

Faculty of Science

Percolation in random directed graphs with an arbitrary degree distribution

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Femke van Ieperen

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Supervisor:

Dr. I. Kryven
Mathematical Institute

Second Reader:

Dr. R. Bisseling
Mathematical Institute

Abstract

Random graphs are probability spaces having graphs obeying predefined constraints as events. Sampling from such a space can be a challenge because of non-trivial dependencies that graph constraints may impose. An exotic observation that has been made in various random graphs is that small changes in the constraints may impose dramatic changes in the graph structure – the phenomenon that is referred to as the phase transition. One example of such a small change is gradual removal of the edges (or vertices), also known as percolation. This work brings the notion of directionality to random graphs with a given degree sequence, by studying two aspects: 1) algorithmic construction of such random graphs and 2) percolation processes on vertices and edges.

For undirected graphs, the percolation threshold for existence of the giant component (a component whose size scales linearly in the total number of vertices) is derived by Jason [1] and Fountoulakis [2] who both built upon earlier results of Molloy and Reed [3] about the existence of a giant component. We derive the percolation threshold for the existence of a giant strongly connected component for edge and vertex percolation by extending the results of Fountoulakis for undirected graphs and combining them with the Cooper and Frieze's [4] existence criteria. By building on the results of Bayati, Kim and Saberi [5], we then develop an algorithm that generates directed random graphs almost uniformly with runtime close to linear in the number of edges. Finally, we illustrate the theoretical results with numerical simulations.

Contents

1 Introduction

Random graphs are probability spaces having graphs obeying predefined constraints as events. It has been observed that a small change in these constraints, may impose dramatic changes in the graph structure. This phenomenon is referred to as the phase transition. One example of such a small change is gradual removal of the edges or vertices, also known as percolation. The study of percolation in random graphs dates back to the 1960's. In this decade Erdős and Rényi [6] studied random graphs on n vertices where each of the possible $\frac{n(n-1)}{2}$ edges is present with probability p. This is equivalent to removing every edge of the complete graph on n vertices with probability $1-p$, *i.e.* applying bond percolation to the complete graph with percolation probability p. The dramatic change in the graph structure observed in this model is the appearance of a giant component. A giant component is a connected component whose size scales linearly with the number of vertices. Erdős and Rényi determined the critical value p_c such that for all $p > p_c$ the graph with high probability contains a giant component and for $p < p_c$ it does not. For this reason p_c is called the percolation threshold.

One can also study the percolation threshold for emergence of a giant component in random undirected graphs with a given degree distribution. Fountoulakis [2] and Jason [1] studied this question independently using different techniques. What their techniques have in common is that they both rely upon Molloy and Reed's theorem [3, 7]. This theorem indicates whether an undirected random graph with a given degree distribution contains a giant component and how large it is. There are many variations on the work of Molloy and Reed, proving the theorem under slightly different conditions or using a new technique, see for example [8–10]. Determining the percolation threshold for random graphs with a given degree distribution is more than a theoretical tool. It has many applications, such as the resilience of networks under breakdowns [11], and the spread of diseases in epidemics [12, 13]. In some applications the networks are better modelled using directed graphs instead of undirected ones.

In this work we will study percolation in random directed graphs with a given degree distribution. Our *main contribution* is a proof for the percolation threshold for emergence of the giant strongly connected component (GSCC). This is achieved by extending the results of Fountoulakis for undirected graphs and combining them with the Cooper and Frieze's existence criteria [4].

Besides investigating percolation, we also propose a *new algorithm* for constructing random directed graphs with a given degree sequence. We then use this algorithm to illustrate the theoretical results on the percolation threshold with numerical simulations. Being able to sample graphs numerically gives access to rich information about such graphs, as the samples may be further analysed with graph algorithms. For example, by using our construction algorithm one may numerically investigate the diameter, cycles, spectral properties, etc. in random directed graphs.

Sampling from a random graph model can be a challenge because of non-trivial dependencies that graph constrains may impose. There are several algorithms for generation of random directed graphs with a given degree distribution. Examples are the repeated configuration model and Markov Chain Monte Carlo algorithms. The repeated configuration model allows to draw graphs uniformly at random, however its runtime is unknown and numerical experiments show that this procedure is not practical even for very small graphs. Indeed, the number of configurations inducing a simple graph with a given degree sequence is unknown. A more popular algorithm to generate simple digraphs uses a Markov Chain [14]. For a general introduction on Markov Chain algorithms, we refer the interested reader to [15]. This algorithm randomises a given graph by swapping edges and reorienting cycles while preserving the degree sequence. Generating the initial graph is not difficult in practice. For example, the Havel-Hakimi algorithm constructs a deterministic simple digraph obeying a given degree sequence [16]. The problem is that for a general degree sequence the mixing time of the Markov chain is unknown. This means that it is not known when the graph is sufficiently independent from the initial guess, *i.e.* when the algorithm has achieved a uniformly random graph. Additionally for those degree sequences that do have a bound on the mixing time, this bound often appears to be a high degree polynomial in n [14].

Neither the repeated configuration model nor the Markov Chain approach is an ideal way to uniformly generate random graphs obeying a given degree sequence. In this work we build an algorithm that will almost uniformly generate random simple directed graphs obeying a given degree sequence with a expected runtime near-linear in m. Our algorithm is a generalisation of an algorithm by Bayati, Kim and Saberi [5] for undirected graphs. While our algorithm does not provide exact uniformity, it is a good trade-off if asymptotic uniformity of the graphs suffices as well.

The rest of the thesis is structured as follows. Before we present the main results concerning the percolation threshold for the GSCC and the construction algorithm for directed random graphs, we first explain the preliminaries. In Section 2 random directed graphs with a given degree distribution are defined. In this section, we also explain the configuration model, which is a method to generate random *multigraphs* obeying a given degree sequence. This model will be used in Sections 3, 4 and 5. Then we discuss the results from Cooper and Frieze [4] on the existence of the GSCC in Section 3. At this point we are ready to study percolation. After defining the site and bond percolation processes, our main result regarding percolation is stated in Section 4. The proof of this result is given in the same section, which concludes our theoretical study of percolation. In the second part of the thesis, the focus shifts to the algorithmic construction of random directed graphs with a given degree sequence. Our new algorithm for constructing random directed graphs with a given degree sequence, which is an extension of the work from Bayati, Kim and Saberi [5] to the directed case, can be found in Section 5. This includes a proof that this algorithm generates graphs distributed up to a factor of $1 \pm o(1)$ of uniformity and the runtime analysis. Combining the algorithm from Section 5 with the results from Section 4, we illustrate the theory with numerical simulations in Section 6. We consider random graphs obeying two simple degree distribution with a small and constant maximum degree. Both types of percolation are investigated. Our work is concluded in Section 7 with a summary of the results and ideas for further research.

2 Random directed graphs with a given degree distribution

The goal of this section is to define random directed graphs with a given degree distribution. In Section 2.1 basic properties of graphs and notation is discussed. These allow us to define *degree sequences* and *degree arrays* in Section 2.2. Section 2.3 discusses the *directed configuration model*, which is a theoretical model for a random directed graph obeying a given degree sequence. This model will be useful in the later sections where we investigate the behaviour of random graphs.

2.1 Basic notation for graphs

A directed graph G_n consists of a set of vertices V and a set of edges $E \subset V \times V$, where n is the number of vertices in the graph, i.e. $|V| = n$. Unless stated otherwise by a graph we mean a directed graph or digraph. The graph is assumed to have m edges, i.e. $|E| = m$. The number of vertices and the number of edges are natural numbers, denoted by $\mathbb{N}$. We also use $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for the set of all nonnegative integers. In a directed graph, each edge has a direction. Given an edge $(u, v) \in E$, we call u its source and v its target. We can only move along an edge from its source to its target. In many applications a graph can only contain one copy of an edge (u, v) and there are no edges from a vertex to itself, also called *self-loops*. Such a graph is called a *simple graph*. If a graph contains multiple copies of an edge, also called *multiedges*, or self-loops or both, it is called a *multigraph*. In general, when we say a graph, we mean a simple directed graph. To emphasise this difference, a simple graph with n vertices is denoted by G_n, while $\tilde{G}_n$ stands for a multigraph on the same amount of vertices.

Given a vertex $v \in V$, its in-degree indicates the number of edges in G_n with v as target, i.e.

$$d_v^- = |\{(u, v) \in E\}| .$$

The out-degree indicates the number of edges with v as source and is thus defined as

$$d_v^+ = |\{(v, w) \in E\}| .$$

Another quantity of a graph that will be of interest later, is its the maximum degree $d_{\max} := \max\{d_{\max}^+, d_{\max}^-\}$, where $d_{\max}^+ = \max_{v \in V} d_v^+$ and $d_{\max}^- = \max_{v \in V} d_v^-$. The in- and the out-degree of every vertex in the graph together form the degree sequence,

$$\vec{d^n} := \begin{pmatrix} d^{n,-} \\ d^{n,+} \end{pmatrix} = \begin{pmatrix} d_1^-, d_2^-, \ldots, d_n^- \\ d_1^+, d_2^+, \ldots, d_n^+ \end{pmatrix} . \tag{2.1}$$

Given a graph its degree sequence is uniquely defined, for example, by adopting a lexicographic order. However, given a degree sequence $\vec{d^n}$, there might be multiple graphs all obeying it. Denote the set of all multigraphs obeying degree sequence $\vec{d^n}$ by $\mathcal{G}_{\vec{d^n}}$. Given a degree sequence $\vec{d^n}$, one may define a 'model' according to which a graph is chosen uniformly at random from $\mathcal{G}_{\vec{d^n}}$. Such a model can be viewed as a graph-valued alternative to the uniform random variable. A random simple graph that obeys a given degree sequence will be denoted by $G_{\vec{d^n}}$. A random *multigraph* obeying the same degree sequence will be denoted by $\tilde{G}_{\vec{d^n}}$.

Our goal is to study limiting behaviour of sequences of random graphs with a given degree sequence. To study this we consider random graphs with a given *degree distribution*. The degree distribution is a bivariate probability distribution on the non-negative integers, prescribing the probability that a uniformly at random chosen vertex has degree (j, k), *i.e.* it has in-degree j and out-degree k. For every $n \in \mathbb{N}$, we draw a degree sequence $\vec{d^n}$ from the bivariate degree distribution, and uniformly choose a simple element from $\mathcal{G}_{\vec{d^n}}$. All degree sequences in such a progression are collected together into a *degree array*, denoted as $\left(\vec{d^n}\right)_{n=1}^{\infty}$. All graphs in such a progression are collected together into a *graph array*, denoted by $\left(G_{\vec{d^n}}\right)_{n=1}^{\infty}$. There is a slight issue with this definition. When choosing degree sequence $\vec{d^n}$ at random, there is no guarantee that

there is a graph that corresponds to it, as it could happen that $\mathcal{G}_{\vec{d^n}} = \emptyset$. Also to be able to study the limiting behaviour, the degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ must satisfy some constraints. Therefore we will restrict sampling of the degree sequences, by introducing a notion of a *graphical* degree sequence and a *feasible* degree array.

2.2 Graphical degree sequences & feasible degree arrays

The first concern is that $\mathcal{G}_{\vec{d^n}}$ might be empty, which we avoid by constraining the degree sequence to be *valid*.

Definition 2.1. A degree sequence $\vec{d^n}$ is *valid* if for all $i \in \{1, 2, \ldots, n\}$, $d_i^-, d_i^+ \in \mathbb{N}_0$ and

$$m := \sum_{i=1}^{n} d_i^- = \sum_{i=1}^{n} d_i^+. \tag{2.2}$$

A graph obeying this degree sequence has m edges.

These constraints follow naturally from the definitions of the in- and out-degree of a vertex. Since the sum of the in-degrees equals the sum of the out-degrees, it will always be possible to draw edges such that the graph obeys the desired degree sequence if self-loops and multi-edges are allowed. Thus, so long as multigraphs are of concern, set $\mathcal{G}_{\vec{d^n}}$ is non-empty for any valid degree sequence.

If there is a simple graph obeying a degree sequence, it is called *graphical*. As we are interested in the behaviour of simple graph, we will consider graphical degree sequences. The following theorem allows to determine whether a degree sequence is graphical or not. It is a generalization of the Erdős-Gallai theorem for undirected graphs (see for example, [17]).

Theorem 2.2 (Fulkerson)**.** *[18, Theorem 4] Let $\vec{d^n}$ be a valid degree sequence. Define $\overrightarrow{d^n}$ to be a ordering of $\vec{d^n}$ such that $d_i^+ \geq d_{i+1}^+$ and $d_i^- \geq d_{i+1}^-$ if $d_i^+ = d_{i+1}^+$ for all $i \in \{1, 2, \ldots, n-1\}$. This is called a positive lexicographical ordering. Furthermore define $\underrightarrow{d^n}$ to be a ordering of $\vec{d^n}$ such that $d_i^- \geq d_{i+1}^-$ and $d_i^+ \geq d_{i+1}^+$ if $d_i^- = d_{i+1}^-$. Then the degree sequence is graphical, i.e. can be represented by a simple graph, if and only if for all $k \in \{1, 2, \ldots, n\}$:*

$$\sum_{i=1}^{k} \min[\overrightarrow{d_i^-}, k-1] + \sum_{i=k+1}^{n} \min[\overrightarrow{d_i^-}, k] \geq \sum_{i=1}^{k} \overrightarrow{d_i^+} \tag{2.3}$$

and

$$\sum_{i=1}^{k} \min[\underrightarrow{d_i^+}, k-1] + \sum_{i=k+1}^{n} \min[\underrightarrow{d_i^+}, k] \geq \sum_{i=1}^{k} \underrightarrow{d_i^-}. \tag{2.4}$$

In Section 2.1 we indicated that a degree array is drawn from a given degree distribution. For all $j, k \geq 0$ denote the number of vertices with degree (j, k) by

$$N_{j,k}(n) = \left|\{i \in V_n | d_i^- = j, d_i^+ = k\}\right|. \tag{2.5}$$

Here the subscript n of the vertex set, indicates that it consists of n elements. The requirement that the degree sequences of the array are drawn from a bivariate probability distribution is equivalent to requiring that $\lim_{n\to\infty} \frac{N_{j,k}(n)}{n}$ converges to this distribution for all $j, k \geq 0$. We will show that

$$\lim_{n\to\infty} \mathbb{P}\left[G_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right] = 1,$$

where $\mathcal{A}\left(\vec{d^n}\right)$ is the set of all multigraphs, which obey the degree sequence $\vec{d^n}$ while also satisfying a desired property, to be specified later. If the limit exists for a given property, then we say that the random graph has this property with high probability (w.h.p) or asymptotically almost surely (a.a.s.). Our primary goal is to study connected components, for which we rely on some stronger assumptions on the degree distribution as formalized by the following definition:

Definition 2.3. A degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is called feasible if for any $n \in \mathbb{N}$, $\vec{d^n}$ is graphical and there exists a bivariate probability distribution $(p_{j,k})_{j,k=0}^{\infty}$ that is independent of n and its first moment are equal, *i.e.*

$$\sum_{j,k=0}^{\infty} j p_{j,k} = \sum_{j,k=0}^{\infty} k p_{j,k}. \tag{2.6}$$

For this probability distribution the following equations must hold:

(i) for every $j, k \geq 0$, $\lim_{n\to\infty} \frac{N_{j,k}(n)}{n} = p_{j,k}$;

(ii) $\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{jN_{j,k}(n)}{n} = \sum_{j,k=0}^{\infty} jp_{j,k} \in (0,\infty)$;

(iii) $\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{jkN_{j,k}(n)}{n} = \sum_{j,k=0}^{\infty} jkp_{j,k} \in (0,\infty)$;

(iv) $\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{j^2N_{j,k}(n)}{n} = \sum_{j,k=0}^{\infty} j^2p_{j,k} \in (0,\infty)$;

(v) $\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{k^2N_{j,k}(n)}{n} = \sum_{j,k=0}^{\infty} k^2p_{j,k} \in (0,\infty)$.

From equation (2.2) there follows that

$$\sum_{j,k=0}^{\infty} kN_{j,k}(n) = \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^- = \sum_{j,k=0}^{\infty} jN_{j,k}(n).$$

One may thus write

$$\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{jN_{j,k}(n)}{n} = \lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{kN_{j,k}(n)}{n},$$

assuming that these limits exist. As this value is required to converge to the first moments of $(p_{j,k})_{j,k=0}^{\infty}$, it is necessary for the probability distribution to satisfy equation (2.6).

The probability distribution $(p_{j,k})_{j,k=0}^{\infty}$ will be called the degree distribution of the degree array; its probability generating function is given by:

$$g(x,y) = \sum_{j,k=0}^{\infty} p_{j,k} x^j y^k. \tag{2.7}$$

The moments of the degree distribution are defined as:

$$\mu_{il} = \sum_{j,k=0}^{\infty} j^i k^l p_{j,k}. \tag{2.8}$$

Equation (2.6) implies that the first moments of the distribution are equal, hence we will write

$$\mu := \mu_{10} = \mu_{01}. \tag{2.9}$$

2.3 Directed configuration model

To study the behaviour of random graphs obeying a feasible degree array, they must be generated. Generating simple graphs obeying a given degree sequence is difficult. If the model is allowed to generate multigraphs as well, the configuration model can be used. The configuration model, first introduced by [19] and refined in [20], is a model for generating undirected graphs with a given degree sequence. Later it has been extended

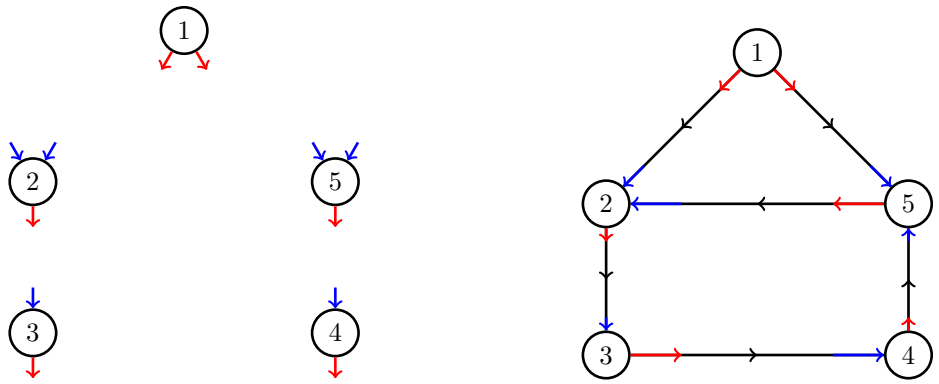


Figure 1: An illustration of the configuration model for the degree sequence $\vec{d^n} = \begin{pmatrix} 0 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}$. On the left we see the stubs of all vertices. In-stubs are blue, out-stubs red. On the right a configuration is shown. Two stubs are matched if they lie on the same edge. This also shows the resulting graph.

to the case of directed graphs [4], which is called the directed configuration model. In this section, we introduce the directed configuration model and show that this model can be used to study the behaviour of simple graphs, despite the fact that it generates multigraphs.

Consider a given valid degree sequence $\vec{d^n}$. The aim is to generate a random multigraph obeying it. For each vertex $i \in V$ it is known how many in-coming and out-going edges it should have. The only freedom left is to define where these edges come from and go to. The configuration model determines this by uniformly choosing a random *configuration*.

Definition 2.4. Let a valid degree sequence $\vec{d^n}$ be given. For all $i \in \{1, 2, \dots, n\}$ define a set W_i^- with d_i^- elements and a set W_i^+ containing d_i^+ elements. Define $W^- = \cup_{i\in\{1,2,\dots,n\}} W_i^-$ and $W^+ = \cup_{i\in\{1,2,\dots,n\}} W_i^+$. Then a configuration is a random perfect bipartite matching of W^- and W^+. A perfect bipartite matching is a set of tuples (a, b) such that each tuple contains one element from W^- and one from W^+ and all elements of W^- and W^+ appear in exactly one tuple.

Now the question is how does a configuration determine the edges of a graph?
For $i \in \{1, 2, \dots, n\}$ an element of W_i^- is called an *in-stub* of i. Similarly an element of W_i^+ is an *out-stub* of i. These stubs play the role of half-edges. An in-stub of i can be regarded as an edge with i as target and no source. Similarly an out-stub is an edge with i as source and no target. Matching an in-stub with an out-stub leads to an edge. The configuration prescribes such a matching for all stubs. In this way, a configuration $\mathcal{M}$ defines a multigraph $\tilde{G}_{\vec{d^n}}$ with vertices $V = \{1, 2, \dots, n\}$ and edges

$$E = \{(i, j) \mid (a, b) \in \mathcal{M}, a \in W_i^+, b \in W_j^-\}. \tag{2.10}$$

The resulting multigraph will satisfy the degree sequence $\vec{d^n}$, as each vertex has $|W_i^-|$ incoming edges and $|W_i^+|$ outgoing edges. This process is illustrated in Figure 1.

The configuration model generates a multigraph $\tilde{G}_{\vec{d^n}}$ by choosing a uniformly random configuration, and therefore, a multigraph as described by equation (2.10). One may now ask what is the probability that the configuration model will generate a specific multigraph $\tilde{G}_{\vec{d^n}}$?
The same graph can be generated by multiple configurations. To see this take $a, a' \in W_i^+$, $b \in W_j^-$ and $c \in W_k^-$. Consider a matching $\mathcal{M}$ with $(a, b), (a', c) \in \mathcal{M}$ and define $\mathcal{M}' = \mathcal{M} \setminus ((a, b), (a', c)) \cup ((a', b), (a, c))$. Then the multigraph induced by $\mathcal{M}$ is the same as the one induced by $\mathcal{M}'$. Since the configuration is chosen uniformly at random, the probability that the configuration model generates $\tilde{G}_{\vec{d^n}}$ depends on the number of configurations that induces this multigraph. Let $\mathrm{CM}_n\left(\vec{d^n}\right)$ be the random variable for the outcome of the configuration model. The following proposition determines this probability.

Proposition 2.5. *Consider a multigraph $\tilde{G}_{\vec{d^n}}$ obeying the degree sequence $\vec{d^n}$. For all pairs $i, j \in V$ let x_{ij} denote the number of copies of the edge (i, j) in the graph. Then there holds*

$$\mathbb{P}\left[CM_n\left(\vec{d^n}\right) = \tilde{G}_{\vec{d^n}}\right] = \frac{1}{m!}\frac{\prod_{i=1}^n d_i^-! \prod_{i=1}^n d_i^+!}{\prod_{1\leq i,j\leq n} x_{ij}!}. \tag{2.11}$$

Proof. This proposition and its proof are adapted for directed graphs from [21, Proposition 7.4] written for the undirected case. First the number of different configurations is determined. This equals the number of perfect bipartite matchings between W^- and W^+. For each element of W^- a match is chosen in W^+. Each element of W^- chooses a match amongst the unmatched elements of W^+. The first element of W^- has m choices for its match. Then the second element can choose its match from the remaining $m-1$ unmatched elements of W^+. Continuing in this fashion, we find $m!$ different perfect matchings. Thus there are $m!$ different configurations. As the configuration is chosen uniformly at random, this implies that

$$\mathbb{P}\left[\mathrm{CM}_n\left(\vec{d^n}\right) = \tilde{G}_{\vec{d^n}}\right] = \frac{1}{m!}N\left(\tilde{G}_{\vec{d^n}}\right),$$

with $N\left(\tilde{G}_{\vec{d^n}}\right)$ the number of different configurations $\mathcal{M}$ inducing the graph $\tilde{G}_{\vec{d^n}}$. From equation (2.10) it follows that the exact matching of the stubs does not matter. As long as an element of W_i^+ is matched to an element of W_j^-, the graph gets an edge (i, j). In other words permuting the stub labels leads to a different configuration that induces the same multigraph. There are $\prod_{i=1}^n d_i^-! \prod_{i=1}^n d_i^+!$ such permutations. However some permutations lead to the same matching $\mathcal{M}$. For $a, a' \in W_i^+$ and $b, b' \in W_j^-$ with $(a, b), (a', b') \in \mathcal{M}$, any permutation swapping a with a' and b with b' leads to the exact same matching. We have to compensate for this by a factor $x_{ij}!$ for all edges. This leads to

$$N\left(\tilde{G}_{\vec{d^n}}\right) = \frac{\prod_{i=1}^n d_i^-! \prod_{i=1}^n d_i^+!}{\prod_{1\leq i,j\leq n} x_{ij}!},$$

completing the proof. □

So, despite the fact that the configuration model generates uniformly random configurations, it does not generate uniformly random multigraphs. That being said, it does generate all simple graphs with equal probability. To see this remark that x_{ij} is $0, 1$ for every pair i, j in a simple graph. Therefore, conditional on the event that configuration model generates a simple graph, a uniformly random simple element of $\mathcal{G}_{\vec{d^n}}$ is chosen.

The configuration model allows us to sample a random element of $\mathcal{G}_{\vec{d^n}}$. Surprisingly, it can be shown that results on uniformly random configurations and their induced multigraphs can be transferred to uniformly generated simple graphs. Let $\mathcal{A}\left(\vec{d^n}\right)$ be the set of all multigraphs, which obey the degree sequence $\vec{d^n}$ while also satisfying a desired property, to be specified later. The goal of the remainder of this section is to show that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right] = 1,$$

implies that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right) \mid \tilde{G}_{\vec{d^n}} \text{ is simple}\right] = 1.$$

The first step is showing that the probability that the configuration model generates a simple graph is bounded away from zero.

Theorem 2.6. *[22, Theorem 4.3] Let $\left(\vec{d^n}\right)_{n=1}^{\infty}$ be a feasible degree array with $d_{\max} = \mathcal{O}\left(\sqrt{n}\right)$. Then the probability that the configuration model generates a simple graph is asymptotically*

$$e^{-\frac{\mu_{11}}{\mu} - \frac{(\mu_{20}-\mu)(\mu_{02}-\mu)}{\mu}} > 0.$$

Proof. The proof of the Theorem follows from the proof [22, Theorem 4.3] which is based on [22, Proposition 4.2]. The main difference is that in [22] the in-degree of a vertex is independent of its out-degree, *i.e.* the bivariate degree distribution is the product of two univariate distributions. It is enough to replace Condition 4.1 and Lemma 5.2 from Ref. [22] with the requirement of a feasible degree array obeying $d_{\max} = \mathcal{O}(\sqrt{n})$ to generalise the proof to the case of an arbitrary bivariate degree distribution. □

Lemma 2.7. *Let $\left(\vec{d^n}\right)_{n=1}^{\infty}$ be a feasible degree array with $d_{\max} = \mathcal{O}(\sqrt{n})$ and let $\mathcal{A}\left(\vec{d^n}\right)$ be a set of multigraphs, all obeying the degree sequence $\vec{d^n}$. If for a random multigraph $\tilde{G}_{\vec{d^n}}$ generated by the configuration model there holds*

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right] = 0,$$

then it is also true that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right) \mid \tilde{G}_{\vec{d^n}} \textit{ is simple}\right] = 0.$$

Proof. By Bayes'rule

$$\mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right) \mid \tilde{G}_{\vec{d^n}} \text{ is simple}\right] \leq \frac{\mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right]}{\mathbb{P}\left[\tilde{G}_{\vec{d^n}} \text{ is simple}\right]}. \tag{2.12}$$

Because we have a feasible degree array with $d_{\max} = \mathcal{O}(\sqrt{n})$, Theorem 2.6 assures that

$$\mathbb{P}\left[\tilde{G}_{\vec{d^n}} \text{ is simple}\right] = (1 + o(1))\, e^{-\frac{\mu_{11}}{\mu} - \frac{(\mu_{20}-\mu)(\mu_{02}-\mu)}{\mu}}.$$

Hence

$$\lim_{n\to\infty} \inf \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \text{ is simple}\right] > 0.$$

This completes the proof as the numerator in equation (2.12) converges to zero. □

Corollary 2.8. *Let $\left(\vec{d^n}\right)_{n=1}^{\infty}$ be a feasible degree array with $d_{\max} = \mathcal{O}(\sqrt{n})$. Take $\mathcal{A}\left(\vec{d^n}\right)$ to be a set of multigraphs, all obeying the degree sequence $\vec{d^n}$. Let $\tilde{G}_{\vec{d^n}}$ be a random multigraph generated by the configuration model. If there holds*

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right] = 1,$$

then it is also true that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right) \mid \tilde{G}_{\vec{d^n}} \textit{ is simple}\right] = 1.$$

Proof. Let $\mathcal{V}\left(\vec{d^n}\right)$ denote the set of all multigraphs on n vertices obeying the degree sequence $\vec{d^n}$ and define

$$\overline{\mathcal{A}}\left(\vec{d^n}\right) = \mathcal{V}\left(\vec{d^n}\right) \setminus \mathcal{A}\left(\vec{d^n}\right).$$

As there holds that $\tilde{G}_{\vec{d^n}} \in \mathcal{V}\left(\vec{d^n}\right)$ by definition and $\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \mathcal{A}\left(\vec{d^n}\right)\right] = 1$ by assumption, the law of total probability implies $\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \overline{\mathcal{A}}\left(\vec{d^n}\right)\right] = 0$. Then Lemma 2.7 implies that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}_{\vec{d^n}} \in \overline{\mathcal{A}}\left(\vec{d^n}\right) \mid \tilde{G}_{\vec{d^n}} \text{ is simple}\right] = 0.$$

Again using that $\mathcal{A}\left(\vec{d^n}\right) \cup \overline{\mathcal{A}}\left(\vec{d^n}\right)$ is the set of all multigraphs obeying the given degree sequence, the claim follows. □

3 Existence of a giant strongly connected component in a directed graph

The goal is to study the influence of percolation on the existence of a giant strongly connected component in random directed graphs obeying a given degree array. In Section 2, it is explained how to generate such random graphs using the configuration model. However this does not provide sufficient knowledge to fully study percolation as we are limited to small systems that are computationally tractable. Plus, the existence of the GSCC in the random graphs itself must be studied first. The goal of this section is to give some preliminary definitions and to introduce the findings of Cooper and Frieze [4] regarding the existence of a giant strongly connected component. In Section 3.1 we will define the GSCC. Here we will also define the strongly connected component. To study the strongly connected component, the notion of in- and out-components are used. Hence we decided to explain all the notions of a connected component in a directed graph in this section. Once the definition of the GSCC is understood, its existence criteria is given, in the form of a theorem by Cooper and Frieze [4]. Additionally it regards the existence of giant in-component and giant out-component. This theorem can be regarded as the directed analogue of the theorem from Molloy and Reed for undirected graphs [3, 7].

3.1 Types of connected components in a directed graph

A connected component of a graph is a set of vertices that is connected to each other. Two vertices are connected if there is a path between them. This allows to define a connected component as a maximal subset C of the vertices such that there is a path between any pair of vertices $u, v \in C$. Here maximal means that no vertex can be added to C without destroying the property that all vertices are connected by a path. In a directed graph each edge has a direction. Hence we can defined two types of paths in a directed graph, those that respect the direction of the edges and those that ignoring the directionality.

Definition 3.1. Consider a directed graph G_n. Any two $v_1, v_k \in V$ are connected by:

- a *directed path*, if there exist distinct vertices $v_2, v_3, \ldots, v_{k-1} \in V$ such that for all $i \in \{2, 3, \ldots, k\}$ $(v_{i-1}, v_i) \in E$. This is called a directed $v_1 - v_k$ path.
- an *undirected path*, if there exist distinct vertices $v_2, v_3, \ldots, v_{k-1} \in V$ such that for all $i \in \{2, 3, \ldots, k\}$ either $(v_{i-1}, v_i) \in E$ or $(v_i, v_{i-1}) \in E$. This is called an undirected $v_1 - v_k$ path.

An undirected path discards the direction of the edges. Thus defining a type of connected component based on undirected paths, leads to the same subsets of the vertices as the connected components when treating the graph as an undirected one.

Definition 3.2. Consider a directed graph G_n. The *weakly connected components* of G_n are the maximal subsets of V such that between any pair of vertices there exits an undirected path.

In an analogous way a connected component can be defined based on directed paths.

Definition 3.3. (Strongly connected component) Consider a directed graph G_n. The *strongly connected components* of G_n are the maximal subsets of V such that between any pair of vertices u, v directed $u - v$ and $v - u$ paths exist simultaneously.

Each each vertex is an element of exactly one weakly and one strongly connected component. This allows us to denote the weakly and strongly connected component containing v by respectively $\mathrm{WCC}(v)$ and $\mathrm{SCC}(v)$. By definition there holds $\mathrm{SCC}(v) \subset \mathrm{WCC}(v)$. Both, the strongly and weakly connected components partition the vertices of the graph. We want to study the strongly connected components of a random graph. Instead of studying these components directly, their characterization in terms of in-components and out-components is used.

Definition 3.4. Consider a directed graph G_n and take $v \in V$. Then

- the*in-component* of v, denoted by $\mathrm{In}(v)$, consists of v itself and all vertices $u \in V$ such that a directed $u - v$ path exists;

- the *out-component* of v, denoted by $\mathrm{Out}(v)$, consists of v itself and all vertices $w \in V$ for which a directed $v - w$ path exists.

Unlike the strong and the weak components, these components are defined based on a specific vertex. From this definition we also note that $u \in \mathrm{In}(v)$ does not imply $\mathrm{In}(u) = \mathrm{In}(v)$. The same observation holds for the out-component as well. This means that these types of components do not partition the vertices of the graph. Furthermore we note that $\mathrm{In}(v), \mathrm{Out}(v) \supset \mathrm{SCC}(v)$ and $\mathrm{In}(v), \mathrm{Out}(v) \subset \mathrm{WCC}(v)$. These components allow to characterize the strongly connected component in the following way.

Lemma 3.5. *Consider a directed graph G_n. For any $v \in V$ there holds $In(v) \cap Out(v) = SCC(v)$.*

Proof. To prove that $\mathrm{In}(v) \cap \mathrm{Out}(v) = \mathrm{SCC}(v)$, it suffices to show $\mathrm{SCC}(v) \subset \mathrm{In}(v) \cap \mathrm{Out}(v)$ and $\mathrm{SCC}(v) \supset \mathrm{In}(v) \cap \mathrm{Out}(v)$. Take $u \in \mathrm{SCC}(v)$. By definition of the strongly connected component this implies that directed $u - v$ and $v - u$ paths exist. Hence $u \in \mathrm{In}(v)$ and $u \in \mathrm{Out}(v)$, which shows that $\mathrm{SCC}(v) \subset \mathrm{In}(v) \cap \mathrm{Out}(v)$. Next take $u \in \mathrm{In}(v) \cap \mathrm{Out}(v)$. This implies that directed $u - v$ and $v - u$ paths exist. For u to be an element of $\mathrm{SCC}(v)$ it must also hold that for any $w \in \mathrm{SCC}(v)$ directed $w - u$ and $u - w$ paths exist. Because $w \in \mathrm{SCC}(v)$, directed $v - w$ and $w - v$ paths are present in G_n. This implies existence of a $u - w$ path by concatenation of the $u - v$ and $v - w$ paths. Similarly the $w - u$ path can be formed by concatenating the $w - v$ path with the $v - u$ path. Thus $u \in \mathrm{SCC}(v)$, which shows that $\mathrm{SCC}(v) \supset \mathrm{In}(v) \cap \mathrm{Out}(v)$. □

For any type of connected component introduced above, a corresponding giant connected component can be defined. As a giant component is a component of which the size scales linearly in the number of vertices, it is defined for a graph array $(G_n)_{n=1}^{\infty}$ rather than for just one graph. We will not consider the giant weakly connected component in this work. For completeness we do introduce its definition.
Let $\mathcal{C}_1^W(G_n)$ be the largest weakly connected component of G_n.

Definition 3.6. The graph array $(G_n)_{n=1}^{\infty}$ contains a *giant weakly connected component*(GWCC) if

$$\lim_{n\to\infty} \frac{|\mathcal{C}_1^W(G_n)|}{n} := c_{wcc} > 0. \tag{3.1}$$

Denoting the largest strongly connected component of G_n by $\mathcal{C}_1^S(G_n)$, also the notion of a giant strongly connected component can be defined.

Definition 3.7. The graph array $(G_n)_{n=1}^{\infty}$ contains a *giant strongly connected component*(GSCC) if

$$\lim_{n\to\infty} \frac{|\mathcal{C}_1^S\left(G_{\vec{d^n}}\right)|}{n} := c_{scc} > 0. \tag{3.2}$$

The values c_{wcc} and c_{scc} are the size of the GWCC respectively GSCC. Here by size we mean the fraction of the vertices contained by the component. As the in-component and out-component are defined based on a vertex, there giants are defined differently.

Definition 3.8. The graph array $(G_n)_{n=1}^{\infty}$ contains a *giant in-component* (GIN) if for a uniformly random vertex v

$$\lim_{n\to\infty} \frac{|\mathrm{In}(v)|}{n} := c_{in} > 0. \tag{3.3}$$

Definition 3.9. The graph array $(G_n)_{n=1}^{\infty}$ contains a *giant out-component* (GOUT) if for a uniformly random vertex v

$$\lim_{n\to\infty} \frac{|\mathrm{Out}(v)|}{n} := c_{out} > 0. \tag{3.4}$$

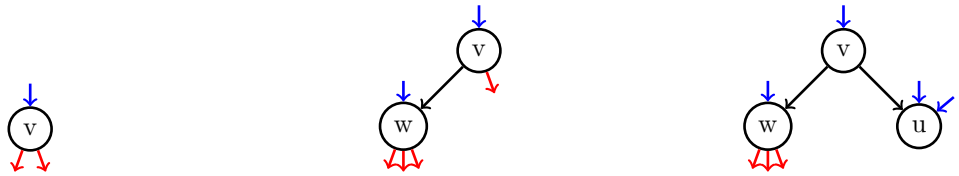


Figure 2: An illustration of the construction of the out-component of v by the directed configuration model. In-stubs are displayed in blue, out-stubs in red.

3.2 Existence of GIN, GOUT & GSCC

The goal of this section is to introduce a theorem by Cooper and Frieze that describes whether a random graph array $\left(G_{\vec{d^n}}\right)_{n=1}^{\infty}$ obeying a given *proper* degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ w.h.p. contains a GSCC [4]. Instead of directly stating the theorem, we first give the intuitive 'proof' of this theorem. This also allows us to explain where the values used in the theorem come from. But first we define a proper degree array.

Definition 3.10. A degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is *proper* if it is feasible and additionally satisfies

1. $d_{\max} \leq \frac{n^{1/12}}{\ln(n)}$;
2. $\rho = \max\left(\sum_{j,k=0}^{\infty} \frac{j^2 k N_{j,k}(n)}{\mu n}, \sum_{j,k=0}^{\infty} \frac{j k^2 N_{j,k}(n)}{\mu n}\right) = o(d_{\max})$.

Cooper and Frieze show that the out-component of each vertex either contains $\mathcal{O}\left(d_{\max}^2 \ln(n)\right)$ vertices or w.h.p. contains $a_0^+ n$ vertices, for some yet unknown constant a_0^+. Similarly they show that the in-component of each vertex either contains $\mathcal{O}\left(d_{\max}^2 \ln(n)\right)$ vertices or w.h.p. contains $a_0^- n$ vertices, for some yet unknown constant a_0^-. Denote the set of all vertices with out-component of size $a_0^+ n$ by L^+ and the set of all vertices with in-component of size $a_0^- n$ by L^-. Lemma 3.5 implies that a strongly connected component is the intersection of the in-component and the out-component of some vertex. Thus a vertex v must be in L^+ and L^- to be in the GSCC, if the GSCC exists. First, we will heuristically investigate the probability that a random vertex is in L^+ (respectively L^-) by investigating the out-components (in-components) of a random graph.

To investigate the out-component of the random graphs, we take a proper degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ and denote its underlying degree distribution by $(p_{j,k})_{j,k=0}^{\infty}$. In Section 2.3 we showed that results on multigraphs induced by uniformly random configurations can be transferred to uniformly random simple graphs. Thus we will consider multigraphs generated by the configuration model. As discussed in Section 2.3, the configuration model generates a uniformly random configuration of W^- and W^+. This gives us the freedom to construct the configuration in such a way that first the out-component of a vertex v is constructed. How can we do this? Pick a vertex v of which we wish to construct its out-component. Choose any out-stub of v and match it to a uniformly random unmatched in-stub. Suppose this in-stub is an element of W_w^-. This implies that we add w to $\mathrm{Out}(v)$. Now also the out-stubs of w can enlarge the out-component of v. Hence at the next step, we take any unmatched out-stub from the set $W_v^+ \cup W_w^+$ and match it to a uniformly random unmatched in-stub. This process is illustrated in Figure 2. It continues until all out-stubs of $\cup_{w \in \mathrm{Out}(v)} W_w^+$ are matched. At that point the out-component of v is completed. Randomly match the remaining unmatched in-stubs with the unmatched out-stubs to complete the configuration.

The construction of the out-component of v, as described above, is similar to a Galton-Watson branching process. A Galton-Watson branching process is a discrete time process. It starts with one individual. At each time-step a random living individual generates an integer amount of off-spring and dies. The amount of off-spring generated is governed by the off-spring distribution $Z(k)$. Such a process either continues on indefinitely, *i.e.* at each time-step at least one individual remains alive, or it becomes extinct at some point, *i.e.* at some time-step all individuals are dead. Let ρ be the probability that the branching process becomes extinct at some point.

Lemma 3.11. *Consider a Galton-Watson branching process with offspring distribution $Z(k)$. The extinction probability ρ of this process is*

- 1 *if* $\mathbb{E}[Z(k)] < 1$;
- *the unique solution in* $[0,1)$ *to the equation* $x = \sum_{k=0}^{\infty} \mathbb{P}[Z(k)=k]\, x^k$ *if* $\mathbb{E}[Z(k)] > 1$.

The parallel between a Galton-Watson branching process and the construction of the out-component arises when identifying an individual with an unmatched out-stub. An unmatched out-stub 'dies' when it is matched to a random in-stub b. The amount of off-spring it generates, equals d_w^+, where w is the vertex such that $b \in W_w^-$. There is one exception: in case w was already an element of $\mathrm{Out}(v)$, the amount of off-spring is zero. This means that the off-spring distribution is approximately $\left(p_k^+\right)_{k=0}^{\infty}$, with p_k^+ the probability that by following a uniformly random chosen in-stub, we find a vertex with out-degree k. This latter probability is given by

$$p_k^+ = \sum_{j=0}^{\infty} \frac{j}{\mu} p_{j,k}. \tag{3.5}$$

Here the division by μ ensures that the probability distribution is normalized. We expect that $v \in L^+$ if the corresponding Galton-Watson branching with off-spring distribution $\left(p_k^+\right)_{k=0}^{\infty}$ becomes extinct. Applying Lemma 3.11 to the distribution, we find that this only can happen if

$$\sum_{k=0}^{\infty} k p_k^+ = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} jk p_{j,k}}{\mu} = \frac{\mu_{11}}{\mu} > 1.$$

Thus we expect that L^+ is empty unless $\frac{\mu_{11}}{\mu} > 1$. In the case $\frac{\mu_{11}}{\mu} > 1$, the probability that the branching process does not become extinct is η^+, where $1-\eta^+$ is the unique solution in $[0,1)$ to

$$\left(1-\eta^+\right) = \sum_{k=0}^{\infty} p_k^+ \left(1-\eta^+\right)^k. \tag{3.6}$$

Recall that the Galton-Watson process always starts with one individual. However the generation of the out-component of v starts with d_v^+ out-stubs. Assuming that each out-stub of v generates a disjoint subset of $\mathrm{Out}(v)$, this can be regarded as d_v^+ independent copies of the branching process. This collection of processes terminates if and only if all of the individual branching processes become extinct. Therefore the probability that the process corresponding to the generation of the out-component of v becomes extinct is

$$\left(1-\eta^+\right)^{d_v^+}.$$

Thus for a random vertex v the probability that $v \in L^+$ is approximately ζ^+, where $1-\zeta^+$ is given by

$$\left(1-\zeta^+\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{j,k} \left(1-\eta^+\right)^k. \tag{3.7}$$

There is nothing special about the out-component in the above reasoning. The same idea can be applied to the in-component, but with a different off-spring distribution. In this case we need to probability that following a uniformly random chosen out-stub, we find a vertex with out-degree j, i.e.

$$p_j^- = \sum_{k=0}^{\infty} \frac{k}{\mu} p_{j,k}. \tag{3.8}$$

A Galton-Watson branching process with off-spring distribution $\left(p_j^-\right)_{j=0}^{\infty}$ has an extinction probability smaller than 1 if

$$\sum_{j=0}^{\infty} j p_j^- = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} jk p_{j,k}}{\mu} = \frac{\mu_{11}}{\mu} > 1.$$

Note that this is the same condition as for the off-spring distribution of the out-component. In other words L^+ and L^- are expected to become non-empty under the same conditions. The probability that this branching process does not become extinct is η^-, where $1-\eta^-$ is the unique solution in $[0,1)$ to

$$\left(1-\eta^-\right) = \sum_{j=0}^{\infty} p_j^- \left(1-\eta^-\right)^j . \tag{3.9}$$

Thus we expect a random vertex v to be in L^- with probability ζ^-, with $1-\zeta^-$ defined by

$$\left(1-\zeta^-\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{j,k} \left(1-\eta^-\right)^j . \tag{3.10}$$

Using these probabilities for a random vertex to be in L^- or L^+ and Lemma 3.5, we determine heuristically whether a giant strongly connected component is present in the graph. For a vertex $v \in L^-$ and a vertex $u \in L^+$ it is very likely that there exists an edge from a vertex $\mathrm{In}(u)$ to one in $\mathrm{Out}(v)$, i.e. $v \in \mathrm{In}(u)$ and $u \in \mathrm{Out}(v)$. Thus if $u, v \in L^-$ and $u, v \in L^+$, it is likely that u and v are in each others in-component and out-component. By Lemma 3.5 this implies that u and v lie in the same strongly connected component. Hence we expect a GSCC consisting of the vertices $L^+ \cap L^-$. This set is expected to be non-empty if $\frac{\mu_{11}}{\mu} > 1$. To determine the probability for a random vertex to be in $L^+ \cap L^-$, i.e. to approximate the size of the GSCC, define

$$\psi = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{j,k} \left(1-\eta^-\right)^j \left(1-\eta^+\right)^k . \tag{3.11}$$

This approximates the probability that the branching process corresponding to the generation of the in-component and the out-component both become extinct. Hence the probability that both process continue on indefinitely can be approximated by

$$c = \zeta^+ + \zeta^- + \psi - 1. \tag{3.12}$$

The above intuitive idea is rigorously proven by Cooper and Frieze for proper degree arrays.

Theorem 3.12 (Existence of a GIN, GOUT and GSCC)**.** *[4, Theorem 1 and 2] Consider a proper degree array* $\left(\vec{d^n}\right)_{n=1}^{\infty}$*. Take a uniformly random sequence of simple graphs* $\left(G_{\vec{d^n}}\right)_{n=1}^{\infty}$ *obeying this degree array. Then the following statements hold.*

1. *If* $\frac{\mu_{11}}{\mu} < 1$*, with high probability the size of the in-component and the out-component of each vertex is* $\mathcal{O}\left(d_{\max}^2 \ln(n)\right)$*.*
2. *If* $\frac{\mu_{11}}{\mu} > 1$ *and* $p_0^+, p_0^- > 0$*, with high probability*
   - *There are* $\zeta^+ n$ *vertices with an out-component containing* $\zeta^- n$ *vertices;*
   - *There are* $\zeta^- n$ *vertices with an in-component containing* $\zeta^+ n$ *vertices;*
   - *There is a unique giant strongly connected component with vertex set* $L^+ \cap L^-$ *of size* $\left(\zeta^+ + \zeta^- + \psi - 1\right) n$*.*

Note that in case $\frac{\mu_{11}}{\mu} < 1$, the theorem only regards the size of the in-components and out-components. Applying Lemma 3.5 it follows that the size of the strongly connected component to which a vertex belongs is upper bounded by the minimum of the size of its in-component and out-component. Hence the fact that w.h.p. the in-component and the out-component do not scale linear in n for any vertex, implies that w.h.p. no GSCC is present. Furthermore we remark in case $\frac{\mu_{11}}{\mu} > 1$ and $p_0^+, p_0^- > 0$ the theorem assures that a GIN and GOUT exist w.h.p. This criterium for the existence of a GSCC is used in Section 4 to study the influence of percolation on the structure of a directed random graph.

4 Percolation in directed graphs

Having defined the notion of a random graph with a given degree distribution in Section 2 and studied the existence of a giant strongly connected component in Section 3, we are ready to study percolation theoretically. This section introduces our new result in the form of a theorem determining the percolation threshold for the existence of a giant strongly connected component for bond and site percolation. This is proven by extending Fountoulakis' method [2] for determining the percolation threshold in undirected graphs to directed ones and combining it with Theorem 3.12 [4]. Section 4.1 states our theorem and all the necessary notations. The remainder of the section presents the proof of the theorem, which is split into two parts: the proof for bond percolation, see Section 4.3, and for site percolation, see Section 4.4. Before the proof is split, Section 4.2 gives some auxiliary definitions and theorems used in the proof for both types of percolation.

4.1 Percolation threshold for GSCC

Percolation is a random process on a graph G_n removing its edges. We consider two types of percolation:

- *Bond percolation* Fix a value $\pi \in (0,1)$. Each edge of G_n is removed independently of the other edges with probability $1-\pi$.
- *Site percolation* Fix a value $\pi \in (0,1)$. For each vertex of G_n all the edges incident to this vertex are removed with probability $1-\pi$, independently of the other vertices. If this happens we say that the vertex is made isolated. Such a vertex may also be referred to as a deleted vertex.

The value π is called the percolation probability. Applying percolation to a graph leads to a random subgraph on the same vertices. Let G_n be the graph to which percolation is applied with percolation probability π, then the subgraph that remains after percolation is denoted by G_n^π. The context will indicate which type of percolation is performed.

Whenever one studies percolation on a graph, a certain property of the graph is investigated as a function of π. We look at the influence of percolation on the existence of a giant strongly connected component in random graphs obeying a proper degree array. In Section 3 we saw that these giant components are only defined for a graph array. Hence we look at the effect percolation has on a limiting behaviour of uniformly random graph array $\left(G_{\vec{d^n}}\right)_{n=1}^{\infty}$ obeying a proper degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$. This results in a percolated graph array denoted by $\left(G^\pi_{\vec{d^n}}\right)_{n=1}^{\infty}$. The main goal is to determine the percolation threshold π_c. This is the value of the percolation probability π such that:

- For $\pi > \pi_c$, the array of percolated graphs $\left(G^\pi_{\vec{d^n}}\right)_{n=1}^{\infty}$ w.h.p. contains a giant strongly connected component.
- For $\pi < \pi_c$, the array of percolated graphs $\left(G^\pi_{\vec{d^n}}\right)_{n=1}^{\infty}$ w.h.p does not contain a giant strongly connected component.

More formally it is defined as

$$\pi_c = \sup\left\{\pi \in [0,1] \;\middle|\; \lim_{n\to\infty} \mathbb{P}\left[\frac{\left|\mathcal{C}_1^S\left(G^\pi_{\vec{d^n}}\right)\right|}{n} = 0\right] = 0\right\}. \tag{4.1}$$

Here the probability is taken with respect to a sequence of probability spaces indexed by $n \in \mathbb{N}$. Given n this probability space contains the graphs that can remain after applying the type of percolation of choice with percolation probability π to any graph obeying $\vec{d^n}$. The probability assigned to each graph equals the probability that it remains after applying percolation to a uniformly random graph obeying $\vec{d^n}$. This definition does not specify the type of percolation applied. As both types of percolation might have different percolation thresholds, they will be denoted by π_c^{bond} and π_c^{site}.

Our main result concerning percolation in a random directed simple graphs with a given degree array is the following theorem.

Theorem 4.1. *Let $\left(\vec{d^n}\right)_{n=1}^{\infty}$ be a proper degree array for which $\mu_{11} > \mu$ and $p_0^-, p_0^+ > 0$. Then the percolation threshold for the existence of a giant strongly connected component is given by*

$$\pi_c^{bond} = \pi_c^{site} = \frac{\mu}{\mu_{11}}. \tag{4.2}$$

Moreover for $\pi < \pi_c^{bond}$ there is an unique value c^{bond}, defined by equation (4.31), such that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\left|\mathcal{C}_1^S\left(G_{\vec{d^n}}^{\pi}\right)\right|}{n} = c^{bond}\right] = 1.$$

Similarly for $\pi > \pi_c^{site}$ there is an unique value c^{site}, defined by equation (4.54), such that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\left|\mathcal{C}_1^S\left(G_{\vec{d^n}}^{\pi}\right)\right|}{n} = c^{site}\right] = 1.$$

This theorem excludes proper degree arrays with $\mu > \mu_{11}$ as for these degree arrays the percolation threshold is not defined. Recall that Theorem 3.12 states that a random graph obeying a proper degree sequence with $\mu > \mu_{11}$ w.h.p. does not contain a giant strongly connected component. Since percolation only removes edges from the graph, it can only decrease the size of the strongly connected components. Thus if the original graph array w.h.p. does not contain a GSCC, the same holds for the percolated graph array. By definition of the percolation threshold, equation (4.1), this shows that the percolation threshold is not defined for these degree arrays.

Our Theorem 4.1 can be regarded as the directed analogue of Theorem 1.1 [2]. Theorem 1.1 determines the percolation threshold for the existence of a giant connected component in undirected graphs for bond and site percolation. We base the proof of our theorem on Fountoulakis' proof of Theorem 1.1 [2]. The idea of this proof is as follows. First, show that conditional on the degree sequence after percolation, a uniformly random configuration remains. Using this fact, the degree distribution after percolation is determined. After showing that additional requirements are fulfilled, this allows to apply Molloy and Reed's theorem for the existence of a giant component in a random graph with given degree array [3] to the percolated configuration. Applying this theorem to the degree array after percolation, allows to determine the percolation threshold.

The proof of Theorem 4.1 adopts a similar path as described above, with the theorem of Molloy and Reed being replaced by Theorem 3.12 by Cooper and Frieze [4]. This implies that we need to show that the degree array remains proper after percolation. Also in the entire proof some changes are made to make it suitable for directed graphs.

4.2 Preliminaries for the proof of Theorem 4.1

Before we proof Theorem 4.1 separately for bond and site percolation, in this section we introduce auxiliary definitions and theorems used in these proofs. Theorem 4.1 gives the percolation threshold for simple graphs. In the proof we consider uniformly random configurations and their induced multigraph obeying a proper degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$ instead of simple graphs. The results on these multigraphs can be transferred to the simple graphs using Corollary 4.1 and a variant thereof, as for any proper degree array there holds $d_{\max} \leq \frac{n^{1/12}}{\ln(n)} = \mathcal{O}\left(\sqrt{n}\right)$. This means that Theorem 3.12 is applied to configurations and their induced multigraphs rather than simple graphs. As the proof of this theorem also uses configurations rather than simple graphs this is not a problem, despite the fact that Theorem 3.12 is stated for uniformly random simple graphs [4].

To work with percolated configurations, we need additional notation. Like in Section 2.3, let W^- and W^+ denote the set of stubs inducing the degree sequence $\vec{d^n}$. Here by inducing the degree sequence $\vec{d^n}$, we mean that any configuration on W^- and W^+ leads to a multigraph obeying $\vec{d^n}$. Define $\vec{d^n}'$ to be the random variable for the degree sequence after percolation. Furthermore by $\vec{d^n_\pi}$ we denote a possible degree sequence after percolation. So $\vec{d^n_\pi}$ is a value that the random variable $\vec{d^n}'$ can take. Percolation removes edges in a graph, thus in a configuration it removes in-stubs together with the out-stubs they are matched to. Let $W^{-,\pi}$ and $W^{+,\pi}$ denote the in-stubs and out-stubs surviving percolation. Conditional on $\vec{d^n}' = \vec{d^n_\pi}$ these stubs are in one-to-one correspondence with $W^-_{\vec{d^n_\pi}}$ and $W^+_{\vec{d^n_\pi}}$. Here $W^-_{\vec{d^n_\pi}}$ and $W^+_{\vec{d^n_\pi}}$ denote the set of stubs inducing the degree sequence $\vec{d^n_\pi}$. The one-to-one correspondence follows from the fact that all vertices have the same amount of in-stubs (respectively out-stubs) in $W^{-,\pi}(W^{+,\pi})$ as in $W^-_{\vec{d^n_\pi}}(W^+_{\vec{d^n_\pi}})$. Thus any mapping sending an in-stub of i of $W^{-,\pi}$ to an in-stub of i in $W^-_{\vec{d^n_\pi}}$ and an out-stub of i of $W^{+,\pi}$ to an out-stub of i in $W^+_{\vec{d^n_\pi}}$ induces a bijection. Let us fix such a bijection. This induces a one-to-one correspondence between the configuration on $(W^{-,\pi}, W^{+,\pi})$ and the configuration on $\left(W^-_{\vec{d^n_\pi}}, W^+_{\vec{d^n_\pi}}\right)$. This one-to-one correspondence will be important later in the proof.

The goal is to prove that the percolation threshold is given by equation (4.2). However, the degree array after percolation is a random variable. This requires the definition of the probability space for the degree array after percolation. This is closely related to the sequence probability spaces over which the probability in equation (4.2) is taken. Recall that we consider percolation applied to a random graph array obeying the degree sequence $\left(\vec{d^n}\right)_{n=1}^\infty$. For the proof we let this be a random multigraph array. Let D_n be the probability space containing all degree sequences $\vec{d^n_\pi}$ that can be obtained by applying percolation to a random configuration on (W^-, W^+). The probability assigned to each $\vec{d^n_\pi}$ is the probability that it is induced by $(W^{-,\pi}, W^{+,\pi})$. The probability space for the degree array $\left(\vec{d^n_\pi}\right)_{n=1}^\infty$ is the product space $D = \prod_{n=1}^\infty D_n$ with product measure ν.

This section is concluded with stating several concentration results that will be used in the proof of Theorem 4.1 for bond and site percolation. Concentration inequalities are important as we deal with two sources of randomness: the initial multigraph is random and percolation randomly removes edges. Hence we will encounter many random variables. It is often useful to restrict their values to a bounded interval with high probability. Concentration inequalities allow to determine such intervals. The first concentration inequality that is used, is Hoeffding's inequality.

Theorem 4.2. *(Hoeffding's inequality)[23] Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose that $a_i \leq X_i \leq b_i$ for all $i \in \{1, 2, \ldots, n\}$ and define $c_i = b_i - a_i$. Furthermore define $S_n = \sum_{i=1}^n X_i$. Then there holds*

$$\mathbb{P}\left[|S_n - \mathbb{E}\left[S_n\right]| > t\right] \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right). \tag{4.3}$$

The second concentration inequality is a corollary of a theorem by McDiarmid.

Theorem 4.3. *[24, Theorem 7.4] Let (V, d) be a finite metric space. Suppose there exists a sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_s$ of increasingly refined partitions with $\mathcal{P}_0$ the trivial partition consisting of V and $\mathcal{P}_s$ the partition where each element of V is a partition element on its own. Take a sequence of positive integers $c_0, c_1, \ldots, c_s$ such that for all $k \in \{1, 2, \ldots, s\}$ and any $A, B \in \mathcal{P}_k$ with C satisfying $A, B \subset C \in \mathcal{P}_{k-1}$ there exists a bijection $\phi : A \to B$ with $d\left(x, \phi(x)\right) \leq c_k$ for all $x \in A$. Let the function $f : V \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq d(x, y)$ for all $x, y, \in V$. Then for X uniformly distributed over V and any $t > 0$ there holds*

$$\mathbb{P}\left[|f(X) - \mathbb{E}\left[f(X)\right]| > t\right] \leq 2\exp\left(-\frac{2t^2}{\sum_{k=0}^s c_k^2}\right).$$

This theorem allows to show the following concentration inequality, that will prove to be useful in Sections 4.3 and 4.4.

Corollary 4.4. *Consider two finite sets A_0 and A_1 of sizes $|A_0| = a_0$ and $|A_1| = a_1$. Define $S = \cup_{i\in\{0,1\}}\{(x,i) \mid x \in A_i\}$. A subset of S containing b_0 elements with $i = 0$ and b_1 elements with $i = 1$ is called a (b_0, b_1)-subset of S. Define V as the space of all (b_0, b_1)-subsets of S. Let $f : V \to \mathbb{R}$ be a function such that for any $B, B' \in V$ there holds $|f(B) - f(B')| \leq |B \triangle B'|$. Here $B \triangle B'$ denotes the symmetric difference, i.e. $B \triangle B' = (B \cup B') \setminus (B \cap B')$. Then for X distributed uniformly over V and any $t > 0$ there holds*

$$\mathbb{P}\left[|f(X) - \mathbb{E}\left[f(X)\right]| > t\right] \leq 2\exp\left(-\frac{t^2}{8(b_0+b_1)}\right). \tag{4.4}$$

Proof. Consider a (b_0, b_1)-subset of S. Assign each element a unique number in the set $\{1, 2, \ldots, b_0+b_1\}$, such that for all elements with $i = 0$ this number is smaller than b_0+1. Note that this implies that for each element with $i = 1$ its number is larger than b_0. A (b_0, b_1)-subset of S with such a numbering is called a (b_0, b_1)-ordering of S. Define W to be the set of all (b_0, b_1)-orderings of S. The function $f : V \to \mathbb{R}$ can be extended to a function $f : W \to \mathbb{R}$ by regarding each (b_0, b_1)-ordering as (b_0, b_1)-subset. This extension respects the relation $|f(x) - f(y)| \leq x \triangle y$, *i.e.* it holds for $x, y \in W$ as well. This is true since for any two orderings x, y their symmetric difference as (b_0, b_1)-orderings is bounded from bellow by their symmetric difference as (b_0, b_1)-subsets. The next step in proving equation (4.4) is applying Theorem 4.3 to the metric space $(W, \triangle)$.

We will now define a sequence of refined partitions on W using the notion of an i-prefix. An i-prefix determines the first i elements of an ordering. This allows for all $k \in \{0, 1, \ldots, b_0 + b_1\}$ to construct the partition $\mathcal{P}_k$ by defining its elements to be the sets of orderings with the same k-prefix. Now the partition $\mathcal{P}_0$ is the trivial partition consisting of W. As each (b_0, b_1)-ordering has $b_0 + b_1$ elements, $\mathcal{P}_{b_0+b_1}$ will be the partition where each element is a single ordering. Next the values c_k need to be determined. Take $B, D \in \mathcal{P}_k$ with C satisfying $B, D \subset C \in \mathcal{P}_{k-1}$. This implies that any ordering in B has the same $k-1$-prefix as an ordering in D. Furthermore these orderings must differ at the k^{th} element. The remaining $b_0 + b_1 - k$ elements can be any element that is not present in the k-prefix that lead to a valid ordering. Denote the k^{th} element of any ordering in B by $a_{B,k}$. Similarly let $a_{D,k}$ denote the k^{th} element of any ordering in D. Define the bijection $\phi : B \to D$ by taking $x \in B$ and mapping its k^{th} element to $a_{D,k}$. If x contains $a_{D,k}$ at some position $l > k$, map the l^{th} element of x to $a_{B,k}$. All the other elements are unchanged by the bijection. By definition this is an element of D. This bijection is illustrated in figure 3.

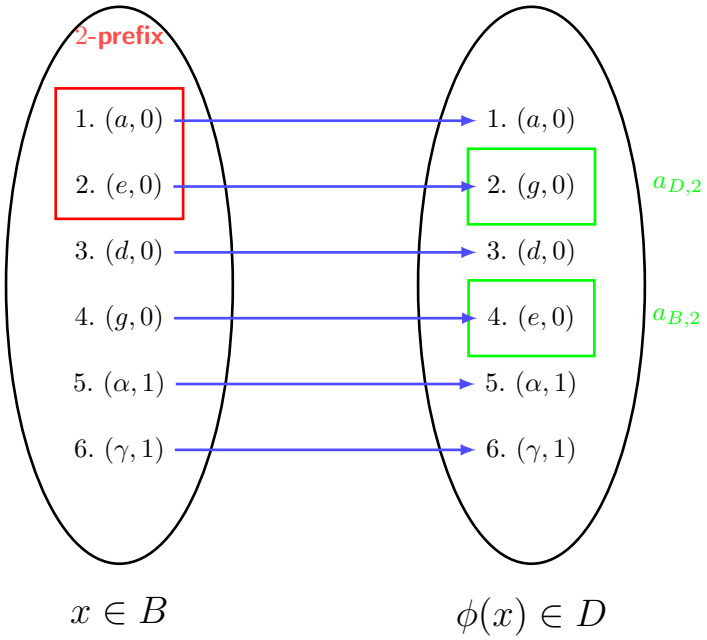


Figure 3: The definition of the bijection $\phi : B \to D$. A 2-prefix is highlighted in *red*. The elements $a_{B,2}$ and $a_{D,2}$ are indicated in *green*. The bijection itself is displayed in *blue*.

According to definition of ϕ for any $x \in B$ we have $|x \triangle \phi(x)| \leq 4$. Thus we may take $c_k = 4$ for all $k \in \{1, 2, \ldots, b_0 + b_1\}$. Applying Theorem 4.3 we find that for distributing X uniformly random over W and any $t > 0$ there holds

$$\mathbb{P}\left[|f(X) - \mathbb{E}\left[f(X)\right]| > t\right] \leq 2\exp\left(-\frac{t^2}{8(b_0+b_1)}\right).$$

Remark that each element in V gives rise to $b_0! + b_1!$ different orderings. All these orderings have the same

value under f. Thus the probability that $f(X) = c$ does not change when we take X to be a uniformly random element of V instead of W. Together with the above equation, this proves the claim. □

4.3 Bond percolation

This section is devoted to proving Theorem 4.1 for bond percolation. This proof is based on [2, Section 3]. It is divided into three steps. First, it will be shown that conditional on the degree sequence after percolation, each configuration on $W^-_{\vec{d^n_\pi}}$ and $W^+_{\vec{d^n_\pi}}$ is equally likely, see Section 4.3.1. In Section 4.3.2, we determine the limit of the expected number of vertices with degree (j, k) after percolation. Combining these results, the proof can be completed by showing that an element of D is ν a.s. proper, which allows us to apply Theorem 3.12, see Section 4.3.3.

4.3.1 A percolated configuration is a uniformly random configuration

Consider a uniformly random configuration on (W^-, W^+). Bond percolation randomly removes matches from this configuration. A configuration on $(W^{-,\pi}, W^{+,\pi})$ remains. It will now be shown that conditional on the degree sequence after percolation, a uniformly random configuration on $(W^{-,\pi}, W^{+,\pi})$ arises. The proof is split into two lemma's. Our lemma's are the directed analogues of Lemma 3.1 and 3.2 [2] for undirected graphs. The first step is showing that conditional on the number of edges after percolation, the surviving stubs are chosen uniformly at random.

Lemma 4.5. *Suppose that l of the m edges survive bond percolation applied to a uniformly random configuration $\mathcal{M}$ on (W^-, W^+). Then the surviving stubs $W^{-,\pi} \subset W^-$ and $W^{+,\pi} \subset W^+$ are uniformly distributed amongst all pairs of subsets of W^- and W^+ of size l.*

Proof. As l edges survive percolation, there holds $|W^{-,\pi}| = |W^{+,\pi}| = l$. The probability that $W^{-,\pi} \subset W^-$ and $W^{+,\pi} \subset W^+$ are the stubs surviving percolation equals the probability that all points in $W^{-,\pi}$ have their match in $W^{+,\pi}$ and that exactly these l matches survive percolation. Since the graph contains m matches of which l survive percolation, the probability exactly those l matches remain is $\frac{1}{\binom{m}{l}}$.
It is left to investigate the probability that all stubs in $W^{-,\pi}$ have their match in $W^{+,\pi}$. This implies that $\mathcal{M}$ must decompose into a perfect bipartite matching of $W^{-,\pi}$ with $W^{+,\pi}$ and a perfect bipartite matching of $W^- \setminus W^{-,\pi}$ with $W^+ \setminus W^{+,\pi}$. Between two sets of size l there are $l!$ perfect bipartite matchings, hence the probability that $\mathcal{M}$ decomposes as desired, is $l!(m-l)!/m!$. Thus the probability that $(W^{-,\pi}, W^{+,\pi})$ are the stubs surviving percolation is

$$\frac{l!\,(m-l)!}{m!}\frac{1}{\binom{m}{l}} = \frac{1}{\binom{m}{l}^2}.$$

This is the probability that $W^{-,\pi}$ is a uniformly random subset of size l of W^- and that $W^{+,\pi}$ is a uniformly random subset of W^+ of size l. □

Using this lemma, the desired statement can be shown.

Lemma 4.6. *Apply bond percolation to a uniformly random configuration on (W^-, W^+) obeying the degree sequence $\vec{d^n}$. Conditional on having degree sequence $\vec{d^n_\pi}$ after bond percolation, i.e. $\vec{d^n}' = \vec{d^n_\pi}$, all configurations of $W^-_{\vec{d^n_\pi}}$ with $W^+_{\vec{d^n_\pi}}$ are equally likely.*

Proof. The goal is to show that all configurations on $W^-_{\vec{d^n_\pi}}$ and $W^+_{\vec{d^n_\pi}}$ have equal probability to arise after percolation, given that $\vec{d^n}' = \vec{d^n_\pi}$. This implies that for any perfect bipartite matching $\mathcal{M}^\pi$ of $W^-_{\vec{d^n_\pi}}$ with $W^+_{\vec{d^n_\pi}}$ there must hold

$$\mathbb{P}\left[\mathcal{M}^\pi | \vec{d^n}' = \vec{d^n_\pi}\right] = \frac{1}{l!}.$$

Here l denotes the sum of the in-degrees of $\vec{d}_\pi^n$. First, rewrite this probability using $\mathbb{P}\left[|W^{-,\pi}| = l \middle| \vec{d^n}' = \vec{d}_\pi^n\right] = 1$ and the law of total probability to obtain

$$\mathbb{P}\left[\mathcal{M}^\pi | \vec{d^n}' = \vec{d}_\pi^n\right] = \mathbb{P}\left[\mathcal{M}^\pi \middle| |W^{-,\pi}| = l, \vec{d^n}' = \vec{d}_\pi^n\right].$$

Applying Bayes' formula to the right hand side of the previous equation gives

$$\mathbb{P}\left[\mathcal{M}^\pi | \vec{d^n}' = \vec{d}_\pi^n\right] = \frac{\mathbb{P}\left[\mathcal{M}^\pi \cap \vec{d^n}' = \vec{d}_\pi^n \middle| |W^{-,\pi}| = l\right]}{\mathbb{P}\left[\vec{d^n}' = \vec{d}_\pi^n \middle| W^{-,\pi}| = l\right]}. \tag{4.5}$$

It will now be shown that this expression equals $\frac{1}{l!}$. First determine the value of $\mathbb{P}\left[\vec{d^n}' = \vec{d}_\pi^n \middle| W^{-,\pi}| = l\right]$. Define $S(\vec{d}_\pi^n)$ to be the collection of pairs of subsets of (W^-, W^+) that induce the degree sequence $\vec{d}_\pi^n$. Recalling that $|W^{-,\pi}| = |W^{+,\pi}|$, we see that for any pair of subsets in $S(\vec{d}_\pi^n)$, both sets must contain l elements. In combination with Lemma 4.5 this implies

$$\mathbb{P}\left[\vec{d^n}' = \vec{d}_\pi^n \middle| |W^{-,\pi}| = l\right] = \frac{|S(\vec{d}_\pi^n)|}{\binom{m}{l}^2}.$$

Next we consider $\mathbb{P}\left[\mathcal{M}^\pi \cap \vec{d^n}' = \vec{d}_\pi^n \middle| |W^{-,\pi}| = l\right]$. Assume that $(W^{-,\pi}, W^{+,\pi})$ are the stubs surviving percolation and that $(W^{-,\pi}, W^{+,\pi}) \in S(\vec{d}_\pi^n)$. We want the probability that the configuration on these stubs induces the configuration $\mathcal{M}^\pi$ on $\left(W^-_{\vec{d}_\pi^n}, W^+_{\vec{d}_\pi^n}\right)$. As we fixed a bijection between $(W^{-,\pi}, W^{+,\pi})$ and $\left(W^-_{\vec{d}_\pi^n}, W^+_{\vec{d}_\pi^n}\right)$, exactly one configuration of $(W^{-,\pi}, W^{+,\pi})$ induces the configuration $\mathcal{M}^\pi$ on $\left(W^-_{\vec{d}_\pi^n}, W^+_{\vec{d}_\pi^n}\right)$. This determines the configuration on $(W^{-,\pi}, W^{+,\pi})$. However we are free to choose the configuration on $(W^- \setminus W^{-,\pi}, W^+ \setminus W^{+,\pi})$. Thus assuming that $(W^{-,\pi}, W^{+,\pi}) \in S(\vec{d}_\pi^n)$, the probability that it induces the configuration $\mathcal{M}^\pi$ on $(W^{-,\pi}, W^{+,\pi})$ is

$$\frac{(m-l)!}{m!}.$$

As Lemma 4.5 states that the remaining stubs after percolation are chosen uniformly random conditional on the number of matches surviving, there follows

$$\mathbb{P}\left[\mathcal{M}^\pi \cap \vec{d^n}' = \vec{d}_\pi^n \middle| |W^{-,\pi}| = l\right] = \frac{(m-l)!}{m!} \frac{|S(\vec{d}_\pi^n)|}{\binom{m}{l}}.$$

Plugging our findings back in equation (4.5), we obtain

$$\begin{aligned}\mathbb{P}\left[\mathcal{M}^\pi | \vec{d^n}' = \vec{d}_\pi^n\right] &= \frac{\binom{m}{l}\binom{m}{l}}{|S(\vec{d}_\pi^n)|} \frac{(m-l)!}{m!} \frac{|S(\vec{d}_\pi^n)|}{\binom{m}{l}} \\ &= \binom{m}{l} \frac{(m-l)!}{m!} = \frac{1}{l!}.\end{aligned}$$

□

4.3.2 The expected number of vertices with degree (j,k) after bond percolation

The next step in the proof of Theorem 4.1 for bond percolation is showing that the limit

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N^\pi_{j,k}(n)\right]}{n} := p^{\text{bond}}_{j,k} \tag{4.6}$$

exists and determining its value for all $0 \leq j,k < \infty$. Here $N_{j,k}^{\pi}(n)$ denotes the number of vertices in the percolated graph or configuration with in-degree j and out-degree k, i.e. with degree (j,k). In Section 4.3.3 we show that ν a.s. $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$ is the degree distribution of the percolated graph. Bearing this in mind, the distribution $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$ is used to determine an educated guess for π_c^{bond} at the end of this section. First the limit of equation (4.6) is shown to exist and the value of $p_{j,k}^{\text{bond}}$ is determined.

For all values of (j,k) with $j > d_{\max}$ or $k > d_{\max}$ or both the limit of equation (4.6) is easily evaluated. Under these condition, the graph does not contain a vertex of degree (j,k) before applying bond percolation by definition of $d_{\max}$. As percolation only decreases the degree of the vertices, this implies that $N_{j,k}^{\pi}(n) = 0$. Thus also for the expected number of vertices of degree (j,k) there holds $\mathbb{E}\left[N_{j,k}^{\pi}(n)\right] = 0$. Hence the limit of equation (4.6) is

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N_{j,k}^{\pi}(n)\right]}{n} = \lim_{n\to\infty} \frac{0}{n} = 0 := p_{j,k}^{\text{bond}}. \tag{4.7}$$

In case $j,k \leq d_{\max}$ it requires more work to show the existence of the limit in equation (4.6). Since equation (4.6) is the directed analogue of equation (3.3) of Fountoulakis [2] we adapt the corresponding proof from this reference. Remark that

$$\mathbb{E}\left[N_{j,k}^{\pi}(n)\right] = \sum_{l=0}^{m} \mathbb{E}\left[N_{j,k}^{\pi}(n) \,\middle|\, |W^{-,\pi}| = l\right] \mathbb{P}\left[|W^{-,\pi}| = l\right]. \tag{4.8}$$

This conditional expectation of $N_{j,k}^{\pi}(n)$ in turn can be written as

$$\mathbb{E}\left[N_{j,k}^{\pi}(n) \,\middle|\, |W^{-,\pi}| = l\right] = \sum_{d^-=j}^{d_{\max}} \sum_{d^+=k}^{d_{\max}} N_{d^-,d^+}(n)\, \mathbb{P}\left[\text{vertex of degree } (d^-,d^+) \text{ has new degree } (j,k) \,\middle|\, |W^{-,\pi}| = l\right].$$

Here (d^-,d^+) is the degree of the vertex before percolation and (j,k) is its degree after percolation. This requires us to determine the probability $\mathbb{P}\left[\text{vertex of degree } (d^-,d^+) \text{ has new degree } (j,k) \,||W^{-,\pi}| = l\right]$. As Lemma 4.5 implies that the surviving stubs are chosen uniformly at random conditional on the size of $W^{-,\pi}$, there holds

$$\mathbb{P}\left[\text{vertex of degree } (d^-,d^+) \text{ has new degree } (j,k) \,\middle|\, |W^{-,\pi}| = l\right] = \binom{d^-}{j} \frac{\binom{m-d^-}{l-j}}{\binom{m}{l}} \binom{d^+}{k} \frac{\binom{m-d^+}{l-k}}{\binom{m}{l}}.$$

This value can be further approximated for $l \in I := \left[m\pi - \ln(n)\sqrt{n}, m\pi + \ln(n)\sqrt{(n)}\right]$. As the edges are removed independently of each other, the size of $W^{-,\pi}$ is the sum of m independent Bernoulli variables, each having expectation π. Applying Theorem 4.2 yields

$$\mathbb{P}\left[\left||W^{-,\pi}| - m\pi\right| > \ln(n)\sqrt{n}\right] \leq \exp\left[-\Omega(\ln^2(n))\right]. \tag{4.9}$$

This implies that $\mathbb{P}\left[l \notin I\right] = o\left(\frac{1}{n^3}\right)$. Fountoulakis [2] shows that for $d_{\max} \leq n^{1/9}$ and $l \in I$ there holds

$$\frac{\binom{2m-d}{2l-j}}{\binom{2m}{2k}} = \pi^j (1-\pi)^{d-j} \left(1 + \mathcal{O}\left(\frac{\ln(n)}{n^{7/18}}\right)\right).$$

As we consider proper degree arrays there holds $d_{\max} \leq \frac{n^{1/12}}{\ln(n)}$. Since $\frac{n^{1/12}}{\ln(n)} < n^{1/9}$ for all $n \geq 3$, we can use the same argument of Fountoulakis to show that

$$\mathbb{P}\left[\text{vertex of degree } (d^-,d^+) \text{ has new degree } (j,k) \,|\, |W^{-,\pi}| = l\right] = \binom{d^-}{j}\binom{d^+}{k} \pi^{j+k} (1-\pi)^{d^-+d^+-j-k} \left(1 + \mathcal{O}\left(\frac{\ln(n)}{n^{7/18}}\right)\right),$$

uniformly for all $d^-, d^+ \leq d_{\max}$ and $l \in I$. This allows to determine $\mathbb{E}\left[N^{\pi}_{j,k}(n) \,||W^{-,\pi}| = l\right]$ for all $l \in I$. In combination with equation (4.8), $\mathbb{P}[l \notin I] = o\left(\frac{1}{n^3}\right)$ and the fact that $N^{\pi}_{j,k}(n) \leq n$, this yields

$$\mathbb{E}\left[N^{\pi}_{j,k}(n)\right] = \left(1 + \mathcal{O}\left(\frac{\ln(n)}{n^{7/18}}\right)\right) \sum_{d^-=j}^{d_{\max}} \sum_{d^+=k}^{d_{\max}} \binom{d^-}{j}\binom{d^+}{k} \pi^{j+k}(1-\pi)^{d^- + d^+ - j - k} + o\left(\frac{1}{n^3}\right), \tag{4.10}$$

using same argument as for $\mathbb{E}[D'_i(n)]$ in [2, p. 344]. With this approximation of $\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]$, we can show that the limit of equation (4.6) exists. This requires for all $\epsilon > 0$ the existence of $\kappa(\epsilon)$ and $N(\epsilon)$ such that for all $n > N$

$$\frac{1}{n} \sum_{\substack{(d^-,d^+)=(0,0) \\ d^- \geq \kappa+1 \text{ or } d^+ \geq \kappa+1}}^{(d_{\max},d_{\max})} N_{d^-,d^+}(n) \binom{d^-}{j}\binom{d^+}{k} \pi^{j+k}(1-\pi)^{d^- + d^+ - j - k} \leq \frac{1}{n} \sum_{\substack{(d^-,d^+)=(0,0) \\ d^- \geq \kappa+1 \text{ or } d^+ \geq \kappa+1}}^{(d_{\max},d_{\max})} N_{d^-,d^+}(n) < \epsilon. \tag{4.11}$$

The left inequality holds by the binomial theorem, which implies that $\sum_{j=0}^{d^-}\binom{d^-}{j}\pi^j(1-\pi)^{d^- - j} = \sum_{k=0}^{d^+}\binom{d^+}{k}\pi^k(1-\pi)^{d^+ - k} = 1$, yielding for all $j \leq d^-, k \leq d^+$ that $\binom{d^-}{j}\binom{d^+}{k}p^{j+k}(1-p)^{d^- + d^+ - j - k} \leq 1$. The right inequality holds since $\lim_{n\to\infty} \frac{N_{j,k}(n)}{n} = p_{j,k}$ for $j, k \geq 0$, which follows from the degree array being proper. Combining equations (4.10) and (4.11) the limit in equation (4.6) can be shown to exists in analogy to the proof of [2, eqaution (3.3)]. Not only does this show the existence of the limit, it will also determine its value:

$$p^{\text{bond}}_{j,k} = \lim_{n\to\infty} \frac{\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]}{n} = \sum_{d^-=j}^{\infty} \sum_{d^+=k}^{\infty} p_{d^-,d^+} \binom{d^-}{j}\binom{d^+}{k} \pi^{j+k}(1-\pi)^{d^- - j + d^+ - k}. \tag{4.12}$$

Thus the desired limit is evaluated. It is expected that $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$ will be the degree distribution of the percolated graph. In Section 4.3.3 this will be shown that this holds ν a.s. However this implies that $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$ must obey equation (2.6) and be normalized. The normalization follows from the binomial theorem and that fact that $(p_{j,k})_{j,k=0}^{\infty}$ is normalized:

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{d^-=j}^{\infty}\sum_{d^+=k}^{\infty} p_{d^-,d^+}\binom{d^-}{j}\binom{d^+}{k}\pi^{j+k}(1-\pi)^{d^- - j + d^+ - k} \\
&\quad= \sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} p_{d^-,d^+} \sum_{j=0}^{d^-}\binom{d^-}{j}\pi^j(1-\pi)^{d^- - j} \sum_{k=0}^{d^+}\binom{d^+}{k}\pi^k(1-\pi)^{d^+ - k} \\
&\quad= \sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} p_{d^-,d^+} = 1.
\end{aligned}$$

Next it is shown that $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$ obeys equation (2.6). Besides the binomial theorem this derivation also uses the equation

$$\sum_{k=0}^{n}\binom{n}{k} k x^k y^{n-k} = xn(x+y)^{n-1},$$

which can be obtained by applying $x\frac{d}{dx}$ to the binomial theorem. We find

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} jp_{j,k}^{\text{bond}} &= \sum_{j=0}^{\infty} j\sum_{k=0}^{\infty}\sum_{d^-=j}^{\infty}\binom{d^-}{j}\pi^j(1-\pi)^{d^--j}\sum_{d^+=k}^{\infty}\binom{d^+}{k}\pi^k(1-\pi)^{d^+-k}p_{d^-,d^+}\\ &= \sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} p_{d^-,d^+}\sum_{j=0}^{d^-} j\binom{d^-}{j}\pi^j(1-\pi)^{d^--j}\sum_{k=0}^{d^+}\binom{d^+}{k}\pi^k(1-\pi)^{d^+-k}\\ &= \pi\sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} d^-p_{d^-,d^+}.\end{aligned}$$

A symmetric argument shows that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} kp_{j,k}^{\text{bond}} = \pi\sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} d^+p_{d^-,d^+}.$$

This proves that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} jp_{j,k}^{\text{bond}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} kp_{j,k}^{\text{bond}} = \pi\mu := \mu^{\pi,\text{bond}}, \tag{4.13}$$

as $(p_{j,k})_{j,k=0}^{\infty}$ satisfies equation (2.6). Thus $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$ indeed is normalized and satisfies equation (2.6).

As we expect that $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$ will be the degree distribution of the percolated graph, we expect from Theorem 3 the percolation threshold will be the value of π such that

$$\sum_{j,k=0}^{\infty} jkp_{j,k}^{\text{bond}} = \sum_{j,k=0}^{\infty} jp_{j,k}^{\text{bond}}. \tag{4.14}$$

Denote this value by $\hat{\pi}^{\text{bond}}$. We will now determine this value. Recall that we just showed

$$\sum_{j,k=0}^{\infty} jp_{j,k}^{\text{bond}} = \pi\mu.$$

In the same way, we find that

$$\sum_{j,k=0}^{\infty} jkp_{j,k}^{\text{bond}} = \pi^2\sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} d^-d^+p_{d^-,d^+} := \mu_{11}^{\pi,\text{bond}}. \tag{4.15}$$

Plugging this into equation 4.14 yields

$$\hat{\pi}^{\text{bond}} = \frac{\sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} d^-p_{d^-,d^+}}{\sum_{d^-=0}^{\infty}\sum_{d^+=0}^{\infty} d^-d^+p_{d^-,d^+}} = \frac{\mu}{\mu_{11}}.$$

It remains to prove that $\hat{\pi}^{\text{bond}} = \pi_c^{\text{bond}}$ and to determine c^{bond}.

4.3.3 Determining π_c^{bond} and c^{bond}

The equality $\hat{\pi}^{\text{bond}} = \pi_c^{\text{bond}}$ can be shown by applying Theorem 3.12 to the percolated multigraph array $\left(\tilde{G}_{\vec{d^n}}^{\pi}\right)_{n=1}^{\infty}$. This also allows us to determine the value of c^{bond} and hence to prove Theorem 4.1 for bond percolation. The percolated multigraph array $\left(\tilde{G}_{\vec{d^n}}^{\pi}\right)_{n=1}^{\infty}$ is obtained by applying bond percolation to a multigraph array obeying the degree array $\left(\vec{d^n}\right)_{n=1}^{\infty}$, which is generated using the directed configuration model.

To apply Theorem 3.12, we must show that $\left(\tilde{G}^{\pi}_{\vec{d^n}}\right)_{n=1}^{\infty}$ is a random multigraph array obeying a proper degree array. Lemma 4.6 implies that $\tilde{G}^{\pi}_{\vec{d^n}}$ is multigraph arising from a uniformly random configuration obeying the degree sequence $\vec{d^n_\pi}$ conditional on $\vec{d^n_\pi}$ being the degree sequence after percolation. Thus conditional on $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty}$ being proper and the degree array after percolation, Theorem 3.12 can be applied to $\left(\tilde{G}^{\pi}_{\vec{d^n}}\right)_{n=1}^{\infty}$. This requires $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty}$ to be proper. It will be shown that an element of D is ν a.s. proper. Hence Theorem 4.1 may be applied to almost all degree arrays $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty}$ to determine π_c^{bond} and c^{bond} for random multigraphs. A variant of Lemma 2.7 is then applied to show that the same holds for a percolated graph array $\left(G^{\pi}_{\vec{d^n}}\right)_{n=1}^{\infty}$ that arises from applying percolation to a uniformly random simple graph array obeying $\left(\vec{d^n}\right)_{n=1}^{\infty}$. In the remainder of this section, we formalize the above argument. This is based on the proof of Theorem 1.1 [2, p. $345-348$]. The first step is showing that each element of D is ν a.s. proper.

Definition 3.10 implies that each proper degree array must be feasible . For any $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty} \in D$ that is feasible, it can be easily shown that it is proper as well, using the fact that $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is proper. So we will first prove that any $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty} \in D$ is ν a.s. feasible. By Definition 2.3 a degree array $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty}$ is feasible if $\left(\frac{N^{\pi}_{j,k}(n)}{n}\right)_{j,k=0}^{\infty}$ and its first, first mixed and second moments converge to those of a bivariate distribution obeying equation (2.6). As we consider multigraphs at this point, for now we replace the constraint of each degree sequence being graphical with each degree sequence being valid. That each $\vec{d^n_\pi}$ is valid, is a direct consequence from the fact that $\vec{d^n}$ is valid, which holds as $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is proper. We may swap these conditions as Theorem 3.12 holds for a proper degree arrays where each degree sequence is valid as well [4]. When conditioning on the graph before percolation being simple later on in the proof, we implicitly replace valid by graphical. At that point the degree sequence $\vec{d^n_\pi}$ will be graphical, as applying percolation to a simple graph must result in another simple graph. Now we will show that ν a.s. $\left(\frac{N^{\pi}_{j,k}(n)}{n}\right)_{j,k=0}^{\infty}$ and its first, first mixed and second moments converge to $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$ and its corresponding moments. The first step is showing that ν a.s.

$$\lim_{n\to\infty} \frac{N^{\pi}_{j,k}(n)}{n} = p^{\text{bond}}_{j,k}, \tag{4.16}$$

for all $j,k \geq 0$. This is shown using the technique Fountoulakis applies [2, p. $346-347$]. By [25, Lemma 6.8] it suffices to prove that $\epsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\left|\frac{N^{\pi}_{j,k}(n)}{n} - p^{\text{bond}}_{j,k}\right| > \epsilon\right] < \infty, \tag{4.17}$$

to show equation (4.16). By definition of $p^{\text{bond}}_{j,k}$ for any fixed $\epsilon > 0$ there is an N such that for all $n > N$

$$\left|\frac{\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]}{n} - p^{\text{bond}}_{j,k}\right| < \frac{\epsilon}{2}.$$

This implies

$$\mathbb{P}\left[\left|\frac{N^{\pi}_{j,k}(n)}{n} - p^{\text{bond}}_{j,k}\right| > \epsilon\right] \leq \mathbb{P}\left[\left|\frac{N^{\pi}_{j,k}(n)}{n} - \frac{\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]}{n}\right| > \frac{\epsilon}{2}\right].$$

To determine this latter probability, recall Lemma 4.5. This states that conditional on $|W^{-,\pi}| = l$, the stubs surviving percolation $(W^{-,\pi}, W^{+,\pi})$ are uniformly distributed amongst all pairs of subsets of (W^{-}, W^{+})

of size l. The value $N^{\pi}_{j,k}(n)$ is a function of $W^{-,\pi} \cup W^{+,\pi}$. Furthermore for two sets $W^{-,\pi} \cup W^{+,\pi}$ and $W^{-,\pi'} \cup W^{+,\pi'}$ their values of $N^{\pi}_{j,k}(n)$ differ by at most the number of elements in the symmetric difference of $W^{-,\pi} \cup W^{+,\pi}$ and $W^{-,\pi'} \cup W^{+,\pi'}$. This implies that for $A_0 = W^-, A_1 = W^+, b_0 = b_1 = l$ and $N^{\pi}_{j,k}(n)$ as function f the requirements of Corollary 4.4 are fulfilled. Applying this Corollary yields

$$\mathbb{P}\left[\left|N^{\pi}_{j,k}(n) - \mathbb{E}\left[N^{\pi}_{j,k}(n)\right]\right| > \frac{n\epsilon}{2} \,|\, |W^{-,\pi}| = l\right] \leq 2\exp\left(\frac{\epsilon^2 n^2}{64 l^2}\right).$$

If $l \in I$, this probability is $o\left(\frac{1}{n^3}\right)$. By equation (4.9) the probability that $l \notin I$ is $o\left(\frac{1}{n^3}\right)$. Combining these observations we find

$$\mathbb{P}\left[\left|N^{\pi}_{j,k}(n) - \mathbb{E}\left[N^{\pi}_{j,k}(n)\right]\right| > n\epsilon\right] = o\left(\frac{1}{n^3}\right). \tag{4.18}$$

For any $\epsilon > 0$ this shows equation (4.17), which in turn proves that the limit in equation (4.16) holds ν a.s.

It remains to show that the first, first mixed and second moments of $\left(\frac{N^{\pi}_{j,k}(n)}{n}\right)_{j,k=0}^{\infty}$ converge ν a.s. to those of $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$, to prove that $\left(\vec{d^n_\pi}\right)_{n=1}^{\infty}$ is ν a.s. feasible . All these moments can be shown to converge ν a.s. using the same argument, which is based on the proof Fountoulakis uses for equation (3.5) [2]. First we investigate the convergence of the first moments. In Section 4.3.2 we saw that $\sum_{j,k=0}^{\infty} j p^{\text{bond}}_{j,k} = \sum_{j,k=0}^{\infty} k p^{\text{bond}}_{j,k}$. As any edge must begin at one vertex and end at another, there also holds $\sum_{j,k=0}^{\infty} j N^{\pi}_{j,k}(n) = \sum_{j,k=0}^{\infty} k N^{\pi}_{j,k}(n)$. This implies that we can restrict ourself to showing that

$$\lim_{n\to\infty} Q'_n := \lim_{n\to\infty} \frac{\sum_{j,k=0}^{\infty} j N^{\pi}_{j,k}(n)}{n} \tag{4.19}$$

ν a.s. exists and equals

$$Q = \sum_{j,k=0}^{\infty} j p^{\text{bond}}_{j,k} = \pi\mu,$$

to show that both first moments converge. To determine the limit in equation (4.19) define

$$X_{\kappa,n} = \frac{1}{n}\sum_{j=0}^{\kappa}\sum_{k=0}^{\kappa} j N^{\pi}_{j,k}(n).$$

Remark that there holds $X_{\kappa,n} \leq Q'_n$. Since $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is a proper degree array, for all $\epsilon > 0$ there exists $\widetilde{\kappa}(\epsilon), N(\epsilon)$ such that for all $\kappa > \widetilde{\kappa}$ and $n > N$

$$\frac{1}{n}\sum_{\substack{(d^-,d^+)=(0,0)\\ d^->\kappa \text{ or } d^+>\kappa}}^{(d_{\max},d_{\max})} j N_{j,k}(n) < \epsilon. \tag{4.20}$$

Thus for $\kappa > \widetilde{\kappa}$ there holds $X_{\kappa,n} \leq Q'_n \leq X_{\kappa,n} + \epsilon$. This implies that if ν a.s.

$$\lim_{n\to\infty} X_{\kappa,n} = \sum_{j=0}^{\kappa}\sum_{k=0}^{\kappa} j p^{\text{bond}}_{j,k} := \tilde{X}_{\kappa}, \tag{4.21}$$

then $\lim_{n\to\infty} Q'_n = Q$ ν a.s. as well [2, (3.5)]. Thus the goal is to prove equation (4.21). Applying Lemma 6.8 [25] we can show that this limit ν a.s. holds, if for any $\epsilon > 0$ there holds

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\left|X_{\kappa,n} - \tilde{X}_{\kappa}\right| > \epsilon\right] < \infty. \tag{4.22}$$

We show this analogous to the proof of equation (4.17). By the definition of $X_{\kappa,n}, \tilde{X}_\kappa$ and $p^{\text{bond}}_{j,k}$, for all $\epsilon > 0$ exists an $\widetilde{N}$ such that for all $n > \widetilde{N}$

$$\left|\mathbb{E}\left[X_{\kappa,n}\right] - \tilde{X}_\kappa\right| < \frac{\epsilon}{2}.$$

Combing this with the reverse triangle inequality, we find for any $\epsilon > 0$

$$\mathbb{P}\left[\left|X_{\kappa,n} - \tilde{X}_\kappa\right| > \epsilon\right] \leq \mathbb{P}\left[\left|X_{\kappa,n} - \mathbb{E}\left[X_{\kappa,n}\right]\right| > \frac{\epsilon}{2}\right].$$

Remark that

$$\left|X_{\kappa,n} - \mathbb{E}\left[X_{\kappa,n}\right]\right| = \frac{1}{n}\sum_{j=0}^{\kappa}\sum_{k=0}^{\kappa} j\left(N^\pi_{j,k}(n) - \mathbb{E}\left[N^\pi_{j,k}(n)\right]\right). \tag{4.23}$$

This implies that for $\epsilon' = \frac{\epsilon}{2\sum_{j\leq\kappa} j}$ there holds

$$\mathbb{P}\left[\left|X_{\kappa,n} - \mathbb{E}\left[X_{\kappa,n}\right]\right| > \frac{\epsilon}{2}\right] \leq \sum_{j\leq\kappa,k\leq\kappa} \mathbb{P}\left[\frac{1}{n}\left|N^\pi_{j,k}(n) - \mathbb{E}\left[N^\pi_{j,k}(n)\right]\right| > \epsilon'\right].$$

Using equation (4.18) we find

$$\mathbb{P}\left[\left|X_{\kappa,n} - \mathbb{E}\left[X_{\kappa,n}\right]\right| > \frac{\epsilon}{2}\right] \leq \sum_{j\leq\kappa,k\leq\kappa} o\left(\frac{1}{n^3}\right) \leq o\left(\frac{1}{n^{2\frac{7}{9}}}\right).$$

Here we used the fact that $d_{\max} = O\left(n^{1/9}\right)$ and that for $j > d_{\max}$ or $k > d_{\max}$ or both, there holds $N^\pi_{j,k}(n) = \mathbb{E}\left[N^\pi_{j,k}(n)\right] = 0$. This shows equation (4.22) and hence proves that Q'_n converges ν a.s. to Q.

By redefining $Q'_n, Q, X_{\kappa,n}, \tilde{X}_\kappa$ the same can be shown to hold for the first mixed moment and the second moments. This also requires to new definition of ϵ'. Defining

$$\epsilon'' = \min\left\{\frac{\epsilon}{2}, \frac{\epsilon}{2\sum_{j\leq\kappa} j}, \frac{\epsilon}{2\sum_{j\leq\kappa,k\leq\kappa} jk}, \frac{\epsilon}{2\sum_{j\leq\kappa} j^2}, \frac{\epsilon}{2\sum_{k\leq\kappa} k^2}\right\},$$

we can show that all the moments of interest and the distribution itself converge simultaneously ν a.s. for an element of D. Thus we have shown that $\left(\vec{d^n_\pi}\right)_{n=1}^\infty$ is ν a.s. feasible. To show that $\left(\vec{d^n_\pi}\right)_{n=1}^\infty$ is ν a.s. proper, we show for each $\left(\vec{d^n_\pi}\right)_{n=1}^\infty \in D$, which is feasible, that $d^\pi_{\max} \leq \frac{n^{1/12}}{\ln(n)}$ and $\rho^\pi = \max\left\{\frac{\sum_{j,k=0}^\infty j^2 k N^\pi_{j,k}(n)}{\mu^\pi n}, \frac{\sum_{j,k=0}^\infty jk^2 N^\pi_{j,k}(n)}{\mu^\pi n}\right\} = o\left(\frac{n^{1/12}}{\ln(n)}\right)$. Here $d^\pi_{\max}$ is the maximum degree of the percolated degree sequence. By Definition 3.10 this suffices to show that $\left(\vec{d^n_\pi}\right)_{n=1}^\infty$ is proper. Remark that we here used the feasibility of the degree array to define ρ. As $\left(\vec{d^n}\right)_{n=1}^\infty$ is proper, there holds $d_{\max} \leq \frac{n^{1/12}}{\ln(n)}$. Percolation can only decrease the in-degree and the out-degree of a vertex, implying $d^\pi_{\max} \leq d_{\max}$. Together these observations show that $d^\pi_{\max} \leq \frac{n^{1/12}}{\ln(n)}$ for any $\left(\vec{d^n_\pi}\right)_{n=1}^\infty \in D$. It remains to show $\rho^\pi = o\left(\frac{n^{1/12}}{\ln(n)}\right)$. First investigate the summation $\sum_{j,k=0}^\infty j^2 k N^\pi_{j,k}(n)$ (respectively $\sum_{j,k=0}^\infty jk^2 N^\pi_{j,k}(n)$). Each vertex of degree (j,k) contributes $j^2k(jk^2)$ to the summation. As the in-degree and the out-degree can only decrease due to percolation, this implies

$$\sum_{j,k=0}^\infty j^2 k N^\pi_{j,k}(n) \leq \sum_{j,k=0}^\infty j^2 k N_{j,k}(n) \quad \text{and} \quad \sum_{j,k=0}^\infty jk^2 N^\pi_{j,k}(n) \leq \sum_{j,k=0}^\infty jk^2 N_{j,k}(n).$$

Equation (4.13) gives that $\mu^{\pi,\text{bond}} = \pi\mu$. Recall that π is constant, i.e. it is assigned a fixed value before the percolation is applied. These observations together with the fact that $\left(\vec{d^n}\right)_{n=1}^\infty$ is proper imply

$$\rho^\pi \leq \max\left\{\frac{\sum_{j,k=0}^\infty j^2 k N_{j,k}(n)}{\pi\mu n}, \frac{\sum_{j,k=0}^\infty jk^2 N_{j,k}(n)}{\pi\mu n}\right\} = \frac{\rho}{\pi} = o\left(\frac{n^{1/12}}{\ln(n)}\right).$$

Thus any $\left(\vec{d}_\pi^n\right)_{n=1}^\infty \in D$ is ν a.s. proper. Let $E \subset D$ be the event over which the degree array is proper. As any element of D is ν a.s. proper, there holds $\nu(E) = 1$.

To complete the proof, we follow the idea of Fountoulakis [2, p. 347 − 348]. In place of [2, Theorem 2.1] we use Theorem 3.12. For any $\left(\vec{d}_\pi^n\right)_{n=1}^\infty \in E$ we may apply Theorem 3.12 to a sequence of random multigraphs $\left(\tilde{G}^\pi_{\vec{d}^n}\right)_{n=1}^\infty$ arising from uniformly random configurations. Recall that Lemma 4.6 implies that this is the case for all n if we condition on $\vec{d^n}' = \vec{d}_\pi^n$. Fix $\left(\vec{d^n}\right)_{n=1}^\infty \in E$ and apply Theorem 3.12 to $\left(\tilde{G}^\pi_{\vec{d}^n}\right)_{n=1}^\infty$. We will distinguish two cases based on the value of π: $\pi < \hat{\pi}^{\text{bond}}$ and $\pi > \hat{\pi}^{\text{bond}}$. Recall that $\hat{\pi}^{\text{bond}} = \frac{\mu}{\mu_{11}}$.

First consider the case $\pi < \hat{\pi}^{\text{bond}}$. Define $\mathcal{A}_\epsilon\left(\vec{d}_\pi^n\right)$ to be the set of all multigraphs obeying $\vec{d}_\pi^n$ for which the largest strongly connected component contains no more than ϵn vertices for $\epsilon \in (0,1)$. As $\frac{\mu_{11}^\pi}{\mu^\pi} = \pi\frac{\mu_{11}}{\mu} < 1$, Theorem 3.12 implies for all ϵ:

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}^\pi_{\vec{d}^n} \in \mathcal{A}_\epsilon\left(\vec{d}_\pi^n\right) \mid \vec{d^n}' = \vec{d}_\pi^n\right] = 1. \tag{4.24}$$

Next consider $\pi > \hat{\pi}^{\text{bond}}$. Define $\mathcal{B}_\epsilon\left(\vec{d}_\pi^n\right)$ to be the set of all graphs whose largest strongly connected component contains ϵn vertices for $\epsilon \in (0,1)$. As $\frac{\mu_{11}^\pi}{\mu^\pi} = \pi\frac{\mu_{11}}{\mu} > 1$ Theorem 3.12 implies that there exists an unique ϵ such that

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{G}^\pi_{\vec{d}^n} \in \mathcal{B}_\epsilon\left(\vec{d}_\pi^n\right) \mid \vec{d^n}' = \vec{d}_\pi^n\right] = 1.$$

Not only does the theorem imply existence of such of ϵ, it also determines this value. This value is the equivalent of $(\zeta^+ + \zeta^- + \psi - 1)$ for the degree distribution $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^\infty$. We will express this value in terms of $(p_{j,k})_{j,k=0}^\infty$. This requires us to determine the probability that a uniformly random in-stub is attached to a vertex of out-degree k in the percolated graph. In analogy to equation (3.5) this is

$$\begin{aligned} p_k^{+,\text{bond}} &= \frac{1}{\mu^{\pi,\text{bond}}}\sum_{j=0}^\infty j p^{\text{bond}}_{j,k} = \frac{1}{\pi\mu}\sum_{d^+=k}^\infty\sum_{d^-=0}^\infty p_{d^-,d^+}\sum_{j=0}^{d^-} j\binom{d^-}{j}\pi^j(1-\pi)^{d^--j}\binom{d^+}{k}\pi^k(1-\pi)^{d^+-k} \\ &= \frac{\pi}{\pi\mu}\sum_{d^+=k}^\infty\sum_{d^-=0}^\infty d^- p_{d^-,d^+}\binom{d^+}{k}\pi^k(1-\pi)^{d^+-k} = \sum_{d^+=k}^\infty p^+_{d^+}\binom{d^+}{k}\pi^k(1-\pi)^{d^+-k}. \end{aligned} \tag{4.25}$$

Similarly the probability that a uniformly random out-stub is attached to a vertex of in-degree j in the percolated graph becomes

$$p_j^{-,\text{bond}} = \sum_{d^-=j}^\infty p^-_{d^-}\binom{d^-}{j}\pi^j(1-\pi)^{d^--j}. \tag{4.26}$$

The distributions $\left(p_j^{-,\text{bond}}\right)_{j=0}^\infty$ and $\left(p_j^{+,\text{bond}}\right)_{j=0}^\infty$ both have expected value $\pi\frac{\mu_{11}}{\mu} > 1$. As $p_0^-, p_0^+ > 0$ there also holds $p_0^{-,\text{bond}}, p_0^{+,\text{bond}} > 0$. Hence the generating functions of these probability distributions will have unique fixed points, $x = f(x)$, that lie in the interval $(0,1)$ [4, Lemma 1]. These are denoted by $\left(1-\eta^{-,\text{bond}}\right)$ and $\left(1-\eta^{+,\text{bond}}\right)$ and given by

$$\begin{aligned} \left(1-\eta^{-,\text{bond}}\right) &= \sum_{j=0}^\infty p_j^{-,\text{bond}}\left(1-\eta^{-,\text{bond}}\right)^j = \sum_{d^-=0}^\infty p^-_{d^-}\sum_{j=0}^{d^-}\binom{d^-}{j}\pi^j(1-\pi)^{d^--j}\left(1-\eta^{-,\text{bond}}\right)^j \\ &= \sum_{d^-=0}^\infty p^-_{d^-}\left(\pi\left(1-\eta^{-,\text{bond}}\right)+1-\pi\right)^{d^-} \end{aligned} \tag{4.27}$$

and

$$\left(1-\eta^{+,\text{bond}}\right)=\sum_{d^+=0}^{\infty}p_{d^+}^{+}\left(\pi\left(1-\eta^{+,\text{bond}}\right)+1-\pi\right)^{d^+}. \tag{4.28}$$

Using η^- and η^+ we can determine the analogues of ζ^+,ζ^- and ψ for the degree distribution $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$. Following the equations (3.10), (3.7) and (3.11) these are defined by

$$\left(1-\zeta^{-,\text{bond}}\right)=\sum_{j,k=0}^{\infty}p_{j,k}^{\text{bond}}\left(1-\eta^{-,\text{bond}}\right)^{j},\quad \left(1-\zeta^{+,\text{bond}}\right)=\sum_{j,k=0}^{\infty}p_{j,k}^{\text{bond}}\left(1-\eta^{+,\text{bond}}\right)^{k} \tag{4.29}$$

and

$$\psi^{\text{bond}}=\sum_{j,k=0}^{\infty}p_{j,k}^{\text{bond}}\left(1-\eta^{-,\text{bond}}\right)^{j}\left(1-\eta^{+,\text{bond}}\right)^{k}. \tag{4.30}$$

Now we can define

$$c^{\text{bond}}=\zeta^{-,\text{bond}}+\zeta^{+,\text{bond}}+\psi^{\text{bond}}-1. \tag{4.31}$$

This is the unique value of ϵ such that

$$\lim_{n\to\infty}\mathbb{P}\left[\tilde{G}_{\vec{d^n}}^{\pi}\in\mathcal{B}_{c^{\text{bond}}}\left(\vec{d_\pi^n}\right)\mid\vec{d_\pi^n}=\vec{d^n}'\right]=1, \tag{4.32}$$

by Theorem 3.12.

Equations (4.32) and (4.24) almost prove Theorem 4.1. There are two minor issues that can be easily resolved. First of all the theorem is stated for a percolated multigraph array $\left(\tilde{G}_{\vec{d^n}}^{\pi}\right)_{n=1}^{\infty}$ without conditioning on the degree array of percolated graphs. As $\nu(E)=1$, the argument of Fountoulakis [2, p. 348] can be applied to show that:

- $\lim_{n\to\infty}\mathbb{P}\left[\tilde{G}_{\vec{d^n}}^{\pi}\in\mathcal{A}_\epsilon\left(\vec{d_\pi^n}\right)\right]=1$ for all $\epsilon\in(0,1)$ if $\pi<\hat{\pi}^{\text{bond}}$;
- $\lim_{n\to\infty}\mathbb{P}\left[\tilde{G}_{\vec{d^n}}^{\pi}\in\mathcal{B}_{c^{\text{bond}}}\left(\vec{d_\pi^n}\right)\right]=1$ and $\lim_{n\to\infty}\mathbb{P}\left[\tilde{G}_{\vec{d^n}}^{\pi}\in\mathcal{B}_{\epsilon}\left(\vec{d_\pi^n}\right)\right]=0$ for all $\epsilon\in(0,1),\epsilon\neq c^{\text{bond}}$ if $\pi>\hat{\pi}^{\text{bond}}$.

The last problem is that we are interested percolation on uniformly random simple graphs instead of random multigraphs. Replace the the graph $\tilde{G}_{\vec{d^n}}$ in Lemma 2.7 and Corollary 2.8 by the graph $\tilde{G}_{\vec{d^n}}^{\pi}$ and condition on the graph to which percolation is applied ($G_{\vec{d^n}}$) being simple. This yields slightly different variants of the lemma and corollary that do not require the additional changes to the proof. Now applying this variant of Corollary 2.8 to the above limits, we derive that

- $\lim_{n\to\infty}\mathbb{P}\left[G_{\vec{d^n}}^{\pi}\in\mathcal{A}_\epsilon\left(\vec{d_\pi^n}\right)\right]=1$ for all $\epsilon\in(0,1)$ if $\pi<\hat{\pi}^{\text{bond}}$;
- $\lim_{n\to\infty}\mathbb{P}\left[G_{\vec{d^n}}^{\pi}\in\mathcal{B}_{c^{\text{bond}}}\left(\vec{d_\pi^n}\right)\right]=1$ and $\lim_{n\to\infty}\mathbb{P}\left[G_{\vec{d^n}}^{\pi}\in\mathcal{B}_{\epsilon}\left(\vec{d_\pi^n}\right)\right]=0$ for all $\epsilon\in(0,1),\epsilon\neq c^{\text{bond}}$ if $\pi>\hat{\pi}^{\text{bond}}$,

completing the proof of Theorem 4.1 for the case of bond percolation.

4.4 Site percolation

It remains to prove Theorem 4.1 for site percolation. This proof has the same structure as for bond percolation. Hence for parts of the proof, we will refer back to Section 4.3. Like in the case of bond percolation, the proof is split into different parts. First in Section 4.4.1 we show that applying site percolation to a uniformly random configuration leads to another uniformly random if we condition on the degree sequence after percolation, like in the case of bond percolation. The next step is determining the limit of the expected number of vertices with degree (j,k) after site percolation, see Section 4.3.2. Combining the results from these sections with elements from Section 4.3, the proof Theorem 4.1 for site percolation is completed in Section 4.4.3. Our proof of Theorem 4.1 for site percolation is based [2, Section 4]. We will conclude this section by introducing notation that is used throughout the proof.

Site percolation randomly deletes vertices. Recall from Section 4.1 that deleting a vertex means that we remove all edges adjacent to this vertex. In the setting of the configuration model this implies that all stubs attached to a deleted vertex are removed. Denote these stubs by $(W^{-,r}, W^{+,r})$. As site percolation removes any edges adjacent to a vertex, also the match of any stub in $(W^{-,r}, W^{+,r})$ will be removed. A stub in $(W^{-,r}, W^{+,r})$ can have its match in the same set, as it might happen that both endpoints of one edge are deleted. Let $(W^{-,m}, W^{+,m})$ contain all the matches of stubs in $(W^{-,r}, W^{+,r})$ that are not connected to a deleted vertex. Thus $W^{-,r} \cup W^{-,m}$ (respectively $W^{+,r} \cup W^{+,m}$) are all in-stubs (out-stubs) removed by site percolation. The stubs that survive percolation are still denoted by $(W^{-,\pi}, W^{+,\pi})$. Remark that this implies

$$W^- = W^{-,\pi} \cup W^{-,r} \cup W^{-,m} \quad \text{and} \quad W^+ = W^{+,\pi} \cup W^{+,r} \cup W^{+,m}. \tag{4.33}$$

These definitions of $(W^{-,r}, W^{+,r})$ and $(W^{-,m}, W^{+,m})$ will be important throughout the proof.

4.4.1 A percolated configuration is a uniformly random configuration

Like in the case of bond percolation, conditional on $\vec{d}_\pi^n$ being the degree sequence after percolation, applying site percolation to a uniformly random configuration on (W^-, W^+) leads to a uniformly random configuration obeying $\vec{d}_\pi^n$. Due to the different nature of site percolation, this requires another proof then in Section 4.3.1. First it will be shown in Lemma 4.7 that conditional on the stubs that are removed by site percolation, the matching on the surviving stubs is uniformly random.

Lemma 4.7. *Apply site percolation to a uniformly random configuration $\mathcal{M}$ on (W^-, W^+). Conditional on the elements of $(W^{-,r}, W^{+,r})$ and $(W^{-,m}, W^{+,m})$, each configuration on $(W^{-,\pi}, W^{+,\pi})$ is equally likely.*

Proof. By equation (4.33) conditioning on the elements of $(W^{-,r}, W^{+,r})$ and $(W^{-,m}, W^{+,m})$, also determines the elements of $(W^{-,\pi}, W^{+,\pi})$. Choosing the elements of $(W^{-,r}, W^{+,r})$ and $(W^{-,m}, W^{+,m})$ furthermore implies that the configuration $\mathcal{M}$ is the union of a configuration on $(W^{-,r} \cup W^{-,m}, W^{+,r} \cup W^{-,m})$ with one on $(W^{-,\pi}, W^{+,\pi})$. As $\mathcal{M}$ is a uniformly random configuration obeying this split and the elements of $(W^{-,\pi}, W^{+,\pi})$ are fixed, the configuration on $(W^{-,\pi}, W^{+,\pi})$ will be a uniformly random one.

□

This lemma allows us to prove that conditional on the degree sequence after percolation, a uniformly random configuration remains.

Lemma 4.8. *Apply site percolation to a uniformly random configuration on (W^-, W^+). Conditional on $\vec{d}^{n\,\prime} = \vec{d}_\pi^n$, any configuration on $\left(W^-_{\vec{d}_\pi^n}, W^+_{\vec{d}_\pi^n}\right)$ is equally likely.*

Proof. Define $l = |W^{-,\pi}|$ and let $S(\vec{d}_\pi^n)$ contain all sets of surviving stubs $(W^{-,\pi}, W^{+,\pi})$ that induce the degrees sequence $\vec{d}_\pi^n$. Fix a matching $\mathcal{M}^\pi$ of $\left(W^-_{\vec{d}_\pi^n}, W^+_{\vec{d}_\pi^n}\right)$. Then it holds that

$$\mathbb{P}\left[\mathcal{M}^\pi \middle| \vec{d}^{n\,\prime} = \vec{d}_\pi^n\right] = \sum_{(A,B)\in S(\vec{d}_\pi^n)} \mathbb{P}\left[\mathcal{M}^\pi \middle| \vec{d}^{n\,\prime} = \vec{d}_\pi^n, \left(W^{-,\pi}, W^{+,\pi}\right) = (A,B)\right] \mathbb{P}\left[\left(W^{-,\pi}, W^{+,\pi}\right) = (A,B) \,|\vec{d}^{n\,\prime} = \vec{d}_\pi^n\right].$$

Remark that $\mathbb{P}\left[\mathcal{M}^\pi \middle| \vec{d^n}' = \vec{d_\pi^n}, (W^{-,\pi}, W^{+,\pi}) = (A,B)\right] = \mathbb{P}\left[\mathcal{M}^\pi | (W^{-,\pi}, W^{+,\pi}) = (A,B)\right]$ as $(A,B) \in S(\vec{d_\pi^n})$ implies that $(W^{-,\pi}, W^{+,\pi})$ must induce the degree sequence $\vec{d_\pi^n}$. Lemma 4.7 and the bijection between $(W^{-,\pi}, W^{+,\pi})$ and $\left(W^-_{\vec{d_\pi^n}}, W^+_{\vec{d_\pi^n}}\right)$ together imply $\mathbb{P}\left[\mathcal{M}^\pi | (W^{-,\pi}, W^{+,\pi}) = (A,B)\right] = \frac{1}{l!}$. Combining these observations with the fact that $\sum_{(A,B)\in S(\vec{d_\pi^n})} \mathbb{P}\left[(W^{-,\pi}, W^{+,\pi}) = (A,B) \,|\vec{d^n}' = \vec{d_\pi^n}\right] = 1$ by definition of $S(\vec{d_\pi^n})$, we obtain

$$\mathbb{P}\left[\mathcal{M}^\pi \middle| \vec{d^n}' = \vec{d_\pi^n}\right] = \frac{1}{l!} \sum_{(A,B)\in S(\vec{d_\pi^n})} \mathbb{P}\left[\left(W^{-,\pi}, W^{+,\pi}\right) = (A,B) \,|\vec{d^n}' = \vec{d_\pi^n}\right] = \frac{1}{l!},$$

completing the proof. □

4.4.2 The expected number of vertices with degree (j,k) after site percolation

The next step in the proof of Theorem 4.1 for site percolation is showing that the limit

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N^\pi_{j,k}(n)\right]}{n} := p^{\text{site}}_{j,k}, \tag{4.34}$$

exists and determining its value for all $0 \le j,k < \infty$, in analogy to the case of bond percolation. This requires us to first show existence of this limit and to determine the value of $p^{\text{site}}_{j,k}$. Based on $\left(p^{\text{site}}_{j,k}\right)^\infty_{j,k=0}$, a value $\hat{\pi}^{\text{site}}$ can be determined analogously to $\hat{\pi}^{\text{bond}}$. In Section 4.4.3 $\hat{\pi}^{\text{site}}$ is shown to be the percolation threshold for site percolation.

First we consider the limit for all degrees (j,k) where the in-degree j or the out-degree k or both are larger than $d_{\max}$. By definition of $d_{\max}$ this implies $N_{j,k}(n) = 0$. As percolation does not increase degree of a vertex, there holds $N^\pi_{j,k}(n) = 0$ as well. Thus for these degrees (j,k) we find

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N^\pi_{j,k}(n)\right]}{n} = 0 = p^{\text{site}}_{j,k}.$$

Next consider the limit of equation (4.34) for degrees (j,k) with $0 \le j,k \le d_{\max}$. This derivation is based on [2, Section 4]. This requires us to bound the value of $\mathbb{E}\left[N^\pi_{j,k}(n)\right]$. Like in the case of bond percolation, we investigate this value by splitting it into the probability that a vertex of degree (d^-,d^+) has degree (j,k) after site percolation and the number of vertices with degree (d^-,d^+) before percolation. Deleted vertices are treated separately from the others. Let $N^{\pi,r}_{d^-,d^+}(n)$ denote the number of vertices of degree (d^-,d^+) before percolation, that are not deleted. Thus $N_{d^-,d^+}(n) - N^{\pi,r}_{d^-,d^+}(n)$ equals the number of vertices of degree (d^-,d^+) that are deleted. As each vertex is deleted with probability $1-\pi$ independently of other vertices, there holds

$$\mathbb{E}\left[N^{\pi,r}_{d^-,d^+}(n)\right] = \pi N_{d^-,d^+}(n). \tag{4.35}$$

This implies that $\mathbb{E}\left[N_{d^-,d^+}(n) - N^{\pi,r}_{d^-,d^+}(n)\right] = (1-\pi) N_{d^-,d^+}(n)$. A deleted vertex will have degree $(0,0)$ after percolation with probability 1. Let $P_{j,k}(d^-,d^+)$ be the probability that vertex of degree (d^-,d^+), which is not deleted, has degree (j,k) after percolation. Then we find for $(j,k) = (0,0)$

$$\mathbb{E}\left[N^\pi_{0,0}(n)\right] = \sum_{d^-=0}^{d_{\max}} \sum_{d^+=0}^{d_{\max}} (1-\pi) N_{d^-,d^+}(n) + \pi P_{0,0}\left(d^-,d^+\right) N_{d^-,d^+}(n) \tag{4.36}$$

and else

$$\mathbb{E}\left[N^\pi_{j,k}(n)\right] = \sum_{d^-=j}^{d_{\max}} \sum_{d^+=k}^{d_{\max}} \pi P_{j,k}\left(d^-,d^+\right) N_{d^-,d^+}(n). \tag{4.37}$$

The next step is determining $P_{j,k}(d^-, d^+)$. Define $s^- = |W^{-,\pi} \cup W^{-,m}|$, $s^+ = |W^{+,\pi} \cup W^{+,m}|$, $r^- = |W^{-,m}|$ and $r^+ = |W^{+,m}|$. Note that there must hold $s^- - r^- = s^+ - r^+$ as $s^- - r^- = |W^{-,\pi}|$, $s^+ - r^+ = |W^{+,\pi}|$ and the remaining configuration on $(W^{-,\pi}, W^{+,\pi})$ forms a directed graph. Let $P_{j,k}(d^-, d^+, s^-, s^+, r^-, r^+)$ denote the probability $P_{j,k}(d^-, d^+)$ conditional on the values s^-, s^+, r^-, r^+. We will now determine this conditional probability. Site percolation combines the independent random processes of deleting vertices and creating a uniformly random configuration on (W^-, W^+). As these processes are independent, we may first determine the elements of $(W^{-,r}, W^{+,r})$ and then randomly create a configuration on (W^-, W^+). Thus conditional on the value r^-(respectively r^+), each subset of $W^- \setminus W^{-,r}$($W^+ \setminus W^{+,r}$) of this size is equally likely to be $W^{-,m}$($W^{+,m}$). This implies

$$P_{j,k}(d^-, d^+, r^-, r^+, s^-, s^+) = \binom{d^-}{d^- - j}\binom{d^+}{d^+ - k}\frac{\binom{s^- - d^-}{r^- - d^- + j}}{\binom{s^-}{r^-}}\frac{\binom{s^+ - d^+}{r^+ - d^+ + k}}{\binom{s^+}{r^+}}. \tag{4.38}$$

To approximate this probability we will show that with high probability s^-, s^+ and r^-, r^+ lie in some bounded interval. This enables us to determine $P_{j,k}(d^-, d^+, r^-, r^+, s^-, s^+)$ for s^-, s^+, r^-, r^+ in these intervals. First consider s^- and s^+. Combining equation (4.35) with the linearity of expectation, we obtain

$$\mathbb{E}\left[s^-\right] = \sum_{d^-=0}^{d_{\max}}\sum_{d^+=0}^{d_{\max}} \pi d^- N^{\pi,r}_{d^-,d^+}(n) = m\pi \quad \text{and} \quad \mathbb{E}\left[s^+\right] = \sum_{d^-=0}^{d_{\max}}\sum_{d^+=0}^{d_{\max}} \pi d^+ N^{\pi,r}_{d^-,d^+}(n) = m\pi.$$

Using $d_{\max} \leq n^{1/9}$ and Hoeffding's inequality we also find

$$\mathbb{P}\left[\left|s^- - \mathbb{E}\left[s^-\right]\right| > n^{2/3}\ln(n)\right] \leq e^{-\Omega\left(\ln^2(n)\right)} \quad \text{and} \quad \mathbb{P}\left[\left|s^+ - \mathbb{E}\left[s^+\right]\right| > n^{2/3}\ln(n)\right] \leq e^{-\Omega\left(\ln^2(n)\right)}. \tag{4.39}$$

This implies that

$$s^-, s^+ \in I' := \left[m\pi - n^{2/3}\ln(n), m\pi + n^{2/3}\ln(n)\right]$$

with probability $1 - e^{-\Omega\left(\ln^2(n)\right)}$. The following Lemma specifies such an interval for r^- and r^+. This Lemma is the directed equivalent of [2, Lemma 4.1].

Lemma 4.9. *Conditional on $s^-, s^+ \in I'$, there holds $r^- \in I := \left[m\pi(1-\pi) - n^{2/3}\ln^2(n), m\pi(1-\pi) + n^{2/3}\ln(n)^2\right]$ with probability $1 - e^{-\Omega\left(\ln^2(n)\right)}$. Also $r^+ \in I$ with probability $1 - e^{-\Omega\left(\ln^2(n)\right)}$.*

Proof. This proof is adapted from the proof of [2, Lemma 4.1]. Since the proof for r^+ is identical to the one for r^- up to switching the roles of in-stubs and out-stubs, only the proof for r^- is presented. As an uniformly random configuration on (W^-, W^+) is considered, the probability that any in-stub is matched to an out-stub in $W^{+,r}$ is $\frac{m - s^+}{m} = (1-\pi)\left(1 + \mathcal{O}\left(n^{-1/3}\ln(n)\right)\right)$ as $s^-, s^+ \in I'$. Since r^- equals the number of in-stubs in $W^- \setminus W^{-,r}$ with a match in $W^{+,r}$, this implies

$$\mathbb{E}\left[r^-\right] = s^-\frac{m - s^+}{m} = m\pi(1-\pi)\left(1 + \mathcal{O}\left(n^{2/3}\ln(n)\right)\right).$$

To complete the proof, we will now show that

$$\mathbb{P}\left[\left|r^- - \mathbb{E}\left[r^-\right]\right| > n^{2/3}\ln^2(n)\right] \leq e^{-\Omega\left(\ln^2(n)\right)}.$$

This is realized by applying Theorem 4.3 to the space of configurations on (W^-, W^+) with the symmetric difference as metric. The value of r^- plays the role of the function f. To partition this space, order the in-stubs of W^-. Define an i-prefix to be the first i in-stubs together with their match. An element of the partition $\mathcal{P}_k$ consists of all configurations with the same k-prefix for all $k \in \{0, 1, \ldots, m\}$. For any $A, B \in \mathcal{P}_k$ such that $A, B \subset C \in \mathcal{P}_{k-1}$ a bijection $\phi : A \to B$ can be defined. Denote the k^{th} pair of a configuration in A by (x, y_A) and the k^{th} pair of a configuration in B by (x, y_B). Then ϕ maps $\mathcal{M} \in A$ to the configuration in B with (x, y_A) replaced by (x, y_B) and with y_A the match of the in-stub in $\mathcal{M}$ matched to y_B. By definition

of ϕ it follows that $|\mathcal{M} - \phi(\mathcal{M})| = 4 := c_k$ for all $k \in \{1, 2, \ldots, m\}$. As the value of r^- also changes by at most the symmetric difference of the two matchings, Theorem 4.3 implies

$$\mathbb{P}\left[|r^- - \mathbb{E}\left[r^-\right]| > n^{2/3}\ln^2(n)\right] \leq 2\exp\left(\frac{n^{4/3}\ln^2(n)}{2m}\right) = e^{-\Omega\left(\ln^2(n)\right)},$$

as $m \leq nd_{\max} \leq n^{10/9}$. □

Fountoulakis [2, Section 4] shows that for $d_{\max} \leq n^{1/9}$ there holds uniformly for $r \in I$ and $s \in I'$:

$$\binom{d}{d-i}\frac{\binom{s-d}{r-d+i}}{\binom{s}{r}} = \binom{d}{d-i}(1-\pi)^{d-i}\pi^i\left(1+\mathcal{O}\left(\frac{\ln^2(n)}{n^{1/3}}\right)\right).$$

Applying this to equation 4.38 implies that uniformly for all $s^-, s^+ \in I'$ and $r^-, r^+ \in I$ there holds

$$P_{j,k}\left(d^-, d^+, r^-, r^+, s^-, s^+\right) = \binom{d^-}{d^- - j}\binom{d^+}{d^+ - k}\pi^{d^- + d^+ - j - k}(1-\pi)^{j+k}\left(1+\mathcal{O}\left(\frac{\ln^2(n)}{n^{1/3}}\right)\right).$$

However we yet know nothing about this probability if one or more of the conditions $s^-, s^+ \in I'$, $r^-, r^+ \in I$ is violated. Instead of determining the probability in this case, we show that it is unlikely that one or more these conditions are violated. Instead of bounding the probability $\mathbb{P}\left[s^- \notin I' \text{ or } s^+ \notin I' \text{ or } r^- \notin I \text{ or } r^+ \notin I\right]$, a condition on $N^{\pi,r}_{d^-,d^+}(n)$ is added. This allows to bound the value of $\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]$. Theorem 4.2 implies that

$$\mathbb{P}\left[N^{\pi,r}_{d^-,d^+}(n) - \mathbb{E}\left[N^{\pi,r}_{d^+,d^-}(n)\right]| > \sqrt{n}\ln(n)\right] < e^{-\Omega\left(\ln^2(n)\right)}. \tag{4.40}$$

In combination with equation (4.35) this implies that

$$N^{\pi,r}_{d^-,d^+}(n) \in I''(d^-, d^+) = \left[\max\left\{\pi N_{d^-,d^+}(n) - \sqrt{n}\ln(n), 0\right\}, \pi N_{d^-,d^+}(n) + \sqrt{n}\ln(n)\right],$$

with probability $1 - e^{-\Omega\left(\ln^2(n)\right)}$. Together with equation (4.39) and Lemma 4.9 there follows:

$$\begin{aligned}
&\mathbb{P}\left[s^- \notin I' \text{ or } s^+ \notin I' \text{ or } r^- \notin I \text{ or } r^+ \notin I \text{ or } N^{\pi,r}_{d^-,d^+}(n) \notin I''\left(d^-, d^+\right)\right] \\
\leq &\mathbb{P}\left[s^- \notin I'\right] + \mathbb{P}\left[s^+ \notin I'\right] + \mathbb{P}\left[r^- \notin I\right] + \mathbb{P}\left[r^+ \notin I\right] + \mathbb{P}\left[N^{\pi,r}_{d^-,d^+}(n) \notin I''\left(d^-, d^+\right)\right] \\
= &o\left(\frac{1}{n^3}\right) + \mathbb{P}\left[r^- \notin I\right] + \mathbb{P}\left[r^+ \notin I\right].
\end{aligned}$$

By the law of total probability $\mathbb{P}\left[r^- \notin I\right]$ equals

$$\begin{aligned}
&\mathbb{P}\left[r^- \notin I | s^- \in I', s^+ \in I'\right]\mathbb{P}\left[s^- \in I', s^+ \in I'\right] + \mathbb{P}\left[r^- \notin I | s^- \notin I', s^+ \in I'\right]\mathbb{P}\left[s^- \notin I', s^+ \in I'\right] \\
+&\mathbb{P}\left[r^- \notin I | s^- \in I', s^+ \notin I'\right]\mathbb{P}\left[s^- \in I', s^+ \notin I'\right] + \mathbb{P}\left[r^- \notin I | s^- \notin I', s^+ \notin I'\right]\mathbb{P}\left[s^- \notin I', s^+ \notin I'\right] = o\left(\frac{1}{n^3}\right).
\end{aligned}$$

Similarly it is shown that $\mathbb{P}\left[r^+ \notin I\right] = o\left(\frac{1}{n^3}\right)$. Thus there holds

$$\mathbb{P}\left[s^- \notin I' \text{ or } s^+ \notin I' \text{ or } r^- \notin I \text{ or } r^+ \notin I \text{ or } N^{\pi,r}_{d^-,d^+}(n) \notin I''\left(d^-, d^+\right)\right] = o\left(\frac{1}{n^3}\right). \tag{4.41}$$

This allows to determine a lower and upper bound for the value $\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]$. As $N^{\pi,r}_{d^-,d^+}(n) \leq N_{d^-,d^+}(n)$ and $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is proper, for all $\epsilon > 0$ there exist $\kappa(\epsilon)$ and $N(\epsilon)$ such that for all $n > N$:

$$\sum_{\substack{(d^-,d^+)=(0,0)\\ d^- \geq \kappa+1 \text{ or } d^+ \geq \kappa+1}}^{(d_{\max},d_{\max})} P_{j,k}\left(d^-, d^+\right) N^{\pi,r}_{d^-,d^+}(n) \leq \sum_{\substack{(d^-,d^+)=(0,0)\\ d^- \geq \kappa+1 \text{ or } d^+ \geq \kappa+1}}^{(d_{\max},d_{\max})} N_{d^-,d^+}(n) \leq \epsilon n. \tag{4.42}$$

In combination with equation (4.37) this implies for $(j,k) \neq (0,0)$

$$\sum_{d^-=j}^{\kappa}\sum_{d^+=k}^{\kappa} P_{j,k}\left(d^-,d^+\right) N^{\pi,r}_{d^-,d^+}(n) \leq \mathbb{E}\left[N^{\pi}_{j,k}(n)\right] \leq \sum_{d^-=j}^{\kappa}\sum_{d^+=k}^{\kappa} P_{j,k}\left(d^-,d^+\right) N^{\pi,r}_{d^-,d^+}(n) + \epsilon n. \tag{4.43}$$

Using equation (4.41) on the left-hand side of the above equation we find

$$\begin{aligned}\mathbb{E}\left[N^{\pi}_{j,k}(n)\right] \geq \sum_{d^-=j}^{\kappa}\sum_{d^+=k}^{\kappa}\sum_{\tilde{r}^-\in I'}\sum_{\tilde{r}^+\in I'}\sum_{\tilde{s}^-\in I}\sum_{\tilde{s}^+\in I}\sum_{\widetilde{d}_{d^-,d^+}\in I''(d^-,d^+)} \widetilde{d}_{d^-,d^+} P_{j,k}\left(d^-,d^+,\tilde{r}^-,\tilde{r}^+,\tilde{s}^-,\tilde{s}^+\right) \\ \cdot \mathbb{P}\left[r^-=\tilde{r}^-, r^+=\tilde{r}^+, s^-=\tilde{s}^-, s^+=\tilde{s}^+, N^{\pi,r}_{d^-,d^+}(n)=\widetilde{d}_{d^-,d^+}\right] + o\left(\frac{1}{n^2}\right).\end{aligned}$$

As equation (4.40) implies that $\sum_{\widetilde{d}_{d^-,d^+}\in I''(d^-,d^+)} \widetilde{d}_{d^-,d^+}\mathbb{P}\left[N^{\pi,r}_{d^-,d^+}(n)=\widetilde{d}_{d^-,d^+}\right] = \mathbb{E}\left[N^{\pi,r}_{d^-,d^+}(n)\right] + o\left(\frac{1}{n^2}\right)$, following [2] we end up with the expression

$$\mathbb{E}\left[N^{\pi}_{j,k}(n)\right] \geq \pi \sum_{d^-=j}^{\kappa}\sum_{d^+=k}^{\kappa} N_{d^-,d^+}(n)\binom{d^-}{d^--j}\binom{d^+}{d^+-k}\pi^{d^-+d^+-j-k}(1-\pi)^{j+k}\left(1+\mathcal{O}\left(\frac{\ln^2(n)}{n^{1/3}}\right)\right) + o\left(\frac{1}{n^2}\right). \tag{4.44}$$

In a similar fashion we can show, using the right-hand side of equation (4.43):

$$\mathbb{E}\left[N^{\pi}_{j,k}(n)\right] \leq \pi \sum_{d^-=j}^{\kappa}\sum_{d^+=k}^{\kappa} N_{d^-,d^+}(n)\binom{d^-}{d^--j}\binom{d^+}{d^+-k}\pi^{d^-+d^+-j-k}(1-\pi)^{j+k}\left(1+\mathcal{O}\left(\frac{\ln^2(n)}{n^{1/3}}\right)\right) + \epsilon n + o\left(\frac{1}{n^2}\right). \tag{4.45}$$

From this it can be shown, see [2, p. 353], that

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[N^{\pi}_{j,k}(n)\right]}{n} = \pi \sum_{d^-=j}^{\infty}\sum_{d^+=k}^{\infty} p_{d^-,d^+}\binom{d^-}{j}\binom{d^+}{k}\pi^{j+k}(1-\pi)^{d^--j+d^+-k} = p^{\text{site}}_{j,k}. \tag{4.46}$$

For $(j,k)=(0,0)$, we need to use equation (4.36) instead of equation (4.37). That $N^{\pi,r}_{d^-,d^+}(n) \leq N_{d^-,d^+}(n)$ and $\left(\vec{d^n}\right)_{n=1}^{\infty}$ is proper also implies that: for all $\epsilon>0$ there exist $\kappa(\epsilon)$ and $N(\epsilon)$ such that for all $n>N$

$$\sum_{\substack{(d^-,d^+)=(0,0)\\ d^-\geq\kappa+1 \text{ or } d^+\geq\kappa+1}}^{(d_{\max},d_{\max})}\left(N_{d^-,d^+}(n) - N^{\pi,r}_{d^-,d^+}(n)\right) \leq \sum_{\substack{(d^-,d^+)=(0,0)\\ d^-\geq\kappa+1 \text{ or } d^+\geq\kappa+1}}^{(d_{\max},d_{\max})} N_{d^-,d^+}(n) \leq \epsilon n.$$

Thus the equivalent of (4.43) for $(j,k)=(0,0)$ becomes

$$\begin{aligned}\sum_{d^-=0}^{\kappa}\sum_{d^+=0}^{\kappa}\left[\left(N_{d^-,d^+}(n) - N^{\pi,r}_{d^-,d^+}(n)\right) + P_{0,0}\left(d^-,d^+\right)N^{\pi,r}_{d^-,d^+}(n)\right] \leq \mathbb{E}\left[N^{\pi}_{0,0}(n)\right] \leq \\ \sum_{d^-=0}^{\kappa}\sum_{d^+=0}^{\kappa}\left[\left(N_{d^-,d^+}(n) - N^{\pi,r}_{d^-,d^+}(n)\right) + P_{0,0}\left(d^-,d^+\right)N^{\pi,r}_{d^-,d^+}(n)\right] + 2\epsilon n.\end{aligned}$$

The same argument as for $(j,k)\neq(0,0)$ can be applied to obtain:

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[N^{\pi}_{0,0}(n)\right]}{n} = (1-\pi) + \pi\sum_{d^-=j}^{\infty}\sum_{d^+=k}^{\infty} p_{d^-,d^+}\binom{d^-}{j}\binom{d^+}{k}\pi^{j+k}(1-\pi)^{d^--j+d^+-k} = p^{\text{site}}_{0,0}. \tag{4.47}$$

Comparing equations (4.46) and (4.47) with equation (4.12) we find:

$$p_{j,k}^{\text{site}} = \begin{cases} \pi p_{j,k}^{\text{bond}}, & (j,k) \neq (0,0), \\ \pi p_{0,0}^{\text{bond}} + 1 - \pi, & (j,k) = (0,0). \end{cases} \tag{4.48}$$

Exploiting this connection between $\left(p_{j,k}^{\text{site}}\right)_{j,k=0}^{\infty}$ and $\left(p_{j,k}^{\text{bond}}\right)_{j,k=0}^{\infty}$, we find that $\left(p_{j,k}^{\text{site}}\right)_{j,k=0}^{\infty}$ is normalized as

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{j,k}^{\text{site}} = 1 - \pi + \pi \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{j,k}^{\text{bond}} = 1.$$

Exploiting equation (4.48) further, we find that

$$\mu^{\pi,\text{site}} = \pi\mu^{\pi,\text{bond}} = \pi^2\mu \quad \text{and} \quad \mu_{11}^{\pi,\text{site}} = \pi\mu_{11}^{\pi,\text{bond}} = \pi^3\mu_{11}. \tag{4.49}$$

This link between the two distribution also implies that $\left(p_{j,k}^{\text{site}}\right)_{j,k=0}^{\infty}$ satisfies equation (2.6).

It is left to determine $\hat{\pi}^{\text{site}}$. As we expect the percolated graph to obey the degree distribution $\left(p_{j,k}^{\text{site}}\right)_{j,k=0}^{\infty}$, from Theorem 3 we expect that the percolation threshold is the value of π such that

$$\sum_{j,k=0}^{\infty} jkp_{j,k}^{\text{site}} = \sum_{j,k=0}^{\infty} jp_{j,k}^{\text{site}}. \tag{4.50}$$

Denote this value by $\hat{\pi}^{\text{site}}$. Combing equations (4.50) and (4.49) we find

$$\hat{\pi}^{\text{site}^2}\mu = \hat{\pi}^{\text{site}^3}\mu_{11}.$$

From this we find $\hat{\pi}^{\text{site}} = \frac{\mu}{\mu_{11}} = \hat{\pi}^{\text{bond}}$. So we expect that the percolation thresholds for site and bond percolation are equal. This can be explained by remarking that the expected degree distribution after site percolation is a rescaled version the degree distribution after bond percolation, expect for $(0,0)$. Hence one expects a GSCC to appear under the same conditions, although it is expected to contain fewer vertices for site percolation.

4.4.3 Determining π_c^{site} and c^{site}

To prove Theorem 4.1 for site percolation, it remains to show that $\pi_c^{\text{site}} = \hat{\pi}^{\text{site}}$ and to determine c^{site}. This is done similar to the proof of Theorem 4.1 for bond percolation in Section 4.3.3. Because of the similarity between these proofs, we only explain the changes that are made in Section 4.3.3 to convert it into the proof for site percolation.

It is obvious that we need to replace $p_{j,k}^{\text{bond}}$ by $p_{j,k}^{\text{site}}$. Where Lemma 4.6 is used in Section 4.3.3, replace it by Lemma 4.8. As Lemma 4.8 proves the exact same statement for site percolation as Lemma 4.6 for bond percolation, swapping these lemma's does not require additional changes. Furthermore equation (4.22) requires a different proof. For this we follow [2, p. 354]. Conditional on a certain realization of $(W^{-,r}, W^{+,r})$ and the values $s^-, s^+ \in I'$, $r^-, r^+ \in I$, the value of $N_{j,k}^{\pi}(n)$ is determined by the random choice of $(W^{-,m}, W^{+,m})$. By changing one element of $(W^{-,m}, W^{+,m})$ the value of $N_{j,k}^{\pi}(n)$ changes by at most 2. Thus Corollary 4.4 can be applied to obtain

$$\begin{aligned} &\mathbb{P}\left[|N_{j,k}^{\pi}(n) - \mathbb{E}\left[N_{j,k}^{\pi}(n)\right]| > \sqrt{n}\ln^2(n) \mid s^-, s^+, r^-, r^+, \left(W^{-,r}, W^{+,r}\right)\right] \\ &\qquad \leq 2\exp\left(\frac{n\ln^2(n)}{\left(m(1-\pi)\pi + n^{2/3}\ln^2(n)\right)}\right) = \exp\left(-\Omega\left(\ln^2(n)\right)\right). \end{aligned}$$

Using Lemma 4.9 and equation (4.39) there follows

$$\mathbb{P}\left[|N^{\pi}_{j,k}(n) - \mathbb{E}\left[N^{\pi}_{j,k}(n)\right]| > \sqrt{n}\ln^2(n)\right] = o\left(\frac{1}{n^3}\right).$$

As κ is bounded this completes the proof of equation (4.22).

The last change we need, is induced by the fact that Theorem 3.12 is now applied to a proper degree array with $\left(p^{\text{site}}_{j,k}\right)_{j,k=0}^{\infty}$ as degree distribution instead of $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$. Hence $\hat{\pi}^{\text{bond}}$ and c^{bond} must be replaced by $\hat{\pi}^{\text{site}}$ and c^{site}. In Section 4.4.2 we already found that $\hat{\pi}^{\text{site}} = \frac{\mu}{\mu_{11}}$. Thus it remains to determine c^{site}. This value is derived identical to c^{bond}, expect for replacing $\left(p^{\text{bond}}_{j,k}\right)_{j,k=0}^{\infty}$ by $\left(p^{\text{site}}_{j,k}\right)_{j,k=0}^{\infty}$. This implies that we first need to determine the probability that a uniformly random out-stub (respectively in-stub) is attached to a vertex with in-degree j (out-degree k) in the configuration after applying site percolation. In analogy to equations (4.26) and (4.25) these probability are given by

$$p_j^{-,\text{site}} = \sum_{k=0}^{\infty} \frac{k}{\mu^{\pi,\text{site}}} p^{\text{site}}_{j,k} = \sum_{d^-=j}^{\infty} p^-_{d^-} \binom{d^-}{j} \pi^j (1-\pi)^{d^- - j} \quad \text{and} \quad p_k^{+,\text{site}} = \sum_{d^+=k}^{\infty} p^+_{d^+} \binom{d^+}{k} \pi^k (1-\pi)^{d^+ - k}. \tag{4.51}$$

Note that $p_j^{-,\text{site}} = p_j^{-,\text{bond}}$ and $p_k^{+,\text{site}} = p_k^{+,\text{bond}}$. While this might seem surprising, there is a logical explanation. A vertex of degree $(0,0)$ does not play any role in this distribution, as it will never be encountered by following a uniformly random in-stub or out-stub. Equation (4.48) implies that for all other degrees there holds $p^{\text{site}}_{j,k} = \pi p^{\text{bond}}_{j,k}$. Hence after normalization the value of $p_j^{-,\text{site}}$ (respectively $p_k^{+,\text{site}}$) equals $p_j^{-,\text{bond}}$($p_k^{+,\text{bond}}$) for all j (k). Since these distributions are equal, there also holds $\eta^{-,\text{bond}} = \eta^{-,\text{site}}$ and $\eta^{+,\text{bond}} = \eta^{+,\text{site}}$, *i.e.* they are also given by equations (4.27) and (4.28). The difference between the two types of percolation arises in the definitions of $\zeta^{-,\text{site}}, \zeta^{+,\text{site}}$ and ψ^{site}. These are given by

$$\left(1-\zeta^{-,\text{site}}\right) = \sum_{j,k=0}^{\infty} p^{\text{site}}_{j,k} \left(1-\eta^{-,\text{site}}\right)^j, \quad \left(1-\zeta^{+,\text{site}}\right) = \sum_{j,k=0}^{\infty} p^{\text{site}}_{j,k} \left(1-\eta^{+,\text{site}}\right)^k \tag{4.52}$$

and

$$\psi^{\text{site}} = \sum_{j,k=0}^{\infty} p^{\text{site}}_{j,k} \left(1-\eta^{-,\text{site}}\right)^j \left(1-\eta^{+,\text{site}}\right)^k. \tag{4.53}$$

Applying equation (4.48) and the fact $\eta^{-,\text{bond}} = \eta^{-,\text{site}}$ and $\eta^{+,\text{bond}} = \eta^{+,\text{site}}$ and recalling equations (4.29) and (4.30), the above equations become

$$\begin{aligned}
\left(1-\zeta^{-,\text{site}}\right) &= \pi \sum_{j,k=0}^{\infty} p^{\text{bond}}_{j,k} \left(1-\eta^{-,\text{bond}}\right)^j + 1 - \pi = \pi\left(1-\zeta^{-,\text{bond}}\right) + 1 - \pi, \\
\left(1-\zeta^{+,\text{site}}\right) &= \pi \sum_{j,k=0}^{\infty} p^{\text{bond}}_{j,k} \left(1-\eta^{+,\text{bond}}\right)^k + 1 - \pi = \pi\left(1-\zeta^{+,\text{bond}}\right) + 1 - \pi \quad \text{and} \\
\psi^{\text{site}} &= \pi \sum_{j,k=0}^{\infty} p^{\text{bond}}_{j,k} \left(1-\eta^{-,\text{site}}\right)^j \left(1-\eta^{+,\text{site}}\right)^k + (1-\pi) = \pi\psi^{\text{bond}} + 1 - \pi.
\end{aligned}$$

Following Theorem 3.12 and exploiting the above equations, we find

$$c^{\text{site}} = \zeta^{-,\text{site}} + \zeta^{+,\text{site}} + \psi^{\text{site}} - 1 = \pi\left(\zeta^{-,\text{bond}} + \zeta^{+,\text{bond}} + \psi^{\text{bond}} - 1\right) = \pi c^{\text{bond}}. \tag{4.54}$$

This relation between c^{site} and c^{bond} also can be intuitively explained, again using equation (4.48). The main difference in the distributions is in vertices with degree $(0,0)$. A vertex of degree $(0,0)$ forms its own strongly connected component. Hence these vertices are not in the GSCC. As $p^{\text{site}}_{j,k} = \pi p^{\text{bond}}_{j,k}$ and $p^{\text{site}}_{0,0} = \pi p^{\text{bond}}_{j,k} + 1 - \pi$, hence we could already have predicted that $c^{\text{site}} = \pi c^{\text{bond}}$. This is the last change that needs to be made to Section 4.3.3 to complete the proof of Theorem 4.1 for site percolation. This completes the proof of Theorem 4.1.

5 Numerical construction of random directed graphs

In the previous section we have proven Theorem 4.1, which determines the percolation threshold for graphs in the asymptotic limit. We want to illustrate this theorem with numerical simulations. This requires us to uniformly sample simple directed graphs obeying a given degree sequence. Additionally, being able to sample such graphs numerically gives access to much richer information about them, as such samples may be further analysed with graph algorithms. For example, to investigate diameter, cycles, spectral properties, etc.

A first guess might be to use the configuration model to generate the desired graphs. Theoretically this works. One can keep repeatedly drawing a uniformly random configuration, until it induces a simple graph. This is called the repeated configuration model. However, as already mentioned in Section 1, numerical experiments show that this procedure is not practical even for very small graphs. A more popular algorithm to generate simple graphs uses a Markov Chain [14]. However for most degree sequences it is not known when the graph is sufficiently independent from the initial guess, i.e. when the algorithm has achieved a uniformly random graph. Additionally for those degree sequences for which it is known, this bound often appears to be a high degree polynomial in n [14].

Neither the repeated configuration model nor the Markov Chain approach is an ideal way to uniformly generate random graphs obeying a given degree sequence. In this work we build an algorithm that will almost uniformly generate random simple directed graphs obeying a given degree sequence with expected runtime near-linear in m. Our algorithm is a generalisation of an algorithm by Bayati, Kim and Saberi [5] for undirected graphs. While our algorithm cannot be used if the graphs must be distributed exactly uniformly, it is a good trade-off between speed and uniformity if almost uniform generation of the graphs suffices. We explain the algorithm in Section 5.1. After presenting it, we prove the claims about its performance. The proof that this algorithm generates graphs distributed within up to a factor of $1 \pm o(1)$ of uniformity is presented in Section 5.2 and is based on [5, Section 7]. Our algorithm might fail to construct a graph, but it is shown that this happens only with probability $o(1)$ in Section 5.3, following [5, Section 5]. This section is completed with a runtime analysis of the algorithm, see Section 5.4, which is based on [5, Section 6].

5.1 The algorithm

The algorithm is based on the configuration model. Generating a uniformly random configuration is not an issue. The problem is that a random configuration might induce a multigraph, which we do not desire. This problem can be remedied by the following procedure: A configuration is generated by sequentially matching a random in-stub to a random out-stub. A match between a given in-stub and out-stub is rejected if this match leads to a self-loop or multi-edge. Then the resulting configuration induces a simple graph. This constraint on accepting the matches, might make it impossible to finish a configuration, for example, if only one in-stub and one out-stub of the same vertex remain. In this case, we reject the partial configuration and start from scratch again. Note that the rejection of specific matches destroys the uniformity of the generated graphs. To cancel out the non-uniformity bias, we accept each admissible match between an in- and an out-stub with a cleverly chosen probability, which will slightly repair the uniformity. The resulting generated graphs can be then assured to be within $1 \pm o(1)$ of uniformity. The pseudo-code of our algorithm is shown in Algorithm 1, which is based on [5, Procedure A] for the undirected graphs.

Recall that for each vertex i, the degree sequence prescribes its in-degree d_i^- and its out-degree d_i^+. The residual in-degree $\hat{d}_i^-$ (respectively out-degree $\hat{d}_i^+$) of the vertex i is the number of unmatched in-stubs(out-stubs) of this vertex. The edges are added to E in the loop at line 5. On each step of the loop, one edge is chosen and added to E. When for all ordered pairs (i,j) such that $i \neq j, (i,j) \notin E$ there holds $\hat{d}_i^+ = 0$ or $\hat{d}_j^- = 0$ or both no edge can be added to E and the algorithm terminates. If the algorithm terminates before m edges have been added to E, it has failed to construct a simple graph obeying the desired degree sequence. Hence it outputs a failure. Else the algorithm returns a simple graph obeying the degree sequence $\vec{d^n}$. In the loop at line 5 also a value P is computed. Assuming that the algorithm does not return a failure, P is the probability that the algorithm generates the edges of the graph $G_{\vec{d^n}}$ in the order it has just constructed them. That the order in which the edges are constructed matters, follows from the fact that the probability

Algorithm 1: generating simple directed graphs obeying a given degree array

input : $\vec{d^n}$ a graphical degree array
output: $G_{\vec{d^n}} = (V, E)$ a digraph obeying $\vec{d^n}$ and N an estimation for the number of simple digraphs obeying $\vec{d^n}$ or a failure

$V = \{1, 2, \dots n\}$// set of vertices
$\hat{d} = \vec{d^n}$// residual degree
$E = \emptyset$// set of edges
$P = 1$ // probability of generating this ordering
while *edges can be added to* E **do**
  Pick $i, j \in V$ with probability P_{ij} proportional to $\hat{d}_i^+ \hat{d}_j^- \left(1 - \frac{d_i^+ d_j^-}{2m}\right)$ amongst all ordered pairs (i, j) with $i \neq j$ and $(i, j) \notin E$;
  Add (i, j) to E, decrease $\hat{d}_i^+$ and $\hat{d}_j^-$ by 1 and set $P = P \cdot P_{ij}$;
if $|E| = m$ **then**
  Output $G_{\vec{d^n}} = (V, E)$, $N = \frac{1}{m!P}$
else
  return failure

P_{ij} depends on the elements of E. It will be shown that asymptotically each ordering of a set of m edges is generated with the same probability. Hence the probability that the algorithm generates the graph $G_{\vec{d^n}}$ is asymptotically $m!P$. As we will show that each graph is generated within a factor of $1 \pm o(1)$ of uniformity, $\frac{1}{m!P}$ is an approximation to the number of simple graphs obeying the degree sequence.

This algorithm can be shown to have the following favourable properties.

Theorem 5.1. *Suppose we are given a graphical degree sequence* $\vec{d^n}$*, for which there exists* $\tau > 0$ *such that* $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$*. Then Algorithm 1 terminates successfully with probability* $1 - o(1)$ *and has an expected runtime of* $\mathcal{O}\left(m d_{\max}\right)$*. Furthermore any graph* $G_{\vec{d^n}}$ *is generated with a probability within factor* $1 \pm o(1)$ *of uniformity.*

The remainder of this section is covered by the proof of Theorem 5.1, which is split into three parts: the uniformity of the generated graphs, the failure probability of the algorithm and its runtime.

5.2 The probability that Algorithm 1 generates a graph $G_{\vec{d^n}}$

This section is devoted to proving that Algorithm 1 generates any graph $G_{\vec{d^n}}$ with a probability within $1 \pm o(1)$ of the uniform probability. This is realized by proving the following Theorem.

Theorem 5.2. *Take a graphical degree sequence* $\vec{d^n}$ *with* $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$ *for some* $\tau > 0$*. Let* $G_{\vec{d^n}}$ *be a random simple graph obeying this degree sequence. Then Algorithm 1 generates* $G_{\vec{d^n}}$ *with probability*

$$[1 + o(1)] \left(\frac{m!}{\prod_{r=0}^{m-1} (m - r)^2} \prod_{i=1}^{n} d_i^+! \prod_{i=1}^{n} d_i^-! \, e^{\frac{\sum_{i=1}^n d_i^- d_i^+}{m} - \frac{\sum_{i=1}^n (d_i^-)^2 + (d_i^+)^2}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \frac{1}{2}} \right).$$

This a generalisation of [5, Lemma 1] to directed graphs, and our proof is guided by the proof of this lemma. The proof is split into four steps. The first step is determining the probability that the algorithm generates a graph $G_{\vec{d^n}}$, see Section 5.2.1. This section reduces the proof of Theorem 5.2 to showing that equations (5.4) and (5.3) hold. Equation (5.4) is proven in Section 5.2.3 and equation (5.3) in Section 5.2.4. In order to show that these equations hold, first ψ_r must be defined in Section 5.2.2.

5.2.1 Analysing the probability of generating a given graph $G_{\vec{d^n}}$

Fix a simple directed graph $G_{\vec{d^n}}$ that obeys the degree sequence $\vec{d^n}$. The goal is to determine the probability that Algorithm 1 outputs $G_{\vec{d^n}}$ on input of $\vec{d^n}$. Denote this probability by $\mathbb{P}_A(G_{\vec{d^n}})$. Let $R(G_{\vec{d^n}}) = \left\{\mathcal{M} \,|\, G_{\mathcal{M}} = G_{\vec{d^n}}\right\}$ be the set of all configurations on (W^-, W^+) that induce the graph $G_{\vec{d^n}}$. Since the output of Algorithm 1 is also a configuration, there holds

$$\mathbb{P}_A(G_{\vec{d^n}}) = \sum_{\mathcal{M}\in R(G_{\vec{d^n}})} \mathbb{P}_A\left(\mathcal{M}\right).$$

Recall from Section 2.3 that any two configurations inducing the same graph, differ only in the labelling of the stubs. As the algorithm ignores the label of a stub, each configuration in $R(G_{\vec{d^n}})$ is generated with equal probability. Note that the probability to match an out-stub of i to an in-stub of j depends on the partial constructed configuration. Hence the order in which the matches are chosen, influences the probability of generating a configuration $\mathcal{M}$. Take a configuration $\mathcal{M} \in R(G_{\vec{d^n}})$ and define $S\left(\mathcal{M}\right)$ to be all the orderings in which the configuration can be created. Because the configuration already determines the match for each in-stub, an ordering of $\mathcal{M}$ can be thought of as a prescription which in-stub gets matched first, which second, etc. As $\mathcal{M}$ contains m in-stubs, there are $m!$ different orderings constructing the configuration $\mathcal{M}$. This implies that

$$\mathbb{P}_A(G_{\vec{d^n}}) = \prod_{i=1}^{n} d_i^-! \prod_{i=1}^{n} d_i^+! \sum_{\mathcal{N}\in S(\mathcal{M})} \mathbb{P}_A\left(\mathcal{N}\right).$$

So we need to investigate $\mathbb{P}_A\left(\mathcal{N}\right)$. Identifying a match between an in-stub with an out-stub by an edge, we can write $\mathcal{N} = \{e_1, e_2, \ldots, e_m\}$. Note that any other element in $S\left(\mathcal{M}\right)$ can be obtained by permuting the elements of $\mathcal{N}$. If the algorithm has constructed the first r elements of $\mathcal{N}$, it is said to be at step $r \in \{0, 1, \ldots, m-1\}$. Step m does not exists, as the algorithm terminates immediately after constructing the m^{th} edge. Let $d_i^{-(r)}$ (respectively $d_i^{+(r)}$) denote the number of unmatched in-stubs (out-stubs) of the vertex i at step r. Define

$$E_r := \left\{(i,j) \mid i,j \in V,\ d_i^{+(r)} > 0,\ d_j^{-(r)} > 0,\ i \neq j,\ (i,j) \notin \{e_1, e_2 \ldots, e_r\}\right\}.$$

This is the set of all edges that can be added to the ordering at step r. It will also be referred to as the set of all eligible edges or pairs at step r. This notation allows to write the probability of generating the ordering $\mathcal{N}$ as

$$\mathbb{P}_A(\mathcal{N}) = \prod_{r=0}^{m-1} \mathbb{P}\left[e_{r+1} | e_1, \ldots, e_r\right],$$

for

$$\mathbb{P}\left[e_{r+1} = (i,j) | e_1, \ldots, e_r\right] = \frac{1 - \frac{d_i^+ d_j^-}{2m}}{\sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)} \left(1 - \frac{d_u^+ d_v^-}{2m}\right)}.$$

Here we slightly abuse the notation as this is the conditional probability that a given out-stub of i is matched with a given in-stub of j, rather than the conditional probability that the edge (i,j) is created. Now the probability that the algorithm generates the graph $G_{\vec{d^n}}$ can be written as

$$\mathbb{P}_A(G_{\vec{d^n}}) = \prod_{i=1}^{n} d_i^-! \prod_{i=1}^{n} d_i^+! \prod_{(i,j)\in G_{\vec{d^n}}} \left(1 - \frac{d_i^+ d_j^-}{2m}\right) \sum_{\mathcal{N}\in S(\mathcal{M})} \prod_{r=0}^{m-1} \frac{1}{(m-r)^2 - \Psi_r(\mathcal{N})}, \tag{5.1}$$

where

$$\Psi_r\left(\mathcal{N}\right) = \sum_{(u,v)\notin E_r} d_u^{+(r)} d_v^{-(r)} + \sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)} \frac{d_u^+ d_v^-}{2m}. \tag{5.2}$$

Recall that the goal is to prove Theorem 5.2. Suppose we manage to show that

$$\sum_{\mathcal{N}\in S(\mathcal{M})} \prod_{r=0}^{m-1} \frac{1}{(m-r)^2 - \Psi_r(\mathcal{N})} = \left[1+o(1)\right] m! \prod_{r=0}^{m-1} \frac{1}{(m-r)^2 - \psi_r}, \tag{5.3}$$

and

$$\prod_{r=0}^{m-1} \frac{1}{(m-r)^2 - \psi_r} = \left[1+o(1)\right] \prod_{r=0}^{m-1} \frac{1}{(m-r)^2} e^{\frac{\sum_{i=1}^n d_i^- d_i^+}{m} - \frac{\sum_{i=1}^n (d_i^-)^2 + (d_i^+)^2}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \frac{\sum_{(i,j)\in G_{\vec{d^n}}} d_i^+ d_j^-}{2m} + \frac{1}{2}}, \tag{5.4}$$

hold for some quantity ψ_r, to be defined in section 5.2.2. For now think of it as the expected value of $\Psi_r(\mathcal{N})$. Combining these two equations with equation (5.1) and using that $1 - x = e^{-x+\mathcal{O}(x^2)}$ we find

$$\mathbb{P}_A(G_{\vec{d^n}}) = \left[1+o(1)\right] m! \prod_{i=1}^n d_i^-! \prod_{i=1}^n d_i^+! \prod_{r=0}^{m-1} \frac{1}{(m-r)^2} e^{\frac{\sum_{i=1}^n d_i^- d_i^+}{m} - \frac{\sum_{i=1}^n (d_i^-)^2 + (d_i^+)^2}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \frac{1}{2}}.$$

This is exactly the statement of Theorem 5.2. Thus proving equations (5.3) and (5.4) suffices to show Theorem 5.2.

5.2.2 Defining the value ψ_r

Before equations (5.3) and (5.4) can be shown to hold, ψ_r must be defined. This is closely related to the expected value of $\Psi_r\left(\mathcal{N}\right)$. For ease of notation, abbreviate $\Psi_r\left(\mathcal{N}\right)$ by Ψ_r whenever $\mathcal{N}$ follows from the context. For further analysis we require to write

$$\Psi_r = \Delta_r + \Lambda_r,$$

with

$$\Delta_r = \sum_{(u,v)\notin E_r} d_u^{+(r)} d_v^{-(r)} \quad \text{and} \quad \Lambda_r = \sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)} \frac{d_u^+ d_v^-}{2m}. \tag{5.5}$$

Note that Δ_r counts the number of *unsuitable pairs*, i.e. the number of pairs of the unmatched in-stubs with out-stubs that induce a self-loop or multi-edge. In the sequel we refer to a combination of an unmatched in-stub and an unmatched out-stub as a *pair*. To simplify the analysis of Δ_r and Λ_r, they are also written as the sum of multiple quantities. As Δ_r counts the number of unsuitable pairs, it is necessary to split this quantity up into the number pairs creating a self-loop

$$\Delta_r^1 = \sum_{i=1}^n d_i^{-(r)} d_i^{+(r)}, \tag{5.6}$$

and the number of pairs creating a multi-edge

$$\Delta_r^2 = \Delta_r - \Delta_r^1. \tag{5.7}$$

This implies $\Delta_r = \Delta_r^1 + \Delta_r^2$. Note that Δ_r^2 counts the number of pairs leading to a double edge, as we assume that $\{e_1, e_2, \ldots, e_r\}$ does not contain a multi-edge.

The quantity Λ_r is split up into two terms, to remove the summation over $(u,v) \in E_r$. The reason for this representation will become apparent once expected value of Λ_r is determined. We write Λ_r as:

$$\Lambda_r = \frac{\Lambda_r^{1^+}\Lambda_r^{1^-} - \Lambda_r^2}{4m} - \frac{\Lambda_r^3}{2m}, \tag{5.8}$$

with

$$\Lambda_r^{1^+} = \sum_{i=1}^{n} d_i^{+(r)} d_i^+, \quad \Lambda_r^{1^-} = \sum_{i=1}^{n} d_i^{-(r)} d_i^-, \tag{5.9}$$

$$\Lambda_r^2 = \sum_{i=1}^{n} d_i^{+(r)} d_i^+ d_i^{-(r)} d_i^- \quad \text{and} \tag{5.10}$$

$$\Lambda_r^3 = \sum_{\substack{(u,v)\notin E_r \\ u\neq v}} d_u^{+(r)} d_v^{-(r)} d_u^+ d_v^-. \tag{5.11}$$

Some of these quantities have simple bounds, which will be important in Section 5.2.4.

Lemma 5.3. *For all $0 \le r \le m-1$ there holds*

(i) $\Delta_r \le (m-r) d_{\max}^2$;

(ii) $\Lambda_r^{1^+} \le d_{\max}(m-r)$, $\Lambda_r^{1^-} \le d_{\max}(m-r)$;

(iii) $\Lambda_r \le \frac{d_{\max}^2}{2m}(m-r)^2$.

Proof. (i) At step r, there are $m-r$ unmatched in-stubs left. Each unmatched in-stub can form a self-loop by connecting to an unmatched out-stub of the same vertex. The number of unmatched out-stubs at each vertex is upper bounded by $d_{\max}$, hence $\Delta_r^1 \le (m-r)d_{\max}$. The vertex to which an unmatched in-stub belongs has at most $d_{\max}-1$ incoming edges. The target of such an edge has at most $d_{\max}-1$ unmatched out-stubs left. Thus the number of out-stubs an unmatched in-stub can be paired with to create a double edge is at most $(d_{\max}-1)^2$. Hence $\Delta_r^2 \le (m-r)(d_{\max}-1)^2$. This implies that $\Delta_r = \Delta_r^1 + \Delta_r^2 \le (m-r)d_{\max}^2$.

(ii) By definition there holds $\Lambda_r^{1^+} = \sum_{i=1}^n d_i^{+(r)} d_i^+$. As $\sum_{i=1}^n d_i^{+(r)} = m-r$ and $d_i^+ \le d_{\max}$ for all i, this implies $\Lambda_r^{1^+} \le d_{\max}(m-r)$. In a similar way it is shown that $\Lambda_r^{1^-} \le d_{\max}(m-r)$.

(iii) By definition there holds $\Lambda_r = \sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)} \frac{d_u^+ d_v^-}{2m} \le \frac{d_{\max}^2}{2m} \sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)}$. As $\sum_{i=1}^n d_u^{+(r)} = m-r$ and $d_v^- \le (m-r)$ for all v, the claim follows.

□

Next we determine the expected value of Ψ_r. The value Ψ_r depends on the first r edges in an ordering $\mathcal{N}$. Let $\mathcal{N}_r$ denote these first r edges of the ordering. The subgraph of $G_{\vec{d^n}}$ containing exactly those r edges is denoted by $G_{\mathcal{N}_r}$. Taking the expected value of Ψ_r over all orderings inducing the same configuration, $G_{\mathcal{N}_r}$ turns into a random subgraph of $G_{\vec{d^n}}$ with exactly r edges. A random subgraph of $G_{\vec{d^n}}$ containing r edges is closely related to a subgraph of $G_{\vec{d^n}}$ where each edge is present with probability $p_r = \frac{r}{m}$. Denote such a random subgraph by G_{p_r}. In the G_{p_r} model we can determine the expected value of Ψ_r, denoted by $\mathbb{E}_{p_r}(\Psi_r)$, by determining the expected values of $\Delta_r^1, \Delta_r^2, \Lambda_r^{1^+}, \Lambda_r^{1^-}, \Lambda_r^2$ and Λ_r^3.

Lemma 5.4. *For each $0 \le r \le m-1$ the following equations hold*

(i) $\mathbb{E}_{p_r}\left[\Delta_r^1\right] = \frac{(m-r)^2}{m^2} \sum_{i=1}^n d_i^+ d_i^-$;

(ii) $\mathbb{E}_{p_r}\left[\Delta_r^2\right] = \frac{r(m-r)^2}{m^3} \sum_{(i,j)\in G_{\vec{d^n}}} (d_i^+ - 1)(d_j^- - 1)$;

(iii) $\mathbb{E}_{p_r}\left[\Lambda_r^{1^-}\Lambda_r^{1^+}\right] = \frac{(m-r)^2}{m^2} \sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2 + \frac{r(m-r)}{m^2} \sum_{(i,j)\in G_{\vec{d^n}}} d_i^+ d_j^-$;

(iv) $\mathbb{E}_{p_r}\left[\Lambda_r^2\right] = \frac{(m-r)^2}{m^2}\sum_{i=1}^n (d_i^-)^2(d_i^+)^2$;

(v) $\mathbb{E}_{p_r}\left[\Lambda_r^3\right] = \frac{r(m-r)^2}{m^3}\sum_{(i,j)\in G_{\vec{d^n}}} d_i^+(d_i^+-1)d_j^-(d_j^--1)$.

Proof. (i) The value $d_i^{+(r)}$ equals the number of edges $(i,\bullet)\in G_{\vec{d^n}}$ such that $(i,\bullet)\notin G_{p_r}$. As $p_r = \frac{r}{m}$ this implies $\mathbb{E}_{p_r}\left[d_i^{\pm(r)}\right] = d_i^{\pm}\frac{m-r}{m}$. As $G_{\vec{d^n}}$ is simple, it contains no self-loops. This implies that $d_i^{-(r)}$ and $d_i^{+(r)}$ are independent. Using the fact that $\Delta_r^1 = \sum_{i=1}^n d_i^{-(r)}d_i^{+(r)}$, we find that $\mathbb{E}_{p_r}\left[\Delta_r^1\right] = \frac{(m-r)^2}{m^2}\sum_{i=1}^n d_i^+d_i^-$.

(ii) The value Δ_r^2 counts the number of pairs leading to a double edge. Pick a random $(i,j)\in G_{\vec{d^n}}$. To form an extra copy of this edge at step r, this edge must be present in G_{p_r}, which happens with probability p_r. A double edge can be created by any pair of edges $(i,k),(l,j)$ that are in $G_{\vec{d^n}}$ but not in G_{p_r}. Instead of adding these edges, the edges (i,j) and (l,k) can be created. The number of combinations of l and k that exist, is $(d_i^{+(r)}-1)(d_j^{-(r)}-1)$. Taking the expected value of this value, summing it over all edges of $G_{\vec{d^n}}$ and multiplying it by the probability p_r that $(i,j)\in G_{p_r}$, the claimed expected value of Δ_r^2 follows.

(iii) Remark that $\Lambda_r^{1^-}\Lambda_r^{1^+} = \sum_{j=1}^n\sum_{i=1}^n d_i^{+(r)}d_j^{-(r)}d_i^+d_j^-$, which implies

$$\mathbb{E}_{p_r}\left[\Lambda_r^{1^-}\Lambda_r^{1^+}\right] = \sum_{j=1}^n\sum_{i=1}^n \mathbb{E}_{p_r}\left[d_i^{+(r)}d_j^{-(r)}\right]d_i^+d_j^-.$$

The random variables $d_i^{+(r)}$ and $d_j^{-(r)}$ are independent, unless $(i,j)\in G_{\vec{d^n}}$. To see this recall that $d_i^{+(r)}$ (respectively $d_j^{-(r)}$) is the sum of d_i^+ (d_j^-) independent Bernoulli variables representing the out-stubs(in-stubs). If $(i,j)\in G_{\vec{d^n}}$ one fixed in-stub of j is to form an edge with a fixed out-stub of i. This implies that those two Bernoulli variables always need to take on the same value. Denote these Bernoulli variables by $d_{i_j}^+$ and $d_{j_i}^-$. Now that we have characterised the dependence between $d_i^{+(r)}$ and $d_j^{-(r)}$, we are ready to determine $\mathbb{E}_{p_r}\left[d_i^{+(r)}d_j^{-(r)}\right] = \mathbb{E}_{p_r}\left[d_i^{+(r)}\right]\mathbb{E}_{p_r}\left[d_j^{-(r)}\right] + \mathrm{Cov}\left(d_i^{+(r)}d_j^{-(r)}\right)$. As already explained in (i) $\mathbb{E}_{p_r}\left[d_i^{+(r)}\right]\mathbb{E}\left[d_j^{-(r)}\right] = \frac{(m-r)^2}{m^2}d_i^+d_j^-$. For the covariance there holds

$$\mathrm{Cov}\left(d_i^{+(r)}d_j^{-(r)}\right) = \begin{cases} 0 & \text{if } (i,j)\notin G_{\vec{d^n}} \\ \mathrm{Cov}\left(d_{i_j}^+d_{j_i}^-\right) & \text{if } (i,j)\in G_{\vec{d^n}} \end{cases}.$$

The covariance of any random variable X and a Bernoulli variable Y with expectation p^* equals $\mathrm{Cov}(X,Y) = (\mathbb{E}[X|Y=1]-\mathbb{E}[X|Y=0])\,p^*(1-p^*)$. Applying this to $X = d_{i_j}^+$ and $Y = d_{j_i}^-$, their covariance becomes $\frac{r(m-r)}{m^2}$. Thus there holds

$$\mathbb{E}_{p_r}\left[d_i^{+(r)}d_j^{-(r)}\right] = \begin{cases} \frac{(m-r)^2}{m^2}d_i^+d_j^- & \text{if } (i,j)\notin G_{\vec{d^n}} \\ \frac{(m-r)^2}{m^2}d_i^+d_j^- + \frac{r(m-r)}{m^2} & \text{if } (i,j)\in G_{\vec{d^n}} \end{cases}.$$

Plugging this back into the expression for $\mathbb{E}_{p_r}\left[\Lambda_r^{1^-}\Lambda_r^{1^+}\right]$ the desired equation follows.

(iv) Recall that $\Lambda_r^2 = \sum_{i=1}^n d_i^{-(r)}d_i^{+(r)}d_i^-d_i^+$. In the proof of (i) we already explained that $\mathbb{E}_{p_r}\left[d_i^{-(r)}d_i^{+(r)}\right] = d_i^+d_i^-\frac{(m-r)^2}{m^2}$. Hence $\mathbb{E}_{p_r}\left[\Lambda_r^2\right] = \frac{(m-r)^2}{m^2}\sum_{i=1}^n {d_i^-}^2{d_i^+}^2$.

(v) From equation (5.11) it follows that $\Lambda_r^3 = \sum_{(i,j)\notin E_r, i\neq j} d_i^+d_j^-d_i^{+(r)}d_j^{-(r)}$. Realizing that $\Delta_r^2 = \sum_{(i,j)\notin E_r,i\neq j} d_i^{+(r)}d_j^{-(r)}$ we can use the proof of (ii). This implies each edge $(i,j)\in G_{\vec{d^n}}$ contributes $\frac{(m-r)^2}{m^2}\frac{rd_i^+(d_i^+-1)d_j^-(d_j^--1)}{m}$ to the sum, proving the claim.

□

The value of ψ_r is an approximation to the value of $\mathbb{E}_{p_r}(\Psi_r)$. For the approximation the following relations are used

- $\sum_{i=1}^n \left(d_i^-\right)^s = \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^-\right)^{s-1} = \mathcal{O}\left(m d_{\max}^{s-1}\right)$;
- $\sum_{i=1}^n \left(d_i^+\right)^t = \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+\right)^{t-1} = \mathcal{O}\left(m d_{\max}^{t-1}\right)$;
- $\sum_{i=1}^n \left(d_i^-\right)^s \left(d_i^-\right)^t = \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^-\right)^{s-1} \left(d_i^+\right)^t = \mathcal{O}\left(m d_{\max}^{s+t-1}\right)$.

Combing this with Lemma 5.4 we find

$$\mathbb{E}_{p_r}\left[\frac{\Lambda_r^{1^-}\Lambda_r^{1^+}}{4m}\right] = \frac{(m-r)^2}{4m^3}\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2 + (m-r)^2\mathcal{O}\left(\frac{r d_{\max}^2}{(m-r)m^2}\right),$$

$$\mathbb{E}_{p_r}\left[\frac{\Lambda_r^2}{4m}\right] = (m-r)^2\mathcal{O}\left(\frac{d_{\max}^3}{m^2}\right) \quad \text{and} \quad \mathbb{E}_{p_r}\left[\frac{\Lambda_r^3}{2m}\right] = (m-r)^2\mathcal{O}\left(r\frac{d_{\max}^4}{m^3}\right).$$

This allows to define ψ_r.

Lemma 5.5. *For all $0 \le r \le m-1$ there holds*

$$\psi_r = (m-r)^2\left[\frac{\sum_{i=1}^n d_i^- d_i^+}{m^2} + \frac{r\sum_{(i,j)\in G_{\vec{d^n}}}\left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{m^3} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^3} + \xi_r\right], \tag{5.12}$$

with $\xi_r = \mathcal{O}\left(\frac{d_{\max}^3}{m^2} + \frac{r d_{\max}^2}{(m-r)m^2} + \frac{r d_{\max}^4}{m^3}\right)$.

The following upper bound on ψ_r will be useful in Sections 5.2.3 and 5.2.4.

Lemma 5.6. *For each $0 \le r \le m-1$ the quantity ψ_r is upper bounded by $\mathcal{O}\left((m-r)^2\frac{d_{\max}^2}{m}\right)$.*

Proof. Combing equation (5.12) with the relation $\sum_{i=1}^n \left(d_i^-\right)^s \left(d_i^-\right)^t = \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^-\right)^{s-1} \left(d_i^+\right)^t = \mathcal{O}\left(m d_{\max}^{s+t-1}\right)$ we find

$$\psi_r = (m-r)^2\mathcal{O}\left(\frac{d_{\max}}{m} + \frac{r d_{\max}^2}{m^2} + \frac{d_{\max}^2}{4m} + \frac{r d_{\max}^2}{(m-r)m^2} + \frac{d_{\max}^3}{m^3} + \frac{r d_{\max}^4}{m^3}\right).$$

The fact that $r \le m$ implies $\frac{r}{m} < 1$ and $\frac{r}{(m-r)m} < 1$. Combing this with $d_{\max}^2 = o(m)$ and the previous equation, one obtains

$$\psi_r = (m-r)^2\mathcal{O}\left(\frac{d_{\max}^2}{m}\right).$$

□

5.2.3 Proving equation (5.4)

With help of Lemma's 5.5 and 5.6 we are now ready to prove equation (5.4). We start by multiplying equation (5.4) by $\prod_{r=0}^{m-1}(m-r)^2$. This leads to

$$\prod_{r=0}^{m-1}\frac{(m-r)^2}{(m-r)^2-\psi_r} = \prod_{r=0}^{m-1}\left(1 + \frac{\psi_r}{(m-r)^2-\psi_r}\right).$$

Applying Lemma 5.5 to the numerator and Lemma 5.6 to the denominator we find

$$= \prod_{r=0}^{m-1} 1 + \frac{\frac{\sum_{i=1}^n d_i^- d_i^+}{m^2} + \frac{r \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{m^3} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^3} + \xi_r}{1 - \mathcal{O}\left(\frac{d_{\max}^2}{m}\right)}$$

$$= \exp\left[\sum_{r=0}^{m-1} \ln\left(1 + \frac{\frac{\sum_{i=1}^n d_i^- d_i^+}{m^2} + \frac{r \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{m^3} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^3} + \xi_r}{1 - \mathcal{O}\left(\frac{d_{\max}^2}{m}\right)}\right)\right].$$

Using $\frac{1}{1-x} = 1 + x + \mathcal{O}\left(x^2\right)$ and $\mathcal{O}\left(\frac{d_{\max}^2}{m}\right) = \mathcal{O}\left(\frac{1}{m^{1/2+2\tau}}\right)$, we obtain

$$= \exp\left[\sum_{r=0}^{m-1} \ln\left(1 + \frac{\sum_{i=1}^n d_i^- d_i^+}{m^2} + \frac{r \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{m^3} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^3} + \xi_r\right)\right].$$

Invoking $\ln(1+x) = x - \mathcal{O}(x^2)$ leads to

$$= \exp\left[\sum_{r=0}^{m-1} \frac{\sum_{i=1}^n d_i^- d_i^+}{m^2} + \frac{r \sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{m^3} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^3} + \mathcal{O}\left(\frac{d_{\max}^4}{m^2} + \frac{r d_{\max}^4}{m^3} + \frac{r d_{\max}^2}{(m-r)m^2}\right)\right]$$

$$= \exp\left[\frac{\sum_{i=1}^n d_i^- d_i^+}{m} + (m-1)\frac{\sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{2m^2} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \mathcal{O}\left(\frac{d_{\max}^4}{m} + \frac{d_{\max}^2}{m}\ln(m)\right)\right]$$

$$= \exp\left[\frac{\sum_{i=1}^n d_i^- d_i^+}{m} + \frac{\sum_{(i,j)\in G_{\vec{d^n}}} \left(d_i^+ - 1\right)\left(d_j^- - 1\right)}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + o(1)\right]$$

$$= \exp\left[\frac{\sum_{i=1}^n d_i^- d_i^+}{m} - \frac{\sum_{i=1}^n (d_i^-)^2 + \sum_{i=1}^n (d_i^+)^2}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \frac{\sum_{(i,j)\in G_{\vec{d^n}}} d_i^+ d_j^-}{2m} + \frac{1}{2} + o(1)\right].$$

Thus we have shown that there holds

$$\prod_{r=0}^{m-1} \frac{(m-r)^2}{(m-r)^2 - \psi_r} = [1 + o(1)] \exp\left[\frac{\sum_{i=1}^n d_i^- d_i^+}{m} - \frac{\sum_{i=1}^n (d_i^-)^2 + \sum_{i=1}^n (d_i^+)^2}{2m} + \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} + \frac{\sum_{(i,j)\in G_{\vec{d^n}}} d_i^+ d_j^-}{2m} + \frac{1}{2}\right],$$

which proves equation (5.4).

5.2.4 Proving equation (5.3)

It remains to show equation (5.3) to prove Theorem 5.2. Defining

$$f(\mathcal{N}) = \prod_{r=0}^{m-1} \frac{(m-r)^2 - \psi_r}{(m-r)^2 - \Psi_r}, \tag{5.13}$$

equation (5.3) becomes equivalent to

$$\mathbb{E}[f(\mathcal{N})] = 1 + o(1), \tag{5.14}$$

which we will show instead.

5.2.4.1 Partitioning $S(\mathcal{M})$

In order to show equation (5.14), the set of orderings $S(\mathcal{M})$ is partitioned. Using this partition, we determine $\mathbb{E}[f(\mathcal{N})]$. The partition is constructed in four steps:

1. For a small number $0 \leq \tau \leq \frac{1}{3}$, such that $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$, define
$$S^*(\mathcal{M}) = \left\{\mathcal{N} \in S(\mathcal{M}) \,|\Psi_r(\mathcal{N}) - \psi_r \leq \left(1 - \frac{\tau}{4}\right)(m-r)^2, \forall\, 0 \leq r \leq m-1\right\}. \tag{5.15}$$
Let $S(\mathcal{M}) \setminus S^*(\mathcal{M})$ be the first element of the partition.

2. Take as second element of the partition
$$\mathcal{A} = \left\{\mathcal{N} \in S^*(\mathcal{M}) \,|\Psi_r(\mathcal{N}) - \psi_r > T_r\left(\ln(n)^{1+\delta}\right), \forall\, 0 \leq r \leq m-1\right\}. \tag{5.16}$$
The family of functions T_r will be defined by equation (5.24). The quantity δ is a small positive constant, take $0 < \delta < 0.1$.

3. The next element of the partition must be a subset of $S^*(\mathcal{M}) \setminus \mathcal{A}$. This will be
$$\mathcal{B} = \left\{\mathcal{N} \in S^*(\mathcal{M}) \setminus \mathcal{A} | \exists 0 \leq r \leq m-1 \,\text{such that}\, m - r \leq \ln(n)^{1+2\delta} \,\text{and}\, \Psi_r(\mathcal{N}) > 1\right\}. \tag{5.17}$$

4. To complete the partition, define as last element
$$\mathcal{C} = S^*(\mathcal{M}) \setminus (\mathcal{A} \cup \mathcal{B}). \tag{5.18}$$

We will show that the following equations hold

$$\mathbb{E}(f(\mathcal{N})\, 1_{\mathcal{A}}) = o(1); \tag{5.19}$$
$$\mathbb{E}(f(\mathcal{N})\, 1_{\mathcal{B}}) = o(1); \tag{5.20}$$
$$\mathbb{E}(f(\mathcal{N})\, 1_{\mathcal{C}}) \leq 1 + o(1); \tag{5.21}$$
$$\mathbb{E}(f(\mathcal{N})\, 1_{\mathcal{C}}) \geq 1 - o(1); \tag{5.22}$$
$$\mathbb{E}\left(f(\mathcal{N})\, 1_{S(\mathcal{M})\setminus S^*(\mathcal{M})}\right) = o(1). \tag{5.23}$$

As

$$\mathbb{E}[f(\mathcal{N})] = \mathbb{E}[f(\mathcal{N})\, 1_{\mathcal{A}}] + \mathbb{E}[f(\mathcal{N})\, 1_{\mathcal{B}}] + \mathbb{E}[f(\mathcal{N})\, 1_{\mathcal{C}}] + \mathbb{E}\left[f(\mathcal{N})\, 1_{S(\mathcal{M})\setminus S^*(\mathcal{M})}\right],$$

the above equations suffice to prove equation (5.14). It remains to show equations (5.19), (5.20), (5.21), (5.22) and (5.23). The first step to proving these equations is defining T_r.

5.2.4.2 The family of functions T_r

Define the family of functions $T_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ indexed by $r \in \{0, 1, \ldots, m-1\}$ by

$$\mathrm{T}_r(\lambda) = \begin{cases} 4\beta_r(\lambda) + 2\min(\gamma_r(\lambda), \nu_r) & \text{if } m - r \geq \lambda\omega \\ \frac{\lambda^2}{\omega^2} & \text{else} \end{cases}, \tag{5.24}$$

with

$$\beta_r(\lambda) = c\sqrt{\lambda\left(m d_{\max}^2 q_r^2 + \lambda^2\right)\left(d_{\max}^2 q_r + \lambda\right)}, \tag{5.25}$$
$$\gamma_r(\lambda) = c\sqrt{\lambda\left(m d_{\max}^2 q_r^3 + \lambda^3\right)\left(d_{\max}^2 q_r^2 + \lambda^2\right)}, \tag{5.26}$$
$$\nu_r = 8 m d_{\max}^2 q_r^3. \tag{5.27}$$

The quantity c is a large positive constant, which will be defined later. Furthermore $\omega = \ln(n)^\delta$ and $q_r = \frac{m-r}{m}$. Recalling the G_{p_r} model from Section 5.2.2, remark that $q_r = 1 - p_r$. Thus q_r is the probability that an edge of $G_{\vec{d^n}}$ is not present in G_{p_r}. The intuition behind the definition of this family of functions will become apparent in Section 5.2.4.3.

Define $\lambda_0 = \omega \ln(n)$ and $\lambda_i = 2^i \lambda_0$ for all $i \in \{1, 2, \ldots, L\}$. Here L is the unique integer such that $\lambda_{L-1} < c d_{\max} \ln(n) \leq \lambda_L$. The following relation between the values $T_r(\lambda_i)$ and $T_r(\lambda_{i-1})$ holds.

Lemma 5.7. *For all $0 \leq r \leq m-1$ and $i \in \{1, 2, \ldots, L\}$ there holds*

$$T_r(\lambda_i) \leq 8T_r(\lambda_{i-1}).$$

Proof. As the function T_r is defined piece-wise, we distinguish three cases.

1. Suppose $m - r < \lambda_i \omega$ and $m - r < \lambda_{i-1}\omega$.
   Then
   $$T_r(\lambda_i) = \frac{\lambda_i^2}{\omega^2} = \frac{4\lambda_{i-1}^2}{\omega^2} < \frac{8\lambda_{i-1}^2}{\omega^2} = 8T_r(\lambda_{i-1}),$$
   showing that $T_r(\lambda_i) \leq 8T_r(\lambda_{i-1})$.

2. Suppose $m - r < \lambda_i \omega$ and $m - r \geq \lambda_{i-1}\omega$.
   Then by definition of T_r there holds $T_r(\lambda_i) = \frac{4\lambda_{i-1}^2}{\omega^2}$ and $T_r(\lambda_{i-1}) \geq 4\beta_r(\lambda_{i-1}) \geq 4c\lambda_{i-1}^2$. Hence we find $T_r(\lambda_i) \leq T_r(\lambda_{i-1})$.

3. Suppose $m - r \geq \lambda_i \omega$ and $m - r \geq \lambda_{i-1}\omega$.
   Then by definition of T_r there holds $T_r(\lambda_i) = 4\beta_r(\lambda_i) + 2\min(\gamma_r(\lambda_i), \nu_r)$ and $T_r(\lambda_{i-1}) = 4\beta_r(\lambda_{i-1}) + 2\min(\gamma_r(\lambda_{i-1}), \nu_r)$. Both $\beta_r(\lambda)$ and $\gamma_r(\lambda)$ are the square root of a summation. Each term of the summation contains no higher power of λ than λ^6. As $\lambda_i = 2\lambda_{i-1}$ and $\sqrt{2^6} = 8$, this implies that $\beta_r(\lambda_i) \leq 8\beta_r(\lambda_{i-1})$ and $\gamma_r(\lambda_i) \leq 8\gamma_r(\lambda_{i-1})$. Hence there holds $T_r(\lambda_i) \leq 8T_r(\lambda_{i-1})$.

This completes the proof, because $m - r \geq \lambda_i\omega$ and $m - r < \lambda_{i-1}\omega$ is impossible as $\lambda_i > \lambda_{i-1}$. □

In order to show equations (5.19) and (5.20) we subpartition $\mathcal{A}$ and $\mathcal{B}$. Define the chain of subsets $A_0 \subset A_1 \subset \ldots \subset A_L \subset S^*(\mathcal{M})$ by

$$A_i = \{\mathcal{N} \in S^*(\mathcal{M}) \,|\Psi_r(\mathcal{N}) - \psi_r < T_r(\lambda_i), \forall 0 \leq r \leq m-1\}. \tag{5.28}$$

To ensure that we cover $S^*(\mathcal{M})$ entirely, define

$$A_\infty = S^*(\mathcal{M}) \setminus A_L = \{\mathcal{N} \in S^*(\mathcal{M}) \,|\exists\, 0 \leq r \leq m-1 \,\text{such that}\, \Psi_r(\mathcal{N}) - \psi_r \geq T_r(\lambda_L)\}. \tag{5.29}$$

Now equation (5.16) implies that

$$\mathcal{A} = S^*(\mathcal{M}) \setminus A_0 = \cup_{i=1}^{L} A_i \setminus A_{i-1} \bigcup A_\infty.$$

Next we partition A_0. The goal of this partition is to write $\mathcal{B}$ as the union of some smaller sets. As $\mathcal{N} \in A_0$ for all $0 \leq r \leq m-1$ such that $r \geq m - \omega\lambda_0$ there holds

$$\Psi_r(\mathcal{N}) < T_r(\lambda_0) + \psi_r = \ln(n)^2 + \psi_r.$$

By Lemma 5.6 there holds for all $m - 1 \geq r \geq m - \omega\lambda_0$, $\psi_r = o(1)$. Hence there exists some n_0 such that for all $n > n_0$:

$$\Psi_r(\mathcal{N}) < \ln(n)^2 + 1.$$

As the goal is to show equation (5.3), we may assume that $n > n_0$. Define K to be the unique integer such that

$$2^{K-1} < \ln(n)^2 + 1 \leq 2^K. \tag{5.30}$$

Then for all $r \geq m - \omega\lambda_0$:

$$\Psi_r \leq 2^K.$$

This allows to define the chain of subsets $B_0 \subset B_1 \subset \ldots \subset B_k = A_0$ by

$$B_j = \left\{\mathcal{N} \in A_0 | \Psi_r(\mathcal{N}) < 2^j, \forall r \geq m - \omega\lambda_0\right\}. \tag{5.31}$$

From equations (5.17) and (5.18) it immediately follows that

$$\mathcal{B} = \cup_{i=1}^{K} B_i \setminus B_{i-1} \quad \text{and} \quad \mathcal{C} = B_0.$$

These descriptions of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ allow us to show equations (5.19), (5.20), (5.21) and (5.22). In Section 5.2.4.3 we prove equation (5.19). Some results from Section 5.2.4.3 are reused in Sections 5.2.4.4 and 5.2.4.5 to prove equations (5.20), (5.21) and (5.22). The remaining equation (5.23) is shown to hold in Section 5.2.4.6. This requires a different technique than for the other equations, as this concerns all orderings not in $S^*(\mathcal{M})$.

5.2.4.3 Proving equation (5.19)

Based on the definition of $\mathcal{A}$ in terms of A_i's and A_∞, we now prove equation (5.19). For this we use the following Lemma's.

Lemma 5.8. *For all $1 \leq i \leq L$ there holds*

(a) $\mathbb{P}[\mathcal{N} \in A_i \setminus A_{i-1}] \leq e^{-\Omega(\lambda_i)}$;

(b) For all $\mathcal{N} \in A_i \setminus A_{i-1}$ there holds $f(\mathcal{N}) \leq e^{o(\lambda_i)}$.

Lemma 5.9. *For a large enough constant c there holds*

(a) $\mathbb{P}[\mathcal{N} \in A_\infty] \leq e^{-\Omega(cd_{\max}\ln(n))}$;

(b) For all $\mathcal{N} \in A_\infty$ there holds $f(\mathcal{N}) \leq e^{72d_{\max}\ln(n)}$.

Together the lemma's imply

$$\begin{aligned}\mathbb{E}[f(\mathcal{N})1_{\mathcal{A}}] &= \sum_{i=1}^{L} \mathbb{E}\left[f(\mathcal{N})1_{A_i \setminus A_{i-1}}\right] + \mathbb{E}[f(\mathcal{N})1_{A_\infty}] \\ &\leq \sum_{i=1}^{L} e^{-\Omega(\lambda_i)} e^{o(\lambda_i)} + e^{-\Omega(cd_{\max}\ln(n))} e^{72d_{\max}\ln(n)} = o(1),\end{aligned}$$

proving equation (5.19). It remains to prove these lemma's. First we proof Lemma 5.8 (a) and Lemma 5.9 (a). This is done by showing the stronger statement

$$\mathbb{P}\left[\mathcal{N} \in A_{i-1}^c\right] \leq e^{-\Omega(\lambda_i)},$$

for all $i \in \{0, 1, \ldots, L\}$. This statement is stronger than the statements of Lemma 5.8 (a) as $(A_i \setminus A_{i-1}) \subset (S(\mathcal{M}) \setminus A_{i-1})$. This also works for Lemma 5.9 (a) since $A_\infty \in A_L^c$ and $\lambda_L \geq cd_{\max}\ln(n)$.

By definition of A_{i-1} there holds

$$A_{i-1}^c \subset \{\mathcal{N} \in S(\mathcal{M}) \,|\exists\, 0 \leq r \leq m-1 \,\text{such that}\, \Psi_r(\mathcal{N}) - \psi_r > T_r(\lambda_{i-1})\}.$$

Using Lemma 5.7 we find

$$A_{i-1}^c \subset \left\{\mathcal{N} \in S(\mathcal{M}) \,|\exists\, 0 \leq r \leq m-1 \,\text{such that}\, \Psi_r(\mathcal{N}) - \psi_r > \frac{T_r(\lambda_i)}{8}\right\}.$$

This implies that to prove Lemma 5.8 (a) and Lemma 5.9 (a), it suffices to show for all $i \in \{0, 1, \ldots, L\}$ and $0 \leq r \leq m-1$:

$$\mathbb{P}\left[|\Psi_r - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}. \tag{5.32}$$

Determining the value of Ψ_r is a challenge. It is more convenient to work in the G_{p_r} model and determine Ψ_{p_r}, because there each edge is present with probability p_r. As we remarked in Section 5.2.3 for a random

$\mathcal{N} \in S(\mathcal{M})$, the graph $G_{\mathcal{N}_r}$ is a random subgraph of $G_{\vec{d^n}}$ with exactly r edges. Denoting the number of edges in G_{p_r} by $E[G_{p_r}]$ we find:

$$\mathbb{P}\left[|\Psi_r - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right] = \frac{\mathbb{P}\left[|\Psi_{p_r} - \psi_r| \geq \frac{T_r(\lambda_i)}{8} \cap |E[G_{p_r}]| = r\right]}{\mathbb{P}[|E[G_{p_r}]| = r]} \leq \frac{\mathbb{P}\left[|\Psi_{p_r} - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right]}{\mathbb{P}[|E[G_{p_r}]| = r]}.$$

Bayati, Kim and Saberi have shown the following bound on the probability that the random graph G_{p_r} contains exactly r edges.

Lemma 5.10. *([5, Lemma 21]) For all $0 \leq r \leq m$ there holds $\mathbb{P}[|E[G_{p_r}]| = r] \geq \frac{1}{n}$.*

Using this Lemma we obtain

$$\mathbb{P}\left[|\Psi_r - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right] \leq n \cdot \mathbb{P}\left[|\Psi_{p_r} - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right].$$

As $\lambda_i = 2^i \ln(n)^{1+\delta} \gg \ln(n)$, there holds $ne^{-\Omega(\lambda_i)} = e^{-\Omega(\lambda_i)+\ln(n)} = e^{-\Omega(\lambda_i)}$. Hence showing

$$\mathbb{P}\left[|\Psi_{p_r} - \psi_r| \geq \frac{T_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)},$$

suffices to prove equation (5.32). As T_r is defined piecewise, we use a different proof for r such that $m-r \geq \omega\lambda_i$ than for r $m - r < \omega\lambda_i$. Let us first investigate the case $m - r < \lambda_i\omega$.

Lemma 5.11. *For all $i \in \{0, 1, \ldots, L\}$ and $0 \leq r \leq m-1$ such that $m - r < \lambda_i\omega$ there holds*

$$\mathbb{P}\left[\Psi_{p_r} - \psi_r \geq \frac{\lambda_i^2}{8\omega^2}\right] \leq e^{-\Omega(\lambda_i)}. \tag{5.33}$$

Proof. Instead of showing the desired inequality, we show the even stronger statement:

$$\mathbb{P}\left[\Psi_{p_r} \geq \frac{\lambda_i^2}{8\omega^2}\right] \leq e^{-\Omega(\lambda_i)}.$$

Combining the fact that $\Psi_{p_r} \leq \frac{\lambda_i^2}{8\omega}$ with $\Psi_{p_r} = \Delta_{p_r} + \Lambda_{p_r}$ we find

$$\Delta_{p_r} \geq \frac{\lambda_i^2}{8\omega^2} - \Lambda_{p_r}.$$

Using Lemma 5.3 this becomes

$$\Delta_{p_r} \geq \frac{\lambda_i^2}{8\omega^2} - \frac{d_{\max}^2 m}{2} q_r^2.$$

As $mq_r = m - r < \omega\lambda_i$ and $\omega^4 d_{\max}^2 < \frac{m}{5}$ for large n there holds

$$\Delta_{p_r} \geq \frac{\lambda_i^2}{8\omega^2} - \frac{d_{\max}^2}{2m}\omega^2\lambda_i^2 \quad \geq \frac{\lambda_i^2}{40\omega^2}.$$

Let G_{q_r} be the complement of G_{p_r} in $G_{\vec{d^n}}$ and define $N_0(u) := \{v \in V \mid (u,v) \in G_{q_r}\} \cup \{u\}$. Denote by $d^+_{G_{q_r}}(u)$ (respectively $d^-_{G_{q_r}}(u)$) the out-degree (in-degree) of u in G_{q_r}. By definition of Δ_{p_r} there holds

$$\Delta_{p_r} \leq \sum_{u \in V} d^+_{G_{q_r}}(u) \sum_{v \in N_0(u)} d^-_{G_{q_r}}(v).$$

Combining this with the lower bound on Δ_{p_r} just derived, we find

$$\frac{\lambda_i^2}{40\omega^2} \leq \Delta_{p_r} \leq \sum_{u \in V} d^+_{G_q}(u) \sum_{v \in N_0(u)} d^-_{G_q}(v). \tag{5.34}$$

We will now show that this equation implies that one of the following statements must hold:

(a) G_q has more than $\frac{\omega^2\lambda_i}{40}$ edges;

(b) for some $u \in V$ there holds $\sum_{v\in N_0(u)} d^-_{G_q}(v) \geq \frac{\lambda_i}{\omega^4}$.

If (a) is violated, there holds $\sum_{u\in V} d^+_{G_q}(u) \leq \frac{\omega^2\lambda_i}{40}$. If (b) is violated, for all $u \in V$ there holds $\sum_{v\in N_0(u)} d^-_{G_q}(v) < \frac{\lambda_i}{\omega^4}$. Hence if (a) and (b) are both violated, we find

$$\Delta_{p_r} \leq \sum_{u\in V} d^+_{G_q}(u) \sum_{v\in N_0(u)} d^-_{G_q}(v) < \frac{\omega^2\lambda_i}{40}\frac{\lambda_i}{\omega^4} = \frac{\lambda_i^2}{40\omega^2}.$$

This violates equation (5.34). Thus it is not possible that (a) and (b) are violated at the same time. This implies that at least one of the statements holds. Using the proof of [5, Lemma 20], the probability that statement (a) holds, is upper bounded by $e^{-\Omega(\lambda_i)}$. This proof is also used to upper bound the probability that statement (b) holds by $e^{-\Omega(\lambda_i)}$. As $\Psi_{p_r} \geq \frac{\lambda_i^2}{8\omega}$ implies that at least one of these statements holds, this completes the proof. □

For $m - r \geq \lambda_i\omega$ a similar relation is proven.

Lemma 5.12. *For all $i \in \{0, 1, \ldots, L\}$ and r such that $m - r \geq \lambda_i\omega$ there holds*

$$\mathbb{P}\left[|\Psi_{p_r} - \psi_r| \geq \frac{4\beta_r(\lambda_i) + 2\min(\nu_r, \gamma_r(\lambda_i))}{8}\right] \leq e^{-\Omega(\lambda_i)}. \tag{5.35}$$

Recall that $\Psi_{p_r} = \Delta^1_{p_r} + \Delta^2_{p_r} + \frac{{\Lambda^1_{p_r}}^+{\Lambda^1_{p_r}}^- - \Lambda^2_{p_r}}{4m} - \frac{\Lambda^3_{p_r}}{2m}$ and that ψ_r is of the order of $\mathbb{E}[\Psi_{p_r}]$, see equation (5.12). Thus to prove Lemma 5.12, it suffices to concentrate $\Delta^1_{p_r}, \Delta^2_{p_r}, {\Lambda^1_{p_r}}^+{\Lambda^1_{p_r}}^-, \Lambda^2_{p_r}$ and $\Lambda^3_{p_r}$ around their expected values with probability $e^{-\Omega(\lambda_i)}$ such that the difference between their sum and the sum of their expected values is smaller than $\frac{4\beta_r(\lambda_i)+2\min(\nu_r,\gamma_r(\lambda_i))}{8}$. This is shown using Vu's concentration inequality.

Theorem 5.13. *[Vu's concentration inequality [26]] Consider independent random variables $t_1, t_2, \ldots, t_n$ with arbitrary distribution in $[0, 1]$. Let $Y(t_1, t_2, \ldots, t_n)$ be a polynomial of degree k with coefficients in $(0, 1]$. For any multi-set A let $\partial_A Y$ denote the partial derivative with respect to the variables in A. Define $\mathbb{E}_j(Y) = \max_{|A|\geq j} \mathbb{E}(\partial_A Y)$ for all $0 \leq j \leq k$. Recursively define $c_1 = 1, d_1 = 2, c_k = 2\sqrt{k}(c_{k-1} + 1), d_k = 2(d_{k-1} + 1)$. Then for any $\mathcal{E}_0 > \mathcal{E}_1 > \ldots > \mathcal{E}_k = 1$ and λ fulfilling*

i) $\mathcal{E}_j \geq \mathbb{E}_j(Y)$;

ii) $\frac{\mathcal{E}_j}{\mathcal{E}_{j-1}} \geq \lambda + 4j\ln(n)$ *for all* $0 \leq j \leq k - 1$;

there holds

$$\mathbb{P}\left[|Y - \mathbb{E}[Y]| \geq c_k\sqrt{\lambda\mathcal{E}_0\mathcal{E}_1}\right] \leq d_k e^{-\lambda/4}.$$

Lemma 5.14. *For all $i \in \{0, 1, \ldots, L\}$ and $0 \leq r \leq m - 1$ there holds:*

(i) $\mathbb{P}\left[\left|\Delta^1_{p_r} - \mathbb{E}\left[\Delta^1_{p_r}\right]\right| \geq \frac{\beta_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}$;

(ii) $\mathbb{P}\left[\left|\Delta^2_{p_r} - \mathbb{E}\left[\Delta^2_{p_r}\right]\right| \geq \frac{\min(\beta_r(\lambda_i)+\gamma_r(\lambda_i),\beta_r(\lambda_i)+\nu_r)}{8}\right] \leq e^{-\Omega(\lambda_i)}$;

(iii) $\mathbb{P}\left[\left|\frac{{\Lambda^1_{p_r}}^-{\Lambda^1_{p_r}}^+ - \Lambda^2_{p_r}}{4m} - \frac{\mathbb{E}\left[{\Lambda^1_{p_r}}^-{\Lambda^1_{p_r}}^+ - \Lambda^2_{p_r}\right]}{4m}\right| \geq \frac{\beta_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}$;

(iv) $\mathbb{P}\left[\left|\frac{\Lambda^3_{p_r}}{2m} - \frac{\mathbb{E}\left[\Lambda^3_{p_r}\right]}{2m}\right| \geq \frac{\min(\beta_r(\lambda_i)+\gamma_r(\lambda_i),\beta_r(\lambda_i)+\nu_r)}{8}\right] \leq e^{-\Omega(\lambda_i)}$.

Proof. To prove each of the above equations, we write the quantity as a polynomial and apply Theorem 5.13 to it. This polynomial will be a function of m Bernoulli variables. Each variable t_e is coupled to an edge $e \in G_{\vec{d^n}}$. If $e \in G_{p_r}$ then t_e is 0, else, i.e if $e \notin G_{p_r}$, $t_e = 1$. Remark that by definition of G_{p_r}, see Section 5.2.2, there holds $\mathbb{E}[t_e] = q_r$ for all e. Also by definition of G_{p_r}, the variables t_e are independent of each other.

(i) Recall that $\Delta^1_{p_r}$ counts the number of pairs creating a self-loop. Each vertex v has d_v^- in-stubs and d_v^+ out-stubs. The number of those out-stubs (respectively in-stubs) that are matched equals the number of outgoing (incoming edges) for v in G_{p_r}. Thus the number of unmatched in-stubs of vertex v is $\sum_{e=(\bullet,v)\in G_{\vec{d^n}}} t_e$. and the number of unmatched out-stubs of v equals $\sum_{e=(v,\bullet)\in G_{\vec{d^n}}} t_e$. The number of ways to create a self-loop at v is $\sum_{e=(v,\bullet)\in G_{\vec{d^n}}} \sum_{f=(\bullet,v)\in G_{\vec{d^n}}} t_e t_f$. Hence we find

$$\Delta^1_{p_r} = \sum_{v\in V} \sum_{e=(v,\bullet)\in G_{\vec{d^n}}} \sum_{f=(\bullet,v)\in G_{\vec{d^n}}} t_e t_f. \tag{5.36}$$

Vu's concentration inequality requires us to upper bound the values $\mathbb{E}_0\left[\Delta^1_{p_r}\right], \mathbb{E}_1\left[\Delta^1_{p_r}\right]$ and $\mathbb{E}_2\left[\Delta^1_{p_r}\right]$. First look at the expectation of $\Delta^1_{p_r}$. Because $G_{\vec{d^n}}$ is simple, for each element of the summation in equation (5.36) e does not equal f. Thus there holds $\mathbb{E}[t_e t_f] = q_r^2$. The summations over v and e in equation (5.36), can be replaced by one summation over all edges in $G_{\vec{d^n}}$. For each edge $e \in G_{\vec{d^n}}$, there are at most $d_{\max}$ edges in $G_{\vec{d^n}}$ with the source of e as target. Hence we find $\mathbb{E}\left[\Delta^1_{p_r}\right] \le m d_{\max} q_r^2$. Suppose we take the partial derivative with respect to one variable t_e for some $e = (u, v)$, then we obtain $\sum_{f=(\bullet,u)\in G_{\vec{d^n}}} t_f + \sum_{f=(v,\bullet)\in G_{\vec{d^n}}} t_f$. This is upper bounded by $2 d_{\max} q_r$. As $\Delta^1_{p_r}$ is a polynomial of degree 2 with all coefficients 1, it is clear that $\mathbb{E}\left[\partial_{t_e}\partial_{t_f}\Delta^1_{p_r}\right] \le 1$ for all e, f. Thus we find

$$\mathbb{E}_0\left[\Delta^1_{p_r}\right] \le \max\left(1, 2d_{\max}q_r, m d_{\max} q_r^2\right), \quad \mathbb{E}_1\left[\Delta^1_{p_r}\right] \le \max\left(1, 2 d_{\max} q_r\right) \quad \text{and} \quad \mathbb{E}_2\left[\Delta^1_{p_r}\right] \le 1.$$

The maximization follows from the definition of $\mathbb{E}_j(Y)$. Define the values

$$\mathcal{E}_0 = 9\lambda_i^2 + 2 m d_{\max} q_r^2, \quad \mathcal{E}_1 = 9\lambda_i + 2 d_{\max} q_r \quad \text{and} \quad \mathcal{E}_2 = 1.$$

We claim that together with $\lambda = \lambda_i$, they fulfil the conditions of Theorem 5.13. It is obvious that $\mathcal{E}_2 \ge \mathbb{E}_2\left[\Delta^1_{p_r}\right]$. Also $\mathcal{E}_1 \ge \mathbb{E}_1\left[\Delta^1_{p_r}\right]$ as $\lambda_i \ge 1$ for all $n \ge 3$. Furthermore $\mathcal{E}_0 \ge \mathbb{E}_0\left[\Delta^1_{p_r}\right]$ as $\lambda_i \ge 1$ and $m q_r = m - r$ implies that $2 m d_{\max} q_r^2 \ge 2 d_{\max} q_r$. This shows the first condition of Theorem 5.13. For the second condition remark that $\lambda_i \ge \ln(n)$ and $\ln(m) \le 2\ln(n)$ as $m \le n^2$. This implies

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \mathcal{E}_1 \ge \lambda_i + 4\ln(m).$$

Furthermore there holds

$$\frac{\mathcal{E}_0}{\mathcal{E}_1} = \lambda_i \left(\frac{9\lambda_i + \frac{2 d_{\max} m q_r^2}{\lambda_i}}{9 + \frac{2 d_{\max} q_r}{\lambda_i}} \right) \ge \lambda_i,$$

showing that the second condition of Theorem 5.13 is fulfilled as well. Thus we may apply Vu's concentration inequality. Applying this we obtain

$$\mathbb{P}\left[\left| \Delta^1_{p_r} - \mathbb{E}\left[\Delta^1_{p_r}\right] \right| \ge c_2 \sqrt{\lambda_i \left(9\lambda_i + 2 d_{\max} q_r\right)\left(9\lambda_i^2 + 2 m d_{\max} q_r^2\right)} \right] \le e^{-\Omega(\lambda_i)}.$$

As for $a > b$ there holds $\mathbb{P}\left[\left|\Delta^1_{p_r} - \mathbb{E}\left[\Delta^1_{p_r}\right]\right| \ge a\right] \le \mathbb{P}\left[\left|\Delta^1_{p_r} - \mathbb{E}\left[\Delta^1_{p_r}\right]\right| \ge b\right]$, taking the value of c in equation (5.25) larger than $8 \cdot 9 c_2$, this completes the proof.

(ii) Recall that $\Delta^2_{p_r}$ counts the number of pairs that create an edge already present in G_{p_r}, i.e a double edge. Pairing an out-stub of u with an in-stub of v only creates a double edge if $(u, v) \in G_{p_r}$, i.e. if for

$e = (u, v)$, $t_e = 1$. Recalling the expressions for the number of unmatched in-stubs and out-stubs at a vertex v from the proof of (i) and defining

$$Q = \left\{(e, f, g) | e, f, g \in G_{\vec{d^n}}, e \neq f, f \neq g, e \neq g, f = (u, v) \text{ for some } u, v \in V, e = (u, \bullet), g = (\bullet, v)\right\},$$

we find

$$\Delta_{p_r}^2 = \sum_{(e,f,g)\in Q} t_e t_g (1 - t_f) = \sum_{(e,f,g)\in Q} t_e t_g - \sum_{e,f,g\in Q} t_e t_g t_f = Y_1 - Y_2.$$

Vu's inequality will be applied to Y_1 and Y_2 separately. Let us first look at Y_1. To upper bound the expected value of Y_1, we need an upper bound on the size of Q. Given f, the source of e and the target of g are fixed. Hence there are at most $d_{\max}^2$ triples in Q with a fixed edge f. As f may be any edge, $|Q| \leq m d_{\max}^2$. Together with $\mathbb{E}[t_e t_g] = q_r^2$ this implies $\mathbb{E}[Y_1] \leq m d_{\max}^2 q_r^2$. Suppose we differentiate Y_1 with respect to $t_{\widetilde{e}}$. Then we obtain the expression

$$\sum_{\substack{(e,f,g)\in Q \\ e=\widetilde{e}}} t_g + \sum_{\substack{(e,f,g)\in Q \\ g=\widetilde{e}}} t_e.$$

As

$$\sum_{\substack{(e,f,g)\in Q \\ e=\widetilde{e}}} 1 \leq d_{\max}^2 \quad \text{and} \quad \sum_{\substack{(e,f,g)\in Q \\ g=\widetilde{e}}} 1 \leq d_{\max}^2,$$

this implies $\mathbb{E}[\partial_{t_{\widetilde{e}}} Y_1] \leq 2 d_{\max}^2 q_r$. Since Y_1 is a polynomial of degree 2 with all coefficients equal to 1, any second derivative can be at most 1. Together these observations imply

$$\mathbb{E}_0[Y_1] \leq \max\left(1, 2 d_{\max}^2 q_r, m d_{\max}^2 q_r^2\right), \quad \mathbb{E}_1[Y_1] \leq \max\left(1, 2 d_{\max}^2 q_r\right) \quad \text{and} \quad \mathbb{E}_2[Y_1] \leq 1.$$

Similar to (i) it can be shown that $\lambda = \lambda_i$ and

$$\mathcal{E}_0 = 9\lambda_i^2 + 2 m d_{\max}^2 q_r^2, \quad \mathcal{E}_1 = 9\lambda_i + 2 d_{\max}^2 q_r \quad \text{and} \quad \mathcal{E}_2 = 1,$$

fulfil the conditions of Theorem 5.13. Applying Vu's inequality and assuming $c \geq 8 \cdot 9 c_2$, we obtain

$$\mathbb{P}\left[|Y_1 - \mathbb{E}[Y_1]| \geq \frac{\beta_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}.$$

Moving on to Y_2, we see that $\mathbb{E}[Y_2] \leq m d_{\max}^2 q_r^3$ as $|Q| \leq m d_{\max}^2$ and $\mathbb{E}[t_e t_f t_g] = q_r^3$. Differentiating Y_2 to with respect $t_{\widetilde{e}}$, we obtain

$$\sum_{\substack{(e,f,g)\in Q \\ e=\widetilde{e}}} t_f t_g + \sum_{\substack{(e,f,g)\in Q \\ f=\widetilde{e}}} t_e t_g + \sum_{\substack{(e,f,g)\in Q \\ g=\widetilde{e}}} t_e t_f.$$

This implies that $\mathbb{E}[\partial_{t_{\widetilde{e}}} Y_1] \leq 3 d_{\max}^2 q_r$. Differentiating Y_2 to with respect $t_{\widetilde{e}}$ and $t_{\widetilde{f}}$ for $\widetilde{e} \neq \widetilde{f}$, we obtain

$$\sum_{\substack{(e,f,g)\in Q \\ e=\widetilde{e} \\ f=\widetilde{f}}} t_g + \sum_{\substack{(e,f,g)\in Q \\ e=\widetilde{e} \\ g=\widetilde{f}}} t_f + \sum_{\substack{(e,f,g)\in Q \\ f=\widetilde{e} \\ g=\widetilde{f}}} t_e + \sum_{\substack{(e,f,g)\in Q \\ f=\widetilde{e} \\ e=\widetilde{f}}} t_g + \sum_{\substack{(e,f,g)\in Q \\ g=\widetilde{e} \\ e=\widetilde{f}}} t_f + \sum_{\substack{(e,f,g)\in Q \\ g=\widetilde{e} \\ f=\widetilde{f}}} t_e.$$

In each summation only one edge is left to choose. As the source, the target or both are fixed for this edge, each summation is upper bounded by $d_{\max} q_r$. By definition of Q there follows that at most two of the summations are non-zero, implying $\mathbb{E}\left[\partial_{t_{\widetilde{e}}} \partial_{t_{\widetilde{f}}} Y_2\right] \leq 2 d_{\max} q_r$. As Y_2 is a polynomial of degree 3 and all its coefficients are 1, any third order partial derivative of Y_2 can be at most 1. Summarizing we find

$$\mathbb{E}_0[Y_2] \leq \max\left(1, 2 d_{\max} q_r, 3 d_{\max}^2 q_r^2, m d_{\max}^2 q_r^3\right), \qquad \mathbb{E}_1[Y_2] \leq \max\left(1, 2 d_{\max} q_r, 3 d_{\max}^2 q_r^2\right),$$
$$\mathbb{E}[Y_2] \leq \max\left(1, 2 d_{\max} q_r\right) \text{ and} \qquad \mathbb{E}_3[Y_2] \leq 1.$$

Vu's inequality can be applied to Y_2 using $\lambda = \lambda_i$ and

$$\mathcal{E}_0 = 85\lambda_i^3 + 3md_{\max}^2 q_r^3, \quad \mathcal{E}_1 = 85\lambda_i^2 + 3d_{\max}^2 q_r^2, \quad \mathcal{E}_2 = 17\lambda_i + 2d_{\max} q_r \quad \text{and} \quad \mathcal{E}_3 = 1,$$

which implies

$$\mathbb{P}\left[|Y_2 - \mathbb{E}[Y_2]| \geq 85c_3\sqrt{\lambda_i\left(\lambda_i^2 + d_{\max}^2 q_r^2\right)\left(\lambda_i^3 + md_{\max}^2 q_r^3\right)}\right] \leq e^{-\Omega(\lambda_i)}.$$

If we choose c large enough, this implies that

$$\mathbb{P}\left[\left|\Delta_{p_r}^2 - \mathbb{E}\left[\Delta_{p_r}^2\right]\right| \geq \frac{\beta_r(\lambda_i) + \gamma_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}.$$

Next remark that

$$\begin{aligned}\left|\Delta_{p_r}^2 - \mathbb{E}\left[\Delta_{p_r}^2\right]\right| &= |Y_1 - Y_2 - \mathbb{E}[Y_1] + \mathbb{E}[Y_2]| \leq |Y_1 - \mathbb{E}[Y_1]| + \mathbb{E}[Y_2] \\ &\leq |Y_1 - \mathbb{E}[Y_1]| + md_{\max}^2 q_r^3 = |Y_1 - \mathbb{E}[Y_1]| + \frac{\nu_r}{8}.\end{aligned}$$

This implies that there also holds

$$\mathbb{P}\left[\left|\Delta_{p_r}^2 - \mathbb{E}\left[\Delta_{p_r}^2\right]\right| \geq \frac{\beta_r(\lambda_i) + \nu_r}{8}\right] \leq e^{-\Omega(\lambda_i)},$$

completing the proof.

(iii) To prove that $\mathbb{P}\left[\left|\frac{\Lambda_{p_r}^{1\ -}\Lambda_{p_r}^{1\ +} - \Lambda_{p_r}^2}{4m} - \frac{\mathbb{E}\left[\Lambda_{p_r}^{1\ -}\Lambda_{p_r}^{1\ +} - \Lambda_{p_r}^2\right]}{4m}\right| \geq \frac{\beta_r(\lambda_i)}{8}\right] \leq e^{-\Omega(\lambda_i)}$, Vu's inequality is applied to $\frac{\Lambda_{p_r}^{1\ +}\Lambda_{p_r}^{1\ -}}{d_{\max}^2}$ and $\frac{\Lambda_{p_r}^2}{d_{\max}^2}$ separately. The structure is almost identical to the proofs of (i) and (ii). First consider

$$\begin{aligned}\frac{\Lambda_{p_r}^{1\ +}\Lambda_{p_r}^{1\ -}}{d_{\max}^2} &= \frac{\sum_{i=1}^n d_i^{-(r)} d_i^- \sum_{i=1}^n d_i^{+(r)} d_i^+}{d_{\max}^2} = \left(\sum_{e=(u,v)\in G_{d\vec{n}}} \frac{d_u^-}{d_{\max}} t_e\right)\left(\sum_{f=(w,z)\in G_{d\vec{n}}} \frac{d_z^+}{d_{\max}} t_f\right) \\ &= \left(\sum_{e=(u,v)\in G_{d\vec{n}}} \frac{d_u^- d_v^+}{d_{\max}^2} t_e^2\right) + \sum_{\substack{e=(u,v)\in G_{d\vec{n}} \\ f=(w,z)\in G_{d\vec{n}} \\ e\neq f}} \frac{d_u^- d_z^+}{d_{\max}^2} t_e t_f = Z_1 + Z_2.\end{aligned}$$

Start with Z_1. This is a polynomial of degree one, as for a Bernoulli variable there holds $t_e^2 = t_e$. Since its coefficients are at most 1, it is clear that any first order partial derivative of Z_1 is upper bounded by 1. The expected value of Z_1 is upper bounded by mq_r. This implies

$$\mathbb{E}_0[Z_1] \leq \max(1, mq_r) \quad \text{and} \quad \mathbb{E}_1[Z_1] \leq 1.$$

Hence

$$\mathcal{E}_0 = mq_r + \lambda_i \quad \text{and} \quad \mathcal{E}_1 = 1,$$

with $\lambda = \lambda_i$ satisfy the constraints of Theorem 5.13. Applying this theorem we find

$$\mathbb{P}\left[|Z_1 - \mathbb{E}[Z_1]| \geq c_1\sqrt{\lambda_i(\lambda_i + mq_r)}\right] \leq e^{-\Omega(\lambda_i)}.$$

Next consider Z_2. This is a sum over all pairs of distinct edges, hence it contains fewer than m^2 terms. Combining this with $\frac{d_u^- d_z^+}{d_{\max}^2} \leq 1$ and $\mathbb{E}[t_e t_f] = q_r^2$, we find $\mathbb{E}[Z_2] \leq m^2 q_r^2$. Taking the partial derivative with respect to a variable t_g and writing $g = (i,j)$ leads to

$$\sum_{\substack{f=(w,z)\in G_{\vec{d^n}} \\ f\neq g}} \frac{d_i^- d_z^+}{d_{\max}^2} t_f + \sum_{\substack{e=(u,v)\in G_{\vec{d^n}} \\ e\neq g}} \frac{d_u^- d_j^+}{d_{\max}^2} t_e.$$

Each term of the summations is upper bounded by q_r. Each summation contains $m-1$ terms. Thus we find: $\mathbb{E}\left[\partial_{t_g} Z_2\right] \leq 2mq_r$. As Z_2 is a second order polynomial with coefficients upper bounded by 1, any second order partial derivative will be at most 1. Combining these observations we find

$$\mathbb{E}_0[Z_2] \leq \max\left(1, 2mq_r, m^2 q_r^2\right), \quad \mathbb{E}_1[Z_2] \leq \max(1, 2mq_r) \quad \text{and} \quad \mathbb{E}_2[Z_2] \leq 1.$$

Similar to the proof of (i) it can be shown that $\lambda = \lambda_i$ and

$$\mathcal{E}_0 = 9\lambda_i^2 + 2m^2 q_r^2, \quad \mathcal{E}_1 = 9\lambda_i + 2mq_r \quad \text{and} \quad \mathcal{E}_2 = 1,$$

satisfy the constraints of Vu's concentration inequality. Applying this inequality yields

$$\mathbb{P}\left[|Z_2 - \mathbb{E}[Z_2]| \geq 9c_2\sqrt{\lambda_i(\lambda_i + mq_r)(\lambda_i^2 + m^2 q_r^2)}\right] \leq e^{-\Omega(\lambda_i)}.$$

As $\sqrt{\lambda_i(\lambda_i + mq_r)} \leq \sqrt{\lambda_i(\lambda_i + mq_r)(\lambda_i^2 + m^2 q_r^2)}$ and $\frac{{\Lambda^1_{p_r}}^+ {\Lambda^1_{p_r}}^-}{4m} = \frac{d_{\max}^2}{4m}(Z_1 + Z_2)$, we obtain

$$\mathbb{P}\left[\left|\frac{{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+}{4m} - \frac{\mathbb{E}\left[{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+\right]}{4m}\right| \geq \frac{d_{\max}^2}{m}(9c_2 + c_1)\sqrt{\lambda_i(\lambda_i + mq_r)(\lambda_i^2 + m^2 q_r^2)}\right] \leq e^{-\Omega(\lambda_i)}.$$

Pulling this factor $\frac{d_{\max}^2}{m}$ inside the root and taking $c > 8(c_1 + 9c_2)$, we also find

$$\mathbb{P}\left[\left|\frac{{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+}{4m} - \frac{\mathbb{E}\left[{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+\right]}{4m}\right| \geq \frac{c}{8}\sqrt{\lambda_i(\lambda_i + d_{\max}^2 q_r)(\lambda_i^2 + md_{\max}^2 q_r^2)}\right] \leq e^{-\Omega(\lambda_i)}.$$

Next we look at

$$\frac{\Lambda^2_{p_r}}{d_{\max}^2} = \sum_{i=1}^n \frac{d_i^{-(r)} d_i^- d_i^{+(r)} d_i^+}{d_{\max}} = \sum_{i=1}^n \frac{d_i^- d_i^+}{d_{\max}^2}\left(\sum_{e=(i,\bullet)\in G_{\vec{d^n}}} t_e\right)\left(\sum_{f=(\bullet,i)\in G_{\vec{d^n}}} t_f\right).$$

Note that this is the same expression as for $\Delta^1_{p_r}$ where the coefficient of each term is replaced by $\frac{\Lambda^2_{p_r}}{d_{\max}^2}$. Hence using the same argument as for (i) we obtain

$$\mathbb{P}\left[\left|\frac{\Lambda^2_{p_r}}{4m} - \frac{\mathbb{E}\left[\Lambda^2_{p_r}\right]}{4m}\right| \geq 9c_2\frac{d_{\max}^2}{4m}\sqrt{\lambda_i(\lambda_i + q_r d_{\max})(\lambda_i^2 + md_{\max} q_r^2)}\right] \leq e^{-\Omega(\lambda_i)}.$$

Again pulling $\frac{d_{\max}^2}{m}$ inside the square root, we find

$$\mathbb{P}\left[\left|\frac{{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+ - \Lambda^2_{p_r}}{4m} - \frac{\mathbb{E}\left[{\Lambda^1_{p_r}}^- {\Lambda^1_{p_r}}^+ - \Lambda^2_{p_r}\right]}{4m}\right| \geq 9c_2\sqrt{\lambda_i(\lambda_i + d_{\max}^2 q_r)(\lambda_i^2 + md_{\max}^2 q_r^2)}\right] \leq e^{-\Omega(\lambda_i)}.$$

As $\beta = c\sqrt{\lambda_i(\lambda_i + d_{\max}^2 q_r)(\lambda_i^2 + md_{\max}^2 q_r^2)}$, this completes the proof if we take $c > 8(18c_2 + c_1)$.

(iv) This argument is exactly the same as for (ii), since there holds

$$\frac{\Lambda_{p_r}^3}{d_{\max}^2} = \sum_{\substack{(e,f,g)\in Q \\ e=(u,v)}} \frac{d_u^+ d_v^-}{d_{\max}^2} t_e \left(1-t_f\right) t_g.$$

Hence we obtain

$$\mathbb{P}\left[\left|\frac{\Lambda_{p_r}^3}{2m} - \frac{\mathbb{E}\left[\Lambda_{p_r}^3\right]}{2m}\right| \geq \frac{d_{\max}^2}{2m} \frac{\min\left(\beta_r(\lambda_i)+\gamma_r(\lambda_i), \beta_r(\lambda_i)+\nu_r\right)}{8}\right] \leq e^{-\Omega(\lambda_i)}.$$

As $\frac{d_{\max}^2}{m} = o(1)$, this completes the proof.

□

Combining all equations of Lemma (5.14) we find

$$\mathbb{P}\left[\left|\Psi_{p_r} - \mathbb{E}\left[\Psi_{p_r}\right]\right| \geq \frac{4\beta_r(\lambda_i) + 2\min(\nu_r, \gamma_r(\lambda_i))}{8}\right] \leq e^{-\Omega(\lambda_i)},$$

for all $i \in \{0,1,\ldots,L\}$ and $0 \leq r \leq m-1$. By definition of ψ_r this shows equation (5.35) and hence it proves Lemma 5.12. This completes the proofs of Lemma 5.8 (a) and 5.9 (a).

Next we prove Lemma 5.8 (b) and 5.9 (b). This requires the following Lemma.

Lemma 5.15. *For all $i \in \{1,2,\ldots,L\}$ and $\mathcal{N} \in A_i \setminus A_{i-1}$ there holds*

$$\sum_{r=0}^{m-1} \frac{\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{(m-r)^2 - \Psi_r\left(\mathcal{N}\right)} \leq o\left(\lambda_i\right).$$

Furthermore for all $\mathcal{N} \in A_0$ there holds

$$\sum_{m-r=\lambda_0\omega}^{m} \frac{\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{(m-r)^2 - \Psi_r\left(\mathcal{N}\right)} \leq o\left(1\right).$$

Proof. The first claim follows by changing the summation $\sum_{m-r=2}^{2m-2}$ into $\sum_{m-r=1}^{m}$ in the proof of Lemma 15(b) [5]. The second claim follows by applying a similar change to the proof of Lemma 18 [5]. □

First we determine an upper bound on $f\left(\mathcal{N}\right)$ for all $\mathcal{N} \in S^*\left(\mathcal{M}\right)$. By definition of $S^*\left(\mathcal{M}\right)$ there holds $\Psi_r\left(\mathcal{N}\right) \leq \left(1-\frac{\tau}{4}\right)(m-r)^2$ for all $0 \leq r \leq m-1$. This implies

$$\begin{aligned} f\left(\mathcal{N}\right) &= \prod_{r=0}^{m-1} \left(1 + \frac{\Psi_r\left(\mathcal{N}\right) - \psi_r}{(m-r)^2 - \Psi_r\left(\mathcal{N}\right)}\right) \\ &\leq \prod_{r=0}^{m-1} \left(1 + \frac{4\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{\tau(m-r)^2}\right). \end{aligned}$$

Using the approximation $1 + x \leq e^x$ this becomes

$$f\left(\mathcal{N}\right) \leq e^{\sum_{r=0}^{m-1} \frac{4\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{\tau(m-r)^2}}. \tag{5.37}$$

Now take $\mathcal{N} \in A_i \setminus A_{i-1}$ for some $i \in \{1,2,\ldots,L\}$. Then we can apply Lemma 5.15 to equation (5.37) to obtain

$$f\left(\mathcal{N}\right) \leq e^{o(\lambda_i)}.$$

This completes the proof of Lemma 5.8 (b).

It remains to prove Lemma 5.9 (b). As $A_\infty \subset S^*(\mathcal{M})$ there holds

$$f(\mathcal{N}) \leq \prod_{r=0}^{m-d_{\max}^2} \left(1 + \frac{4\max(\Psi_r(\mathcal{N}) - \psi_r, 0)}{\tau(m-r)^2}\right) \prod_{r=m-d_{\max}^2+1}^{m-1} \frac{(m-r)^2 - \psi_r}{(m-r)^2 - \Psi_r(\mathcal{N})}.$$

As $0 < \Psi_r(\mathcal{N}), \psi_r < (m-r)^2$, this implies

$$f(\mathcal{N}) \leq \left(d_{\max}^4\right)^{d_{\max}^2} \prod_{r=0}^{m-d_{\max}^2} \left(1 + \frac{4\Psi_r(\mathcal{N})}{\tau(m-r)^2}\right).$$

By Lemma 5.3 there holds $\Psi_r = \Delta_r + \Lambda_r \leq (m-r)d_{\max}^2 + \frac{d_{\max}^2}{2m}(m-r)^2 \leq 2(m-r)d_{\max}^2$. Inserting this gives

$$f(\mathcal{N}) \leq \left(d_{\max}^4\right)^{d_{\max}^2} \prod_{r=0}^{m-d_{max}^2} \left(1 + \frac{8d_{\max}^2}{\tau(m-r)}\right).$$

As $(1+x) \leq e^x$, we find

$$f(\mathcal{N}) \leq e^{4d_{\max}^2 \ln(d_{\max}) + \frac{8}{\tau}\sum_{i=d_{\max}^2}^{m} \frac{d_{\max}^2}{i}} \leq e^{4d_{\max}^2 \ln(d_{\max}) + \frac{8}{\tau}\ln(m) - \frac{8}{\tau}\ln(d_{\max}^2)}.$$

Using that $\tau \leq \frac{1}{3}$ and $m \leq nd_{\max}$, we obtain

$$\begin{aligned} f(\mathcal{N}) &\leq e^{4d_{\max}^2 \ln(d_{\max}) + 24\ln(m)} \leq e^{4d_{\max}^2 \ln(d_{\max}) + 24\ln(nd_{\max})} \\ &\leq e^{24d_{\max}^2 \ln\left(nd_{\max}^2\right)} \leq e^{24d_{\max}^2 \ln\left(n^3\right)} = e^{72d_{\max}^2 \ln(n)}. \end{aligned}$$

This proves Lemma 5.9 (b). This completes the proofs of Lemma's 5.8 and 5.9 and hence it completes the proof of equation (5.19).

5.2.4.4 Proving equation (5.20)

The next step is showing that equation (5.20) holds. To this end, we first prove the following Lemma.

Lemma 5.16. *For all $1 \leq j \leq K$*

(a) $\mathbb{P}[\mathcal{N} \in B_j \setminus B_{j-1}] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)}$;

(b) For all $\mathcal{N} \in B_j \setminus B_{j-1}$ it holds that $f(\mathcal{N}) \leq e^{\mathcal{O}(2^j)}$.

Proof. (a) The probability that $\mathcal{N} \in B_j \setminus B_{j-1}$ is upper bounded by the probability that $\mathcal{N} \in B_{j-1}^c := S(\mathcal{M}) \setminus B_{j-1}$. Hence if we show that

$$\mathbb{P}\left[\mathcal{N} \in B_j^c\right] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)},$$

the claim is proven. Remark that

$$B_{j-1}^c \subset \left\{\mathcal{N} \in S(\mathcal{M}) \,|\, \exists\, r \,\text{such that}\, m - r \leq \omega\lambda_0 \,\text{and}\, \Psi_r \geq 2^{j-1}\right\}.$$

Only those values of r are considered for which $m - r \leq \omega\lambda_0$. Let $\mathbb{P}\left[\Psi_r \geq 2^{j-1}\right]$ denote the probability that $\Psi_r \geq 2^{j-1}$ holds for one arbitrary r such that $m - r \leq \omega\lambda_0$. Suppose that

$$\mathbb{P}\left[\Psi_r \geq 2^{j-1}\right] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)}.$$

As $\omega\lambda_0 \ll \ln(n)^2$ this implies that

$$\begin{aligned}\mathbb{P}\left[\mathcal{N} \in B_{j-1}^c\right] &\leq \ln^2(n)\mathbb{P}\left[\Psi_r \geq 2^{j-1}\right] \leq \ln^2(n)e^{-\Omega\left(2^{j/2}\ln(n)\right)} \\ &= e^{-\Omega\left(2^{j/2}\ln(n)\right)+2\ln(\ln(n))} = e^{-\Omega\left(2^{j/2}\ln(n)\right)}.\end{aligned}$$

Thus it remains to show

$$\mathbb{P}\left[\Psi_r \geq 2^{j-1}\right] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)},$$

to complete the proof. Fix an arbitrary r such that $m - r < \omega\lambda_0$. This implies that $\Psi_r \leq 2^{j-1}$. In combination with $\Psi_r = \Delta_r + \Lambda_r$ there follows

$$\Delta_r = \Psi_r - \Lambda_r \geq 2^{j-1} - \Lambda_r.$$

Lemma 5.3 implies that

$$\Delta_r \geq 2^{j-1} - \frac{d_{\max}^2 m}{2} q_r^2.$$

As $m - r \leq \omega\lambda_0 < 2^{j-1}\omega\lambda_0$ and $d_{\max}^2\omega^2\lambda_0^2 < m$,

$$\begin{aligned}\Delta_r &\geq 2^{j-1} - \frac{2^{j-1}d_{\max}^2}{2m}\omega^2\lambda_0^2. \\ &\geq 2^{j-1} - \frac{2^{j-1}}{2} = 2^{j-2}.\end{aligned}$$

The remainder of the proof is the same as in Lemma 5.11, but with equation (5.34) replaced by

$$2^{j-2} \leq \Delta_{p_r} \leq \sum_{u\in V} d_{G_q}^+(u) \sum_{v\in N_0(u)} d_{G_q}^-(v).$$

This can be shown to imply that one of the following statements must hold

(a) G_q has more than $2^{j/2-1}$ edges;
(b) for some $u \in V$ there holds $\sum_{v\in N_0(u)} d_{G_q}^-(v) \geq 2^{j/2-1}$.

The probability that either of those statements holds, is upper bounded by $e^{-\Omega\left(2^{j/2}\ln(n)\right)}$. This is proven using the same argument as in the proof of Lemma 5.11. Since r is arbitrary this shows $\mathbb{P}\left[\Psi_r \geq 2^{j-1}\right] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)}$ for all r such that $m - r < \omega\lambda_0$, completing the proof.

(b) For all $1 \leq j \leq K$ there holds $B_j \subset S^*(\mathcal{M})$. Thus equation (5.37) yields

$$f(\mathcal{N}) \leq e^{\sum_{r=0}^{m-1} \frac{4\max(\Psi_r(\mathcal{N})-\psi_r,0)}{\tau(m-r)^2}},$$

for all $\mathcal{N} \in B_j \setminus B_{j-1}$. By definition of B_j, there holds

$$\sum_{m-r=1}^{\omega\lambda_0} \frac{\max\left(\Psi_r(\mathcal{N}) - \psi_r, 0\right)}{(m-r)^2} \leq \sum_{m-r=1}^{\omega\lambda_0} \frac{2^j}{(m-r)^2} = \mathcal{O}\left(2^j\right).$$

As $B_j \subset A_0$, Lemma 5.15 implies

$$\sum_{m-r=\omega\lambda_0}^{m} \frac{4\max\left(\Psi_r(\mathcal{N}) - \psi_r, 0\right)}{\tau(m-r)^2} = o(1).$$

Hence for all $\mathcal{N} \in B_j$ it holds

$$f(\mathcal{N}) \leq e^{\mathcal{O}\left(2^j\right)+o(1)} = e^{\mathcal{O}\left(2^j\right)}.$$

□

Now we prove equation (5.20). Lemma 5.16 implies that for all $B_j \setminus B_{j-1}$

$$\mathbb{E}\left[f\left(\mathcal{N}\right)1_{B_j\setminus B_{j-1}}\right] \leq e^{-\Omega\left(2^{j/2}\ln(n)\right)}e^{\mathcal{O}\left(2^j\right)}.$$

Recall that $j \leq K$. In combination with equation (5.30) this yields $2^{\frac{j-1}{2}} \leq \ln(n)$. Hence there holds

$$\mathbb{E}\left[f\left(\mathcal{N}\right)1_{\mathcal{B}}\right] = \sum_{j=1}^{K}\mathbb{E}\left[f\left(\mathcal{N}\right)1_{B_j\setminus B_{j-1}}\right] \leq \sum_{j=1}^{K} e^{-\Omega\left(2^{j/2}\ln(n)\right)}e^{\mathcal{O}\left(2^j\right)} = o(1),$$

proving equation (5.20).

5.2.4.5 Proving equations (5.21) and (5.22)

In this section, we bound the expected value of $f\left(\mathcal{N}\right)$ for all $\mathcal{N} \in \mathcal{C}$. First we derive an upper bound, i.e. we show equation (5.21). To prove this equation, it suffices to show for all $\mathcal{N} \in \mathcal{C}$,

$$f\left(\mathcal{N}\right) \leq 1 + o(1).$$

As $\mathcal{C} \subset S^*\left(\mathcal{M}\right)$, in analogy to equation (5.37), there holds

$$\begin{aligned} f\left(\mathcal{N}\right) &= \prod_{r=0}^{m-1}\left(1 + \frac{\Psi_r\left(\mathcal{N}\right) - \psi_r}{(m-r)^2 - \Psi_r\left(\mathcal{N}\right)}\right) \\ &\leq \prod_{m-r=1}^{\lambda_0\omega}\left(1 + \frac{4\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{\tau(m-r)^2}\right)e^{\sum_{m-r=\lambda_0\omega+1}^{m}\frac{4\max\left(\Psi_r\left(\mathcal{N}\right)-\psi_r,0\right)}{\tau(m-r)^2}}. \end{aligned}$$

Because $\mathcal{C} \subset A_0$, Lemma 5.15 implies

$$\sum_{m-r=\lambda_0\omega+1}^{m}\frac{4\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{\tau(m-r)^2} = o(1).$$

Hence for all $\mathcal{N} \in \mathcal{C}$,

$$f\left(\mathcal{N}\right) \leq \prod_{m-r=1}^{\lambda_0\omega}\frac{4\max\left(\Psi_r\left(\mathcal{N}\right) - \psi_r, 0\right)}{\tau(m-r)^2}e^{o(1)}.$$

By definition of $\mathcal{C}$, $\Psi_r\left(\mathcal{N}\right) \leq 1$ for all $m - r \leq \omega\lambda_0$. Thus we find

$$\begin{aligned} f\left(\mathcal{N}\right) &\leq \prod_{m-r=1}^{\lambda_0\omega}\left(\frac{4}{\tau(m-r)^2}\right)e^{o(1)} \\ &\leq \left(1 + \mathcal{O}\left(\frac{4\lambda_0\omega}{\tau}\prod_{m-r=1}^{\lambda_0\omega}\frac{1}{(m-r)^2}\right)\right)e^{o(1)} \\ &\leq e^{o(1)}\left(1 + o(1)\right) = 1 + o(1), \end{aligned}$$

proving equation (5.21).

Next we derive a lower bound on $\mathbb{E}\left[f\left(\mathcal{N}\right)1_{S^*\left(\mathcal{M}\right)}\right]$. As $\mathcal{C} \subset S^*\left(\mathcal{M}\right)$ this will prove equation (5.22). Take any ordering $\mathcal{N} \in S^*\left(\mathcal{M}\right)$. Lemma 5.12 implies that

$$\mathbb{P}\left[\left|\Psi_r\left(\mathcal{N}\right) - \psi_r\right| \geq 4\beta_r\left(\lambda_0\right) + 2\min\left(\gamma_r\left(\lambda_0\right), \nu_r\right)\right] \leq e^{-\Omega(\lambda_0)} < e^{-\ln(n)^{1+\delta}} = \left(\frac{1}{n}\right)^{1+\delta} = o(1), \tag{5.38}$$

holds for all r such that $m-r \geq \omega\lambda_0$. Thus the probability that $|\Psi_r(\mathcal{N}) - \psi_r| \geq 4\beta_r(\lambda_0) + 2\min(\gamma_r(\lambda_0), \nu_r)$ holds for at least one r is small. Now consider an ordering $\mathcal{N} \in S^*(\mathcal{M})$ such that for all r with $m-r \geq \omega\lambda_0$ there holds

$$|\Psi_r(\mathcal{N}) - \psi_r| \leq 4\beta_r(\lambda_0) + 2\min(\gamma_r(\lambda_0), \nu_r). \tag{5.39}$$

Recall that $\mathcal{N} \in S^*(\mathcal{M})$ implies $\Psi_r(\mathcal{N}) \leq \left(1 - \frac{\tau}{4}\right)(m-r)^2$. Combining this with the definition of $f(\mathcal{N})$, we find

$$\begin{aligned} f(\mathcal{N}) &\geq \prod_{m-r=\omega\lambda_0^3}^{m} \left(1 - \frac{\Psi_r(\mathcal{N}) - \psi_r}{(m-r)^2 - \Psi_r(\mathcal{N})}\right) \prod_{m-r=1}^{\omega\lambda_0^3+1} \left(1 - \frac{\psi_r}{(m-r)^2 - \Psi_r(\mathcal{N})}\right) \\ &\geq \prod_{m-r=\omega\lambda_0^3+1}^{m} \left(1 - \frac{4}{\tau}\frac{4\beta_r(\lambda_0) + 2\min(\gamma_r(\lambda_0), \nu_r)}{(m-r)^2}\right) \prod_{m-r=1}^{\omega\lambda_0^3} \left(1 - \frac{4}{\tau}\frac{\psi_r}{(m-r)^2}\right). \end{aligned}$$

The definition of T_r and Lemma 5.15 imply $\sum_{m-r=\omega\lambda_0^3+1}^{m} \frac{4}{\tau}\frac{4\beta_r(\lambda_0)+2\min(\gamma_r(\lambda_0),\nu_r)}{(m-r)^2} = o(1)$. Combining this with the approximation $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$, we obtain

$$f(\mathcal{N}) \geq e^{-o(1)} \prod_{m-r=1}^{\omega\lambda_0^3} \left(1 - \frac{4}{\tau}\frac{\psi_r}{(m-r)^2}\right).$$

To approximate the remaining product, we apply Lemma 5.6. In combination with the approximation $1 - x \geq e^{-2x}$ and $\lambda_0^3 \omega d_{\max}^2 = o(m)$, we find

$$f(\mathcal{N}) \geq e^{-2o(1)} \geq 1 - o(1).$$

Now for each $\mathcal{N} \in S^*(\mathcal{M})$ we have shown that either $f(\mathcal{N}) \geq 1 - o(1)$ or that its probability is upper bounded by $o(1)$. Together this completes the proof of equation (5.22). Remark that in fact we have proven

$$\mathbb{E}\left[f(\mathcal{N})\, 1_{S^*(\mathcal{M})}\right] \geq 1 - o(1).$$

The proofs of this section together with Section 5.2.4.3 and 5.2.4.4 show the following Corollary.

Corollary 5.17. *For sufficiently large c in the definition of λ_L there holds*

$$\mathbb{E}\left[\exp\left(\frac{1}{\tau^2}\sum_{r=0}^{m-1} \frac{\max(\Psi_r(\mathcal{N}) - \psi_r, 0)}{(m-r)^2}\right)\right] = 1 + o(1).$$

This will be used to prove equation (5.23).

5.2.4.6 Proving equation (5.23)

In order to prove equation (5.4), it remains to show equation (5.23). This concerns the expected value of $f(\mathcal{N})$ for the orderings in $S(\mathcal{M}) \setminus S^*(\mathcal{M})$. Equation (5.15) implies that for any $\mathcal{N} \in S(\mathcal{M}) \setminus S^*(\mathcal{M})$, there exists at least one $0 \leq r \leq m-1$ such that the inequality

$$\Psi_r(\mathcal{N}) \leq \left(1 - \frac{\tau}{4}\right)(m-r)^2 \tag{5.40}$$

is violated. This inequality can only be violated for specific values of r. To determine these values, suppose that the above inequality is violated and investigate what this implies for Δ_r. Recalling $\Psi_r = \Delta_r + \Lambda_r$ and using Lemma 5.3 to bound Λ_r, we obtain

$$\Delta_r > \Psi_r - \frac{d_{\max}^2}{2m}(m-r)^2.$$

As $d_{\max}^4 = o(m)$, there will be a n_0 such that for all $n > n_0$ there holds $\frac{d_{\max}^2}{m} < \frac{\tau}{2}$. Assuming that $n > n_0$, we find

$$\Delta_r > \Psi_r - \frac{\tau}{4}(m-r)^2.$$

By violation of equation (5.40) this becomes

$$\Delta_r > \left(1 - \frac{\tau}{2}\right)(m-r)^2. \tag{5.41}$$

Lemma 5.3 states that $\Delta_r \leq (m-r)d_{\max}^2$. Hence we can deduce that

$$(m-r)\left(1-\frac{\tau}{2}\right) \leq d_{\max}^2, \quad \text{implying} \quad m - r \leq \frac{2d_{\max}^2}{2-\tau}.$$

Thus equation (5.40) can only be violated if $m - r \leq \frac{2d_{\max}^2}{2-\tau}$. This allows us to partition

$$S(\mathcal{M}) \setminus S^*(\mathcal{M}) = \sum_{t=1}^{\frac{2d_{\max}^2}{2-\tau}} S_t(\mathcal{M}),$$

with $S_t(\mathcal{M})$ the set of all orderings $\mathcal{N}$ violating equation (5.40) for the first time at $r = m - t$, i.e. for all $r < m - t$ equation (5.40) still holds. To prove equation (5.23), it suffices to show that

$$\mathbb{E}\left[f(\mathcal{M})\, 1_{S_t}\right] \leq \mathcal{O}\left(\frac{1}{m^{t\tau}}\right) \tag{5.42}$$

for all $t \in \{1, 2, \ldots, \frac{2d_{\max}^2}{2-\tau}\}$ as $\sum_{t=1}^{\infty} \frac{1}{m^{t\tau}} = o(1)$. Thus the goal is to prove equation (5.42).

By definition of Ψ_r there holds $(m-r)^2 - \Psi_r = \sum_{(u,v)\in E_r} d_u^{+(r)} d_v^{-(r)} \left(1 - \frac{d_u^+ d_v^-}{2m}\right)$. As $\mathcal{N} \in S(\mathcal{M})$, the algorithm will finish. Hence at step r there must be at least $m - r$ suitable pairs left, implying $(m-r)^2 - \Psi_r \geq (m-r)\left(1 - \frac{d_{\max}^2}{2m}\right)$. Thus it holds that

$$\frac{(m-r)^2}{(m-r)^2 - \Psi_r} \leq \frac{(m-r)}{1 - \frac{d_{\max}^2}{2m}} = (m-r)\left(1 + \mathcal{O}\left(\frac{d_{\max}^2}{2m}\right)\right).$$

For $m - r \leq \frac{2d_{\max}^2}{2-\tau}$ this becomes

$$\frac{(m-r)^2}{(m-r)^2 - \Psi_r} \leq m - r + 1,$$

as $\frac{d_{\max}^4}{m} = o(1)$. Now we find that

$$\prod_{r=m-t}^{m-1} \frac{(m-r)^2 - \psi_r}{(m-r)^2 - \Psi_r} \leq \prod_{r=m-t}^{m-1} \frac{(m-r)^2}{(m-r)^2 - \Psi_r} \leq \prod_{r=m-t}^{m-1} m - r + 1 = (t+1)! \leq t^t(t+1).$$

For all $r < m - t$ equation (5.40) does hold. Hence analogous to equation (5.37) it can be shown that

$$\prod_{r=0}^{m-t} \frac{(m-r)^2 - \psi_r}{(m-r)^2 - \Psi_r} \leq \exp\left[\frac{4}{\tau} \sum_{r=0}^{m-1} \frac{\max(\Psi_r - \psi_r, 0)}{(m-r)^2}\right].$$

Combing these two observations we find

$$f(\mathcal{N})\, 1_{S_t} = 1_{S_t} \prod_{r=0}^{m-r} \frac{(m-r)^2 - \psi_r}{(m-r)^2 - \Psi_r} \leq 1_{S_t} \exp\left[\frac{4}{\tau} \sum_{r=0}^{m-1} \frac{\max(\Psi_r - \psi_r, 0)}{(m-r)^2}\right] t^t(t+1).$$

Next we take the expected value of the above equation and apply Hölder's inequality to obtain:

$$\mathbb{E}\left[f\left(\mathcal{N}\right)1_{S_t}\right] \leq \mathbb{E}\left[1_{S_t}\right]^{1-\tau}\mathbb{E}\left[1_{S_t}\exp\left[\frac{4}{\tau^2}\sum_{r=0}^{m-1}\frac{\max(\Psi_r-\psi_r,0)}{(m-r)^2}\right]\right]^{\tau}t^t(t+1).$$

Using Corollary 5.17 this becomes

$$\mathbb{E}\left[f\left(\mathcal{N}\right)1_{S_t}\right] \leq \mathbb{E}\left[1_{S_t}\right]^{1-\tau}\left[1+o(1)\right]t^t(t+1).$$

Now to prove equation (5.42), it remains to show

$$\mathbb{P}\left[\mathcal{N}\in S_t\right]^{1-\tau}t^t(t+1) \leq \left[1+o(1)\right]\frac{1}{m^{\tau t}}. \tag{5.43}$$

This requires an upper bound on $\mathbb{P}\left[\mathcal{N}\in S_t\right]$. To obtain this bound, first we show that if $\mathcal{N}\in S_t$, then $G_{\mathcal{N}_r}$ contains a vertex with a special property. The probability of such a vertex existing, is used to upper bound $\mathbb{P}\left[\mathcal{N}\in S_t\right]$.

Assume that $\mathcal{N}\in S_t$. Define $r=m-t$ and $\Gamma(u)=\{v\in V|(u,v)\in G_{\mathcal{N}_r}\}$. By definition of Δ_r, this allows us to write

$$\Delta_r=\sum_{u\in V}d_u^{+(r)}\sum_{v\in\Gamma(u)\cup\{u\}}d_v^{-(r)} \quad\text{and}\quad (m-r)^2=\sum_{u\in V}d_u^{+(r)}\sum_{v\in V}d_v^{-(r)}.$$

Because $\mathcal{N}\in S_t$, equation (5.41) must hold. Inserting the above expressions for Δ_r and $(m-r)$ into this equation yields

$$\sum_{u\in V}d_u^{+(r)}\sum_{v\in\Gamma(u)\cup\{u\}}d_v^{-(r)}>\left(1-\frac{\tau}{2}\right)\sum_{u\in V}d_u^{+(r)}\sum_{v\in V}d_v^{-(r)}>(1-\tau)\sum_{u\in V}d_u^{+(r)}\sum_{v\in V}d_v^{-(r)}.$$

This implies that there exists a vertex $u\in V$ such that

$$d_u^{+(r)}>0 \quad\text{and}\quad \sum_{v\in\Gamma(u)\cup\{u\}}d_v^{-(r)}>(1-\tau)\sum_{v\in V}d_v^{-(r)}=(1-\tau)t. \tag{5.44}$$

Thus we have shown that if $\mathcal{N}\in S_t$, there must exists a vertex u obeying equation (5.44). Hence the probability that $G_{\mathcal{N}_r}$ contains such a vertex u upper bounds $\mathbb{P}\left[\mathcal{N}\in S_t\right]$.

Next we derive an upper bound on the probability that the vertex u obeys equation (5.44). Recall that $G_{\mathcal{N}_r}$ contains the first r edges of the ordering $\mathcal{N}$. Adding the remaining t edges of $\mathcal{N}$, creates the graph $G_{\vec{d}^n}$. Let l denote the number of these t edges with target in $\Gamma(u)\cup\{u\}$, then $l=\sum_{v\in\Gamma(u)\cup\{u\}}d_v^{-(r)}$. Furthermore define $k:=d_u^+-|\Gamma(u)|=d_u^{+(r)}$. Equation (5.44) holds if and only if $k\geq 1$ and $l\geq(1-\tau)t$. To upper bound the probability that u satisfies equation (5.44), we upper bound the probability that $k\geq 1$ and $l\geq(1-\tau)t$ for a random ordering $\mathcal{N}\in S\left(\mathcal{M}\right)$. That $\mathcal{N}$ is a random element of $S\left(\mathcal{M}\right)$ implies that the m edges are known, but their ordering is random. To obtain a fixed value of k, exactly k of the d_u^+ edges with u as source must be in $\mathcal{N}\setminus\mathcal{N}_r$. Choosing these edges determines $\Gamma(u)$. To also obtain the desired value of l, exactly l edges with target in $\Gamma(u)\cup\{u\}$ must be in $\mathcal{N}\setminus\mathcal{N}_r$. There are $\sum_{v\in\Gamma(u)\cup\{u\}}\left(d_v^--1\right)+d_u^-$ edges to choose from, since for each $v\in\Gamma(u)$ the edge with v as the target and u as the source is already in $\mathcal{N}_r$. The remaining $t-l-k$ edges not in $\mathcal{N}_r$ may be chosen freely amongst all edges that do not have u as a source or an element of $\Gamma(u)\cup\{u\}$ as target. Thus the probability to get a specific combination of k and l is

$$\frac{\binom{d_u^+}{k}\binom{\sum_{v\in\Gamma(u)}\left(d_v^--1\right)+d_u^-}{l}\binom{m-d_u^+-\sum_{v\in\Gamma(u)\cup}\left(d_v^--1\right)-d_u^-}{t-l-k}}{\binom{m}{t}}.$$

This allows to upper bound the probability that the vertex u satisfies equation (5.44) by

$$\sum_{k\geq 1, l\geq (1-\tau)t} \frac{\binom{d_u^+}{k}\binom{(d_u^+-k+1)d_{\max}}{l}\binom{m-d_u^+-\sum_{v\in\Gamma(u)}\left(d_v^- -1\right)-d_u^-}{t-l-k}}{\binom{m}{t}}.$$

For $\mathcal{N}\in S_t$ there needs to be some vertex satisfying equation (5.44). As in the above argument we fixed a random vertex u, we find:

$$\mathbb{P}\left[\mathcal{N}\in S_t\right] \leq \sum_{u\in V}\sum_{k\geq 1, l\geq (1-\tau)t} \frac{\binom{d_u^+}{k}\binom{(d_u^+-k+1)d_{\max}}{l}\binom{m-d_u^+-\sum_{v\in\Gamma(u)}\left(d_v^- -1\right)-d_u^-}{t-l-k}}{\binom{m}{t}}.$$

Remark that $\binom{m}{k}\leq \frac{m^k}{k!}$. Furthermore, as $t=\mathcal{O}\left(d_{\max}^2\right)$ and $\mathcal{O}\left(d_{\max}^4\right)=o(m)$ there holds

$$\binom{m}{t} = \left[1+o(1)\right]\frac{m^t}{t!}.$$

This yields

$$\begin{aligned}\mathbb{P}\left[S_t\right] &\leq \sum_{u\in V}\sum_{k\geq 1, l\geq (1-\tau)t} \left[1+o(1)\right]\frac{{d_u^+}^k\left((d_u^+-k+1)(d_{\max})\right)^l m^{t-l-k}t!}{m^t k!l!(m-l-k)!}\\ &= \sum_{u\in V}\sum_{k\geq 1, l\geq (1-\tau)t} \left[1+o(1)\right]\frac{\left(\frac{d_u^+}{m}\right)^k\left(\frac{(d_u^+-k+1)d_{\max}}{m}\right)^l t!}{k!l!(m-l-k)!}.\end{aligned}$$

Next we approximate the summation over k and l. Since adding t edges completes the ordering: $\sum_{u\in V} {d_u^+}^{(r)} = \sum_{u\in V} {d_u^-}^{(r)} = t$. This implies that $k\in\{1,2,\dots t\}$ and that l is an integer in the interval $[(1-\tau)t, t]$. Thus the summation consists of at most $t\tau$ terms. Remark that as $l,k\leq t=\mathcal{O}\left(d_{\max}^2\right)=\mathcal{O}\left(m^{1/2}\right)$, $\left(\frac{d_u^+}{m}\right)=\mathcal{O}\left(\frac{1}{m^{3/4}}\right)$ and $((d_u^+-k+1)(d_{\max}))=\mathcal{O}\left(\frac{1}{m^{1/2}}\right)$, the term inside the summation is maximal for $k=1$ and $l=(1-\tau)t$. This gives

$$\begin{aligned}\mathbb{P}\left[S_t\right] &\leq \left[1+o(1)\right]\tau t\sum_{u\in V}\left(\frac{d_u^+}{m}\right)\left(\frac{d_u^+ d_{\max}}{m}\right)^{(1-\tau)t}\binom{t}{t\tau}\\ &\leq \left[1+o(1)\right]2^t t\left(\frac{d_{\max}^2}{m}\right)^{(1-\tau)t}\sum_{v\in V}\left(\frac{d_u^+}{m}\right)\\ &\leq \left[1+o(1)\right]2^t t\left(\frac{d_{\max}^2}{m}\right)^{(1-\tau)t}.\end{aligned}$$

Here we used that $\tau\leq\frac{1}{3}$, $\binom{m}{k}\leq 2^m$ and $\sum_{u\in V} d_u^+ = m$. Plugging this into equation (5.43) yields

$$\mathbb{P}\left[\mathcal{N}\in S_t\right]^{1-\tau} t^t(t+1) \leq \left[1+o(1)\right]t^t(t+1)\left(2^t t\left(\frac{d_{\max}^2}{m}\right)^{(1-\tau)t}\right)^{1-\tau}.$$

As $t\leq\frac{2d_{\max}^2}{2-\tau}$, there holds

$$\mathbb{P}\left[\mathcal{N}\in S_t\right]^{1-\tau} t^t(t+1) \leq \left[1+o(1)\right](t+1)t^{1-\tau}\left(\frac{2\cdot 2^{1-\tau}}{2-\tau}\frac{d_{\max}^{4-4\tau+2\tau^2}}{m^{1-2\tau+\tau^2}}\right)^t.$$

Since $\tau\leq\frac{1}{3}$, for any $x\geq 1$, $x^{1-\tau}\leq x$. Thus we find

$$\mathbb{P}\left[\mathcal{N}\in S_t\right]^{1-\tau} t^t(t+1) \leq \left[1+o(1)\right](t+1)t\left(\frac{4}{2-\tau}\frac{d_{\max}^{4-4\tau+2\tau^2}}{m^{1-2\tau+\tau^2}}\right)^t.$$

Inserting the relation $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$ yields

$$\mathbb{P}\left[\mathcal{N} \in S_t\right]^{1-\tau} t^t(t+1) \leq \left[1+o(1)\right](t+1)t\left(\frac{4}{2-\tau}m^{-3\tau+3.5\tau^2-3\tau^3}\right)^t.$$

Using $t = o\left(m^{1/2}\right)$ and that $\frac{4}{2-\tau}$ is constant whereas m goes to infinity with n

$$\begin{aligned}\mathbb{P}\left[\mathcal{N} \in S_t\right]^{1-\tau} t^t(t+1) &\leq \left[1+o(1)\right] o\left(m^{1/2}\right)\mathcal{O}\left(m^{-3\tau+3.5\tau^2-3\tau^3}\right)^t \\ &= \mathcal{O}\left(m^{-\tau t}\right).\end{aligned}$$

This completes the proof of equation (5.42) and hence it shows that equation (5.23) holds. This completes the prove of equation (5.14) and hence shows that equation (5.3). Together with the results from Section 5.2.1, 5.2.3 this completes the proof of Theorem 5.2.

5.3 The probability of failure of Algorithm 1

The next step in proving Theorem 5.1, is showing that Algorithm 1 terminates successfully with probability $1-o(1)$. We prove the equivalent statement: the probability the algorithm fails is $o(1)$. The proof is based on [5, Section 5]. The algorithm fails at step s, if every pair of an unmatched in-stub with an unmatched out-stub is unsuitable, i.e. it creates a self-loop or double edge when the edge corresponding to the pair is added to $G_{\mathcal{N}_s}$. First we investigate for which steps $s \in \{0,1,\ldots,m-1\}$ the algorithm can fail. Then we upper bound the number of vertices that have unmatched stubs left when the algorithm fails. This allows to determine the probability that the algorithm fails with a given amount of unmatched stubs left. Combining all these results, we show that the probability the algorithm fails is $o(1)$.

First determine at which steps that algorithm might fail.

Lemma 5.18. *If Algorithm 1 fails at step s, then $m-s \leq d_{\max}^2$.*

Proof. At step s, there are $(m-s)^2$ pair of unmatched stubs. If the algorithm fails at step s, all these pairs are unsuitable. The number of unsuitable pairs at step s is Δ_s. By Lemma 5.3 there holds $\Delta_s \leq d_{\max}^2(m-s)$. Thus if the algorithm fails at step s, there must hold $(m-s)^2 \leq d_{\max}^2(m-s)$. □

The number of vertices that have unmatched stubs when the algorithm fails is bounded as well. Suppose a vertex $v \in V$ has unmatched in-stub(s) left when the algorithm fails. Since the number of unmatched in-stubs equals the number of unmatched out-stubs, this implies that there are also unmatched out-stubs. Because the algorithm fails, any pair of an unmatched in-stub and an unmatched out-stub induces a double edge or self-loop. Hence only v and vertices that are the source of an edge with v as target can have unmatched out-stub(s). As v has at least one unmatched in-stub, there are at most $d_{\max}-1$ edges with v as target. Thus at most $d_{\max}$ vertices have unmatched out-stub(s). By a symmetric argument it is shown that at most $d_{\max}$ vertices have unmatched in-stub(s) when a failure occurs.

Let $A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}$ be the event that the algorithm fails at step s with the only the vertices having unmatched in-stubs being $v_{i_1},\ldots,v_{i_{k^-}}$, each having $d_{i_l}^{-(s)}$ unmatched in-stubs, and the only vertices having unmatched out-stubs being $v_{j_1},\ldots,v_{j_{k^+}}$, each having $d_{j_l}^{+(s)}$ unmatched out-stubs. As k^- (respectively k^+) denotes the number of vertices with unmatched in-stubs(out-stubs) left, there holds $k^-, k^+ \leq d_{\max}$. This allows to write the probability that Algorithm 1 fails as

$$\mathbb{P}\left[\,\text{failure}\,\right] = \sum_{m-s=1}^{d_{\max}^2}\ \sum_{k^-=1}^{\max(m-s,d_{\max})}\ \sum_{k^+=1}^{\max(m-s,d_{\max})}\ \sum_{i_1,\ldots,i_{k^-}=1}^{n}\ \sum_{j_1,\ldots,j_{k^+}=1}^{n}\mathbb{P}\left[A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}\right]. \tag{5.45}$$

Here the summation $\sum_{i_1,\dots,i_{k^-}=1}^{n}$ denotes the sum over all possible subsets $B \subset \{1,2,\dots,n\}$ of size k^- such that $\sum_{i\in B} d_i^{-(s)} = m-s$ and $\sum_{i\notin B} d_i^{-(s)} = 0$. The goal is to show that $\mathbb{P}[\,\text{failure}\,] = o(1)$. To achieve this, first we determine an upper bound for $\mathbb{P}\left[A_{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)},d_{j_1}^{+(s)},\dots d_{j_{k^+}}^{+(s)}}\right]$.

Lemma 5.19. *The probability of the event $A_{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)},d_{j_1}^{+(s)},\dots d_{j_{k^+}}^{+(s)}}$ is upper bounded by*

$$e^{o(1)} d_{\max}^{2k^+k^- - 2k^\pm} \frac{\prod_{i\in K^+} {d_i^+}^{d_i^{+(s)}} \prod_{i\in K^-} {d_i^-}^{d_i^{-(s)}}}{m^{k^+k^- - k^\pm} m^{m-s} m^{m-s}} \binom{m-s}{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)}} \binom{m-s}{d_{j_1}^{+(s)},\dots,d_{j_{k^+}}^{+(s)}}. \tag{5.46}$$

Proof. For notational convenience define

$$K^- = \{i_1, i_2, \dots i_{k^-}\}, \quad K^+ = \{j_1, j_2, \dots j_{k^+}\} \quad \text{and} \quad K^\pm = K^- \cap K^+.$$

When the event $A_{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)},d_{j_1}^{+(s)},\dots d_{j_{k^+}}^{+(s)}}$ occurs, the algorithm has constructed a partial graph $G_{\mathcal{M}_s}$. This partial graph obeys the degree sequence $\widetilde{\vec{d^n}}$ defined by

$$\widetilde{d_i^-} = \begin{cases} d_i^- & \text{if } i \notin K^- \\ d_i^- - d_i^{-(s)} & \text{if } i \in K^- \end{cases}, \quad \widetilde{d_i^-} = \begin{cases} d_i^+ & \text{if } i \notin K^+ \\ d_i^+ - d_i^{+(s)} & \text{if } i \in K^+ \end{cases}.$$

The probability of $A_{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)},d_{j_1}^{+(s)},\dots d_{j_{k^+}}^{+(s)}}$ equals the number of graphs $G_{\mathcal{M}_s}$ obeying the degree sequence $\widetilde{\vec{d^n}}$ leading to a failure times the probability that the algorithm constructs this partial graph. To upper bound the number of graphs obeying $\widetilde{\vec{d^n}}$ leading to a failure, note that such a graph must contain the edge (i,j) for all $i \in K^+, j \in K^-, i \neq j$. Thus a graph obeying $\widetilde{\vec{d^n}}$ leading to a failure, contains a subgraph obeying the degree sequence $\overline{d_{K^-,K^+}}^{(s)}$, which is defined by

$$\overline{d_i^-}^{(s)} = \begin{cases} d_i^- & \text{if } i \notin K^- \\ d_i^- - d_1^{-(s)} - k^+ & \text{if } i \in K^-, i \notin K^+ \\ d_i^- - d_1^{-(s)} - k^+ + 1 & \text{if } i \in K^-, i \in K^+ \end{cases} \quad \text{and} \quad \overline{d_i^+}^{(s)} = \begin{cases} d_i^+ & \text{if } i \notin K^+ \\ d_i^+ - d_1^{+(s)} - k^- & \text{if } i \in K^+, i \notin K^- \\ d_i^+ - d_1^{+(s)} - k^- + 1 & \text{if } i \in K^+, i \in K^- \end{cases}.$$

The number of graphs obeying the degree sequence $\overline{d_{K^-,K^+}}^{(s)}$ upper bounds the number of partial graphs inducing the event $A_{d_{i_1}^{-(s)},\dots,d_{i_{k^-}}^{-(s)},d_{j_1}^{+(s)},\dots d_{j_{k^+}}^{+(s)}}$. Denote by $\mathcal{L}\left(\vec{d^n}\right)$ the space of simple graphs obeying the degree sequence $\vec{d^n}$. As the uniform distribution on a set S assigns each element a probability of $\frac{1}{|S|}$, Theorem 5.2 implies that for any degree sequence d with $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$ there holds

$$|\mathcal{L}(d)| \leq \frac{\prod_{r=0}^{m-1}(m-r)^2}{m! \prod_{i=1}^n d_i^+! \prod_{i=1}^n d_i^-!} e^{-\frac{\sum_{i=1}^n d_i^- d_i^+}{m} + \frac{\sum_{i=1}^n (d_i^-)^2 + (d_i^+)^2}{2m} - \frac{\sum_{i=1}^n (d_i^-)^2 \sum_{i=1}^n (d_i^+)^2}{4m^2} - \frac{1}{2} + o(1)}. \tag{5.47}$$

We want to apply this to the degree sequence $\overline{d_{K^-,K^+}}^{(s)}$. A graph obeying this degree sequence has $s - k^+k^- + k^\pm$ edges, with $k^\pm = |K^\pm|$. Thus we must show that $d_{\max} = \mathcal{O}\left((s - k^-k^+ + k^\pm)^{1/4-\tau}\right)$. Lemma 5.18 states that $m - s \leq d_{\max}^2$, implying $m \leq d_{\max}^2 + s$. Combining this with $d_{\max}^4 = o(m)$ yields $d_{\max}^2\left(d_{\max}^2 - 1\right) \leq s$. For $d_{\max} > 1$ this implies $s > 3d_{max}^2$. As $k^+k^- \leq d_{\max}^2$, we now find $m < 2(s - k^-k^+ + k^\pm)$, i.e. $m = \mathcal{O}(s - k^-k^+ + k^\pm)$. If $d_{\max} = 1$, then $m \leq s+1$ and $k^+k^- \leq 1$. As the algorithm cannot fail at the first step, $s > 0$. This implies that $m \leq 3(s - k^+k^- + k^\pm)$. Now $m = \mathcal{O}(s - k^-k^+k^\pm)$ implies that $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right) = \mathcal{O}\left((s - k^-k^+k^\pm)^{1/4-\tau}\right)$. Thus we may apply equation (5.47) to $\overline{d_{K^-,K^+}}^{(s)}$. This

yields

$$\left|\mathcal{L}\left(\overline{d_{K^-,K^+}}^{(s)}\right)\right| \leq \frac{(s-k^+k^-+k^\pm)!}{\prod_{i=1}^n \overline{d_i^+}^{(s)}!\prod_{i=1}^n \overline{d_i^-}^{(s)}!} \exp\left(-\frac{\sum_{i=1}^n \overline{d_i^-}^{(s)}\overline{d_i^+}^{(s)}}{s-k^+k^-+k^\pm} + \frac{\sum_{i=1}^n\left[(\overline{d_i^-}^{(s)})^2+(\overline{d_i^+}^{(s)})^2\right]}{2\left(s-k^+k^-+k^\pm\right)} - \frac{\sum_{i=1}^n(\overline{d_i^-}^{(s)})^2\sum_{i=1}^n(\overline{d_i^+}^{(s)})^2}{4\left(s-k^+k^-+k^\pm\right)^2} - \frac{1}{2} + o(1)\right).$$

Next the probability $\mathbb{P}_A\left(G_{\mathcal{M}_s}\right)$ that the algorithm constructs a partial graph $G_{\mathcal{M}_s}$ is upper bounded. This will only depend on the degree sequence of the partial graph and not on its edges. Following the derivation in Sections 5.2.1 and 5.2.3 we find

$$\begin{aligned}
\mathbb{P}_A\left(G_{\mathcal{M}_s}\right) &= \frac{\prod_{i=1}^n d_i^+!\prod_{i=1}^n d_i^-!}{\prod_{i\in K^+} d_i^{+(s)}!\prod_{i\in K^-} d_i^{-(s)}!}\sum_{\mathcal{N}_s\in S(\mathcal{M}_s)}\mathbb{P}_A\left(\mathcal{N}_s\right)\\
&= \frac{\prod_{i=1}^n d_i^+!\prod_{i=1}^n d_i^-!}{\prod_{i\in K^+} d_i^{+(s)}!\prod_{i\in K^-} d_i^{-(s)}!}s!\exp\left(-\sum_{(i,j)\in G_{\mathcal{M}_s}}\frac{d_i^+d_j^-}{2m}+o(1)\right)\prod_{r=0}^{s-1}\frac{1}{(m-r)-\psi_r\left(G_{\mathcal{M}_s}\right)}\\
&= \frac{\prod_{i=1}^n d_i^+!\prod_{i=1}^n d_i^-!}{\prod_{i\in K^+} d_i^{+(s)}!\prod_{i\in K^-} d_i^{-(s)}!}s!\prod_{r=0}^{s-1}\frac{1}{(m-r)^2}\cdot\\
&\quad\exp\left(\frac{s\sum_{i=1}^n d_i^-d_i^+}{m^2}-\frac{s^2\sum_{i=1}^n\left[(d_i^-)^2+(d_i^+)^2\right]}{2m^3}+\frac{s\sum_{i=1}^n(d_i^-)^2\sum_{i=1}^n(d_i^+)^2}{4m^3}+\frac{s^2}{2m^2}+o(1)\right).
\end{aligned}$$

The fraction of factorials accounts for the number of different configurations leading to the same graph $G_{\mathcal{M}_s}$. This equals6 the number of permutations of the stub labels. However for $i\in K^-$ there are only $\frac{d_i^-!}{d_i^{-(s)}!}$ permutations of the labels of the in-stubs of v_i that lead to a different configuration. To see this remark that changing the label of an in-stub that remains unmatched with another in-stub that remains unmatched does not change the configuration. By the same argument for $i\in K^+$ there are only $\frac{d_i^+!}{d_i^{+(s)}!}$ ways to permute the labels of the out-stubs of v_i.

Now we can determine

$$\mathbb{P}\left[A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}\right] \leq \mathbb{P}\left[G_{\mathcal{M}_s}\right]\left|\mathcal{L}\left(\bar{d}_{k^-,k^+}^{(s)}\right)\right|.$$

First we examine the product of the exponents in the approximations of $\mathbb{P}\left[G_{\mathcal{M}_s}\right]$ and $\left|\mathcal{L}\left(\bar{d}_{k^-,k^+}^{(s)}\right)\right|$. Hence we consider

$$\begin{aligned}
&\exp\left(-\frac{\sum_{i=1}^n \overline{d_i^-}^{(s)}\overline{d_i^+}^{(s)}}{s-k^+k^-+k^\pm} + \frac{\sum_{i=1}^n\left[(\overline{d_i^-}^{(s)})^2+(\overline{d_i^+}^{(s)})^2\right]}{2\left(s-k^+k^-+k^\pm\right)} - \frac{\sum_{i=1}^n(\overline{d_i^-}^{(s)})^2\sum_{i=1}^n(\overline{d_i^+}^{(s)})^2}{4\left(s-k^+k^-+k^\pm\right)^2} - \frac{1}{2} + o(1)\right)\\
&\quad\exp\left(\frac{s\sum_{i=1}^n d_i^-d_i^+}{m^2}-\frac{s^2\sum_{i=1}^n\left[(d_i^-)^2+(d_i^+)^2\right]}{2m^3}+\frac{s\sum_{i=1}^n(d_i^-)^2\sum_{i=1}^n(d_i^+)^2}{4m^3}+\frac{s^2}{2m^2}+o(1)\right)\\
&\quad= \exp\left(\frac{s}{m}\mathcal{O}\left(d_{\max}\right)+\frac{s}{m}\mathcal{O}\left(d_{\max}^2\right)+o(1)\right)\exp\left(-\mathcal{O}\left(d_{\max}\right)-\mathcal{O}\left(d_{\max}^2\right)+o(1)\right)\\
&= \exp\left[\mathcal{O}\left(d_{\max}^2\left(\frac{s}{m}-1\right)\right)\right] = \exp\left[\mathcal{O}\left(d_{\max}^2\left(\frac{m-d_{\max}^2}{m}-1\right)\right)\right] = \exp\left[\mathcal{O}\left(\frac{d_{\max}^4}{m}\right)\right] = e^{o(1)}.
\end{aligned}$$

Here we used that $m > s \geq m - d_{\max}^2$. This leads to

$$\begin{aligned}
&\mathbb{P}\left[A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}\right] \leq \mathbb{P}\left[G_{\mathcal{M}_s}\right]\left|\mathcal{L}\left(\bar{d}^{(s)}_{k^-,k^+}\right)\right| \\
&\leq e^{o(1)}\frac{\prod_{i\in K^+} d_i^+!\prod_{i\in K^-} d_i^-!\prod_{i\in K^+,i\in K^-}\left(d_i^+ - d_i^{+(s)} - k^-\right)\left(d_i^- - d_i^{-(s)} - k^+\right)}{\prod_{i\in K^+}\left(d_i^+ - d_i^{+(s)} - k^-\right)!d_i^{+(s)}!\prod_{i\in K^-}\left(d_i^- - d_i^{-(s)} - k^+\right)!d_i^{-(s)}!}\frac{(s-k^+k^-+k^\pm)!s!(m-s)!(m-s)!}{m!m!} \\
&\leq e^{o(1)}\frac{\prod_{i\in K^+} d_i^{+\,d_i^{+(s)}+k^-}\prod_{i\in K^-} d_i^{-\,d_i^{-(s)}+k^+}}{\prod_{i\in K^+,i\in K^-} d_i^+ d_i^-}\frac{1}{\prod_{j=0}^{k^+k^--k^\pm+1} s-j}\frac{s!s!}{m!m!}\binom{m-s}{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)}}\binom{m-s}{d_{j_1}^{+\,(s)},\ldots,d_{j_{k^+}}^{+\,(s)}} \\
&\leq e^{o(1)}d_{\max}^{2k^+k^--2k^\pm}\prod_{i\in K^+} d_i^{+\,d_i^{+(s)}}\prod_{i\in K^-} d_i^{-\,d_i^{-(s)}}\frac{1}{\prod_{j=0}^{k^+k^--k^\pm+1} s-j}\frac{s!s!}{m!m!}\binom{m-s}{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)}}\binom{m-s}{d_{j_1}^{+\,(s)},\ldots,d_{j_{k^+}}^{+\,(s)}}.
\end{aligned}$$

To get this upper bound equal to equation (5.46), it remains to bound $\frac{s!}{m!}$ and $\frac{\prod_{j=0}^{k^+k^--k^\pm+1} s-j}{m^{k^+k^--k^\pm}}$. First look at $\frac{s!}{m!}$. Using that $m - s = \mathcal{O}\left(d_{\max}^2\right)$, we find

$$\begin{aligned}
\frac{m!}{s!} &= (s+1)(s+2)\cdots(m-1)m = m^{m-s}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{m-s-1}{m}\right) \\
&= m^{m-s}\left(1-\prod_{i=1}^{m-s-1}\frac{i}{m}+\mathcal{O}\left((m-s)^2\frac{(m-s)^2}{m^2}\right)\right) \\
&\geq m^{m-s}e^{-\sum_{i=1}^{m-s-1}\frac{i}{m}+\mathcal{O}\left(\frac{d_{\max}^8}{m^2}\right)} = m^{m-s}e^{-\frac{(m-s)(m-s-1)}{2m}+\mathcal{O}\left(\frac{d_{\max}^8}{m^2}\right)} \\
&= m^{m-s}e^{-\mathcal{O}\left(\frac{d_{\max}^4}{m}\right)}.
\end{aligned}$$

This implies

$$\frac{s!}{m!} \leq \frac{1}{m^{m-s}}e^{\mathcal{O}\left(\frac{d_{\max}^4}{m}\right)} = \frac{1}{m^{m-s}}e^{o(1)}.$$

Next consider $\frac{1}{\prod_{j=0}^{k^+k^--k^\pm+1} s-j}$. Using that $m - s \leq d_{\max}^2, k^+, k^- \leq d_{\max}$ and $0 \leq k^\pm \leq \min\left(k^-, k^+\right)$, we obtain

$$\begin{aligned}
\prod_{j=0}^{k^+k^--k^\pm+1} s-j &\geq \prod_{j=0}^{k^+k^--k^\pm+1} m-d_{\max}^2-j = m^{k^+k^--k^\pm}\prod_{j=0}^{k^+k^--k^\pm+1}\left(1-\frac{d_{\max}^2+j}{m}\right) \\
&= m^{k^+k^--k^\pm}\left(1-\prod_{j=1}^{k^+k^--k^\pm+1}\frac{d_{\max}^2+j}{m}+\mathcal{O}\left(\frac{d_{\max}^8}{m^2}\right)\right) \\
&\geq m^{k^+k^--k^\pm}e^{-\frac{(d_{\max}^2+k^+k^-+k^\pm+1)(d_{\max}^2+k^+k^--+k^\pm+2)}{2m}+\mathcal{O}\left(\frac{d_{\max}^8}{m^2}\right)} \\
&= m^{k^+k^--k^\pm}e^{-\mathcal{O}\left(\frac{d_{\max}^4}{m}\right)}.
\end{aligned}$$

This implies

$$\frac{1}{\prod_{j=0}^{k^+k^--k^\pm+1} s-j} \leq \frac{1}{m^{k^+k^--k^\pm}}e^{\mathcal{O}\left(\frac{d_{\max}^4}{m}\right)} = \frac{1}{m^{k^+k^--k^\pm}}e^{o(1)}.$$

Thus the upper bound on the probability of $A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}$ becomes

$$e^{o(1)}d_{\max}^{2k^+k^--2k^\pm}\frac{\prod_{i\in K^+} d_i^{+\,d_i^{+(s)}}\prod_{i\in K^-} d_i^{-\,d_i^{-(s)}}}{m^{k^+k^--k^\pm}m^{m-s}m^{m-s}}\binom{m-s}{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)}}\binom{m-s}{d_{j_1}^{+\,(s)},\ldots,d_{j_{k^+}}^{+\,(s)}}.$$

□

Combining equation (5.45) with the Lemma 5.19, we are able to show the desired result.

Lemma 5.20. *The probability that Algorithm 1 returns a failure is* $o(1)$.

Proof. Lemma 5.19 implies that

$$\mathbb{P}\left[A_{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)},d_{j_1}^{+\,(s)},\ldots d_{j_{k^+}}^{+\,(s)}}\right] \leq e^{o(1)} d_{\max}^{2k^+k^- - 2k^\pm} \frac{\prod_{i\in K^+} {d_i^+}^{d_i^{+\,(s)}} \prod_{i\in K^-} {d_i^-}^{d_i^{-\,(s)}}}{m^{k^+k^- - k^\pm} m^{m-s} m^{m-s}} \binom{m-s}{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)}} \binom{m-s}{d_{j_1}^{+\,(s)},\ldots,d_{j_{k^+}}^{+\,(s)}}.$$

The fraction $\left(\frac{d_{\max}^2}{m}\right)^{k^+k^- - k^\pm}$ is either 1 if $k^+k^- = k^\pm$ or smaller than $\frac{d_{\max}^2}{m}$ if $k^+k^- \neq k^\pm$. As $k^\pm \leq \min(k^-, k^+)$, $k^+k^- = k^\pm$ implies that $k^+ = k^- = 1$. Together $k^+ = k^- = 1$ and the conditions under which the algorithm can fail imply that $K^+ = K^-$. First we consider this case. As $K^+ = K^- = K^\pm = 1$ there holds $d_{i_1}^{-\,(s)} = d_{i_1}^{+\,(s)} = m - s$, hence we find

$$\mathbb{P}\left[A_{d_{i_1}^{-\,(s)},d_{i_1}^{+\,(s)}}\right] \leq e^{o(1)} \frac{{d_{i_1}^+}^{m-s} {d_{i_1}^-}^{m-s}}{m^{m-s} m^{m-s}} = o(1).$$

Next assume that $k^+k^- \neq k^\pm$. This implies that $\left(\frac{d_{\max}^2}{m}\right)^{k^+k^- - k^\pm} \leq \frac{d_{\max}^2}{m}$. By definition of the summations $\sum_{i_1,\ldots,i_{k^-}=1}^{n}$ and $\sum_{j_1,\ldots,j_{k^+}=1}^{n}$, we can apply the multinomial theorem to obtain

$$\sum_{k^-=1}^{\max(m-s,d_{\max})} \sum_{i_1,\ldots,i_{k^-}=1}^{n} \prod_{i\in K^-} {d_i^-}^{d_i^{-\,(s)}} \binom{m-s}{d_{i_1}^{-\,(s)},\ldots,d_{i_{k^-}}^{-\,(s)}} = \left(d_1^- + \ldots + d_n^-\right)^{m-s}$$

and

$$\sum_{k^+=1}^{\max(m-s,d_{\max})} \sum_{j_1,\ldots,j_{k^+}=1}^{n} \prod_{i\in K^+} {d_i^+}^{d_i^{+\,(s)}} \binom{m-s}{d_{j_1}^{+\,(s)},\ldots,d_{j_{k^+}}^{+\,(s)}} = \left(d_1^+ + \ldots + d_n^+\right)^{m-s}.$$

Plugging this into equation (5.45) yields

$$\begin{aligned}\mathbb{P}\left[\,\text{failure}\,\right] &\leq o(1) + e^{o(1)} \frac{d_{\max}^2}{m} \sum_{m-s=1}^{d_{\max}^2} \frac{\left(d_1^+ + \ldots d_n^+\right)^{m-s} \left(d_1^- + \ldots d_n^-\right)^{m-s}}{m^{m-s} m^{m-s}} \\ &\leq o(1) + \frac{d_{\max}^2}{m} \sum_{m-s=1}^{d_{\max}^2} 1 = o(1) + \mathcal{O}\left(\frac{d_{\max}^4}{m}\right) = o(1).\end{aligned}$$

□

This proves the claim of Theorem 5.1 on the probability that Algorithm 1 fails.

5.4 Running time Algorithm 1

All that is left to complete the proof of Theorem 5.1, is to show that the expected running time of Algorithm 1 is $\mathcal{O}(md_{\max})$. For this we follow [5, Section 6]

Lemma 5.21. *Algorithm 1 can be implemented so that its expected running time is* $\mathcal{O}(md_{\max})$ *for graphical degree sequences* $\vec{d^n}$ *with* $d_{\max} = \mathcal{O}\left(m^{1/4-\tau}\right)$ *for some* $\tau > 0$.

Proof. Our implementation of Algorithm 1 is based on the implementation that Bayati, Kim and Saberi [5] use to generate undirected graphs. This implementation in turn is based on Steger and Wormald' implementation of their algorithm to generate undirected regular random graphs. For the sake of completeness we include the entire analysis bellow. Steger and Wormald invented the three phase procedure to pick an edge

(i,j) at step r with probability proportional to ${d_i}^{(r)}{d_j}^{(r)}$. Bayati, Kim and Saberi changed the acceptance criteria so that each edge is accepted with probability proportional to ${d_i}^{(r)}{d_j}^{(r)}\left(1-\frac{d_i d_j}{4m}\right)$. We change this acceptance probability to $d_i^{+(r)}d_j^{-(r)}\left(1-\frac{d_i^+ d_j^-}{2m}\right)$. Also do we change the criteria which determine the phase of the algorithm.

For notional convenience let E denote the set of edges constructed by the algorithm so far. Let N be an array containing for each vertex v: the set of vertices w such that $(v,w)\in E$. This array allows to determine all vertices w for v in time $\mathcal{O}(d_{\max})$. It can be updated in constant time.

In the first phase a random unmatched in-stub and a random unmatched out-stub are selected. Using the array N it can be checked in $\mathcal{O}(d_{\max})$ whether this is an eligible pair (i.e. if it does not create a self-loop or double edge when added to E). If eligible, the pair is accepted with probability $1-\frac{d_i^+ d_j^-}{2m}$ and the edge (i,j) is added to E. Select edges according to this procedure, until the number of unmatched in-stubs (which equals the number of unmatched out-stubs) drops bellow $2d_{\max}^2$. This marks the end of phase 1. Checking if a pair is eligible requires $\mathcal{O}(d_{\max})$ comparisons. Each eligible pair is accepted with probability at least $\frac{1}{2}$. At most one half of the stub pairs is ineligible. To see this recall that Lemma 5.3 (a) implies that at most $(m-r)d_{\max}^2$ of the $(m-r)^2$ pairs are ineligible. Thus as long as $(m-r)>2d_{\max}^2$ at least half of the stub pairs is eligible. Hence creating one edge in phase 1 has an expected computational complexity of $\mathcal{O}(d_{\max})$, making the total expected runtime of this phase $\mathcal{O}(md_{\max})$.

When phase 1 ends, phase 2 begins. In this phase we select a pair of vertices instead of a pair of stubs. This requires us to keep track of the vertices which have unmatched in-stubs left and of vertices with unmatched out-stubs left. These sets are constructed in $\mathcal{O}(n)$ and can be updated in constant time. Draw a uniformly random vertex j from the set of vertices with unmatched in-stubs and a uniformly random vertex i from the set of vertices with unmatched out-stubs. Accept i (respectively j) with probability $\frac{d_i^{+(r)}}{d_i^{+(r)}}\left(\frac{d_j^{-(r)}}{d_j^{-(r)}}\right)$. If both vertices are accepted, it is checked if (i,j) is an eligible edge. This can still be done in $\mathcal{O}(d_{\max})$. If the edge is eligible, it is accepted with probability $1-\frac{d_i^+ d_j^-}{2m}$. Phase 2 ends when the number of vertices with unmatched in-stubs or the number of vertices with unmatched out-stubs is less than $2d_{\max}$. As every vertex with unmatched in-stubs(respectively out-stubs) has at most $d_{\max}$ unmatched in-stubs(out-stubs), this assures that the edge is eligible with probability at least $\frac{1}{2}$. An eligible edge is accepted with probability at least $\frac{1}{2}$. To get a pair of accepted vertices an expected number of $\mathcal{O}\left(d_{\max}^2\right)$ redraws are needed. Thus the construction of one edge is expected to take $\mathcal{O}\left(d_{\max}^2\right)$. As there are only $2d_{\max}^2$ unmatched in-stubs at the start of phase 2, at most $d_{\max}^2$ edges are created in this phase. Thus the expected running time of phase 2 is $\mathcal{O}\left(d_{\max}^4\right)$.

Phase 3 is the final phase. At the beginning of this phase, the set of all remaining eligible edges is created. Denote this set of edges by $\widetilde{E}$. At the start of phase 3 there are only $2d_{\max}$ vertices left with unmatched in-stubs or with unmatched out-stubs. Hence there are at most $2d_{\max}^2$ vertices with unmatched out-stubs or in-stubs. Thus $\widetilde{E}$ contains no more than $4d_{\max}^3$ edges. For each possible edge it is checked in time $\mathcal{O}(d_{\max})$ if it does not create a double edge or self-loop. Thus constructing $\widetilde{E}$ takes $\mathcal{O}\left(d_{\max}^4\right)$. The rest of phase 3 consist of picking a random element of $\widetilde{E}$ and accepting it with probability $\frac{d_i^{+(r)}d_j^{-(r)}}{d_i^+ d_j^-}\left(1-\frac{d_i^+ d_j^-}{2m}\right)$. This leads to an expected number of $\mathcal{O}\left(d_{\max}^2\right)$ repetitions to accept one edge. If an edge is accepted, it is removed from $\widetilde{E}$ and the values of $d_i^{+(r)}$ and $d_j^{-(r)}$ are updated. After selecting an element of $\widetilde{E}$, it must be checked if $d_i^{+(r)}>0$ and $d_j^{-(r)}>0$. If this is not the case, the edge is not added to E and removed from $\widetilde{E}$. This continues until $\widetilde{E}$ is empty or $|E|=m$. This has expected running time of order $\mathcal{O}\left(d_{\max}^5\right)$ as there are $\mathcal{O}\left(d_{\max}^3\right)$ edges that are expected to be discarded or accepted in $\mathcal{O}\left(d_{\max}^2\right)$. Thus the total running time of the algorithm becomes

$$\mathcal{O}(md_{\max})+\mathcal{O}(n)+\mathcal{O}\left(d_{\max}^4\right)+\mathcal{O}\left(d_{\max}^5\right).$$

As $d_{\max}=\mathcal{O}\left(m^{1/4-\tau}\right)$ this is $\mathcal{O}(md_{\max})$.

However we also must compute P_{ij} at each step. Let $P_{ij}^{(r)}$ denote the probability that the edge (i,j) is added to E at step r. There holds:

$$P_{ij}^{(r)} = \frac{d_i^{+\,(r)} d_j^{-\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right)}{(m-r)^2 - \Psi_r(\mathcal{N})}.$$

The value $d_i^{+\,(r)} d_j^{-\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right)$ can be computed in constant time. To determine the denominator of this value remark that

$$\begin{aligned}
&\left[(m-r+1)^2 - \Psi_{r+1}(\mathcal{N})\right] - \left[(m-r)^2 - \Psi_r(\mathcal{N})\right] \\
&= \sum_{(u,v)\in E_{r+1}} d_u^{+\,(r+1)} d_v^{-\,(r+1)} \left(1 - \frac{d_u^+ d_v^-}{2m}\right) - \sum_{(u,v)\in E_r} d_u^{+\,(r)} d_v^{-\,(r)} \left(1 - \frac{d_u^+ d_v^-}{2m}\right) \\
&= -d_i^{+\,(r)} d_j^{-\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right) - \sum_{\substack{(i,v)\in E_r \\ v\neq j}} d_v^{-\,(r)} \left(1 - \frac{d_i^+ d_v^-}{2m}\right) - \sum_{\substack{(u,j)\in E_r \\ u\neq i}} d_u^{+\,(r)} \left(1 - \frac{d_u^+ d_j^-}{2m}\right) \\
&= -d_i^{+\,(r)} d_j^{-\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right) - 2(m-r) + \frac{d_i^+}{2m} \sum_{k=1}^n d_k^- d_k^{-\,(r)} + \frac{d_j^-}{2m} \sum_{k=1}^n d_k^+ d_k^{+\,(r)} \\
&+ \sum_{(i,v)\in G_{\mathcal{N}_r}} d_v^{-\,(r)} \left(1 - \frac{d_i^+ d_v^-}{2m}\right) + \sum_{(u,j)\in G_{\mathcal{N}_r}} d_u^{+\,(r)} \left(1 - \frac{d_u^+ d_j^-}{2m}\right) + d_i^{-\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right) + d_j^{+\,(r)} \left(1 - \frac{d_i^+ d_j^-}{2m}\right).
\end{aligned}$$

Each of these terms can be updated at each step in $\mathcal{O}(d_{\max})$ operations. This allows us to determine the value of $P_{ij}^{(r)}$ at each step using $\mathcal{O}(d_{\max})$. As the construction of one edge also takes at least $\mathcal{O}(d_{\max})$ in every phase, this does not change the complexity of the algorithm. The initial value is

$$\Psi_0(\mathcal{N}) = m^2 - \sum_{i=1}^n d_i^- d_i^+ - \frac{\sum_{i=1}^n {d_i^-}^2 \sum_{i=1}^n {d_i^+}^2 - \sum_{i=1}^n {d_i^-}^2 {d_i^+}^2}{2m},$$

which can be computed in $\mathcal{O}(n)$. As $n \leq m$ this does not change the order of the expected running time and hence this completes the proof. □

This lemma completes the proof of Theorem 5.1.

6 Numerical simulations

In this section, we apply the construction algorithm to illustrate Theorem 4.1. Section 6.1 describes the simulation setup and how we numerically determine the percolation threshold and the size of the GSCC. Section 6.2 introduces the sample degree distributions for which the simulations are performed. Then, the results of the simulations are compared with the theory given by Theorem 4.1, which is split into two parts: the bond and site percolation are discussed in respectively Sections 6.3 and 6.4.

6.1 The simulation

The goal of the simulation is to numerically illustrate Theorem 4.1, which identifies the values $\pi_c^{\text{bond}}, \pi_c^{\text{site}}, c^{\text{bond}}$ and c^{site}. We start by choosing the value of π and the type of percolation applied (bond/site). Then, we apply a procedure that generates random graphs obeying a given degree sequence and removes each edge/isolates each vertex with probability $1-\pi$. By analysing the sizes of strongly connected components of the percolated graph, we obtain the estimated value of c^{bond} (or c^{site}) as the relative size of the largest strongly connected component, *i.e.* $\frac{1}{n}\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$. Determining the percolation threshold numerically is a little more involving, as it requires one to decide whether or not a giant component is present by looking at a finite graph. Equation (4.1) defines the percolation threshold to be the supremum over all values of $\pi \in [0,1]$ for which $\mathbb{P}\left[\frac{1}{n}\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right| = 0\right]$ converges to zero as n goes to infinity. However the value of $\frac{1}{n}\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ is lower bounded by $\frac{1}{n}$. As the value of n remains finite in the simulation, we cannot rely on this value being zero. Thus we need another method to numerically determine the percolation threshold. We look at the behaviour of the size of the second largest strongly connected component. As Theorem 4.1 is proven by applying Theorem 3 to the percolated graph, it will contain w.h.p a unique giant strongly connected component for $\pi > \pi_c$ and no giant if $\pi < \pi_c$. So the second largest connected component is expected to reach its maximal relative size at the percolation threshold. If π increases but remains smaller than π_c, this value will grow as more edges lead to larger connected components. However for $\pi > \pi_c$ a giant component arises by connecting smaller components into one larger component. Hence the size of the second largest component will decrease. So to determine the numerical percolation threshold $\widetilde{\pi}^{\text{bond}}$ (or $\widetilde{\pi}^{\text{site}}$), apply percolation for different values of π. The value of the percolation probability that leads to the largest second largest component will be $\widetilde{\pi}^{\text{bond}}$ (or $\widetilde{\pi}^{\text{site}}$).

It remains to explain the procedure to obtain percolated random graphs obeying a given degree sequence. Given the degree sequence, Algorithm 1 can be used to (almost) uniformly sample a graph obeying this degree sequence. Then bond (respectively site) percolation on this graph can be simulated using a random number generator. Generate a random number in $[0,1]$ for each edge (vertex) in the graph. If this is smaller than $1-\pi$, delete the edge (isolate the vertex). The result is the desired percolated graph.
To get more reliable numerical values for $\pi_c^{\text{bond}}, \pi_c^{\text{site}}, c^{\text{bond}}$ and c^{site}, we generate `ngraphs` simple graphs obeying this degree sequence using Algorithm 1. To each graph we apply the percolation procedure `npercolation` times. This leads to `ngraphs·npercolation` values for the size of the largest and second largest strongly connected component. The desired numerical values are determined using the average for the size of the largest and second largest strongly connected component over these `ngraphs·npercolation` graphs.

Above we considered a graph with a given degree sequence. But Theorem 4.1 regards proper degree arrays converging to some degree distribution $(p_{j,k})_{j,k=0}^{\infty}$. As input for our simulations, we choose the degree distribution (a bivariate probability mass function) and the number of vertices n. We then draw n elements from this distribution, representing the in-degree and the out-degree of each vertex. We repeat this process until we obtain graphical degree sequence. We use the following Theorem to check whether a degree sequence is graphical.

Theorem 6.1. *[27, Theorem 2] Consider a degree sequence $\vec{d^n}$ ordered such that for all $i \in \{1,2,\ldots,n-1\}$ there holds $d_i^- \geq d_{i+1}^-$. Then the degree sequence is graphical if and only if $\sum_{i=1}^n d_i^- = \sum_{i=1}^n d_i^+$ and for all*

$1 \leq k \leq n$ with $d_k^- > d_{k+1}^-$ and $n = k$ there holds

$$\sum_{i=1}^{k} \min\left(d_i^+, k-1\right) + \sum_{i=k+1}^{n} \min\left(d_i^+, k\right) \geq \sum_{i=1}^{k} d_i^-.$$

Note that this reduces the number of equations we need to check with respect to Theorem 2.2. As the results should be the same for all different proper degree sequences with the same degree distribution, we consider just one degree sequence for each value of n. This concludes the explanation of the simulation.

6.2 Degree distributions

In this section, we introduce the two sample degree distributions, for which we run simulations to determine $\pi_c^{\text{bond}}, \pi_c^{\text{site}}, c^{\text{bond}}$ and c^{site} numerically. First, let us consider the uniform degree distribution. Denoting the maximum degree of this distribution by Δ, its probability mass function is given by

$$p_{j,k} = \begin{cases} \frac{1}{(\Delta+1)^2-1} & \text{if } j,k \leq \Delta \text{ and } (j,k) \neq (0,0) \\ 0 & \text{else} \end{cases}.$$

This distribution is constant, expect for exclusion of $(0,0)$. Remark that this distribution is almost the product of two univariate uniform distributions on $\{0,1,\ldots,\Delta\}$, expect for assigning $(0,0)$ a probability of 0. The first and first mixed moments of this distribution are

$$\mu = \sum_{j,k=0}^{\Delta} j p_{j,k} = \frac{(\Delta+1)^2 \Delta}{2\left((\Delta+1)^2 - 1\right)} \quad \text{and} \quad \mu_{11} = \sum_{j,k=0}^{\Delta} jk p_{j,k} = \frac{(\Delta+1)^2 \Delta^2}{4\left((\Delta+1)^2 - 1\right)}.$$

Theorem 4.1 may only be applied if $\mu_{11}/\mu > 1$. As there holds

$$\frac{\mu_{11}}{\mu} = \frac{\Delta}{2} > 1,$$

this implies that we need to take $\Delta > 2$. Theorem 4.1 implies that

$$\pi_c^{\text{bond}} = \pi_c^{\text{site}} = \frac{2}{\Delta}.$$

Also the size of the GSCC follows from Theorem 4.1. First consider c^{bond}. This requires the values of $\eta^{-,\text{bond}}$ and $\eta^{+,\text{bond}}$, given by equations (4.27) and (4.28). For the uniform degree distribution they are the unique solutions in $(0,1)$ to

$$\left(1-\eta^{-,\text{bond}}\right) = \sum_{j,k=0}^{\Delta} p_{j,k} \left(\pi\left(1-\eta^{-,\text{bond}}\right) + 1 - \pi\right)^j \quad \text{and} \quad \left(1-\eta^{+,\text{bond}}\right) = \sum_{j,k=0}^{\Delta} p_{j,k} \left(\pi\left(1-\eta^{+,\text{bond}}\right) + 1 - \pi\right)^k.$$

Remark that these equations are in fact the same. Hence it holds that $\eta^{-,\text{bond}} = \eta^{+,\text{bond}}$. The value of $\eta^{-,\text{bond}}$ is determined using a numerical solver. From this value, $\zeta^{-,\text{bond}}, \zeta^{+,\text{bond}}$ and ψ^{bond} are determined using equations (4.29) and (4.30). In turn these values allow us to determine the value of c^{bond} using equation (4.31). From the value of c^{bond}, c^{site} can easily be obtained using equation (4.54). The numerical values of c^{bond} and c^{site} are shown in Sections 6.3 and 6.4 and allow for a direct comparison with the simulation results.

The uniform degree distribution is almost degenerate, i.e. there holds $\mathbb{P}\left[d_v^- = j, d_v^+ = k\right] = \mathbb{P}\left[d_v^- = j\right]\mathbb{P}\left[d_v^+ = k\right]$ except for $(j,k) = 0$. One might argue that a directed graph with a degenerate distribution can be modelled by an undirected graph with degree distribution $\mathbb{P}\left[d_v = l\right] = \sum_{j+k=l} \mathbb{P}\left[d_v^- = j, d_v^+ = k\right]$. To illustrate that our theorem also applies to non-degenerate degree distributions, we also study a modification of the uniform degree distribution for $\Delta = 4$. This modified uniform distribution is not degenerate and the probabilities for

the total degree, irrespective of edge direction, $\mathbb{P}[d_v = l] = \sum_{j+k=l} \mathbb{P}[d_v^- = j, d_v^+ = k]$ are the same as in the uniform degree distribution with $\Delta = 4$. Such a probability mass function is given by

$$p_{0,2} = \frac{1}{24}, \quad p_{0,3} = \frac{1}{24}, \tag{6.1}$$

$$p_{1,0} = \frac{1}{12}, \quad p_{1,1} = \frac{1}{12}, \quad p_{1,3} = \frac{1}{24}, \quad p_{1,4} = \frac{1}{12}, \tag{6.2}$$

$$p_{2,1} = \frac{1}{8}, \quad p_{2,2} = \frac{1}{24}, \quad p_{2,4} = \frac{1}{24}, \tag{6.3}$$

$$p_{3,1} = \frac{1}{8}, \quad p_{3,3} = \frac{1}{12}, \quad p_{3,4} = \frac{1}{12}, \tag{6.4}$$

$$p_{4,1} = \frac{1}{12}, \quad p_{4,4} = \frac{1}{24}. \tag{6.5}$$

A simple calculation shows that

$$\sum_{j,k=0}^{\infty} jp_{j,k} = \sum_{j,k=0}^{\infty} kp_{j,k} = 2.0833 \quad \text{and} \quad \sum_{j,k=0}^{\infty} jkp_{j,k} = 4.4167.$$

This shows that a random graph obeying this degree distribution w.h.p. will contain a GSCC and hence the percolation threshold exists. By Theorem 4.1 the percolation threshold equals

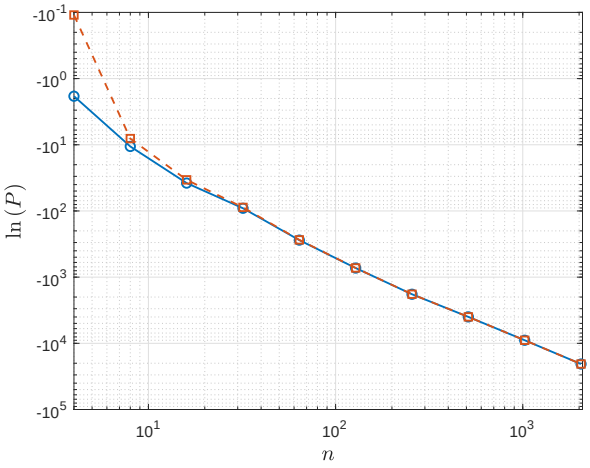
$$\pi_c^{\text{bond}} = \pi_c^{\text{site}} = 0.4717.$$

The values of c^{bond} and c^{site} are determined using the same equation as for the uniform degree distribution. Remark that for the modified degree distribution there holds $\eta^{-,\text{bond}} \neq \eta^{+,\text{bond}}$.

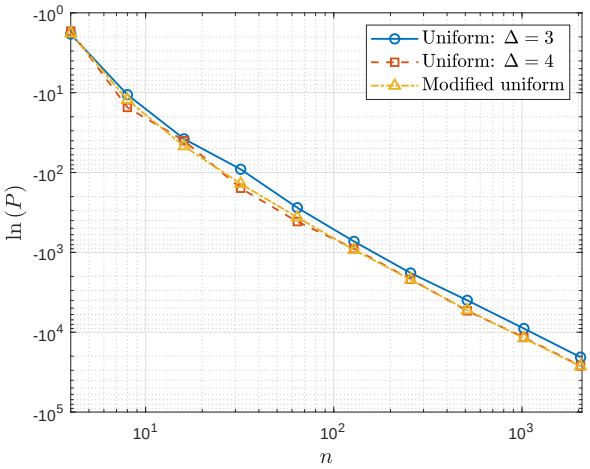
6.3 Simulating bond percolation

In this section, we present the outcomes of the simulations performed regarding bond percolation. As degree distribution we use: the uniform distribution for $\Delta = 3$ and $\Delta = 4$ and the modified uniform distribution, as introduced in Section 6.2. Simulations are performed for graphs with 10000 vertices and for graphs with $n = 2, 4, 8, \ldots, 2048$ vertices. For all values of n, we determine $\widetilde{\pi}^{\text{bond}}$ and $\mathcal{C}_1^S\left(G_{\vec{d}^n}^{\pi}\right)/n$. The goal of the first simulation is to compare the numerical values for a graph on $n = 10000$ vertices with the asymptotic theoretical results. As the gap between these values is not too big, we also want to look at the convergence speed. This is studied using the simulations for $n = 2, 4, 8, \ldots, 2048$. To determine $\widetilde{\pi}^{\text{bond}}$ we apply the procedure described in Section 6.1, considering as value for π: all values between $\pi_c^{\text{bond}} - 0.15$ and $\pi_c^{\text{bond}} + 0.15$ using steps of size 0.01. So we consider $\pi_c^{\text{bond}} - 0.15, \pi_c^{\text{bond}} - 0.14, \ldots, \pi_c^{\text{bond}} + 0.14, \pi_c^{\text{bond}} + 0.15$. The numerical value of c^{bond} is determined for all values of π between 0.45 and 0.95 in steps of 0.50. For all simulations we use `ngraphs`= 100 and `npercolation`= 100.

Before we look at the values regarding percolation, first we look at the additional output of Algorithm 1. Recall from Section 5.1 that the algorithm outputs a sequence of edges describing a graph together with the probability $P_{\mathcal{N}}$ that the algorithm generates precisely this sequence of edges. Equation (5.3) implies that all orderings leading to the same graph are generated with asymptotically equal probability. Thus the probability that the algorithm generates a given graph can be approximated by $P := m!P_{\mathcal{N}}$, since there are $m!$ different orderings for the edges of a graph. For each value of n, we determined the minimum and the maximum P in the ensemble of 100 different graphs obeying the degree sequence. The logarithm of this value is shown for the uniform degree distribution with $\Delta = 3$ in Figure 4a for $n = 2, 4, 8, \ldots, 2048$. We see that $\log_{10}(\ln P)$ seems to have a linear dependence on $\log_{10}(n)$. For small n there is a big discrepancy between the maximum and the minimum. When n increases, this difference quickly becomes small. Upon a closer inspection of the values, it turns out that the absolute difference between the maximum and the minimum decreases slowly, but remains of the same order of magnitude. However, as the value of $\ln(P)$ rapidly decreases, this means that the relative difference quickly decreases as well. We expect that this decrease in the difference is caused by the fact that the algorithm generates graphs within $1 \pm o(1)$ of uniformity. In principle



(a) The minimum and maximum value of $\ln(P)$ over 100 graphs are shown as a function of n for the uniform degree distribution with $\Delta = 3$. For small n there is a big difference between the minimum and the maximum, but this quickly becomes invisible with increase of n.



(b) The minimum value of $\ln(P)$ over 100 graphs is shown as a function of n for different underlying degree distributions. The behaviour is roughly the same for all distributions.

Figure 4: The value P is an approximation to the probability that Algorithm 1 generates a given graph $G_{\vec{d^n}}$. The minimum value of $\ln(P)$ over 100 graphs is shown as a function of n for different underlying degree distributions. The behaviour is roughly the same for all distributions.

the same graphs can be created for the remaining two distributions. However, as the difference between the maximum and minimum shows the same behaviour for these distributions, this will not provide additional insight. Instead we show the minimum for the three distributions together, see Figure 4b. This allows to compare the probability for the different degree sequences. This is mainly interesting, with regard to the approximation $N = \frac{1}{m!P}$ for the number of simple directed graphs obeying a given degree sequence. In the figure we see that the uniform distribution with $\Delta = 3$ is slightly larger, implying that there will be fewer graphs obeying it than for $\Delta = 4$ and the modified degree distribution. This is most likely caused by the fact that a larger degree at each vertex leads to more choices for the neighbours of the vertex.

Now, we turn to percolation. We generate graphs on 10000 vertices. The behaviour of the value $\left|\mathcal{C}_2^S\left(G^{\pi}_{\vec{d^n}}\right)\right|/n$ for these graphs is shown in Table 1 for the uniform distribution and in Table 2 for the modified uniform distribution. Here we selected a subinterval of $[\pi_c^{\text{bond}} - 0.15, \pi_c^{\text{bond}} + 0.15]$ for each distribution that captures the important behaviour. For all distributions the relative size of the second largest component shows similar behaviour. First it increases monotonically with π. At some point it reaches a maximum, after which it monotonically decreases. Recall from Section 6.1 that this is the behaviour we expected. For the uniform distribution with $\Delta = 3$ we find $\widetilde{\pi}^{\text{bond}} = 0.71$, whereas $\pi_c^{\text{bond}} = \frac{2}{3}$. For $\Delta = 4$, there holds $\widetilde{\pi}^{\text{bond}} = 0.53$, where $\pi_c^{\text{bond}} = 0.5$. For modified uniform distribution we find a numerical percolation threshold of $\widetilde{\pi}^{\text{bond}} = 0.50$ where $\pi_c^{\text{bond}} = 0.4717$. This amounts to a difference between the theoretical and numerical values of approximately 0.03 for all distributions. Remark that this does not contradict our theorem, as the theory considers the asymptotic limit.

Next we compare the value of c^{bond} with the average size of the largest strongly connected component of the simulated graph. For the uniform distribution the results are shown in Table 3, and for the modified uniform distribution in Table 4. Remark that for all values of $\pi < \pi_c^{\text{bond}}$ the theoretical prediction is 0, as w.h.p. no giant strongly connected component is present. However the numerical value is simply the size of the largest strongly connected component. As such a component contains at least one vertex, even if no giant component is present, this value does not become zero. This is an unfair comparison as also theoretically the largest strongly connected component will contain at least one vertex, but it does not scale linearly in n. Remark that the numerical value is small in this case, *i.e.* it contains fewer then 20 vertices. For the values

Table 1: The average relative size of the second largest strongly connected component after bond percolation is shown for different values of π. The results are obtained by simulating graphs on 10000 vertices with a degree sequence drawn from the uniform distribution for $\Delta = 3$ and $\Delta = 4$.

(a) The results for $\Delta = 3$. A maximum is observed for $\pi = 0.71$.

π	$\left\|\mathcal{C}_2^S\left(G_{\vec{d^n}}^{\pi}\right)\right\|/n$
0.65	$4.83e-04$
0.66	$6.02e-04$
0.67	$7.54e-04$
0.68	$9.29e-04$
0.69	$1.14e-03$
0.70	$1.34e-03$
0.71	$1.45e-03$
0.72	$1.44e-03$
0.73	$1.30e-03$
0.74	$1.11e-03$
0.75	$9.26e-04$
0.76	$7.70e-04$
0.77	$6.65e-04$
0.78	$5.77e-04$
0.79	$5.19e-04$
0.80	$4.73e-04$

(b) The results for $\Delta = 4$. A maximum is observed for $\pi = 0.53$.

π	$\left\|\mathcal{C}_2^S\left(G_{\vec{d^n}}^{\pi}\right)\right\|/n$
0.45	$2.31e-04$
0.46	$2.70e-04$
0.47	$3.40e-04$
0.48	$4.30e-04$
0.49	$5.69e-04$
0.50	$7.55e-04$
0.51	$9.66e-04$
0.52	$1.17e-03$
0.53	$1.33e-03$
0.54	$1.27e-03$
0.55	$1.08e-03$
0.56	$9.20e-04$
0.57	$7.23e-04$
0.58	$5.85e-04$
0.59	$5.18e-04$
0.60	$4.50e-04$

Table 2: The average relative size of the second largest strongly connected component after bond percolation is shown for different values of π. The results are obtained by simulating graphs on 10000 vertices with a degree sequence drawn from the modified uniform distribution. A maximum is observed for $\pi = 0.50$.

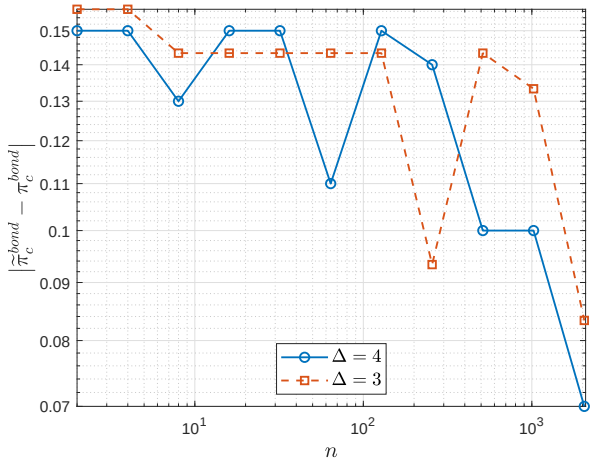
π	$\left\|\mathcal{C}_2^S\left(G_{\vec{d^n}}^{\pi}\right)\right\|/n$
0.42	$2.19e-04$
0.43	$2.77e-04$
0.44	$3.35e-04$
0.45	$4.49e-04$
0.46	$6.03e-04$
0.47	$8.30e-04$
0.48	$1.08e-03$
0.49	$1.32e-03$
0.50	$1.37e-03$
0.51	$1.25e-03$
0.52	$1.02e-03$
0.53	$7.90e-04$
0.54	$6.26e-04$
0.55	$5.32e-04$
0.56	$4.69e-04$
0.57	$4.23e-04$

Table 3: The average relative size of the largest strongly connected component after bond percolation is compared with c^{bond} for different values of π. The results are obtained by simulating graphs on 10000 vertices obeying a degree sequence drawn from the uniform distribution for $\Delta = 3$ and $\Delta = 4$.

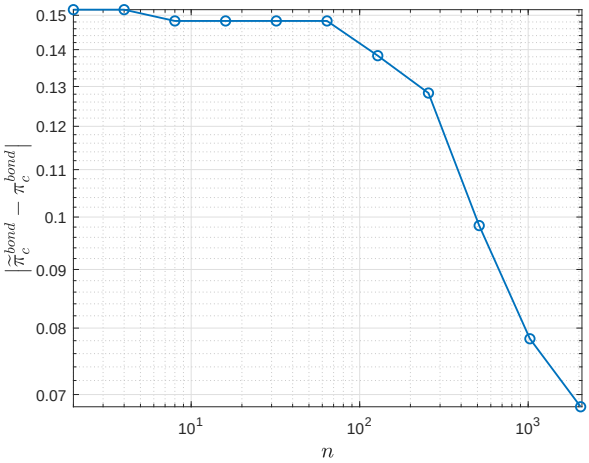
π	$\Delta = 3$		$\Delta = 4$	
	$\left\lvert \mathcal{C}_1^S \left(G^{\pi}_{\vec{d^n}} \right) \right\rvert / n$	c^{bond}	$\left\lvert \mathcal{C}_1^S \left(G^{\pi}_{\vec{d^n}} \right) \right\rvert / n$	c^{bond}
0.45	$1.79e-04$	$0.00e+00$	$5.71e-04$	$0.00e+00$
0.50	$2.28e-04$	$0.00e+00$	$2.53e-03$	$0.00e+00$
0.55	$3.34e-04$	$0.00e+00$	$2.75e-02$	$3.13e-02$
0.60	$5.85e-04$	$0.00e+00$	$9.38e-02$	$9.66e-02$
0.65	$1.58e-03$	$0.00e+00$	$1.69e-01$	$1.71e-01$
0.70	$1.07e-02$	$1.15e-02$	$2.42e-01$	$2.45e-01$
0.75	$5.50e-02$	$5.76e-02$	$3.10e-01$	$3.12e-01$
0.80	$1.19e-01$	$1.20e-01$	$3.71e-01$	$3.73e-01$
0.85	$1.86e-01$	$1.86e-01$	$4.24e-01$	$4.26e-01$
0.90	$2.51e-01$	$2.51e-01$	$4.70e-01$	$4.72e-01$
0.95	$3.12e-01$	$3.11e-01$	$5.10e-01$	$5.12e-01$

Table 4: The average relative size of the largest strongly connected component after bond percolation is compared with c^{bond} for different values of π. The results are obtained by simulating graphs on 10000 vertices obeying a degree sequence drawn from the modified uniform distribution.

π	$\left\lvert \mathcal{C}_1^S \left(G^{\pi}_{\vec{d^n}} \right) \right\rvert / n$	c^{bond}
0.45	$1.09e-03$	$0.00e+00$
0.50	$1.24e-02$	$1.47e-02$
0.55	$8.12e-02$	$8.49e-02$
0.60	$1.75e-01$	$1.77e-01$
0.65	$2.71e-01$	$2.73e-01$
0.70	$3.62e-01$	$3.63e-01$
0.75	$4.46e-01$	$4.45e-01$
0.80	$5.21e-01$	$5.20e-01$
0.85	$5.89e-01$	$5.88e-01$
0.90	$6.52e-01$	$6.51e-01$
0.95	$7.11e-01$	$7.09e-01$



(a) The value $\left|\pi_c^{\text{bond}} - \widetilde{\pi}^{\text{bond}}\right|$ is shown as a function of n for graphs obeying a uniform degree distribution for $\Delta = 3, 4$.



(b) The value $\left|\pi_c^{\text{bond}} - \widetilde{\pi}^{\text{bond}}\right|$ is shown as a function of n for graphs obeying a modified uniform degree distribution.

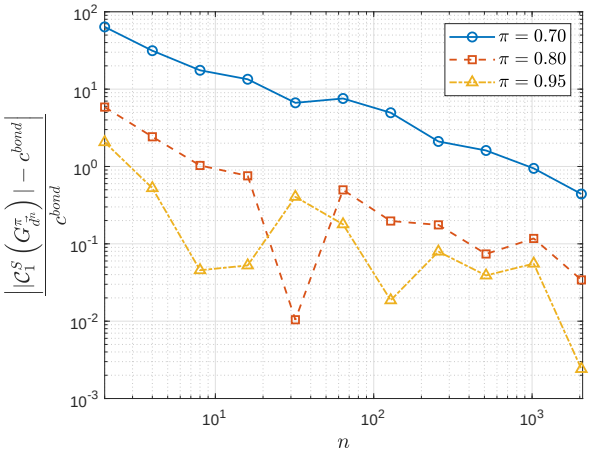
Figure 5: The behaviour of $\left|\pi_c^{\text{bond}} - \widetilde{\pi}^{\text{bond}}\right|$ is shown for the uniform and modified uniform distribution as a function of n.

of $\pi > \pi_c^{\text{bond}}$, we see that the difference between the theory and the simulation decreases, as π increases. When c^{bond} reaches values of the order 10^{-1} the absolute error remains almost constant. A last observation regarding these results is that for $\pi > \pi_c^{\text{bond}}$, the value of c^{bond} is always larger than the average size of largest SCC for the simulated graphs for the uniform distribution. However for the modified uniform distribution this is not the case.

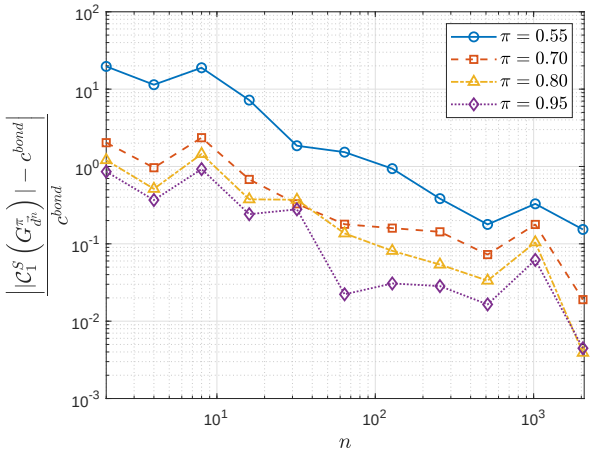
The difference between the numerical and theoretical value of c^{bond} is small, especially when π increases. This raises the question about how fast the numerical values converge to the theoretical ones. To offer some insight, we perform the same simulations as for $n = 10000$ for graphs on $n = 2, 4, 8, \ldots, 2048$ vertices using the same three degree distributions. Instead of the actual values produced by the simulation, we will show the difference between the numerical and theoretical value as a function of n, since we want to study the convergence.

First look at the percolation threshold π_c^{bond}. We expect that the difference between $\widetilde{\pi}^{\text{bond}}$ and π_c^{bond} decreases, when the number of vertices increases. Recall that this difference was approximately 0.03 for $n = 10000$. Hence we do not expect the difference to drop bellow 0.03. Furthermore it is upper bounded by approximately 0.15 as we run the simulation for $\pi \in [0.51, 0.81]$ for $\Delta = 3$, for $\pi \in [0.35, 0.65]$ for $\Delta = 4$ and for $\pi \in [0.32, 0.62]$ for the modified degree distribution. The results are shown in Figure 5. For the uniform degree distribution an erratic "burn-in" regime is observed, followed by eventually decrease of the error for larger values of n. For the modified uniform distribution we do not observe the erratic behaviour. Instead we see that the difference remains almost constant, until at $n = 128$ a decrease sets in. The difference between the values roughly halves when the number of vertices increases by a power of 2.

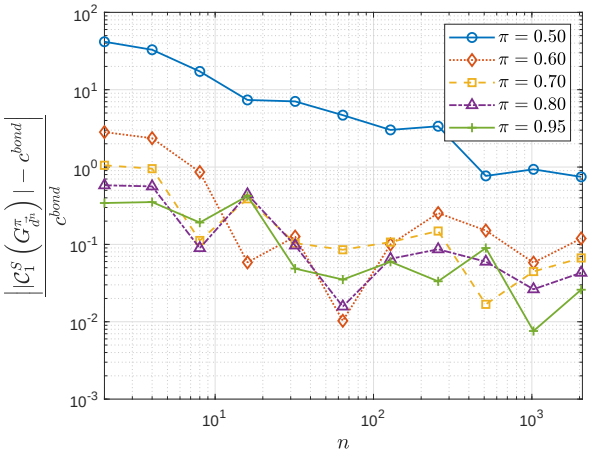
Next we look at the relative difference between c^{bond} and the relative size of the largest strongly connected component in the graph as a function of n. We choose to show the relative difference and not the absolute, as for different values of π, the order of magnitude of c^{bond} changes. The results are shown in Figure 6. For all three distributions we see that the relative difference decreases if n increases. Also the agreement between simulation and theory is better for larger values of π. This is consistent with the behaviour Tables 3 and 4 display. For small values of n the difference is very large. This is mainly due to the fact $\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ can only take values in $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$. Remark for the values of π close to the percolation threshold that the error remains fairly large. Another important remark is that we in fact look at a convergence of two intertwined



(a) The relative difference of c^{bond} and $\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ is shown as a function of n. The degree distribution is a uniform distribution with $\Delta = 3$.



(b) The relative difference of c^{bond} and $\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ is shown as a function of n. The degree distribution is a uniform distribution with $\Delta = 4$.



(c) The relative difference of c^{bond} and $\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ is shown as a function of n. The degree distribution is the modified uniform distribution.

Figure 6: The relative difference of c^{bond} and $\left|\mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right|$ is shown as a function of n for multiple values of π. The actual data points are shown with markers. The lines connecting the markers provide a visual aid to highlight the trend. For larger values of π, the relative difference is smaller.

Table 5: The average relative size of the second largest strongly connected component after site percolation is shown for different values of π. The results are obtained by simulating graphs on 10000 vertices with a degree sequence drawn from the uniform distribution for $\Delta = 3$ and $\Delta = 4$.

(a) The results for $\Delta = 3$. A maximum is observed for $\pi = 0.72$.

π	$\left\lvert \mathcal{C}_2^S \left(G_{\vec{d^n}}^{\pi} \right) \right\rvert / n$
0.65	$4.30e-04$
0.66	$5.24e-04$
0.67	$6.35e-04$
0.68	$7.68e-04$
0.69	$9.31e-04$
0.70	$1.07e-03$
0.71	$1.19e-03$
0.72	$1.24e-03$
0.73	$1.23e-03$
0.74	$1.14e-03$
0.75	$1.05e-03$
0.76	$8.18e-04$
0.77	$7.22e-04$
0.78	$6.08e-04$
0.79	$5.43e-04$
0.80	$4.95e-04$

(b) The results for $\Delta = 4$. A maximum is observed for $\pi = 0.54$.

π	$\left\lvert \mathcal{C}_2^S \left(G_{\vec{d^n}}^{\pi} \right) \right\rvert / n$
0.45	$2.31e-04$
0.46	$2.64e-04$
0.47	$3.22e-04$
0.48	$3.88e-04$
0.49	$4.90e-04$
0.50	$6.12e-04$
0.51	$7.48e-04$
0.52	$8.70e-04$
0.53	$1.01e-03$
0.54	$1.03e-03$
0.55	$1.02e-03$
0.56	$9.15e-04$
0.57	$7.63e-04$
0.58	$6.36e-04$
0.59	$5.43e-04$
0.60	$4.64e-04$

processes. The size of the largest component should converge to c^{bond} with n. However this is true for uniformly random generated graphs. Recall that Algorithm 1 generates graphs that are asymptotically drawn from a uniform distribution. As Theorem 4.1 concerns uniformly random graphs, this also may enlarge the error for small n.

6.4 Simulating site percolation

In this section, we present the outcomes of the simulations performed regarding site percolation. For site percolation we have performed the same simulations as for bond percolation, but only for graphs obeying a uniform degree distribution. The modified degree distribution is omitted here, as we saw in the case of bond percolation that both distribution show fairly similar results.

First we consider the value of $\widetilde{\pi}^{\text{site}}$ for the graphs on $n = 10000$ vertices. The average sizes of the second largest strongly connected component are shown in Table 5. Again we selected a subinterval of $[\pi_c^{\text{bond}} - 0.15, \pi_c^{\text{bond}} + 0.15]$ for each distribution that captures the important behaviour of this value. Similarly to bond percolation, the value first monotonically increases. Then it reaches its maximum and after that it monotonically decreases. As numerical percolation threshold we find $\widetilde{\pi}^{\text{site}} = 0.72$ for $\Delta = 3$ and $\widetilde{\pi}^{\text{site}} = 0.54$ for $\Delta = 4$. As $\pi_c^{\text{site}} = \frac{2}{\Delta}$, we find that the numerical percolation threshold is approximately $\pi_c^{\text{site}} + 0.4$. Recalling that $\pi_c^{\text{bond}} = \pi_c^{\text{site}}$ and comparing with Table 1, we observe an increase of 0.1 in the difference with respect to bond percolation. Comparing with this table we also see that the general behaviour of $\left\lvert \mathcal{C}_2^S \left(G_{\vec{d^n}}^{\pi} \right) \right\rvert$ is the same for both types of percolation. However the value of $\left\lvert \mathcal{C}_2^S \left(G_{\vec{d^n}}^{\pi} \right) \right\rvert$ is lower for site percolation. Equation (4.48) explains this. After site percolation there are more isolated vertices, while for other degrees the probability is the same up to normalization factor π.

Next we compare the value of c^{site} and the fraction of vertices in the largest connected component of the simulated graphs. These results are shown in Table 6. Like for bond percolation, $c^{\text{site}} = 0$ implies that no giant strongly connected component is present. For those values we see that the components in the simulated graph are very small, *i.e.* the contain approximately 15 vertices or less. Like for bond percolation, the differ-

Table 6: The average relative size of the largest strongly connected component after site percolation is compared with c^{site} for different values of π. The results are obtained by simulating graphs on 10000 vertices with a degree sequence drawn from the uniform distribution for $\Delta = 3$ and $\Delta = 4$. The agreement between theory and simulation improves when π increases.

π	$\Delta = 3$		$\Delta = 4$	
	$\left\lvert \mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right\rvert /n$	c^{site}	$\left\lvert \mathcal{C}_1^S\left(G^{\pi}_{\vec{d^n}}\right)\right\rvert /n$	c^{site}
0.45	$1.80e-04$	$0.00e+00$	$5.18e-04$	$0.00e+00$
0.50	$2.26e-04$	$0.00e+00$	$1.73e-03$	$0.00e+00$
0.55	$3.27e-04$	$0.00e+00$	$1.22e-02$	$1.72e-02$
0.60	$5.46e-04$	$0.00e+00$	$5.10e-02$	$5.80e-02$
0.65	$1.33e-03$	$0.00e+00$	$1.03e-01$	$1.11e-01$
0.70	$6.75e-03$	$8.07e-03$	$1.63e-01$	$1.71e-01$
0.75	$3.60e-02$	$4.32e-02$	$2.25e-01$	$2.34e-01$
0.80	$8.81e-02$	$9.57e-02$	$2.89e-01$	$2.98e-01$
0.85	$1.50e-01$	$1.58e-01$	$3.53e-01$	$3.62e-01$
0.90	$2.17e-01$	$2.26e-01$	$4.15e-01$	$4.25e-01$
0.95	$2.86e-01$	$2.96e-01$	$4.77e-01$	$4.86e-01$

ence between the simulation and the theory decreases, if π increases. Again we observe that the theoretical value upper bounds the numerical one for bond percolation. However the difference between the theoretical and numerical value is larger than for bond percolation, see Table 3. We expect that the difference between both types of percolation can explain this. When applying bond percolation a vertex only looses degree if an adjacent edge is removed, which happens with probability $1-\pi$. Site percolation can change the degree of a vertex in two ways: the vertex may be isolated, which happens with probability $1-\pi$ or the vertex looses an adjacent edge because the other vertex of this edge is deleted. We expect that the probability that a vertex has degree (j,k) after bond percolation converges faster to the asymptotic probability than the probability that a not deleted vertex has degree (j,k) after site percolation.

Next we look at the convergence diagrams for site percolation. These are constructed using graphs on $n = 2, 4, 8, \ldots, 2048$ vertices, like in Section 6.3. First consider the percolation threshold. Again for $\Delta = 3$ we consider $\pi \in [0.51, 0.81]$ and $\Delta = 4$ $\pi \in [0.35, 0.65]$ in steps of 0.01. The results are shown in Figure 7. For $\Delta = 4$ we see a very big jump for $n = 4$ in the graph. We suspect that the degree sequence here is close to regular, with each vertex having in- and out-degree 3. Remark that as we consider simple graphs only, $n = 4$ implies that the in- and out-degree of each vertex is upper bounded by 3. If the degree sequence is regular with degree 3, between any pair of vertices there is a direct edge in both directions. This means that a lot of edges need to be removed to get to strongly connected components. Furthermore for $\Delta = 4$ we often seem to reach the maximum error that can appear in the simulation. This figure suggest that the error starts to decrease at $n = 256$. To be able to conclude this, a simulation over a broader range of π needs to be performed so that the simulation no longer upper bounds the error. The final error , i.e. for $n = 2048$ is larger than in Figure 5. This could be expected from the fact that this is also the case for $n = 10000$.

Our final simulation regards the relative error between c^{site} and the largest strongly connected component in the graph as a function of n for multiple values of π. The results are shown in Figure 8. For all combinations of Δ and π a rapid decrease is observed for small values of n. Like in Section 6.3, we expect that this is due to the increase in possible outcomes of the simulation, i.e. the value of $\left\lvert \mathcal{C}_2^S\left(G^{\pi}_{\vec{d^n}}\right)\right\rvert$ can take only n different values. The further π is from the percolation threshold, the better estimate does c^{site} provide for the largest component of a simulated finite graph.

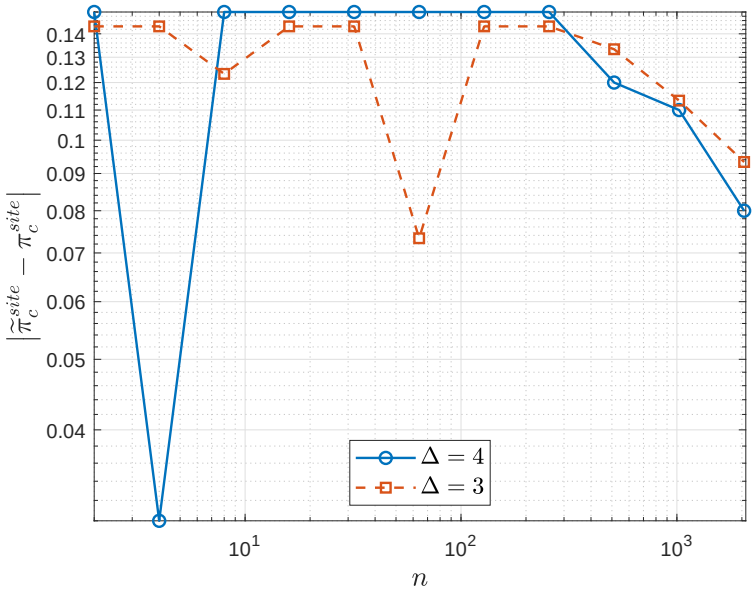
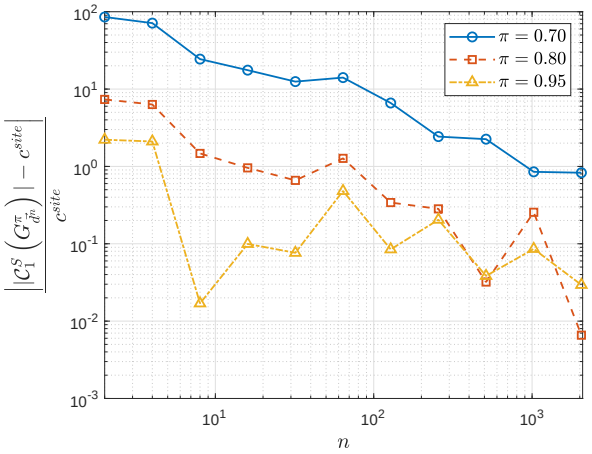
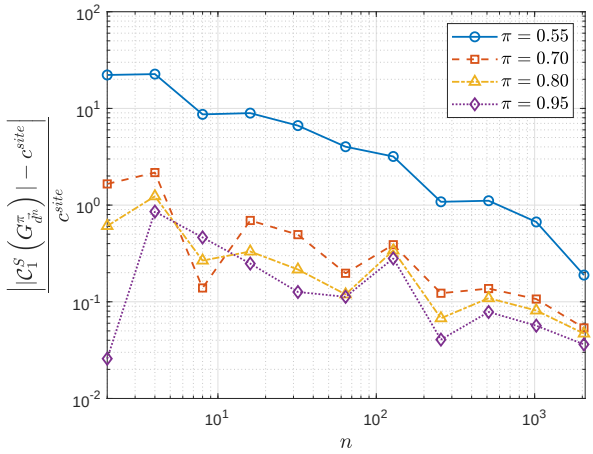


Figure 7: The value $\left|\pi_c^{\text{site}} - \widetilde{\pi}^{\text{site}}\right|$ as a function of n for graphs with a uniform degree distribution for $\Delta = 3, 4$. After fluctuation, a decrease seems to set in at $n = 256$.



(a) The relative difference between c^{site} and $\left|\mathcal{C}_1^S\left(G_{\vec{d}^n}^{\pi}\right)\right|$ is shown as a function of n. The degree distribution is a uniform one with $\Delta = 3$.



(b) The relative difference between $\left|\left|\mathcal{C}_1^S\left(G_{\vec{d}^n}^{\pi}\right)\right|\right|$ and c^{bond} is shown as a function of n. The degree distribution is a uniform one with $\Delta = 4$.

Figure 8: The difference between c^{bond} and $\left|\mathcal{C}_1^S\left(G_{\vec{d}^n}^{\pi}\right)\right|$ is shown as a function of n for multiple values of π. The solid lines connecting the markers provide a visual aid to highlight the trend. The difference between the theory and simulation on average decreases with n.

7 Conclusion

In this work we investigated bond and site percolation in random directed graphs (digraphs) with a given degree distribution. The first main result is that, we determined the percolation thresholds for the existence of a giant strongly connected component in random simple digraphs obeying a proper degree array, as well as the expression for the size of the giant component. These properties turned out to be the identical for all proper degree arrays with the same underlying degree distribution. The second main result concerns the algorithmic construction of random simple digraphs obeying a given degree sequence. We propose a new algorithm to generate such random digraphs. We show that the algorithm generates all digraphs obeying the degree sequence with a probability that is asymptotically uniform, namely it is within $1 \pm o(1)$ of the uniform distribution. Furthermore we show that the algorithm has an expected runtime near linear in the number of edges, and fails to construct the graph with probability $o(1)$.

Using our algorithm we numerically simulated bond and site percolation. From these simulations, the numerical percolation threshold and the numerical size of the giant strongly connected component are determined. Comparing these values with the theoretical results in the asymptotic limit, we find that the size of the giant component converges faster than the percolation threshold. Also the values for bond percolation seem to converge faster than for site percolation.

As further work, more thorough numerical simulations regarding percolation can be performed to determine the convergence speed of the percolation threshold and the size of the giant strongly connected component. This requires to determine, besides the average values, also the standard deviation of the numerical values of interest. These simulations can also be performed using different degree distributions, to see if the convergence speed changes if the degree array has an increasing maximum degree $d_{\max}$.

Our algorithm itself can be used for many different studies. For example, it approximates the number of simple digraphs obeying a given degree sequence. As determining the number of such digraphs is difficult, it could be of interest to study whether the approximation of the algorithm can be improved or if an estimation of the error in this approximation can be determined. Furthermore the algorithm allows to generate random bipartite or hypergraphs obeying a given degree sequence, as these types of graphs can be modelled by a digraph. A bipartite graph can be obtained from a directed one by assigning each vertex only a non-zero in- or out-degree. A hypergraph can be modelled by replacing hyper-edges with labelled vertices, and declaring the edge direction to point from unlabelled vertices to labelled vertices. In these random graphs it is not interesting to look at strongly connected components as they will consist of just one vertex.
In this case the existence of a giant weakly connected component is more important, which also is of interest in a random digraph. Numerically this can be easily studied using our algorithm. On the theoretical side, the same method we use to determine the percolation for the giant strongly connected component could be used for the giant weakly connected component. However this requires a result about the existence of a giant weakly connected component in these random graphs. We expect that such a result can be obtained by comparing the creation of a weakly connected component with a multi-type Galton-Watson branching process.

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