Symmetries of String Theory and Newton Cartan Geometry

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ABSTRACT

The string theory has some beautiful properties, which have bewildered researchers since it's inception. One of such properties is the presence of a duality symmetry, which relates large objects to very small ones. The duality is also called topological *or* T Duality. T Duality informally says, strings can not differentiate between objects of radii R and that of Radii $1/R$. This allows strings to wind the same way around the two said objects, leading to the same winding mode quantum numbers. The same modes imply same observables for both cases, which are otherwise very different. A symmetry of this nature is obscure in other theories in Physics. There has been extensive research to explore this duality as an explicit symmetry of a field theory. One such attempt is the Double Field Theory. Turning towards another area of research in the subject, these rather complex formulations of double field theory contain the non relativistic Newton Cartan string, which sounds unlikely, but holds true. The Newton Cartan string theory was first written to discover simpler physics, but the surprising nature of string theory brings us to this junction. The thesis begins on the intersection of the two, and we proceed as follows.

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Chapter One

Inception of String Theory

1.1 History

At the turn of the nineteenth century, Lord Kelvin made some rather bold statements that would not go down well with the history of physics in the years and decades to follow. Kelvin is rumored to have said, physics is largely solved, and what remains are two clouds. He was referring to the failure of Michelson Morley experiment, and blackbody radiation. The two ambiguities he implied towards, were discovered by the researchers in early twentieth century. They became the two cornerstones of science of the century - firstly quantum mechanics and second, special relativity. Inside a decade or two, the paradigm of physics and research in physics shifted into a new direction. The next few decades were spent on studying and building quantum mechanics extensively. The quest to explain blackbody radiation led to the postulates of a quantum theory. The foundations of quantum mechanics were established during the first half of the twentieth century by Max Planck, Niels Bohr, Schrödinger, and others. In 1905, while working at a swiss patent office, Einstein published his research about special relativity. Special relativity as well, on the other hand gave rise to general relativity. Both these theories - quantum theory and relativity, were successful in not just describing existing phenomena in greater detail, but also predicting new phenomena and leading to new discoveries. But there was an underlying inconsistency- the two theories were not compatible with each other. In the pursuit of a theory of quantised gravity, in early 1970s, a new theory came into existence - **String theory**.

The string theory is one of the most popular theories and a front line contender for a modern scientist's approach to a theory of everything. It is a theory that aims at unification of all fundamental forces, gravity, electromagnetism and weak and strong interactions. The inception of string theory was in late 1960s, when physicists made attempts to explain the $SU(3)$ or strong interaction forces of nature. Even though the approach did not succeed in explaining the strong forces, the theory became an accepted topic for discussion and research, as it consisted another very promising feature - string theory could reconcile gravity with quantum physics. String theory has extra dimensions that are needed to achieve mathematical consistency. For example, bosonic string theory is described with 26 dimensions, and the superstring theory is described with 10 dimensions.

1.2 Fundamentals of String Theory

One of the most fundamental differences between any particle theory and string theory is, strings are linear objects. A particle sweeps out a worldline on Minkowski space. Whereas, a string sweeps out a worldsheet. The background of Minkowski spacetime is referred to as target space. String theory describes the propagation of this worldsheet on the target space. The length scale of string theory, as determined by other fundamental scales is referred to as the Planck length. At distances larger than the Planck length, a string behaves like an ordinary particle whose properties are determined by the vibrations of the inherent string. Since the theory is laden with extra dimensions, it becomes important to offer an interpretation of the same. Compactification is one of such ways to explain the presence of extra dimensions. The assumption is, some of the existing dimensions correspond to some form of gauging which results in them curling upon themselves, like circles. In a theory where the compact dimensions become very small, say, comparable to the least possible length scales, one can effectively ignore their presence. For a relativistic bosonic string, to be quantised in a flat Minkowski background, we have 26 dimensions. Upon addition of fermionic modes to the theory, one arrives at the so- called superstring theory. In the superstring theory, the number of critical dimensions is limited to 10. The lowest state of superstring theory is the massless state. This contains among many, three massless fields - $G, B \text{and} \phi$, a topic which is discussed in fair light later in the second chapter. The choice of adding fermions results in five independent classifications of the theory, namely Type I, Type II A, Type II B, Heterotic $SO(32)$,and Heterotic $E8 \times E8$. These different classes of string theory were believed to be independent for a long time, until it was realised they are linked via two special, non trivial dualities. A duality refers to a symmetry where two apparently different systems show equivalence. The two theories can be moulded into each other upon mathematical transformations. These 5 theories are found to be connected using S and T duality. We explore T duality in the next chapter, and make an attempt to make this non trivial duality manifest.

Chapter Two

Introduction to Newton Cartan

2.1 NC Geometry

What is NC geometry and why is it important?

Einstein discovered the general theory of relativity in the beginning of the nineteenth century. This discovery led to a new understanding of space and time, and opened doors to many new concepts and questions, which are being answered to this day. A decade before general relativity, Einstein had shattered common beliefs of absolute nature of time, which used to be a common point of agreement of philosophers and scientists alike. The theory of General Relativity established gravity as a property of spacetime, and showed that spacetime is curved. It was the first frame independent formulation of gravity. General Relativity was verified in multiple observations and paved way for new horizons of research. General Relativity successfully not only extends the applicability of gravity, but also reduces to Newtonian physics in appropriate limits. The same indicates that Newtonian physics should not be considered wrong, but just falls short under certain regimes of study. This allows for a reformulation of Einstein's gravity under Newtonian limit, *i.e* sending the speed of light in vacuum to infinity. The covariant, or frame independent reformulation of Newtonian gravity was done by Cartan in $1920s$[6]. He showed the way to use differential geometry to rewrite Newtonian gravity, and obtained covariant equations of motion for the theory. The Minkowski spacetime in relativistic theories uses Riemannian geometry Poincaré symmetries to describe the theory. In comparison, the tangent space on Newton Cartan spacetime makes use of Galilean symmetries instead.

2.1.1 Motivation for this thesis

The motivation to study the Newton Cartan geometry is threefold.

- The NC gravity might serve as a way to study quantum gravity. For years, the quest of a modern physicist has been to understand and unite all fundamental forces of nature. At current understanding, it remains to make quantum physics consistent with General Relativity. General Relativity, as it is, is not very well defined at the Planck scale. Due to this, in the quantum regime, the quantum fluctuations of spacetime play a dominant role. A classical theory can be quantised following the rules of renormalisation techniques under certain conditions. For a theory to have a quantum version, it is required that it should be asymptotically free i, e it should be defined by a finite number of parameters. Gravity, upon quantisation tends to have infinitely many parameters, or coefficients in its perturbative renormalisation. This is a reason why relativity can not be quantised like other theories. Lately, Newton Cartan gravity has been developed in string theory. Embedding NC gravity in string backgrounds, which makes Newton Cartan gravity one of the potential candidates to study non relativistic quantum gravity. Using this, it might be insightful to study the full relativistic quantum gravity.

- NC gravity finds applications in low energy quantum field theories. In areas of condensed matter physics, very often relativistic gravity does not yield useful applications. The reason behind this is, at low energies, systems exhibit Galilean symmetries. This allows Newton Cartan gravity to offer a better explanation of things compared to Einstein's gravity. To study and calculate partition functions, or say, the energy momentum tensor, a coupling with a metric is required. Since, the gravity is non relativistic, and hence a Newton Cartan metric can be useful in explaining the underlying physics.

- Upon its inception in 1916, General Relativity was put to three classical tests[5] proposed by Einstein. Some of these tests were based on phenomena unexplained earlier

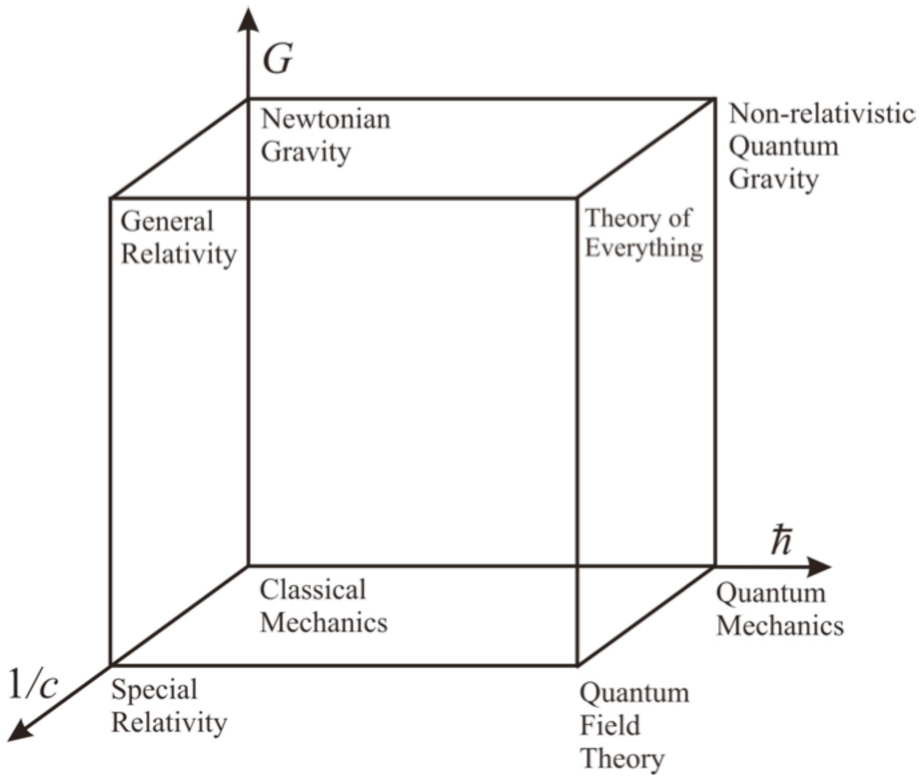

Figure 2.1 A cube representing the basic physical theories, and the various pathways to arrive at quantum gravity[19]. The cube is parametrised by fundamental constants along its axes. These constants represent the theory they first characterised. One can look starting at the origin, there are three fundamental theories -SR, Quantum physics, and Newtonian gravity. by reaching at the corner trying to couple the theories, one can move towards Quantum gravity. *Eg.* special relativity and quantum mechanics give rise to quantum field theory, as so on.

- The perihelion precession of the orbit of Mercury,
- Deflection of light due to the gravitational field of sun, and
- Gravitational red-shift of light.

Upon the covariant reformulation, Newton Cartan gravity satisfies all of the aforementioned tests. It explains all things similar to Einsteins gravity, and just falls short of predicting the gravitational waves. This further establishes Newton Cartan gravity as a low energy limit of General relativity.

In this thesis, we propose an action for the Newton Cartan theory, under various parametrisations.

Chapter Three

Double Field Theory

3.1 T Duality

One of the many things that sets it apart from other theories is the presence of a duality symmetry, aka the T duality[23]. String theory suggests compactification of dimensions as represented in the following figure. Following from compactification along a given direction, the worldsheet curls up to a cylindrical form. Let's take the radius of the cylinder to be R. T duality postulates, a circle of radius R has indistinguishable physics from one of radius $1/R$. Their equivalence is studied as an operator map between the two theories, while respecting all commutation relations. To study T duality on string theory, it is needed to look at what this duality means for a two dimensional theory, where one dimension is curled. Let the curled co-ordinate be given by x, such that, $x \sim x + 2\pi R$ and let y represent the other coordinate, here, which is the length of this cylinder. This cylinder is shown in the figure. We investigate various possible strings, starting with open strings. String (a) is the simplest case,

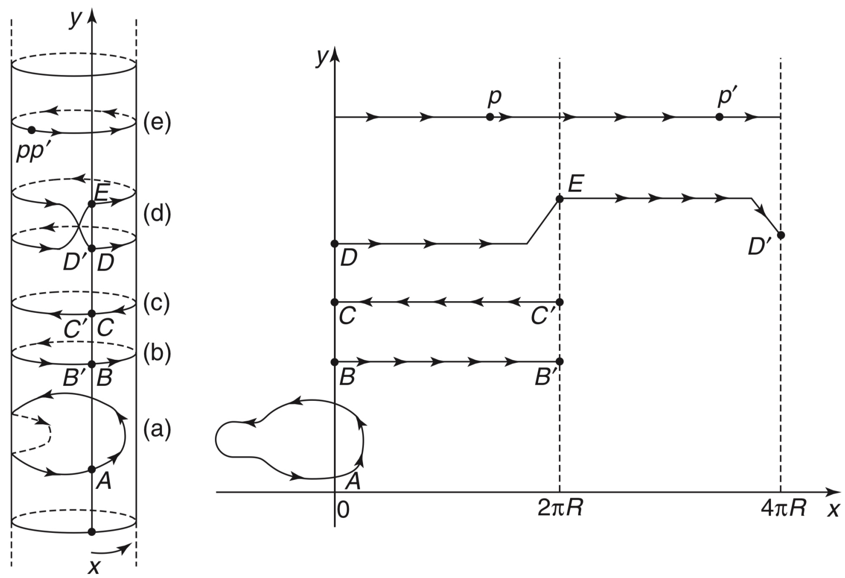

Figure 3.1 T Duality depicted by various closed strings on cylinder on the left, and same represented on the surface of cylinder [23]

which does not wrap around the cylinder. Its winding number is zero. The second is (b), which winds around the worldsheet once, and its winding number is $+1$. The winding number of (c) is -1 as it winds around the worldsheet once, exactly like (b), but in opposite direction. . The string (d) is wrapped around twice, and so is (e) The winding number for these strings is 2. Taking a look at the the mass-spectrum for a closed string on a circle of radius R

$$M^2 = (N + \tilde{N} - 2) + p^2 \frac{l_s^2}{R^2} + \tilde{p}^2 \frac{l_s^2}{\tilde{R}^2} \tag{3.1}$$

where l_s is the string length and $\tilde{R}$ is the radius dual to R, given by $\tilde{R} = \frac{l_s^2}{R}$ The mass spectrum is invariant under the transformations

$$\frac{R}{l_s} \leftrightarrow \frac{\tilde{R}}{l_s} = \frac{l_s}{R}, \quad p \leftrightarrow \tilde{p} \tag{3.2}$$

. This symmetry holds for any observable in DFT.

3.2 Early work on DFT

Double Field Theory was first discussed as an extension of string field theory. In a closed string field theory, the string field depends on all zero- modes and so, can be expanded to give an infinite set of fields on a doubled torus.

What is String Field Theory? It is a gauge invariant formulation of string dynamics, around any background. T duality is realised as an explicit field symmetry of this theory [16]. Let us consider a string theory described in a space with D dimensions. The spacetime has both compactified and non compactified dimensions. Let x^μ represent non compactified and x^a represent compactified dimensions. Together, they are represented as as, $x^i = x^a + x^\mu$, $i = 0, 1...D-1$. A basic difference between a string and a particle from field theories is, a string is itself the fundamental building block, unlike fields that give rise to particles upon quantisation in field theories. A string has the capability to wind around objects and dimensions. Let the compactified dimension has a radius given by R. Under the exchange of $R \rightarrow \frac{1}{R}$, this winding number exchanges in value with the value of momentum. If a string has momentum p and winding number w, for a radius R, then it acquires a momentum w and winding number$p$ for compactification radius $\frac{1}{R}$. This establishes winding should be treated as a new quantum number of the string. The winding numbers must also associated with a set of coordinates under quantisation, and this results in a second set of dual to the compact coordinates, given by $\tilde{x}_a$. The momentum associated with the compact coordinates x^a, is given by given by p_a. For winding, the winding modes associated with x_a are represented by w^a. This new set of coordinates, which arise due to winding modes, is also referred to as the new doubled set of coordinates. The non compact set of coordinates is also ethically doubled, to give rise to $\tilde{x}_\mu$. This ensures that all initial coordinates are doubled under the Doubled Field approach. One can think of non compact coordinates as compactifications with infinitely large radii. When the radius of compactification is extremely large, the dual coordinates do not play any role. Thus, coordinates dual to non compact modes can be dropped with ease. These coordinates do not play any part and do not correspond to any significance. Thus, any physical field ϕ in Double Field Theory can be sufficiently described as $\phi(x^a, \tilde{x}_a, x^\mu)$.

3.2.1 Construction of a generalised geometry

String theory depicts multiple vibrating modes for the string. These various vibrating modes lead to different physics. In the first excited state, string theory can be broken down to three irreducible representations, or three massless fields such that the string oscillations can be identified with these fields. The fields are, $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. The first field, $G_{\mu\nu}$ corresponds to a massless spin 2 particle - which results in gravity. The second, $B_{\mu\nu}$ is an anti symmetric field, also called Kalb Ramond field. ϕ represents a scalar field, called dilaton. Thus, the massless sector of string theory is parameterised by these three fields. For simplicity, the massless modes are considered in double field theory. So, the fields for consideration on DFT are $G_{\mu\nu}(x^a, x_\mu)$, $B_{\mu\nu}(x^a, x_\mu)$ and $\phi(x^a, x_\mu)$, where they depend on both compactified and non compactified coordinates. It is a common practice to use conformal gauge[20] for simpler results. The string theory low energy action in conformal gauge is given by [20]

$$S = \frac{1}{4\pi\alpha\prime}\int d^2\sigma G_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu \tag{3.3}$$

The DFT action consists of an integration over the dual fields x_a as well -

$$S = \int dx^a d\tilde{x}_a dx^\mu \mathcal{L}(x^a, \tilde{x}_a, x^\mu) \tag{3.4}$$

It can be given as,

$$S = -\frac{1}{4\pi}\int d^2\sigma \left(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\right) \tag{3.5}$$

where,

$$\eta^{\alpha\beta} = \text{diag}(-1, 1), \qquad \epsilon^{01} = -1, \qquad \partial_\alpha = (\partial_\tau, \partial_\sigma) \tag{3.6}$$

$$X^i = (X^a, X^\mu) \qquad X^a \sim X^a + 2\pi, \qquad i = 0, \ldots, D-1 \tag{3.7}$$

The matrices for G_{ij}, B_{ij} span over both compactified and non compactified dimensions -

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad G^{ij}G_{jk} = \delta^i_k$$

The fields do not simply retain the transformation properties of from string theory. The reason is introduction of new coordinates. The spacetime is now expressed in doubled $2D$ dimensions, instead of D dimensions. The fields G_{ij}, B_{ij} retain transformation properties of only D out of the $2D$ dimensions, depending only the usual coordinates, or depending only on dual coordinates or a mix of both. The sensible thing to demand is, the transformation of fields under doubled coordinates. One can simply think of a generalised metric arising from the simple combinations of G_{ij} and B_{ij}. One such simple combination is,

$$E_{ij} = G_{ij} + B_{ij} = \begin{pmatrix} \hat{E}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad \text{with} \quad \hat{E}_{ab} = \hat{G}_{ab} + \hat{B}_{ab} \tag{3.8}$$

But this combination fails to serve the desired transformation properties. A generalised metric is needed, which preserves the usual transformation properties of string theory fields. The choice of a generalised metric is not obvious.

3.2.2 Metric and Lie derivatives

The generalised metric in DFT is given as, $\mathcal{H}^{\mathcal{MN}}$ $M, N... = 1. =, 2, ..., 2D$, which combines the metric tensor and Kalb-Ramond fields as follows (D is the number of dimensions)

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \tag{3.9}$$

The Double Field Theory has two local symmetries from string theory -

- Diffeomorphisms, parameterised by $\xi^i \in T(M)$, or the tangent bundles in the manifold $\mathcal{M}$

- Gauge transformations of b_{ij}, which are given by one forms, $\tilde{\xi}_i \in T^*(M)$ or dual to the tangent bundles in $\mathcal{M}$

In an attempt to formulate a generalised geometry, both the vectors and the one form are treated at the same footing, so that it makes sense to add them to an object living in a space which is the sum of a tangent and its dual. This results in,

$$\xi + \tilde{\xi} \in T(M) \oplus T^*(M) \tag{3.10}$$

The generalised metric is an $O(D,D)$ non degenerate, as each of its components is non degenerate. An $O(D,D)$ invariant metric, η is used to raise and lower $O(D,D)$ indices. It is defined with constant off-diagonal indices-

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It can be easily checked that $\eta\mathcal{H}\eta = \mathcal{H}^{-1}$. Th identify it as a metric, the following index convention is used -

$$\mathcal{H} \leftrightarrow \mathcal{H}^{MN}$$
$$\mathcal{H}^{-1} \leftrightarrow \mathcal{H}_{MN}$$

where the $O(D,D)$ indices M, N run over $2D$ values.

3.2.3 Strong Constraint

Since the metric now is a $2D \times 2D$ matrix, constructed out of two $D \times D$ matrices, constraints need to be imposed to make the degrees of freedom consistent. The constraint is called "Strong Constraint" or "Section Condition". In string theory, any given state must satisfy a condition- all states must be annihilated by $L_0 - \tilde{L}_0$ operator, where $L_0 + \tilde{L}_0$ is the canonical Hamiltonian of string theory. The Hamiltonian constraint sets it to zero, and thus, one arrives at $L_0 = \tilde{L}_0$. This condition, also called the level matching condition, gives rise to equal contributions to mass from left and right moving modes. The level matching condition can be reformulated in Double field theory to obtain the strong constraint mentioned above. Strong constraint is expressed as

$$\partial_A \partial^A = 0 \tag{3.11}$$

The third and last field of mass-less mode of string theory, the scalar dilaton ϕ. The dilaton is by demand, invariant under both gauge transformations and diffeomorphisms on both standard coordinates and dual coordinates. There is no linear relation of dilaton that satisfies the property for both set of coordinates. Non- linearly, the dilaton is generalised as the double field dilaton, d and can be expressed as

$$e^{-2d} = e^{-2\phi}\sqrt{-g} \tag{3.12}$$

It can also be understood in the following manner. The coordinates on a doubled spacetime do not represent the actual physical points in a *onetoone* correspondence. The reason for this is a symmetry, called coordinate gauge symmetry observed on such spacetimes. A physical point should be *onetoone* identified with a gauge orbit in a coordinate spacetime. The symmetry can be represented as $x^A \sim x^A + \phi\partial^A\xi$ for any two arbitrary DFT fields ϕ and xi. This symmetry also holds for the diffeomorphism parameters $V^A \sim V^A + \phi\partial^A\xi$. It implies there are more than one transformation rules for the same diffeomorphism[17]. The Coordinate gauge symmetry is also an equivalence to the strong constraint in DFT.

3.2.4 Deriving the metric in DFT

The generalised DFT metric is made up of four components, as shown in (3.9). The components are generalised objects, arising from mass-less string theory sector. In it's most general form, the metric is given by[17],

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y_i^{\mu}X_{\lambda}^{i} - \bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\lambda}^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu} + X_{\kappa}^{i}Y_i^{\nu} - \bar{X}_{\kappa}^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^{\nu} & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X_{(\kappa}^{i}B_{\lambda)\rho}Y_i^{\rho} - 2\bar{X}_{(\kappa}^{\bar{\imath}}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^{\rho} \end{pmatrix} \tag{3.13}$$

This further can be formulated to be,

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} H & Y_i\left(X^i\right)^T - \bar{Y}_i\left(\bar{X}^i\right)^T \\ X^i\left(Y_i\right)^T - \bar{X}^i\left(\bar{Y}_i\right)^T & K \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} \tag{3.14}$$

Where, the kernels of H and K are spanned by $\left\{X^i_\mu, \bar{X}^i_\nu\right\}$ and $\left\{Y^\mu_j, \bar{Y}^\nu_j\right\}$ respectively.

$$H^{\mu\nu}X^i_\nu = 0, \quad H^{\mu\nu}\bar{X}^{\bar{\imath}}_\nu = 0; \quad K_{\mu\nu}Y^\nu_j = 0, \quad K_{\mu\nu}\bar{Y}^\nu_{\bar{\jmath}} = 0$$

H, K and Z are porposed solutions to the generalised metric $\mathcal{H}$ under certain constraints designated by the theory. They generate the metric in (3.9).

3.2.5 Buscher Rules

Buscher rules describe how the metric g and other fields change under the application of T -duality. When the theory has a $U(1)$ isometry, the coordinates can be split into isometric and non isometric. It is possible to then rewrite the theory in a form invariant to the isometry, by gauging the symmetry. following from the symmetry property, the theory now appears in its dual form, and the transformation can be effectively shown by Buscher Rules. They are used to perform along a given direction. They are given in terms of transformations of the metric. These rules are derived using the equation of motion and the Bianchi identities. Let us consider a spacetime metric g_{ij} and a two form b_{ij}. for the kth position along a diagonal matrix[1],

$$g_{kk} \rightarrow \frac{1}{g_{kk}}, \quad g_{ki} \rightarrow \frac{b_{ki}}{g_{kk}}, \quad g_{ij} \rightarrow g_{ij} - \frac{g_{ki}g_{kj} - b_{ki}b_{kj}}{g_{kk}}$$

$$b_{ki} \rightarrow \frac{g_{ki}}{g_{kk}}, \quad b_{ij} \rightarrow b_{ij} - \frac{g_{ki}b_{kj} - b_{ki}g_{kj}}{g_{kk}}$$

3.3 Deriving the geometry

3.3.1 Connection and Riemann Tensor

The generalised metric, made up of $B_{\mu\nu}$ field, and the metric $g_{\mu\nu}$ is given by,

$$\mathcal{H}_{MN} = \begin{bmatrix} g^{ij} & -g^{ik}b_{kl} \\ \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{ij} \end{bmatrix} \tag{3.15}$$

where the capital indices run over $2D$ fundamental $O(D,D)$indices, and small indices over$D$ spacetime dimensions. $\mathcal{H}_{MN}$ satisfies

$$\mathcal{H}^{MK}\mathcal{H}_{KN} = \delta^M_N, \qquad \mathcal{H}^{KN} = \eta^{MK}\eta^{NL}\mathcal{H}_{KL}, \tag{3.16}$$

The connection and rest of DFT geometry is derived starting from defining covariant derivatives. Generalised $O(D,D)$ tensors transform under diffeomorphisms and gauge symmetries via generalised Lie derivatives. Generalised Lie derivative is defined on a vector as[12],

$$\delta_\xi A^M = \widehat{\mathcal{L}}_\xi A^M \equiv \xi^N \partial_N A^M + \left(\partial^M \xi_N - \partial_N \xi^M\right) A^N$$
$$\delta_\xi A_M = \widehat{\mathcal{L}}_\xi A_M \equiv \xi^N \partial_N A_M + \left(\partial_M \xi^N - \partial^N \xi_M\right) A_N \tag{3.17}$$

On defining a covariant derivative, the demand is that covariant derivative transforms as a generalised tensor.

$$\nabla_M A_N \equiv \partial_M A_N - \Gamma^K_{MN} A_K \tag{3.18}$$

The second term consists the connection Γ, which is not generalised tensor. The Riemann tensor for DFT is defined as

$$[\nabla_M, \nabla_N] A_K = -R_{MNK}{}^L A_L - T_{MN}{}^L \nabla_L A_K \tag{3.19}$$

Where,

$$R_{MNK}{}^L = \partial_M \Gamma^L_{NK} - \partial_N \Gamma^L_{MK} + \Gamma^L_{MQ}\Gamma^Q_{NK} - \Gamma^L_{NQ}\Gamma^Q_{MK}$$

$$T^K_{MN} = 2\Gamma^K_{[MN]}$$

$R_{MNK}{}^L$ is anti symmetric in the last two indices. Under generalised lie derivatives, the Riemann tensor R or the torsion, T defined above do not transform as a tensor. Hence, a generalised Riemann tensor is defined to suit the transformation properties. The generalised Riemann tensor is,

$$\mathcal{R}_{MNKL} \equiv R_{MNKL} + R_{KLMN} + \Gamma_{QMN}\Gamma^Q{}_{KL} \tag{3.20}$$

Constraints on the connection

To derive the connection in terms of physical fields, it should satisfy the following constraints[12]

- η_{MN} and $\mathcal{H}_{MN}$ are covariantly constant.
- The partial derivatives in definition of Lie derivative should be replaced with covariant derivative.

- The generalised Lie derivatives should stay the same under a change from $\partial \to \nabla$.

These constraints result in symmetry properties on the connections. For example, the covariant constancy makes the connection anti symmetric in its last two indices, and also imposes Bianchi identities. To understand the implication of these constraints, we digress to look at projectors[13]. Since the relation

$$\mathcal{H}\eta\mathcal{H} = \eta^{-1} \quad \Leftrightarrow \quad (\mathcal{H}\eta)^2 = (\eta\mathcal{H})^2 = 1$$

holds, it can be used to define

$$P = \frac{1}{2}(1 - \mathcal{H}\eta), \qquad \bar{P} = \frac{1}{2}(1 + \mathcal{H}\eta), \qquad P^T = \frac{1}{2}(1 - \eta H) \qquad \bar{P}^T = \frac{1}{2}(1 + \eta H) \tag{3.21}$$

Projectors make analysis of generalised geometry easier. They are regarded very fundamental objects in Double Field Theory. Expressing the metric in terms of the projectors.

$$P_M^N = \frac{1}{2}\left(\delta_M^N - \mathcal{H}_M^N\right), \quad \bar{P}_M^N = \frac{1}{2}\left(\delta_M^N + \mathcal{H}_M^N\right) \tag{3.22}$$

Projectors satisfy a usual projector's properties-

$$P\partial_A P = P\partial_A P\bar{P} \qquad \partial_A PP = \bar{P}\partial_A PP$$

$$P^2 = P \qquad \bar{P} = 1 - P$$

Projectors allow us to project onto a 'left-handed' or 'right-handed' subspace[10]. This is the analogue of the factorized tangent space group $GL(D) \times$

$GL(D)$ in the frame formulation, and equivalence of the two formalisms then requires the projectors to be covariantly constant. Else, Since both $\eta$and$\mathcal{H}$ are covariantly constant, so are the projectors. Using projectors, the connection can be written as,

$$\Gamma_{MNK} = \Gamma_{\underline{MNK}} + \Gamma_{\underline{MN}\bar{K}} + \Gamma_{\underline{M}\bar{N}\underline{K}} + \Gamma_{\underline{M}\bar{N}\bar{K}} + \Gamma_{\bar{M}\underline{NK}} + \Gamma_{\bar{M}\underline{N}\bar{K}} + \Gamma_{\bar{M}\bar{N}\underline{K}} + \Gamma_{\bar{M}\bar{N}\bar{K}} \quad (3.23)$$

Using symmetries of connection, and properties of projectors, the exactly determined part of connection in terms of projection operators is,

$$\widehat{\Gamma}_{MNK} = -2\left(P\partial_M P\right)_{[NK]} - 2\left(\bar{P}_{[N}P\bar{P}_{K]}{}^{Q} - P^{P}_{[N}P^{Q}_{K]}\right)\partial_P P_{QM} \quad (3.24)$$

$$+ \frac{4}{D-1}\left(P_{M[N}P^{Q}_{K]} + \bar{P}_{M[N}\bar{P}^{Q}_{K]}\right)\left(\partial_Q d + \left(P\partial^P P\right)_{[PQ]}\right) \quad (3.25)$$

and in terms of physical fields, it is,

$$\widehat{\Gamma}_{MNK} = \frac{1}{2}\mathcal{H}_{KQ}\partial_M\mathcal{H}^{Q}_{N} + \frac{1}{2}\left(\delta^{P}_{[N}\mathcal{H}^{Q}_{K]} + \mathcal{H}^{P}_{[N}\delta^{Q}_{K]}\right)\partial_P\mathcal{H}_{QM} \quad (3.26)$$

$$+ \frac{2}{D-1}\left(\eta_{M[N}\delta^{Q}_{K]} + \mathcal{H}_{M[N}\mathcal{H}^{Q}_{K]}\right)\left(\partial_Q d + \frac{1}{4}\mathcal{H}^{PM}\partial_M\mathcal{H}_{PQ}\right) \quad (3.27)$$

The constraints respectively result in a connection following usual properties - it is anti-symmetric in last two indices, obeys Bianchi identities, can be expressed in terms of, and the value of trace of the connection[12]. The generalised Riemann tensor is not fully determined, owing to the fact that the connection is not fully determined. Upon contracting the Riemann tensor to obtain the Ricci tensor, it becomes evident that even though the former is not determined, the Ricci tensor is

$$\mathcal{R}_{MN}{}^{MN} = \eta^{MK}\eta^{NL}\mathcal{R}_{MNKL} = \mathcal{R}_{\underline{MN}}{}^{\underline{MN}} + \mathcal{R}_{\bar{M}\bar{N}}{}^{\bar{M}\bar{N}}$$

which can be further solved to obtain,

$$\mathcal{R} \equiv \mathcal{R}^{\underline{MN}}{}_{\underline{MN}} = -\mathcal{R}^{\bar{M}\bar{N}}{}_{\bar{M}\bar{N}}$$

Where $\mathcal{R}$ is called the scalar curvature. Using the expression for generalised Riemann tensor expressed in terms of physical fields, the expression for Ricci scalar takes the form[11]

$$\begin{aligned}\mathcal{R} \equiv 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\end{aligned} \quad (3.28)$$

3.3.2 Obtaining the Action

The action in DFT is defined as,

$$S = \int dx d\tilde{x} e^{-2d}\mathcal{R} \quad (3.29)$$

It is gauge invariant in nature, and under a variation of metric, it varies as,

$$\delta S = \int dx d\tilde{x} e^{-2d}\delta\mathcal{H}^{MN}\mathcal{K}_{MN}$$

Where

$$\begin{aligned}\mathcal{K}_{MN} \equiv & \frac{1}{8}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{4}\left(\partial_L - 2\left(\partial_L d\right)\right)\left(\mathcal{H}^{LK}\partial_K \mathcal{H}_{MN}\right) + 2\partial_M \partial_N d \\ & - \frac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L \mathcal{H}_{N)K} + \frac{1}{2}\left(\partial_L - 2\left(\partial_L d\right)\right)\left(\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \mathcal{H}^{K}_{(M}\partial_K \mathcal{H}^{L}_{N)}\right)\end{aligned}$$

The field equation is,

$$\mathcal{R}_{MN} = 0$$

and

$$\mathcal{R}_{MN} \equiv \frac{1}{4}\left(\delta_M{}^P - \mathcal{H}_M{}^P\right)\mathcal{K}_{PQ}\left(\delta^Q{}_N + \mathcal{H}^Q{}_N\right) + \frac{1}{4}\left(\delta_M{}^P + \mathcal{H}^P{}_M\right)\mathcal{K}_{PQ}\left(\delta^Q{}_N - \mathcal{H}^Q{}_N\right)$$

Chapter Four

Details of Newton Cartan gravity

4.1 The geometry of Newton Cartan

The task of deriving the differential geometry formulation starts with the classical equation of motion of a particle[2]. For a classical particle,

$$\ddot{x}^i(t) + \frac{\partial \phi(x)}{\partial x^i} = 0 \tag{4.1}$$

where $x^i(t)$ are spatial coordinates. Under a four dimensional spacetime, this can be rewritten as,

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0 \tag{4.2}$$

where x^μ runs over (t, x^i). This kind of connection can be defined for non degenerate metrics, like in General Relativity. But, NC geometry does not have a metric that can incorporate both space and time components, because of its non relativistic nature. As one approaches the limit $c \to \infty$, the Minkowski metric becomes

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3/c^2 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix} \tag{4.3}$$

gives us two non-invertible or degenerate metrics. Thus, the single spacetime metric is split into a covariant temporal and contravariant spatial component, referred to as $\tau_{\mu\nu}$ and $h^{\mu\nu}$ respectively[7]. Since the metric $\tau_{\mu\nu}$ has three zero components, it can be rewritten simply as a vector, such that $\tau_{\mu\nu} = \tau_\mu \tau_\nu$. This is called the time-like vielbein. The two metrics satisfy orthogonality condition, *i.e* $h^{\mu\nu}\tau_\nu = 0$. Due to lack of a Lorentz spacetime metric, the indices cannot be simply raised or lowered. Thus, the inverse spatial metric $h^{\mu\nu}$, and inverse temporal metric τ^ν are introduced, with the following properties -

$$h^{\mu\nu}h_{\nu p} = \delta^\mu_\rho - \tau^\mu\tau_\rho, \qquad \tau^\nu\tau_\mu = \delta^\nu_\mu, \qquad h^{\mu\nu}\tau_\nu = h_{\mu\nu}\tau^\nu = 0 \tag{4.4}$$

At this point, it is useful to define $\bar{h}_{ij}$, $\hat{v}^i$, and $\tilde{\Phi}$ as

$$\bar{h}_{ij} \equiv h_{ij} - \tau_i m_j - \tau_j m_i, \qquad \hat{v}^i \equiv \tau^i - h^{ij}m_j, \qquad \tilde{\Phi} \equiv -\tau^i m_i + \frac{1}{2}h^{ij}m_i m_j \tag{4.5}$$

This gives us the metric as,

$$g^{\mu\nu} = \begin{pmatrix} h^{ij} & -\hat{v}^i \\ -\hat{v}^j & 2\tilde{\Phi} \end{pmatrix} \tag{4.6}$$

4.1.1 Covariant Derivative and Connections

The geodesic equations of NC gravity correspond to the classical equation of motion for particles [18]. To derive the geometry, one can start from defining the covariant derivatives. A covariant derivative acting on a tensor, $A^{\nu}{}_{\rho}$ is,

$$D_{\mu}A^{v}{}_{\rho} = \partial_{\mu}A^{v}{}_{\rho} + \Gamma^{v}_{\sigma\mu}A^{\sigma}{}_{\rho} - \Gamma^{\sigma}_{\rho\mu}A^{v}{}_{\sigma} \tag{4.7}$$

Like in the case of Riemannian geometry, a reasonable demand is, the derivative should exhibit metric compatibility[21] which implies,

$$\nabla_{p}h^{\mu\nu} = 0, \quad \nabla_{\rho}\tau_{\mu} = 0. \tag{4.8}$$

But since the manifold is not lorentzian, it is not possible to attain both metric compatibility and a fully torsion-less derivative. The derivative is defined up to a two form we call $F_{\mu\nu}$[21]. To determine the derivative uniquely, one demands that the torsion is purely temporal.

The Newton Cartan gravity aims at a causal non relativistic geometry. Newton causality can be incorporated by a constraint on the timelike vielbein. The time difference between two events can be written as,

$$T = \int_{\mathcal{C}} \mathrm{d}x^{\mu}\tau_{\mu} \tag{4.9}$$

If the time difference is independent of the path $\mathcal{C}$, it can be identified as absolute time. This can be attained if the curl of timelike vielbein is set to vanish. This is referred to as 'zero torsion condition'. A necessary and sufficient condition for Newtonian causality comes from understanding the hypersurface orthogonality condition[3]. It is based on $Frobenius'$ Condition which can be written as, $\tau \wedge d\tau$=0.

This condition ensures that there are no points on the spacetime which can

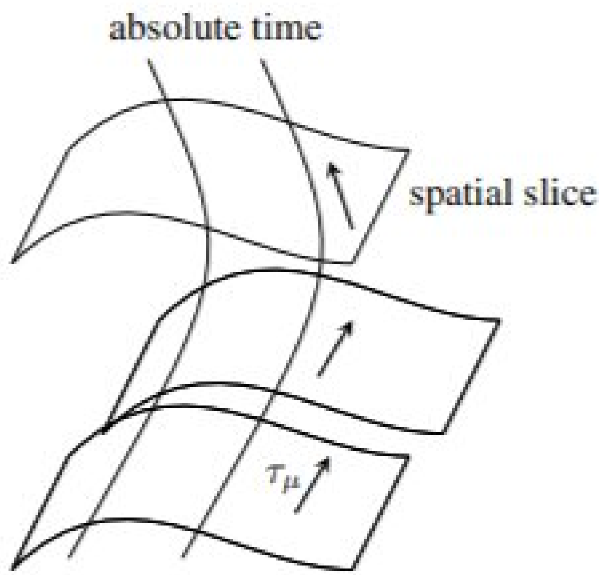

Figure 4.1 Folliation of spacetime [22]

be reached with spacelike lines. Thus, the required causality is preserved[9]. τ satisfies hypersurface orthogonality if there exists a foliation of spacetime into hypersurfaces of equal-time slices such that τ is orthogonal to all such hypersurfaces. The condition takes the form

$$\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$$

where square brackets denote anti-symmetry. This is also known as the twistless torsion or temporal torsion Newton Cartan geometry. Torsion Newton Cartan gravity is defined using the one form τ_μ, the spatial symmetric tensor $h_{\mu\nu}$ and the $U(1)$ connection term m_μ. The $U(1)$ invariant term corresponds to mass conservation laws in Galilean theories[14].

Newton Cartan geometry can also be obtained by Null reduction of a one higher dimension theory of General Relativity[4]. Consider the following general parametrisation in a $d+1$ dimensional spacetime, with metric g

$$ds^2 = g_{ij}dx^i dx^j = 2\tau_\mu dx^\mu \left(du - m_\mu dx^\mu\right) + h_{\mu\nu}dx^\mu dx^\nu \tag{4.10}$$

where u is the null direction. The rank of $h_{\mu\nu}$ is $d-1$. This kind of parametrisation automatically reduces the manifold to a Newton Cartan manifold of d dimensions. The field m_μ is a $U(1)$ term. NC is completely described by the spatial metric $h_{\mu\nu}$,temporal metric τ_μ, and $U(1)$ gauge field m_μ.

$$\boxed{N.C.Geometry = LorentzSpacetime + NullReduction}$$

Owing to this, the most general connection takes the form

$$\Gamma^\sigma_{\mu\nu} = \tau^\sigma \partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\sigma\rho}\left(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}\right) + h^{\sigma\lambda}K_{\lambda(\mu}\tau_{\nu)} \tag{4.11}$$

for an arbitrary two from $K_{\mu\nu}$. The connection is uniquely determined after gauging the Bargmann algebra[2]. The Bargmann algebra is a central extension of the Galilean algebra, and is discussed in the following subtopic. It acquires the form

$$K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]} \tag{4.12}$$

The defined inverse of τ_ν, v^ν can be thought of as analogous to velocity. This implies that an acceleration could be defined as

$$a^\mu \equiv v^v D_v v^\mu \quad \text{or simply,} \quad a_r = \hat{v}^t F_{tr}$$

$F_{\mu\nu}$ can be thought of as a field strength tensor of a $U(1)$ connection[15]. Since the torsion must be temporal, as a condition prior agreed to, it can be conveniently written as,

$$2\Gamma^{\mu}_{[\rho\sigma]} = -2\hat{v}^{\mu}\partial_{[\rho}\tau_{\sigma]} = -\hat{v}^{\mu}F_{\rho\sigma} \tag{4.13}$$

The fields in NC geometry obey the transformation rules under diffeomorphisms and Galilean symmetries, as boosts and a $U(1)$ gauge transformation.

4.2 Bargmann algebra

The NC geometry is described using the central extension of Galilean algebra, known as Bargmann algebra. The algebra is spanned by the usual generators, $J_{ij}, G_i, H, M, \text{and} P_j$, This algebra can be expressed via the following commutation relations -

$$\begin{aligned}
[J_{ij}, J_{kl}] &= 4\delta_{[i[k}J_{l]j]} \\
[J_{ij}, G_k] &= -2\delta_{k[i}G_{j]} \\
[G_i, P_j] &= -\delta_{ij}M \\
[J_{ij}, P_k] &= -2\delta_{k[i}P_{j]} \\
[G_i, H] &= -P_i
\end{aligned} \tag{4.14}$$

Where H is Hamiltonian, the generator of time translations, P_i is momentum, the generator of linear translations, J_{ij} is angular momentum, the generator of rotations. M is the central charge, and corresponds to mass. A central charge is an operator which commutes with all other operators of the alegbra. When $M \to 0$, this reduces to regular Galilean algebra. The gauging of Bargmann algebra is done by first associating gauges fields with each of its generator.

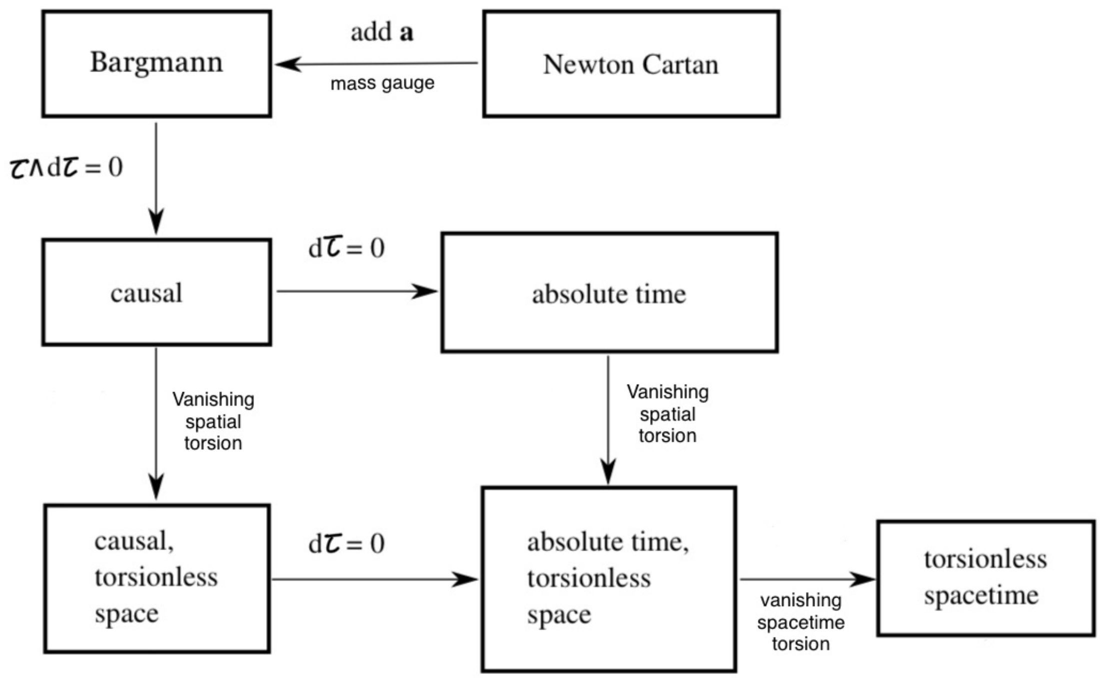

Figure 4.2 A pictorial representation of Bargmann spacetime and various conditions imposed on them[9]

Upon imposing constraints on curvature of these gauge fields, and solving, one obtains the fundamental fields of the gauge theory of Newton Gravity [18].

4.3 Generalised NC Metric

Now, inverse metric $g^{\mu\nu}$ can be dualised on the null isometric direction u. The (3.9) is the standard DFT metric. For simplicity, B is taken to be absent. This reduces the generalised metric to $diag(g^{\mu\nu}, g_{\mu\nu})$. The Newton Cartan metric, derived in (4.6) can be inserted in the DFT metric. The NC metric generalised in DFT parametrisation can be achieved as follows Following out Buscher transformations[1] along the null direction.

From [4] the generalised Newton Cartan metric can be given as given as,

$$(\mathcal{H}_{\text{NC}})_{MN} = \begin{pmatrix} \bar{h}_{ij} & 0 & 0 & \tau_i \\ 0 & 2\tilde{\Phi} & -\hat{v}^j & 0 \\ 0 & -\hat{v}^i & h^{ij} & 0 \\ \tau_j & 0 & 0 & 0 \end{pmatrix} \tag{4.15}$$

Once this metric is achieved, this can be used with the DFT action to give the equations of motion.

Chapter Five

Action and Equations of Motion

We here describe the actions and equations of motions we observed for various settings. We started out with exploring the NC geometry and used the technique developed in DFT to arrive at the action. We can then either vary the action with respect to all the fields, or write the equations of motion directly from DFT. Comparing both the processes, we observe that we do not obtain all equations of motion by just varying the action. We here enlist the results obtained for a general connection. A general connection can always be split into a symmetric and an anti-symmetric component. Upon using such a general connection to obtain the results, it can be proven that the the action and equations of motion are devoid of a contribution from the symmetric part. Thus, under a general connection, the action and equations of motion are covariant. After proving covariance, the general connection can be parameterised for torsion, using the relation (4.12). The relations for acceleration in terms of $F_{\mu\nu}$ By setting the constraints for TNC, the action and equations of motion for TNC are obtained. To obtain the Action and equations of motion for twistless torsion Newton Cartan, metric compatibility is used. It might a useful reminder that both metric compatibility and fully determined covariant derivative do not co-exist in TNC geometry, as previously explained. Torsion Newton Cartan is characterised by the two form $F_{\mu\nu}$ of the timelike vector τ.

$$F \equiv d\tau \tag{5.1}$$

5.1 Action and Equations of Motion for Torsional Newton Cartan

The action for torsional Newton Cartan is expressed in terms of the Field strength tensor.

The action is

$$S = \int d^d x e^{-2d} (h^{mn} \mathcal{R}_{mn} - \frac{5}{2} a^2 - 2h^{mn} D_m a_n)$$

The Equations of Motion are -

Equation of motion for the metric $h_{\mu\nu}$

$$8\mathcal{R}_{mn} - 4a_m a_n - D_m a_n - 3D_n a_m - 2\bar{h}_{mn} D_p a_q h^{pq} + 2\hat{v}^p D_p a_n \tau_m + 2D_p D_n \hat{v}^p \tau_m$$

$$+ 8a_p D_n \hat{v}^p \tau_m + 2a_m D_p \hat{v}^p \tau_m - 2\bar{h}_{nr} D_p D_q \hat{v}^r h^{pq} \tau_m - 6a_p \bar{h}_{nr} D_q \hat{v}^r h^{pq} \tau_m$$

$$- 2\hat{v}^p D_p a_m \tau_n + 2D_p D_m \hat{v}^p \tau_n + 8a_p D_m \hat{v}^p \tau_n - 2a_m D_p \hat{v}^p \tau_n - 2\bar{h}_{mr} D_p D_p \hat{v}^r h^{pq} \tau_n$$

$$- 6a_p \bar{h}_{mr} D_q \hat{v}^r h^{pq} \tau_n + 4\hat{v}^p D_q D_p \hat{v}^q \tau_m \tau_n + 24a_p D_q \Phi h^{pq} \tau_m \tau_n + 12D_p a_q h^{pq} \tau_m \tau_n \Phi$$

$$+ 32a^2 \tau_m \tau_n \phi = 0$$

Equation of motion for the inverse metric $h^{\mu\nu}$

$$4D_p a_q h^{mq} h^{np} + 4a_p a_q h^{mq} h^{np} + 2D_p a_q h^{mn} h^{pq} - 8\mathcal{R}_{pq} h^{mq} h^{np} = 0$$

Equation of motion for the temporal one form τ_μ

$$D_m a_n h^{mn} + \frac{a^2}{2} = 0$$

Equation of motion for Newton potential

$$R_{mn} \hat{v}^n \hat{v}^m - \frac{7}{2} a_m \hat{v}^n D_n \hat{v}^m - 2a_m D_n \Phi h^{mn} - 4a^2 - a^2 \Phi = 0$$

Where Φ is the Newton Potential.

Equation of Motion

$$\frac{1}{8} F_{mn} F_{pq} h^{mp} h^{nq} = 0 \tag{5.2}$$

5.2 Action and equations of motion for twistless torsion Newton Cartan geometry.

Since the theory is derived from a reduction of GR, this action is the Null Reduction of the Einstein Hilbert action in General Relativity. It has an acceleration term a, which stands for geodesic acceleration and parametrises the twistless torsion. This action does not give all equations of motion upon the vairation. Ther results here are verified from [8].

$$S = \int d^d x e^{-2d} (h^{mn} \mathcal{R}_{mn} + \frac{a^2}{2}) \tag{5.3}$$

The equations of motion are obtained directly from the results in DFT. Analogous to the Einstein's Equation,

$$\mathcal{R}_{(mn)} + 2D_{(m}D_{n)}\phi = h^{tp}\bar{h}^{t}_{p(m}D_{n)}a_t + \frac{a_m a_n}{2} - a^2\Phi\tau_m\tau_n \tag{5.4}$$

Equation of motion for torsion,

$$D \cdot a + a^2 = 2(a \cdot D\phi) \tag{5.5}$$

Newton's law modified with advection and mass.

$$D^2\Phi + 3(a \cdot D\Phi) + m^2_\Phi \Phi = \rho_\kappa \tag{5.6}$$

Where, the Newton Potential Mass m^2_Φ, extrinsic curvature tensor $\mathcal{K}$ and curvature density ρ are introduced as,

$$m_{\Phi}^2 \equiv a^2 + 4a \cdot D\phi$$

$$\rho_{\mathcal{K}} \equiv \mathcal{K}_{rs}\mathcal{K}_{tw}h^{rt}h^{sw} - \hat{v}^n D_n \left(\mathcal{K}_{rs}h^{rs}\right)$$

$$\mathcal{K}_{mn} \equiv -\frac{1}{2}\mathcal{L}_{\hat{v}}\bar{h}_{mn} = -\frac{1}{2}\left[\hat{v}^t D_t \bar{h}_{mn} + \bar{h}_{mt}D_n\hat{v}^t + \bar{h}_{nt}D_m\hat{v}^t - 4\Phi a_{(m}\tau_{n)}\right]$$

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