MAStER's THESIS

# Algebraic structure of the bosonic string with Newtonian background 

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#### Abstract

I derive the symmetry group of the bosonic string with a non-relativistic background which takes the form of a covariant Newton-Cartan geometry. This is done by studying a Polyakov-type action obtained by performing a dimensional reduction of a 1-dimension higher relativistic background along a null Killing vector field. The constraints that arise from this construction then act as additional fields of the theory which have to be taken into account when deriving the symmetries of the non-relativistic bosonic string action. I confirm that the resulting symmetries form a closed group with the resulting Bargmann algebra- the central extension of the Galilean algebra, which correctly describes nonrelativistic motion of objects.


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## 1 Introduction

The first couple of decades of the past century, with the advent of General Relativity (GR) and Quantum Mechanics, was probably the most remarkable period of time for Physics. It did not take long after Paul Dirac formulated the quantum properties of the electromagnetic field for others to realize that gravity, the other known force at the time, should also be quantised. Just a few years after, in the 1930s, Leon Rosenfeld published the first paper on quantum gravity. He immediately recognised that there were constraints between the canonical momenta in GR and their relation to diffeomorphism invariance. This was the first of the many problems that arise when trying to define a consistent theory of quantum gravity, next it was Heisenberg in 1939 that pointed out that the gravitational constant was dimensional and that would pose further issues when quantising the theory. Since then research in quantum gravity has been split into three main directions: "covariant", "canonical" and using the Feynman functional integral. Each of these lines of thought have resulted in well defined theories: String Theory, Loop Quantum Gravity and Euclidean Quantum Gravity respectively. The former has been the most consistent of the three and has turned out to be an extremely rich theory, especially when considering the AdS/CFT correspondence. One of the many applications of the correspondence is to take certain regions of relativistic String Theory as a limit to a non-relativistic region of it.

The theory of non-relativistic gravity that has Galilean symmetry was developed by Ellie Cartan [1] just a few years after Einstein first published his papers on GR. He realised that the profound connection between the geometric structure of space-time and gravity is not unique to special relativity and Lorentz invariance. He then developed a covariant framework that accurately describes Newtonian Physics in the language of differential geometry. Subsequent work on the subject [2,3] has proven that the symmetry structure of this theory is not the Galilean algebra, but its central extension: the Bargmann algebra. This algebra consists of the Galilean boosts, translations, rotations and in addition the $\mathrm{U}(1)$ central extension that is physically related to the mass conservation of the theory. Just as it can be shown that GR can be formulated as a gauge theory of the Poincare algebra, Newton-Cartan gravity can be obtained by gauging the Bargman algebra. Finally, this framework can be used in conjunction with String Theory by embedding the strings in this non-relativistic space instead of the usual pseudo-Riemannian one.

Here I present my work on the algebraic structure of the bosonic string embedded in a NC space-time background. I work on the properties of of the worldsheet theory consider whether the NC space-time symmetries can be realised on the worldsheet in terms of a
current algebra. This is done by calculating the conserved currents and their charges in turn obtained by transforming a non-relativistic Polyakov-type action. This action is obtained via a dimensional reduction of the relativistic theory along a null Killing field. Then, by means of the Sugawara construction I calculate the operator product expansions between said currents.

## 2 General relativity and the Poincare symmetry

### 2.1 Special Relativity

After the discovery of the Maxwell equations it became clear that Galilean transformations were not universal since those equations were not invariant under Galilean boosts. Soon after Lorentz discovered the correct transformations under which the equations of electromagnetism were invariant, but no one at the time knew what they meant. That is, until Einstein came up with his famous postulate, stating that the speed of light is constant in every inertial frame. In addition if one also considers the principle of relativity, that the laws of physics must remains the same in every inertial frame, one can straightforwardly derive the Lorentz transformations without any other assumptions:

$$
\begin{align*}
x^{\prime} & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}(x-v t) \\
y^{\prime} & =y \\
z^{\prime} & =z  \tag{2.1}\\
c t^{\prime} & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}\left(c t-\frac{v}{c} x\right)
\end{align*}
$$

Observing these transformations one can see that for velocities $v>c$ the transformations become imaginary and the discussion of inertial inertial frames falls apart, thus making the theory inconsistent in that region. One can therefore conclude that the speed of light $c$ is the absolute maximum speed any object or a piece of information can travel at.

A better and much more insightful way to look at these transformations is in the context of differential geometry. From eq. (2.1) one can already see that the spatial and temporal directions transform into each other. And that was another of Einsteins profound conclusions: that time and space can be treated equally. This then lead Minkowski to state that time and space can be considered as dimensions of a four dimensional vector space $\mathbf{R}^{4}$ with an indefinite inner product:

$$
\begin{equation*}
(x, y) \equiv x \cdot y=-x_{0} y_{y}+x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3} \tag{2.2}
\end{equation*}
$$

Which can also be expressed in matrix form using the summation convention:

$$
\begin{equation*}
x \cdot y=\eta_{\mu \nu} x^{\mu} x^{\nu} \tag{2.3}
\end{equation*}
$$

where we define the Minkowski metric:

$$
\eta_{\mu \nu}=\left(\begin{array}{cccc}
-1 & 0 & 0 & 0  \tag{2.4}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

The space-time interval on the Minkowski manifold is then given by:

$$
\begin{equation*}
d s^{2}=\eta_{\mu \nu} d x^{\mu} d x^{\nu} \tag{2.5}
\end{equation*}
$$

This line element is invariant under Lorentz transformations in addition to space-time translations. Together they form the full group of transformations of Special Relativity: the Poincare group. One can also express the transformation laws in matrix form:

$$
\begin{equation*}
x^{\prime \mu}=\lambda^{\mu}{ }_{\nu} x^{\nu}+\zeta^{\mu} \tag{2.6}
\end{equation*}
$$

where we denote Lorentz transformations by $\lambda^{\mu}{ }_{\nu} \in S O(3,1)$ and space-time translations by $\zeta^{\mu} \in \mathbf{R}^{4}$. Then for the transformation of the metric we have:

$$
\begin{equation*}
\lambda^{\mu}{ }_{\alpha} \lambda^{\nu}{ }_{\beta} \eta_{\mu \nu}=\eta_{\alpha \beta} \tag{2.7}
\end{equation*}
$$

from which we conclude that the metric is invariant under Lorentz transformations and indeed the Lorentz group can be thought as the one that leaves the metric invariant.

### 2.2 General Relativity

General relativity is the theory of gravity consistent with Special Relativity which makes use of the geometrical approach described in the previous section. This new approach is needed as Newton's theory of gravity is inconsistent with the speed limit of propagation of any kind of information. If one considers just the Poisson equation in order to describe how the gravitational field behaves, they will end up with instantaneous changes in the field when the source is perturbed.

### 2.2.1 The geometric structure of General Relativity

The single pillar on which GR was built on, which happens to be "Einstein's happiest thought" is the principle of equivalence, which states that locally observers free falling in a gravitational field experience the same laws of Physics as those undergoing uniform acceleration. Or in
simpler words: inertial mass and the gravitational "charge" of objects are equal. This is a profound consequence of the structure of space-time i.e. that it is a pseudo-Riemannian manifold and that gravity itself is a manifestation of the curvature of said manifold. The properties of this space-time manifold are that it is everywhere differentiable and its metric is non-degenerate.

In order to introduce curvature to the manifold one has to let the metric be a function of the space-time coordinates:

$$
\begin{equation*}
\eta_{\mu \nu} \rightarrow g_{\mu \nu}(x) \tag{2.8}
\end{equation*}
$$

On the other hand it is always possible to find an orthonormal basis $\left\{e_{\mu}\right\}$ of the tangent space $T_{p}(M)$ of the manifold $M$ so that locally the metric is flat. In conjunction with the condition for differentiability or smoothness of the manifold we have:

$$
\begin{equation*}
\left.g_{\mu \nu}\right|_{p}=\eta_{\mu \nu} \tag{2.9}
\end{equation*}
$$

The equivalence principle is also deeply rooted in this very important property of differentiable manifolds.


Choosing a basis in which the manifold is flat at given point $p$ does not necessarily mean that it will be flat everywhere. However, since we are always free to choose a basis and the laws of Physics must remain the same under such change, as it can be totally arbitrary, this must be also a symmetry of the theory: it is the symmetry of general coordinate transformations or diffeomorphisms.

The metric transforms under diffeomorphisms as:

$$
\begin{equation*}
g_{\mu \nu}^{\prime}\left(x^{\prime}\right)=\frac{d x^{\prime \alpha}}{d x^{\mu}} \frac{d x^{\prime \beta}}{d x^{\nu}} g_{\alpha \beta}(x) \tag{2.10}
\end{equation*}
$$

from which we can see that the line element (2.5) is left invariant. The derivative operator, however, is not invariant under diffeomorphisms, therefore a covariant derivative is needed,
much like in gauge theories. For this we introduce the connection on the manifold so that acting with the covariant derivative on a vector we have:

$$
\begin{equation*}
\nabla_{\mu} V^{\nu}=\partial_{\mu} V^{\nu}+\Gamma_{\mu \rho}^{\nu} V^{\rho} \tag{2.11}
\end{equation*}
$$

which can be generalised to tensors of any rank.
The covariant derivative is not only invariant under diffeomorphisms, but also provides us with a map from the tangent space $T_{p}(M)$ at a given point to the tangent space at any other point $T_{q}(M)$, or as a matter of fact, for any vector field at different points on the manifold. This in essence is what allows the connection to act as a derivative on the curved manifold. This map is called parallel transport and the condition for it can be expressed mathematically as the geodesic equation:

$$
\begin{equation*}
\frac{d^{2} x^{\mu}}{d \tau^{2}}+\Gamma_{\nu \rho}^{\mu} \frac{d x^{\nu}}{d \tau} \frac{d x^{\rho}}{d \tau}=0 \tag{2.12}
\end{equation*}
$$

for some affine parameter $\tau$. This equation governs the motion of objects on the manifold and is a generalization of a "straight line" on curved space-time.

In GR the connection is uniquely defined by requiring:

- Metric compatibility: $\nabla_{\mu} g_{\nu \rho}$
- Zero torsion: $\Gamma_{[\mu \nu]}^{\rho}$

From which we can express the connection in terms of the metric:

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}=\frac{1}{2} g^{\rho \sigma}\left(\partial_{\mu} g_{\sigma \nu}+\partial_{\nu} g_{\sigma \mu}-\partial_{\sigma} g_{\mu \nu}\right) \tag{2.13}
\end{equation*}
$$

Now, since we have fully constructed the covariant derivative we can determine the curvature or Riemann tensor from its commutator:

$$
\begin{equation*}
\left[\nabla_{\rho}, \nabla_{\sigma}\right] V^{\mu}=R_{\nu \rho \sigma}^{\mu} V^{\nu} \tag{2.14}
\end{equation*}
$$

Then, by taking its trace once and twice we get the Ricci tensor and scalar respectively:

$$
\begin{equation*}
R_{\mu \nu}=R_{\mu \rho \nu}^{\rho} \quad R=g^{\mu \nu} R_{\mu \nu} \tag{2.15}
\end{equation*}
$$

Using these one can construct the simplest possible action that is Lorentz invariant and also invariant under diffeomorphisms, called the Einstein-Hilbert action:

$$
\begin{equation*}
S=\int d x^{4} \sqrt{-g} R \tag{2.16}
\end{equation*}
$$

where $g$ is the determinant of the metric. The equations of motion obtained by varying this action under infinitesimal changes of the metric are the Einstein equations in vacuum:

$$
\begin{equation*}
R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=0 \tag{2.17}
\end{equation*}
$$

where $G$ is Newton's constant. To include matter one has to modify the action for some arbitrary matter fields:

$$
\begin{equation*}
S=\int d x^{4} \sqrt{-g}\left(R+\mathcal{L}_{M}\right) \tag{2.18}
\end{equation*}
$$

from which follows that the Einstein field equations with matter are:

$$
\begin{equation*}
R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=8 \pi G T_{\mu \nu} \tag{2.19}
\end{equation*}
$$

where $T_{\mu \nu}$ is the stress-energy tensor for the matter fields given by:

$$
\begin{equation*}
T_{\mu \nu}=-\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g} \mathcal{L}_{M}\right)}{\delta g^{\mu \nu}} \tag{2.20}
\end{equation*}
$$

This whole construction looks awfully similar to how gauge theories are constructed, so naturally one might ask themselves if GR can be derived as a classical gauge theory of its symmetry group, the Poincare group. The answer is yes and we explore how to do it in later chapters.

### 2.2.2 Vielbein formalism

Before discussing gravity as a gauge theory it is worthwhile to discuss briefly the vielbein formalism, which reformulates GR in terms of new fields, the vielbein, instead of the metric. This proves to be quite useful as explained below.

A natural basis for the tangent space $T_{p}(M)$ is given by partial derivatives with respect to the coordinates at the point $p, \hat{e}_{\mu}=\partial_{\mu}$. Then, similarly for the cotangent space $T_{p}^{*}(M)$ it is given by the gradient of the coordinate functions at the point $p, \hat{e}^{\mu}=d x^{\mu}$. This, however, is arbitrary and we are free to choose any basis. It is very convenient to choose a non-coordinate basis so that they are orthonormal and also that locally the Riemannian manifold is flat [4]:

$$
\begin{equation*}
g\left(\hat{e}_{a}, \hat{e}_{b}\right)=\eta_{a b} \tag{2.21}
\end{equation*}
$$

The vectors comprising this basis are called vielbein or, in four space-time dimensions, tetrads or vierbein. Just like how we cannot always choose a coordinate basis that covers the whole manifold, we can neither do it with the orthonormal basis, but we can always work in patches. Physically this is just defining a free-falling frame that does not exprience the effects of gravity.

The usefulness of this basis comes from the ability to express any vector or tensor by a linear combination of the vielbeins.

$$
\begin{equation*}
e_{\mu}{ }^{a} V^{\mu}=V^{a} \tag{2.22}
\end{equation*}
$$

The vielbeins are an $n \times n$ invertible matrix and we denote the inverse by switching the indices

$$
\begin{equation*}
e^{\mu}{ }_{a}=g^{\mu \nu} \eta_{a b} e_{\nu}{ }^{b} \tag{2.23}
\end{equation*}
$$

therefore the following identities hold:

$$
\begin{equation*}
e_{\nu}{ }^{a} e_{a}{ }^{\mu}=\delta_{\nu}^{\mu} \quad e_{\mu}{ }^{a} e_{b}{ }^{\mu}=\delta_{b}^{a} \tag{2.24}
\end{equation*}
$$

Then, equation (2.21) can be expressed in terms of the vielbeins:

$$
\begin{equation*}
g_{\mu \nu} e^{\nu}{ }_{a} e^{\mu}{ }_{a}=\eta_{a b} \tag{2.25}
\end{equation*}
$$

The vielbeines also transform under Lorentz transformations:

$$
\begin{equation*}
e^{\prime \mu}{ }_{a}=\Lambda_{a}{ }^{b} e^{\mu}{ }_{b} \tag{2.26}
\end{equation*}
$$

so that when substituting into equation (2.26) we recover the correct transformation for the metric (2.7)

Taking a covariant derivative in the non-coordinate basis also requires using a different connection: the spin connection. Its name comes from the fact that one can take covariant derivatives of spinors, which is not possible using the Levi-Civita connection. This is in fact where a lot of the usefulness of the vielbein formalism comes from, one can use it to describe spinor fields in General Relativity.

The covariant derivative in the non-coordinate basis then takes the form:

$$
\begin{equation*}
\nabla_{\mu} V_{b}^{a}=\partial_{\mu} V_{b}^{a}+\omega_{\mu}{ }^{a}{ }_{c} X_{b}^{c}-\omega_{\mu}{ }^{c}{ }_{b} X^{a}{ }_{c} \tag{2.27}
\end{equation*}
$$

A very useful and general result can be derived by performing covariant differentiation on a vector field in the coordinate and non-coordinate bases and equating them.

In the coordinate basis we have:

$$
\begin{equation*}
\nabla V=\left(d_{\mu} V^{\nu}+\Gamma_{\mu \rho}^{\nu} V^{\rho}\right) \hat{e}^{\mu} \hat{e}_{\nu} \tag{2.28}
\end{equation*}
$$

and in the non-coordinate basis:

$$
\begin{align*}
\nabla V & =\left(\partial_{\mu} V^{a}+\omega_{\mu}{ }^{a}{ }_{b} V^{b}\right) \hat{e}^{\mu} \hat{e}_{a}  \tag{2.29}\\
& =\left(\partial_{\mu}\left(e_{\nu}{ }^{a} V^{\nu}\right)+\omega_{\mu}{ }^{a}{ }_{b} e_{\rho}{ }^{b} V^{\rho}\right) \hat{e}^{\mu} e^{\sigma}{ }_{a} \hat{e}_{\sigma}  \tag{2.30}\\
& =\left(\partial_{\mu} V^{\nu}+e^{\nu}{ }_{a} \partial_{\mu} e_{\rho}{ }^{a} V^{\rho}+e^{\nu}{ }_{a} e_{\rho}{ }^{b} \omega_{\mu}{ }_{a}{ }_{b} V^{\rho}\right) \hat{e}^{\mu} \hat{e}_{\nu} \tag{2.31}
\end{align*}
$$

which finally, after comparing the two equations, gives:

$$
\begin{equation*}
\omega_{\mu}{ }^{a}{ }_{b}=e_{\nu}{ }^{a} e^{\rho}{ }_{b} \Gamma_{\mu \rho}^{\nu}-e^{\rho}{ }_{b} \partial_{\mu} e_{\rho}{ }^{a} \tag{2.32}
\end{equation*}
$$

This is a general result and is often called the vielbein postulate.

### 2.3 General Relativity as a gauge theory

The construction of a curvature tensor from the commutator of the covariant derivative (2.14) and then using a scalar constructed from it to write down an action (2.18) is exactly the same procedure that we do in Yang-Mills theory. The only underlying difference is that the underlying geometry is a fibre bundle instead of the geometry of space-time as in GR.

To derive GR as a gauge theory one starts with the Poincare algebra [5, 6], which is the algebra of the symmetry group of relativity:

$$
\begin{align*}
& {\left[P_{a}, P_{b}\right]=0} \\
& {\left[M_{a b}, P_{c}\right]=-2 \eta_{c[a} P_{b]}}  \tag{2.33}\\
& {\left[M_{a b}, M_{c d}\right]=4 \eta_{[a[c} M_{d] b]}}
\end{align*}
$$

where $P_{a}$ are the generators for space-time translations and $M_{a b}$ are the generators for Lorentz transformations.

We wish to associate a gauge field to each of these transformations: let $e_{\mu}{ }^{a}$ be the gauge field for translations with a space-time dependant parameter $\zeta^{a}(x)$ and $\omega_{\mu}{ }^{a b}$ with parameters $\lambda^{a b}$ for Lorentz transformations. With the following transformation properties for the fields [2]:

$$
\begin{align*}
\delta e_{\mu}{ }^{a} & =\partial_{\mu} \zeta^{a}-\omega_{\mu}{ }^{a b} \zeta^{b}+\lambda^{a b} e_{\mu}{ }^{b} \\
\delta \omega_{\mu}^{a b} & =\partial_{\mu} \lambda^{a b}+2 \lambda^{c[a} \omega_{\mu}{ }^{b] c} \tag{2.34}
\end{align*}
$$

With the curvature tensors:

$$
\begin{align*}
R_{\mu \nu}{ }^{a}(P) & =2\left(\partial_{[\mu} e_{\nu]}{ }^{a}-\omega_{[\mu}{ }^{a b} e_{\nu}{ }^{b}\right) \\
R_{\mu \nu}{ }^{a b}(M) & =2\left(\partial_{[\mu} \omega_{\nu]}{ }^{a b}-\omega_{[\mu}{ }^{c a} \omega_{\nu]}{ }^{b c}\right) \tag{2.35}
\end{align*}
$$

To make contact with gravity we want to express local space-time translations with general coordinate transformations, since this is the actual gauge symmetry of GR. In mathematical terms this means to say that we want to interpret the gauge field $e_{\mu}{ }^{a}$ as the vielbein. To do so, curvature constraints have to be imposed and one has to make use of the general identity for a gauge field:

$$
\begin{equation*}
0=\delta_{g c t}\left(\xi^{\nu}\right) A_{\mu}^{a}+\xi^{\nu} R_{\mu \nu}{ }^{a}-\sum_{\{c\}} \delta\left(\xi^{\nu} A_{n}{ }^{c}\right) A_{m}{ }^{a} \tag{2.36}
\end{equation*}
$$

where $A_{m}{ }^{a}$ is a gauge field, $R_{\mu \nu}$ its associated curvature tensor and the parameter for general coordinate transformations $\xi^{\nu}$ can be expressed as:

$$
\begin{equation*}
\xi^{\nu}=e_{a}^{\mu} \zeta^{a} \tag{2.37}
\end{equation*}
$$

The derivation of this identity is shown in appendix A.
The sum in (2.36) can then be split into the translations and Lorentz transformation terms, so that we get:

$$
\begin{equation*}
\delta_{P}\left(\zeta^{b}\right) e_{m}{ }^{a}=\delta_{g c t}\left(\xi^{\rho}\right) e_{m}{ }^{a}+\xi^{\rho} R_{\mu \rho}{ }^{a}(P)-\delta_{M}\left(\xi^{\rho} \omega_{\rho}{ }^{a b}\right) e_{m}{ }^{a} \tag{2.38}
\end{equation*}
$$

Which tells us that performing a translation actually corresponds to a diffeomorphism and a Lorentz transformation in addition to some curvature term. We can further constrain this result by constraining the curvature tensor for translations (2.35). We note that the
gauge field $e_{\mu}{ }^{a}$ only transforms under translations and not Lorentz transformations we are led to impose:

$$
\begin{equation*}
R_{\mu \nu}{ }^{a}(P)=0 \tag{2.39}
\end{equation*}
$$

Now we are able to solve (2.38) and express the gauge field $\omega_{\mu}^{a b}$ also in terms of the vielbein and its inverse:

$$
\begin{equation*}
\omega_{\mu}^{a b}=-2 e^{\rho[a} \partial_{[\mu} e_{\rho]}^{b]}+e_{\mu}^{c} e^{\rho a} e^{\sigma b} \partial_{[\rho]} e_{\sigma]}^{c} \tag{2.40}
\end{equation*}
$$

Then the Levi-Civita connection of GR can be introduced by making use of the vielbein postulate (2.32):

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}=e_{a} D_{\nu} e_{\mu}^{a} \tag{2.41}
\end{equation*}
$$

Finally to put this theory on-shell one has to impose Einstein's equations, retrieving General Relativity as a gauge theory.

### 2.4 Quantum gravity

General Relativity has proven to be one of the most consistent theories in Physics that we have had, it has been tested thoroughly for the past hundred years and its predictions have always agreed with experiment. But GR is a classical theory, what happens when we quantize it?

One way to do this it to expand the Einstein-Hilbert action (2.18) around a flat Minkowski metric and another is to just construct the Lagrangian for a massless spin-2 particle invariant under diffeomorphisms and Lorentz transformations [7]. Either way the leading kinetic terms in the Lagrangian are:

$$
\begin{equation*}
\mathcal{L}_{k i n}=\frac{1}{4} h_{\mu \nu} \partial^{2} h^{\mu \nu}-\frac{1}{2} h_{\mu \nu} \partial^{\mu} \partial_{\rho} h_{\frac{1}{2}}^{\nu \rho} h \partial_{\mu} \partial_{\nu} h^{\mu \nu}-\frac{1}{4} h \partial^{2} h \tag{2.42}
\end{equation*}
$$

where $h_{\mu \nu}$ is the fluctuation of the metric and $h$ its determinant.
The interacting terms are, however:

$$
\begin{equation*}
\mathcal{L}_{\text {int }} \sim \sqrt{G} \partial^{2} h^{3} \tag{2.43}
\end{equation*}
$$

which can be shown to be non-renormalizable.
The theory is still consistent though and predictions can still be made about the quantum corrections due to gravity for light objects. These, however, are so small and negligible that
one can not possibly hope to ever measure. For very massive objects like black holes the theory is non-perturbative as shown by (2.43) and a UV completion of it is needed. This comes to show us that a novel approach is needed and one such is String Theory.

## 3 String Theory

In String theory the point particles of standard Physics are generalised to one-dimensional objects: strings. Since these objects have a physical dimension they can be dynamic and have excitation modes. These excitation modes can be shown to correspond to the properties of not only the elementary particles of the Standard Model, but also of the graviton and some new exotic particles.

These strings turn out to exhibit a very special set of symmetries, including conformal symmetry, which opens up the possibility of using the mathematical tools of conformal field theories (CFTs) which is extremely powerful.

### 3.1 The bosonic string

### 3.1.1 The point particle

The action for a relativistic point particle in $d+1$ dimensions is:

$$
\begin{equation*}
S=-m \int \sqrt{-g_{\mu \nu} \frac{d x^{\mu}}{d \tau} \frac{d x^{\nu}}{d \tau}} d \tau \tag{3.1}
\end{equation*}
$$

where $\tau$ is some parameter that we use to label the position on the particle's path and the action itself can be interpreted as the length of the particle's path.

We note that the parameter $\tau$ is not unique and we are free to redefine it as we wish without changing the action. This is reparametrization invariance and is a gauge symmetry of the theory. This gauge symmetry essentially reduces the degrees of freedom of the action by one, which is what one expects since letting $\tau=x^{0}$ gives the correct number of degrees of freedom for a classical point particle.

We also note that this action is invariant under transformations of the Poincare group (2.6) as expected for a relativistic action.

Since the whole action is proportional to the mass of


Figure 1: The world-line of a point particle [8] the particle $m$ it is clear that for massless particles it needs to be modified. One way to do it is by introducing a new field $e$, essentially acting as a vielbein on the world-line for the one dimensional theory:

$$
\begin{equation*}
S=\frac{1}{2} \int\left(e^{-1} g_{\mu \nu} \frac{d x^{\mu}}{d \tau} \frac{d x^{\nu}}{d \tau}-e m^{2}\right) d \tau \tag{3.2}
\end{equation*}
$$

To see that this action is equivalent to (3.1) one can solve the equation of motions for $e$ :

$$
\begin{equation*}
e(\tau)=\frac{1}{m} \sqrt{-g_{\mu \nu} \frac{d x^{\mu}}{d \tau} \frac{d x^{\nu}}{d \tau}} \tag{3.3}
\end{equation*}
$$

Substituting this equation of motion into (3.2) we see that we retrieve (3.1).

### 3.1.2 The Nambu-Goto action

We wish to generalise the point particle action (3.1) to a one-dimensional object that is also embedded in a Riemannian space-time. We previously noted that the action of the point particle is just the length of its path. Analogously we can construct the action of a string by demanding it represent the area that a string sweeps while propagating trough space-time called the worldsheet.

To do this we note that the worldsheet is a curved manifold in itself, therefore we can express its metric as the pullback of the Riemannian metric of the background:

$$
\begin{equation*}
\gamma_{\alpha \beta}=\frac{\partial X^{\mu}}{\partial \sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial \sigma^{\beta}} g_{\mu \nu} \tag{3.4}
\end{equation*}
$$

The action of the string then can be expressed as the area of the worldsheet given by:


Figure 2: The worldsheet of a one-dimensional closed string [8]

$$
\begin{equation*}
S=-T \int d \sigma^{2} \sqrt{-\gamma} \tag{3.5}
\end{equation*}
$$

where $T$ is the tension of the string and $\gamma$ is the determinant of the worldsheet metric.
Writing it out explicitly we have:

$$
\gamma_{\alpha \beta}=\left(\begin{array}{cc}
\dot{X}^{2} & \dot{X} \cdot X^{\prime}  \tag{3.6}\\
\dot{X} \cdot X^{\prime} & X^{\prime 2}
\end{array}\right)
$$

with $\dot{X}=\partial X / \partial \sigma^{0}$ and $X^{\prime}=\partial X / \partial \sigma^{1}$.

Substituting (3.6) into (3.5) we get:

$$
\begin{equation*}
S=-T \int d \sigma^{2} \sqrt{\left(\dot{X} \cdot X^{\prime}\right)^{2}-\dot{X}^{2} \cdot X^{\prime 2}} \tag{3.7}
\end{equation*}
$$

which is called the Nambu-Goto action.

Just like the action for the point-particle, this action is invariant under both reparametrisations and Poincare transformations.

### 3.1.3 The Polyakov action

The Nambu-Goto action is quite inconvenient to work with because of the square root in it. For this reason, much like in the point particle case, a new field can be introduced without actually changing the physics of the string it describes.

This action is called the Polyakov action and is given by:

$$
\begin{equation*}
S=-T \int d \sigma^{2} \sqrt{-g}^{\alpha \beta} h_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \tag{3.8}
\end{equation*}
$$

where $g^{\alpha \beta}$ is the novel field, $g$ its determinant, $h_{\mu \nu}$ is the metric of the Riemannian background and now $X^{\mu}$ acts as the dynamic fields on the worldsheet.

Solving the equations of motion for the fields $X^{\mu}$ we have:

$$
\begin{equation*}
\partial_{\alpha}\left(\sqrt{-g} g^{\alpha \beta} \partial_{\beta} X^{\mu}\right)=0 \tag{3.9}
\end{equation*}
$$

and for $g^{\alpha \beta}$ :

$$
\begin{equation*}
g_{\alpha \beta}=2 f(\sigma) \gamma^{\alpha \beta} \tag{3.10}
\end{equation*}
$$

where the function $f(\sigma)$ is given by:

$$
\begin{equation*}
f(\sigma)^{-1}=g^{\alpha \beta} h_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \tag{3.11}
\end{equation*}
$$

Substituting (3.11) into the Polyakov action (3.8) we see that we retrieve the NambuGoto action (3.7). Substituting in the equations of motion (3.10) we see that it is also left unchanged, thus taking into consideration these two results we have proven that the two actions are equivalent.

We note that the worldsheet metric $\gamma^{\alpha \beta}$ and the fields $g^{\alpha \beta}$ only differ by the conformal factor $f$ which is a profound consequence of the theory being two-dimensional on the worldsheet. In other words in addition to being reparametrisation and Poincare invariant the
theory is invariant under rescalings that preserve the angles between lines, this is called Weyl invariance.

Using the invariance under reparametrisations and Weyl invariance we can completely fix the worldsheet metric. We can therefore set it to be flat and will consider so from now on.

### 3.2 Conformal field theory

A conformal field theory is a field theory which is invariant under a coordinate change that leave the metric invariant up to a scalar function.

Explicitly we have:

$$
\begin{equation*}
\sigma^{\alpha} \rightarrow \sigma^{\alpha}(\sigma) \tag{3.12}
\end{equation*}
$$

which results in:

$$
\begin{equation*}
g_{\alpha \beta}(\sigma) \rightarrow \Omega^{2}(\sigma) g_{\alpha \beta}(\sigma) \tag{3.13}
\end{equation*}
$$

The diffeomorphism and Weyl invariance of the bosonic string action taken together result in it being a theory with conformal symmetry.

### 3.2.1 Operator Product Expansion

The operator product expansion (OPE) is a mathematical construction that allows us to express the product of local operators at nearby points as a sum over operators at one of these points.

Explicitly this statement takes the form:

$$
\begin{equation*}
\left\langle\mathcal{O}_{i}(\sigma) \mathcal{O}_{J}\left(\sigma^{\prime}\right)\right\rangle=\sum_{k} C_{i j}^{k}\left(\sigma-\sigma^{\prime}\right)\left\langle\mathcal{O}_{i}\left(\sigma^{\prime}\right)\right\rangle \tag{3.14}
\end{equation*}
$$

where the coefficients $C_{i j}^{k}$ are c-number functions and the OPE will only depend on the operators $\mathcal{O}_{i}(\sigma), \mathcal{O}_{j}\left(\sigma^{\prime}\right)$ and their separation, and will be independent of the identity. Usually the time-ordered correlation function notation is omitted, but it must still continue to be understood as such.

OPEs are singular as $z \rightarrow w$ and it is these singular terms that are of interest since they hold all the useful information, such as transformations under symmetries. OPEs are the equivalent operation to a commutation relation in a CFT.

We can now make use the Ward identities to derive the propagator $\left\langle X^{\mu}(\sigma) X^{\nu}\left(\sigma^{\prime}\right)\right.$.

Starting from:

$$
\begin{align*}
0 & =\int \mathcal{D} X \frac{\delta}{\delta X(\sigma)}\left(e^{-S} X\left(\sigma^{\prime}\right)\right) \\
& =\int \mathcal{D} X e^{-S}\left(-T \partial^{2} X(\sigma) X\left(\sigma^{\prime}\right)+\delta\left(\sigma-\sigma^{\prime}\right)\right) \tag{3.15}
\end{align*}
$$

From which it follows that:

$$
\begin{align*}
\left\langle\partial^{2} X(\sigma) X\left(\sigma^{\prime}\right)\right\rangle & =-\frac{1}{T} \delta\left(\sigma-\sigma^{\prime}\right) \\
& =-\frac{1}{2 T} \partial^{2} \ln \left(\sigma-\sigma^{\prime}\right)^{2} \tag{3.16}
\end{align*}
$$

Then the propagator is given by:

$$
\begin{equation*}
X(\sigma) X\left(\sigma^{\prime}\right)=-\frac{1}{2 T} \ln \left(\sigma-\sigma^{\prime}\right)^{2} \tag{3.17}
\end{equation*}
$$

Which can also be split into a left-moving, holomorphic and an right-moving, antiholomorphic parts:

$$
\begin{equation*}
X(z) X(w)=-\frac{1}{2 T} \ln (z-w) \quad \bar{X}(\bar{z}) \bar{X}(\bar{w})=-\frac{1}{2 T} \ln (\bar{z}-\bar{w}) \tag{3.18}
\end{equation*}
$$

By using the OPE for the propagator is then possible to derive the OPEs between the conserved currents of the theory and any other field.

### 3.2.2 Radial quantisation

One way to quantise the bosonic string is through radial quantisation, which consists of defining a qauntum field theory on the plane. This process starts by going to complex coordinates:

$$
\begin{equation*}
w=\sigma^{0}+i \sigma^{1} \quad \bar{w}=\sigma^{0}+i \sigma^{1} \tag{3.19}
\end{equation*}
$$

and then mapping the infinitely long cylinder onto the complex plane $\mathbb{R} \times S^{1} \rightarrow \mathbb{C}$. This allows us to to split the components of the theory into holomoprhic and anti-holomorphic parts, therefore allowing us to use complex analysis to study the theory. This map is realised by the following change in coordinates:

$$
\begin{equation*}
z=e^{-i w} \quad \bar{z}=e^{-i \bar{w}} \tag{3.20}
\end{equation*}
$$



Figure 3: Map from the cylinder onto the complex plane [9]
Whereas the Hamiltonian generated time translations on the cylinder, on the plane they are generated by the dilatation operator $D=z \partial+\bar{z} \bar{\partial}$. Therefore the Hilbert space defined on the plane is made of circles with constant radius instead of constant time slices as on the cylinder. In two dimensional CFT this way of quantisation is allows for the use of short distance operator expansions.

### 3.2.3 The Virasoro generators

By varying the Polyakov action (3.8) with respect to the worldsheet metric we can find the holomorphic and anti-holomorphic parts of the stress-energy tensor:

$$
\begin{equation*}
T=-\frac{1}{2 \alpha^{\prime}} \partial X \partial X \quad \bar{T}=-\frac{1}{2 \alpha^{\prime}} \bar{\partial} X \bar{\partial} X \tag{3.21}
\end{equation*}
$$

where $\alpha^{\prime} \propto 1 / T$ (in this expression $T$ is the tension not to be confused with the holomorphic part of the stress-energy tensor).

The stress-energy tensor can then be expanded in a Laurent series:

$$
\begin{equation*}
T(z)=\sum_{m=-\infty}^{\infty} \frac{L_{m}}{z^{m+2}}, \quad \bar{T}(\bar{z})=\sum_{m=-\infty}^{\infty} \frac{\tilde{L}}{\bar{z}^{m+2}} \tag{3.22}
\end{equation*}
$$

Alternatively we can invert these in order to express $L_{m}$ in terms of $T$ :

$$
\begin{equation*}
L_{m}=\frac{1}{2 \pi i} \oint d z z^{m+1} T(z), \quad \tilde{L}_{m}=\frac{1}{2 \pi i} \oint d \bar{z} \bar{z}^{m+1} \bar{T}(\bar{z}) \tag{3.23}
\end{equation*}
$$

These are the conserved charges of the conformal transformations $z \rightarrow z+z^{m+1}$ and $\bar{z} \rightarrow \bar{z}+\bar{z}^{m+1}$ called the Virasoro generators.

Of partucular interest are:

- $L_{-1}$ and $\tilde{L}_{-1}$ that generate translations
- $L_{0}$ and $\tilde{L}_{0}$ that generate scalings and rotations
- the combination $L_{0}+\tilde{L}_{0}$ called the dilation operator, which is the analogue for the Hamiltonian on the plane


### 3.2.4 The Virasoro algebra

Now that we have the Virasoro charges we can compute their algebra, whose representations will classify the states on the string.

The commutator of the Virasoro generators is then:

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=\left(\oint \frac{d z}{2 \pi i} \oint \frac{d w}{2 \pi i}-\oint \frac{d w}{2 \pi i} \oint \frac{d z}{2 \pi i}\right) z^{m+1} w^{n+1} T(z) T(w) \tag{3.24}
\end{equation*}
$$

After performing the two contour integrals one obtains the Virasoro Algebra:

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\frac{c}{12} m\left(m^{2}-1\right) \delta_{m+n, 0} \tag{3.25}
\end{equation*}
$$

where $c$ is a central term that commutes with all of the generators and is associated with the dimension of the Riemannian background. For the bosonic string that is $d=26$.

With the Virasoro algebra one can then build up the quantum states of the theory [8, 9].

## 4 Newtonian gravity and Galilean symmetry

Before we embed the bosonic string into a non-relativistic baclground we first need to describe and understand what we mean by that. Since String Theory is a geometrical, covariant theory we need a non-relativistic geometrical background that has the properties of Newtonian gravity and is covariant. Such reformulation of non-relativistic gravity already exists and it was actually formulated just a few years after GR by Élie Cartan [1]. This theory is called Newton-Cartan gravity and is Galilean invariant.

### 4.1 Galilean relativity

A "non-relativistic" theory is one whose formulation is covariant under the Galilei group, which is the relativity group that governs "non-relativistic" theories. As a matter of fact the Galilean group is equally relativistic to the Poincare group and the former should not be considered only as a limit of the latter as the only difference between the two being their relativity group. It is only for historical reasons that Einstein's theory is "relativistic" and Galilean ones "non-relativistic". In this section I follow the review of the subject in [10].

The Galilean group in three space and one time dimensions is consists of ten elements parametrised by:

$$
\begin{equation*}
g=\left(\zeta^{0}, \boldsymbol{\zeta}, \mathbf{V}, \lambda\right) \tag{4.1}
\end{equation*}
$$

where the parameters correspond to time translations, space translations, Galilean boosts and spacial rotations respectively. The inverse are then:

$$
\begin{equation*}
g^{-1}=\left(-\zeta^{0},-R^{-1}\left(\boldsymbol{\zeta}-\mathbf{V} \zeta^{0}\right), \lambda^{\prime} \lambda\right) \tag{4.2}
\end{equation*}
$$

with the identity element:

$$
\begin{equation*}
\mathbb{1}_{g}=(0, \mathbf{0}, \mathbf{0}, \mathbf{1}) \tag{4.3}
\end{equation*}
$$

The group law is:

$$
\begin{equation*}
g^{\prime \prime}=g * g^{\prime}=\left(\zeta^{0 \prime}+\zeta^{0}, \zeta^{\prime}+\lambda^{\prime} \boldsymbol{\zeta}+\mathbf{V}^{\prime} \zeta^{0}, \mathbf{V}^{\prime} \zeta^{0}, \mathbf{V}^{\prime}+\lambda^{\prime} \mathbf{V}, \lambda^{\prime} \lambda\right) \tag{4.4}
\end{equation*}
$$

The action of $g$ on a space-time vector $(t, \mathbf{x})$ can be expressed as the action a $5 \times 5$ matrix:

$$
\left(\begin{array}{ccc}
\lambda & \mathbf{V} & \boldsymbol{\zeta}  \tag{4.5}\\
0 & 1 & \zeta^{0} \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\mathbf{x} \\
t \\
1
\end{array}\right)=\left(\begin{array}{c}
\lambda \mathbf{x}+\mathbf{V} t+\boldsymbol{\zeta} \\
t+\zeta^{0} \\
1
\end{array}\right)
$$

The additional element 1 on both the transformation matrix and the space-time vector is needed to represent the group of affine transformations in a homogeneous, matrix from. This is needed only if an origin for space-time is not set. Since flat space-time is not a vector space, but rather an affine one, any coordinate change in an affine space will result in an non-homogeneous transformation. Therefore the above change amends this.

The group algebra is then:

$$
\begin{array}{rlrl}
{\left[J_{i j}, G_{k}\right]} & =-2 \delta_{k[i} G_{j]} & {\left[J_{i j}, P_{k}\right]} & =-2 \delta_{k[i} P_{j]} \\
{\left[G_{i}, H\right]} & =-P_{i} & {\left[J_{i j}, J_{k l}\right]=4 \delta_{[i[k} J_{l] j]}}
\end{array}
$$

where $P_{i}$ generates space translations, $H$ time translations, $G_{i}$ Galilean boosts, $J_{i j}$ rotations.

### 4.2 Newton-Cartan gravity

First we will restrict the discussion to $d=4$ space-time dimensions, one for time and three for the spatial dimensions. In this section we will review Newton-Cartan (NC) gravity as a geometric theory that mimics the framework of GR as close as possible [2, 1, 11]. Our end result in this section is to see how we can obtain e covariant Poission equation for gravity.

First we consider the classical equation of motion for a point particle in a Newtonian potential:

$$
\begin{equation*}
\ddot{X}(t)+\frac{\partial \phi(x)}{\partial x^{i}}=0 \tag{4.7}
\end{equation*}
$$

where $x(i), i=(1,2,3)$ are the coordinates in space, $t$ is the time and $\phi(x)$ is the gravitational potential. From Newtonian dynamics we know that this potential satisfies the Poisson equation:

$$
\begin{equation*}
\partial^{2} \phi(x)=4 \pi G \rho(x) \tag{4.8}
\end{equation*}
$$

where $\rho(x)$ is the mass density.

The equations of motion (4.7) and (4.8) are invariant under transformations of the Galilean group:

$$
\begin{equation*}
t \rightarrow t+\xi^{0} \quad x^{i} \rightarrow \lambda^{i}{ }_{j} x^{j}+v^{i} t+\xi^{i} \tag{4.9}
\end{equation*}
$$

where the parameters $\xi^{0}, \xi^{i}$ are for translations in time and space respectively, $\lambda^{i}{ }_{j} \in S O(3)$ for spatial rotations and $v^{i}$ for Galilean boosts.

From a Newtonian perspective equations (4.7) describes objects following curved paths in flat space-time. Just as in GR we wish to amend this and describe this motion as following a geodesic in curved space-time. This means that we have to rewrite (4.7) as the geodesic equation (2.12). To do so we set the coordinates $\left\{x^{\mu}\right\}=\left(t, x^{i}\right)$ and the non-zero components of the connection to be:

$$
\begin{equation*}
\Gamma_{00}^{i}=\delta^{i j} \partial_{j} \phi \tag{4.10}
\end{equation*}
$$

From (4.10) one can now derive Riemann tensor for this geometry, whose non-zero components are:

$$
\begin{equation*}
R_{0 j 0}^{i}=\delta^{i k} \partial_{k} \partial_{j} \phi \tag{4.11}
\end{equation*}
$$

From which we can retrieve the Poisson equation (4.8) by imposing:

$$
\begin{equation*}
R_{00}=4 \pi G \rho \tag{4.12}
\end{equation*}
$$

To covariantise (4.12) we need the metric of the space-time manifold. The metric in NC gravity and unlike GR is degenerate, to see this we can take the $c \rightarrow \infty$ limit of the Minkowski metric:

$$
\eta_{\mu \nu} / c^{2}=\left(\begin{array}{cc}
-1 & 0  \tag{4.13}\\
0 & \mathbb{1} / c^{2}
\end{array}\right) \quad \eta^{\mu \nu}=\left(\begin{array}{cc}
-1 / c^{2} & 0 \\
0 & \mathbb{1}
\end{array}\right)
$$

where we end up with a temporal metric $\tau_{\mu \nu}=\tau_{\mu} \tau_{\nu}$ with $\tau_{\mu}$ the temporal vielbein and the spatial metric $h^{\mu \nu}$. These satisfy:

$$
\begin{equation*}
h^{\mu \nu} \tau_{\mu}=0 \tag{4.14}
\end{equation*}
$$

And we can define their inverse by imposing [12]:

$$
\begin{align*}
h^{\mu \nu} h_{\nu \rho} & =\delta_{\nu}^{\mu}-\tau^{\mu} \tau_{\rho} \quad \tau^{\nu} \tau_{\nu}=1 \\
h_{\mu \nu} \tau^{\nu} & =0 \tag{4.15}
\end{align*}
$$

To relate the metric and connection we impose metric compatibility:

$$
\begin{equation*}
\nabla h^{\mu \nu}=0 \quad \nabla \tau_{\nu}=0 \tag{4.16}
\end{equation*}
$$

From (4.16) follows that $\tau_{\mu}=\partial_{\mu} t(x)$ is a gradient of a scalar field, in the case of NC gravity that is the absolute Newtonian time $t$.

The metric compatibility conditions (4.31), however, do not uniquely define the connection $\Gamma_{\mu \nu}^{\rho}$ as the connection is still arbitrary up to an additional anti-symmetric field [13]:

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho \rho}=\Gamma_{\mu \nu}^{\rho}+h^{\rho \sigma} K_{\sigma(\mu} \tau_{\nu)} \tag{4.17}
\end{equation*}
$$

The most general connection that satisfies the metric compatibility condition (4.31) is then [13]:

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}=\tau^{\rho} \partial_{(\mu} \tau_{\nu)}+\frac{1}{2} h^{\rho \sigma}\left(\partial_{\nu} h_{\sigma \mu}+\partial_{\mu} h_{\sigma \nu}-\partial_{\sigma} h_{\mu \nu}\right) h^{\rho \sigma} K_{\sigma(\mu} \tau_{\nu)} \tag{4.18}
\end{equation*}
$$

In adapted coordinates where $x^{0}=t, \tau_{\mu}=\delta_{\mu}^{0}$ the connection components are then given by [13]:

$$
\begin{align*}
\Gamma_{00}^{i} & =h^{i j}\left(\partial_{0} h_{j 0}-\frac{1}{2} \partial_{j} h_{00}+K_{j 0}\right) \equiv h^{i j} \Phi \\
\Gamma_{0 j}^{i} & =h^{i k}\left(\frac{1}{2} \partial_{0} h_{j k}-\partial_{[j} h_{k] 0}+\frac{1}{2} K_{j k}\right) \equiv h^{i j}\left(\frac{1}{2} \partial_{0} h_{j k}+\omega_{j k}\right)  \tag{4.19}\\
\Gamma_{j k}^{i} & =\frac{1}{2} h^{i l}\left(\partial_{k} h_{l j}+\partial_{j} h_{l k}-\partial_{l} h_{j k}\right) \\
\Gamma_{\mu \nu}^{0} & =0
\end{align*}
$$

Then a covariant form of the Poisson equations can be constructed as:

$$
\begin{equation*}
R_{\mu \nu}=4 \pi G \rho \tau_{\mu} \tau_{\nu} \tag{4.20}
\end{equation*}
$$

To make full contact with Newtonian gravity (4.20) has to reduce to the Poisson equation. To do so one has impose further constraints, given by conditions on the Riemann tensor as demonstrated in [14]:

$$
\begin{equation*}
h^{\rho[\sigma} R_{(\nu \lambda) \rho}^{\mu]}=0 \tag{4.21}
\end{equation*}
$$

which in adapted coordinates equates to:

$$
\begin{equation*}
\partial_{0} \omega_{\mu i}-\partial_{[\mu} \Phi_{i]} \quad \partial_{[k} \omega_{\mu i]}=0 \tag{4.22}
\end{equation*}
$$

from which $K_{\mu \nu}$ can be further constrained:

$$
\begin{equation*}
K_{\mu \nu}=2 \partial_{[\mu} m_{\nu]} \tag{4.23}
\end{equation*}
$$

where $m$ is a vector field given by $m_{\mu}=\partial_{\mu} m(x)$ for $m(x)$ scalar function.
The second condition is for $\omega_{i j}$ to just depend on time so that it satisfies (4.19). As shown in [12] these are:

$$
\begin{align*}
& h^{\rho \lambda} R^{\mu}{ }_{\nu \rho \sigma}(\Gamma) R^{\nu}{ }_{\mu \lambda \xi}(\Gamma)=0  \tag{4.24}\\
& \tau_{[\lambda} R^{\mu}{ }_{\nu] \rho \sigma}(\Gamma)=0  \tag{4.25}\\
& h^{\sigma[\lambda} R_{\nu \rho \sigma}^{\mu{ }_{\nu}}(\Gamma)=0 \tag{4.26}
\end{align*}
$$

all lead to $\omega_{i j}=\omega_{i j}(t)$ independently. Then after performing a coordinate change [13] $x \rightarrow \lambda^{i}{ }_{j}(t) x^{j}$ where $\lambda^{i}{ }_{j} \in S O(3)$ we end up with:

$$
\begin{equation*}
\Gamma_{00}^{i}=\delta^{i j} \partial_{j} \Phi \tag{4.27}
\end{equation*}
$$

in the new coordinate system. Now we can identify $\Phi$ with the Newton potential so that we recover the Poisson equation:

$$
\begin{equation*}
R_{00}=\partial_{i} \Gamma_{00}^{i}=\delta^{i j} \partial_{i} \partial_{j} \Phi=4 \pi G \rho \tag{4.28}
\end{equation*}
$$

### 4.3 Newton-Cartan gravity as a gauge theory

Just like Genral Relativity can be obtained by gauging the Poincare algebra Newton-Cartan gravity can also be derived as a gauge theory of a symmetry group. Intuitively one might think that the group that needs to be gauged is the Galilean group, that is not entirely true and in fact a central extension of the Galilean algebra is needed, called the Bargmann algebra.

### 4.3.1 Symmetry group of Newton-Cartan gravity

The necessity for a central extension to the Galilean algebra can be most easily seen by considering the transformation properties of the classical point particle with a Lagrangian:

$$
\begin{equation*}
\mathcal{L}=\frac{m \dot{x}^{2}}{2} \tag{4.29}
\end{equation*}
$$

That under Galilean boosts transforms as a total derivative:

$$
\begin{equation*}
\delta \mathcal{L}=\frac{d}{d t} m\left(\delta_{i j} x^{i} v^{i}+\frac{1}{2} \delta_{i j} v^{i} v^{j} t\right) \tag{4.30}
\end{equation*}
$$

where $v^{i}$ is again the Galilean boost parameter.
There is no way of removing this total derivative terms and even though it does not change the equations of motion classically, it has profound consequences when the theory is quantised [10]. To see how this term gives rise to a central extention we can explicitly compute the Poisson bracket between a Galilean boost and a space translation:

$$
\begin{equation*}
\left\{G_{i}, P_{j}\right\}=-m \delta_{i j} v^{i} x^{j} \tag{4.31}
\end{equation*}
$$

It is then straightforward to check that the expression on the right-hand side of (4.31) has a zero Poisson bracket with all of the generators. From this we conclude that the algebra of Newtonian gravity is the Bargmann algebra:

$$
\begin{array}{rlr}
{\left[J_{i j}, G_{k}\right]} & =-2 \delta_{k[i} G_{j]} & {\left[J_{i j}, P_{k}\right]=-2 \delta_{k[i} P_{j]}} \\
{\left[G_{i}, H\right]} & =-P_{i} & {\left[J_{i j}, J_{k l}\right]=4 \delta_{[i[k} J_{l] j]}} \\
{\left[G_{i}, P_{j}\right]} & =-\delta_{i j} M & \tag{4.32}
\end{array}
$$

where we have the same generators as in (??) with the addition of $M$, the central extension.
As we will see the central extension is the $\mathrm{U}(1)$ symmetry that is responsible for mass conservation in the theory. This is not, however, unique to the point particle and every non-relativistic theory with mass conservation exhibits it, including field theories with nonrelativistic backgrounds [3].

The underlying reason for this is because the Galilean algebra itself permits a central extension and therefore the Poisson brackets or commutation algebra will always realize the centrally extended algebra. This is not the case for the Poincare algebra in $D=1+3$ dimensions, for example, since the algebra does not permit a central extension. Any central term added to the algebra can be absorbed by redefinition of the generators.

### 4.3.2 Newton-Cartan gravity as a gauge theory of the Bargmann algebra

Gauging the Bargmann algebra follows the same procedure as in section 2.3, we assign to each symmetry transformation a gauge field:

- Space translations $\zeta^{i}(x)$ : spatial vielbein $e_{\mu}{ }^{i}$
- Time translations $\zeta^{0}(x)$ : temporal vielbein $\tau_{\mu}$
- Rotations $\lambda^{i j}: \omega_{\mu}{ }^{i j}$
- Galilean boosts $\lambda^{0 i}: \omega_{\mu}{ }^{i 0}$
- Central extension $\mathrm{U}(1)$ transformation $\sigma: m_{\mu}$

With transformation properties given by:

$$
\begin{align*}
\delta \tau_{\mu} & =\partial_{\mu} \zeta^{0} \\
\delta e_{\mu}{ }^{i} & =D_{\mu} \zeta^{i}+\lambda^{i j} e_{\mu}{ }^{j}+\lambda_{\mu}^{i 0}-\zeta^{0} \omega_{\mu}{ }^{i 0} \\
\delta \omega_{\mu}{ }^{i 0} & =D_{\mu} \lambda^{i 0}+\lambda^{i j} \omega_{\mu}{ }^{j 0}  \tag{4.33}\\
\delta \omega_{\mu}{ }^{i j} & =D_{\mu} \lambda^{i j} \\
\delta m & =\partial_{\mu} \sigma-\zeta^{i} \omega_{\mu}{ }^{i 0}+\lambda^{i 0} e_{\mu}{ }^{i}
\end{align*}
$$

where $D_{\mu}$ is a covariant derivative under rotations that depends only on $\omega_{\mu}{ }^{i j}$.
The curvature tensors for each of the gauge fields are then given by [2]:

$$
\begin{align*}
R_{\mu \nu}(H) & =2 \partial_{[\mu} \tau_{\nu]}  \tag{4.34}\\
R_{\mu \nu}{ }^{i}(P) & =2\left(D_{[\mu} e_{\nu]}{ }^{i}-\omega_{[\mu}{ }^{i 0} \tau_{\nu]}\right.  \tag{4.35}\\
R_{\mu \nu}{ }^{i j}(J) & =2\left(\partial_{[\mu} \omega_{\nu]}{ }^{i j}-\omega_{[\mu}{ }^{k i} \omega_{\nu]}{ }^{j k}\right)  \tag{4.36}\\
R_{\mu \nu}{ }^{i j}(G) & =2 D_{[\mu} \omega_{\nu]}{ }^{i 0}  \tag{4.37}\\
R_{\mu \nu}(M) & =2\left(\partial_{[\mu} m_{\nu]}+e_{[\mu}{ }^{j} \omega_{\nu]}{ }^{j 0}\right. \tag{4.38}
\end{align*}
$$

From (4.34) follows that we can set $\tau_{\mu}=\partial_{\mu} t$ where $t$ is the absolute time, in accordance with the same result obtained from (4.16) in the previous section.

Using the general identity for gauge fields (2.36) we associate the parameters for space and time translations with the parameter for general coordinate transformations:

$$
\begin{equation*}
\xi^{\mu}=e_{i}^{\mu} \zeta^{i}+\tau^{\mu} \zeta^{0} \tag{4.39}
\end{equation*}
$$

Analogously to the constraint (2.39) we can impose:

$$
\begin{equation*}
R_{\mu \nu}^{i}(P)=R_{\mu \nu}(H)=R_{\mu \nu}(M)=0 \tag{4.40}
\end{equation*}
$$

In addition one can use the Bianchi identities to obtain further relations between the curvatures:

$$
\begin{equation*}
R_{[\rho \nu}^{i j}(J) e_{\nu]}^{j}=-R_{[\rho \nu}^{i 0}(G) \tau_{\nu]} \quad e_{\rho]}^{i} R_{\rho \nu]}^{i 0}(G)=0 \tag{4.41}
\end{equation*}
$$

Since only $e_{\mu}{ }^{i}, \tau_{\mu}$ and $m$ transform under space and time translations, and we wish to only express the spin connection $\omega_{\mu}{ }^{i j}$ in terms of the other fields. It is important to note that we want to keep $m$ as an independent fields since it will account for mass conservation in the theory.

To solve for $\omega_{\mu}{ }^{i j}$ use the condition for the momentum curvature [2] as follows:

$$
\begin{equation*}
R_{\mu \nu}^{i}(P) e^{i}+R_{\rho \mu}^{i}(P) e^{i}-R_{\nu \rho}^{i}(P) e^{i}=0 \tag{4.42}
\end{equation*}
$$

from which we obtain:

$$
\begin{equation*}
\omega_{\mu}^{k l}=\partial_{[\mu} e_{\nu]}^{[k} e^{\nu l]}+e_{\mu}^{i} \partial_{[\nu} e_{\rho]}^{i} e^{\nu[k} e^{\rho l]}-\tau_{\mu} e^{\rho[k} \omega_{\mu}^{l] 0} \tag{4.43}
\end{equation*}
$$

Then from contracting the curvature for the $m$ field:

$$
\begin{equation*}
R_{\mu \nu}(M) e_{i}^{\mu}=0 \quad R_{\mu \nu}(M) \tau^{\mu}=0 \tag{4.44}
\end{equation*}
$$

in conjunction with (4.42) and (4.43) we have:

$$
\begin{equation*}
\omega_{\mu}^{i 0}=e^{\nu i} \partial_{[\mu} m_{\nu]}+e^{\nu i} \tau^{\rho} e^{i} \partial_{[\nu} e_{\rho]}^{j}+\tau_{\mu} \tau^{\nu} e^{\rho i} \partial_{[\nu} m_{\rho]}+\tau^{\nu} \partial_{[\mu} e_{\nu]}^{i} \tag{4.45}
\end{equation*}
$$

To obtain Newton-Cartan gravity for this approach the connection also needs to be defined, for this one has to make use of the vielbein (2.32) postulate (2.32) for the spatial and temporal vielbeins [2]:

$$
\begin{align*}
& \partial_{\mu} e_{\nu}^{i}-\omega_{\mu}^{i j} e_{\nu}^{j}-\omega_{\mu}^{i 0} \tau_{\nu}-\Gamma_{\nu \mu}^{\rho} e_{\rho}^{j}=0  \tag{4.46}\\
& \partial_{\mu} \tau_{\nu}-\Gamma_{\nu \mu}^{\rho} \tau_{\rho}=0 \tag{4.47}
\end{align*}
$$

From (4.46) and (4.48) the following expression for the connection is obtained:

$$
\begin{equation*}
\Gamma_{\nu \mu}^{\rho}=\tau^{\rho} \partial_{(\mu} \tau_{\nu)}+e^{\rho}{ }_{i}\left(\partial_{(\mu} e_{\nu)}{ }^{i}-\omega_{(\mu}{ }^{i j} e_{\nu)}{ }^{j}-\omega_{(\mu}{ }^{i 0} \tau_{\nu)}\right) \tag{4.48}
\end{equation*}
$$

The connection (4.48) is now uniquely defined, unlike (4.18), therefore by comparing (4.18) with (4.48) $K_{\mu \nu}$ can be fixed:

$$
\begin{align*}
K_{\mu \nu} & =\omega_{[\mu}{ }^{i 0} e_{\nu]}{ }^{j}  \tag{4.49}\\
& =2 \partial_{[\mu} m_{\nu]} \tag{4.50}
\end{align*}
$$

where to get from (4.49) to (4.50) the condition $R_{\mu \nu}(M)=0$ is used. Then the Riemann tensor constructed from (4.48) can be given in terms of the curvatures $R_{\mu \nu}{ }^{i j}(G)$ and $R_{\mu \nu}{ }^{i j}(J)$ :

$$
\begin{equation*}
R_{\nu \rho \sigma}^{\mu}(\Gamma)=-e_{i}^{\mu}\left(R_{\rho \sigma}{ }^{i 0}(G) \tau_{\nu}+R_{\rho \sigma}{ }^{i j}(J) e_{\nu j}\right) \tag{4.51}
\end{equation*}
$$

The conditions (4.24), (4.25) and (4.41) are all equivalent to:

$$
\begin{equation*}
R_{\mu \nu}{ }^{i j}(J)=0 \tag{4.52}
\end{equation*}
$$

which can be used to finally construct the covariant equation of motion:

$$
\begin{equation*}
\tau^{\mu} e^{\nu(i} R_{\mu \nu}{ }^{j) 0}(G)=\delta^{k(j} R_{0 k 0}^{i)}(\Gamma) \tag{4.53}
\end{equation*}
$$

This equation of motion equates to the only non-zero component of (4.11) of the Riemann tensor for the covariantised Poisson equation of Newton-Cartan gravity.

## 5 The bosonic string in Newton-Cartan background

### 5.1 The bosonic string with Newton-Cartan background

There has been work done on Galilean invariant bosonic strings for quite some time now [15] since it can be applied in conjunction with the holographic principle in order to study the properties of other sectors of String Theory, including relativistic ones [16]. Only in the past decade was String Theory studied as non-relativistic theory embedded in a Newton-Cartan background. There are two ways to make this embedding of the bosonic string. One is to take the usual $c \rightarrow \infty$ limit of the Polyakov action as demonstrated in [17, 18, 19]. And the second one, which I make use of is by dimensional reduction along a null Killing vector field.

### 5.1.1 Dimensional reduction on along null Killing field

The special properties of taking the dimensional reduction along a null Killing field can be most easily seen by setting the metric to [20]:

$$
G_{M N}=\left(\begin{array}{cc}
h_{m n} & \xi_{m}  \tag{5.1}\\
\xi_{n} & \xi^{2}
\end{array}\right)
$$

where $G_{M N}$ and $h_{m n}$ are the $D+1$ and $D$ dimensional metrics respectively, $\xi=(0,0 \ldots, 1)$ is the null Killing vector and $v$ is the extra dimension we are constraining. And where also this form of the metric is valid for arbitrary $\xi^{2}$. Then its inverse can be parametrised as:

$$
G^{M N}=\left(\begin{array}{cc}
h^{m n} & N^{m}  \tag{5.2}\\
N^{n} & N^{u}
\end{array}\right)
$$

Setting $\xi$ to be a null Killing field:

$$
\begin{equation*}
\xi^{2}=0 \tag{5.3}
\end{equation*}
$$

this essentially means that one component of the metric is set so zero and the equations of motion will be reduced by one. From (5.1) and (5.3) we can clearly see that the metric ends up to be degenerate since:

$$
\begin{equation*}
h^{m n} \xi_{n}=0 \tag{5.4}
\end{equation*}
$$

Which is the reason why this theory leads to a generally covariant Galilean theory. As already shown in the previous section this type of metric complex $h^{m n}, u_{m}$ leads to a NewtonCartan geometry, a degenerate contravariant metric $h^{m n}$ with a zero eigenvector $u_{m}$.

### 5.1.2 The Polyakov-type action for string with NC background

Making use of (5.1) and (5.3) we the relativistic $(d+2)$ dimensional metric with the null isometry in the form:

$$
\begin{equation*}
g_{M N} d x^{M} d x^{N}=2 \tau_{m} d x^{m}\left(d u-m_{n} d x^{n}\right)+h_{m n} d x^{m} d x^{n} \tag{5.5}
\end{equation*}
$$

where $h_{m n}, \tau_{m}$ and $m$ are the spatial metric, temporal vielbein and $\mathrm{U}(1)$ field of NC gravity respectively, $u$ is again used to index the null direction and $x^{m}$ are the coordinates in the $(\mathrm{d}+1)$ dimensions. It is then possible to derive a Polyakov-type action on the worldsheet embedded in NC geometry by substituting (5.5) in the relativistic Polaykov action (3.8) [21, 22]:

$$
\begin{equation*}
\mathcal{L}=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}} \gamma^{\alpha \beta}\left(h_{\alpha \beta}-\tau_{\alpha} m_{\beta}-m_{\alpha} \tau_{\beta}\right)-\frac{\sqrt{-\gamma}}{2 \pi \alpha^{\prime}} \gamma^{\alpha \beta} \tau_{\alpha} \partial_{\beta} X^{u} \tag{5.6}
\end{equation*}
$$

where $\gamma^{\alpha \beta}$ is the worldsheet metric, $\gamma$ its determinant and $h_{\alpha \beta}=h_{m n} \partial_{\alpha} X^{m} \partial_{\beta} X^{n}, \tau_{\alpha}=$ $\tau_{m} \partial_{\alpha} X^{m}$ and $m_{\alpha}=m_{n} \partial_{\alpha} X^{n}$ are the pullbacks of $h_{m n}, \tau_{n}$ and $m_{n}$ onto the worldsheet respectively.

The strings considered are closed and obey $X^{m}\left(\sigma^{0}, \sigma^{1}+2 \pi\right)=X^{m}\left(\sigma^{0}, \sigma^{1}\right)$, and are without winding. Then following [21] we want to find an action for the closed string in a NC background only in terms of $h_{m n}, \tau_{n}$ and $m_{n}$. Therefore we want to fix the momentum on the null direction.

The worldsheet momentum current of the string in the null direction is given by:

$$
\begin{equation*}
P_{u}^{\alpha}=\frac{\partial \mathcal{L}}{\partial \partial_{\alpha} X^{u}}=-\frac{1}{2 \pi \alpha^{\prime}} \sqrt{-\gamma} \gamma^{\alpha \beta} \tau_{\beta} \tag{5.7}
\end{equation*}
$$

with the total along $u$ momentum being:

$$
\begin{equation*}
P=\int_{0}^{2 \pi} d \sigma^{1} P_{u}^{0} \tag{5.8}
\end{equation*}
$$

In this formulation the momentum is conserved on-shell, to fix it off-shell we consider a dual formulation with the Lagrangian:

$$
\begin{equation*}
\mathcal{L}=-\frac{1}{4 \pi \alpha^{\prime}}\left[\sqrt{-\gamma} \gamma^{\alpha \beta} \bar{h}_{\alpha \beta}+\left(\sqrt{-\gamma} \gamma^{\alpha \beta} \tau_{\beta}-\epsilon^{\alpha \beta} \partial_{\alpha} \eta\right) A_{\alpha}\right] \tag{5.9}
\end{equation*}
$$

where $\eta$ is a scalar novel field and $A_{\alpha}$ is a Lagrange multiplier that enforces the conservation of the momentum $P=\frac{\epsilon^{\alpha \beta} \partial_{\beta} \eta}{2 \pi \alpha^{\prime}}$ in the $u$ direction off-shell. Here we also use the convenient combination:

$$
\begin{equation*}
\bar{h}_{\alpha \beta}=h_{\alpha \beta}-m_{\alpha} \tau_{\beta}-\tau_{\alpha} m_{\beta} \tag{5.10}
\end{equation*}
$$

which is invariant under local Galilean boosts. To confirm that (5.6) and (5.9) are equivalent we can solve the equation of motion for $\eta$ to obtain $A_{\alpha}=\partial_{\alpha} X^{u}$ and substitute it in. Then introduce the vielbein on the worldsheet $e_{\alpha}{ }^{a}$ and its inverse $e^{\alpha}{ }_{a}=\epsilon^{\alpha \beta} e_{\beta}{ }^{b} \epsilon_{b a}$ that satisfy:

$$
\begin{equation*}
\eta_{a b} e_{\alpha}{ }^{a} e_{\beta}{ }^{b}=\gamma_{\alpha \beta} \quad \eta^{a b} e^{\alpha}{ }_{a} e^{\beta}{ }_{b}=\gamma^{\alpha \beta} \tag{5.11}
\end{equation*}
$$

This allows us to rewrite the constraints as:

$$
\begin{align*}
& \epsilon^{\alpha \beta}\left(e_{\alpha}{ }^{0}+e_{\alpha}{ }^{1}\right)\left(\tau_{\beta}+\partial_{\beta} \eta\right)=0 \\
& \epsilon^{\alpha \beta}\left(e_{\alpha}{ }^{0}-e_{\alpha}{ }^{1}\right)\left(\tau_{\beta}-\partial_{\beta} \eta\right)=0 \tag{5.12}
\end{align*}
$$

Then the Lagrange multipliers can be redefined as:

$$
\begin{equation*}
A_{\alpha}=m_{\alpha}+\frac{1}{2}\left(\lambda_{+}-\lambda_{-}\right) e_{\alpha}^{0}+\left(\lambda_{+}+\lambda_{-}\right) e_{\alpha}^{1} \tag{5.13}
\end{equation*}
$$

Substituting (5.25) into (5.9) we obtain the Polyakov-type Lagrangian of the bosonic string in NC background as proposed in [21]:

$$
\begin{equation*}
\mathcal{L}=-\frac{e}{4 \pi l_{s}^{2}}\left[e_{+}^{\alpha} e_{-}^{\beta} \bar{h}_{\alpha \beta}+\lambda_{+} e_{-}^{\beta}\left(\partial_{\beta} \eta+\tau_{\beta}\right)+\lambda_{-} e_{+}^{\beta}\left(\partial_{\beta} \eta-\tau_{\beta}\right)\right] \tag{5.14}
\end{equation*}
$$

where $e_{ \pm}^{\alpha}=e_{\alpha}^{0} \pm e_{\alpha}^{1}$.
The transformation properties for each of the objects from the metric structure are:

$$
\begin{align*}
\delta \tau_{m} & =\mathcal{L}_{\xi} \tau_{m} \\
\delta e_{m}^{i} & =\mathcal{L}_{\xi} e_{m}^{i} e_{m}^{i}+\lambda^{i} \tau_{m}+\lambda^{i}{ }_{j} e_{m}^{j} \\
\delta \tau^{m} & =\lambda^{i} e_{i}^{m}  \tag{5.15}\\
\delta e_{i}^{m} & =\mathcal{L}_{\xi} e_{i}^{m} \\
\delta m_{n} & =\mathcal{L}_{\xi} m_{n}+\lambda_{i} e_{n}^{i}+\partial_{n} \sigma
\end{align*}
$$

Using the transformations (5.15) one can check that the action based on (5.14) is invariant under diffeomorphisms, local Galilean boosts, local rotations and local $\mathrm{U}(1)$ transformations.

### 5.2 The Bargmann algebra for the bosonic string

### 5.3 Conserved currents

In order to derive the correct conserved currents we first need to know if and how the Lagrange multipliers transform under the Galilean transformations.

### 5.3.1 Transformation properties of the Lagrange multipliers

Before looking at the Lagrange multipliers it is useful to first see whether the field $X^{u}$ in the null direction transforms. As we already saw the Lagrange multipliers are given in part in terms of it so it is crucial to know if it contributes.

To do so I look at how the unconstrained Lagrangian transforms:

$$
\begin{equation*}
\mathcal{L}=-T \sqrt{-\gamma} \gamma^{\alpha \beta}\left(\frac{1}{2} \bar{h}_{m n} \partial_{\alpha} X^{n} \partial_{\beta} X^{m}+\tau_{m} \partial_{\alpha} X^{m} \partial_{\beta} X^{u}\right) \tag{5.16}
\end{equation*}
$$

Under boosts, the fields transform as $X^{i} \rightarrow X^{i}+\lambda^{i} X^{0}$ and $\delta X^{0}=0$. Then let $X^{u} \rightarrow X^{u}+\delta X^{u}$.

Transforming the Lagrangian we get:

$$
\begin{align*}
\delta \mathcal{L} & =-T \sqrt{-\gamma} \gamma^{\alpha \beta}\left[\bar{h}_{0 i} \partial_{\alpha} X^{0} \partial_{\beta}\left(\lambda^{i} X^{0}\right)+\bar{h}_{j i} \partial_{\alpha} X^{j} \partial_{\beta}\left(\lambda^{i} X^{0}\right)\right. \\
& \left.+\tau_{i} \partial_{\alpha}\left(\lambda^{i} X^{0}\right) \partial_{\beta} X^{u}+\tau_{m} \partial_{\alpha} X^{m} \partial_{\beta}\left(\delta X^{u}\right)\right] \\
& =-T \sqrt{-\gamma} \gamma^{\alpha \beta}\left[\bar{h}_{m i} \partial_{\alpha} X^{m} \partial_{\beta}\left(\lambda^{i} X^{0}\right)+\tau_{i} \partial_{\alpha}\left(\lambda^{i} X^{0}\right) \partial_{\beta} X^{u}\right.  \tag{5.17}\\
& \left.+\tau_{m} \partial_{\alpha} X^{m} \partial_{\beta}\left(\delta X^{u}\right)\right]
\end{align*}
$$

For the equation of motion for $X^{\mu}$ we have:

$$
\begin{equation*}
\partial_{\alpha}\left[-T \sqrt{-\gamma} \gamma^{\alpha \beta}\left(\frac{1}{2} 2 \bar{h}_{m n} \partial_{\beta} X^{n} \delta_{\mu}{ }^{m}+\tau_{m} \partial_{\beta} X^{u} \delta_{\mu}{ }^{m}+\partial_{\beta} X^{m} \delta_{\mu}{ }^{u}\right)\right]=0 \tag{5.18}
\end{equation*}
$$

where $\mu=(m, u)$ and $m$ goes over the non-null directions and $u$ is the null direction.
We can integrate by parts (5.17) to get:

$$
\begin{align*}
\delta \mathcal{L} & =\partial_{\beta}\left[-T \sqrt{-\gamma} \gamma^{\alpha \beta}\left(\bar{h}_{m i} \partial_{\alpha} X^{m}+\tau_{i} \partial_{\beta} X^{u}\right)\right] \lambda^{i} X^{0}  \tag{5.19}\\
& +\tau_{m} \partial_{\alpha} X^{m} \partial_{\beta}\left(\delta X^{u}\right) \tag{5.20}
\end{align*}
$$

From which we conclude that $X^{u}$ does not transform under Galilean boosts.

Then taking this into consideration I derive how $\lambda_{ \pm}$transform. Rearranging the terms in (5.25) gives:

$$
\begin{equation*}
A_{\alpha}=m_{\alpha}+\frac{1}{2} \lambda_{+} e_{\alpha}^{+}-\frac{1}{2} \lambda_{-} e_{\alpha}^{-} \tag{5.21}
\end{equation*}
$$

Contracting separately with $e_{+}^{\alpha}$ and $e_{-}^{\alpha}$ we get the pair of equations:

$$
\begin{align*}
& e_{+}^{\alpha}\left(A_{\alpha}-m_{\alpha}\right)=\lambda_{+}  \tag{5.22}\\
& e_{-}^{\alpha}\left(m_{\alpha}-A_{\alpha}\right)=\lambda_{-} \tag{5.23}
\end{align*}
$$

Where we use that:

$$
\begin{equation*}
e_{-}^{\alpha} e_{\alpha}^{-}=e_{+}^{\alpha} e_{\alpha}^{+}=2 \tag{5.24}
\end{equation*}
$$

Taking the variations of equations (53) and (54) we have:

$$
\begin{align*}
\delta \lambda_{-} & =e_{-}^{\alpha} \delta\left(\partial_{\alpha} X^{u}-m_{n} \partial_{\alpha} X^{n}\right) \\
& =e_{-}^{\alpha}\left(-\delta m_{n} \partial_{\alpha} X^{n}-m_{n} \partial_{\alpha} \delta X^{n}\right)  \tag{5.25}\\
& =e_{-}^{\alpha}\left(-\lambda^{i} \delta_{i j} e_{n}^{j} \partial_{\alpha} X^{n}-m_{i} \partial_{\alpha} \delta\left(\lambda^{i} X^{0}\right)\right) \\
\delta \lambda_{+} & =e_{+}^{\alpha} \delta\left(-\partial_{\alpha} X^{u}+m_{n} \partial_{\alpha} X^{n}\right) \\
& =e_{+}^{\alpha}\left(+\delta m_{n} \partial_{\alpha} X^{n}-m_{n} \partial_{\alpha} \delta X^{n}\right)  \tag{5.26}\\
& =e_{+}^{\alpha}\left(+\lambda^{i} \delta_{i j} e_{n}^{j} \partial_{\alpha} X^{n}+m_{i} \partial_{\alpha} \delta\left(\lambda^{i} X^{0}\right)\right)
\end{align*}
$$

Using the transformations (5.25) and (5.26) we can now derive the conserved charges for each of the symmetry transformations.

### 5.3.2 The conserved currents

To derive each of the generators we vary the Lagrangian (5.14) under:

- translations: $X^{i} \rightarrow \zeta^{i}$ and $X^{0} \rightarrow \zeta^{0}$
- rotations: $X^{i} \rightarrow \lambda_{j}{ }^{i} X^{j}$
- Galilean boosts: $X^{i} \rightarrow \lambda^{i} X^{0}$

Starting with space translations the transformation $\delta X^{i}=\zeta^{i}$ for the conserved currents we have:

$$
\begin{equation*}
P_{i}^{\alpha}=\frac{e}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{\alpha \beta} \bar{h}_{i n} \partial_{\beta} X^{n}-2 m_{i}\left(\gamma^{\alpha \beta} \tau_{m} \partial_{\beta} X^{m}-\epsilon^{\alpha \beta} \partial_{\beta} \eta\right)\right] \tag{5.27}
\end{equation*}
$$

For time translations $\delta X^{0}=\zeta^{0}$ :

$$
\begin{equation*}
H^{\alpha}=\frac{e}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{\alpha \beta} \bar{h}_{0 n} \partial_{\beta} X^{n}+\tau_{0}\left(\lambda_{-} e_{+}^{\beta}-\lambda_{+} e_{-}^{\beta}\right)-2 m_{0}\left(\gamma^{\alpha \beta} \tau_{m} \partial_{\beta} X^{m}-\epsilon^{\alpha \beta} \partial_{\beta} \eta\right)\right] \tag{5.28}
\end{equation*}
$$

For rotations $\delta X^{i}=\lambda_{j}{ }^{i} X^{j}$ :

$$
\begin{equation*}
J_{j}^{i \alpha}=\frac{e}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{\alpha \beta} \bar{h}_{j n} \partial_{\beta} X^{n}-2\left(\gamma^{\alpha \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{\alpha \beta} \partial_{\beta} \eta\right) m_{j}\right] X^{i} \tag{5.29}
\end{equation*}
$$

For Galilean boosts $\delta X^{i}=\lambda^{i} X^{0}$ :

$$
\begin{align*}
G_{i}^{\alpha} & =\frac{e}{4 \pi \alpha^{\prime}}\left[\left(-2 \gamma^{\alpha \beta}\right) \bar{h}_{i m} \partial_{\beta} X^{m} X^{0}+2 \gamma^{\alpha \beta} \tau_{m} \partial_{\beta} X^{m}\left(\delta_{i j} X^{j}+m_{i} X^{0}\right)\right. \\
& \left.+2 \epsilon^{\alpha \beta} \partial_{\beta} \eta\left(\delta_{i j} X^{j}+m_{i} X^{0}\right)\right] \tag{5.30}
\end{align*}
$$

### 5.3.3 Poisson brackets

In order to compute the Poisson brackets we first calculate the canonical momenta:

$$
\begin{align*}
\Pi_{m}=\frac{\partial \mathcal{L}}{\partial X^{m}} & =\frac{e}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{0 \beta} \bar{h}_{m n} \partial_{\beta} X^{n}+2 \gamma^{0 \beta}\left(\partial_{\beta} X^{u}-m_{n} \partial_{\beta} X^{n}\right)\right. \\
& \left.-2 \gamma^{0 \beta} \tau_{n} m_{m} \partial_{\beta} X^{n}+2 \epsilon^{0 \beta} m_{m} \partial_{\beta} \eta\right] \tag{5.31}
\end{align*}
$$

We are interested in the conserved charges so we set:

$$
\begin{array}{rlrl}
P_{i} & \equiv P_{i}{ }^{0} & H & \equiv H^{0} \\
J_{j}{ }^{i} & \equiv J_{j}{ }^{i 0} & G_{i} & \equiv G_{i}{ }^{0} \tag{5.32}
\end{array}
$$

$$
\begin{align*}
&\left\{J_{j}^{i}, J_{k}^{l}\right\}=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}} 4\left[-2 \gamma^{0 \beta} \bar{h}_{j n} \partial_{\beta} X^{n} \delta_{m 0}-2\left(\gamma^{0 \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) m_{j}\right] \delta_{m}^{i} X^{l} \delta_{k}^{m} \\
&=4 \delta^{i}{ }_{k} J_{j}^{l}  \tag{5.33}\\
&\left\{J_{j}{ }^{i}, G_{k}\right\}=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}} 2\left[-2 \gamma^{0 \beta} \bar{h}_{j n} \partial_{\beta} X^{n} \delta_{m 0}-2\left(\gamma^{0 \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) m_{j}\right] \delta_{m}{ }^{i} X^{0} \delta_{k}^{m} \\
&+2\left(\gamma^{0 \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) \delta_{j k} X^{i} \\
&=-2 G_{j} \delta_{k}^{i}  \tag{5.34}\\
&\left\{\begin{array}{l}
\left.J_{j}{ }^{i}, P_{k}\right\}=
\end{array}\right.  \tag{5.35}\\
&=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}} 2\left[-2 \gamma^{0 \beta} \bar{h}_{j n} \partial_{\beta} X^{n} \delta_{m 0}-2\left(\gamma^{0 \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) m_{j}\right] \delta_{m}^{i} \delta_{k}^{m}  \tag{5.36}\\
&=-2 \delta_{k}^{i} P_{j}
\end{align*}
$$

Where we have an extra factor of 2 for each $J_{j}{ }^{i}$ because it is antisymmetric with two lower indices.

$$
\begin{align*}
\left\{G_{i}, H\right\} & =-\frac{1}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{0 \beta} \bar{h}_{i n} \partial_{\beta} X^{n}-2 m_{i}\left(\gamma^{0 \beta} \tau_{m} \partial_{\beta} X^{m}-\epsilon^{0 \beta} \partial_{\beta} \eta\right)\right] \\
& =-P_{i} \tag{5.37}
\end{align*}
$$

$$
\begin{equation*}
\left\{G_{i}, P_{j}\right\}=2\left(\gamma^{0 \beta} \tau_{m} \partial_{\beta} X^{m}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) \delta_{i j} \tag{5.38}
\end{equation*}
$$

From which we find that the central extension must be given by:

$$
\begin{equation*}
M=-2\left(\gamma^{0 \alpha} \tau_{m} \partial_{\alpha} X^{m}+\epsilon^{0 \alpha} \partial_{\alpha} \eta\right) \tag{5.39}
\end{equation*}
$$

And confirm that the conserved charges indeed satisfy the Bragmann algebra (4.32) since $M$ commutes trivially with the rest of the conserved charges, as expected.

### 5.4 Algebra of the currents

In this section we want to compute the OPEs between the conserved currents and see if the NC space-time symmetries can be realised on the worldsheet in terms of a current algebra.

### 5.4.1 The Sugawara construction

The currents derived in 5.3.2 are of particular interest as not only their zero modes form a Lie algebra, but there is also an infinite dimensional extension of this known as Kac-Moody algebra: the algebra of the currents [23, 24]:

$$
\begin{equation*}
J^{a}(\sigma) J^{b}\left(\sigma^{\prime}\right) \sim \frac{k \delta^{a b}}{2\left(\sigma-\sigma^{\prime}\right)^{2}}+\frac{i f_{c}^{a b} J^{c}\left(\sigma^{\prime}\right)}{\left(\sigma-\sigma^{\prime}\right)}+\ldots \tag{5.40}
\end{equation*}
$$

where $J^{a}(\sigma)$ are the currents of the theory, the parameter $k$ is called the level of the KacMoody algebra and is analogous to the central extension $c$ of the Virasoro algebra and $f^{a b}{ }_{c}$ are the structure constants of the algebra.

This is of particular interest since full dynamics of the theory should be formulated in terms of the currents, in other words the stress-energy tensor should be expressed in terms of the currents $[25,26]$. This procedure is called the Sugawara construction and can be expressed as [27]:

$$
\begin{equation*}
T(\sigma)=\frac{1}{k+\tilde{h}_{G}} \sum_{A=1}^{\operatorname{dim} G}: J^{A}(\sigma) J^{A}(\sigma): \tag{5.41}
\end{equation*}
$$

where $T(\sigma)$ is the stress-energy tensor and $\tilde{h}_{G}$ is the dual Coxeter number and takes specific integer values depending on the type of algebra.

### 5.4.2 OPEs of the conserved currents

In order to compute the OPEs between the currents we first need the propagators for the dynamic fields of the theory. They are given by [28]:

$$
\begin{aligned}
\left\langle X^{i}(\sigma) X^{j}\left(\sigma^{\prime}\right)\right\rangle & =\Delta_{2} h^{i j} \ln \left(|\Delta \sigma|^{2}\right) \\
\left\langle X^{m}(\sigma) \lambda_{ \pm}\left(\sigma^{\prime}\right)\right\rangle & =\delta_{0}^{m} \frac{\mp 2 \Delta_{2}}{\left(\sigma-\sigma^{\prime}\right)_{ \pm}} \\
\left\langle\eta(\sigma) \lambda_{ \pm}\left(\sigma^{\prime}\right)\right\rangle & =\frac{-2 \Delta_{2}}{\left(\sigma-\sigma^{\prime}\right)_{ \pm}} \\
\left\langle\lambda_{ \pm}(\sigma) \lambda_{ \pm}\left(\sigma^{\prime}\right)\right\rangle & =\frac{4 \Delta_{2}}{\left(\sigma-\sigma^{\prime}\right)_{ \pm}} \\
\left\langle\lambda_{-}(\sigma) \lambda_{-}\left(\sigma^{\prime}\right)\right\rangle & =-4 \pi \Delta_{2} \delta\left(\sigma-\sigma^{\prime}\right)
\end{aligned}
$$

where $\Delta_{2}$ is a constant that does not have any significance. Using (5.42) I calculate the following OPEs:

$$
\begin{align*}
P_{i}^{\alpha} P_{j}^{\beta} & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[\frac{1}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}} 4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \delta_{\rho}^{+} \delta_{\xi}^{+} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\right. \\
& +\frac{1}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\left(4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \delta_{\rho}^{-} \delta_{\xi}^{-} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\right] \tag{5.42}
\end{align*}
$$

$$
\begin{align*}
H^{\alpha} H^{\beta} & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[\frac{1}{\sigma_{-}-\sigma_{-}^{\prime}} e_{+}^{\alpha} e_{+}^{\beta} \tau_{0} \tau_{0}+\frac{1}{\sigma_{+}-\sigma_{+}^{\prime}} e_{-}^{\alpha} e_{-}^{\beta} \tau_{0} \tau_{0}\right. \\
& +\frac{1}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}\left(4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \delta_{\rho}^{+} \delta_{\xi}^{+} \bar{h}_{0 m} \bar{h}_{0 n} h^{m n}+4\left(\gamma^{\alpha \rho} e_{-}^{\beta}-\gamma^{\beta \rho} e_{-}^{\alpha}\right) \bar{h}_{0 m} \tau_{0} \delta_{\rho}^{+} \delta_{0}^{m}\right. \\
& \left.+4\left(\epsilon^{\beta \rho} e_{-}^{\alpha}-\epsilon^{\alpha \rho} e_{-}^{\beta}\right) \tau_{0} m_{0} \delta_{\rho}^{+}+4 \gamma^{\alpha \rho} e_{-}^{\beta} \tau_{n} \tau_{0} m_{0} \delta_{\xi}^{+} \delta_{0}^{n}\right)  \tag{5.43}\\
& +\frac{1}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\left(4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \delta_{\rho}^{-} \delta_{\xi}^{-} \bar{h}_{0 m} \bar{h}_{0 n} h^{m n}-4\left(\gamma^{\alpha \rho} e_{+}^{\beta}+\gamma^{\beta \rho} e_{+}^{\alpha}\right) \bar{h}_{0 m} \tau_{0} \delta_{\rho}^{-} \delta_{0}^{m}\right. \\
& \left.\left.-4\left(\epsilon^{\beta \rho} e_{+}^{\alpha}-\epsilon^{\alpha \rho} e_{+}^{\beta}\right) \tau_{0} m_{0} \delta_{\rho}^{-}-4 \gamma^{\alpha \rho} e_{+}^{\beta} \tau_{n} \tau_{0} m_{0} \delta_{\rho}^{-} \delta_{0}^{n}\right)+4 \pi \tau_{0} \tau_{0}\left(e_{+}^{\alpha} e_{-}^{\beta}+e_{-}^{\alpha} e_{+}^{\beta}\right) \delta\left(\sigma-\sigma^{\prime}\right)\right]
\end{align*}
$$

$$
\begin{align*}
G^{\alpha}{ }_{i}(\sigma) G^{\beta}{ }_{j}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} 4 \Delta_{2}\left[\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) X^{0}(\sigma)\right. \\
& \left.+\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}} \partial_{+} X^{0}(\sigma)+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}} \partial_{-} X^{0}(\sigma)\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma)\right] \tag{5.44}
\end{align*}
$$

$$
\begin{align*}
J_{i}^{\alpha}{ }_{i}^{k} J^{\beta}{ }_{j}{ }^{l}= & \left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2} \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n}\left[\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}\right)^{2}}\right) h^{m n} X^{k}(\sigma) X^{l}(\sigma)\right.  \tag{5.45}\\
+ & \frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}} h^{m n} X^{k}(\sigma) \partial_{+} X^{l}(\sigma)+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}} h^{m n} X^{k}(\sigma) \partial_{-} X^{l}(\sigma)  \tag{5.46}\\
+ & \left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) h^{k n} \partial_{\rho} X^{m}(\sigma) X^{l}(\sigma)+\partial_{\rho} X^{m} \partial_{\xi} X^{n}(\sigma) h^{k l} \ln \left(|\Delta \sigma|^{2}\right)  \tag{5.47}\\
+ & \left(\frac{\delta_{0}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma^{\prime}\right)^{2}}\right) h^{m n} h^{k l} \ln \left(|\Delta \sigma|^{2}\right) \Delta_{2}  \tag{5.48}\\
+ & \left.\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) h^{m n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) h^{k n} \Delta_{2}\right]  \tag{5.49}\\
G_{i}^{\alpha}(\sigma) H^{\beta}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{0 n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{0}(\sigma)\right. \\
& +4 \gamma^{\alpha \rho} \bar{h}_{0 m} \tau_{0} \delta_{0}^{m}\left(\frac{e_{-}^{\beta} \delta_{\rho}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{e_{+}^{\beta} \delta_{\rho}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{0}(\sigma) \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{0 n} \tau_{0} \delta_{i k} h^{k n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} X^{0}(\sigma)  \tag{5.50}\\
& \left.-4 \gamma^{\alpha \rho} \tau_{0}^{2} \delta_{i k}\left(\frac{e_{-}^{\beta} \delta_{\rho}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{e_{+}^{\beta} \delta_{\rho}^{-}}{\left(\sigma_{-}-\sigma^{\prime}\right)^{2}}\right) X^{k}(\sigma)\right]
\end{align*}
$$

$$
\begin{aligned}
G_{i}^{\alpha}(\sigma) P_{j}^{\beta}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{0}(\sigma)\right. \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{j n} \delta_{i k} \tau_{0} h^{k n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} X^{0}(\sigma) \\
& \left.-4 \epsilon^{\alpha \beta} \gamma^{\beta \xi} \bar{h}_{j n} \delta_{i k} h^{k n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} \eta(\sigma)\right]
\end{aligned}
$$

$$
\begin{align*}
G_{i}^{\alpha}(\sigma) J^{\beta}{ }_{j}{ }^{k}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}\right)^{2}}\right) X^{0}(\sigma) X^{k}(\sigma)\right. \\
& +4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} X^{n}(\sigma) \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} m_{j} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{0}(\sigma) X^{0}(\sigma) \\
& -4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \bar{h}_{i m} m_{j}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} \eta(\sigma) X^{0}(\sigma)  \tag{5.51}\\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{j n} \delta_{i l} \tau_{0} h^{l n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma) \\
& -\epsilon^{\alpha \beta} \gamma^{\beta \xi} \delta_{i l} \bar{h}_{j n} h^{l n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} \eta(\sigma) X^{k}(\sigma) \\
& \left.+4 \epsilon^{\alpha \beta} \epsilon^{\beta \xi} \delta_{i l} m_{j} h^{l k} \partial_{\xi} \eta(\sigma) \partial_{\rho} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right)\right]
\end{align*}
$$

$$
\begin{align*}
P_{i}^{\alpha}(\sigma) H^{\beta}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{0 n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right)\right. \\
& -4 \gamma^{\alpha \rho} m_{i} \tau_{0}^{2}\left(\frac{e_{-}^{\beta} \delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{e_{+}^{\beta} \delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right)-4 \gamma^{\alpha \rho} \bar{h}_{i 0} \tau_{0}\left(\frac{e_{-}^{\beta} \delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{e_{+}^{\beta} \delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \\
& \left.+4 \epsilon^{\alpha \beta} m_{i} \tau_{0}\left(\frac{e_{-}^{\beta} \delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{e_{+}^{\beta} \delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right)\right] \tag{5.52}
\end{align*}
$$

$$
P_{i}^{\alpha}(\sigma) J^{\beta}{ }_{j}^{k}\left(\sigma^{\prime}\right)=\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{k}(\sigma)\right.
$$

$$
+4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{n}(\sigma)
$$

$$
\begin{equation*}
+4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} m_{j} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{0}(\sigma) \tag{5.53}
\end{equation*}
$$

$$
\begin{equation*}
+4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \bar{h}_{i m} m_{j} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} \eta(\sigma) \tag{5.54}
\end{equation*}
$$

$$
\begin{align*}
J_{j}^{\beta}{ }_{j}(\sigma) H^{\alpha}\left(\sigma^{\prime}\right) & =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{0 m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}\right)^{2}}\right) X^{k}(\sigma)\right. \\
& +4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{0 m} \bar{h}_{j n} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{n}(\sigma) \\
& +4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{0 m} m_{j} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{0}(\sigma)  \tag{5.55}\\
& -4 \gamma^{\beta \xi} m_{j} \tau_{0}^{2}\left(\frac{e_{-}^{\alpha} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{e_{+}^{\alpha} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{k}(\sigma) \\
& -4 \epsilon m_{j} \tau_{0}\left(\frac{e_{-}^{\alpha} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{e_{+}^{\alpha} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{k}(\sigma)
\end{align*}
$$

where I have used the equations of motion for $\eta[28]$ on the final results for the OPEs in order to cancel out some of the unwanted singular terms that are proportional to $\ln \left(|\Delta \sigma|^{2}\right)$ :

$$
\begin{align*}
& 0=e_{-}^{\alpha} \partial_{\alpha} X^{m} \tau_{m}+e_{-}^{\alpha} \partial_{\alpha} \eta  \tag{5.56}\\
& 0=e_{+}^{\alpha} \partial_{\alpha} X^{m} \tau_{m}-e_{+}^{\alpha} \partial_{\alpha} \eta \tag{5.57}
\end{align*}
$$

And extensively make use the identities between the worldsheet metric and vielbein:

$$
\begin{align*}
\gamma^{\alpha \beta} & =e_{+}^{\alpha} e_{-}^{\beta}+e_{-}^{\alpha} e_{+}^{\beta}  \tag{5.58}\\
\epsilon^{\alpha \beta} & =e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta} \tag{5.59}
\end{align*}
$$

## 6 Outlook and discussion

With these results I conclude the thesis. There are several questions that can be probed using them.

The first and most obvious one is to actually perform the calculation for the stress-energy tensor as it would be interesting to see whether it is well defined and consistent with other results [28]. With the stress-energy tensor at hand one will be able to find the central charge of the theory and therefore its dimension directly by computing the OPE of the stress-energy tensor with itself in analogy with the theory of the relativistic bosonic string. This can be very easily checked as the dimension of the bosonic string embedded in a Newton-Cartan background is known to be $d=25$ predicted both by the dimensional reduction procedure and calculated explicitly using the Faddeev-Popov procedure [28]. This is an important step in ultimately finding whether the dynamics of this sector could be described correctly in terms of a current algebra.

As we saw there are several terms in the resulting OPEs that are not first and second order singular. At first this may look concerning, but there are other current algebras in which they arise [29]. There, however, the logarithmic and delta terms do not contribute when calculating the weights of the field, so it will be interesting to see if this is the case in for this particular theory. This procedure can then be done without enforcing the equations of motion for the $\eta$ field on the OPEs as it could be possible for the terms that cancel out on it to also not contribute when calculating the weights and stress-energy tensor.

Second all of the calculations that I have performed are without matter fields. The theory permits a Kalab-Ramond and dilaton terms with their own dynamics where they affect the equations of motion (5.56). It would be interesting to see if the correct form of the Bargamnn algebra is retrieved by including the matter fields. This in turn will result in modfied equations of motion [28] and additional terms will also appear in the final results for the OPEs. Then it has to be investigated whether the equations of motion need to be enforced again.

In conclusion I successfully derived the Bargmann algebra for the bosonic string embedded in Newton-Cartan background. This was done by first calculating the conserved charges that result from a Polyakov-type action without matter fields included. Then I calculated the operator product expansions of the conserved currents that can result as a basis for further research in the field.

## A Gauge algebra identity derivation

In this appendix the proof for the identity between a gauge field transformation and a general coordinate transformation (2.35) is laid out. This is a general identity for any gauge field that transforms under diffeomorphisms.

We start with a gauge field $B_{\mu}{ }^{a}$ on which a Lie algebra $\mathfrak{g}$ is realised:

$$
\begin{equation*}
\left[T_{a}, T_{b}\right]=f_{b c}{ }^{a} T_{c} \tag{A.1}
\end{equation*}
$$

where $T_{a}$ and $f_{b c}{ }^{a}$, which are anti-symmetric in its lower indices, are the elements and structure constants of $\mathfrak{g}$ respectively.

This gauge field has a space-time index and therefore transforms under diffeomorphisms:

$$
\begin{align*}
\delta(\xi)_{g c t} B_{\mu}{ }^{a} & =\mathcal{L}_{\xi} B_{\mu}{ }^{a}  \tag{A.2}\\
& =\xi^{\lambda} \partial_{\lambda} B_{\mu}{ }^{a}+B_{\mu}{ }^{a} \partial_{\lambda} \xi^{\lambda}  \tag{A.3}\\
& =\xi^{\lambda} \partial_{[\lambda} B_{\mu]}{ }^{a}+\partial_{\lambda}\left(\xi^{\lambda} B_{\mu}{ }^{a}\right)  \tag{A.4}\\
& =\xi^{\lambda}\left(\partial_{[\lambda} B_{\mu]}{ }^{a}+f_{b c}{ }^{a} B_{\lambda}{ }^{b} B_{\mu}{ }^{c}\right)+\partial_{\lambda}\left(\xi^{\lambda} B_{\mu}{ }^{a}\right)+f_{c b}{ }^{a} B_{\lambda}{ }^{b} \xi^{\lambda} B_{\mu}{ }^{c}  \tag{A.5}\\
& =\xi^{\lambda} R_{\lambda \mu}{ }^{a}-\sum_{\{c\}} \delta\left(\xi^{\lambda} B_{\lambda}{ }^{c}\right) B_{\mu}{ }^{a} \tag{A.6}
\end{align*}
$$

where $\mathcal{L}_{\xi}$ is the Lie derivative with respect to $\xi^{\lambda}$. To get from (A.3) to (A.4) add and subtract $\xi^{\lambda} \partial_{\mu} B_{\lambda}{ }^{a}$ and combine into the anti-symmetrised and total derivative terms respectively. On (A.5) add and subtract $f_{c b}{ }^{a} B_{\lambda}{ }^{b} \xi^{\lambda} B_{\mu}{ }^{c}$ and to get to (A.6) recognise the first term on (A.5) to be exactly the curvature for a non-abelian gauge field and for the second term make the ansatz:

$$
\begin{equation*}
\epsilon=\xi^{\lambda} B_{\lambda}{ }^{c} \tag{A.7}
\end{equation*}
$$

where $\epsilon$ is the gauge transformation parameter.

## B Galilean boost current derivation

Here I give the derivation for the Galilean boost current from the Lagrangian (5.14) as an example.

Taking the variation of the Lagrangian with respect to Galilean boosts where:

$$
\begin{equation*}
\delta X^{i}=\lambda^{i} X^{0} \quad \delta m_{n}=\delta_{i j} m^{i} e_{n}^{j} \tag{B.1}
\end{equation*}
$$

results in:

$$
\begin{align*}
\delta \mathcal{L} & =\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\left[\left(e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta}\right) \bar{h}_{m i} \partial_{\alpha} X^{m} \partial_{\beta}\left(\delta X^{i}\right)\right. \\
& \left.+\left[\left(e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta}\right) \tau_{0} \partial_{\alpha} X^{0}+\left(e_{+}^{\alpha} e_{-}^{\beta}+e_{-}^{\alpha} e_{+}^{\beta}\right) \partial_{\alpha} \eta\right]\left(\delta m_{i} \partial_{\beta} X^{i}-m_{i} \partial_{\beta}\left(\delta X^{i}\right)\right)\right]  \tag{B.2}\\
& =\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\left[\left(e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta}\right) \bar{h}_{m i} \partial_{\alpha} X^{m} \partial_{\beta} X^{0} \lambda^{i}\right. \\
& \left.+\left[\left(e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta}\right) \tau_{0} \partial_{\alpha} X^{0}+\left(e_{+}^{\alpha} e_{-}^{\beta}+e_{-}^{\alpha} e_{+}^{\beta}\right) \partial_{\alpha} \eta\right]\left(-\lambda^{i} \delta_{i j} \partial_{\beta} X^{j}-m_{i} \partial_{\beta} X^{0} \lambda^{i}\right)\right]  \tag{B.3}\\
& =\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}} \partial_{\beta}\left[-2 \gamma^{\alpha \beta} \bar{h}_{m i} \partial_{\alpha} X^{m} \partial_{\beta} X^{0}-2\left(\gamma^{\alpha \beta} \tau_{0} \partial_{\alpha} X^{0}+\epsilon^{\alpha \beta} \partial_{\alpha} \eta\right)\left(\delta_{i j} \partial_{\beta} X^{j}+m_{i} \partial_{\beta} X^{0}\right)\right. \\
& \left.+2\left(\gamma^{\alpha \beta} \tau_{0}+\epsilon^{\alpha \beta} \partial_{\alpha} \eta X^{j}\right)\right] \lambda^{i}  \tag{B.4}\\
& -\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}  \tag{B.5}\\
& -\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}  \tag{B.6}\\
\partial_{\beta} & {\left[-2 \gamma^{\alpha \beta} \bar{h}_{m i} \partial_{\alpha} X^{m}-2\left(\gamma^{\alpha \beta} \tau_{0}+\epsilon^{\alpha \beta} \tau_{0} \partial_{\alpha} X^{0}+\epsilon^{\alpha \beta} \partial_{\alpha} \eta\right)\left(\delta_{i j} \partial_{\beta} X^{j}+m_{i}\right)\right] \lambda^{i} X^{0} }
\end{align*}
$$

where I use the identities:

$$
\begin{align*}
-2 \gamma^{\alpha \beta} & =e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta}  \tag{B.7}\\
-2 \epsilon^{\alpha \beta} & =e_{+}^{\alpha} e_{-}^{\beta}-e_{-}^{\alpha} e_{+}^{\beta} \tag{B.8}
\end{align*}
$$

And the equation of motion for $X^{m}$ is given by:

$$
\begin{equation*}
\partial_{\alpha}\left[-2 \gamma^{\alpha \beta} \bar{h}_{m i} \partial_{\alpha} X^{m}+2 m_{i}\left(\gamma^{\alpha \beta} \tau_{n} \partial_{\alpha} X^{n}+\epsilon^{\alpha \beta} \partial_{\alpha} \eta\right)\right]=0 \tag{B.9}
\end{equation*}
$$

We see that the total derivative term (B.4) gives precicely the Noether current and (B.5), and (B.6) vanish on the equations of motion. The rest of the currents are derived the same way, but are more straightforward as the do not involve transforming any of the gauge fields.

## C Poisson brackets of the conserved charges

Here I calculate the partial derivatives of the conserved charges with respect to the fields $X^{m}$ and their canonical momenta $\Pi_{m}$ which I use to calculate the Poisson brackets.

$$
\begin{gather*}
\frac{\partial P_{j}}{\partial X^{m}}=0 \quad \frac{\partial P_{j}}{\partial \Pi_{m}}=\delta_{j}^{m}  \tag{C.1}\\
\frac{\partial H}{\partial X^{m}}=0 \quad \frac{\partial H}{\partial \Pi^{m}}=\delta_{m 0}  \tag{C.2}\\
\frac{\partial G_{i}}{\partial \Pi_{m}}=X^{0} \delta_{j}^{m}  \tag{C.3}\\
\frac{\partial G_{i}}{\partial X^{m}}=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{\alpha \beta} \bar{h}_{i n} \partial_{\beta} X^{n} \delta_{m 0}+2 \gamma^{\alpha \beta} \tau_{m} \partial_{\beta} X^{m}\left(\delta_{i m}-m_{i} \delta_{m}^{0}\right)\right. \\
\left.-2 \epsilon^{\alpha \beta} \partial_{\beta} \eta\left(\delta_{i m}-m_{i} \delta_{m}^{0}\right)\right]  \tag{C.4}\\
\frac{\partial J_{j}{ }^{i}}{\Pi_{m}}=X^{i} \delta_{j}^{m}  \tag{C.5}\\
\frac{\partial J_{j}{ }^{i}}{X^{m}}=\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\left[-2 \gamma^{0 \beta} \bar{h}_{j n} \partial_{\beta} X^{n} \delta_{m 0}-2\left(\gamma^{0 \beta} \tau_{n} \partial_{\beta} X^{n}-\epsilon^{0 \beta} \partial_{\beta} \eta\right) m_{j}\right] \delta_{m}{ }^{i} \tag{C.6}
\end{gather*}
$$

## D OPEs of the currents

Here I give two examples of the calculations that I have performed for the OPEs obtained in 5.4.2

$$
\begin{align*}
& G_{i}{ }^{\alpha}(\sigma) J^{\beta}{ }_{j}^{k}\left(\sigma^{\prime}\right) \\
& =\left(\frac{e}{4 \pi \alpha^{\prime}}\right)^{2}\left[\left(-2 \gamma^{\alpha \rho}\right) \bar{h}_{i m} \partial X^{m}(\sigma) X^{0}(\sigma)+2 \gamma^{\alpha \rho} \tau_{0} \partial_{\rho} X^{0}(\sigma)\left(\delta_{i j} X^{j}(\sigma)+m_{i} X^{0}(\sigma)\right)\right.  \tag{D.1}\\
& \left.+2 \epsilon^{\alpha \rho} \partial_{\rho} \eta\left(\delta_{i j} X^{J}(\sigma)+m_{i} X^{0}(\sigma)\right)\right]\left[-2 \gamma^{\beta \xi} \bar{h}_{j n} \partial_{\xi} X^{n}\left(\sigma^{\prime}\right)\right. \\
& \left.-2\left(\gamma^{\beta \xi} \tau_{0} \partial_{\xi} X^{0}\left(\sigma^{\prime}\right)-\epsilon^{\beta \xi} \partial_{\xi} \eta\left(\sigma^{\prime}\right)\right) m_{j}\right] X^{k}\left(\sigma^{\prime}\right)  \tag{D.2}\\
& =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{0}(\sigma) X^{k}(\sigma)\right.  \tag{D.3}\\
& +4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} X^{n}(\sigma) \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} m_{j} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{0}(\sigma) X^{0}(\sigma) \\
& -4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \bar{h}_{i m} m_{j} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} \eta(\sigma) X^{0}(\sigma) \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \delta_{i l} \bar{h}_{j n} \tau_{0} h^{l k} \partial_{\rho} X^{0}(\sigma) \partial_{\xi} X^{0}(\sigma) \ln \left(|\Delta \sigma|^{2}\right)-4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \delta_{i l} m_{j} \tau_{0} h^{l k} \partial_{p} X^{0}(\sigma) \partial_{\xi} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right) \\
& -4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{j n} \delta_{i l} \tau_{0} h^{l n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma) \\
& -4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \delta_{i l} m_{j} \tau_{0} h^{l k} \partial_{p} X^{0}(\sigma) \partial_{\xi} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right) \\
& -\epsilon^{\alpha \beta} \gamma^{\beta \xi} \delta_{i l} \bar{h}_{j n} h^{l n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} \eta(\sigma) X^{k}(\sigma) \\
& \left.-4 \epsilon^{\alpha \rho} \gamma^{\beta \xi} \delta_{i l} \bar{h}_{j n} h^{l k} \partial_{\rho} X^{0}(\sigma) \partial_{\xi} X^{0}(\sigma) \ln \left(|\Delta \sigma|^{2}\right)+4 \epsilon^{\alpha \beta} \epsilon^{\beta \xi} \delta_{i l} m_{j} h^{l k} \partial_{\xi} \eta(\sigma) \partial_{\rho} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right)\right] \tag{D.4}
\end{align*}
$$

$=\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} \Delta_{2}\left[4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n}\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) X^{0}(\sigma) X^{k}(\sigma)\right.$
$+4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} X^{n}(\sigma)$
$-4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} m_{j} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} X^{0}(\sigma) X^{0}(\sigma)$
$-4 \gamma^{\alpha \rho} \epsilon^{\beta \xi} \bar{h}_{i m} m_{j} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\xi} \eta(\sigma) X^{0}(\sigma)$
$-4 \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{j n} \delta_{i l} \tau_{0} h^{l n}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma)$
$-\epsilon^{\alpha \beta} \gamma^{\beta \xi} \delta_{i l} \bar{h}_{j n} h^{\ln }\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) \partial_{\rho} \eta(\sigma) X^{k}(\sigma)$
$\left.+4 \epsilon^{\alpha \beta} \epsilon^{\beta \xi} \delta_{i l} m_{j} h^{l k} \partial_{\xi} \eta(\sigma) \partial_{\rho} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right)\right]$

$$
\begin{align*}
& G^{\alpha}{ }_{i}(\sigma) G^{\beta}{ }_{j}\left(\sigma^{\prime}\right)=  \tag{D.6}\\
& =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2}\left[-2 \gamma^{\alpha \rho} \bar{h}_{i m} \partial_{\rho} X^{m}(\sigma) X^{0}(\sigma)+2 \gamma^{\alpha \rho} \tau_{0} \partial_{\rho} X^{0}(\sigma)\left(\delta_{i k} X^{k}(\sigma)+m_{i} X^{0}(\sigma)\right)\right. \\
& \left.+2 \epsilon^{\alpha \rho} \partial_{\rho} \eta\left(\delta_{i k} X^{k}(\sigma)+m_{i} X^{0}(\sigma)\right)\right]\left[-2 \gamma^{\beta \xi} \bar{h}_{j n} \partial_{\xi} X^{n}\left(\sigma^{\prime}\right) X^{0}\left(\sigma^{\prime}\right)\right. \\
& \left.+2 \gamma^{\beta \xi} \tau_{0} \partial_{\xi} X^{0}\left(\sigma^{\prime}\right)\left(\delta_{j l} X^{l}\left(\sigma^{\prime}\right)-m_{j} X^{0}\left(\sigma^{\prime}\right)\right)+2 \epsilon^{\beta \xi} \partial_{\xi} \eta\left(\delta_{j l} X^{l}\left(\sigma^{\prime}\right)+m_{j} X^{0}\left(\sigma^{\prime}\right)\right)\right] \\
& =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} 4\left[\gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} \partial_{\rho} X^{m}(\sigma) X^{0}(\sigma) \partial_{\xi} X^{n}\left(\sigma^{\prime}\right) X^{0}\left(\sigma^{\prime}\right)\right.  \tag{D.7}\\
& -\gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \tau_{0} \delta_{j k} \partial_{\rho} X^{m}(\sigma) X^{0}(\sigma) \partial_{\xi} X^{0}\left(\sigma^{\prime}\right) X^{k}\left(\sigma^{\prime}\right)-\gamma^{\alpha \rho} \epsilon^{\beta \xi} \bar{h}_{i m} \delta_{j k} \partial_{\rho} \overleftarrow{X}^{m}(\sigma) X^{0}(\sigma) \partial_{\xi} \eta\left(\sigma^{\prime}\right) X^{k}\left(\sigma^{\prime}\right) \\
& -\gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{j m} \delta_{i k} \tau_{0} \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma) \partial_{\xi} X^{m}\left(\sigma^{\prime}\right) X^{0}\left(\sigma^{\prime}\right)+\gamma^{\alpha \rho} \gamma^{\beta \xi} \tau_{0} \tau_{0} \delta_{i k} \delta_{j l} \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma) \partial_{\xi} X^{0}\left(\sigma^{\prime}\right) X^{l}\left(\sigma^{\prime}\right) \\
& +\gamma^{\alpha \rho} \epsilon^{\beta \xi} \tau_{0} \delta_{i k} \delta_{j l} \partial_{\rho} X^{0}(\sigma) X^{k}(\sigma) \partial_{\xi} \eta\left(\sigma^{\prime}\right) X^{l}\left(\sigma^{\prime}\right)-\epsilon^{\alpha \rho} \gamma^{\beta \xi} \delta_{i j} \bar{h}_{j m} \partial_{\rho} \eta(\sigma) X^{k}(\sigma) \partial_{\xi} X^{m}\left(\sigma^{\prime}\right) X^{0}\left(\sigma^{\prime}\right) \\
& + \epsilon ^ { \alpha \rho } \gamma ^ { \beta \xi } \delta _ { i k } \delta _ { j l } \tau _ { 0 } \partial _ { \rho } \eta ( \sigma ) \longdiv { X ^ { k } ( \sigma ) \partial _ { \xi } X ^ { 0 } ( \sigma ^ { \prime } ) X ^ { l } ( \sigma ^ { \prime } ) + \epsilon ^ { \alpha \rho } \epsilon ^ { \beta \xi } \delta _ { i k } \delta _ { j l } \partial _ { \rho } \eta ( \sigma ) \longdiv { X ^ { k } ( \sigma ) \partial _ { \xi } \eta ( \sigma ^ { \prime } ) X ^ { l } ( \sigma ^ { \prime } ) ] } ] . ] ~ ] . ~ } \\
& =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} 4 \Delta_{2}\left[\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) X^{0}(\sigma)\right.  \tag{D.8}\\
& +\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}} \partial_{+} X^{0}(\sigma)+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}} \partial_{-} X^{0}(\sigma)\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) \\
& -\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} \tau_{0} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} X^{0}(\sigma) \\
& -\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} \tau_{0} h^{m k}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\rho} X^{0}(\sigma) \\
& -\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}-e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} \eta(\sigma) \\
& -\left(e_{+}^{\alpha} e_{-}^{\rho}-e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} h^{m k}\left(\frac{\delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\rho} \eta(\sigma) \\
& +\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \tau_{0} \tau_{0} \delta_{i k} \delta_{j l} h^{k l} \partial_{\rho} X^{0}(\sigma) \partial_{\xi} X^{0}\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right) \\
& +\left(e_{+}^{\alpha} e_{-}^{\rho}-e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}-e_{-}^{\beta} e_{+}^{\xi}\right) \delta_{i k} \delta_{j l} h^{k l} \partial_{\rho} \eta(\sigma) \partial_{\xi} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right) \\
& +\left(e_{+}^{\alpha} e_{-}^{\rho}-e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \delta_{i k} \delta_{j l} \tau_{0} h^{k l} \partial_{\rho} \eta(\sigma) \partial_{\xi} X^{0}\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right) \\
& \left.+\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}-e_{-}^{\beta} e_{+}^{\xi}\right) \delta_{j k} \delta_{i l} \tau_{0} h^{k l} \partial_{\xi} \eta(\sigma) \partial_{\rho} X^{0}\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right)\right]
\end{align*}
$$

$$
\begin{align*}
& =\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} 4 \Delta_{2}\left[\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) X^{0}(\sigma)\right.  \tag{D.9}\\
& +\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}} \partial_{+} X^{0}(\sigma)+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}} \partial_{-} X^{0}(\sigma)\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) \\
& -\left(e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(-e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} h^{m k}\left(\frac{\delta_{\rho}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}}+\frac{\delta_{\rho}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\xi} \eta(\sigma) \\
& -\left(-e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}+e_{-}^{\beta} e_{+}^{\xi}\right) \bar{h}_{i m} \delta_{j k} h^{m k}\left(\frac{\delta_{\xi}^{+}}{\left.\frac{\sigma_{+}-\sigma_{+}^{\prime}}{}+\frac{\delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}}\right) X^{0}(\sigma) \partial_{\rho} \eta(\sigma), ~\left(\sigma^{\prime}\right)}\right.
\end{align*}
$$

$$
\begin{align*}
& +\left(-e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(-e_{+}^{\beta} e_{-}^{\xi}+e^{\beta} e_{+}^{\xi}\right) \delta_{i k} \delta_{j l} h^{k l} \partial_{\rho} X^{0}(\sigma) \partial_{\xi} X^{0}\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right) \\
& +\overline{\left(e_{+}^{\alpha} e_{-}^{\rho}-e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}-e^{\beta} e_{+}^{\xi}\right) \delta_{i k} \delta_{j l} h^{k l} \partial_{\rho} \eta(\sigma) \partial_{\xi} \eta(\sigma) \ln \left(|\Delta \sigma|^{2}\right)} \\
& +\left(e_{+}^{\alpha} e_{-}^{\rho}-e_{-}^{\alpha} e_{+}^{\rho}\right)\left(-e_{+}^{\beta} e_{-}^{\xi}+e^{\beta} e_{+}^{\xi}\right) \delta_{i k} \delta_{j l} \tau_{0} h^{k l} \partial_{\rho} \eta(\sigma) \partial_{\xi} \eta\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right) \\
& +\left(-e_{+}^{\alpha} e_{-}^{\rho}+e_{-}^{\alpha} e_{+}^{\rho}\right)\left(e_{+}^{\beta} e_{-}^{\xi}-e^{\beta} e_{+}^{\xi}\right) \delta_{j k} \delta_{i l} h^{k l} \partial_{\xi} \eta(\sigma) \partial_{\rho} \eta\left(\sigma^{\prime}\right) \ln \left(|\Delta \sigma|^{2}\right) \\
& \overline{=\left(\frac{\sqrt{-\gamma}}{4 \pi \alpha^{\prime}}\right)^{2} 4 \Delta_{2}\left[\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\left(\sigma_{+}-\sigma_{+}^{\prime}\right)^{2}}+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\left(\sigma_{-}-\sigma_{-}^{\prime}\right)^{2}}\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma) X^{0}(\sigma), ~(\sigma)\right.}  \tag{D.10}\\
& \left.+\left(\frac{\delta_{\rho}^{+} \delta_{\xi}^{+}}{\sigma_{+}-\sigma_{+}^{\prime}} \partial_{+} X^{0}(\sigma)+\frac{\delta_{\rho}^{-} \delta_{\xi}^{-}}{\sigma_{-}-\sigma_{-}^{\prime}} \partial_{-} X^{0}(\sigma)\right) \gamma^{\alpha \rho} \gamma^{\beta \xi} \bar{h}_{i m} \bar{h}_{j n} h^{m n} X^{0}(\sigma)\right]
\end{align*}
$$

Where on (B.7) only the non-zero contractions are shown; to get from (B.8) to (B.9) the equations of motion (5.56) are used and finally from (B.9) to (B.10) the terms from the third on cancel out. The rest of the OPEs are more straightforward and do not require use of the equations of motion.

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