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Master Thesis

Theoretical Physics

Wormholes, Energy Conditions, Topological Censorship, and all that

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Abstract

We consider Einstein-Maxwell theory with a negative cosmological constant and study the possible existence of traversable wormholes in Asymptotically (locally) AdS spacetimes. It has been argued in the literature that such solutions are foliations of warped AdS_3 spacetimes and are supersymmetric solutions to N=2 d=4 gauged supergravity. In this talk, I will briefly go through our journey through the the geometric and physical aspects of the solutions, provide some results, as well as some failed attempts, and suggest some open questions. We find that traversability, as measured from the boundary, seems to hold, with the null energy condition being satisfied. The total electric and magnetic charge are zero in the global cover. We interpret this as two semiwormhole solutions glued together. The conformal mass blows up because of the non-compactness of the boundaries. We were not able to produce similar constructions in 5d, but found a new solution which embeds the 4d solution in 5d. In the future, we intend to test the wormholeness of the 4d solution, generalize the construction and study the new 5d solution.

Acknowledgements

I would like to thank several people for being supportive during this year.

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July 15, 2020

To women in physics

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Introduction

General Relativity does not exclude non trivial topology as one of its possible solutions. Wormholes made their first appearance in a paper by Misner and Wheeler in 1957 [1], after having been proposed, as a concept, by Weyl earlier. One can think of them either as shortcuts between two regions of spacetime, or as tubes that connect two disconnected regions. They have not yet been fully understood classically, or in the semi-classical regime. Wormholes can exist as vacuum solutions of the equations of motion, or can be accompanied and supported by matter. This matter can either be regular, i.e. with positive energy, or exotic, i.e. with negative energy. We are particularly interested in the possible existence of wormholes with regular matter. However, it is quite hard to find such solutions that are stable and can give a significant contribution to the path integral of quantum gravity. This is mainly due to the fact that, roughly speaking, the wormhole guides matter to come closer at the throat, i.e. the region where the size of the wormhole is minimum. As a result, matter with positive energy at the throat will lead to gravitational collapse. An indirect way to assure stability is via supersymmetry.[1] If such stable solutions exist, then it is possible to send a signal through the wormhole, which means very roughly that they have the property of traversability. A class of traversable wormhole solutions to free Einstein-Maxwell theory with a negative cosmological constant has been recently proposed in [2] and proven to be supersymmetric in [3].

It is remarkable that such wormholes supported by regular matter exist as solutions to such a simple theory. We study this class of solutions with several ambitions in mind. One of them is to find how they are embedded in string theory, which is the candidate theory as the theory of quantum gravity. Another goal is to examine closely the non-exoticity of the matter accompanying the wormhole and explain why the throat does not collapse. In fact, regular matter means that the energy conditions are satisfied. Energy conditions are restrictions on the stress energy tensor of general relativity such that it describes physical matter, with positive energy density. However, energy conditions usually rule out exotic phenomena, such as traversable wormholes. Finally, our aim is to take a closer look at the physical properties of these solutions and better interpret them.

After introducing some preliminary concepts in chapter 1, we move on to elab-

[1] It is beyond the scope of this thesis to explain that argument in detail

orating on wormholes, in general, as well as on the particular model of interest in chapter 2. Afterwards, we present our first attempt of finding a higher dimensional origin of these wormholes in chapter 3. There, we use some ansätze to find an embedding in five dimensions and we find in the literature that the solutions consistently uplift to eleven dimensional supergravity, as introduced in [4]. In the following chapter, namely chapter 4, we compute that the null energy condition, as well as the traversability as measured from the boundary, are satisfied. This immediately leads us to an observation that there is an apparent contradiction to the Topological Censorship theorem, which , in principle, forbids such traversable wormholes that obey this condition. Moving on, in chapter 5, we calculate all the conserved charges, those associated to the Killing isometries and to electromagnetism, as an attempt to study all the physical properties of the solutions. Even though we find that they are all zero, there is interesting physics in the vacuum structure of quantum gravity. We find that a picture of two semiwormholes with opposite physical properties is a consistent interpretation for the model and provides interesting diagnostics that will be further discussed in the end. We conclude with some open questions and an outlook of possible further research.

1. Preliminaries

1.1 Einstein-Maxwell theory

Classical Electrodynamics and General Relativity can both be described under the veil of geometry. Matter fields and electromagnetism can be viewed as manifestations of the bending of space.[1] The relation of the electromagnetic field with geometry can be seen in 1.1

In this thesis, we consider free[1] Einstein-Maxwell's theory in curved spacetime with cosmological constant Λ, which has the following action

$$S = \int d^4x\sqrt{-g}\left[(R + 2\Lambda) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is the antisymmetric electromagnetic field strength and $g = det(g_{\mu\nu})$ is the determinant of the metric. The first term in the action is the Einstein-Hilbert term with a cosmological constant. The second term consists the dynamical term for the gauge field A_μ.

Maxwell's equations in curved spacetime and Einstein's field equations that emerge upon varying this action with respect to the fields, i.e. A_μ and $g_{\mu\nu}$, plus the Bianchi identity, are the following

$$\text{Homogenous Maxwell's equations:} \quad \nabla_{[\alpha}F_{\mu\nu]} = 0 \tag{1.2}$$

$$\text{Inhomogenous Maxwell's equations:} \quad \nabla_\mu F^{\mu\nu} = 0 \quad (J^\nu = 0) \tag{1.3}$$

Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{\mu_0 c^4}\left(F_{\mu\rho}g^{\rho\sigma}F_{\sigma\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\sigma}_{\rho}\right) \tag{1.4}$$

where J^ν is the source term of the electromagnetic field.[2] Equation (1.2) is just the Bianchi identity giving Gauss's law for magnetism and Faraday's law. Equation (1.3) give Gauss's law and Ampère–Maxwell law. In Einstein's field equations (1.4)

[1] the term "free" here means that there are no sources

[2] If the sources in Maxwell's equations are zero, nothing prevents the field lines to have a strength. It will be a divergenceless non vanishing field strength.

in the LHS there is the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar, and $\Lambda g_{\mu\nu}$ is the cosmological constant term. In the RHS, this expression is the energy momentum tensor $T_{\mu\nu}$ of Maxwell fields.

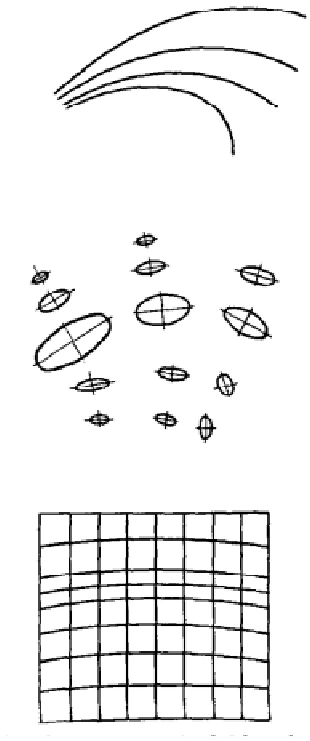

Figure 1.1: The impact of the electromagnetic field in geometry, and vice verca. Top: field lines of force. Middle: Maxwell stress tensor caused by these field lines. This stress tensor sources the gravitational field and is equal to the contracted curvature tensor of the spacetime continuum, up to a multiplicative constant. Bottom: The distorted metric of spacetime. The imprint of the field lines is so specific, that one can go back and reconstruct through the geometry all the properties of the field lines. In this picture, one can see the interpretation of electromangetism out of pure geometry. Figure taken from [1]

Maxwell's equations in curved background have a more compact formulation by means of exterior calculus and differential forms, which we choose to omit here.

The Maxwell tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.5}$$

and its dual

$$\star F_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{1.6}$$

where $\star$ is the hodge star operator. Some useful invariants are the following

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu} \tag{1.7}$$

$$F\tilde{F} \equiv \frac{1}{2}F_{\mu\nu} \star F^{\mu\nu} \tag{1.8}$$

1.2 AdS spacetimes

Anti-de Sitter space[3] takes its name after the Dutch mathematician, physicist and astronomer Willem de Sitter. It is a maximally symmetric Lorentzian manifold of constant negative curvature. The relevant one of constant positive curvature is called de Sitter space.

Maximally symmetric spaces are those that have a maximal amount of Killing isometries. For dimension n, that number is $\frac{n(n+1)}{2}$, with flat Minkowski being the first and easiest example. Its group of isometries is the Poincaré group. Anti-de Sitter space is a homogeneous space that can be defined as a quadric surface in a flat vector space. As a first familiar example of such a quadric surface one can consider the $n-$sphere, $\boldsymbol{S^n}$, of radius R , which can be defined as the positive definite quadric surface in flat euclidean space of one dimension higher,$\boldsymbol{R^{n+1}}$.

$$X_1^2 + X_2^2 + \ldots + X_{n+1}^2 = R^2 \tag{1.9}$$

The metric would be

$$ds^2_{S^n} = dX_1^2 + \ldots + dX_{n+1}^2 \tag{1.10}$$

On this quadric surface, the group SO$(n+1)$ takes a pair of two antipodal points $\vec{X} = (X_1, \ldots, X_{n+1})$ and $\vec{X} = (-X_1, \ldots, -X_{n+1})$ and maps it to some other pair on the $n-$sphere.

In the exact same way, $\boldsymbol{AdS_n}$ spacetime can be defined as a hyperbolic quadric surface with two timelike coordinates, negative radius of curvature,l, embedded in $\boldsymbol{R^{(2,n-2)}}$.

$$X_1^2 + X_2^2 + \ldots + X_{n-1}^2 - U^2 - V^2 = -l^2 \tag{1.11}$$

with metric

$$ds^2_{AdS_n} = dX_1^2 + \ldots + dX_{n-1}^2 - dU^2 - dV^2 \tag{1.12}$$

The group SO$(2, n-1)$ leaves the null quadric invariant, or takes a pair two antipodal points $\vec{X} = (X_1, \ldots, X_{n+1})$ and $\vec{X} = (-X_1, \ldots, -X_{n+1})$ on the surface and maps them to another pair. This is the group of isometries of AdS$_n$, with $\frac{n(n+1)}{2}$ generators. The topology of $\boldsymbol{AdS_n}$ is $\boldsymbol{R^{n-1}} \otimes \boldsymbol{S^1}$.

For completeness, let us note that de Sitter spacetime, with radius of curvature k, is a one sheeted hyperboloid

$$X_1^2 + X_2^2 + \ldots + X_n^2 - X_{n+1}^2 = k^2 \tag{1.13}$$

[3]In fact, the first name of anti-de Sitter space, was just de Sitter of second kind.

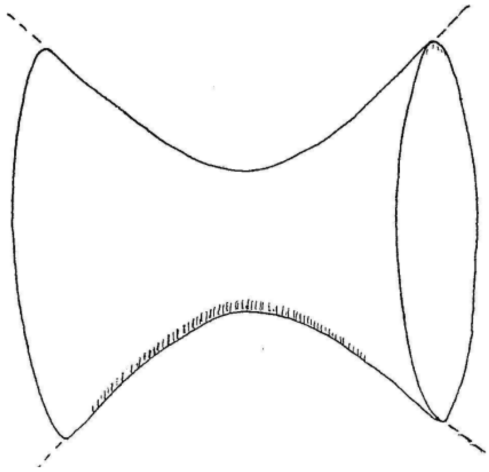

Figure 1.2: (1 + 1) Anti-de Sitter space embedded in (1 + 2) flat space. This picture is slightly counter intuitive, since in our eyes it seems like a hyperboloid of dimension 2 embedded in a Euclidean space of dimension 3. In fact, what this picture tries to show is a generalized hyperbolic surface in a Minkowski-like space with two timelike coordinates. This embedding suffers from closed timelike curves. In order to avoid that one takes the universal cover, which "unwraps" the embedding. Figure taken from [5]

embedded in Minkowski $\boldsymbol{R^{(1,n+1)}}$ with

$$ds^2_{dS_n} = dX_1^2 + \ldots + dX_n^2 - dX_{n+1}^2 \tag{1.14}$$

The group of isometries is $\mathrm{SO}(1, n-1)$ this time. In the same fashion, de-Sitter space also has maximal amount of isometries. The topology of $\boldsymbol{dS_n}$ is $\boldsymbol{S^{n-1}} \otimes \boldsymbol{R}$.

1.2.1 AdS as solutions to Einstein's field equations

Anti-de Sitter metrics, as well as de Sitter ones, serve as exact solutions to vacuum Einstein's field equations with cosmological constant, whose sign derermines the sign of the curvature[4]. More precisely, anti-de Sitter are homogeneous spacetimes of constant negative curvature, while de-Sitter ones correspond to positive curvature. Constant curvature is reflected locally in the Riemann tensor, which takes the form

$$R_{abcd} = \frac{R}{12}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{1.15}$$

where R is the Ricci scalar. By definition, the Weyl tensor, which is the traceless part of the Riemann tensor, is zero. $C_{abcd} = R_{ab} - \frac{1}{4}Rg_{ab} = 0$. The Ricci scalar

[4]In the $(-,+,+,+)$ signature

is everywhere constant by virtue of the constracted Bianchi identities. Einstein's equations in four dimensions take the following form

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\frac{1}{4}Rg_{ab} \tag{1.16}$$

where $\Lambda = \frac{1}{4}R$ is the cosmological constant that appears in the Einstein-Hilbert action.

Let us suppose that the LHS of Einstein's equations are geometry related quantities and in the RHS are the matter related quantities. With this prescription, one can think of these solutions as non vacuum, involving a perfect fluid with constant density, $\rho = R/32\pi$, and constant pressure $P = -R/32\pi$. However, such fluids would be required to have either negative energy, or negative pressure.[6] In the semiclassical regime, this term on the RHS can be thought of as vacuum energy of quantum fields. According to this, the computed value is many orders of magnitude greater than its experimental bound, leading to the, so called, "cosmological constant problem". Therefore, the interpretation of Λ can be made through the geometry. It can be thought of as a fundamental constant of Nature, whose smallness (if non-zero), is as problematic as the smallness of the other fundamental constants of Nature, e.g. the Planck length.[7] For that reason, in this thesis we regard it as a constant related to the geometry and place it in the LHS of Einstein's field equations.

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0 \tag{1.17}$$

The current observed value is slightly positive, which corresponds to positive curvature, i.e. de Sitter spacetime. A negative value for the cosmological constant implies negative constant curvature, i.e. anti-de Sitter spacetime.

1.2.2 Conformal Compactification

Conformal compactification is a concept first proposed by R. Penrose in [8] that is now being used in order to study asymptotic properties of spacetimes. This aspect of it was not completely understood at the time. In a series of three lectures in Les Houches summer school[9], he explained the technique in detail for each of the cases $\Lambda > 0$ and $\Lambda < 0$. For the case $\Lambda = 0$, he published a lengthy analysis of the asymptotic behaviour of zero rest-mass fields with this technique.[10]

The geometrical construction, whose definitions were taken from [11], is the following

- Let a smooth Lorentzian manifold $(\mathcal{M}, g)$ be the "physical" spacetime, of which we want to study the assymptotic properties.

- There is an "unphysical" spacetime, which is a smooth manifold $\bar{\mathcal{M}}$, with boundary $\mathcal{I}$. The bulk of it is equivalent to the physical $\mathcal{M}$.

- There is a function Ω, defined in such a way that brings us from the boundary to the physical spacetime. This is a positive function on $\mathcal{M}$, it is smooth on $\bar{\mathcal{M}}$, zero on the boundary $\Omega|_{\mathcal{I}}$ and defined up to a scalar function such that $d\Omega|_{\mathcal{I}} \neq 0$.
- The conformally rescaled metric $\bar{g} = \Omega^2 g$, with these conditions, is a smooth non degenerate Lorentzian metric on the unphysical spacetime $\bar{\mathcal{M}}$.

Conformal compactification is made possible for specific spacetimes, and the possibility is determined by the fall-off of the Weyl tensor at infinity. The conformal boundary will have different components depending on different ways of going from the physical spacetime to infinity.

It is important to note that the quantities we aim to study in the physical spacetime through this technique, should be invariant under conformal rescalings. This is due to the fact that the physical spacetime, does not include the boundary.

This can be regarded as a technique of bringing the infinity to a closer coordinate distance. However, the word compactification is not meant in the usual sense. The unphysical spacetime need not be compact, with the exception of Minkowski space.

1.3 Causality

1.3.1 Causality in Minkowski

Let us begin with some useful definitions. Let a coordinate system on Minkowski[5] spacetime M_4 be $x^\mu = (t, \vec{x})$,$\mu = 0, 1, 2, 3$,

- A path is usually described by $x^\mu(\lambda)$ is usually parametrized by a parameter s.
- It is required that the tangent vector $\frac{dx^\mu(\lambda)}{d\lambda}$ is non vanishing and paths transformed under affine reparametrizations are considered equivalent.
- A causal path is defined as the one whose tangent vector $\frac{dx^\mu(\lambda)}{d\lambda}$ is either timelike or null.
- A causal diamond $D(q, p)$ is the region of all the causal paths between two points q, p. It is the intersection of the causal future of p and the causal past of q.

The proper time elapsed along a causal path from a point p to a point q is

$$\tau = \int_0^1 d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} = \int_0^1 d\lambda \sqrt{\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{d\vec{x}}{d\lambda}\right)^2} \tag{1.18}$$

[5]The use of the concept of Minkowski, here, is just for simplicity. The concepts can be generalized to arbitrary Lorentzian signature manifolds

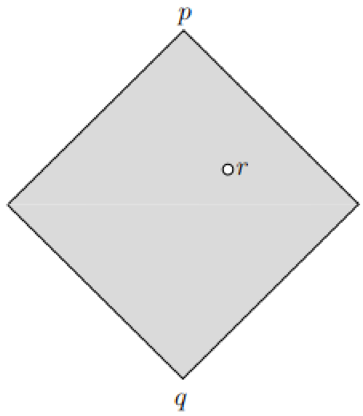

Figure 1.3: A causal diamond $D(p, q)$ in Minkowski spacetime, being the intersection of causal future of q and the causal past of p. Note that it can be defined for more general cases.Figure taken by [12]

If the causal diamond between those two points q, p is compact then, there is a path that extremizes the proper time. And then, such a causal path that extremizes the proper time is called a geodesic. If this compactness fails, then there is no geodesic between the two points, q and p.[12]

Timelike paths are usually parametrized by the proper time, while for null paths, the proper time elapsed is zero. In order to parametrize null paths, one uses an affine parameter.

1.3.2 Causality in AdS

Starting from the definition of AdS$_2$ as a hyperbolic hypersurface embedded in $R^{(2,1)}$.

$$-u^2 - v^2 + w^2 = -l^2 \tag{1.19}$$

where l is the radius of curvature. The line element is

$$ds^2 = -du^2 - dv^2 + dw^2 \tag{1.20}$$

One can make the following change of coordinates

$$u = \sqrt{l^2 + w^2} \cos t, \quad v = \sqrt{l^2 + w^2} \sin t, \quad t \simeq t + 2\pi, -\infty < w < +\infty \tag{1.21}$$

one can take the universal cover and think of t as a real variable, in order to avoid closed timelike curves. Then upon this change of coordinates, the metric takes the form

$$ds^2 = -\left(R^2 + w^2\right) dt^2 + \frac{R^2}{R^2 + w^2} dw^2 \tag{1.22}$$

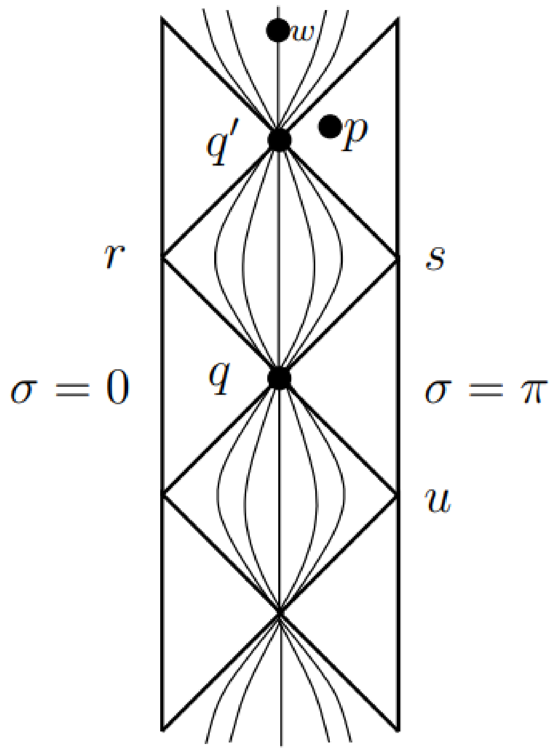

Figure 1.4: The conformal diagram of AdS_2. An observer on the boundary will measure that a lightray travels from one conformal boundary to the other in finite time. In this diagram q and p are causally connected but there is no geodesic between the two, because the proper time elapsed along the path can be arbitrarily large. For AdS_n the conformal diagram is half of this diagram, $\sigma < \pi/2$, where each point in this diagram corresponds to a codimension 1 sphere. Figure taken by [12]

The change of coordinates

$$\sin\sigma = \frac{l}{\sqrt{l^2 + w^2}}, \quad \text{where} \quad 0 < \sigma < \pi \quad \text{for} \quad -\infty < w < +\infty \tag{1.23}$$

brings the metric (1.22) of global AdS_2 to its conformally flat form as follows

$$ds^2 = \frac{l^2}{\sin\sigma}(-dt^2 + d\sigma^2), \quad -\infty < t < +\infty, \quad 0 < \sigma < \pi \tag{1.24}$$

.

Physical AdS spacetime is non compact and the boundary in figure 1.4 is not included in it. A causal diamond of two conformally related manifolds is independent of the rescaling. This means that causality works in the same manner in both of them, and it is sufficient if we just look at the conformal diagram 1.4. In that diagram if the two points that define the causal diamond $D(p, q)$ are sufficiently close, then the causal diamond is compact. In the case that the causal diamond is big enough, so that it reaches the boundary, then due to the fact that AdS is non-compact, the causal diamond is non-compact.

1.3.3 Global Hyperbolicity in AdS

Let us first provide some definitions. A spacelike hypersurface S of a manifold M is called achronal if it intersects timelike geodesics no more than once. The future/past domain of dependence of a spacelike hypersurface is the points on the manifold that belong to the causal future/past of S. An achronal spacelike hypersurface that its future and past domain of dependence produce the whole manifold is called a Cauchy hypersurface. A Cauchy hypersurface is an initial value hypersurface.

A manifold is called *globally hyperbolic* if it admits a Cauchy hypersurface. Then, there exists a natural foliation $M = R \times S$, with which the initial data hypersurface can be evolved to produce the whole manifold.[see figure 1.5][6]

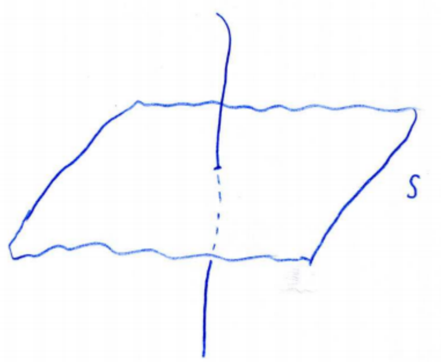

Figure 1.5: M is globally hyperbolic $\implies$ $M = R \times S$. Figure taken from [13]

In anti-de Sitter spacetime, the conformal infinity is timelike. Specifying the initial data on a spacelike hypersurface would not determine the future evolution deterministically, unless there are boundary conditions associated with the conformal infinity. In the above description, using the causal diamonds, there are conditions to make the diamonds compact. It is true and well known that the physical AdS spacetime does not include the boundary and its causal diamonds are non-compact, leading to issues with the initial value problem and global hyperbolicity. Another viewpoint via the conformal diagram (1.4) is the following: every

[6]If a spacetime is globally hyperbolic, there exists a foliation [see appendix D for more information] $M = R \times S$, where if M contains a wormhole, S should contain one as well.

horizontal line on this diagram is a constant t surface, which evolves upwards. If there are no conditions imposed on the conformal boundary, it is obvious that the inital value problem is not well defined.[7]

However, if one studies the conformally rescaled manifold together with its boundary, i.e. the conformally compactified AdS, then with appropriate conditions this is globally hyperbolic. This will, of course, not be a physical manifold, but if the quantities being calculated with this scheme are independent of the rescaling, then one can consider that there is no issue with global hyperbolicity. This is also referred to in the literature as *global hyperbolicity in the AdS sense*.

[7]The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is considered as the D'Alambertian operator acting on the metric. Einstein's equations are like wave equations on 2-tensors with source term $T_{\mu\nu}$. For the wave equation to be well defined, global hyperbolicity is important.

2. Wormhole solutions

2.1 Realization of wormholes in the literature

As a physics student, that has not taken any topology course, the first encounter with wormholes should be in a General Relativity course, where the Schwarszchild black hole solution is studied extensively. So, let us begin with a qualitative review of that wormhole.

The maximal analytic extension of the Schwarschild solution gives the Kruskal diagram 2.1, where right and left there are two assymptotically flat causally disconnected regions, and up and down are a black hole and a white hole. The causal disconnectedness can be easily seen with the help of the light cones. If an observer is in either of the assymptotic regions, they only end in the singularity to their future.

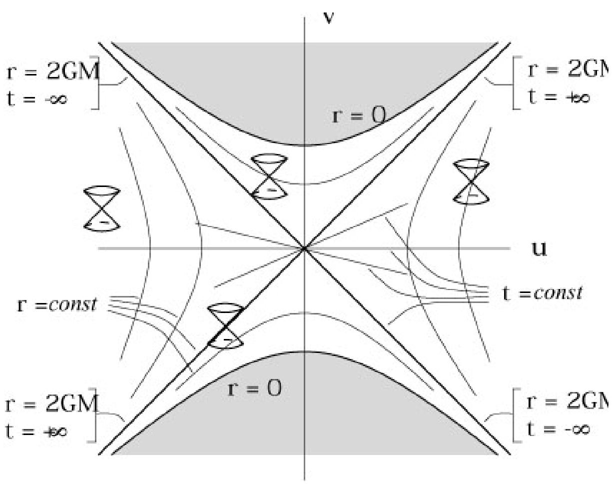

Figure 2.1: Kruskal diagram of the maximal analytic extension of Schwarszchild's black hole solution. The shaded regions are the black hole(up) and the white hole(down) singularities, the horizon at $R = 2GM$, the constant t slices and the constant r slices are shown in the figure. Null paths in this diagram are all oriented at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ angles. The future and past timelike infinity are also shown. Figure taken from [14]

The Schwarszchild spacetime appears to have a wormhole at the $t = 0$ slice, with topology R $\times$ S^2 which is depicted in the following picture 2.2.[1] In that $t = 0$ slice, while going to smaller and smaller values of the radial coordinate $r-$, the size of the $2-$ sphere at each constant r decreases. The fact that it does not shrink to zero is interpreted as a wormhole throat.

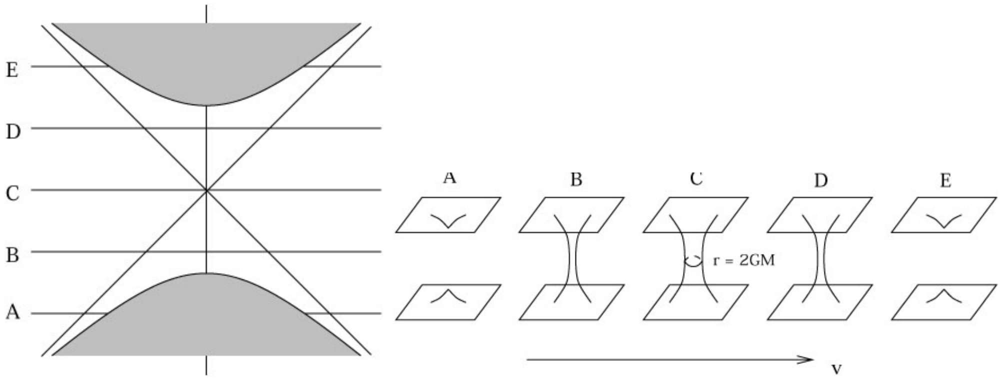

Figure 2.2: Taking horizontal slices in the Kruskal diagram we can see the formation of a wormhole at the $t = 0$ slice, with the topology $\mathrm{R} \times \mathrm{S}^2$. In the right figure one dimension is suppressed, so each circle corresponds to a sphere. The wormhole has a throat of minimal finite size placed at $r = 2GM$. Figure taken from [14]

2.2 The peculiar property of traversability

The notion of traversability is an important property that, crudely speaking, means that an entity can travel through the throat of the wormhole at finite time. A more rigorous definition of traversability says that "*If a spacetime contains a causal path that begins and ends at spatial infinity and cannot be continuously deformed to a causal curve that lies entirely in the spatial boundary region, the spacetime contains a* ***traversable*** *wormhole.*" [15]. Some useful definitions taken from [15] will follow here

- *intra-universe wormhole:* a wormhole that connects two regions of spacetime that are in the same universe
- *inter-universe wormhole:* a wormhole that connects two regions of spacetime that belong to different universes
- *short wormhole:* a wormhole creating a causal path that takes shorter to go through than to go around

[1]Not all horizontal slices correspond to constant time, but to constant Kruskal coordinate v. Only slice C does is a constant $t = 0$ slice.

- *long wormhole:* a wormhole creating a causal path that takes longer to go through than to go around
- *eternal wormhole:* given a spacetime that admits a time-function, if there is no topology change with respect to time and a wormhole exists on every constant-time hypersurface, then the wormhole is *eternal*
- *eternally traversable wormhole:* an eternal wormhole is traversable for all time

Some immediate rational questions might arise at this point. For example, what does it really mean that there are two separate universes, when they are connected by an *inter-universe* wormhole?

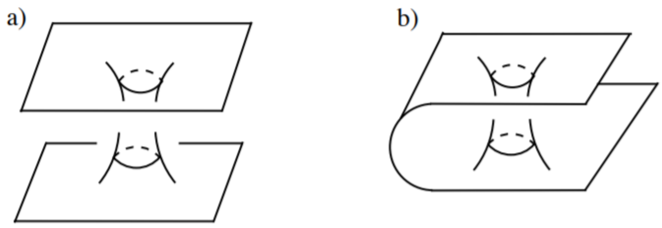

Figure 2.3: (a) Is an inter-universe wormhole that connects two disconnected assymptotic regions. (b) The boundary is identified and the wormholes are intra-universe. Topologically, taking the cover of picture (b) one can have two copies of the picture (a).

A suitable remark at this point is that the wormhole of figure 2.2, that appears in the Schwarszchild solution is neither traversable, nor eternal. Traversable wormholes are, at least classically, not allowed. The reason is that they would lead to causality violations. More specifically, inter-universe wormholes are not allowed. However, in the case of intra-universe traversable wormholes, they are allowed if they are long. In the semi classical regime, they are allowed if they are supported by exotic matter. A more detailed discussion about these matters will continue in chapter 4.

2.3 Notable wormhole constructions in the literature

Introducing wormholes with the example of the Schwarszchild solution might be misleading. Wormholes can be contained in spacetimes with or without black holes. They are multiply connected geometries that can be solutions to Einstein's field equations, with or without fields, with in turn can be either classical or quantum.

Wormholes were first proposed classically by Weyl, without really having that name yet. His hypothesis involved the possibility that sourceless electromagnetic field energy can cause the spacetime to curve, giving mass in some region, where the field lines are trapped inside a one dimensional tube.[see figure 2.6] The term "wormhole" was given by J. Wheeler in the analysis with C. Misner in the Annals of Physics[1]. Inside this analysis, inspired by Weyl's idea, he states that they are the "handles" of multiply connected geometries. A topological definition of what a wormhole is, does not apply in this thesis, because the definitions that exist in literature concern Minkowski spacetime. A definition is contained in Visser's book for Lorentzian traversable wormholes.[16] A geometrical definition is the one depicted in the figure 2.2. *"A wormhole is a region of spacetime that contains a "world tube", i.e. the time evolution of a closed surface, that cannot be continuously deformed to a world line, namely, the time evolution of a point."*[17]

The starting point for the study of wormholes was the Schwarschild eternal black hole solution, where a wormhole makes its appearance at the $t = 0$ slice. As mentioned above, the Schwarszchild wormholes are not eternal. Namely, they collapse very quickly to a black hole and a white hole. Those are also known as Einstein-Rosen bridges and are not traversable. Inspired by Schwarchild wormholes, Kip Thorne first introduced the idea that negative energy matter, also referred to as exotic matter, can maintain the throat open, preventing gravitational collapse. In fact, Stephen Hawking in [18], Kip Thorne in [19] and other physicists argued that quantum effects, such as the Casimir effect in quantum field theory, allow negative energy density, which makes it possible to stabilize the throat.

The first traversable Lorentzian wormholes were introduced independently by Ellis in [20] and Bronnikov in [21], in 1973. These papers present solutions to gravity with a minimally coupled scalar field, which has negative polarity instead of positive. This kind of matter renders the throat open and produces a solution known as the Ellis drainhole, which is horizonless, singularity-free, geodesically complete and traversable. A bit later, in 1988, Thorne and his PhD student Morris introduce the same solution for educational purposes, without knowing the existence of Ellis's and Bronnikov's papers. This remained in the literature as the Morris-Thorne wormhole. Last but not least, another class of traversable Lorentzian wormholes supported by negative energy matter was analyzed in Visser's book in 1989 [16]. Wormholes that are solutions to extended theories of gravity can be proven traversable without requiring the existence of matter, but are not in the interests of this thesis.

Recent advances in the wormhole literature become more and more interesting. In 2013, Susskind and Maldacena in [22] conjectured a connection between traversability and quantum teleportation, known as the ER = EPR conjecture. This conjecture, makes the study of traversable wormholes quite promising. A notable construction is that of Gao, Jafferis and Wall in [23]. In this paper, they work on an eternal BTZ black hole solution, where an interaction through a double trace deformation between the two boundary CFTs creates a quantum matter stress energy tensor with enough negative energy to support the throat and allow traversability. Another example of traversable wormholes is introduced by Maldacena in [24]. The wormholes in that paper are solutions to Einstein-Maxwell theory with charged massless fermions, which are responsible for the negative energy that supports the throat. They are assymptotically flat, but can also be made assympotically AdS.

Wormholes have been fully understood classically in the literature. Traversability and stability are two properties that seem to be closely related. It has been argued that these properties require matter with negative energy to support the throat of the wormhole from collapsing. In the following section, a solution which was proposed by Anabalon in [2] is being presented, where non-exotic matter supports the throat. In this thesis, we will analyze some aspects of these solutions and how they reflect to the already existing literature.

2.4 Introducing the model

Recall that the action of Einstein-Maxwell theory contains an Einstein-Hilbert term with or without a cosmological constant and a dynamical term for the gauge field, which is minimally coupled to the gravitational field.

$$S = \int d^4x\sqrt{-g}\left[(R+2\Lambda)+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.1}$$

where R is the Ricci scalar, g is the determinant of the metric tensor field, Λ is the cosmological constant term and $F^{\mu\nu}$ is the electromagnetic field strength tensor. This action can have many solutions, one of which is of course empty AdS, for negative cosmological constant and the electromagnetic field turned off.

This theory admits an assymptotically locally AdS solution that contains a wormhole with Maxwell fields. The remarkable feature of those solutions is that the vacuum version of them, as we shall see is dynamical. Let us introduce the model

The metric field $g_{\mu\nu}$ is of the form

$$ds^2 = \frac{4l^4}{\sigma^2 f(r)}dr^2 + h(r)\left(-\cosh^2\theta dt^2 + d\theta^2\right) + f(r)\left(du+\sinh\theta dt\right)^2 \tag{2.2}$$

with $\{t,r\}\in\mathbb{R}$, $\theta\in[\,0,+\infty\,)$, the coordinate u is identified, $u+\alpha=u$.

where

$$f(r) = \frac{4l^2}{\sigma^2}\frac{r^4+(6-\sigma)r^2+mr+\sigma-3}{r^2+1} - \frac{Q^2+P^2}{r^2+1} \quad , \quad h(r) = \frac{l^2}{\sigma}(r^2+1)$$

The matter section of the spacetime is characterized by an electromagnetic field with field strength tensor $F_{\mu\nu}$. The non-zero components of it are

$$F_{ru} = -F_{ur} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2} \tag{2.3}$$

$$F_{rt} = -F_{tr} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2} \sinh\theta = F_{ru}\sinh\theta \tag{2.4}$$

$$F_{\theta t} = -F_{t\theta} = \frac{-2Qr + P(1-r^2)}{r^2+1} \cosh\theta \tag{2.5}$$

The gauge field that produces this field strength is the following

$$A = \Phi(r)\,(du + \sinh\theta dt)\,, \qquad \Phi(r) = \frac{2Qr + P(1-r^2)}{r^2+1} \tag{2.6}$$

Einstein-Maxwell's equations are satisfied for this solution. In this thesis we shall see what the parameters in this solution mean. We expect that m is linked to the mass, σ is just a parameter defining the model and contributes to the volume, l is the radius of curvature of AdS and, thus, a physical scale. Our expectation is that Q and P are related to static electric and magnetic charges[2], which will be proven wrong.[3]

The rest of the details for the model used for the calculations are included in Appendix C.

2.4.1 On the definition of the metric

The different spacetimes are characterised by a set of parameters $\{\sigma, X, m\}$. Those, of course, cannot take arbitrary values, whereas they should agree with several physical constraints.

Translation of the constraints onto the parameters

Some thoughts on what these constraints could be are listed below:

1. $f(r)$ should not have real roots in order to avoid coordinate singularities. Its denominator being always positive, its roots are the same as its numerator's roots. Since its numerator is a quartic polynomial, if its discriminant is positive then it either has all roots real or not real. Then, there are a couple of more inequalities that should be satisfied, which lead us to the case of all roots being real.

[2] As we shall see later on, these are not even charges.

[3] At this point it is useful to note that there is nothing wrong with including a magnetic charge. It is a theoretical solution and the magnetic monopole charge enters the metric in the same way as the electric one. The exact same comment applies for the charged Reissner-Nordstrom black hole.

2. setting $Q = P = 0$ in the discriminant inequality should give the exact same constraints as in the uncharged case. Those are $3 < \sigma < 12$ and $|m| < \frac{2}{3\sqrt{3}} (12 - \sigma) \sqrt{\sigma - 3}$
3. the measure which shows the size of the throat should be positive. This essentially gives $\frac{4l^2}{\sigma^2}(\sigma - 3) > (Q^2 + P^2)$

As it has already been pointed out, if $f(r)$ has no real roots, it will have a constant sign, since it is a continuous function. In our case, this sign will be positive, namely $f(r) > 0$, because the dominant term in the polynomial has a positive sign.

Determining the sign of a quartic function

The analysis requires the study of quartic polynomials. This is because in $f(r)$, for example, the denominator is always positive, thus what is left to study is the numerator, which is a fourth order polynomial. If we demand that it has no real roots, it is the same as demanding that it has a constant sign. We need the non-real roots to avoid singularities and as a gift we get automatically the positivity of $f(r)$.

The form of our polynomial is $r^4 + cr^2 + dr + e$ and for its study one needs to define the discriminant Δ.

$$\Delta = 256e^3 - 128c^2e^2 + 144cd^2e - 27d^4 + 16c^4e - 4c^3d^2 \tag{2.7}$$

In order to have all roots either all real or all imaginary, a positive discriminant is required, $\Delta > 0$. To restrict ourselves to the case that all roots are imaginary, so that a coordinate singularity is avoided in the metric, the constraints, in total, translate to:

$$\left((\Delta > 0) \wedge \left(\left(c < 0 \wedge e > \frac{c^2}{4}\right)\right) \vee (c \geq 0)\right) \vee \left((\Delta = 0) \wedge \left(e = \frac{c^2}{4}\right) \wedge (d = 0)\right)$$

Final Bounds

After taking into account all those physical constraints we end up with the following allowed intervals for $\left\{\sigma, m, X \equiv \frac{3(Q^2+P^2)}{l^2}\right\}$. Note that these parameters are those that define different spacetimes with similar properties. Thus, finding bounds on them, such that the spacetimes are well defined, meaning no coordinate singularities, and obeying the energy conditions such that traversability is not allowed.[4]

For the charged case the positivity of $f(r)$ gives:

[4]Here, in fact, we are trying to probe the validity of the NEC as the condition to rule out traversability. Namely, what we are trying to achieve is to prove that a spacetime can obey the NEC and, simoultaneously, have a traversable wormhole

- $(\Delta > 0) \wedge \left(c < 0 \wedge e > \frac{c^2}{4}\right)$ gives:

$$\begin{aligned}
&(\sigma > 6) \wedge \left(\sigma - 3 - \frac{X\sigma^2}{12} > \frac{(6-\sigma)^2}{4}\right) \implies \\
&0 < X < \frac{(-144 + 48\sigma - 3\sigma^2)}{\sigma^2} \implies \\
&\left(\frac{24 + 12\sqrt{1-X}}{3+X} > \sigma > 6\right) \quad \wedge \quad (0 \leq X < 1)
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\Delta > 0 \implies \\
&|m| < \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{18\left(-48 + 24\sigma - (3+X)\sigma^2\right) + \sigma^3\left(1 + 3X + (1-X)\sqrt{1-X}\right)}
\end{aligned} \tag{2.9}$$

- $(\Delta > 0) \wedge (c \geq 0)$ gives:

$$\begin{aligned}
&0 < \sigma \leq 6 \\
&0 \leq X < \frac{12}{\sigma^2}(\sigma - 3) \implies \left(6 \geq \sigma > \frac{6 - 6\sqrt{1-X}}{X}\right) \wedge (0 \leq X < 1) \\
&|m| < \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{18\left(-48 + 24\sigma - (3+X)\sigma^2\right) + \sigma^3\left(1 + 3X + (1-X)\sqrt{1-X}\right)}
\end{aligned} \tag{2.10}$$

- $(\Delta = 0) \wedge \left(e = \frac{c^2}{4}\right) \wedge (d = 0)$ gives:

$$\left(\sigma = \frac{24 \pm 12\sqrt{1-X}}{3+X}\right) \wedge (X < 1) \wedge (m = 0) \tag{2.11}$$

If we combine the first two sets of constraints,

$$\begin{aligned}
&X < 1 \\
&\frac{24 + 12\sqrt{1-X}}{3+X} \geq \sigma > \frac{6 - 6\sqrt{1-X}}{X} \\
0 \leq\ &|m| < \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{18\left(-48 + 24\sigma - (3+X)\sigma^2\right) + \sigma^3\left(1 + 3X + (1-X)\sqrt{1-X}\right)}
\end{aligned} \tag{2.12}$$

The third constraint is a separate case. The proof that those constraints agree exactly with the constrants (3.9) of [3] can be found in the appendix (which I haven't written yet).
For the uncharged case the positivity of $f(r)$ gives:

- $(\Delta > 0) \wedge \left(c < 0 \wedge e > \frac{c^2}{4}\right)$ gives:

$$6 < \sigma < 12 \tag{2.13}$$
$$|m| < \frac{2}{3\sqrt{3}}\sqrt{-432 + 216\sigma - 27\sigma^2 + \sigma^3} \tag{2.14}$$

- $(\Delta > 0) \wedge (c \geq 0)$ gives:

$$\left(((3 < \sigma < 4) \vee (4 < \sigma \leq 6)) \wedge \left(|m| < \frac{2}{3\sqrt{3}}\sqrt{-432 + 216\sigma - 27\sigma^2 + \sigma^3}\right)\right) \tag{2.15}$$
$$\vee \left((\sigma = 4) \wedge \left(0 < |m| < \frac{16}{3\sqrt{3}}\right)\right) \tag{2.16}$$

- $(\Delta = 0) \wedge \left(e = \frac{c^2}{4}\right) \wedge (d = 0)$ gives:

$$(\sigma = 4 \vee \sigma = 12) \wedge (m = 0) \tag{2.17}$$

Considered altogether they give:

$$3 < \sigma \leq 12 \qquad 0 \leq |m| < \frac{2}{3\sqrt{3}}\sqrt{-432 + 216\sigma - 27\sigma^2 + \sigma^3} \tag{2.18}$$

which are consistent with each other, in the sense that setting $X = 0$ to the first set of constraints gives the second set of constraints. The constraint for $|m|$ in the charged case is exactly as the one in (3.9) of [3] after manipulations. Both sides of the inequality (2.12) for σ are equivalent to those in (3.9) of [3].[5]

Comments on the positivity of $f(r)$

The function $f(r)$ can not have real zeros, since in that way the metric is ill defined and the distances blow up. We concluded that no zeros means positive sign. This positivity plays a role in various aspects regarding the spacetime solution.

First of all, it is a pathological case when a spacetime has closed timelike curves, unless one wants to plan an itinerary to the past, kill their grandparents to end their own existence, leaving physicists puzzled with many paradoxes.[6] The positivity of $f(r)$ plays a role in proving that this spacetime solution is free of closed

[5] The fact that $\frac{1}{1+X+\sqrt{1-X}} = \frac{1+X-\sqrt{1-X}}{X(X+3)}$ was extensively used to show that they are equivalent.see appendix

[6] Well, the absence of closed timelike curves is not the only unphysical aspect. The universe having negative curvature or containing traversable wormholes is also not so realistic, but at least it is not pathological. Space travel is also not promised. This solution can only be seen as just a theoretical tool.

timelike curves. See proof in [2]. As proven in this thesis the null energy condition satisfied, thanks to the positivity of this function $f(r)$, which means that the matter contained in this spacetime is regular and non-exotic. Positivity of the throat size is also an important issue in the case that we want to avoid negative lengths and $f(r)$ being positive assures it.

2.4.2 Construction and Group Structure

Construction

The construction of [2] is based on *squashing* AdS_3 and, then embedding and *warping* it into AdS_4. But let us begin with some definitions in the direction of understanding all those terms, that might look scary at first sight.

The term *squashed* refers to the lorentzian analog of the squashed three sphere, studied in [25]. The 3−sphere is a homogenous space that can be seen as a hopf fibration with base manifold a 2− sphere and hopf fiber a circle, i.e. $\boldsymbol{S^3}/\boldsymbol{S^1} = \boldsymbol{S}^2$.[See appendix B] It, roughly speaking, means that the 3−sphere can be seen as a 2−sphere, attaching a circle S^1 of different size at each point. Now, qquashing it means to deform it along each fiber [See appendix B]. A lorentzian analog of this is writing AdS_3 as a hopf fibration, $\boldsymbol{AdS_3}/\mathbb{R} = AdS_2$. Squashing it along one fiber, one can get

$$ds^2_\lambda = \frac{1}{4}\left(-\cosh^2\theta dt^2 + d\theta^2 + \lambda\left(du + \sinh\theta dt\right)^2\right) \tag{2.19}$$

with $\{t, u\} \in \mathbb{R}$, $\theta \in [\,0, +\infty\,)$.[7]

Setting this parameter λ to zero gives empty AdS_2. It is an important remark that the analog is highly non trivial for higher dimensions.[8] Setting $\lambda = 1$ we just have empty, homogenous and isotropic AdS_3. For other λ the AdS_3 is homogenous, but not isotropic.

Embedding the empty version of this construction (2.19) in empty AdS_4 is a straightforward task.[9]

$$ds^2 = \frac{l^2 dr^2}{r^2+1} + \frac{l^2}{4}\left(r^2+1\right)\left[-\cosh^2\theta dt^2 + d\theta^2 + \left(du + \sinh\theta dt\right)^2\right] \tag{2.20}$$

with $\{t, r, u\} \in \mathbb{R}$, $\theta \in [\,0, +\infty\,)$.

Now, the term *warping* refers to a generalization of the Cartesian product of the y geometry and the x geometry, with the exception that the x part is warped,

[7]The positivity of θ is mentioned in [25] and has to do with the fact that θ is the radial coordinate of the AdS_3 slices

[8]It was one of the failed attempts of this thesis to try to do the same construction in one dimension higher.

[9]See also chapter 3 for more details on how this parametrization of AdS_4 works and how to add extra coordinates

meaning that it is rescaled by a scalar function of the other coordinates y, $f(y)$. A warped geometry is a Riemannian or Lorentzian manifold whose metric tensor can be written in form

$$ds^2 = g_{ab}(y)dy^a dy^b + f(y)g_{ij}dx^i dx^j \tag{2.21}$$

Warping the metric (2.20) of empty AdS_4 in those weird coordinates with arbitrary functions of the radial coordinate r of AdS_4, means that we make it anisotropic by stretching it as a function of r along several directions. One ends up with the metric of the wormhole solution (2.2), which has the metric similar to (2.19) at the conformal coundary. In fact the induced metric on the conformal boundary is calculated as

$$ds^2_\sigma = \frac{l^2}{\sigma}\left(-\cosh^2\theta dt^2 + d\theta^2 + \frac{4}{\sigma}\left(du + \sinh\theta dt\right)^2\right) \tag{2.22}$$

As a matter of fact, the Schwarschild solution or the FLRW metric are examples of warped geometries as well.

Group structure

The function $f(r)$ breaks the isometry group of the AdS_3, $\mathrm{SO}(2,2) \simeq \mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$ to $\mathbb{R}\times\mathrm{SO}(1,2)$. That describes the isometry breaking of the disconnected boundaries $r \to \pm\infty$.

Global AdS_4 has the isometry group $SO(2,3)$. In the wormhole solution, the warping functions $f(r)$ and $h(r)$ together with the fields break this isometry group of each $r-$slice to its subgroup $\mathrm{SO}(2,1)\times\mathrm{U}(1)$[10], generated by the Killing vectors. Since, the functions only depend on r, they are unaffected by the Killing vecors, shown in appendix C. The solution is thought of as warped AdS_3 embedded in AdS_4.

2.4.3 Distinction from empty AdS

The metric of empty AdS in these weird coordinates seems to have a wormhole as well. There is, though, a global parametrization leading to the embedding coordinates that define empty AdS, as we shall see later on in chapter 3. Both solutions solve the same equations of motion and have the same Ricci scalar everywhere $R = -\frac{12}{l^2}$ and the same cosmological constant. Full information about the curvature fails to be captured in the Ricci scalar. Similar example is the Schwarschild solution, which is Ricci flat, but has a curvature singularity in the center.

Hence, it is useful to plot the Kretschmann or Weyl invariants for the empty and the wormhole parametrization to see the differences. The higher curvature invariants are nowhere singular in the wormhole spacetime, and are zero in the

[10]Recall that u is identified

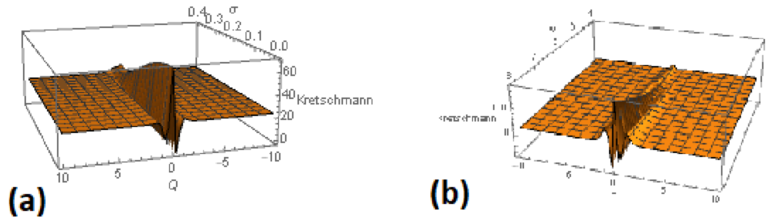

Figure 2.4: (a) Kretschmann invariant for different σ and r for $Q = 0.4, m = 0.4, l = 1, P = 0$ (b) Kretschmann invariant for different Q and r for $\sigma = 5, m = 0.4, l = 1, P = 0$. σ defines different spacetime solutions, while Q defines different charges for a specific σ

empty case. They show that the wormhole spacetime is highly curved close to the center, where a throat is expected to hold. The Kretschmann invariant is

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{2.23}$$

and the Weyl invariant,

$$W = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \tag{2.24}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and there expressions are included in the appendix. For the empty AdS solution in any coordinates, also in these ones, the invariants are as follows

$$K = \frac{12}{l^4} \tag{2.25}$$

$$W = 0 \tag{2.26}$$

For the AdS wormhole solution the Kretschmann invariant is depicted in the figure 2.4, while the Weyl invariant is non-zero and has a similar plot. The Weyl tensor being non-zero already tells us that this is not an empty AdS solution.

2.4.4 Arguments concerning the wormholeness

As it has been highlighted above, the realization of wormholes in the literature involve a "radial"[11] coordinate and a minimum positive volume of the spatial sections for each constant value of this radial coordinate. In the maximal analytic extension of the Schwarchild solution, we saw that the $t = 0$ slice includes a wormhole. The spatial sections of constant t, r have a minimal volume instead of being zero. This is interpreted as a wormhole. In the $AdS-$ like slicing of this

[11]The term "radial" is used for spherically symmetric spacetimes. Here there is axial symmetry, but the word radial means that small values of this coordinate brings us to the bulk of the spacetime, while plus and minus infinity corresponds to the boundary

solution, there is the exact same picture. The difference lies in the fact that in each slice of constant r, instead of a compact sphere, there is a non-compact warped AdS_3. The non-zero size of those sections indicate a non vanishing throat.

In [2] the graph representing the existence of such a throat for the values of the parameters that are allowed is the determinant of the spatial sections of constant t, r.

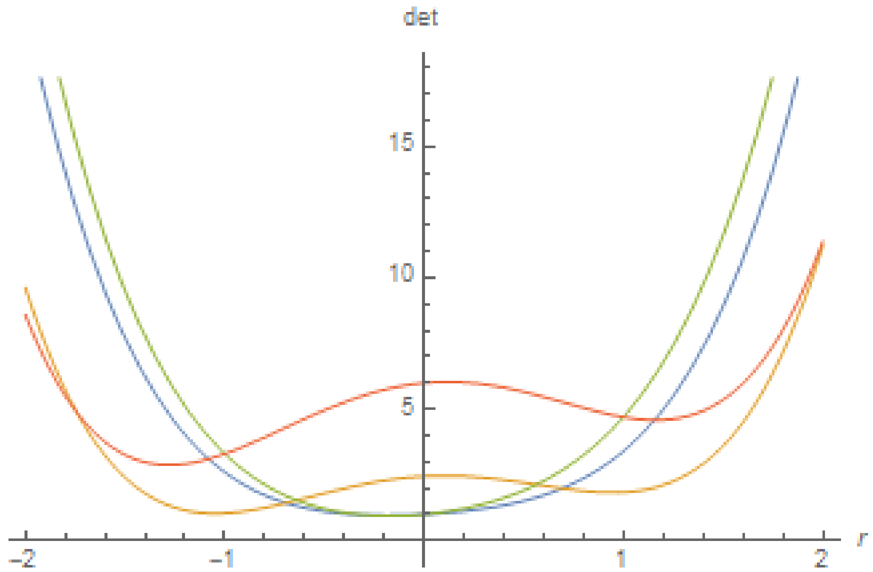

Figure 2.5: Determinant of spatial sections of constant t, r with respect to r, i.e $det = f(r)h(r)l^{-4}\sigma^3/4$. All plots are made for $l = 1, m = 0.4, P = 0$. Plot 1 & 2 are for $\sigma = 5, Q = 0.4$ and for $\sigma = 8, Q = 0.4$. Plots 3 & 4 are for $\sigma = 4, Q = 0$ and $\sigma = 9, Q = 0$. With these plots we have an indication of wormholeness for both the charged and uncharged case. For $\sigma > 0$ there seems to be two antithroats and a throat. This means that the size of the throat becomes minimum for small positive values of r, then experiences a local maximum around zero, and then a local minimum again, for negative r.

The remarkable aspect is that there seems to be a wormhole even without matter supporting the throat. Miracoulously, vacuum solution for $Q, P = 0$ seems to contain a wormhole as well. This is clearly a feature of AdS spacetime, where, due to the negative curvature, it is made possible to construct a wormhole maintained by the geometry itself. This stands as a counterexample to the traditional approach of wormholes introduced in the historical paper by Missner and Wheeler [1]. In this paper they discuss assymptotically flat wormholes in Einstein-Maxwell theory, where the throat can be supported by the existence of electromagnetic field lines.

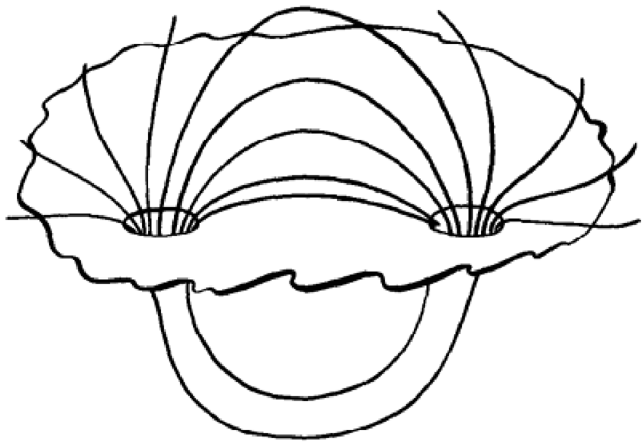

Figure 2.6: Historical schematic figure of a wormhole supported by Maxwell electromagnetic fields. This is a constant time slice, where one spatial dimension is suppressed. The two dimensional curved and multiply connected space is embedded in a three dimensional Euclidean space in this picture. The two-space inside the tunnel looks the same as the two-space elsewhere. The field lines are trapped inside the wormhole topology, they obey Maxwell's sourceless equations and are free of singularities. An observer inside the tunnel would faslingly notice two point charges, without accounting for the topology. In fact, seeing it from outside, electromagnetic flux going into any bounded surface goes out of the surface intact. This picture corresponds to unquantized classical charge, and has nothing to do with the charge thought in elementary particle physics. Electromagnetic energy around the mouth of the wormhole gives mass to the region. This is what is referred to as the Weyl idea of using empty curved space to describe gravitation without gravitation, electromagnetism without electromagnetism, charge without charge and mass without mass. Figure and explanation taken by [1]

Ideas on confirming the wormholeness

It is, obviously, a rather challenging task to verify the existence of wormhole topology in the spacetime solution. Our understanding is that plotting the size of the spatial cross sections and identifying a coordinate are not enough to prove that there is a non contractible cycle in the spacetime. It might be the case that this wormholeness picture is just an artefact of the coordinates. For example, the determinant of the spatial cross sections for the empty case, $\sigma = 4, f(r) = h(r)$ is also nowhere vanishing, but as we shall see this case is just empty AdS written in complicated coordinates.

In order to understand the womrholeness nature of the spacetime solution, one needs to calculate quantities that are independent of the coordinates, i.e. a topological invariant. As a matter of fact, the metric cannot capture all topological aspects of the manifold. A topological invariant of interest, in order to prove that there is a non-contractible cycle, is the euler characteristic. This can be computed as an integral over some geometrical forms by virtue of the Gauss-Bonnet theorem. This theorem is a special case of the Atiyah-Singer index theorem, which relates topology and geometry.

Another idea to predict the wormholeness, would be by studying Raychaudhuri's equation. This equation studies the behaviour of the flow of a timelike or null geodesic congruense,[12] which initiate orthogonally to a spacelike hypersurface. It can, basically, be derived by Einstein's field equations in a particular frame. Through this equation, if the null energy condition is satisfied[13], one can predict whether null geodesics, starting orthogonally to a spacelike hypersurface, will end up focusing. In black hole spacetimes, that is the way to predict coordinate singularities, but not curvature ones. The coordinate system collapses if they focus. In the case of wormholes, Raychaudhuri's equation can give a prediction on whether the null geodesics will focus close to the throat, and there should be, in principle, something to keep them apart. Since, in this spacetime, there does not seem to be any exotic matter to keep the throat apart, if the expansion parameter shows that they should focus, we need to interpret this. So, what keeps the null geodesics coming through the throat apart? Our naive intuition tells us that the non-compactness of each slice allows null geodesics to focus in a neighbourhood close to the throat $r \simeq r_0$, only in one direction, but be arbitrarily far apart in the rest of the directions. That picture could explain why they do not coincide.

An alternative way of examining the wormholeness topology would be by calculating the fundamental group of the manifold. This would show whether there are any holes. However, that would require writing the warped product manifold as a group first, which might be impossible. Writing it as a quotient of two groups should not be possible, since this is some property that homogenous spaces have. The calculation of the fundamental group is not an easy task even for simple manifolds. Hence, it might not even be an option for us.

[12] There is a version of it for general timelike or null paths. But we are particularly interested in a congruence of null geodesics.

[13] There will be a detailed discussion about energy conditions in due time.

3. Higher dimensional origin

3.1 Motivation from String Theory

String theory is believed to be a theory of quantum gravity, in the sense that it always involves a massless spin-2 particle, that are interpreted as the graviton[26]. It is, basically, a conformal field theory on the two dimensional worldsheet of a string[1] coupled to background fields that enjoy Lorentz symmetry in D dimensions. The Polyakov action on the world sheet has the following form

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}(X) \tag{3.1}$$

where $\sigma^\alpha,\ \alpha = 0, 1$ parametrize the worldsheet Σ of the string, $h_{\alpha\beta}$ is the metric on Σ and h its determinant, α' is a constant related to the tension of the string, which has units (length)^2, $X^M,\ M = 0, \ldots, D-1$ are functions attached on each point of the worldsheet Σ, with target space a spacetime manifold $\mathcal{M}$ with metric $G_{MN}(X)$.

The background fields that are important are the massless ones, and this has to do with quantizing the theory and, simoultaneously, preserving Lorentz symmetry. These are, in fact, irreducible representations of the little group $SO(D-2)$ of the Lorentz group $SO(1, D-1)$, which involve a symmetric tensor $G_{\mu\nu}$, thought of as the background spacetime, an antisymmetric one, $B_{\mu\nu}$, which is called the Kalb-Ramond field and a scalar one, ϕ, which is referred to as the dilaton, and they all generalize the idea of a coupling constant. In superstring theory, one can, in principle, have all, or some, of these bosonic background fields turned on and some extra fermionic fields that are there to preserve supersymmetry. The fact that (super)string theory is a two dimensional conformal field theory on the worldsheet is crucial, because it is a Weyl invariant theory and its symmetry group has infinite generators, that might be able to produce all the known particles. Another important remark, is that in order to properly quantize this conformal field theory, weyl invariance should be maintained in the quantum regime. This, essentially, translates to requiring that the beta functions, related to the background fields to be zero. The beta functions express the running of the coupling constants with

[1] in analogy to the one dimensional worldline of a point particle

respect to energy. With this requirement, it turns out that superstrings can be properly quantized and propagate in backgrounds with $D = 10$. This is also referred to as the *critical dimension* of superstring theory. For another theory called M-theory, this dimension is $D = 11$. Hence, superstring theory or M-theory are candidates capable of unifying in high energies all the interaction theories that describe our world.

However, spacetime that we live in and observe is, of course, four dimensional. In order to resolve the discrepancy between the ten dimensions of superstring theory and the four dimensions of our observations, the idea of compactification comes into the game. For a more detailed and didactic view of the idea of compactification in string theory see [27, 28]. The fundamental idea of compactification is that there exist solutions of the $D = 10$ theory that can be consistently decomposed, for example, as a product manifold $\mathcal{M}_{10} = M_4 \times \mathcal{K}_6$, where M_4 corresponds to the four dimensional spacetime and $\mathcal{K}_6$ to a compact manifold, also called the internal manifold on which the reduction happens. In order to explain why the extra dimensions are not observed in our four dimensional world, they should be much smaller in size than the characteristic length scales known from particle accelerators. In this way, one can consistently reduce the theory to a lower dimensional theory, which will have an imprint coming from the 6 dimensional compact manifold. The program that is being used to compactify the extra dimensions is based on the Kaluza-Klein dimensional reduction, and is a generalization of it.

The Kaluza-Klein idea was an attempt to unify four dimensional gravity and electromagnetism via describing both interactions as a purely geometrical gravity theory in one dimension higher. I will try to briefly discuss the Kaluza-Klein hypothesis and the generalization of the idea to dimensional reduction of any theory in due course, for two reasons. Firstly, this hypothesis was one of our ansatze, as well, to try to embed the AdS wormhole solution into 5D. Simoultaneously, it provides the base idea related to how compactifications are done in string theory, which is useful so as to understand how the wormhole solution can be embeded to string theory. Recall, that our motivation is that this wormhole solution being a bosonic solution of a $D = 4, \mathcal{N} = 2$ AdS supergravity action, suggests that it is a stable solution, and, thus, an important contribution to the path integral of quantum gravity. Finding its higher dimensional origin is important, not only in order to see the bigger picture of it geometrically, but also to find its non-perturbative UV completion.

A four dimensional solution can have many UV completions. The term *consistent* being used in Kaluza-Klein compactifications is linked to the fact that all solutions of the lower dimensional theory should be solutions of the higher dimensional one. In fact, after the dimensional reduction, the imprint that the internal manifold has on the four dimensional one, is an infinite tower of "light" and "heavy" fields. The four dimensional solution obeys the equations of motion of the "light" fields only. Therefore, the on shell "light" fields should by no means source the "heavy" ones, such that after the reduction one can just turn them off and have the desired lower dimensional theory that gives the correct equations of motion. The consistent uplifting of all solutions of $D = 4, \mathcal{N} = 2$ AdS supergravity theories to $D = 11$ M-Theory[2] has been studied generically and consistently in [4]. In this paper, they prove that any minimal gauged supergravity can be uplifted on an arbitrary seven-dimensional Einstein-Sasaki manifold. Moreover, in [29] they study another consistent and generic uplift of AdS $D = 4, \mathcal{N} = 2$ solutions to type IIB supergravity in two steps. Starting from a braneworld Kaluza Klein ansatz, the general solutions are consistently uplifted to $D = 5, \mathcal{N} = 4$ gauged supergravity first and then to type IIB supergravity.

3.1.1 A brief review of Kaluza-Klein reduction on a circle

In this subsection, a very brief explanation of the Kaluza-Klein reduction is in order.

The Kaluza-Klein hypothesis

Firstly, it is useful to explain the principal idea of describing gravity and electromagnetism in four dimensions as pure gravity theory in one dimension higher. The Kaluza-Klein hypothesis uses the $4D$ metric, the electromagnetic $4-$vector, A_μ and an auxiliary field ϕ, accompanied by a cylinder condition, in order to construct a $5D$ metric that solves the vacuum Einstein's equations. The ansatz is as follows

$$\tilde{g}_{MN} \equiv \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{3.2}$$

The line element can be written as

$$ds^2 = \tilde{g}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + \phi^2 \left(A_\nu dx^\nu + dx^5 \right)^2 \tag{3.3}$$

where $M, N = 0, \ldots, 5$ and $\mu, \nu = 0, \ldots, 4$.

The cylinder condition suggests that the metric does not depend on the fifth coordinate

$$\frac{\partial \tilde{g}_{MN}}{\partial x^5} = 0 \tag{3.4}$$

The $5D$ action is the Einstein-Hilbert action in five dimensions is

$$S = \int d^5x \sqrt{-\tilde{g}} \tilde{R} \tag{3.5}$$

[2]M-Theory and string theory are dual to each other. The critical dimension for M-Theory is $D = 11$.

The five dimensional vacuum equations of motion decompose to the four dimensional ones as follows

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 0 \implies \tag{3.6}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\phi^2\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\partial_\rho\partial^\rho\phi\right) \tag{3.7}$$

The remaining field equations are

$$\tilde{R}_{55} \implies \partial_\mu\partial^\mu\phi = \frac{1}{4}\phi^3F^{\alpha\beta}F_{\alpha\beta} \tag{3.8}$$

$$\tilde{R}_{5M} = 0 = \frac{1}{2}g^{\mu N}\nabla_\mu\left(\phi^3F_{MN}\right) \tag{3.9}$$

where quantities that have a tilde on top are associated to the $5D$ metric and the quantities that don't have a tilde are associated to the $4D$ metric.

Reduction of a scalar field theory in 5D Minkowski

Conversly thinking, the Kaluza-Klein hypothesis is a dimensional reduction of a gravitational theory, and it will shortly be obvious why this cylinder condition mentioned above is as effective as identifying the fifth coordinate on a circle and requiring that its size is zero.

Let us, now, see what happens for the simplest dimensional reduction of a theory with matter from $5D$ to $4D$.

The $5D$ action of a massless scalar field in Minkowski background is

$$S_0 = -\frac{1}{2}\int d^5x\partial_M\phi\partial^M\phi \tag{3.10}$$

The metric on the $5D$ spacetime is $\eta_{MN} = \text{diag}(-,+,+,+,+)$, the five dimensional manifold can be written as a product $\mathcal{M}_5 = M_4 \times S^1$, with M_4 being the four dimensional Minkowski space. We will shortly see that the cylinder condition in the hypothesis above is imposed for a reason. In the case that it is not imposed, we shall see the imprint of the reduction from $5D$ to $4D$, on the $4D$ lower dimensional theory.

The $5D$ equations of motion for the scalar field are

$$\partial_M\partial^M\phi = 0 \implies \partial_\mu\partial^\mu\phi + \frac{\partial^2\phi}{\partial x_5^2} = 0 \tag{3.11}$$

where $\mu = 0,\ldots,3$ and $x^5 \in [0, 2\pi R]$ is identified, on a circle S^1 of radius R. Since the fifth coordinate is periodic, one can Fourier expand

$$\phi(x, x_5) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{\infty}\phi_n(x)e^{inx_5/R} \tag{3.12}$$

Substituting this to the equation of motion (3.11) for the scalar field ϕ, and using the orthonormality property of the eigenfunctions $\frac{1}{\sqrt{2\pi R}}e^{inx_5/R}$ of the operator $\frac{\partial^2}{\partial x_5^2}$ on the circle S^1, one gets the equations of motion of the modes in four dimensions

$$\partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2}\phi_n = 0 \tag{3.13}$$

The corresponding four dimensional action for the scalar fields after integrating over the fifth coordinate becomes

$$S_0 = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d^4x \left[\partial_\mu \phi_n \partial^\mu \phi_n^\star + \frac{n^2}{R^2}\phi_n^\star \phi_n\right] \tag{3.14}$$

These scalar fields in the four dimensional action ϕ_n are massive, with masses $\frac{n^2}{R^2}$. Taking the limit $R \to 0$ to make the extra dimension small enough so that it is not observable in $4D$, only the zero mode of the scalar field survives. The light modes do not source the heavy modes, so one can discard the heavy modes and keep only the light ones in order to take the four dimensional theory. The zero mode is the one that survives after the dimensional reduction. The exact same result of leaving out the heavy modes can be obtained from the cylinder condition of requiring that $\phi(x_M)$ is independent of x_5.[3] In higher dimensional reductions it is not always the case that the light modes are independent of the heavy modes, or similarly, that the light modes do not depend on the coordinates of the internal manifold, on which the reduction happens. This ensures the consistency of the reduction, as has also been mentioned above, in the manner that all of the lower dimensional theory solutions are solutions of the full higher dimensional theory as well.

3.2 Embedding the vacuum wormhole solution in 5D

One of the methods that we initially used to embed the 4D charged wormhole solution to charge-free 5D was the Kaluza-Klein idea. However, it was a quick realization that this would not work, since the auxiliary scalar field used in the Kaluza-Klein idea had to obey some specific equation, which was not possible to be satisfied in our case. The idea that seems to work for embedding the charge-free 4D wormhole to charge-free 5D is inspired by braneworld scenarios, see [30], which can also thought as a Kaluza-Klein compactification.

[3] It is like separating the variables in (3.11), so the equations of motion in four dimensions do not include any mass term.

3.2.1 The Kaluza-Klein ansatz

The Kaluza-Klein ansatz generalizes a four dimensional field theory of gravity and electromagnetism into a five dimensional field theory of gravity without cosmological constant term. Using the Kaluza-Klein ansatz for the $5D$ metric we tried to uplift the 4D charged wormhole solution in the a similar manner, adapted to a negative cosmological constant term in both the $4D$ and the $5D$ equations of motion.

We used exactly the same ansatz (3.2). In our case, (3.6) will have one more term corresponding to the cosmological constant term in five dimensions. They suggest that $\phi = 1$, but that is not possible because of (3.8). One quickly realizes that the attempt greatly fails to embed the charged wormhole solution to a vacuum AdS_5 solution, without introducing any new fields in the four dimensional theory.

3.2.2 The AdS braneworld ansatz

Another attempt that seemed to work out for the uplift of the uncharged wormhole into $5D$, satisfying the equations of motion of AdS_5, uses an ansatz inspired from braneworld scenarios. These scenarios suggest that the strings have their endpoints on the $D = 4$ manifold, seen as a $4 - brane$ and they propagate in the rest of the dimensions. The fifth dimension is seen as the worldvolume of the $4 - brane$. Before introducing the ansatz that has been generically worked out in [29] for an exact embedding of any vacuum AdS_4 solution to $5D$, it is useful to understand how the foliation works in the empty AdS case first.

Embedding empty AdS_4 in empty AdS_5

This particular $AdS-$like slicing of empty AdS_4 in slices of AdS_3 at each constant radial coordinate, that has been used in [2], can be generalized. It is always possible to add a new coordinate to AdS_n, in order to go to AdS_{n+1}. The new coordinate can be interpreted as the new radial coordinate of the AdS_{n+1}. This is a property that homogeneous spaces have and it has to do with the fact that one can write the manifold as a quotient of two groups. AdS_n is the lorentzian analogue of the S^n sphere, which can always be embedded to an S^{n+1} sphere.

The Embedding Coordinates

AdS_n spacetime can be viewed as a Lorentzian hypersurface embedded in a flat spacetime of one dimension higher, namely $(n + 1)$, which has two timelike coordinates. The equation of the hypersurface is the following:

$$- X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + \ldots + X_n^2 = -1 \tag{3.15}$$

where the radius of AdS_4, is taken to be equal to one, $l = 1$. Changing the parametrization one can reveal some more properties of the spacetime. In the 4D case of [3], the parametrization is the following:

$$\begin{aligned}
X_0 &= [s_t sh_u sh_\theta + c_t ch_u ch_\theta]\sqrt{r^2+1} \\
X_1 &= [s_t ch_u ch_\theta - c_t sh_u sh_\theta]\sqrt{r^2+1} \\
X_2 &= [s_t sh_u ch_\theta - c_t ch_u sh_\theta]\sqrt{r^2+1} \\
X_3 &= [s_t ch_u sh_\theta + c_t sh_u ch_\theta]\sqrt{r^2+1} \\
X_4 &= r
\end{aligned} \tag{3.16}$$

where $s_t = \sin\frac{t}{2}, c_t = \cos\frac{t}{2}, sh_u = \sinh\frac{u}{2}, ch_u = \cosh\frac{u}{2}, sh_\theta = \sinh\frac{\theta}{2}, ch_\theta = \cosh\frac{\theta}{2}$

Note that this is not the standard way to write empty AdS spacetime. The first difference from the metric that we are familiar with is that it is a foliation of AdS_4 with empty AdS_3 in each radial coordinate, instead of having a term of the form $d\Omega_2$ for each radial coordinate. The second and most important difference is those off diagonal terms, which accomondate the existence of a wormhole once one tries to warp this spacetime.

We will try to add one extra coordinate to this parametrization, so that it still obeys the equation of the hyperboloid. Adding the extra coordinate ρ in the trivial way and converting the coordinate $r \equiv sh_\phi$ gives:

$$\begin{aligned}
X_0 &= [s_t sh_u sh_\theta + c_t ch_u ch_\theta] ch_\phi \sqrt{\rho^2+1} \\
X_1 &= [s_t ch_u ch_\theta - c_t sh_u sh_\theta] ch_\phi \sqrt{\rho^2+1} \\
X_2 &= [s_t sh_u ch_\theta - c_t ch_u sh_\theta] ch_\phi \sqrt{\rho^2+1} \\
X_3 &= [s_t ch_u sh_\theta + c_t sh_u ch_\theta] ch_\phi \sqrt{\rho^2+1} \\
X_4 &= sh_\phi \sqrt{\rho^2+1} \\
X_5 &= \rho
\end{aligned} \tag{3.17}$$

where $s_t = \sin\frac{t}{2}, c_t = \cos\frac{t}{2}, sh_u = \sinh\frac{u}{2}, ch_u = \cosh\frac{u}{2}, sh_\theta = \sinh\frac{\theta}{2}, ch_\theta = \cosh\frac{\theta}{2}, sh_\phi = \sinh\frac{\phi}{2} \equiv r, ch_\phi = \cosh\frac{\phi}{2} \equiv \sqrt{r^2+1}$

The renaming of the coordinate r has been made so that it can be better shown that we can interpret the coordinate ρ as the new radial coordinate. This leads to an empty AdS_5 solution written in the $AdS-$ like slicing choice of coordinates.

The line element of the metric written in these coordinates $(t, \rho, \phi, \theta, u)$ is

$$ds^2 = \frac{l^2 d\rho^2}{\rho^2+1} + \frac{l^2}{4}\left(\rho^2+1\right)\left[d\phi^2 + \cosh^2\frac{\phi}{2}\left(-\cosh^2\theta dt^2 + d\theta^2 + (du + \sinh\theta dt)^2\right)\right]$$

This solution is just empty AdS_5 written in some weird coordinates. It obeys the vacuum AdS_5 Einstein equations and the Kretschmann invariant is, as expected, everywhere zero. In this picture, AdS_4 serves as an embedding of the squashed AdS_3 and we have added a new coordinate ρ that embeds this whole thing in AdS_5. Note that here no warping has been made, which essentially means that it is indeed just empty wormhole-free AdS_5.

Our initial idea was to make a construction of a wormhole in this AdS_5 solution, in a similar fashion as was done in [2]. The idea of [2] is based on the squashed AdS_3, embedded in AdS_4, which is illustrated in the coordinate change (3.16). The squashed AdS_3 is the lorentzian analogue of the squashed 3−sphere, which corresponds to a description of the 3−sphere as a hopf fibration with base manifold the 2−sphere and a circle of different size at each point. A more detailed analysis is illustrated in [25], where there is a comment mentioning that the generalization to higher dimensions in the lorentzian case is higly non-trivial. Our initial idea was to use this parametrization of empty AdS_4 and AdS_5 in order to squash AdS_4, embed it in empty AdS_5 and then warp it by arbitrary functions of the new radial coordinate in order to break the isometries of empty AdS_5 and eventually construct a 5D wormhole. This idea did not work out in the end, and it is highly likely that it is due to the fact that there cannot exist squashed AdS_4. In fact, our attempts were always leading us to the empty case, in the sense that the two arbitrary functions had to be the same, in order for the solution to obey the equations of motion.

The ansatz for embedding the wormhole solution

Another possible attempt, using the knowledge from this parametrization, is the search for a higher dimensional embedding. As has already been mentioned above, we used an ansatz based on braneworld scenarios[30] and it will shortly be obvious how this connects to the standard way of adding new coordinates that has been illustrated above. The ansatz for adding a new coordinate y is the following

$$ds^2 \equiv \bar{g}_{AB} dx^A dx^B = e^{2\zeta(y)} g_{\mu\nu} dx^\mu dx^\nu + l^2 dy^2 \tag{3.18}$$

where μ, ν run from $0, 1, 2, 3$ and $g_{\mu\nu}$ is the $4d$ metric, while A, B run from $0, 1, 2, 3, 4$ and $\bar{g}_{AB}$ is the 5d metric. In order for this solution to satisfy the vacuum equations of motion of AdS_5, namely

$$\bar{R}_{AB} - \frac{1}{2} R^{(5)} \bar{g}_{AB} + \frac{3}{10} \bar{R} \bar{g}_{AB} = 0 \tag{3.19}$$

one ends up with a differential equation of the form

$$e^{-2\zeta(y)} = \zeta''(y) \tag{3.20}$$

which is a non-linear second order differential equation, which can be manipulated in the following way

$$\begin{aligned}
e^{-2\zeta(y)} &= \zeta''(y) \\
-\frac{2}{l^2}\zeta'(y)e^{-2\zeta(y)} &= -2\zeta'(y)\zeta''(y) \\
\frac{d}{dy}\left(e^{-2\zeta(y)}\right) &= -\frac{d}{dy}\left(\zeta'(y)^2\right) \\
e^{-2\zeta(y)} &= -\zeta'(y)^2 + C \\
\zeta''(y) &= -\zeta'(y)^2 + C
\end{aligned}$$

A general solution of the final differential equation is

$$\zeta(y) = c_1 + \ln\left(\cosh\left(\sqrt{C}(y + c_2)\right)\right) \tag{3.21}$$

Since we are searching for any $\zeta(y)$ that can solve this differential equation and not necessarily the most general one, we are free to choose whatever integration constants we prefer, as long as it does not make the function trivial. We will choose $C = 1, c_1 = c_2 = 0$. Therefore, a solution to this differential equation is

$$\zeta(y) = \ln\left(\cosh y\right) \implies e^{2\zeta(y)} = \cosh^2 y \tag{3.22}$$

Note that taking $C = 1$ is like requiring that the radius of AdS_5 is unit. The line element (3.18) becomes

$$ds^2 \equiv \bar{g}_{AB}dx^A dx^B = \cosh^2 y g_{\mu\nu}dx^\mu dx^\nu + l^2 dy^2 \tag{3.23}$$

We have confirmed that this is a solution to the AdS_5 equations of motion (3.19). If we, now, make the following change of coordinates

$$\rho = \sinh y \implies d\rho = \cosh y \; dy \implies \frac{d\rho^2}{\rho^2 + 1} = dy^2 \tag{3.24}$$

where $\cosh^2 y = \rho^2 + 1$, the line element of the 5d metric can be written as

$$ds^2 = \frac{l^2 d\rho^2}{\rho^2 + 1} + \left(\rho^2 + 1\right) g_{\mu\nu}dx^\mu dx^\nu \tag{3.25}$$

where $\rho \in R$, is the radial coordinate of *assymptotically locally* AdS_5, which has a warped 4d wormhole of the kind presented in [2] at each $\rho =$ const slice. We characterize this solution as *assymptotically locally* AdS_5 because it satisfies the vacuum equations of motion (3.19) of AdS_5, it has non-trivial topology in the bulk and its boundary will not be topologically $S^3 \times R$.As shown in the previous section, this is just the standard way of adding new coordinates to move from empty AdS_n to AdS_{n+1}. The next step is to try to embed the charged 4D solution to 5D.

4. Traversability and Energy Conditions

There is a deep controversy concerning the class of wormhole solutions in [2, 3]. In general, energy conditions are defined in such a way that they rule out pathological behaviour, such as causal contact of disconnected regions[1], also known as traversability. The contradiction lies in the fact that this wormhole solution seems to obey the conditions, while at the same time is traversable. In this chapter, we will go through the energy condition theorems, with concequences about wormholes, that exist in the literature, provide some more controversial constructions, prove the energy conditions for the wormhole spacetime of [2, 3] and present some indications for the traversability property.

Geometric and Physical form of the Null Energy Condition

For Einstein equations with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

the null energy condition takes the form

$$R_{\mu\nu}u^\mu u^\nu = 8\pi T_{\mu\nu}v^\mu v^\nu \geq 0 \tag{4.1}$$

since, for any null vector v it is true that $g_{\mu\nu}u^\mu u^\nu = 0$. The inequality involving $R_{\mu\nu}$ is the geometric form of the energy condition, whereas the one involving $T_{\mu\nu}$ is the matter form. From this, one can readily see that the NEC is satisfied automatically vacuum solutions with cosmological constant, irrespective of its sign. It is obvious that the energy condition will solely depend on the matter coupled to the spacetime.

4.1 Topological censorship theorem

The *topological censorship theorem* claims that "in a globally hyperbolic, *assymptotically* flat spacetime satisfying the null energy condition, every causal curve

[1] The connectedness or disconnectedness of the boundary is not something proven yet. We will discuss the implications in either case.

from past null infinity to future null infinity is fixed endpoint homotopic to a causal curve in a topologically trivial neighbourhood of infinity" [15, 31]. In other words, "no causal path can go through any nontrivial topology" [32].

A straightforward result, by definition, is that when NEC is obeyed in a spacetime, then a wormhole in that spacetime cannot be traversable. Classical matter respects the NEC, so in spacetimes coupled with only classical matter, traversable wormholes are ruled out automatically, as an immediate concequence of the topological censorship theorem. The proof by Friedmann in [31] concerns *assymptotically* flat spacetimes.

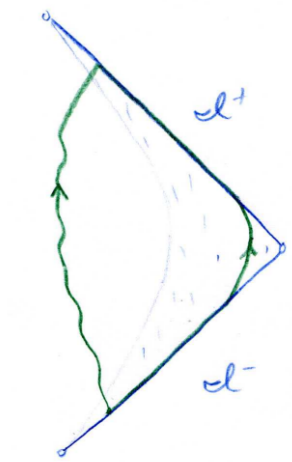

Figure 4.1: An assymptotically flat manifold is one that admits a conformal compactification, as explained in the relevant section in this thesis. A causal curve starting from past null infinity, $\mathscr{J}^- \simeq R \times S^2$, and ending on future null infinity $\mathscr{J}^+ \simeq R \times S^2$. According to the topological censorship theorem, for spacetimes that obey the ANEC, it can be continuously deformed to a curve belonging entirely in a simply connected neighbourhood of the boundary $\mathscr{J} = \mathscr{J}^- \cup \mathscr{J}^+$. Figure taken from Galloway's talk on Topological Censorship in AdS [13]

4.1.1 Topological censorship in AdS

A generalization of the theorem for *assymptotically* locally AdS (AlAdS) spacetimes, motivated by AdS-CFT cooresponcence, was introduced by Galloway, Schleich, Witt and Woolgar in [33, 34]. In this paper, they prove that if (M, g_{ab}) is an assymptotically locally AdS spacetime which obeys the ANEC, with the *domain of outer communications*[2] being globally hyperbolic in the AdS sense[3], then any

[2]the region outside of any black holes or white holes that is in causal contact with infinity

[3]AdS can be made globally hyperbolic with appropriate boundary conditions.

curve, either causal or not, with end points at the boundary can be continuously deformed to a curve lying on the boundary. A corollary of the theorem is that even if conformal infinity has multiple components, the distinct components can not be in causal contact.

4.2 Energy conditions

Einstein's equations for General Relativity are simply a complicated and highly non-linear set of differential equations that directly relate curvature with the presence of matter. A space-time metric can always be a solution, since one can always find an energy momentum tensor by varying the action with respect to the metric. Restrictions are needed, in order to make this energy momentum tensor of matter reasonable. Energy conditions are constraints imposed on it. This can be viewed as a relation that it should satisfy as an attempt to capture the idea that "energy should be positive"[35].[4]In fact, the global properties of a spacetime solution to Einstein's equations can be directly reflected in the energy conditions that are chosen for the local energy density. The fact that there are so many types of energy conditions in the bibliography is a result of the urge to find the most general one that forbids all the exotic phenomena, such as traversable wormholes in our case. Each time that examples with exotic properties that do not violate a condition are found, an improved condition "takes the reins" from the old one.

One example of this, is the energy conditions required in order to avoid the existence traversable wormholes. In the classical regime, classical matter, should obey the pointwise null energy condition (NEC), namely $T_{\mu\nu}k^\mu k^\nu \geq 0$ for all null geodesics, where k^μ corresponds to the tangent vector. In the semiclassical regime, quantum matter on classical background, is allowed to have negative energy up to the point that the average energy along any null geodesic is positive. In this case, ruling out traversability requires the Averaged null energy condition (ANEC) over complete null geodesics to hold, namely $\int_{-\infty}^{+\infty} T_{\mu\nu}k^\mu k^\nu d\lambda \geq 0$, where k^μ corresponds to the tangent vector parametrized by λ. In fact, an even weaker form than that of the ANEC is also introduced by the name Self-Consistent Achronal Averaged Null Energy Condition(SCAANEC). It states that there is no self-consistent solution in semiclassical gravity in which ANEC is violated, on a complete achronal null geodesic [32]. As a matter of fact, the proposal of genuinely requiring an achronal null geodesic was first made by Wald and Yurtserver in [36]. Later, the term self-consistent was introduced in the ANEC discussion for the first time by Penrose,

[4]However, this is not entirely true since throughout the whole history of general relativity, there is no adequate definition of a "gravitational" stress-energy tensor. The problem lies in the fact that stress-energy of purely "gravitational" systems can not be localized, which will be further discussed in the chapter about the conserved charges in General Relativity. This can be possibly seen as an obstruction on the physical content of the standard pointwise energy conditions.

Sorkin and Woolgar in [37] and it refers to self-consistent solutions which had been studied by Flanagan and Wald in [38] before.

Some definitions that need to be added to our list are those of *achronality* and *self-consistency*.

- *Achronal set*: A set S of a manifold M is called *achronal* if no timelike curve intersects the set more than once.
- *Achronal null geodesic*: It is a curve that obeys the geodesic equation of motion, has a null tangent vector and does not meet any timelike curves twice.
- *Self-Consistent spacetime*: The term roughly refers to a spacetime solution to Einstein's equations with the expectation value of the stress energy tensor on the right hand side[12].

SCAANEC is believed to be the energy condition that prohibits exotic phenomena, such as closed timelike curves and wormholes connecting different assymptotic regions of spacetime[5], in the semiclassical regime[32]. Accordingly, this should be the energy condition to be violated if one needs to examine the traversability property of a wormhole that links two disconnected assymptotic regions.

Examples of traversable wormholes supported by nonexotic matter

It is being argued in [39] that the requirement that traversable wormholes should violate the NEC can be removed in spacetimes with torsion or in conformally transformed spacetimes. The interpretation is that "normal" matter might satisfy the null energy condition, whilst the geometry can behave like matter via an induced "geometric" energy-momentum tensor that can violate the NEC. In particular, in conformally transformed wormholes and wormholes in spacetimes with torsion, the prescribed "exoticity", that matter fields, Proca and Rarita-Schwinger fields in this case, should carry, becomes realized within the geometry of the spacetime itself. In Riemann-Cartan spacetimes, torsion induces an effective energy-momentum tensor that can serve as the desired exotic matter that produces a traversable wormhole. In conformally transformed Riemannian spacetimes, the deformation of the metric induces an energy-momentum tensor that plays the role of exotic energy condition violating matter. It is, of course, true that these cases do not apply to the example presented in this thesis, since the spacetime is torsion-free. Moreover, for Maxwell fields the NEC, according to [39], should still be satisfied regardless of the fact that the spacetime has torsion or not. As a consequence, the controversy is still there.

[5]In the case of connected boundaries SCAANEC forbids short wormholes. This means that there should always be a shorter causal path going around and not through the wormhole

In addition, in [40] traversable wormhole solutions are presented, as a higher dimensional extension of the Morris-Thorne type of wormholes. In that case, the throat is supported by ordinary matter, while the extra dimension is responsible for the violation of NEC. Another example is in [41], where wormhole solutions in $f(R,T)$ extended theories of gravity involving nonexotic matter are traversable, with the violation happening due to the extra degrees of freedom of the extended theory. There are some more traversable wormhole constructions, linked to non-commutative geometry[42, 43, 44, 45] and alternative mimetic theories of gravity[46], but let us restrict ourselves to solutions of non-modified theory of gravity. The wormhole paradigm of [2, 3] is solution to General Relativity with negative cosmological constant, seems to contain a traversable wormhole and, simultaneously, the matter obeys the NEC.[6]

In all of the aforementioned controversial attempts for traversability there was some effort to justify why matter is nonexotic, by showing that the violation of the condition happens due to some notion that takes the role of exotic matter, but is not usually interpreted as matter. Therefore, it is left as an open question to attempt to interpret the traversability of the class of wormhole in [2, 3]. Something needs to violate the energy condition, in order to support the throat long enough so that it is traversable, and if it is not ordinary matter, in this case we are expecting that it is something related to the geometry.[7]

4.2.1 Geometric and physical interpretation of NEC

In order to understand what the meaning of the null energy condition, one needs to connect it to the experience of an observer that hypothetically travels along null geodesics.

Using the geodesic deviation equation one can try to calculate an average radial acceleration A_r of a geodesic γ at a point p, roughly speaking, as the averaged sum over the magnitudes of the radial component of the relative acceleration in orthogonal directions to that of γ. A detailed proof of this statement can be found in the technical appendix of [35].

For null geodesic tangent vectors v this average radial acceleration translates to

$$A_r = -\frac{1}{2}R_{\mu\nu}v^\mu v^\nu \tag{4.2}$$

which is the LHS of the geometric form of the null energy condition and upon imposing Einstein's equations translates to

$$A_r = -\frac{8\pi}{2}T_{\mu\nu}v^\mu v^\nu \tag{4.3}$$

[6]In the charge-free case of [2] there is no matter at all. NEC of matter is tautologically satisfied

[7]At this point, it is remarkable to stress out that we think of the cosmological constant term in Einstein equations as part of the geometry, rather than vacuum energy of quantum fields.

When the null energy condition is satisfied, the average radial acceleration is negative or zero, which geometrically means that null geodesic congruences are convergent.

The physical interpretation of the NEC is something that one should be cautious about. It is, basically, claimed that, once NEC is satisfied pointwise, particles moving along null geodesics should observe that the action of "gravity" on neighbouring particles moving along null geodesics, as well, is locally "non-repulsive". However, the convergence of all null geodesics at a point, imposed by the NEC, does not imply the convergence of all timelike geodesics at the same point. A timelike observer may still see repulsive gravity, at a small neighbourhood of a point that satisfies the NEC. An important remark, here, is that energy density, as part of the decomposition of the energy-momentum tensor, is observer-dependent, so that, even if NEC is obeyed, no accurate physical assumption can be really made about the nature of energy density.[8]

4.3 Proof of NEC for the model

In this section, a detailed proof for the null energy condition of the wormhole solution will be presented, first for a radial null path and then for a general null path. The averaged null energy condition is automatically satisfied if the null energy condition is.

4.3.1 Towards the calculation of the NEC

Recall that the null energy condition is satisfied for vacuum solutions with cosmological constant, and only depends on the matter content, in spacetimes coupled to matter.

The energy momentum tensor in that case has the form

$$T_{\mu\nu} = \left(F_{\mu\rho} g^{\rho\sigma} F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\sigma}_{\rho} \right) \tag{4.4}$$

In order to check the energy condition for the model we are studying, we will first calculate the quantity

$$\begin{aligned}
\frac{1}{2} T_{\mu\nu} v^\mu v^\nu &= 2F_{ru}F_{rt}g^{ut}v^r v^r + F_{ru}F_{ru}g^{uu}v^r v^r + F_{ur}F_{ur}g^{rr}v^u v^u \\
&+ F_{rt}F_{rt}g^{tt}v^r v^r + F_{rt}F_{\theta t}g^{tt}v^r v^\theta + F_{tr}F_{tr}g^{rr}v^t v^t \\
&+ F_{tr}F_{ur}g^{rr}v^t v^u + F_{\theta t}F_{\theta t}g^{tt}v^\theta v^\theta + F_{\theta t}F_{rt}g^{tt}v^\theta v^r \\
&+ 2F_{\theta t}F_{ru}g^{tu}v^\theta v^r + F_{t\theta}F_{t\theta}g^{\theta\theta}v^t v^t + F_{ur}F_{tr}g^{rr}v^u v^t
\end{aligned}$$

[8]This is believed to be the core problem regarding the fact that energy conditions seem to be non-derivable from fundamental principles, which is essential because they play a very important role in a lot of theorems (singularity theorems, positive energy theorems etc.) in General Relativity.[35]

where the fact that $g_{ab}v^a v^b = 0$ directly eliminates one term of the stress-energy tensor of electromagnetism.

Moving on to the calculation,

$$\begin{aligned}
T_{\mu\nu}k^\mu k^\nu = {} & 2F_{ru}^2 \frac{\tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \frac{h(r) - f(r)\tanh^2\theta}{f(r)h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \frac{\sigma^2 f(r)}{4l^4}\left(\frac{du}{d\lambda}\right)^2 \\
& - F_{ru}^2 \frac{tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 - F_{ru}F_{\theta t}\frac{\mathrm{sech}\theta\tanh\theta}{h(r)}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + F_{ru}^2 \sinh^2\theta \frac{\sigma^2 f(r)}{4l^4}\left(\frac{dt}{d\lambda}\right)^2 \\
& + F_{ru}^2 \sinh\theta \frac{\sigma^2 f(r)}{4l^4}\frac{dt}{d\lambda}\frac{du}{d\lambda} - F_{\theta t}^2 \frac{\mathrm{sech}^2\theta}{h(r)}\left(\frac{d\theta}{d\lambda}\right)^2 - F_{\theta t}F_{ru}\sinh\theta \frac{\mathrm{sech}^2\theta}{h(r)}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} \\
& + 2F_{\theta t}F_{ru}\frac{\mathrm{sech}\theta\tanh\theta}{h(r)}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} + F_{\theta t}^2 \frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 + F_{ru}^2 \sinh\theta \frac{\sigma^2 f(r)}{4l^4}\frac{du}{d\lambda}\frac{dt}{d\lambda}
\end{aligned}$$

4.3.2 Taking a radial null path for simplicity

One can consider a tangent null vector to a radial geodesic, parametrized as $x^\mu = (t(\lambda), r(\lambda), \frac{\pi}{2}, 0)$, which can be written as $k^\mu = \left(\dot{t}, \dot{r}, 0, 0\right)$. This tangent vector should also obey

$$g_{\mu\nu}k^\mu k^\nu = 0 \tag{4.5}$$

so that it is null and

$$\frac{d}{d\lambda}(g_{\alpha\beta}k^\beta) - \frac{1}{2}g_{\mu\nu,\alpha}k^\mu k^\nu = 0 \tag{4.6}$$

so that it is a tangent vector to a geodesic. Note, that if the first constraint holds, meaning that the path is null, then the geodesic equation which serves as the second constraint mentioned should also hold. This can be seen intuitively if one thinks of the fact that geodesic paths are defined to be the shortest paths, while null paths are also by definition the shortest ones, but for a light-like particle.

Radial null path condition

The null condition becomes:

$$g_{tt}k^t k^t + g_{rr}k^r k^r = 0 \implies \tag{4.7}$$

and its individual components are

$$(-h(r)\cosh^2\theta + f(r)\sinh^2\theta)\left(\frac{dt}{d\lambda}\right)^2 + \frac{4l^4}{\sigma^2 f(r)}\left(\frac{dr}{d\lambda}\right)^2 = 0 \tag{4.8}$$

Geodesic equation of a radial null path

The geodesic equation, split in components, translates as:

$$\frac{d}{d\lambda}(g_{\alpha\beta}k^{\beta}) - \frac{1}{2}g_{\mu\nu,\alpha}k^{\mu}k^{\nu} = 0 \implies$$
$$\begin{aligned} &\frac{d}{d\lambda}(g_{tt}k^{t}) = 0 \\ &\frac{d}{d\lambda}(g_{rr}k^{r}) - \frac{1}{2}g_{rr,r}k^{r}k^{r} - \frac{1}{2}g_{tt,r}k^{t}k^{t} = 0 \end{aligned} \tag{4.9}$$

Let us work on the geodesic equations for $t(\lambda)$ and $r(\lambda)$.

For $t(\lambda)$ we get:

$$\begin{aligned} &\frac{d}{d\lambda}\left(\left(-h(r)\cosh^2\theta_0 + f(r)\sinh^2\theta_0\right)\frac{dt}{d\lambda}\right) = 0 \\ &\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)\frac{dt}{d\lambda} = C, \qquad C \geq 0 \end{aligned} \tag{4.10}$$

and, thus, due to the null path condition, it is true that

$$\frac{4l^4}{\sigma^2 f(r)}\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)\left(\frac{dr}{d\lambda}\right)^2 = C^2 \tag{4.11}$$

For $r(\lambda)$ we get:

$$\begin{aligned} &\frac{d}{d\lambda}\left(\frac{4l^4}{\sigma^2 f(r)}\frac{dr}{d\lambda}\right) - \frac{1}{2}\frac{d}{dr}\left(\frac{4l^4}{\sigma^2 f(r)}\right)\left(\frac{dr}{d\lambda}\right)^2 + \\ &\qquad + \frac{1}{2}\frac{d}{dr}\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)\left(\frac{dt}{d\lambda}\right)^2 = 0 \\ &\frac{4l^4 f'(r)}{\sigma^2 f^2(r)}\left(\frac{dr}{d\lambda}\right)^2 + \frac{4l^4}{\sigma^2 f(r)}\frac{d^2 r}{d\lambda^2} - \frac{1}{2}\frac{4l^4 f'(r)}{\sigma^2 f^2(r)}\left(\frac{dr}{d\lambda}\right)^2 + \\ &\qquad + \frac{1}{2}\left(h'(r)\cosh^2\theta_0 - f'(r)\sinh^2\theta_0\right)\left(\frac{dt}{d\lambda}\right)^2 = 0 \\ &\frac{d^2 r}{d\lambda^2} + \frac{\sigma^2 C^2}{8l^4}\left[\frac{f'(r)}{\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)} + f(r)\frac{h'(r)\cosh^2\theta_0 - f'(r)\sinh^2\theta_0}{\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)^2}\right] = 0 \\ &\frac{d^2 r}{d\lambda^2} + \frac{\sigma^2 C^2}{8l^4}\frac{\left(f(r)\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)\right)'}{\left(h(r)\cosh^2\theta_0 - f(r)\sinh^2\theta_0\right)^2} = 0 \end{aligned} \tag{4.12}$$

Note that each null geodesic path is always a null path anyway. It is sufficient to just use (4.7) for the proof of the energy condition.

4.3.3 Simplified conditions

The energy-momentum tensor version of the energy condition becomes:

$$
\begin{aligned}
T_{\mu\nu}k^\mu k^\nu = {} & 2F_{ru}^2 \frac{\tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \frac{h(r) - f(r)\tanh^2\theta}{f(r)h(r)}\left(\frac{dr}{d\lambda}\right)^2 \\
& - F_{ru}^2 \frac{\tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \sinh^2\theta \frac{\sigma^2 f(r)}{4l^4}\left(\frac{dt}{d\lambda}\right)^2 + F_{\theta t}^2 \frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 \geq 0
\end{aligned}
$$

Substituting the null path constraint gives:

$$
\begin{aligned}
& \frac{\sigma^2 F_{ru}^2}{4l^4}\left(h(r)\cosh^2\theta - f(r)\sinh^2\theta\right)\left(\frac{dt}{d\lambda}\right)^2 + \\
& \qquad + F_{ru}^2 \sinh^2\theta \frac{\sigma^2 f(r)}{4l^4}\left(\frac{dt}{d\lambda}\right)^2 + F_{\theta t}^2 \frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 \geq 0 \\
\implies & \frac{\sigma^2 F_{ru}^2}{4l^4} h(r)\cosh^2\theta \left(\frac{dt}{d\lambda}\right)^2 + F_{\theta t}^2 \frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 \geq 0
\end{aligned}
$$

In this expression each of the terms is positive. This means that pointwise NEC is automatically satisfied along this radial null path.

4.3.4 Taking a general null path

For a general null path $v^\mu = \left(\dot{t}, \dot{r}, \dot{u}, \dot{\theta}\right)$

$$
g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + 2g_{tu}\frac{du}{d\lambda}\frac{dt}{d\lambda} + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + g_{uu}\left(\frac{du}{d\lambda}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 = 0 \tag{4.13}
$$

$$
\begin{aligned}
\left(h(r)\cosh^2\theta - f(r)\sinh^2\theta\right)\left(\frac{dt}{d\lambda}\right)^2 = {} & \frac{4l^4}{\sigma^2 f(r)}\left(\frac{dr}{d\lambda}\right)^2 + 2f(r)\sinh\theta \frac{du}{d\lambda}\frac{dt}{d\lambda} + \\
& + f(r)\left(\frac{du}{d\lambda}\right)^2 + h(r)\left(\frac{d\theta}{d\lambda}\right)^2
\end{aligned} \tag{4.14}
$$

The energy momentum tensor of Electrodynamics in terms of the field strength is:

$$
T_{\mu\nu} = -\frac{1}{\mu_0}\left(F_{\mu\rho}g^{\rho\sigma}F_{\sigma\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^\sigma_\rho\right) \tag{4.15}
$$

In our solution, the charged matter content is described by the following,

$$
\begin{aligned}
F_{ru} &= -F_{ur} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2} \\
F_{rt} &= -F_{tr} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2}\sinh\theta = F_{ru}\sinh\theta \\
F_{\theta t} &= -F_{t\theta} = \frac{2Qr + P(1-r^2)}{r^2+1}\cosh\theta
\end{aligned}
$$

The non-zero metric components are the following:

$$
g_{tt} = -h(r)\cosh^2\theta + f(r)\sinh^2\theta, \quad g_{tu} = g_{ut} = f(r)\sinh(\theta),
$$
$$
g_{uu} = f(r), \quad g_{rr} = \frac{4l^4}{\sigma^2 f(r)}, \quad g_{\theta\theta} = h(r)
$$

The non-zero inverse metric components are the following:

$$
g^{tt} = -\frac{\mathrm{sech}^2\theta}{h(r)}, \quad g^{tu} = g^{ut} = \frac{\mathrm{sech}\theta\tanh\theta}{h(r)},
$$
$$
g^{uu} = \frac{h(r) - f(r)\tanh^2\theta}{f(r)h(r)}, \quad g^{rr} = \frac{\sigma^2 f(r)}{4l^4}, \quad g^{\theta\theta} = \frac{1}{h(r)}
$$

where,

$$
f(r) = \frac{2}{q^2\sigma^2}\frac{r^4 + (6-\sigma)r^2 + mr + \sigma - 3}{r^2+1} - \frac{Q^2+P^2}{r^2+1} \quad , \quad h(r) = \frac{1}{2q^2\sigma}(r^2+1)
$$

The energy condition takes the following form:

$$
\begin{aligned}
\frac{1}{2}\mu_0 T_{\mu\nu}v^\mu v^\nu =& 2F_{ru}F_{rt}g^{ut}v^r v^r + F_{ru}F_{ru}g^{uu}v^r v^r + F_{ur}F_{ur}g^{rr}v^u v^u \\
&+ F_{rt}F_{rt}g^{tt}v^r v^r + F_{rt}F_{\theta t}g^{tt}v^r v^\theta + F_{tr}F_{tr}g^{rr}v^t v^t \\
&+ F_{tr}F_{ur}g^{rr}v^t v^u + F_{\theta t}F_{\theta t}g^{tt}v^\theta v^\theta + F_{\theta t}F_{rt}g^{tt}v^\theta v^r \\
&+ 2F_{\theta t}F_{ru}g^{tu}v^\theta v^r + F_{t\theta}F_{t\theta}g^{\theta\theta}v^t v^t + F_{ur}F_{tr}g^{rr}v^u v^t
\end{aligned}
$$

This sum should indeed have 14 non-zero terms.

- For $\rho = r$: we need $\sigma = r$ and μ, ν can be either r or t, so this corresponds to 4 non-zero terms.
- For $\rho = u$: μ will be for sure r and σ is either u or t. If $\sigma = t$, then ν can be either θ or r. So, this is in total 3 non-zero terms.
- For $\rho = t$, μ can be either r or θ and σ can be either u or t. For $\sigma = t$, ν can be either θ or r. So, this in total yields 6 non-zero terms.

- For $\rho = \theta$, σ should be θ and μ, ν should both be t. So, this is only one non-zero term.

In total, we have $4 + 3 + 6 + 1 = 14$ non-zero terms in the sum.

Now, if we only keep F_{ru} and $F_{\theta t}$, by substituting $F_{rt} = F_{ru} \sinh\theta$, and, simoultaneously, plug in the expressions for the inverse metric components, the energy condition will become:

$$
\begin{aligned}
\frac{1}{2}\mu_0 T_{\mu\nu} v^\mu v^\nu =& 2F_{ru}^2 \frac{\tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \frac{h(r) - f(r)\tanh^2\theta}{f(r)h(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2 \frac{\sigma^2 f(r)}{4l^4}\left(\frac{du}{d\lambda}\right)^2 \\
& - F_{ru}^2 \frac{tanh^2\theta}{h(r)}\left(\frac{dr}{d\lambda}\right)^2 - F_{ru}F_{\theta t}\frac{\mathrm{sech}\theta\tanh\theta}{h(r)}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + F_{ru}^2\sinh^2\theta\frac{\sigma^2 f(r)}{4l^4}\left(\frac{dt}{d\lambda}\right)^2 \\
& + F_{ru}^2 \sinh\theta\frac{\sigma^2 f(r)}{4l^4}\frac{dt}{d\lambda}\frac{du}{d\lambda} - F_{\theta t}^2\frac{\mathrm{sech}^2\theta}{h(r)}\left(\frac{d\theta}{d\lambda}\right)^2 - F_{\theta t}F_{ru}\sinh\theta\frac{\mathrm{sech}^2\theta}{h(r)}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} \\
& + 2F_{\theta t}F_{ru}\frac{\mathrm{sech}\theta\tanh\theta}{h(r)}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} + F_{\theta t}^2\frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 + F_{ru}^2\sinh\theta\frac{\sigma^2 f(r)}{4l^4}\frac{du}{d\lambda}\frac{dt}{d\lambda}
\end{aligned}
$$

If one observes closely, some terms cancel out and, if one uses the null condition (4.14), the expression can be simplified as follows:

$$
\begin{aligned}
\frac{1}{2}\mu_0 T_{\mu\nu} v^\mu v^\nu =& F_{ru}^2\frac{1}{f(r)}\left(\frac{dr}{d\lambda}\right)^2 + F_{ru}^2\frac{\sigma^2 f(r)}{4l^4}\left(\frac{du}{d\lambda}\right)^2 \\
& + F_{ru}^2\sinh^2\theta\frac{\sigma^2 f(r)}{4l^4}\left(\frac{dt}{d\lambda}\right)^2 + 2F_{ru}^2\sinh\theta\frac{\sigma^2 f(r)}{4l^4}\frac{dt}{d\lambda}\frac{du}{d\lambda} \\
& + F_{\theta t}^2\frac{1}{h(r)}\left(\frac{dt}{d\lambda}\right)^2 - F_{\theta t}^2\frac{\mathrm{sech}^2\theta}{h(r)}\left(\frac{d\theta}{d\lambda}\right)^2 \\
=& F_{ru}^2\frac{\sigma^2}{4l^4}\left[h(r)\cosh^2\theta\left(\frac{dt}{d\lambda}\right)^2 - h(r)\left(\frac{d\theta}{d\lambda}\right)^2\right] \\
& + F_{\theta t}^2\frac{1}{h(r)}\left[\left(\frac{dt}{d\lambda}\right)^2 - \mathrm{sech}^2\theta\left(\frac{d\theta}{d\lambda}\right)^2\right] \\
=& \left[F_{ru}^2\frac{\sigma^2 h(r)}{4l^4} + F_{\theta t}^2\frac{1}{h(r)\cosh^2\theta}\right]\left[\cosh^2\theta\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{d\theta}{d\lambda}\right)^2\right]
\end{aligned}
$$

The quantity $\left[\cosh^2\theta\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{d\theta}{d\lambda}\right)^2\right]$ is positive. We can prove it by using the null condition (4.14) as shown below:

$$
h(r)\left[\cosh^2\theta\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{d\theta}{d\lambda}\right)^2\right] = f(r)\left(\frac{du}{d\lambda} + \sinh\theta\frac{dt}{d\lambda}\right)^2 + \frac{4l^4}{\sigma^2 f(r)}\left(\frac{dr}{d\lambda}\right)^2 \geq 0 \tag{4.16}
$$

The LHS of this equation is positive, due to the fact that demanding $f(r)$ to have no singularities, makes it automatically positive everywhere. Thus, the RHS is also positive.

The sign of the quantity $\left[F_{ru}^2 \frac{\sigma^2 h(r)}{4l^4} + F_{\theta t}^2 \frac{1}{h(r)\cosh^2\theta}\right]$ will be positive, since everything in this expression is. This, essentially, means that pointwise NEC is satisfied in this spacetime, independently of the choice of parameters. Hence, ANEC is automatically satisfied for this spacetime solution.

4.4 Indications for traversability of the model

Crossing time from one boundary to the other

What is being computed in [2] as the time that is needed for a light ray to travel from one boundary to the other, as seen by a geodesic observer with $r = \theta = 0, u = \text{const}$, is the time that it takes for a light ray to cross a complete radial null path $\theta = \phi = 0$.[9] What is invariant, as always, is ds^2. The proper time that this observer measures between two events that happen at $t = t_A$ and $t = t_B$ is

$$\Delta\tau = \int_A^B \sqrt{-ds^2} = \int_{t_A}^{t_B} \sqrt{h(0) - f(0)}dt = \sqrt{P^2 + Q^2 - \frac{3l^2(\sigma - 4)}{\sigma^2}} \int_{t_A}^{t_B} dt \tag{4.17}$$

This geodesic observer wants to measure the time it takes for light to travel from $r \to -\infty$ to $r \to +\infty$. Along the radial null path that the lightray takes, the proper time elapsed is, of course zero, $ds^2 = 0$.

$$ds^2 = 0 \implies dt = \frac{2l^2}{\sigma} \frac{dr}{\sqrt{f(r)h(r)}} \tag{4.18}$$

This means that if the event A corresponds to the light ray starting at one boundary and the event B is when it reaches the other boundary, the time interval between those two events measured by the geodesic observer is

$$\Delta\tau = \sqrt{P^2 + Q^2 - \frac{3l^2(\sigma - 4)}{\sigma^2}} \frac{2l^2}{\sigma} \int_{-\infty}^{+\infty} \frac{dr}{\sqrt{f(r)h(r)}} \tag{4.19}$$

The geodesic observer will observe that a light ray along that path will travel from one boundary to the other at finite time. Numerically integrating this integral and plotting it for different allowed σ and X and for fixed $m = 0.4, l = 1$, we can see that this crossing time is finite.[see figure 4.2]

[9] I have not proven myself that those paths are allowed geodesic paths. This statement is made in [2, 3]

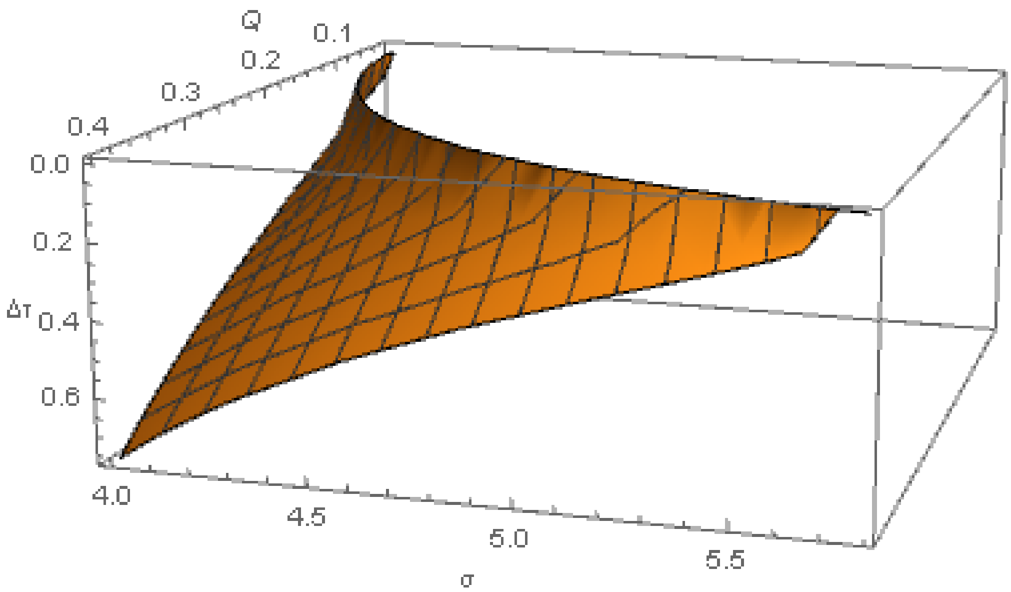

Figure 4.2: Numerical integration of the integral for the crossing time $\Delta\tau$ for different spacetimes defined by different allowed values of σ, Q, with $l = 1, P = 0, m = 0.4$. This figure just shows that the integral is indeed finite.

4.4.1 Discussion

Overall, the ingredients we have are the following

- The Null Energy Condition is satisfied for this spacetime
- The time that an observer at $r = \theta = 0, u = \text{const}$ measures for a light ray going from one boundary to the other is finite.
- Traversability and the Null Energy Condition do not agree due to the Topological Censorship Theorem
- Either traversability is wrong or something should violate the condition

Traversability that is interesting is the one measured by an assymptotic observer. That observer has coordinates (t, r, θ, u)

For such an observer the time that it takes for a radial null path, with $\theta = 0, u = \text{const}$, to go from one boundary to the other is found as

$$ds^2 = 0 \implies \Delta\tau = \int dt \tag{4.20}$$

$$\Delta\tau = \frac{2l^2}{\sigma} \int_{-\infty}^{+\infty} \frac{dr}{\sqrt{f(r)h(r)}} \tag{4.21}$$

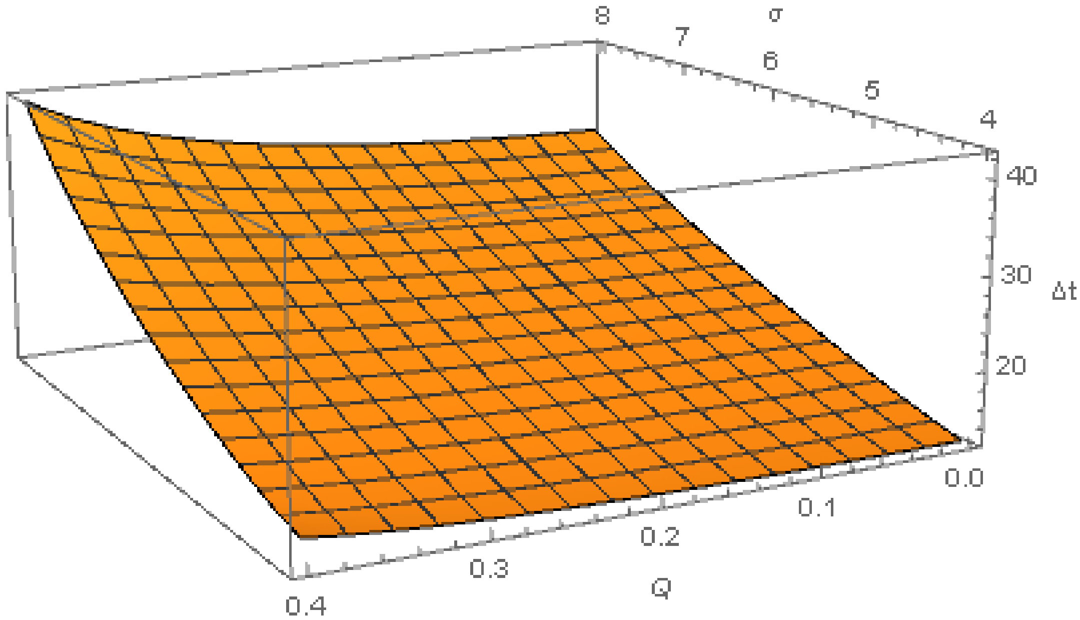

Figure 4.3: Crossing time measured by an assymptotic observer. The same values as in figure 4.2 have been used. For $\sigma = 4, Q = 0$ there crossing time is zero. It seems that in that case the boundaries are connected. Recall that $\sigma = 4$ and $f(r) = h(r)$ corresponds to the empty AdS case. However, just the $\sigma = 4$ condition does not lead you directly to empty AdS in the bulk, since if one plots higher curvature invariants for that spacetime with $\sigma = 4$ they are not everywhere flat as in empty AdS$_4$. However, the boundary metric for $\sigma = 4$ becomes exactly that of empty AdS$_3$.

The crossing time as measured from the boundary seems to be finite as well, and independent from the perspective of the observer. An observer sitting on either of the conformal boundaries will measure the same crossing time.

An interesting remark is that in empty AdS the affine parameter of a null geodesic reaching the conformal boundary is infinite, even though the time measured by an observer on the boundary is finite. This is explained clearly in [12]. This shall not be confusing since, in any case, the interesting crossing time is the one measured by an observer.

5. Physical Properties

5.1 Conserved Charges in General Relativity

Defining conserved charges, energy, angular momentum etc, in General Relativity is a highly non-trivial topic, which has received plenty of attempts, with the very first one being by Einstein himself. Energy conservation, for example, is one of the basic principles in physics. However, in general relativity, local energy density of matter measured by an observer is not an observer-independent quantity, and, thus, far from being conserved. The main issue lies in the fact that local covariant conservation law of the matter energy momentum tensor does not imply a global conservation law, in the general curved case. Conversly thinking, even though it seems as if in curved spacetime, one needs some local energy conservation of both the "gravitational field" plus matter, "gravitational field" energy can not be localized. This can be seen as an immediate concequence of the Principle of Equivalence, which implies that all physical effects of the gravitational field can be locally eliminated by choosing an appropriate freely falling reference frame. For that reason, energy conservation can only be well defined in the whole spacetime. As for the notion of conservation being used, for the energy of stationary spacetimes, this is the conserved quantity associated to the time translation generator. That is the assymptotically timelike Killing vector $\boldsymbol{\partial}_t$, which is congugate to the energy in the canonical approach.

While in special relativity the continuity equation for the energy momentum tensor of matter $\partial_\mu T^{\mu\nu}_{\text{matter}} = 0$ is a conservation equation, in the general curved case the covariant generalization of the continuity equation $\nabla_\mu T^{\mu\nu}_{\text{matter}} = 0$, which is required due to the Principle of General Covariance, is not a conservation equation anymore. More specifically in special relativity, it is straightforward to define the energy as the conserved quantity associated with time translational invariance on a spacelike Cauchy hypersurface, namely $E = \int_\Sigma T_{\mu\nu} n^\mu t^\nu$, where t^ν is the timelike Killing field and n^μ is the normal to the hypersurface. The conservation of this quantity is manifest, by virtue of the continuity equation and the Killing equation, and it is independent of the choice of finite hypersurface in that case. In the case of a gravitational field, on the other hand only the total energy, computed by integrating over the whole volume of timelike hypersurfaces of the whole spacetime, is well defined and independent of the choice of coordinates. In fact, local conservation laws require the relation $\partial_\mu T^{\mu\nu}_{\text{matter}} = 0$ to hold, which is only true in Minkowski spacetime.

Conserved quantities are naturally related to the symmetries of the vacuum background spacetime at asymptotic infinity. For instance, for *assymptotically flat* spacetimes this is the Poincaré group, while for *assymptotically* AdS_4 it is the $SO(2,3)$ group.[1] This is why we will be interested in the assymptotic behaviour of fields. Historically, there exist many approaches to the definition of conserved quantities, with two most important ones being the *pseudotensor* approach and the Noether's approach. It is important to note that different approaches and derivations work for different backgrounds.

In the *pseudotensor* approach, also referred to as the traditional approach, the direction was to search for a gravitational energy-momentum *pseudotensor* to add to the matter energy momentum tensor, in order to compensate for the non-vanishing conservation law $\partial_\mu \left(\sqrt{-g} T^{\mu\nu}_{\text{matter}}\right) \neq 0$. Attempts by Landau-Lifshitz in [47] and Abbott-Deser in [48] work for *assymptotically* flat spacetimes and spacetimes with arbitrary[2] assymptotic behaviour respectively. A special case of the Abbott-Deser approach is the Arnowitt-Deser-Misner (ADM) mass formula, which was first derived by canonical methods and is still extensively used in *assymptotically* flat metrics in cartesian coordinates or in *assymptotically* AdS metrics in static coordinates.

Noether's approach is not totally unrelated to the previous approaches. However, it is more general in the sense that one can define a Noether current for every vector field, either Killing or not. The invariance of the action under General Coordinate transformations, combined with the Bianchi identity, which serves as a gauge identity and is related to the invariance, suggests the covariant conservation of a Noether current $j^\mu_N(\xi)$ associated with any vector field ξ^μ. This Noether current can be massaged to be written as a divergence of an antisymmetric tensor, namely $j_N(\xi)^{\mu\nu} = 2\nabla^{[\mu}\xi^{\nu]}$. Using this, one can derive a well defined conserved quantity and express it as a boundary integral with the aid of Stoke's theorem [see Appendix A]. In this way one can define a conserved quantity associated to any vector field, also known as Komar's formula, which was introduced in [49].

$$E[\xi] = -\frac{2}{\chi^2}\int_{\partial\Sigma} d^{d-2}\Sigma_{\mu\alpha}\nabla^\mu \xi^\alpha$$

where, for $d = 4$, $d^2\Sigma_{\mu\alpha}$ is the directed two-surface element of the spatial sections of the boundary and χ^2 is the euler characteristic.

All those currents and conserved quantities of the theory correspond to gauge symmetries, i.e. general coordinate transformations, and, hence, are not all physi-

[1] The case of the wormhole spacetime that we are studying is slightly more special, in the sense that it is *assymptotically locally* AdS_4 and preserves only some of the isometries of AdS_4.

[2] By arbitrary here we mean either *assymptotically* flat or *assymptotically* Anti-de Sitter or *assymptotically* de-Sitter, which are maximally symmetric solutions to vacuum Einstein equations.

cal. The directions that one needs to plug into Komar's formula in order to derive conserved physical quantities are the Killing vector fields, along which the metric tensor solution is left invariant, i.e. the Lie derivative of the metric tensor is zero along these directions. Depending on the symmetries of the spacetime solutions, these quantities can often be the total mass and angular momentum. Komar's formula is extensively used and gives consistent results in the Schwarzschild and Reissner-Nordstrom solutions, but it has several shortcomings. More precisely, it does not give the right result for the angular momentum in the Kerr solution and it requires background subtraction regularization by hand in *assymptotically* AdS spacetimes, in order to give the right result. In principle, it can be corrected by including total derivative boudnary terms in the Noether current.

There exist, historically, a class of approaches dedicated to *assymptotically* (locally) AdS spacetimes in the literature. Some remarkable papers were those by Ashtekar and Magnon [50] about conserved quantities in four dimensional A(l)AdS, which was later generalized to higher dimensions by Ashtekar and Das[51]. In these papers, it was shown that conserved charges in *assymptotically* AdS spacetimes can be expressed as a $d-2$ surface integral of a quantity that involves the electric part of the assymptotic Weyl tensor and the assymptotic Killing field. This approach is called the Ashtekar-Magnon-Das (AMD) formula, and it will be used in this text. A more formal proof was provided by the approach of Wald in [52], which goes by the name "covariant phase space formalism". The AMD formula should, in principle, be compared and confirmed by other approaches involving canonical formalism and linearized expansion in AdS background by Henneaux and Teitelboim in [53] or by Balasubramanian and Kraus in [54]. In this text, those comparisons will not be considered, since it is already relatively complex to apply the AMD formula to *assymptotically* locally AdS spacetimes.

This introduction was inspired by the following readings [55, 56, 7, 57]

5.1.1 Towards the application of the AMD formula

In order to avoid confusion, it is important to note that the ADM and AMD formalisms are two different approaches. ADM stands for Arnowitt, Deser and Misner, who worked on the canonical (Hamiltonian) formulation of gravity, with the perspective of quantizing it in the standard canonical way. Even though this failed, the decomposition of the spacetime in a foliation of constant time leaves, Σ_t, that is used in this formalism is useful in numerical relativity and in the definition of a conserved notion of energy, also known in literature as ADM mass. The ADM mass is defined as a function of the deviation of the metric from it's assymptotic form. Nevertheless, this canonical approach only works for spacetimes that are assymptotically flat and Assymptotically AdS in static coordinates. A derivation of the canonical approach in AdS background can be found in [53]. In contrast, AMD stands for Ashtekar-Magnon-Das[50, 51], who worked on defining conserved charges in Assymptotically (locally) AdS spacetimes.

Ashtekar, Magnon and Das based their method on techniques of conformal compactification introduced by Penrose in [8]. Those techniques are possible for spacetimes that have a particular fall-off of the Weyl tensor at infinity. At this point, the definition of *Assymptotically* AdS spacetimes (AAdS) and *Assymptotically* locally AdS spacetimes (AlAdS) will be handy for a better understanding of the following sections.

Assymptotically AdS spacetimes

Recall the following definitions from [51, 58] A d-dimensional space-time $(\hat{M}, \hat{g}_{\alpha,\beta})$ is said to be *assymptotically anti-de Sitter* if there exists a manifold M with boundary ∂M, equipped with a metric $g_{\alpha\beta}$ and a diffeomorphism from $\hat{M}$ onto $M - \partial M$ of M (with which $\hat{M}$ and $M - \partial M$ are identified) and the interior of M such that:

1. there exists a function Ω on M for which $g_{\alpha\beta} = \Omega^2 \hat{g}_{\alpha\beta}$ on $\hat{M}$
2. ∂M is topologically $S^{d-2} \times R$, Ω vanishes on ∂M, but its gradient $\nabla_\alpha \Omega$ is nowhere vanishing on ∂M
3. On $\hat{M}$, $\hat{g}_{\alpha\beta}$ satisfies $\hat{R}_{\alpha\beta} - \frac{1}{2}\hat{R}\hat{g}_{\alpha\beta} + \Lambda \hat{g}_{\alpha\beta} = 8\pi G_{(d)} \hat{T}_{\alpha\beta}$, where Λ is the negative cosmological constant. $G_{(d)}$ is Newton's constant in $d-$dimensions, and the matter stress-energy $\hat{T}_{\alpha\beta}$ is such that $\Omega^{2-d}\hat{T}_{\alpha\beta}$ admits a smooth limit to ∂M
4. the Weyl tensor of $g_{\alpha\beta}$ is such that $\Omega^{4-d} C_{\alpha\beta\gamma\delta}$ is smooth on M and vanishes at ∂M

A more detailed explanation on why these conditions are required for *assymptotically anti-de Sitter* spacetimes can be found [51].

Assymptotically locally AdS spacetimes

Upon relaxing these conditions, it follows that for the definition of *assymptotically locally anti-de Sitter* spacetimes, one needs as all of the above, except for the fact that the boundary should contain a topological S^{d-2} sphere. A more detailed review can be found in [58]. These conformal completion techniques bring the boundary $\partial\mathcal{M}$ of a D-dimensional AAdS space $\mathcal{M}$ to a finite distance and formulate a new unphysical manifold with a conformal boundary. They can also be applied to AlAdS spacetimes, where the boundary does not have a topology of $S^{d-2} \times R$, provided that there is a "good" radial coordinate that brings you effectively to the bulk from the conformal boundary.

5.1.2 Applying AMD formalism

The metric of the wormhole solution is time independent and has a Killing vector which is always timelike, provided that $\sigma > 4$[3]. The canonical foliation of constant time hypersurfaces exists, is well defined and is going to be used later on in this text. The fact that there are cross-terms in the metric does not allow us to apply standard techniques for spherical boundaries. Our boundary is warped and spherical symmetry is lost. Our case falls in the category of Assymptotically locally AdS spacetimes, where the boundary is not topologically $S^{d-2} \times R$. The coordinate r is a "good" radial coordinate and the Weyl tensor is smooth in the bulk and vanishes on the boundary. In that context, everything is consistent in order to follow techniques applied for Assymptotically (locally) AdS spacetimes. These techniques were established in [50, 51] and an application of them can be found in the appendix of [59] or in [57].

We will do a conformal rescaling of the metric

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{5.1}$$

where $\Omega = 0, \quad d\Omega \neq 0$ at the boundary. In fact, Ω is defined up to a scalar function that is nonzero on the boundary. We choose $\Omega = \frac{l}{r}$, which is a good radial coordinate in the sense that it leads us to the bulk of the spacetime for small r.

The *physical*$(1,3)$ Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ has the property of being invariant under conformal rescalings. The *unphysical* Weyl tensor $\bar{C}^{\mu}{}_{\nu\rho\sigma}$ is just the *physical* one, but with indices raised and lowered by the conformally rescaled metric $\bar{g}_{\mu\nu}$. However, the $(1,3)$ Weyl tensor is invariant

$$\bar{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma} \tag{5.2}$$

The normal 1-form to the boundary is

$$m_\mu = \partial_\mu \Omega = -\frac{l}{r^2}\boldsymbol{dr} \tag{5.3}$$

The conformally rescaled one is

$$\bar{m}_\mu = \Omega m_\mu \tag{5.4}$$

It will only have r component and if we raise the index with the rescaled metric it takes the form

$$\bar{m}^r = -\frac{1}{\Omega}\frac{\sigma^2 f(r)}{4l^3 r^2} \tag{5.5}$$

To compute any conserved quantity associated with a Killing field we need to define the quantity

$$\bar{E}^{\mu}{}_{\nu} = l^2 \frac{\Omega^{3-d}}{d-3} \bar{m}^\rho \bar{m}^\sigma \bar{C}^{\mu}{}_{\rho\nu\sigma} \overset{d=4}{\Longrightarrow} \bar{E}^{\mu}{}_{\nu} = \Omega^{-3} E^{\mu}{}_{\nu} \tag{5.6}$$

This quantity is nothing but the projection of the Weyl tensor on the conformal boundary, also referred to as the electric part of the Weyl tensor.[3] The *physical* Weyl tensor is the traceless part of the Riemann curvature tensor and it contains information about the curvature that is not affected by the equations of motion. For that reason, the Weyl tensor is usually considered as a representative of "purely gravitational" degress of freedom. This idea is obvious in the case of *Anti*-de Sitter spacetime, for which the Weyl tensor is identically zero. Hence, the vacuum energy of AdS spacetime is not included in this quantity and this is why this method does not need any background subtraction, as compared to other approaches. Therefore, the charges that we are computing only provide information about the additional structure on top of the vacuum assymptotic solution, which in our case is the wormhole in the bulk.

According to [51, 50] the following quantity linked to a Killing field ξ^ν is proven to be conserved and independent of the conformal factor Ω.[4].

$$Q[\xi] = \frac{l}{8\pi} \int_S \bar{E}^\mu{}_\nu \xi^\nu d\bar{S}_\mu \tag{5.7}$$

where

$$d\bar{S}_\mu = \bar{n}_\mu d\bar{S} = \Omega n_\mu d\bar{S} \tag{5.8}$$

is the directed area element of the spatial section on the conformal boundary, which in our case is not a 2-sphere. Here n_μ is the unit 1-form in the direction $\boldsymbol{dt}$. In order to compute the surface element on the conformal boundary we need the determinant of the rescaled induced metric $h_{\mu\nu}$. For a conformally rescaled 2d metric with a conformal factor Ω, the determinant is $\sqrt{\bar{h}} = \Omega^2\sqrt{h}$. This means that

$$d\bar{S}_\mu = \Omega^3 n_\mu dS \tag{5.9}$$

where dS is the usual surface element on the boundary, from which we can see that the charge in (5.7) is indeed independent of the conformal rescaling, namely

$$\bar{E}^\mu{}_\nu \xi^\nu d\bar{S}_\mu = E^\mu{}_\nu \xi^\nu dS_\mu$$

The conformal rescaling essentially serves as a factor that makes every individual quantity of the integrand convergent, so that one can perform an expansion around infinity.

The AMD mass, in particular, will be

$$Q[K] = \frac{l}{8\pi} \int_S \bar{E}^\mu{}_t K^t d\bar{S}_\mu \tag{5.10}$$

[3]the term "electric part" is used as an analogy to the electromagnetic field strength tensor, but it does not refer to electric charges or anything related to electromagnetism. It just indicates which components of the *unphysical* Weyl tensor are extracted due to this contraction

[4]There is a comment in [51] mentioning that this charge is conserved, even if the boundary is not topologically $S^2 \times R$

We will need to contract $\bar{E}^{\mu}_{\nu}$ with the timelike Killing vector K^{μ} as follows

$$\bar{E}^{t}{}_{t}K^{t} = l^2 \frac{1}{\Omega}\frac{1}{\Omega^2}\left(\frac{\sigma^2 f(r)}{4l^3 r^2}\right)^2 C^{t}{}_{rtr} = \frac{l}{\Omega^3}\left(\frac{\sigma^2 f(r)}{4l^3 r^2}\right)^2 C^{t}{}_{rtr} \tag{5.11}$$

It is important to note that the Killing vector need not be rescaled, since it belongs to the asymptotic conformal isometries. It only depends on the equivalence class of metrics with conformal rescalings as the equivalence relation, that the boundary metric is part of, and not in any particular representative of the class.

The corresponding electric part of the Weyl tensor has been computed using the *Mathematica* package *xAct'xTensor* are the following

$$\begin{aligned} C^{t}{}_{rtr} =& \frac{h(r)(-8l^4 + \sigma^2 f'(r)h'(r) - \sigma^2 h(r)f''(r))}{12 f(r)(h(r))^2} \\ &+ \frac{f(r)(8l^4 - \sigma^2(h'(r))^2 + \sigma^2 h(r)h''(r))}{12 f(r)(h(r))^2} \end{aligned} \tag{5.12}$$

The constant time hypersurface normal is

$$n^{\mu} = \left(-\frac{\text{sech}\theta}{\sqrt{h(r)}}, 0, 0, \frac{\tanh\theta}{\sqrt{h(r)}}\right), \qquad n_{\mu} = \left(\sqrt{h(r)}\cosh\theta, 0, 0, 0\right) \tag{5.13}$$

Then,

$$\bar{E}^{\mu}{}_{t}K^{t}d\bar{S}_{\mu} = l\left(\frac{\sigma^2 f(r)}{4l^3 r^2}\right)^2 C^{t}{}_{rtr}h(r)\sqrt{f(r)}\cosh\theta d\theta du \tag{5.14}$$

where $C^{t}{}_{rtr}$ is the one shown in the expression (5.12).

Taking the limit $r \to \pm\infty$ in equation (5.10) should, in principle, commute with performing the integral, due to the fact that the variable with respect to which we are taking the limit is different from the variables being integrated. The integral, then, takes the following form

$$Q_{\pm}[K] = \pm\frac{l}{8\pi}\int_0^{\alpha} du \int_{-\infty}^{\infty} d\theta \frac{l^2 m}{\sigma^2}\cosh\theta \tag{5.15}$$

The indefinite integral of $\cosh\theta$ is $\sinh\theta$, which blows up at infinity. This infinity should not be worrying, since it is explained due to the infinite volume of the AdS boundary. The plus sign corresponds to the boundary at $r \to \infty$,while the minus sign corresponds to the boundary at $r \to -\infty$.

The total charge is the one coming from the contribution of both components of the boundary

$$Q[K] = Q_{+}[K] + Q_{-}[K] = 0 \tag{5.16}$$

If the same procedure is applied for the Killing vector associated with the angular momentum, i.e. $J^{\mu} = (0, 0, 0, 1)$, the result is zero. The angular momentum is conserved but its magnitude is zero. One can predict this result if one takes a quick look at the metric, since all the physical parameters have been recognized and there is no extra parameter that could be associated with some angular momentum. The result of the calculation will be zero by virtue of the directions of the problem. More specifically, the direction at which we need to project the tensor $\bar{E}^{\mu}_{\nu}$ gives identically zero.

$$Q[J] = \frac{l}{8\pi}\int_S \bar{E}^{\mu}{}_{u} J^{u} d\bar{S}_{\mu} = 0 \tag{5.17}$$

As expected, the spacetime is not rotating.

Similarly, we will try to plug in the two remaining Killing vectors in the formula (5.10), which, in principle should provide us with a conserved quantity related to each one of the Killing vectors, ξ_2, ξ_3, which can be found in Appendix C. The corresponding AMD conserved quantities are

$$Q[\xi_2] = \frac{l}{8\pi}\int_S \bar{E}^{\mu}{}_{\nu}\xi_3^{\nu} d\bar{S}_{\mu} \tag{5.18}$$

$$Q[\xi_3] = \frac{l}{8\pi}\int_S \bar{E}^{\mu}{}_{\nu}\xi_3^{\nu} d\bar{S}_{\mu} \tag{5.19}$$

$$\tag{5.20}$$

which upon substituting and taking the limit to $r \to \pm\infty$ become

$$Q_{\pm}[\xi_2] = \mp\frac{\alpha l^3 \sin t}{8\pi\sigma^2} m \int_{-\infty}^{\infty} \sinh\theta d\theta \tag{5.21}$$

$$Q_{\pm}[\xi_3] = \mp\frac{\alpha l^3 \cos t}{8\pi\sigma^2} m \int_{-\infty}^{\infty} \sinh\theta d\theta \tag{5.22}$$

That gives

$$Q[\xi_2] = 0 \tag{5.23}$$

$$Q[\xi_3] = 0 \tag{5.24}$$

5.1.3 Komar mass

In the literature, for the case of assymptotically (locally) AdS spacetimes, both the Komar formula and the AMD formula are being used and compared. They are essentially going to give the same result. However, the Komar formula needs some background subtraction by hand first.

The Komar integral associated with the total energy is the following

$$E_R = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x \sqrt{\gamma|_{t,r=\mathrm{const}}}\, n_{\mu}\sigma_{\nu}\nabla^{\mu}K^{\nu} \tag{5.25}$$

where Σ is a spacelike hypersurface and $\partial\Sigma$ its boundary at infinity, $\gamma^{(2)}$ is the induced metric on the boundary, σ_{μ} is the unit vector normal to the surface $\partial\Sigma$, n_{ν} is the unit vector normal to Σ, $K^{\nu} = (1,0,0,0)$ is the timelike Killing vector.

First, we consider a spacelike hypersurface,Σ_t, of constant t which has a normal vector

$$n^\mu = \frac{g^{0\mu}}{\sqrt{-g^{00}}} = \left(-\frac{\mathrm{sech}\theta}{\sqrt{h(r)}}, 0, 0, \frac{\tanh\theta}{\sqrt{h(r)}}\right) \tag{5.26}$$

The normal vector to the boundary at spacelike infinity will be

$$\sigma_\nu = \frac{1}{\sqrt{g^{11}}}(0,1,0,0) = \left(0, \frac{2l^2}{\sigma\sqrt{f(r)}}, 0, 0\right) \tag{5.27}$$

such that $\sigma^\nu\sigma_\nu = +1$

The induced metric at spacelike infinity is

$$\left(\gamma|_{t,r=\text{const}}\right)_{ij} dx^i dx^j = \frac{l^2}{\sigma}\left(d\theta^2 + \frac{4}{\sigma}du^2\right) \tag{5.28}$$

Then

$$d^2x\sqrt{\gamma|_{t,r=\text{const}}} = \frac{2l}{\sigma}d\theta du \tag{5.29}$$

For the Mass Komar integral we need

$$\begin{aligned}
n_\mu\sigma_\nu\nabla^\mu K^\nu &= n^\mu\sigma_\nu\left(\partial_\mu K^\nu + \Gamma^\nu_{\rho\mu}K^\rho\right)\\
&= n^t\sigma_r\left(\partial_t K^r + \Gamma^r_{tt}K^t\right) + n^u\sigma_r\left(\partial_u K^t + \Gamma^r_{tu}K^t\right)\\
&= \frac{1}{2}\frac{\mathrm{sech}\theta}{\sqrt{h(r)}}\frac{2l^2}{\sigma\sqrt{f(r)}}\frac{\sigma^2 f(r)}{4l^4}\partial_r\left(-h(r)\cosh^2\theta + f(r)\sinh^2\theta\right)\\
&- \frac{1}{2}\frac{\tanh\theta}{\sqrt{h(r)}}\frac{2l^2}{\sigma\sqrt{f(r)}}\frac{\sigma^2 f(r)}{4l^4}\partial_r f(r)\sinh\theta\\
&= -\frac{1}{2}\frac{\sigma}{2l^2}\frac{\sqrt{f(r)}}{\sqrt{h(r)}}\partial_r h(r)\cosh\theta
\end{aligned}$$

The integral will look like

$$E_R = -\frac{2l}{\sigma}\lim_{r\to\infty}\int d\theta du \quad \frac{\sigma}{4l^2}\frac{\sqrt{f(r)}}{\sqrt{h(r)}}\partial_r h(r)\cosh\theta \tag{5.30}$$

This limit is divergent because of the infinite volume of AdS_4. Recall that the empty AdS_4 case corresponds to $\sigma = 4$ and $f(r) = h(r)$. Upon background subtraction we get

$$\bar{E}_R = \frac{1}{2l}\lim_{r\to\infty}\int d\theta du\left[-\frac{\sqrt{f(r)}}{\sqrt{h(r)}}\partial_r h(r)\cosh\theta + \partial_r h(r)|_{\sigma=4}\cosh\theta\right] \tag{5.31}$$

5.1.4 Physical Discussion

The charges defined by the AMD formula seem to be divergent for non-compact boundaries, but this divergence only has to do with the infinite volume of AdS. Recall that this solution is constructed in such a way that the boundary is warped AdS_3.

This infinity exists in the on-shell action as well, where it originates from the determinant of the metric [see appendix C]. It is, thus, convenient to define a total mass per unit volume of the spatial sections of the warped AdS_3 boundary. The charge can then be written as

$$Q_{\pm}[K] \sim \quad \pm\, m \ \mathrm{Vol}\left(wAdS_3\right)_E \tag{5.32}$$

where $\mathrm{Vol}\left(wAdS_3\right)_E$ is the volume of the warped AdS_3 boundary computed upon Wick rotating the time coordinate and periodically identifying it. The proportionality constant above, therefore, contains the inverse temperature β.[5]

This volume will be equal to

$$\mathrm{Vol}\left(wAdS_3\right)_E = 2\frac{l^2}{\sigma}\int_0^{\beta} d\tau \int_0^{\alpha} du \int_{-\infty}^{\infty} \cosh\theta d\theta \tag{5.33}$$

Another remark is that one of the reasons why we needed to do this calculation was in order to relate the conformal mass to the parameter m in the metric. The parameter itself is not a physical quantity, if one does not compute the associated charge. However, the conserved charge defined by the AMD formula is considered as the conformal mass. This calculation shows that the parameter m can be now interpreted as a physical parameter.

5.2 Computing the Electric and Magnetic charges

In this section we will use Stokes theorem to express the conserved electric and magnetic charges as a surface integral at the boundary of a spacelike constant time hypersurface of the spacetime. Those charges are somewhat different from the previous ones, in the sense that they are related to the matter content of the spacetime.

Recall that the metric has the form

$$ds^2 = \frac{4l^4}{\sigma^2 f(r)}dr^2 + h(r)\left(-\cosh^2\theta dt^2 + d\theta^2\right) + f(r)\left(du + \sinh\theta dt\right)^2 \tag{5.34}$$

[5]Moreover, the factor $\frac{1}{8\pi}$ is omitted on purpose here, since it is some normalization factor from the original AMD paper that was presenting the example of AAdS with boundary that was topologically $R\times S^2$, so after the volume integration there would be a factor $\frac{1}{2}$ left.So the correct factor to have there is just this $\frac{1}{2}$ remaining after deviding by the volume.

with $\{t,r\} \in \mathrm{R}$, $\theta \in [\,0,+\infty\,)$, the coordinate u is identified, $u+\alpha = u$, and

$$f(r) = \frac{4l^2}{\sigma^2}\frac{r^4+(6-\sigma)r^2+mr+\sigma-3}{r^2+1} - \frac{Q^2+P^2}{r^2+1} \quad , \quad h(r) = \frac{l^2}{\sigma}(r^2+1)$$

and the matter content is described by

$$F_{ru} = -F_{ur} = \frac{2(r^2-1)Q-4rP}{(r^2+1)^2} \tag{5.35}$$

$$F_{rt} = -F_{tr} = \frac{2(r^2-1)Q-4rP}{(r^2+1)^2}\sinh\theta = F_{ru}\sinh\theta \tag{5.36}$$

$$F_{\theta t} = -F_{t\theta} = \frac{-2Qr+P(1-r^2)}{r^2+1}\cosh\theta \tag{5.37}$$

The rest of the details are included in the appendix 5.2.4.

5.2.1 Hypersurface normals and induced metrics

First, we consider a spacelike hypersurface,Σ_t, of constant t which has a normal vector

$$n^\mu = \frac{g^{0\mu}}{\sqrt{-g^{00}}} = \left(-\frac{\mathrm{sech}\theta}{\sqrt{h(r)}},0,0,\frac{\tanh\theta}{\sqrt{h(r)}}\right), \qquad n_\mu = \left(\sqrt{h(r)}\cosh\theta,0,0,0\right) \tag{5.38}$$

such that

$$\begin{aligned} n_\mu n^\mu &= g_{00}(n^0)^2 + 2g_{02}n^0n^2 + g_{22}(n^2)^2 \\ &= \left(-h(r)\cosh^2\theta + f(r)\sinh^2\theta\right)\frac{\mathrm{sech}^2\theta}{h(r)} - 2f(r)\sinh\theta\frac{\mathrm{sech}\theta}{\sqrt{h(r)}}\frac{\tanh\theta}{\sqrt{h(r)}} \\ &+ \frac{f(r)}{h(r)}\tanh^2\theta = -1 \end{aligned}$$

This is because if we have some local coordinates (x^0,x^1,x^2,x^3) and a hypersurface $x^0=$ const, then if a vector X is tangent to the hypersurface, g^{0a} will be normal to the hypersurface. This is due to the fact that their inner product reads $g_{ab}g^{0a}X^b = X^0$, which will vanish if X is tangent to the hypersurface of constant x^0.

The induced metric for the constant $t-$slices,

$$\gamma_{ij}|_{t=\mathrm{const}}dx^idx^j = \frac{4l^4}{\sigma^2 f(r)}dr^2 + h(r)d\theta^2 + f(r)du^2 \tag{5.39}$$

The closed surface bounding the region Σ_t can be thought of as consisting by two parts, two surfaces of constant $\{t,r\}$,one pointing at $r\to+\infty$ and one pointing at

$r \to -\infty$,$\partial\Sigma_1$ and $\partial\Sigma_2$, and one surface of constant $\{t, \theta\}$, $\partial\Sigma_3$. We can vizualize this surface as a cylinder and we will take it to infinity in order to integrate over the whole spacetime. The normal vector to the constant t, r surface at spacelike infinity will be

$$\boldsymbol{\sigma}_\nu = \frac{1}{\sqrt{g^{11}}}(0,1,0,0) = \left(0, \frac{2l^2}{\sigma\sqrt{f(r)}}, 0, 0\right) \tag{5.40}$$

such that $\sigma^\nu \sigma_\nu = +1$

The induced metric at the surfaces of constant r is

$$\gamma_{ij}|_{t,r=const} dx^i dx^j = h(r) d\theta^2 + f(r) du^2 \tag{5.41}$$

Then,

$$d^2x\sqrt{\gamma|_{t,r=const}} = \sqrt{h(r)f(r)}d\theta du \tag{5.42}$$

The directed two-surface element to the constant $\{t, r\}$ two-surfaces,$\partial\Sigma_1$ and $\partial\Sigma_2$, will be

$$dS^{(1,2)}_{\mu\nu} = \pm 2 n_{[\mu}\sigma_{\nu]}\sqrt{\gamma|_{t,r=const}}d\theta du \tag{5.43}$$

with the "$-$" corresponding to the case $r \to \infty$ and the "$+$" to the case $r \to -\infty$.

The normal vector to the constant θ slice is

$$\boldsymbol{\rho}^\nu = \left(0, 0, \frac{1}{\sqrt{h(r)}}, 0\right) \tag{5.44}$$

such that $\rho_\nu \rho^\nu = +1$. The square root of the determinant of the induced metric is given by

$$\sqrt{\gamma|_{t,\theta=\text{const}}} = \frac{2l^2}{\sigma} \tag{5.45}$$

The induced metric at each slice of constant θ at the boundary is

$$\gamma_{ij}|_{t,\theta=const} dx^i dx^j = \frac{4l^4}{\sigma^2 f(r)} dr^2 + f(r) du^2 \tag{5.46}$$

The directed two-surface element to the constant $\{t, \theta\}$ two-surface,$\partial\Sigma_3$, will be

$$dS^{(3)}_{\mu\nu} = -2 n_{[\mu}\rho_{\nu]}\sqrt{\gamma|_{t,\theta=const}}dr du \tag{5.47}$$

5.2.2 Applying Stokes theorem

We have a non-vanishing antisymmetric tensor $F^{\mu\nu}$, defined in the region Σ_t and bounded by the surface $\partial\Sigma$, which consists of $\partial\Sigma_1, \partial\Sigma_2, \partial\Sigma_3$. Its divergence $\nabla_\mu F^{\mu\nu}$ is zero everywhere, since it satisfies the sourceless Maxwell's equations in curved spacetime.

We are now all set to apply Stokes theorem, as defined in Appendix A. The total electric charge is defined as

$$\begin{aligned} Q_e &= -\int_{\Sigma} \nabla_\nu F^{\mu\nu} d\Sigma_\mu = -\frac{1}{2}\oint_{\partial\Sigma} F^{\mu\nu} dS_{\mu\nu} \\ &= -\frac{1}{2}\int_{\partial\Sigma_1} F^{\mu\nu} dS^{(1)}_{\mu\nu} - \frac{1}{2}\int_{\partial\Sigma_2} F^{\mu\nu} dS^{(2)}_{\mu\nu} - \frac{1}{2}\int_{\partial\Sigma_3} F^{\mu\nu} dS^{(3)}_{\mu\nu} \end{aligned} \tag{5.48}$$

The relevant expression for the magnetic charge should be

$$\begin{aligned} Q_m &= -\frac{1}{4}\oint_{\partial\Sigma} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dS_{\mu\nu} \\ &= -\frac{1}{4}\int_{\partial\Sigma_1} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dS^{(1)}_{\mu\nu} - \frac{1}{4}\int_{\partial\Sigma_2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dS^{(2)}_{\mu\nu} - \frac{1}{4}\int_{\partial\Sigma_3} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dS^{(3)}_{\mu\nu} \end{aligned} \tag{5.49}$$

The first two terms in each one of these two expressions,(5.48) and (5.49), do not give any contribution, since

$$\begin{aligned} n_\mu \sigma_\nu F^{\mu\nu} &= n^\mu \sigma^\nu F_{\mu\nu} = n^t \sigma^r F_{tr} + n^u \sigma^r F_{ur} \\ &= \frac{\mathrm{sech}\theta}{\sqrt{h(r)}} \frac{2l^2}{\sigma\sqrt{f(r)}} F_{ru} \sinh\theta - \frac{\tanh\theta}{\sqrt{h(r)}} \frac{2l^2}{\sigma\sqrt{f(r)}} F_{ru} = 0 \end{aligned} \tag{5.50}$$

and

$$\begin{aligned} \frac{1}{2} n_\mu \sigma_\nu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} &= \frac{1}{2}\sqrt{-g}\, n_t \sigma_r \bar{\epsilon}^{tr\rho\sigma} F_{\rho\sigma} + \frac{1}{2}\sqrt{-g}\, n_u \sigma_r \sqrt{-g}\, \bar{\epsilon}^{ur\rho\sigma} F_{\rho\sigma} \\ &= \frac{1}{2}\sqrt{-g}\left(g_{uu} n^u + g_{ut} n^t\right) \sigma_r \left(\bar{\epsilon}^{urt\theta} F_{t\theta} + \bar{\epsilon}^{ur\theta t} F_{\theta t}\right) \\ &= \left(f(r) \frac{\tanh\theta}{\sqrt{h(r)}} - f(r)\sinh\theta \frac{\mathrm{sech}\theta}{\sqrt{h(r)}} \right) \frac{2l^2}{\sigma\sqrt{f(r)}} \frac{F_{\theta t}}{\sqrt{-g}} \\ &= 0 \end{aligned} \tag{5.51}$$

Here $\bar{\epsilon}_{\mu\nu\rho\sigma}$ is the fully antisymmetric tensor in 4d or Levi-Civita symbol, which is not really a tensor, while $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\bar{\epsilon}_{\mu\nu\rho\sigma}$ transforms as a tensor and is called the Levi-Civita tensor[60].

The fact that these terms give zero is due to the direction of the field lines, which will be studied in the next section.

The third term of (5.48) and (5.49) will give respectively

$$
\begin{aligned}
Q_e &= -\frac{1}{2}\int_{\partial\Sigma_3} F^{\mu\nu} dS^{(3)}_{\mu\nu} = -2\int_0^a du \int_{-\infty}^{\infty} dr \sqrt{\gamma|_{t,\theta=\text{const}}}\, n^\mu \rho^\nu F_{\mu\nu} \\
&= -\frac{4l^2\alpha}{\sigma}\int_{-\infty}^{\infty} dr n^t \rho^\theta F_{t\theta} = \frac{4l^2\alpha}{\sigma}\int_{-\infty}^{\infty} dr \ \text{sech}\theta \frac{F_{t\theta}}{h(r)} \\
&= \frac{4l^2\alpha}{\sigma}\int_{-\infty}^{\infty} dr \frac{\Phi(r)}{h(r)} \cosh\theta \ \text{sech}\theta \\
&= 4\alpha \int_{-\infty}^{\infty} dr \frac{-2Qr + P(1-r^2)}{(r^2+1)^2} \\
&= 4\alpha \int_{-\infty}^{\infty} dr \frac{-2Qr + P(1-r^2)}{(r^2+1)^2} = 0
\end{aligned}
\tag{5.52}
$$

and

$$
\begin{aligned}
Q_m &= -\frac{1}{4}\int_{\partial\Sigma_3} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} dS^{(3)}_{\mu\nu} \\
&= -\int_0^a du \int_{-\infty}^{\infty} dr \sqrt{\gamma|_{t,\theta=\text{const}}}\, n^t \rho^\theta \epsilon_{t\theta r u} F^{ru} \\
&= -\int_0^a du \int_{-\infty}^{\infty} dr \sqrt{\gamma|_{t,\theta=\text{const}}} \sqrt{-g}\, n^t \rho^\theta \bar{\epsilon}_{t\theta r u} F^{ru} \\
&= -\frac{8\alpha l^4}{\sigma^2}\int_{-\infty}^{+\infty} dr h(r) \cosh\theta \frac{\text{sech}\theta}{\sqrt{h(r)}} \frac{1}{\sqrt{h(r)}} \frac{\sigma^2 f(r)}{4l^4} \frac{1}{f(r)} \partial_r \Phi(r) \\
&= -2\alpha \int_{-\infty}^{+\infty} dr f(r) \partial_r \Phi(r) \\
&= -4\alpha \int_{-\infty}^{\infty} dr \frac{(r^2-1)Q - 2Pr}{(r^2+1)^2} = 0
\end{aligned}
\tag{5.53}
$$

5.2.3 Understanding the field line profiles

The tensor field on a manifold M can be split in such a way that it respects the foliation given to the manifold. The metric is already in the form of the canonical foliation, defined in appendix D. Using this foliation of constant t slices, one can decompose the field strength $F_{\mu\nu}$, in order to extract information about the electric and magnetic fields.

At any point $p \in \Sigma_t$, in some local coordinates, $F_{\mu\nu}(t, \mathbf{x})$ can be decomposed according to the $3+1-$formalism, into the electric $E(t, \mathbf{x})$ and magnetic $B(t, \mathbf{x})$ field on the spacelike surface. For more information see 5.2.4 and [61]. These, will take the form

$$
E_\mu(t, \mathbf{x}) = F_{\mu\nu}(t, \mathbf{x}) n^\nu, \qquad B_\mu(t, \mathbf{x}) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} n^\nu(t, \mathbf{x}) F^{\rho\sigma}(t, \mathbf{x}) \tag{5.54}
$$

where $\epsilon_{\mu\nu\rho\sigma} = \frac{\bar{\epsilon}_{\mu\nu\rho\sigma}}{\sqrt{-g}}$ is the four dimensional Levi-Civita symbol. The field strength can be locally written as

$$F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \epsilon_{\mu\nu\rho\sigma} n^\rho B^\sigma \tag{5.55}$$

Let us identify those quantities for a 4d metric

$$g_{ij} = \gamma_{ij}, \qquad g_{00} = -N^2 + \beta_k \beta^k, \qquad g_{0i} = g_{i0} = \beta_i \tag{5.56}$$

and the inverse metric will be decomposed as

$$g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{N^2}, \qquad g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{\beta^i}{N^2} \tag{5.57}$$

The displacement vector will be $\boldsymbol{\beta} = f(r) \sinh\theta \boldsymbol{\partial_u}$ and it lives on the hypersurface Σ_t. The normal vector $\boldsymbol{n}$ can be found by demanding that the inner product

$$\boldsymbol{n} \cdot \boldsymbol{\beta} = 0 \tag{5.58}$$

so in our case it will be

$$n^\mu = \left(-\frac{1}{N}, 0, 0, \frac{\sinh\theta}{N}\right) = \left(-\frac{\text{sech}\theta}{\sqrt{h(r)}}, 0, 0, \frac{\tanh\theta}{\sqrt{h(r)}}\right) \tag{5.59}$$

where the lapse function is $N = \sqrt{h(r)}\cosh\theta$. The normalisation, here, is such, that the normal vector is a unit vector.

Thus, the decomposition of the fields reads

$$\begin{aligned} \boldsymbol{E}_\mu(t, \boldsymbol{x^i}) &= F_{\mu t}(t, \boldsymbol{x^i}) n^t + F_{\mu u}(t, \boldsymbol{x^i}) n^u && (5.60) \\ &= \big(0, F_{rt} n^t + F_{ru} n^u, F_{\theta t} n^t, 0\big) && (5.61) \\ &= \left(0, 0, -\frac{1}{N} F_{\theta t}, 0\right) && (5.62) \end{aligned}$$

The electric field will be,

$$\boldsymbol{E}_\theta(r) = -\frac{\sqrt{\sigma}}{l} \frac{P(r^2-1) - 2Qr}{(r^2+1)^{\frac{3}{2}}} \tag{5.63}$$

Moving on to the magnetic field

$$\begin{aligned} \boldsymbol{B}_\mu(t, \boldsymbol{x^i}) &= \epsilon_{\mu t r u} n^t F^{ru} + \epsilon_{\mu t \theta u} n^t F^{\theta u} + \epsilon_{\mu u r t} n^u F^{rt} + \epsilon_{\mu u \theta t} n^u F^{\theta t} && (5.64) \\ &= \left(0, 0, \frac{\text{sech}\theta}{\sqrt{h(r)}} \sqrt{-g} F^{ru}, 0\right) && (5.65) \end{aligned}$$

where the factor of $\frac{1}{2}$ drops out because of the combination of the antisymmetry of the Levi-Civita tensor and of the field strength tensor.

The magnetic field will be,

$$\boldsymbol{B}_\theta(r) = -\frac{\sigma}{2l^2}\frac{Q(r^2-1)-2Pr}{(r^2+1)^{\frac{3}{2}}} \tag{5.66}$$

Thus,[6] locally, on each one of the constant time leaves, the profile of the lines does not have a radial component, which leads the Komar integral to be zero on the boundary. The field lines are in the θ direction, which is a coordinate at each constant $r-$ slice. However, there is no θ dependence and there is only $r-$ dependence, meaning that their strength changes according to where we are in the manifold. As can be seen, the field lines don't blow up anywhere.

5.2.4 Physical Discussion

The boundary of the constant t hypersurface consists of three components of sending $r \to \pm\infty$ and of sending $\theta \to +\infty$. The contribution of sending $r \to \pm\infty$ is zero because of the orientation of the field lines in the θ direction. The contribution on the constant θ surface is non zero, but once we integrate over all r the result is zero.

Well, as stated in [1] about the Weyl idea of sourceless fields *"But there is nowhere that one can put his finger and say, "This is where some charge is located." Lines of force never end. This freedom from divergence by no means prevents changes in field strengths. Lines of forces which are not trapped into the topology can be continuously shrunk to extinction, as in familiar examples of electromagnetic induction and electromagnetic waves. However, lines of force which are trapped in wormholes cannot diminish in number. The flux out of the mouth of a wormhole cannot change with time, no matter how violent the disturbances in the electromagnetic field, no matter how roughly the metric changes, no matter how rapidly corresponding wormholes recede or approach, up to the moment when they actually coalesce and change the topology."*. In fact, this discussion in [1] is mainly about intra-universe wormholes and closed field lines. The connectedness or disconnectedness of the boundary is still unclear. Hence, in the first case scenario, the global solution as mentioned in [3] is indeed the implementation of Weyl's hypothesis, with the distrinction that the throat can also be supported on its own in the anti-de Sitter background.

On the other hand, if one integrates from $r \to 0^+$ to $r \to +\infty$ and from $r \to -\infty$ to $r \to 0^-$, the result is two opposite finite charges. The picture we have from this result is that the solution can be constructed by patching of two semiwormholes of opposite charges.[7] Roughly speaking, cutting the spacetime in half and considering only half of the wormhole solution, e.g. $r > 0$, then the other half ($r < 0$) acts like

[6]Note, that here the $\sqrt{-g}$ is there because $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\bar{\epsilon}_{\mu\nu\rho\sigma}$, where $\bar{\epsilon}_{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor with entries $(+1,-1,0)$ depending on the even and odd permutations or the repetition of the indices.

[7]The conformal mass calculation adds up to this semiwormhole picture, since the contributions

a static electromagnetic source at $r = 0$ for the "plus" semiwormhole. The mirror picture holds for the "minus" semiwormhole. Note that those two semiwormholes are not separate solutions. Namely, they serve as an interpretation and as an indication of the disconnectedness of the boundary. The latter, is a fact that has not been proven yet. In the possible case of disconnectedness, the Weyl idea still holds for inter-universe wormholes with open sourceless field lines ranging from $-\infty$ to $+\infty$.

The interpretation of semiwormholes was inspired by a recent paper by S. Andriolo, T.C. Huang, T. Noumi, H. Ooguri and G. Shiu[62]. In this paper they present the Giddings-Strominger wormhole, with axionic charge. The picture is not completely relevant to the AdS wormhole of [2], since the wormhole of [62] is Euclidean and assymptotically flat. The total charge is zero and the vizualization is through this story of two glued semiwormholes. [See figure 5.1]

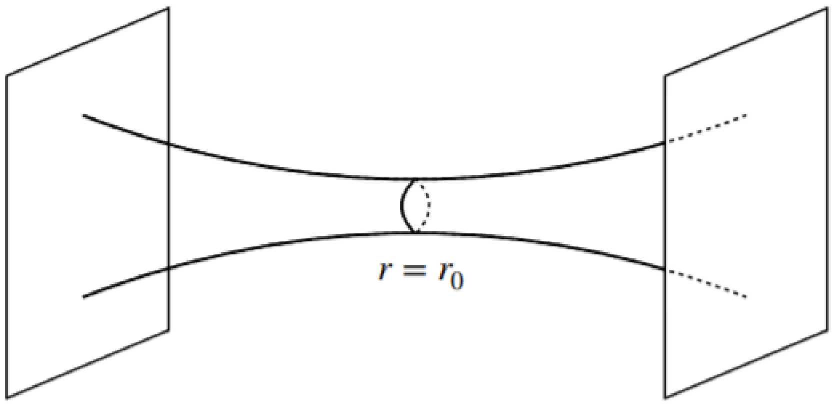

Figure 5.1: Euclidean Giddings-Strominger wormhole connecting two assymptotically flat regions, with zero total axionic charge, seen as two semiwormholes glued together, with opposite charges and the same action. The gluing is done at a 3$-$sphere of minimal size, corresponding to the throat. If one drops the word "axionic" and replaces the word "flat" by "locally AdS", then this is the result that we have computed. Figure taken by [62]

from the two boundaries are of opposite sign. Same reasoning applies to the calculation of the rest of the charges related to Killing isometries.

Conclusion

In this thesis, we attempted to better understand stable wormhole solutions in four dimensions and the possibility of them being supported by non-exotic matter. Using as our initial point the solutions to free Einstein-Maxwell theory with a negative cosmological constant, which were introduced in [2] and proven supersymmetric in [3], we examine closely some physical properties and observe some interesting implications. More specifically, we explicitly compute the null energy condition and we find that it is satisfied, which seems to contradict the Topological Censorship theorem for AlAdS spacetimes. Moreover, we found that all the associated conserved charges are zero. However, there is interesting topological structure in the spacetime, since if we consider only half of the solution some of the charges are non-vanishing. To be more precise, the total mass is zero, whilst considering only half of the spacetime reveals that the parameter m in the metric is related to it. On the other hand, the total angular momentum is calculated as zero, but this result indeed means that there is no angular momentum in the solution. Moreover, we compute the electric and magnetic charges, which are by definition zero because of the divergenceless of the field lines. However, computing them in the same manner as if there were monopoles in the spacetime, we extract valuable information. Our interpretation is that the way in which the integration ends up giving a zero result, encourages a picture of two semi-wormholes, which serves as an indication, yet not a proof, of the disconnectedness of the boundary. Last but not least, we find in the literature how these supersymmetric wormholes can be embedded in string theory. One way in which this can be done is via consistent Kaluza-Klein compactifications over seven dimensional Einstein-Sasaki manifolds[4].

These particular results unlock various questions that possibly deserve some consideration in the future. First of all, this apparent contradiction of the AlAdS wormhole solution in [2] to the Topological Censorship theorem can lead to some possible outcomes. Before going to those outcomes, one needs to verify that the solution obeys all of the conditions for the theorem. There is one condition under the name "the generic condition" in [33] which is left unexplored. Once it is confirmed that the theorem applies for this solution, the possibilities are the following: (i) for connected boundary, traversability is allowed, but there is always a shorter path through the boundary, (ii) for disconnected boundary, either something needs to violate the condition or the theorem should be revisited. An immediate question that arises is whether this contradiction to the Topological Censorship theorem is a general result of breaking the spherical symmetry and having non compact slices in a wormhole solution. Moreover, one can test the wormholeness of the wormhole by calculating the Raychaudhuri expansion parameter in order to predict the focusing of null geodesics. Another potential avenue is to study the solution in the context of AdS-CFT correspondence. One is required to employ holographic renormalization techniques and ask what are the boundary effects that cause the non-trivial topology in the bulk of the spacetime. It is surprising how such a simple model solution of a very well understood theory, leads to so many intriguing and unanswered contemporary physical questions.[8]

[8]

"-*Quand sait-on que c'est fini?*
-*A un moment, on s'arrête.*"
—Portrait de la jeune fille en feu

Conventions

- $\mu_0 = c = G = 1$
- The metric signature we are using is $(-, +, +, +)$
- We consider that a Euclidean signature is an independent picture with respect to a Lorentzian signature. Results in the literature that concern Euclidean signature wormholes do not apply in this text.
- Greek indices $\mu, \nu, \kappa, \lambda \ldots$ indicate spacetime coordinates, taking values $\{0, 1, 2, 3\}$ in a $4d$ spacetime.
- Latin indices $a, b, c, \ldots$ are being used for spatial coordinates in this text, taking values $\{1, 2, 3\}$.[9]
- Capital Latin indices $A, B, C, \ldots$ are being used for coordinates on two-surfaces that are submanifolds of spacelike hypersurfaces of the spacetime.
- Covariant vectors or tensors are those with indices up $V^\lambda, T^{\mu\nu}, \ldots$. The basis of the tangent space T_pM, of some manifold at some point p, in the coordinate system (t, r, θ, u) that we are using is given by $(\boldsymbol{\partial_t}, \boldsymbol{\partial_r}, \boldsymbol{\partial_\theta}, \boldsymbol{\partial_u})$
- Contravariant vectors or tensors are those with indices down $V_\mu, T_{\mu\nu}, \ldots$. The basis of T_p^*M at some point p, is given by the, so called, dual basis $(\boldsymbol{dt}, \boldsymbol{dr}, \boldsymbol{d\theta}, \boldsymbol{du})$
- In index free notation a vector is written as $\boldsymbol{n}$, in bold symbols.
- The covariant derivative acting on a vector based on this signature is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

- Accordingly,

$$\nabla^\mu V_\nu = \partial^\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$$

[9] Sometimes it is be more convenient to write the indices using the chosen coordinates, e.g. t, r, θ, u in this text. We consider these two conventions of writing the indices equivalent, and both are being used since each one of them serves a different purpose. Those are *Latin* indices but they refer to the coordinates.

- The generalization of this gives the action of the covariant derivative to a tensor of arbitrary rank (k,l) as follows

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots\end{aligned}$$

- The Levi-Civita tensor is

$$\epsilon_{\mu\nu\rho\sigma} = \epsilon_{abcd} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d$$

and it takes the values $\pm e = \pm\sqrt{-g}$

Dictionary

- The word space and spacetime both refer to a spacetime, unless stated otherwise.
- Vacuum solution is usually used in this text to refer to the wormhole solution without matter.
- Empty AdS refers to the empty from wormholes, maximally symmetric exact solution to Einstein's field equations with negative cosmological constant. This one is also a vacuum solution, but to avoid confusion, it is convenient to call it by the name empty AdS.
- The term "radial" is used for spherically symmetric spacetimes. Here there is axial symmetry, but the word radial means that small values of this coordinate brings us to the bulk of the spacetime, while plus and minus infinity corresponds to the boundary
- Global hyperbolicity in AdS means global hyperbolicity in the AdS sense, as explained in the introduction. Roughly speaking, including the boundary, in the conformally rescaled manifold, and imposing boudnary conditions yields the spacetime globally hyperbolic.
- The word boundary of AdS usually refers to the conformal boundary.
- Disconnected boundary means that there are two components of the boundary and there is no causal path that belongs entirely on the boundary and takes you from component of it to another. The itinerary should be through the bulk.

Appendix A

Hypersurfaces and the Stokes Theorem

The term hypersurface usually refers to a three dimensional submanifold in a four dimentional spacetime[63]. However, the concept can be generalized in any dimensions greater than 2, where it is called just a surface.

It is useful to know how to study the *intrinsic* geometry of the hypersurface, how to compute the induced metric on it and how to define a surface element so that one can integrate vector fields over it. A hypersurface can be characterized as timelike, spacelike or null depending on whether the normal vector on it is spacelike, timelike or null respectively. For the purpose of this thesis we will demonstrate how it is done for timelike and spacelike hypersurfaces, while for null ones it is slightly more complicated and it is not necessary for us.

Definition

A hypersurface Σ, can be chosen either by restricting the coordinates by an equation of the form

$$\Phi(x^\alpha) = 0$$

or by using parametric equations of the form

$$x^\alpha = x^\alpha \left(y^a\right)$$

where $y^a (a = 1, 2, 3)$ are the coordinates intrinsic to the hypersurface.

Normal vector

The vector $\Phi_{,\alpha}$ is normal to the hypersurface, since the value of this defining expression for the hypersurface Σ changes only along the direction of the normal vector to it. The unit normal is given by

$$n_\alpha = \frac{\epsilon \Phi_{,\alpha}}{|g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu}|^{1/2}}$$

where

$$\epsilon \equiv \begin{cases} -1 & \text{if} \quad \Sigma \quad \text{is spacelike} \\ +1 & \text{if} \quad \Sigma \quad \text{is timelike} \end{cases}$$

Induced metric

The metric with which one can measure distances on the hypersurface Σ can be computed by constraining the line element to displacements that happen only on the hypersurface. If the parametric equations of the hypersurface are $x^\alpha = x^\alpha(y^a)$, then the vectors $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ are tangent to Σ, while $e^\alpha_a n_\alpha = 0$ for a non-null hypersurface. This means that for distances on Σ

$$\begin{aligned} ds^2_\Sigma &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b} dy^b\right) \\ &= h_{ab} dy^a dy^b \end{aligned}$$

where $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ is the induced metric, or *first fundamental form*, on Σ. It is also referred to as a *three-tensor*, since it behaves as a scalar under spacetime coordinate transformations $x^\alpha \to x^{\alpha'}$, but as a tensor under transformations $y^a \to y^{a'}$ of the hypersurface coordinates.

Integration on hypersurfaces

Directed hypersurface element

The invariant three dimensional volume element on a non-null hypersurface reads

$$d\Sigma = |h|^{1/2} d^3y$$

with $h \equiv \det[h_{ab}]$ and it is referred to as the surface element. The combination $n_\alpha d\Sigma$ is a *directed* surface element that points in the direction vertical to the hypersurface. The convention usially used is the following

$$d\Sigma_\alpha = \epsilon n_\alpha d\Sigma$$

where $\epsilon = 1$ for timelike and $\epsilon = -1$ for spacelike hypersurfaces, so that $n^\alpha d\Sigma_\alpha$ is positive for spacelike hypersurfaces and negative for timelike ones.

Directed two-surface element

One can define similar quantities for a two-dimensional surface S embedded in a three-dimensional spacelike hypersurface Σ.

- In the same sense that Σ is described by a constraint equation $\Phi(x^\alpha) = 0$, and/or by parametric equations $x^\alpha(y^a)$,the two-surface S as a submanifold of Σ will be described by a constraint equation $\psi(y^a) = 0$, and/or parametric relations $y^a(\theta^A)$, where θ^A are the coordinates on the two-surface.

- Analogously, $n_\alpha \propto \partial_\alpha \Phi$ was the unit normal vector pointing to the future, and so, $\sigma_a \propto \partial_a \psi$ will be the outgoing normal direction.
- Additionally, since the vectors $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ lie on the hypersurface Σ, those that lie on S will be $e^a_A = \frac{\partial y^\alpha}{\partial \theta^A}$.
- Lastly, the induced metric on Σ is $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ and for completeness $g^{\alpha\beta} = -n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b$, which means that the metric on the two-surface S will be $\gamma_{AB} = h_{ab} e^a_A e^b_B$ and $h^{ab} = \sigma^a \sigma^b + \gamma^{AB} e^a_A e^b_B$

Now, the parametric equations describing the coordinates of the spacetime in terms of the coordinates on Σ and the coordinates on Σ in terms of the coordinates on S can be combined as a composition of functions in order to give a relation $x^\alpha(\theta^A)$. The latter will be used to embed S in the spacetime. Hence, the tangent vectors to S, using the chain rule, translate

$$e^\alpha_A = \frac{\partial x^\alpha}{\partial \theta^A} = \frac{\partial x^\alpha}{\partial y^a} \frac{\partial y^a}{\partial \theta^A} = e^\alpha_a e^a_A$$

while the normal to S is

$$\sigma^\alpha \equiv \sigma^a e^\alpha_a, \qquad \sigma^\alpha n_\alpha = 0$$

Thus, S admits two normal vectors that are also normal to each other, one timelike n^α and one spacelike σ^α. The spacelike one will be connected to a gradient, $\sigma_\alpha \propto \partial_\alpha \Psi$, with $\Psi(x^\alpha)$ defined as $\Psi|_\Sigma = \psi$. The induced metric is

$$\gamma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$$

and

$$g^{\alpha\beta} = -n^\alpha n^\beta + \sigma^\alpha \sigma^\beta + \gamma^{AB} e^\alpha_A e^\beta_B$$

With all these definitions for the embedding of the 2d surface in the 4d specatime, one can now define the directed surface element

$$dS_{\alpha\beta} = -2 n_{[\alpha} \sigma_{\beta]} \sqrt{\gamma} d^2\theta$$

where $\gamma = \det[\gamma_{AB}]$ and the square brackets denote the antisymmetrization of the indices.

Gauss-Stokes theorem

For any vector field V^μ defined within any finite region $\mathcal{A}$ of the spacetime manifold, bounded by a closed hypersurface $\partial\mathcal{A}$ it is true that

$$\int_{\mathcal{A}} \nabla_\mu V^\mu \sqrt{-g} d^4x = \oint_{\partial\mathcal{A}} V^\mu d\Sigma_\mu \tag{67}$$

Here, $d\Sigma_\mu$ is the surface element that has been defined above. This is known as Gauss's theorem.

For any 3d region Σ, bounded by a closed two surface $\partial\Sigma$, for any *antisymmetric* tensor field $F^{\mu\nu}$ within Σ, it is true that

$$\int_\Sigma \nabla_\nu F^{\mu\nu} d\Sigma_\mu = \frac{1}{2}\oint_{\partial\Sigma} F^{\mu\nu} dS_{\mu\nu} \tag{68}$$

with $dS_{\mu\nu}$ being the two-surface element defined above. This is known as Stokes theorem.

Appendix B

Fibre Bundle

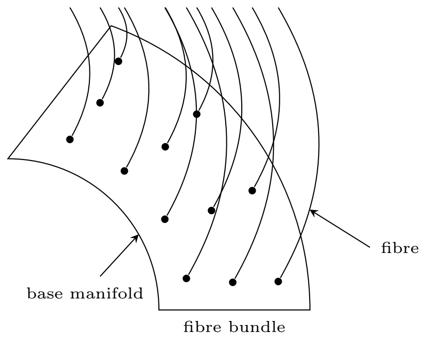

Figure 2: A picture representing a fibre bundle. As stated in the text and analyzed in [25] AdS_3 can be written as a fibre bundle with base manifold an AdS_2 and spacelike fibres, R, attached at each point. Then the squashing/stretching is being done along the spacelike fibres to make AdS_3 anisotropic. This stretched AdS_3 is the boundary of the assymptotically locally AdS_4 wormhole.

Appendix C

The Warped AdS Wormhole Solution

We are using the metric of the form

$$ds^2 = \frac{4l^4}{\sigma^2 f(r)} dr^2 + h(r)\left(-\cosh^2\theta dt^2 + d\theta^2\right) + f(r)\left(du + \sinh\theta dt\right)^2 \tag{69}$$

with $\{t, r\} \in \mathrm{R}$, $\theta \in [\, 0, +\infty \,)$, the coordinate u is identified, $u + \alpha = u$

The non-zero metric components are the following.

$$g_{tt} = -h(r)\cosh^2\theta + f(r)\sinh^2\theta, \quad g_{tu} = g_{ut} = f(r)\sinh(\theta),$$
$$g_{uu} = f(r), \quad g_{rr} = \frac{4l^4}{\sigma^2 f(r)}, \quad g_{\theta\theta} = h(r)$$

The non-zero inverse metric components are the following:

$$g^{tt} = -\frac{\mathrm{sech}^2\theta}{h(r)}, \quad g^{tu} = g^{ut} = \frac{\mathrm{sech}\theta\tanh\theta}{h(r)},$$
$$g^{uu} = \frac{h(r) - f(r)\tanh^2\theta}{f(r)h(r)}, \quad g^{rr} = \frac{\sigma^2 f(r)}{4l^4}, \quad g^{\theta\theta} = \frac{1}{h(r)}$$

with

$$f(r) = \frac{4l^2}{\sigma^2}\frac{r^4 + (6-\sigma)r^2 + mr + \sigma - 3}{r^2+1} - \frac{Q^2+P^2}{r^2+1} \quad , \quad h(r) = \frac{l^2}{\sigma}(r^2+1)$$

The matter section of the spacetime is characterized by an electromagnetic field that produces that particular $\frac{Q^2+P^2}{r^2+1}$ imprint in the metric

$$A = \Phi(r)\left(du + \sinh\theta dt\right), \qquad \Phi(r) = \frac{-2Qr + P(1-r^2)}{r^2+1} \tag{70}$$

The non-zero components of the corresponding field strength tensor $F_{\mu\nu}$ are

$$F_{ru} = -F_{ur} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2} \tag{71}$$

$$F_{rt} = -F_{tr} = \frac{2(r^2-1)Q - 4rP}{(r^2+1)^2}\sinh\theta = F_{ru}\sinh\theta \tag{72}$$

$$F_{\theta t} = -F_{t\theta} = \frac{-2Qr + P(1-r^2)}{r^2+1}\cosh\theta \tag{73}$$

There are some bounds on the set of parameters $\left\{\sigma, m, X \equiv \frac{3(Q^2+P^2)}{l^2}\right\}$ such that the spacetimes that this metric describes are well defined in terms of singularities and energy conditions. The bounds we found are matched with the bounds of [3] and are equivalent.

The Ricci scalar of this spacetime is the same as empty AdS_4

$$R = -\frac{12}{l^2} \tag{74}$$

From the Ricci scalar one can not extract enough information about the curvature. One has to look at the higher curvature invariants. The differences between the empty AdS solution and the wormhole solution will be made visible by the following invariants.

An important invariant of this spacetime is the Kretsmann invariant

$$\begin{aligned} K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} =& \frac{1}{2l^8(1+r^2)^6}[-24l^3m(P^2+Q^2)\sigma^2 r(5-10r^2+r^4) \\ &+ 192l^5m(-4+\sigma)r(3-10r^2+3r^4) \\ &+ (P^2+Q^2)^2\sigma^4(7-34r^2+7r^4) \\ &+ 24l^6m^2(-1+15r^2-15r^4+r^6) \\ &- 48l^4(-33+16\sigma-2\sigma^2) \\ &+ 6(79+5(-8+\sigma)\sigma)r^2 - 15(33+2(-8+\sigma)\sigma)r^4 \\ &+ 2(6+(-8+\sigma)\sigma)r^6 - 15r^8 - 6r^{10} - r^{12}) \\ &- 48l^2(P^2+Q^2)(-4+\sigma)\sigma^2(1+5r^2(-2+r^2))] \end{aligned} \tag{75}$$

The contraction of the Weyl tensor with itself is

$$\begin{aligned} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} =& \frac{3}{l^8(1+r^2)^6}[-4l^3m(P^2+Q^2)\sigma^2 r(5-10r^2+r^4) \\ &+ (P^2+Q^2)^2\sigma^4(1-6r^2+r^4) \\ &+ 32l^5m(-4+\sigma)r(3-10r^2+3r^4) \\ &+ 4l^6m^2(-1+15r^2-15r^4+r^6) \\ &- 16l^4(-4+\sigma)^2(-1+15r^2-15r^4+r^6) \\ &- 8l^2(P^2+Q^2)(-4+\sigma)\sigma^2(1+5r^2(-2+r^2))] \end{aligned} \tag{76}$$

As can be clearly seen, neither of those two invariants diverge. The Kretschmann invariant is constant at infinity, while the Weyl invariant vanishes at infinity, as expected due to the boundary being a warped empty from wormholes AdS_3. The invariant from the electromagnetic field strength is

$$F_{\mu\nu}F^{\mu\nu} = \frac{2\sigma^2\left((Q^2-P^2)(1-6r^2+r^4)) - 8PQr(-1+r^2)\right)}{l^4(1+r^2)^4} \tag{77}$$

The action together with the Gibbons-Hawking-York boundary term will be

$$\begin{aligned} S = S_{bulk} + S_{GHY} = &\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R - 2\Lambda + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ &- \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Theta \end{aligned} \tag{78}$$

where Θ is the trace of the extrinsic curvature on the boundary, h is the determinant of the induced metric on the boundary.

$$\begin{aligned} \Theta &= \frac{(\sigma(h(r)f'(r) + 2f(r)h'(r))}{2l^2\sqrt{f(r)}h(r)} \\ &= \frac{2l^3(m + 3mr^2) + 4l^2r(3 + 3r^4 - 2r^2(-7+\sigma)) - (P^2+Q^2)r\sigma^2}{(l^2(1+r^2)\sqrt{(4l^2(r^4 - r^2(-6+\sigma) + lmr + \sigma - 3) - (P^2+Q^2)\sigma^2)(1+r^2))}} \end{aligned}$$

The square root of the determinant of the induced metric at the boundary is

$$\sqrt{-h} = \frac{2l^3}{\sigma^2}\cosh\theta \tag{79}$$

The Killing vectors will not be ten anymore, since this solution is no longer the maximally symmetric one. However, some of the Killing isometries are maintained. The four Killing vectors are the following

$$\begin{aligned} \boldsymbol{\xi}_1 &= \boldsymbol{\partial}_t \\ \boldsymbol{\xi}_2 &= \sin t\boldsymbol{\partial}_\theta + \tanh\theta\cos t\boldsymbol{\partial}_t + \frac{\cos t}{\cosh\theta}\boldsymbol{\partial}_u \\ \boldsymbol{\xi}_3 &= \cos t\boldsymbol{\partial}_\theta - \tanh\theta\sin t\boldsymbol{\partial}_t - \frac{\sin t}{\cosh\theta}\boldsymbol{\partial}_u \\ \boldsymbol{\xi}_4 &= \boldsymbol{\partial}_u \end{aligned} \tag{80}$$

The first three generate the group $SO(2,1)$, reflecting the isometries preserved by the warped AdS_3 boundary, whereas the fourth one is abelian and commutes with the rest, reflecting the axial symmetry around the $\theta = 0$ axis.

Appendix D

The 3+1- formalism in General Relativity

If (M, g) is a 4 dimensional spacetime manifold M equipped with a smooth metric g, of signature $(-,+,+,+)$, and it is **time-orientable** and **hyperbolic**[10], then there exists a globally defined scalar field $\mathbf{t}$, that defines a foliation $M \simeq R \times \Sigma$ such that the leaves Σ_t of constant t are:

$$\Sigma_t = p \in \mathcal{M} : \mathbf{t}(p) = t = constant \tag{81}$$

This is also known as the $3+1-$ formalism in General Relativity[61] and it is different from the $1+3-$ formalism, which refers to congruences of one dimensional curves, rather than congruences of 3 dimensional hypersurfaces[64]. On such a Manifold M we can define two vector fields and one scalar function to describe evolution. Those are the following

$$\mathbf{n} := -N\nabla\mathbf{t}, \qquad \mathbf{m} := N\mathbf{n}, \qquad N := [-g_{\mu\nu}\nabla^\mu\mathbf{t}\nabla^\nu\mathbf{t}]^{-\frac{1}{2}} \tag{82}$$

where $\nabla\mathbf{t}$ must be timelike throughout the whole M such that we can evolve the spacelike submanifold of M. The point p evolves in the direction of the vector field ∂_t, defined by

$$\partial_t := \mathbf{m} + \boldsymbol{\beta}, \qquad g_{\mu\nu}\beta^\mu n^\nu = 0 \tag{83}$$

∂_t is the derivative along the adapted time and $\beta \in \mathcal{T}_p(\mathcal{M})$ is the displacement vector of the origin of the coordinates between two infinitesimally close leaves.

Local physical measurements $p \in \Sigma_t$ on the spacelike hypersurface of constant t. We can select coordinates $x^\mu = (t, x^i)$ where $i = 1, 2, 3$ refer to the coordinates on the spacelike hypersurface.

Any geometrical object on such manifolds, respects its topological structure and it is possible to be decomposed following the foliation mentioned above. Thus, for example, the induced metric on the spacelike hypersurface can take the form,

$$\gamma_{ij} = g_{ij} \tag{84}$$

which is just a canonical reduction.

[10]Even though AdS is not globally hyperbolic, it can be made as such, by imposing appropriate boundary conditions at all times. Thus, there exists such a foliation and we can still work with it[34].

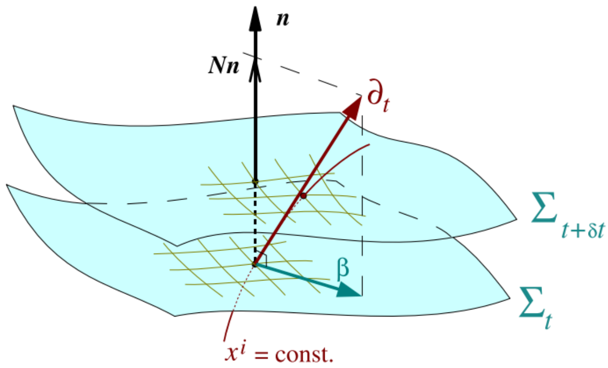

Figure 3: Coordinates on the spacelike hypersurface Σ_t adapted to the foliation. Each line $x_i = \text{const}$ defines the timelike vector $\boldsymbol{\partial}_t$ and the shift vector $\boldsymbol{\beta}$ of the spacetime coordinates $x^\mu = (t, x^i)$. Figure taken from [64]

A more general way of interpreting this is by acting with a projector on the tensor g.

$$P^\mu_\nu := g^\mu_\nu + n^\mu n_\nu, \qquad P_{\mu i} n^\mu = 0 \tag{85}$$

Due to the fact that Σ_t can be seen as a hypersurface embedded in a manifold M one dimension higher, its extrinsic curvature, or *second fundamental form* on the hypersurface, can be defined as follows

$$K_{ij} := P^\mu_i P^\nu_j \nabla_\mu n_\nu, \qquad K := \gamma^{ij} K_{ij} \tag{86}$$

where ∇_μ is the four dimensional covariant derivative. Here, metric compatibility is assumed in both the metric on M and the induced metric on Σ_t. Following that, the line element can be decomposed as follows

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \tag{87}$$

This method can now be applied in other objects on M, in order to project them on Σ_t, such as the electromagnetic field, for example. In this way, one can decompose it with respect to the foliation. Let us identify those quantities for a 4d metric

$$g_{ij} = \gamma_{ij}, \qquad g_{00} = -N^2 + \beta_k \beta^k, \qquad g_{0i} = g_{i0} = \beta_i \tag{88}$$

and the inverse metric will be decomposed as

$$g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{N^2}, \qquad g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{\beta^i}{N^2} \tag{89}$$

The displacement vector will be $\boldsymbol{\beta} = \frac{g^{0i}}{N^2}\boldsymbol{\partial_i}$ and it lives on the hypersurface Σ_t. The normal vector $\boldsymbol{n}$ can be found by demanding that the inner product

$$\boldsymbol{n} \cdot \boldsymbol{\beta} = g_{\mu\nu} \beta^\mu n^\nu = 0 \tag{90}$$

This decomposition is extensively used in Numerical Relativity and in ADM formalism.

Appendix E

Proving that the constraints (3.9) of [3] are the same as those in (2.12)

We need to prove that this set of constraints

$$\begin{aligned}
&0 \le X < 1 \\
&\frac{24+12\sqrt{1-X}}{3+X} \ge \sigma > \frac{6-6\sqrt{1-X}}{X} \\
&0 \le |m| < \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{18\left(-48+24\sigma-(3+X)\sigma^2\right)+\sigma^3\left(1+3X+(1-X)\sqrt{1-X}\right)}
\end{aligned} \tag{91}$$

is the same as this set of constraints

$$\begin{aligned}
&0 \le X < 1 \\
&\frac{12+12\sqrt{1-X}}{1+X+\sqrt{1-X}} \ge \sigma > \frac{12-6\sqrt{1-X}}{1+X+\sqrt{1-X}} \\
&0 \le |m| < \frac{\sqrt{2}}{3\sqrt{3}}\frac{\sigma(6-\sigma)\sqrt{1-X}+24\sigma-\sigma^2\left(1+X\right)-72}{\sqrt{\sigma\left(1+\sqrt{1-X}\right)-6}}
\end{aligned} \tag{92}$$

We will use the fact that

$$\frac{1}{1+X+\sqrt{1-X}} = \frac{1+X-\sqrt{1-X}}{X(X+3)} \tag{93}$$

Let us first prove that the bounds on σ are the same:

$$\begin{aligned}
\frac{12-6\sqrt{1-X}}{1+X+\sqrt{1-X}} &= \frac{(1+X-\sqrt{1-X})(12-6\sqrt{1-X})}{X(X+3)} \\
&= \frac{18+6X-18\sqrt{1-X}-6X\sqrt{1-X}}{X(X+3)} \\
&= \frac{6(3+X)-6(3+X)\sqrt{1-X}}{X(X+3)} \\
&= \frac{6-6\sqrt{1-X}}{X}
\end{aligned} \tag{94}$$

and

$$\begin{aligned}\frac{12+12\sqrt{1-X}}{1+X+\sqrt{1-X}} &= \frac{(1+X-\sqrt{1-X})(12+12\sqrt{1-X})}{X(X+3)}\\ &= \frac{24X+12X\sqrt{1-X}}{X(X+3)}\\ &= \frac{24+12\sqrt{1-X}}{(X+3)}\end{aligned} \tag{95}$$

Those expressions are indeed equivalent. For the bound on $|m|$, since the quantities are positive for positive X, we can take the squares to prove that they are equal.

$$\begin{aligned}&\left(\frac{\sigma(6-\sigma)\sqrt{1-X}+24\sigma-\sigma^2\left(1+X\right)-72}{\sqrt{\sigma\left(1+\sqrt{1-X}\right)-6}}\right)^2\\ &= \frac{\sigma^2(6-\sigma)^2(1-X)+24^2\sigma^2+\sigma^4\left(1+X\right)^2+72^2}{\sigma\left(1+\sqrt{1-X}\right)-6}\\ &+ \frac{48\sigma^2(6-\sigma)\sqrt{1-X}+144\sigma^2\left(1+X\right)-48\sigma^3(1+X)}{\sigma\left(1+\sqrt{1-X}\right)-6}\\ &+ \frac{-2\sigma^3(6-\sigma)(1+X)\sqrt{1-X}-2\times 24\times 72\sigma-144\sigma(6-\sigma)\sqrt{1-X}}{\sigma\left(1+\sqrt{1-X}\right)-6}\end{aligned} \tag{96}$$

The numerator of (96) is

$$\begin{aligned}&\sigma^2(6-\sigma)^2(1-X)+24^2\sigma^2+\sigma^4\left(1+X\right)^2+72^2\\ &+48\sigma^2(6-\sigma)\sqrt{1-X}+144\sigma^2\left(1+X\right)-48\sigma^3(1+X)\\ &-2\sigma^3(6-\sigma)(1+X)\sqrt{1-X}-24\times 144\sigma-144\sigma^2(6-\sigma)\sqrt{1-X}\\ &=\sigma^4\left(1+2X+X^2+2(1+X)\sqrt{1-X}+(1-X)\right)\\ &+\sigma^3\left(-2(1-X)-48(1+X)-48\sqrt{1-X}-12(1+X)\sqrt{1-X}\right)\\ &+\sigma^2\left(6(1-X)+24^2+6\times 48\sqrt{1-X}+144(1+X)-6\times 144\sqrt{1-X}\right)\\ &+\sigma\left(-24\times 144-6\times 144\sqrt{1-X}\right)\\ &+72^2\end{aligned} \tag{97}$$

Now it is left to prove that this expression is the same as the numerator of (96)

$$
\begin{aligned}
&\left(\sigma\left(1+\sqrt{1-X}\right)-6\right)\left(\sigma^3\left(1+3X+(1-X)\sqrt{1-X}\right)+18\left(24\sigma-(3+X)\sigma^2-48\right)\right)\\
&=\sigma\left(1+\sqrt{1-X}\right)\sigma^3\left(1+3X+(1-X)\sqrt{1-X}\right)\\
&+\sigma\left(1+\sqrt{1-X}\right)18\left(24\sigma-(3+X)\sigma^2-48\right)\\
&-6\sigma^3\left(1+3X+(1-X)\sqrt{1-X}\right)-6\times 18\left(24\sigma-(3+X)\sigma^2-48\right)\\
&=\sigma^4\left(2+2\sqrt{1-X}+X+2X\sqrt{1-X}+X^2\right)\\
&+\sigma^3\left(-18(3+X)-18(3+X)\sqrt{1-X}-6\left(1+3X+(1-X)\sqrt{1-X}\right)\right)\\
&+\sigma^2\left(18\times 24+18\times 24\sqrt{1-X}-6\times 18(3+X)\right)\\
&\sigma\left(-18\times 48-18\times 48\sqrt{1-X}-6\times 18\times 24\right)\\
&+6\times 18\times 48
\end{aligned}
\tag{98}
$$

The expressions in (97) and (98) are indeed the same.

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