Utrecht University

Theoretical Physics

Weyl invariance of the bosonic string in torsional Newton-Cartan geometry

Master Thesis

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Abstract

We consider the closed bosonic string in Newton-Cartan geometry with Kalb-Ramond and dilaton matter. This theory is quantized with the background field quantization method. The theory is then no longer invariant under a Weyl transformation: there is an anomaly. To mend this we impose conditions on the spacetime fields. The Weyl anomaly can be parametrized by β-functions of which we will focus on the dilaton β-function, as this was the only β-function not computed in a previous treatment. We find the contribution to the dilaton β-function by considering two-loop Feynman diagrams. These diagrams are given and a general strategy to calculate them is provided.

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1 Introduction

In 1915 Einstein presented his theory of general relativity. Here, spacetime is dynamical and curvature in spacetime causes gravity. In the same period of time physicists started understanding the microscopic world better and better by formulating the theory of quantum mechanics. This understanding then took a flight in the next 50 years, resulting in the standard model of particle physics of which the last missing piece, the Higgs boson, was discovered experimentally in 2011. One big open question, however, is how to combine Einstein's theory of curved spacetime with the microscopic quantum mechanical world. Whereas the first normally describes massive and long-range phenomena, the second generally describes phenomena on a small scale. In extreme situations, such as in black holes, these two regimes meet. We then need a theory combining quantum mechanics with general relativity: quantum gravity.

Before Einstein published his theory of general relativity , he started with a more modest theory: special relativity. Inspired by Maxwell his theory on electromagnetism, Einstein postulated that the speed of light is the same for all observers, regardless of their motion. One then gets a theory that is invariant under Lorentz transformations (rotations and boosts). Whereas uniting general relativity with quantum mechanics remains problematic, special relativity was incorporated with quantum mechanics in quantum field theory. One can think of general relativity as a combination of Newtonian gravity and special relativity, whereas quantum field theory combines special relativity with quantum mechanics. The ultimate goal would be to have a theory encompassing gravity, special relativity, and quantum mechanics. The idea of these three basic theories; quantum mechanics, special relativity, and Newtonian gravity, that can be unified in different combinations, is visualized by the Bronstein cube in Figure 1. This cube is of course a huge simplification of the idea it represents: one cannot express all of modern theoretical physics in one picture, but it serves as a nice way to visualize the different approaches.[1] The traditional way to quantum gravity is to take this cube literally and try to quantize general relativity[2]. While many useful insights have been found this way, one quickly realizes that arriving at a theory of quantum gravity is not as straightforward as simply quantizing spacetime.

A more sophisticated approach is taken by string theory, which does not simply go from corner to corner in the cube of Figure 1. We will treat string theory in detail in Section 3, where we will see that a quantized field which can be associated to the spacetime metric, appears naturally. String theory gives us a theory of quantum gravity [3], but it comes with its own limitations [2]. The consistency of the theory requires the spacetime dimension to be $D = 26$ in bosonic string theory and $D = 10$ or $D = 11$ in superstring theory. The extra dimensions can then be compactified to arrive at our observed four-dimensional spacetime. This compactification can be done in many ways, such that it is not (yet) possible to unambiguously arrive at a four-dimensional theory from the original D-dimensional string theory. In this thesis, we will only focus on a perturbative approach to string theory, but in strong coupling regimes, one needs a non-perturbative description. There is no general way to describe string theory in a non-perturbative fashion. From QCD we know that interesting effects such as quark confinement cannot be explained by a perturbative approach. One would expect that to totally understand string theory, a non-perturbative description is needed [3]. These limitations are still a very active field of research.

[1] See [1] for a modern approach to the Bronstein cube.

[2] Here quantize means to use whatever quantization method is available. A review of the different methods can be found in [2].

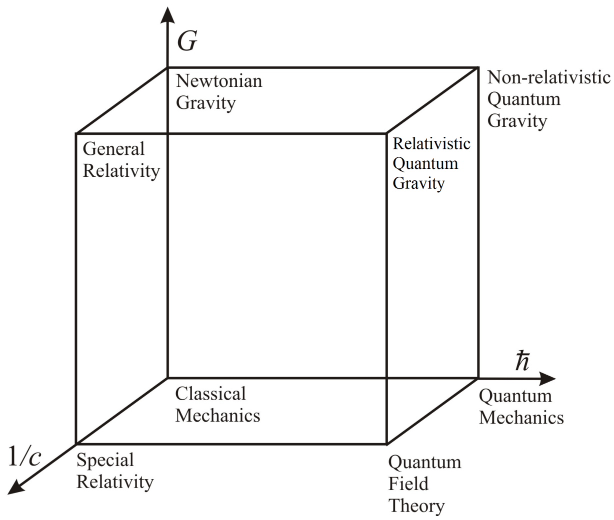

Figure 1: The Bronstein cube, showing the possible combinations of the three regimes in physics that we want to combine. Newtonian gravity is characterized by Newton's constant G, quantum mechanics by Planck's constant $\hbar$, and special relativity by the inverse speed of light $1/c$. The figure is taken from https://beyondspacetime.net/2013/09/29/live-blogging-butterfield-2/

One recent approach that might give some insight into these issues is to take a step back and look at the Bronstein cube (Figure 1) again. Two corners of this cube, the one combining special relativity and Newtonian gravity resulting in general relativity and the one combing quantum mechanics and special relativity resulting in quantum field theory, are together responsible for almost all the achievements in high energy physics in the twentieth century. Would taking a look at the third corner: the one combining Newtonian gravity with quantum mechanics, lead to equally interesting results?

When special relativistic effects are ignored, one can investigate how to combine quantum mechanics and gravity into a theory of non-relativistic quantum gravity. It is possible to examine a dynamical spacetime that is not Lorentz invariant, but has a classical Galilean invariance. This line of thought was first followed by Élie Cartan in 1924 [4]. He described gravity as an emergent phenomenom from curved spacetime in much the same way as Einstein did, but did this with an underlying Galilean invariance instead of Lorentz invariance. The resulting theory gives an underlying geometrical structure to Newtonian gravity called Newton-Cartan (NC) geometry.

In recent years there has been a renewed interest in Newton-Cartan geometry in the context of string theory. This started with a paper by Gomis and Ooguri in 2000 [5], in which string theory is described in an underlying NC geometry. The hope is that the absence of special relativistic effects, such as gravitational waves, will lead to a better understanding of the problems of ordinary string theory described above. It also turns out that a lot of the triumphs of general relativity do not rely on special relativity. When Einstein published his theory of general relativity, he also presented three tests on the theory. Namely the precision of the perihelion of Mercury's orbit, deflection of light by the sun, and gravitational redshift. Quickly after the publication of Einstein's ideas, all three of these were confirmed by experiments and general relativity continues to withstand these tests to date [6]. It turns out that all three of these effects can also be explained by Newton-Cartan geometry, as shown by Hansen et al. in [7]. They are thus manifestations of space-time being dynamical and not of the absolutivity of the speed of light. Recently, it has been shown that more advanced results from general relativity also hold in NC geometry. These include the Schwarzschild metric and the Friedmann–Lemaître–Robertson–Walker spacetime [8]. Furthermore, non-relativistic geometry and Newton-Cartan geometry also sees applications in condensed matter physics, for example in describing the fractional quantum Hall effect [9]. Thus, NC gravity and NC string theory might be interesting on their own and not just as an approximation to relativistic quantum gravity and string theory.

Newton-Cartan geometry can be extended to include torsion [10, 11]. To find the most general non-relativistic bosonic string we will examine the string in torsional Newton-Cartan (TNC) geometry. We will find that the geodesic equation of the bosonic string only allows a special kind of torsion called twistless torsion. In [12] it is argued that a conformal field theory where torsion is not twistless violates causality.

As we will see in Section 3, the bosonic string is invariant under worldsheet reparametrizations. This is called Weyl invariance and it is the main focus of this thesis. By demanding that the quantized bosonic string is also Weyl invariant, one arrives at Einstein's equation [3]

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ and R are the spacetime Ricci tensor and scalar respectively, $G_{\mu\nu}$ is the spacetime metric and $T_{\mu\nu}$ is the energy-momentum tensor on spacetime. The goal of this thesis is to examine the consequences of Weyl invariance of the quantized bosonic string in NC geometry. This Weyl invariance can be parametrized in so-called β-functions: functions that all have to be zero separately for the theory to be Weyl invariant. In the Riemannian bosonic string, there are three such functions, which we compute in Section 3 following the excellent review by Callan and Thorlacius [13]. In the TNC case, there are six β-functions of which five were computed in [14]. In this thesis we will focus on the last of these functions: the dilaton β-function. This function is more complicated to compute than the other five because the worldsheet can no longer be taken as flat and one needs to go one order in perturbation theory higher than for the other functions. We mostly follow [14] in that we take the same Lagrangian and quantize it with the same background field quantization method. Where [14] examined the Weyl invariance of the TNC bosonic string using an approach involving operator product expansions, we will examine it through a perturbation theory with Feynman diagrams, much in the same way as in the review by Callan and Thorlacius [13].

This thesis is organized as follows: In Section 3 we will start with a review of bosonic string theory and the Weyl anomaly. Then, we will do an extensive calculation of the Weyl anomaly. For this calculation and for the same calculation in TNC geometry, we will heavily rely on integral expressions that we determine in Appendix B. In Section 4.1 we will review Newton-Cartan geometry. This is followed by describing the bosonic string in this geometry, by performing a null-reduction on a one dimensional higher Lorentz invariant spacetime. In the subsequent sections, we repeat the steps of the Riemannian bosonic string in the TNC case. This leads to an expanded action in Section 6, from which Feynman diagrams can be constructed. Using these Feynman diagrams one can determine all the contributions to the Weyl anomaly. In this thesis we will focus on the contributions that were not yet computed in [14]: the part parametrized by the dilaton β-function. Finally, the diagrams contributing to this function are presented in Section 8. Their integral expressions are listed in Appendix C and a general way of calculating these integrals is provided in Appendices B and C.

2 The bosonic string

String theory starts with the radical assumption that the basic building blocks of the universe are not zero-dimensional point particles, but one-dimensional strings. We will label the spacelike coordinate of this string by σ. Strings can be either open or closed, but in this thesis, we will only consider closed strings. Furthermore, we will take σ to be periodic, such that $\sigma \in [0, 2\pi)$ and $\sigma(2\pi) = \sigma(0)$. The time coordinate of the string is labeled by τ. Together σ and τ sweep out a two-dimensional surface called the worldsheet (see Figure 2). σ and τ are the flat worldsheet coordinates. We will refer to general worldsheet coordinates by ξ^α, where the first few letters of the Greek alphabet are used for curved worldsheet coordinates. The surface swept out by σ and τ can be described by the two-dimensional worldsheet metric $\gamma_{\alpha\beta}$. This two-dimensional worldsheet is embedded in a D-dimensional spacetime and we can define a map from the worldsheet to spacetime: $X^\mu(\xi)$. One can then define the action of the string as the area of the swept out surface by using the pull-back of the Minkowski metric,

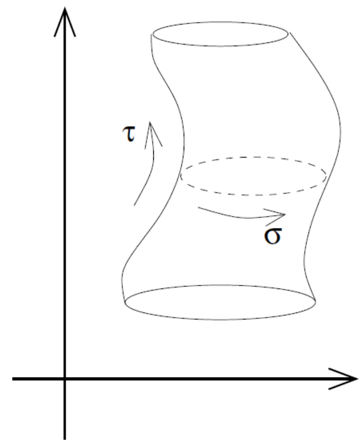

Figure 2: The worldsheet swept out by a closed string as it moves in time. The Figure is taken from David Tong's lecture notes [3]

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}. \tag{2}$$

The action is given by

$$S = -T\int d^2\sigma \sqrt{-\det g} = -T\int d^2\sigma\sqrt{-\dot{X}^2X'^2 + (\dot{X}\cdot X')^2}\,, \tag{3}$$

which is called the Nambu-Goto action for a relativistic string. Here, T is the tension of the string and is usually written as

$$T = \frac{1}{2\pi\alpha'}, \tag{4}$$

where $\alpha' = l_s^2$, the squared string length l_s. There is one symmetry in the Nambu-Goto action (3) that is worth mentioning. If the worldsheet coordinates are changed by

$$\xi^a \to \xi^a + v^a(\xi), \tag{5}$$

the action (3) does not change. This is called reparametrization invariance.

The square root in the Nambu-Goto action makes it difficult to quantize the theory. We can eliminate this square root by using the worldsheet metric γ_{ab} as a dynamical field in the action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}. \tag{6}$$

This is the Polyakov action. It is also invariant under reparametrizations (5). By using the equations of motion found by variating this action, one can eliminate the worldsheet metric from Equation (6) and recover the Nambu-Goto action (3). Hence, both actions are physically the same; by adding γ^{ab} we have created a redundancy in the description of the theory: a gauge symmetry. This symmetry manifests itself by considering local changes of scale in the worldsheet metric

$$\gamma_{ab} \to e^{f(\xi)}\gamma_{ab}. \tag{7}$$

The change in the worldsheet metric under this transformation is

$$\delta\gamma_{ab} = -\delta f(\xi)\gamma_{ab}. \tag{8}$$

The Polyakov action (6) is invariant under the transformation (8). This symmetry is called Weyl invariance and will be the main study of this thesis. We will make use of this symmetry by choosing a gauge: we use the redundant degrees of freedom to make the theory easier to describe. We will then find that the theory is no longer invariant under Weyl transformations (8) after quantization. This phenomenon where a classical symmetry fails to hold at the quantum level is called an anomaly. The Weyl anomaly in bosonic string theory is related to the appearance of negative norm states, called ghosts [3]. A theory with negative norm states is a big problem as it causes states with a negative probability, which is unphysical. Put differently, a theory with negative norm states is no longer unitary. To avoid all of this, we want the theory to retain Weyl invariance at the quantum level. This can be done by putting restrictions on the background fields of the theory. For the Polyakov action (6), the calculation is relatively simple. Including the Faddeev-Popov ghosts (see David Tong's lecture notes [3]) leads to the restriction $D = 26$. Quantizing the Polyakov string and setting the contribution to the Weyl anomaly to zero, gives a leading order requirement that [3]

$$\alpha' R_{\mu\nu} = 0, \tag{9}$$

where $R_{\mu\nu}$ is the Ricci tensor of the spacetime metric $G_{\mu\nu}$. This tells us spacetime should be Ricci flat; it should obey the Einstein vacuum equations. Accordingly, quantizing string theory and demanding consistency of Weyl symmetry gives us the equations of gravity that we are so familiar with: this is a great result!

This is not all however, Equation (9) is only the leading order contribution and there are infinitely more terms in an α' power series. And what about matter? The normal Einstein equation does not have zero on the right-hand side, but the energy-momentum tensor. To get a spacetime with matter we need to add matter to our string action.

In a quantum field theory, renormalization brings in all terms with the same dimension as S_P. We can now add terms to the Polyakov action that are both reparametrization invariant and have worldsheet dimension two, of which two such terms exist[3]. The first involves the antisymmetric Kalb-Ramond field $B_{\mu\nu}$ and is given by

$$S_{AS} = \frac{1}{4\pi\alpha'} \int d^2\xi \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}, \tag{10}$$

where ϵ^{ab} is the Levi-Civita symbol. The second term involves the dilaton scalar field $\phi(X)$,

$$S_D = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} R_\gamma \phi(X). \tag{11}$$

Note that S_{AS} is Weyl invariant, but S_D is not. As we will see later, α' is the loop counting parameter. It then immediately follows that S_D will enter at the first loop order of S_P and S_{AS}. The Weyl anomaly will thus receive contributions from the tree level of S_D and from the loop level of S_P and S_{AS}. This also means that for the first-order to S_D, we need second order connections

[3] One could also add a dimension zero term involving the Tachyon, but as we will work in dimensional regularization, this term will not show up since it only contains quadratic divergences. Dimensional regularization only produces logarithmic divergences and therefore the Tachyon term can be set to zero at the start and be neglected[15].

coming from S_P and S_{AS}. Setting the total contribution to zero for each order of α', would cancel the Weyl anomaly. This gives conditions on the the coupling functions $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ which provide equations of motion that the coupling functions should obey for the Weyl anomaly to cancel.

We have now encountered the three massless, first excited states of bosonic string theory[3], but what about higher excited states? These have an associated vertex operator of worldsheet dimension greater than 2. Terms like this can always be generated by redefining the dimension 2 vertex operators of the metric, the Kalb-Ramond field and the dilaton [15], such that these higher excited states can be ignored.

It is convenient to examine the bosonic string in the conformal gauge. Using reparametrization invariance (5), we can choose the two degrees of freedom in ξ^a such that two of the three degrees of freedom of the worldsheet metric are removed,

$$\gamma_{ab} = 2e^{f(\xi)}\delta_{ab}. \tag{12}$$

This is called the conformal gauge. Note that the worldsheet metric is now completely determined by the factor $f(\xi)$. The line element of the worldsheet can be written in the following form by using complex coordinates $(z, \bar{z})$,

$$ds^2 = 2e^f dz d\bar{z}. \tag{13}$$

The metric then takes the form

$$\gamma_{ab} = e^f \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

Then the worldsheet Ricci scalar in the conformal gauge can be written as

$$\sqrt{\gamma}R_\gamma = -2\partial_z\partial_{\bar{z}}f. \tag{15}$$

Furthermore the covariant derivative on a vector also takes a simple form,

$$\begin{aligned} \nabla_{\bar{z}}v^z &= \partial_{\bar{z}}v^z, \\ \nabla_z v^z &= \partial_z v^z + \partial_z f v^z. \end{aligned} \tag{16}$$

We will now derive some results for the quantum bosonic string that will be of use later on. Consider a general classical action $A[X]$. The generating functional can then be written as

$$\begin{aligned} Z[\gamma] &= \int \mathcal{D}X e^{-A[X]} \\ &\equiv e^{-W[\gamma]}. \end{aligned} \tag{17}$$

Performing the path integral over the fields X^μ gives an effective action $W[\gamma]$. This effective action should be invariant under worldsheet reparametrizations (5). That is

$$\int d^2\xi \frac{\delta W}{\delta\gamma^{ab}}(\nabla^a v^b + \nabla^b v^a) = 0. \tag{18}$$

In the conformal gauge we can use Equation (16) to write this as

$$\int d^2\xi \Big[\frac{\delta W}{\delta\gamma^{zz}}\nabla^z v^z + \frac{\delta W}{\delta\gamma^{\bar{z}\bar{z}}}\nabla^{\bar{z}}v^{\bar{z}} - \frac{\delta W}{\delta f}(\nabla_z v^z + \nabla_{\bar{z}}v^{\bar{z}})\Big] = 0. \tag{19}$$

By partial integration and the fact that this should hold for any of the arbitrary functions v^z and $v^{\bar{z}}$, we find that

$$\nabla_z\Big(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta f}\Big) = \nabla^z\Big(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{zz}}\Big). \tag{20}$$

For the antiholomorphic $\bar{z}$ we find a similar equation.

3 The Weyl anomaly in Riemannian geometry

In this section we determine the β-functions characterizing the Weyl anomaly in the bosonic string on a Riemannian manifold. To this end we closely follow the TASI lecture notes of Callan and Thorlacius[13] and try to reproduce their results. While Callan and Thorlacius skip over some of the more complicated details in calculating Feynman diagrams, here and in Appendix B these are treated in detail.

3.1 The background field quantization

All three of the coupling functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$ transform covariantly under general spacetime coordinate transformations. We want to choose a perturbative expansion where this spacetime symmetry remains manifest. This can be done by using the covariant background field expansion as in [13, 16]. We separate the fields in a background part and a quantum part,

$$X^\mu(\xi) = X_0^\mu(\xi) + \tilde{Y}^\mu(\xi). \tag{21}$$

Now the path integral can be performed over the quantum fields $\tilde{Y}^\mu$ in (17). Since this is a linear coordination transformation the Jacobian does not change.

The quantum fields $\tilde{Y}^\mu(\xi)$ are defined as a coordinate difference between the full fields $X^\mu(\xi)$ and the background fields $X_0^\mu(\xi)$. Therefore they do not transform as a vector under general coordinate transformations. To keep Lorentz invariance explicit we need to replace $\tilde{Y}^\mu$ with a spacetime vector in the path integral. A natural choice for this is the tangent vector to the spacetime geodesic $\lambda^\mu(s)$ that connects X_0^μ with $X_0^\mu + \tilde{Y}^\mu$. The affine parameter s can be chosen such that $\lambda^\mu(0) = X_0^\mu(\xi)$ and $\lambda^\mu(1) = X_0^\mu(\xi) + \tilde{Y}^\mu(\xi)$. We then define $Y^\mu \equiv \dot{\lambda}^\mu(0)$, which is the tangent vector of $\lambda^\mu(s)$ at X_0^μ. We can now use the geodesic equation for $\lambda^\mu(s)$ [17]

$$\frac{d^2\lambda^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0 \tag{22}$$

and its derivatives to expand $\lambda^\mu(s)$ around $s = 0$ [13],

$$\lambda^\mu(s) = X_0^\mu(\xi) + Y^\mu s - \frac{1}{2}\Gamma^\mu_{\rho\sigma}Y^\rho Y^\sigma s^2 - \frac{1}{6}\nabla'_\tau\Gamma^\mu_{\rho\sigma}Y^\tau Y^\rho Y^\sigma s^3 - \frac{1}{24}\nabla'_\rho\nabla'_\sigma\Gamma^\mu_{\tau\lambda}Y^\rho Y^\sigma Y^\tau Y^\lambda s^4. \tag{23}$$

Here the covariant derivative ∇' only works on the lower indices of the connection. At $s = 1$ Equation (23) defines the quantum field $\tilde{Y}^\mu$ as a power series of the new covariant field Y^μ. If we would have done this derivation with Y^μ as the original field, the expansion (23) tells us that $\Gamma^\mu_{\rho\sigma} = 0$ and $\nabla'_{(\sigma_j}...\nabla'_{\sigma_{j-2}}\Gamma^\mu_{\sigma_2\sigma_1)} = 0$. Because of this, the coordinates Y^μ are called Riemann normal coordinates. In these coordinates the Riemann tensor simplifies to

$$R^\mu{}_{\nu\sigma\rho} = \partial_\sigma\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\sigma}. \tag{24}$$

One can then derive the following simplification in the Riemann normal coordinate system [13, 18],

$$\partial_\nu\Gamma^\mu_{\rho\sigma} = \frac{1}{3}(R^\mu{}_{\sigma\nu\rho} + R^\mu{}_{\rho\nu\sigma}). \tag{25}$$

In the normal coordinate system we can expand an arbitrary tensor as

$$T_{\mu_1\ldots\mu_n}(X_0+Y)=\sum_{m=0}^{\infty}\frac{1}{m!}\big(\partial_{\nu_1}\ldots\partial_{\nu_m}T_{\mu_1\ldots\mu_n}(X_0)\big)Y^{\nu_1}\ldots Y^{\nu_m}. \tag{26}$$

This expression can be covariantized by writing the partial derivatives as covariant derivatives minus connection terms. In the Riemann normal coordinate system any derivatives of the connection can be written in terms of the Riemann curvature tensor using Equation (25) and $\Gamma^{\mu}_{\rho\sigma}=0$. Following [18] we can expand a rank $(0,2)$ tensor to second-order in Y^{μ}. With every order Y we get the same number of spacetime derivatives. As in general relativity, we will only consider contributions up to second-order in spacetime derivatives on the geometric functions.

$$T_{\mu\nu}(X_0+\tilde{Y})=T_{\mu\nu}(X_0)+\nabla_\rho T_{\mu\nu}(X_0)Y^\rho+\frac{1}{2}\Big[\nabla_\rho\nabla_\sigma T_{\mu\nu}-\frac{1}{3}R^{\lambda}{}_{\rho\mu\sigma}T_{\lambda\nu}-\frac{1}{3}R^{\lambda}{}_{\rho\nu\sigma}T_{\mu\lambda}\Big]Y^\sigma Y^\rho. \tag{27}$$

This expansion was done in the normal coordinate system, but since it is completely covariant it is valid in any coordinate system. This expression can now be used to expand the metric and the Kalb-Ramond field to second-order. Since the covariant derivative is metric compatible, the expansion of the spacetime metric is particularly simple.

$$G_{\mu\nu}(X_0+\tilde{Y})=G_{\mu\nu}(X_0)+\frac{1}{3}R_{\mu\rho\sigma\nu}Y^\rho Y^\sigma. \tag{28}$$

We also want to expand $\partial_\alpha X^\mu=\partial_a(X_0^\mu+\tilde{Y}^\mu)$ and to achieve this we take the ∂_α derivative of Equation (23). This gives

$$\partial_\alpha(X_0^\mu+\tilde{Y}^\mu)=\partial_\alpha X_0^\mu+\nabla_\alpha Y^\mu+\frac{1}{3}R^{\mu}{}_{\rho\sigma\lambda}\partial_\alpha X_0^\lambda Y^\rho Y^\sigma. \tag{29}$$

Any higher-order terms only involve three or more spacetime derivatives on the metric, which we do not treat here. We can now expand the Polyakov action in the Riemann normal coordinates Y^μ to second-order in spacetime derivatives:

$$\begin{aligned}S_P=\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{\gamma}\gamma^{\alpha\beta}\Big[&\partial_\alpha X_0^\mu\partial_\beta X_0^\nu G_{\mu\nu}(X_0)+2G_{\mu\nu}\partial_\alpha X_0^\mu\nabla_\beta Y^\nu+\partial_\alpha X_0^\mu\partial_\beta X_0^\nu R_{\mu\rho\sigma\nu}Y^\rho Y^\sigma\\&+G_{\mu\nu}\nabla_\alpha Y^\mu\nabla_\beta Y^\nu+\frac{4}{3}R_{\mu\rho\sigma\nu}\partial_\alpha X_0^\mu\nabla_\beta Y^\nu Y^\rho Y^\sigma+\frac{1}{3}R_{\mu\rho\sigma\nu}\nabla_\alpha Y^\mu\nabla_\beta Y^\nu Y^\rho Y^\sigma\Big].\end{aligned} \tag{30}$$

If we choose our background fields X_0^μ to obey the classical equations of motions, then the linear term in Y^μ in the above expansion can be set to zero. The term involving $G_{\mu\nu}(X_0)\nabla_\alpha Y^\mu\nabla_\beta Y^\nu$ is the kinetic term for the Y^μ fields. It is, however, not diagonal for a general spacetime metric, and thus the propagator cannot be determined straight away. To get around this problem we introduce a vielbein e^i_μ, with which we can refer vectors to a local Euclidean frame: $Y^i=e^i_\mu(X_0)Y^\mu$. Using this vielbein we can write the spacetime metric as

$$G_{\mu\nu}(X_0)=e^i_\mu(X_0)e^j_\mu(X_0)\delta_{ij}. \tag{31}$$

The kinetic term is now diagonal in the local Euclidean frame coordinates,

$$G_{\mu\nu}(X_0)\nabla_a Y^\mu\nabla_b Y^\nu=(\nabla_a Y)^i(\nabla_b Y)^j\delta_{ij} \tag{32}$$

and the covariant derivative now involves a spin connection ω^{ij}_μ: $(\nabla_a)^i=\partial_a Y^i+\omega^{ij}_\mu\partial_a X_0^\mu Y^j$. The combination $\omega^{ij}_\mu\partial_a X_0^\mu\equiv A^{ij}_a(X_0)$ can be seen as a gauge potential of the SO($D-1$,1) general coordinate invariance in spacetime. Any physical results should not depend on this gauge potential. We do break the general coordinate invariance by defining the propagator for the Y^i fields, but in the end all insertions of the gauge potential A_a^{ij} have to combine to give gauge covariant objects. We now change variables in the path integral to the local Euclidean frame coordinates Y^i. The path integral measure is spacetime coordinate invariant, such that we are free to do this. In practice this just replaces any Greek indices $\mu, \nu, ...$ on the quantum fields to Latin indices $i, j, ...$ Since all these indices are contracted, each resulting vielbein has a corresponding inverse vielbein and nothing else changes.

The action involving the antisymmetric Kalb-Ramond field (10) can also be expanded in Riemann normal coordinates. It reads

$$\begin{aligned} S_{AS}[X_0+\tilde{Y}] = \frac{1}{4\pi\alpha'}\int d^2\xi\epsilon^{ab}\Big[&\partial_a X_0^\mu \partial_b X_0^\nu B_{\mu\nu} + 2\nabla_a Y^\mu \partial_b X_0^\nu B_{\mu\nu} + \partial_a X_0^\mu \partial_b X_0^\nu (\nabla_\rho B_{\mu\nu}) Y^\rho \\ &+ \nabla_a Y^\mu \nabla_b Y^\nu B_{\mu\nu} + 2\partial_a X_0^\mu \nabla_b Y^\nu (\nabla_\lambda B_{\mu\nu}) Y^\lambda \\ &+ \frac{1}{2}\partial_a X_0^\mu \partial_b X_0^\nu Y^\rho Y^\lambda \big(\nabla_\rho \nabla_\lambda B_{\mu\nu} + R^\sigma{}_{\rho\lambda\mu} B_{\sigma\nu} + R^\sigma{}_{\rho\lambda\nu} B_{\mu\sigma}\big)\Big]. \end{aligned} \tag{33}$$

We can once again change from general coordinates Y^μ to local Euclidean frame coordinates Y^i. Since all of these indices contract, the corresponding vielbeins cancel and we can rewrite (33) with local Euclidean frame indices on all the quantum fields Y. We can rewrite this into a $U(1)$-invariant form by introducing the field strength $H = dB$,

$$S_{AS}[X_0+\tilde{Y}] = \mathcal{O}(< Y^2)+\frac{1}{4\pi\alpha'}\int d^2\xi\epsilon^{ab}\Big[H_{\mu ij}\partial_a X_0^\mu \nabla_b Y^i Y^j + \frac{1}{2}\nabla_i H_{\mu\nu j}\partial_a X_0^\mu \partial_b X_0^\nu Y^i Y^j + \frac{1}{4}H_{ijk}Y^i \nabla_a Y^j \nabla_b Y^k\Big]. \tag{34}$$

Here we have only written down the $\geq \mathcal{Y}^\ni$ that appear in the diagrams we will cover.

Since the dilaton is a scalar in spacetime, expanding its action is simple,

$$S_D[X_0+\tilde{Y}] = \frac{1}{4\pi}\int d^\xi\sqrt{\gamma}\Big(R_\gamma\phi(X_0) + R_\gamma\nabla_i\phi(X_0)Y^i + \frac{1}{2}R_\gamma\nabla_i\nabla_j\phi(X_0)Y^i Y^j\Big) + \mathcal{O}(Y^3). \tag{35}$$

3.2 Calculation of the Weyl anomaly

Classically the actions S_P and S_{AS} are invariant under a change in the metric $\delta\gamma^{ab} = -f\gamma^{ab}$. We will see that this no longer holds at the quantum level. To find the quantum expectation value of the Weyl anomaly we write

$$\left\langle \frac{\delta S}{\delta f}\right\rangle = -\left\langle \gamma^{ab}\frac{\delta S}{\delta\gamma^{ab}}\right\rangle = -\frac{\sqrt{\gamma}}{4\pi}\left\langle T^a_{\ a}\right\rangle. \tag{36}$$

We see that the Weyl anomaly manifests itself in the trace of the energy-momentum tensor. On the classical level T_{ab} is traceless, but on the quantum level this is no longer the case. We can make an ansatz for how the trace of the energy-momentum tensor should look like by noting that it should have the following criteria [15]:

- Worldsheet dimension 2
- Spacetime dimension 0

- Scalar under worldsheet diffeomorphisms
- Invariant under spacetime diffeomorphisms
- $U(1)_B$ invariant
- Local in X_0 and polynomial in derivatives of X_0
- Invariant under constant shifts in ϕ

The trace of T_{ab} can then be parametrized by three unknown functions[4]

$$T^a_a = \partial_a X^\mu_0 \partial_b X^\nu_0 \left(\beta^G_{\mu\nu}(X_0)\gamma^{ab} + \beta^B_{\mu\nu}(X_0)\epsilon^{ab}\right) + \beta^\phi(X_0)R_\gamma. \tag{37}$$

$\beta^G_{\mu\nu}, \beta^B_{\mu\nu}$ and β^ϕ have the same structure around them as $G_{\mu\nu}, B_{\mu\nu}$ and ϕ have in their respective actions and therefore get the suggestive label. These β-functions are related but not equal to the familiar β-functions from renormalization group theory. We will, however, not focus on the renormalization group and simply view them as functions to be determined. See, for example, the review by Callan and Thorlacius [13] for a precise treatment of this relation to the renormalization group. These β-functions can now be determined by calculating contributions to T^a_a and grouping these in $\beta^G_{\mu\nu}, \beta^B_{\mu\nu}$ and β^ϕ. Each β-function is then put to zero for each order of α', providing us with consistency equations on the fields $G_{\mu\nu}, B_{\mu\nu}$ and ϕ.

In the conformal gauge the trace of the energy-momentum tensor takes the form $T^a_a = 2T_{+-}$. The energy-momentum conservation Equation (20) can be written as

$$\nabla^{\bar{z}}\langle T_{\bar{z}z}\rangle + \nabla^z\langle T_{zz}\rangle = 0. \tag{38}$$

The Fourier transform of this results in

$$q_+\langle T_{+-}\rangle + q_-\langle T_{++}\rangle = 0. \tag{39}$$

We can then find the contribution of the Weyl anomaly to the trace of the energy-momentum tensor, $\langle T_{+-}\rangle$, by considering $\langle T_{++}\rangle$ and linking the two through energy-momentum conservation (39).

3.3 The symmetric and anti-symmetric β-functions

The contribution to $\langle T_{++}\rangle$ from S_P to first-order can be computed by inserting $\langle T_{++}\rangle$ in the first-order diagram coming from the expanded Polyakov action (30). The energy-momentum tensor in the conformal gauge is given by $T_{ab} = (\nabla_a Y)^i(\nabla Y)^j\delta_{ij}$, where the covariant derivative has a piece involving the spin connection. Any term involving the spin connection would have to combine into a curvature tensor to form a covariant result. However, this would give a contribution that is fourth-order in spacetime derivatives as the one-loop diagram coming from S_P already has a Riemann tensor. This one-loop diagram with the energy-momentum insertion is shown in Figure 3. The contribution of this diagram is

$$\int \frac{d^2l}{2\pi}\frac{-l_+(l_+ + q_+)}{l^2(l+q)^2}\left\{R_{\mu kl\nu}\partial_a X^\mu_0 \partial^a X^\nu_0\right\}(q)\delta^{kl} = \left\{R_{\mu\nu}\partial_a X^\mu_0 \partial^a X^\nu_0\right\}(q)\int \frac{d^2l}{2\pi}\frac{l_+(l_+ + q_+)}{l^2(l+q)^2}. \tag{40}$$

[4] One could construct one more term: $\beta^V_\mu(X_0)\gamma^{ab}\nabla_a\partial_b X^\mu_0$. However, since X^μ_0 is invariant under the addition of any vector (Lorentz invariance), this term can be ignored. [15]

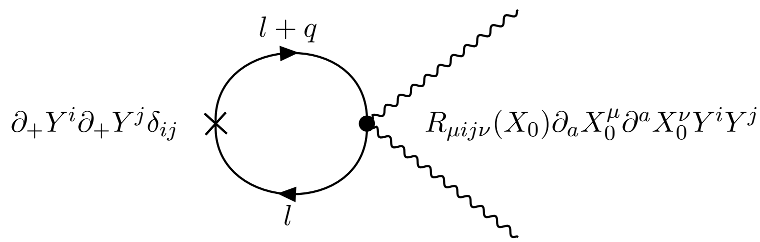

Figure 3: One-loop contribution from S_P to $\langle T_{+-}\rangle$. The straight lines denote the quantum fields Y^i, the wiggly lines denote the classical background fields X_0^μ. The energy-momentum tensor T_{+-} is inserted at the cross with momentum q. The diagrams in this thesis are made with the help of the TikZ-Feynman LaTeX package [19].

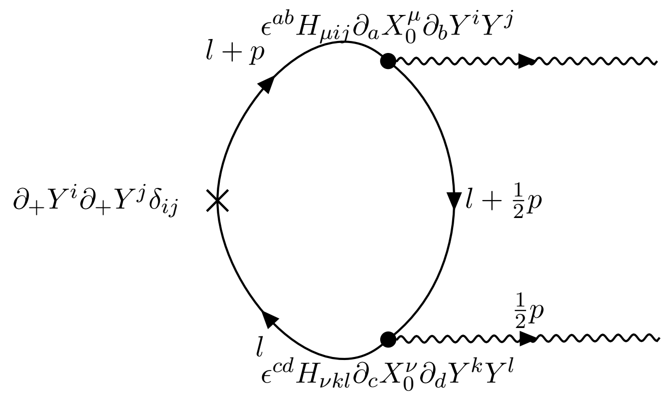

Figure 4

Here {} denotes the Fourier transform. Since we will not use these terms that need to be Fourier transformed, we leave them like this until we transform back to position space at the end of the calculation. This integral can be solved using the integrals in Appendix B. We then find the following expression for it

$$-\frac{1}{4}\frac{q_+}{q_-}\left\{R_{\mu\nu}\partial_a X_0^\mu \partial^a X_0^\nu\right\}(q), \tag{41}$$

where we have used that $q^2 = 2q_+q_-$. The contribution from this diagram to $\langle T_{+-}\rangle$ can now be determined through the energy-momentum conservation (39). Once this has been done, we go back to position space,

$$\langle T_{+-}(\xi)\rangle = \frac{1}{4}R_{\mu\nu}(X_0)\partial_a X_0^\mu(\xi)\partial^a X_0^\nu(\xi). \tag{42}$$

As this term is proportional to $\gamma^{ab}\partial_a X_0^\mu \partial_b X_0^\nu$ it contributes to $\beta^G_{\mu\nu}$ in Equation (37). Similarly, we can construct two terms from S_{AS} that contribute to the Weyl anomaly at lowest order. These are shown in diagrams 4 and 5.

For the diagram in Figure 4 we would need a three-point standard integral. So far we have failed to find a standard integral for this, some links to the literature and possible approaches are discussed in Appendix B.2.

Another first loop diagram from S_{AS} (33) is given in Figure 5. Here, the momentum has been

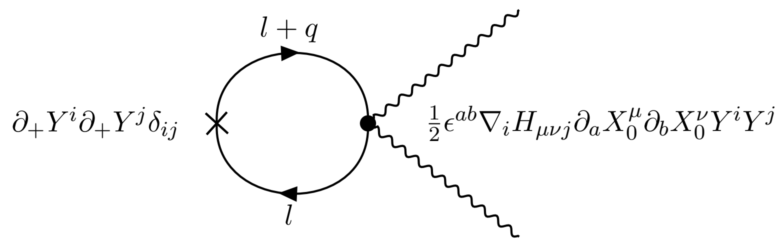

Figure 5: One of two contributions of S_{AS} to $\langle T_{+-}\rangle$.

inserted in the same way as for the S_P diagram (Figure 3). Its contribution is

$$-\frac{1}{2}\left\{\nabla^k H_{\mu\nu k}\partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab}\right\}\int \frac{d^2 l}{2\pi}\frac{l_+(l_+ + q_+)}{l^2(l+q)^2}. \tag{43}$$

The momentum loop is exactly the same as in Equation (40). We thus find the following contribution to the anomaly,

$$\langle T_{+-}(\xi)\rangle = -\frac{1}{8}\epsilon^{ab}\nabla^k H_{\mu\nu k}\partial_a X_0^\mu \partial_b X_0^\nu. \tag{44}$$

We now turn to the dilaton action (11). It is not Weyl invariant and it enters the calculations at one order of α' higher than contributions from S_P and S_{AS}. The tree-level contribution of the dilaton will then cancel the one-loop results that we found in the above treatment. For simplicity we will take our worldsheet to be flat, the dilaton action would then disappear as it depends on the worldsheet curvature R_γ. The energy-momentum tensor, however, is found by variating the worldsheet metric. So even when we take our metric to be flat, $\gamma^{ab} = \eta^{ab}$, there will still be some contribution coming from variating the dilaton action.

$$T^D_{ab} = \frac{4\pi}{\sqrt{\gamma}}\frac{\delta S_D}{\delta \gamma^{ab}}\Big|_{\gamma_{ab}=\eta_{ab}} = \partial_a\partial_b\phi - \delta_{ab}\Box_\gamma\phi, \tag{45}$$

where $\Box_\gamma$ denotes the d'Alembertian on the worldsheet. The trace of this part of the energy-momentum tensor is non-vanishing,

$$T^D_{-+} = \Box_\gamma\phi(X). \tag{46}$$

To couple this to our results of the previous section we need this trace in the background field X_0^μ. Equation (46) then becomes

$$\Box_\gamma\phi(X_0) = \Box X_0^\mu \partial_\mu\phi(X_0) + \partial_a X_0^\mu \partial^a X_0^\nu \partial_\mu\partial_\nu\phi(X_0). \tag{47}$$

This can be rewritten using the equation of motion for X^μ, as we are on a flat worldsheet the only relevant actions are S_P (6) and S_{AS} (10). After variating these actions we contract with the spacetime metric to find

$$\Box X_0^\mu = -\Gamma^\mu_{\lambda\sigma}\partial_a X_0^\lambda \partial^a X_0^\sigma + \frac{1}{2}H^\mu_{\lambda\sigma}\partial_a X_0^\lambda \partial_b X_0^\sigma \epsilon^{ab}. \tag{48}$$

Inserting this into (47) we arrive at a spacetime covariant result,

$$\Box_\gamma\phi(X_0) = \nabla_\mu\nabla_\nu\phi(X_0)\partial_a X_0^\mu \partial^a X_0^\mu + \frac{1}{2}\nabla^\lambda\phi(X_0)H_{\lambda\mu\nu}(X_0)\partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab} \tag{49}$$

This now has the same form as the results of the previous section. Taking (49) together with the results (42), (44) and Figure 4 we find for the trace of the energy-momentum tensor.

$$\langle T_{+-}\rangle = \frac{1}{4}\big(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + \dots\big)\partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{8}\big(\nabla^\lambda H_{\mu\nu\lambda} - 2\nabla^\lambda\phi H_{\lambda\mu\nu}\big)\epsilon^{ab}\partial_a X_0^\mu \partial_b X_0^\nu. \tag{50}$$

The functions inside the brackets are the beta functions that we used to parametrize the Weyl anomaly in equation (37). We used the dots to denote the unknown result coming from the diagram of Figure 4 For the theory to be anomaly free we require each of them to vanish, namely

$$\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + \dots = 0, \tag{51}$$
$$\beta^B_{\mu\nu} = \nabla^\lambda H_{\mu\nu\lambda} - 2\nabla^\lambda\phi H_{\lambda\mu\nu} = 0. \tag{52}$$

3.4 The dilaton β-function

As we saw in Equation (37), there is one more function to be found: the dilaton β-function. As the dilaton enters one loop counting parameter α' higher then the metric and the Kalb-Ramond field we have to consider contributions up to $\mathcal{O}(\alpha')$ here, these come from the two-loop order of S_P and S_{AS} and the one-loop order of S_D. In our previous arguments we could take the worldsheet to be flat, but to take into account the contribution coming from the dilaton action (35) itself we need to take the worldsheet to be curved. A curved worldsheet brings in complications however, but we can consider the following trick to circumvent these complications. We can always write a two dimensional metric as a conformal scale factor times the flat metric, $\gamma_{ab} = e^f\delta_{ab}$. As we saw in equation (36), Weyl invariance implies that the trace of the energy-momentum tensor is zero. At the very least the first variation of the expectation value of the energy-momentum trace should be zero. Evaluating this on a flat worldsheet gives

$$\frac{\delta}{\delta f}\langle T_{-+}(0)\rangle_{e^f\delta_{ab}}\Big|_{f=0} = -\frac{1}{Z}\int \mathcal{D}X\frac{\delta S}{\delta f}e^{-S}T{-}{+}(0)\Big|_{f=0} - \frac{\partial\log(Z)}{\partial f}\langle T_{-+}(0)\rangle\Big|_{f=0}. \tag{53}$$

The second term contains the trace of the energy-momentum tensor on a flat worldsheet, which is zero if we demand the β-functions (51) and (52) to hold. By using Equation (36) to write the variation of the action to f as the energy-momentum tensor we find

$$\frac{\delta}{\delta f}\langle T_{-+}(0)\rangle_{e^f\delta_{ab}}\Big|_{f=0} = -\frac{\sqrt{\gamma}}{2\pi}\langle T_{-+}T_{-+}\rangle_{\delta_{ab}}. \tag{54}$$

We will now attempt to find the dilaton β-function by considering the energy-momentum two-point function and then we can use Equation (54) to get to the anomaly. There is one $\mathcal{O}(\alpha'^0)$ diagram contributing to β^ϕ, given in Figure 6. This diagram corresponds to the following loop integral

$$\langle T_{++}(-q)T_{++}(q)\rangle = 2D\int d^n l\frac{(l_+ + q_+)^2 l_+^2}{l^2(l+q)^2}. \tag{55}$$

Using integrals from Appendix B.1 we find that this is equal to

$$\langle T_{++}(-q)T_{++}(q)\rangle = \frac{\pi D}{6}\frac{q_+^3}{q_-}. \tag{56}$$

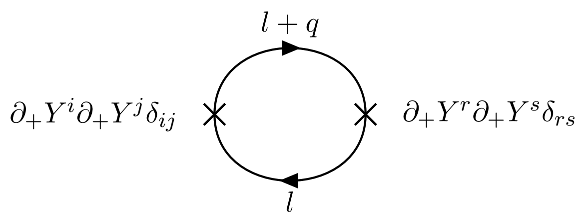

Figure 6: $\mathcal{O}(\alpha'^0)$ contribution to β^ϕ.

Using the momentum conservation equation (39) twice this becomes

$$\langle T_{+-}(-q) T_{+-}(q) \rangle = \frac{\pi D}{6} q_+ q_-. \tag{57}$$

In position space the q^2 factor becomes the d'Alembertian of the delta function

$$\langle T_{+-}(\xi) T_{+-}(0) \rangle = -\frac{\pi D}{12} \Box \delta(\xi). \tag{58}$$

We now use Equation (54) to find

$$\frac{\delta}{\delta f(\xi)} \langle T_{+-}(0) \rangle_{e^f \delta_{ab}} \Big|_{f=0} = -\frac{1}{4\pi} \langle T_{+-}(\xi) T_{+-}(0) \rangle_{\delta_{ab}}. \tag{59}$$

Using Equation (58) to integrate (59) we find for the trace of the energy-momentum tensor,

$$\langle T_{+-}(\xi) \rangle_{e^f \delta_{ab}} = \frac{D}{48} \Box f. \tag{60}$$

As we saw in Equation (15) the d'Alembertian of the scale factor can be written as the worldsheet Ricci scalar, we then find for the contribution to the Weyl anomaly of Diagram 6

$$\langle T_{+-}(\xi) \rangle_{e^f \delta_{ab}} = -\frac{D}{24} \sqrt{\gamma} R_\gamma. \tag{61}$$

This is the only $\mathcal{O}(\alpha'^0)$ contribution to the dilaton β-function. To set it to zero it appears that the dimension of spacetime should be set to zero. This is not true however, as we have not taken the ghost fields into account. As the ghost fields do not couple to the ordinary matter of the bosonic string, we do not consider them here. If one does compute the contribution of the ghost fields to the Weyl anomaly (a clear treatment of this is given in David Tong's lecture notes [3]) a second term to Equation (61) is found which puts the dimension of the spacetime to 26. Accordingly, consistency of Weyl invariance at the quantum level gives us the critical dimension of bosonic string theory.

We now turn to $\mathcal{O}(\alpha')$ contributions to β^ϕ. These come from the two-loop level of S_P and S_{AS} and the one-loop level of S_D. Let's start with diagrams coming from S_P, in the expanded action (30) there is one three-point vertex and one four-point vertex. Only the four-point vertex gives contributions to β^ϕ. Two diagrams can be constructed with this vertex, they are given in Figure 7 and 8. The diagram in Figure 7 has the following contribution to the Weyl anomaly,

$$\frac{\{R\}\alpha'}{4\pi} p^2 \int d^n l \frac{l_+ (l-p)_+}{l^2 (p-l)^2} \int d^n q \frac{q_+ (q-p)_+}{q^2 (p-q)^2}, \tag{62}$$

where we properly symmetrized the vertex to get the correct result. Note that there was a factor of $\frac{1}{3}$ coming from the four-point vertex a that the diagram has a symmetry factor of 24 associated

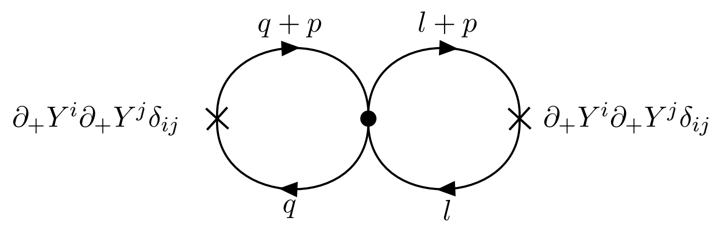

Figure 7: Two-loop contribution of S_P to β^ϕ. The vertex in the middle has an associated vertex factor of $\frac{\alpha'}{12\pi} R_{ijkl} Y^j Y^k \partial^a Y^i \partial_a Y^l$. Again p is the external momentum and l and q the loop momenta.

with it. It is straightforward to solve these two integrals by using the expressions in Appendix B.1. After expanding for small $\epsilon = 2 - n$ we find

$$\langle T_{++}(p) T_{++}(-p) \rangle = \alpha' \pi \{R\} \frac{p_+^4}{p^2} + \mathcal{O}(\epsilon). \tag{63}$$

Going through the same steps as in Equations (58) to (61) we find the contribution to the Weyl anomaly of Figure 7 to be

$$\langle T_{+-}(\xi) \rangle = \frac{\alpha' \pi}{8} R_G \sqrt{\gamma} R_\gamma. \tag{64}$$

Note that both the worldsheet Ricci scalar, R_γ and the spacetime Ricci scalar, R_G appear in this expression.

The second diagram coming from S_P is given in Figure 8. It is divergent so we will have to cancel this divergence with a counterterm later. The loop integral is relatively easy and we quickly find the following contribution

$$\langle T_{++}(q) T_{++}(-q) \rangle = -\frac{4}{3} \{R\} \frac{\alpha'}{2\pi} \int d^n q \frac{1}{q^2} \int d^n l \frac{l_+^2 (l-p)_+^2}{l^2 (p-l)^2} = -\{R\} S \frac{\alpha' p_+^4}{9 p^2}. \tag{65}$$

The $\delta(\epsilon)$ divergence coming from S in Equation (65) can be canceled by inserting a two-point counterterm vertex in place of the vacuum loop in Figure 8. We will also encounter a convergence in the H^2 proportional part of the Weyl anomaly, this can also be absorbed by the same counterterm. We will thus first determine the two-loop diagrams coming from the expanded antisymmetric action (33) before we turn to this counterterm.

S_{AS} produces two diagrams that contribute to the α' level of the dilaton beta function. The first is given in Figure 9. The vertices in this diagram have to be symmetrized to get the right result. Each vertex has an additional factor of $\frac{1}{3}$ associated to it. At first sight, this diagram is harder to solve than what we saw before. However, by computing the worldsheet contractions first, some of the denominators can be removed to simplify matters. There are four momentum terms in the numerator coming from the worldsheet derivatives in the vertices. Each vertex also adds an epsilon tensor to the equation, these can be written as

$$\epsilon^{ab} \epsilon^{cd} = f(n) (\gamma^{ad} \gamma^{bc} - \gamma^{ac} \gamma^{bd}). \tag{66}$$

Here the dimension dependent function $f(n)$ appears because the epsilon tensor is only defined in two dimensions. By going to $n = 2 - \epsilon$ dimensions through dimensional regularization some regularization scheme needs to be added to this relation. Here we follow Metsaev and Tseytlin [20] and set $f(n) = 1 + \epsilon f_1 + \mathcal{O}(\epsilon^2)$, where we keep f_1 arbitrary for now. Note that the $\mathcal{O}(\epsilon)$ term gives

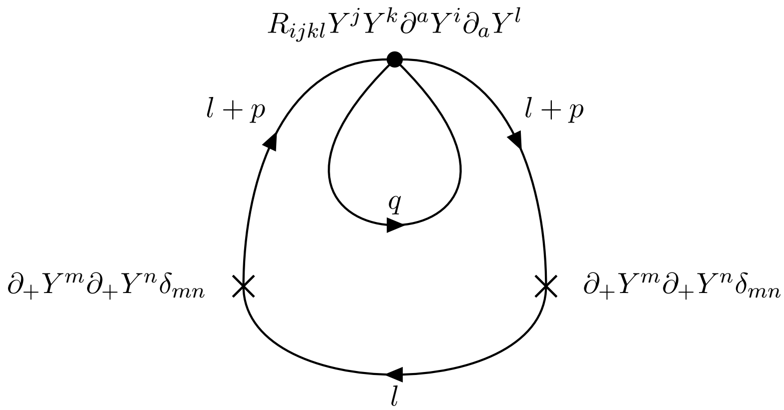

Figure 8: Diagram constructed from fourth-order vertex term in Equation (30), note that it is proportional to the divergent vacuum loop, which we parameterized by the function S. (259)

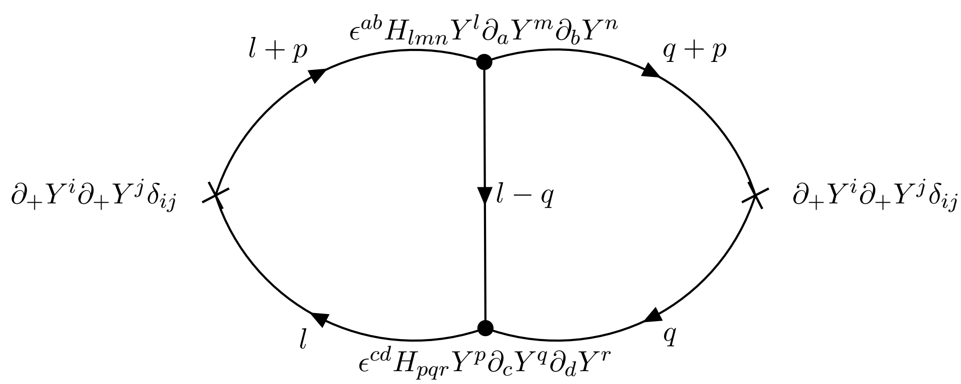

Figure 9: One of two contributions of S_{AS} to the α' order of β^{ϕ}. The loop momenta are l and q, p is the external momentum with which $\langle T_{++}\rangle$ is inserted.

a finite contribution when one encounters divergent results that are then subtracted. After writing the contracting terms in the numerator in terms of the denominator powers we find the following expression for the diagram in Figure 9,

$$\frac{1}{4}\frac{1}{9}\frac{\alpha'}{2\pi}f(n)\{H^2\}\int d^2l\int d^2q\frac{l_+(l+p)_+q_+(q+p)_+}{(l+p)^2l^2(l-q)^2q^2(q+p)^2}\Big[(l-q)^2\big(l^2-2p^2+(l+p)^2 + q^2+(q+p)^2-(l-q)^2\big)+q^2(l+p)^2-q^2(q+p)^2+l^2(q+p)^2-l^2(l+p)^2\big)\Big], \quad (67)$$

where $H^2 = H_{ijk}H^{ijk}$. Each term in the square brackets cancels at least one denominator. To keep the calculation clear we will split this integral in three parts. We denote the integral proportional to the $(l-q)^2(..)$ term in the square brackets by J_1. Note that the second and fourth term are the same when we relabel l and q, the same goes for the third and last term. We will denote the integral resulting from the $q^2(l+p)^2$ term by J_2 and the integral resulting from the $q^2(q+p)^2$ term by J_3. In J_1 the $(l-q)^2$ term cancels the connecting propagator and these integrals are reduced to products of one-loop integrals. Some of these have to be zero by dimensional analysis, we are left with

$$\begin{aligned} J_1 &= f(n)\{H^2\}\frac{\alpha'}{8\pi}\frac{1}{9}\int d^2l\int d^2q\big(l_+(l+p)_+q_+(q+p)_+\big)\left[-\frac{l^2+q^2+2l\cdot q}{l^2(p-l)^2q^2(p-q)^2}-2p^2\frac{1}{(p-l)^2l^2q^2(p-q)^2}\right] \\ &= -\frac{3\alpha'}{\pi}f(n)\{H^2\}\frac{(n-2)^2(I_2[1]p^2-2S)^2p_+^4}{32(n-1)^2p^2}, \end{aligned} \quad (68)$$

with the scalar integrals $I_2[1]$ and S as defined in Equations (251) and (255). The J_2 integral can be expressed in the so-called sunset integrals (Appendix B.3),

$$\begin{aligned} J_2 &= f(n)\{H^2\}\frac{\alpha'}{8\pi}\frac{1}{9}\int d^2l\int d^2q\frac{l_+(l+p)_+q_+(q+p)_+}{(l+p)^2(l-q)^2q^2} \\ &= \{H^2\}\frac{\alpha'}{8\pi}\frac{1}{9}\big(I_s[l_+^2q_+^2]-2p_+I_s[l_+^2q_+]+p_+^2I_s[l_+q_+]\big), \end{aligned} \quad (69)$$

with the I_s integrals expressed in Equation (281), (282) and (283). The final integral, J_3, can be expressed as

$$\begin{aligned} J_3 &= -f(n)\{H^2\}\frac{\alpha'}{8\pi}\frac{1}{9}\int d^2l\int d^2q\frac{l_+(l+p)_+q_+(q+p)_+}{q^2(l-q)^2(q+p)^2} \\ &= -f(n)\{H^2\}\frac{\alpha'}{\pi}\frac{n(n-2)(I_2[1]p^2-2S)Sp_+^4}{36(n^2-1)p^2}. \end{aligned} \quad (70)$$

We can now take the results (68), (69) and (70) together to find the total contribution of the diagram in Figure 9 to the Weyl anomaly. After performing the epsilon expansion and setting $f(n) = 1-f_1\epsilon$ we find

$$\langle T_{++}(p)T_{++}(-p)\rangle = -\frac{\{H^2\}\alpha'\pi p_+^4}{72p^2}\Big(\frac{1}{\epsilon}+\frac{7-3f_1}{3}+\gamma_E+2\log(p^2)+\log(\pi)\Big), \quad (71)$$

where γ_E is the Euler-Mascheroni constant.

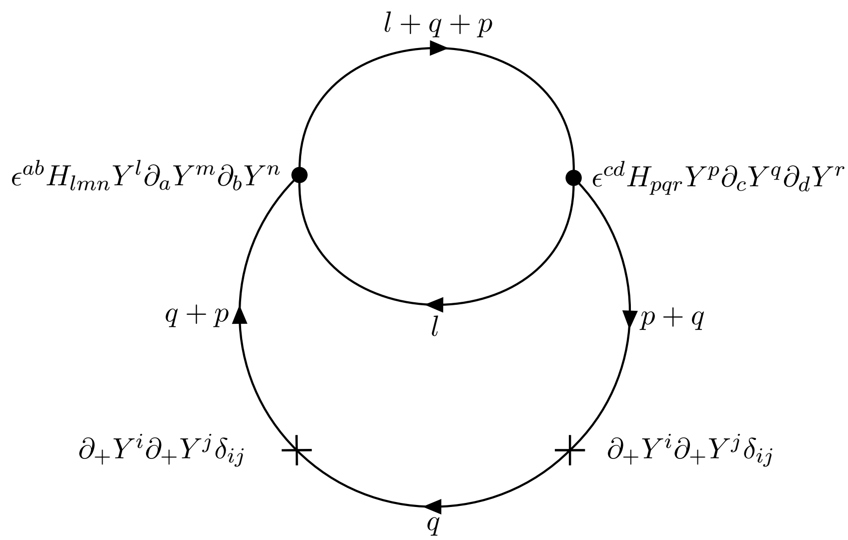

Figure 10: Second contribution of S_{AS} to the α' order of β^ϕ. The loop momenta are l and q. $\langle T_{++}\rangle$ is inserted with external momentum p.

Another diagram with the same S_{AS} vertex is given in Figure 10. It has the following contribution,

$$\{H^2\}\epsilon^{ab}\epsilon^{cd}\int d^nl\int d^nq\frac{(l+q+p)_al_b(q+p)_cl_dq_+(q+p)_+q_+(q+p)_+}{q^2(q+p)^2(q+p)^2l^2(l+q+p)^2}. \tag{72}$$

We can treat the integral over l as a normal one-loop integral with external momentum $q+p$. We again shift the momentum from l to $-l$, the l integral is then of the following form,

$$\int d^nl\frac{(q+p-l)_al_bl_d}{l^2(p+q-l)^2}. \tag{73}$$

The $l_al_bl_d$ contribution is symmetric in a and b and thus cancels when contracted with the antisymmetric ϵ^{ab}. By Lorentz invariance the integral with $(q+p)_al_bl_d$ in the numerator is a combination of a term proportional to its external momentum, $(p+q)_b(p+q)_d$, and a term containing the worldsheet metric γ_{bd}. The entire integral is multiplied with $(p+q)_c$, such that the momentum contribution vanishes when contracted with the epsilon tensors. The part proportional to the metric can be found using the integral expressions from Appendix B.1, such that the l integral reduces to

$$\int d^nl\frac{(q+p)_al_bl_d}{l^2(p+q-l)^2}=(q+p)_a\frac{2S-I_2[1]p^2}{4(n-1)}\gamma_{bd}. \tag{74}$$

This worldsheet metric then contracts with the epsilon tensors through $\gamma_{bd}\epsilon^{ab}\epsilon^{cd}=-f(n)\gamma^{ac}$, as can be derived from Equation (66). Then γ^{ac} combines with $(q+p)_a(q+p)_c$ to cancel a propagator. The q integral can then be solved with the integrals in Appendix B.1. The total contribution of Figure 10 is then found to be

$$\begin{aligned}-f(n)\{H^2\}\int d^nq\frac{1}{4(n-1)}&\left(\frac{\Pi(n)G(n,1,1)}{q^2(q+p)^{2(1+\epsilon/2)}}-\frac{2S}{q^2(p+q)^2}\right)(q_+(q+p)_+q_+(q+p)_+\\ &=\frac{-f(n)\{H^2\}}{4(n-1)}\Big(\Pi(n)G(n,1,1)I_2[q_+(q_+-p_+)q_+(q_+-p_+),1,1+\epsilon/2]\\ &\quad-2SI_2[q_+((q_+-p_+)q_+(q_+-p_+),1,1]\Big).\end{aligned} \tag{75}$$

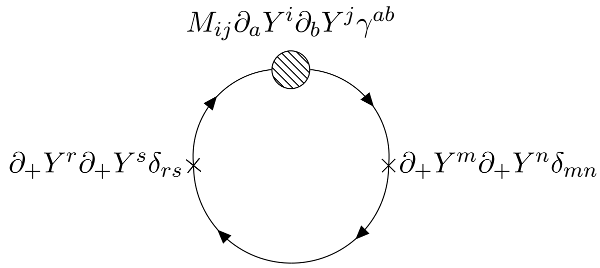

Figure 11: Counterterm to remove H^2 dependent divergences. The blob denotes the vertex, proportional to some tensor M_{ij}, that we are free to determine

We can now take this result together with the integrals resulting from Figure 9 (Equations (68), (69) and (70) to find the total H^2 proportional contribution to $\langle T_{++}T_{++}\rangle$. This turns out to still be convergent. To deal with this we can construct a counterterm diagram, which is given in Figure 11. After solving its Feynman integral we find for it

$$\langle T_{++}T_{++}\rangle = \big\{M\big\}\pi^2\alpha'\frac{p_+^4}{p^2}, \tag{76}$$

where $M \equiv M_{ij}\delta^{ij}$. We can now choose M such that it cancels the divergences coming from the other diagrams and other unwanted terms. To this end we will parametrize M as

$$M = (h_\epsilon \frac{1}{\epsilon} + h_\gamma \gamma_E + h_s S)H^2 + r_s SR. \tag{77}$$

Taking the results of Equation (65), (71), (75) and (76) together we find in total

$$\begin{aligned}\langle T_{++}T_{++}\rangle = -\pi\alpha' H^2 \frac{16h_\epsilon + 3}{24p^2} p_+^4 \frac{1}{\epsilon} - \frac{\pi\alpha' p_+^4}{144p^2}\Big(H^2\big[-18f_1 + 6\gamma_E(16h_\gamma - 8h_\epsilon - 3) + 24S(4h_s + 2h_\epsilon - 1)\\ + 80h_\epsilon + 51 - 12\log p(3 + 8h_\epsilon) + 6\log\pi(16h_\pi - 8h_\epsilon - 3) + 16R(6Sr_S + S + 54) + 1728R\Big) + \mathcal{O}(\epsilon).\end{aligned} \tag{78}$$

Now we see that by picking the following values for the parameters of M gets rid of the divergent terms

$$h_\epsilon = -\frac{3}{16}, \quad h_\gamma = h_\pi = \frac{3}{32}, \quad h_S = \frac{11}{32}, \quad r_S = -\frac{1}{6}. \tag{79}$$

We then get as finite result

$$\langle T_{++}T_{++}\rangle = \frac{\pi\alpha' p_+^4}{8p^2}\big(-2H^2 + f_1 H^2 - 48R + H^2\log p\big). \tag{80}$$

The $\log p$ cannot be renormalized away and its dimension is wrong: we cannot have a logarithm of a non-scalar object. To fix the dimensionality problem a mass scale μ should be added to all diagrams [21], in practice this would replace $\log p$ by $\log p/\mu$, which is dimensionless. We then have a Weyl anomaly that is dependent on the arbitrary external momentum p. This is a result of an infrared divergence, as it will diverge for $p \to 0$. To handle infrared divergences adequately an infrared regulator mass should be added to the problem. This is done for example in [20], there it is found that this regulator mass drops out at the end and makes not contribution to the end result. Here we

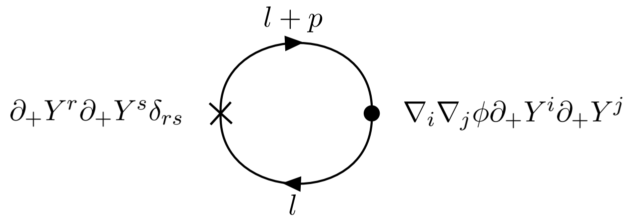

Figure 12: higher-order T_{++} insertion in one-loop diagram.

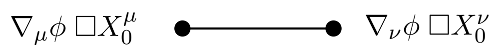

Figure 13: two-point function of T^D_{+-} with itself.

chosen to follow the review by Callan and Thorlacius [13] and not take these infrared divergences into account at all. By going through the steps as we did to get from Equation (58) to Equation (61) we find the contribution of the two-loop diagrams (Figures 7, 9 and 10) to the Weyl anomaly ,

$$\langle T_{+-}(\xi)\rangle_{e^f\delta_{ab}} = \frac{\alpha'}{64}\big(-2H^2 + f_1 H^2 - 48R\big)\sqrt{\gamma}R_\gamma. \tag{81}$$

There is also a one-loop diagram contributing to the α' order of the dilaton β-function. Until now we have taken the lowest order in the insertions of T_{++}, but consider now diagram 12, where one of the inserted energy-momentum tensors has instead a term coming from its background field expansion.

The one-loop diagram contributing to β^ϕ involves an energy-momentum insertion in a graph with the vertex of the expanded dilaton action (35). The diagram is given in Figure 12.It can be expressed as

$$\langle T_{++}(p)T_{++}(-p)\rangle = -4\pi\alpha'\{\nabla^2\phi\}\frac{p_+^4}{p^2}. \tag{82}$$

We subsequently make use of the energy-momentum conservation equation (38) and then go through the same steps as we did to get from Equation (58) to Equation (61). Thereupon, we find as contribution of Figure 12 to the Weyl anomaly

$$\langle T_{+-}(\xi)\rangle = -\frac{1}{2}\alpha'\nabla^2\phi\sqrt{\gamma}R_\gamma. \tag{83}$$

There is also a contribution of the tree-level two-point function of T^D_{+-} with itself, shown in Figure 13. We consider X_0^μ as a quantum field here, its propagator cancels one of the d'Alembertians, the Weyl anomaly contribution of this diagram is

$$\langle T_{+-}(\xi)\rangle = \frac{1}{2}\alpha'(\nabla\phi)^2\sqrt{\gamma}R_\gamma. \tag{84}$$

We now have all contributions to the dilaton β-function, together Equation (81), (83) and (84) give us

$$\beta^\phi = \alpha'\big(-\frac{1}{8}H^2 + \frac{1}{16}f_1 H^2 - 3R + 2(\nabla\phi)^2 - 2\nabla^2\phi\big). \tag{85}$$

4 The bosonic string in Newton-Cartan geometry

4.1 Newton-Cartan geometry

Newton-Cartan geometry is characterized by a symmetric spatial metric h_{mn}, or equivalently, a pair of vielbeins $\{\tau_s, e^i_s\}$ that define the metric through $h_{mn} = e^i_m e^j_n \delta_{ij}$. From these an inverse vielbein pair $\{-v^m, e^n_i\}$ can be constructed, which is defined by its action on the normal vielbeins: $\tau_s e^s_i = 0$ and $\tau_s v^s = -1$. The inverse vielbein e^m_i defines the symmetric inverse spatial metric as $h^{mn} = e^m_i e^j_n \delta^{ij}$. The Galilean data $\{h_{mn}, t_m, h^{mn}, v^m\}$ obeys the following completeness relation,

$$h_{nm}h^{rm} - v^r\tau_n = \delta^r_n. \tag{86}$$

We would now like to construct a covariant derivative that is compatible with the Galilean data $\{\tau_m, h^{mn}\}$, e.g. $D_m\tau_n = 0$ and $D_m h^{rs} = 0$. The most general form of this metric compatible connection is[11]

$$\Gamma^m_{rs} = -v^m\partial_r\tau_s + \frac{1}{2}h^{mn}(\partial_r h_{sn} + \partial_s h_{rn} - \partial_n h_{rs}) + h^{mn}n_{(r}K_{s)n}, \tag{87}$$

where K_{mn} is an arbitrary two-form. This connection has torsion,

$$T^m_{\ rs} \equiv \Gamma^m_{rs} - \Gamma^m_{sr} = v^m(\partial_s\tau_r - \partial_r\tau_s). \tag{88}$$

Because of this torsion we refer to the background as Torsional Newton-Cartan (TNC) geometry. One can also derive the identity [11]

$$K_{mn} = 2h_{s[n}D_{m]}v^s. \tag{89}$$

Upon introducing the shorthand notations

$$\bar{h}_{rs} = h_{rs} - \tau_r m_s - m_s\tau_r, \tag{90}$$

$$\hat{v}^m = v^m - h^{ms}m_s. \tag{91}$$

The connection can be written in the form [14]

$$\Gamma^m_{rs} = -\hat{v}^m\partial_r\tau_s + \frac{1}{2}h^{mt}\big(\partial_r\bar{h}_{st} + \partial_s\bar{h}_{rt} - \partial_t\bar{h}_{rs}\big). \tag{92}$$

The Riemann tensor of this connection is defined in the usual way,

$$R^m_{\ npq} = \partial_p\Gamma^m_{qn} + \Gamma^m_{ps}\Gamma^s_{qn} - (p \longleftrightarrow q). \tag{93}$$

To derive the geodesic equation on a TNC background consider the action of a point particle in a TNC background [22],

$$S = \frac{m}{2}\int d\lambda \frac{1}{N}\bar{h}_{mn}\dot{x}^m\dot{x}^n, \tag{94}$$

where $N \equiv \tau_s\dot{x}^s$. Variating this action with respect to the field x^p gives the equations of motion for a point particle moving on a TNC background, or in other words; the geodesic equation,

$$\ddot{x}^n\bar{h}_{pn} = \Big[\frac{1}{2}\partial_p\bar{h}_{mn} - \partial_m\bar{h}_{pn} + \frac{\tau_p\partial_r\bar{h}_{mn} - \bar{h}_{mn}F_{pr}}{2N}\dot{x}^r - \frac{\dot{N}}{N^2}\tau_p\bar{h}_{mn}\Big]\dot{x}^m\dot{x}^n + \frac{1}{N}\big(\dot{N}\bar{h}_{pn}\dot{x}^n + \tau_p\ddot{x}^m\dot{x}^n\bar{h}_{mn}\big), \tag{95}$$

where we have defined $F = d\tau$. Contracting with h^{pt} greatly simplifies the geodesic equation,

$$\ddot{x}^t + \Gamma^t_{mn}\dot{x}^m\dot{x}^n = \frac{\dot{N}}{N}\dot{x}^t - \frac{\bar{h}_{mn}F_{pr}h^{tp}}{2N}\dot{x}^r\dot{x}^m\dot{x}^n, \tag{96}$$

with the metric compatible connection Γ^t_{mn} as defined in (92).

4.2 The bosonic string in Newton-Cartan geometry

We can arrive at a $D+1$ dimensional Newton-Cartan geometry from a $D+2$ dimensional Lorentz invariant geometry via a null reduction, this was first done in [23]. In the remainder of this section we closely follow the treatment of Gallegos, Gürsoy and Zinnato [14]. Let G_{MN} denote the spacetime metric of the $D+2$ dimensional manifold. We now impose a null isometry on the u direction, the metric can then be decomposed as

$$G_{MN}dx^M dx^N = 2\tau_m(du - m_s dx^s) + h_{mn}dx^m dx^n, \tag{97}$$

where the lower case Latin indices take values from 1 to $D+1$. In this decomposition we recognize the fields that characterize a Newton-Cartan spacetime: $\bar{h}_{mn}$ and τ_m. Using the inner product (97) we can find the form of the Polyakov action for the bosonic string(6),

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-\gamma}\gamma^{\alpha\beta}(\bar{h}_{\alpha\beta} + 2\tau_\alpha\partial_\beta X^u), \tag{98}$$

where $\bar{h}$ was defined in Equation (90) and the Greek indices on the Newton-Cartan fields denote their pullback to the worldsheet, e.g. $h_{\alpha\beta} = h_{mn}\partial_\alpha X^m\partial_\beta X^n$. The string has non-zero momentum along the null direction X^u and by demanding this momentum to be conserved, we demand that the Newton-Cartan string is independent of the null direction u. The momentum current can be determined by

$$P_u^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^u)} = -\frac{\sqrt{-\gamma}}{2\pi\alpha'}\gamma^{\alpha\beta}\tau_\beta. \tag{99}$$

We now consider a Lagrangian where this momentum is conserved off-shell,

$$\mathcal{L}_P = -\frac{1}{4\pi\alpha'}\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \frac{1}{4\pi\alpha'}\big(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\alpha - \epsilon^{\alpha\beta}\partial_\alpha\eta\big)A_\beta. \tag{100}$$

Here we have introduced a Lagrange multiplier A_α which conserves the momentum off-shell through its constraints

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\beta = -\epsilon^{\alpha\beta}\partial_\beta\eta. \tag{101}$$

Using Equation (99) these constraints give $P_u^\alpha = \frac{1}{4\pi\alpha'}\epsilon^{\alpha\beta}\partial_\beta\eta$. From which we see that P_u^α is conserved, $\partial_\alpha P_u^\alpha = 0$. Thus the Lagrangian (100) enforces off-shell momentum conservation. This Lagrangian can be seen to be equal to the one from Equation (98) if one solves the equation of motion for η. This gives $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$, which is solved by $A_\alpha = \partial_\alpha\chi$ with χ a scalar field. Upon identifying $\chi = X^u$ we recover the original Lagrangian (98).

We can rewrite the constraints (101) by introducing the worldsheet zweibein $e_\alpha^{\ a}$ and its inverse,

$$e^\alpha_{\ a} = \frac{1}{e}\epsilon^{\alpha\beta}e_\beta^{\ b}\epsilon_{ba}, \tag{102}$$

where $e = \epsilon^{\alpha\beta}e_\alpha^{\ 0}e_\beta^{\ 1}$. These zweibeins can be used to go from curved worldsheet indices $\{\alpha,\beta\}$ to flat worldsheet indices $\{a,b\}$. The worldsheet metric and its inverse can then be written as

$$\gamma_{\alpha\beta} = \eta_{ab}e_\alpha^{\ a}e_\beta^{\ b} \quad , \quad \gamma^{\alpha\beta} = \eta^{ab}e^\alpha_{\ a}e^\beta_{\ b}. \tag{103}$$

Using these definitions we have $\sqrt{-\gamma} = e$. The constraints (101) can then be written as

$$e\eta^{ab}e^\alpha_{\ a}e^\beta_{\ b}\tau_\beta = -\epsilon^{\alpha\beta}\partial_\beta\eta. \tag{104}$$

We can also consider the antisymmetric tensor on a flat worldsheet $\epsilon^{\alpha\beta} = e\epsilon^{ab}e^{\alpha}{}_{a}e^{\beta}{}_{b}$, here e appears because the Levi-Civita symbol ϵ^{ab} is not a tensor but a tensor density [17]. With this the constraints (104) can be written as

$$\eta^{ab}e^{\beta}{}_{b}\tau_{\beta} = -\epsilon^{ab}e^{\beta}{}_{b}\partial_{\beta}\eta. \tag{105}$$

Following [23] we redefine the two components of the Lagrange multiplier as

$$A_{\alpha} = m_{\alpha} + \frac{1}{2}(\lambda_{+} - \lambda_{-})e_{\alpha}{}^{0} + \frac{1}{2}(\lambda_{+} + \lambda_{-})e_{\alpha}{}^{1}, \tag{106}$$

such that $\lambda_{\pm}$ now take on the role of Lagrange multipliers. With the redefinition (106), the constraint (105) and by writing the worldsheet tensors as zweibeins, the Lagrangian (100) can be written as

$$\mathcal{L}_{P} = \frac{e}{4\pi\alpha'}\Big(- \gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \lambda_{+}e^{\alpha}_{-}(\partial_{\alpha}\eta + \tau_{\beta}) + \lambda_{-}e^{\alpha}_{+}(\partial_{\alpha}\eta - \tau_{\alpha})\Big), \tag{107}$$

where $e^{\alpha}_{\pm} \equiv e^{\alpha}_{0} \pm e^{\alpha}_{1}$. This is a Polyakov-type Lagrangian for the bosonic string in a Newton-Cartan background without matter.

This action can be generalized to include the matter actions S_{AS} and S_{D} that we encountered in Section 3. We start with a null reduction of $S_{P} + S_{AS}$, the dilaton term can be added later. We again use the inner product (97) to write the Polyakov action (6) and the Kalb-Ramond action (10) in a Newton-Cartan background,

$$\mathcal{L}_{P} + \mathcal{L}_{AS} = -\frac{1}{4\pi\alpha'}\Big(\sqrt{-\gamma}\gamma^{\alpha\beta}(\bar{h}_{\alpha\beta} + 2\gamma^{\alpha\beta}\tau_{\alpha}\partial_{\beta}X^{u}) + \epsilon^{\alpha\beta}(\bar{B}_{\alpha\beta} - 2\aleph_{\alpha}\partial_{\beta}X^{u})\Big), \tag{108}$$

where

$$\aleph_{\alpha} = B_{u\alpha}, \tag{109}$$
$$\bar{B}_{\alpha\beta} = B_{\alpha\beta} - 2\aleph_{[\alpha}m_{\beta]}. \tag{110}$$

The momentum along the null direction can again be computed as

$$P^{a}_{u} = \frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}X^{u})} = \frac{1}{2\pi\alpha'}\big(- \sqrt{-\gamma}\gamma^{\alpha\beta}\tau_{\beta} + \epsilon^{\alpha\beta}\aleph_{\beta}\big). \tag{111}$$

This momentum can be conserved off-shell through the constraint of the Lagrange multiplier A_{α},

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\Big(\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta} + 2\big(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_{\alpha} - \epsilon^{\alpha\beta}\partial_{\alpha}\eta - \epsilon^{\alpha\beta}\aleph_{\alpha}\big)A_{\beta}\Big). \tag{112}$$

Which is again identical to the original Lagrangian (108) through the equation of motion of the η field. This constraint reads,

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_{\alpha} - \epsilon^{\alpha\beta}\partial_{\alpha}\eta - \epsilon^{\alpha\beta}\aleph_{\alpha} = 0. \tag{113}$$

By making the same field field redefinition as in (106) and by using the worldsheet zweibeins (102) we can write the Lagrangian (112) as

$$\mathcal{L}_{\text{matter}} = \frac{e}{4\pi\alpha'}\Big[- \gamma^{\alpha\beta}\bar{h}_{\alpha\beta} - \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta} + \lambda_{+}e^{\alpha}_{-}\big(\partial_{\alpha}\eta + \aleph_{\alpha} + \tau_{\alpha}\big) + \lambda_{-}e^{\alpha}_{+}\big(\partial_{\alpha}\eta + \aleph_{\alpha} - \tau_{\alpha}\big)\Big]. \tag{114}$$

We would like to rewrite this Lagrangian a little bit further before we start quantizing it. Just as for the Riemannian bosonic string we will use dimensional regularization to investigate the

Weyl anomaly. A requirement for using dimensional regularization is that the theory is manifestly covariant. The $\lambda_\pm$ terms in the above theory are not manifestly covariant as they have an explicit $+$ and $-$ coordinate. This will cause problems when we want to go from 2 worldsheet dimensions to $n = 2 - \epsilon$ worldsheet dimensions. By recognizing the $+$ and $-$ terms in (114) as lightcone coordinates we can use write (114) in a manifestly covariant form. To do this we can write

$$(\aleph_\alpha + \partial_\alpha \eta)(\lambda_+ e^\alpha_- + \lambda_- e^\alpha_+) = -\frac{1}{2}(\aleph_\alpha + \partial_\alpha \eta)\gamma^{ab}\lambda_a e^\alpha_b. \tag{115}$$

and

$$\tau_\alpha(\lambda_+ e^\alpha_- - \lambda_- e^\alpha_+) = -\frac{1}{2}\tau_\alpha \epsilon^{ab}\lambda_a e^\alpha_b. \tag{116}$$

The complete TNC Lagrangian can now be written as

$$\mathcal{L}_{\text{matter}} = \frac{e}{4\pi\alpha'}\Big[-\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} - \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta} - \frac{1}{2}(\aleph_\alpha + \partial_\alpha\eta)\gamma^{ab}\lambda_a e^\alpha_b - \frac{1}{2}\tau_\alpha\epsilon^{ab}\lambda_a e^\alpha_b\Big]. \tag{117}$$

Here the Latin letters a and b denote the flat worldsheet coordinates, we will take them to be in lightcone coordinates for the remainder of this thesis.

4.3 Symmetries of the TNC action

In the $D + 2$ dimensional Lorentz invariant spacetime the Kalb-Ramond field introduces a U(1) symmetry in the action of the bosonic string. The action is invariant under

$$\delta_\Lambda B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M. \tag{118}$$

This symmetry is still present after the null reduction (97). The TNC background fields then transform as [14],

$$\delta_\Lambda \bar{B}_{mn} = \partial_m \Lambda_n - \partial_n \Lambda_m \tag{119}$$
$$\delta_\Lambda \aleph_m = \partial_m \Lambda_u, \tag{120}$$

where u is the null direction of the $D + 2$ dimensional Lorentz manifold G_{MN}. Just as in the Lorentzian case, the action (114) is invariant under (119). To have invariance under (120) one needs to impose that the worldsheet field η transforms as $\delta_\Lambda \eta = -\Lambda_u$[14].

A different U(1) symmetry comes from the m_s field. The action (117) is invariant under [14]

$$m_s \to m_s + \partial_s \sigma. \tag{121}$$

A good final check on our expanded actions in Section 5 6 and the TNC β-functions would be to check for U(1) invariance, but this is a time consuming task which is not carried out in this thesis.

5 Quantizing the TNC bosonic string

5.1 Background field expansion

Along the same lines as Section 3.1 we will now do the background field quantization [13, 14] for the string embeddings X^μ, λ_a and η. Let

$$\begin{aligned} X^m &= X_0^m + l_s \tilde{Y}^m, \\ \lambda_a &= \lambda_a^0 + l_s \tilde{\Lambda}_a, \\ \eta &= \eta_0 + l_s \tilde{E}. \end{aligned} \tag{122}$$

Here X_0^m, λ_a^0 and η^0 denote the classical (background) fields and $\tilde{Y}^m$, $\tilde{\Lambda}_a$ and $\tilde{E}$ denote the quantum perturbations on them. As before $\tilde{Y}^m$ is not covariant and we have to replace it somehow. Since $\tilde{\Lambda}_a$ and $\tilde{E}$ are spacetime scalars they automatically transform the right way and we do not have to make any replacements for these fields. To replace $\tilde{Y}^m$ we again consider the geodesic connecting X_0^m and $X_0^m + \tilde{Y}^m$ as in Section 3.1. This time we consider the geodesic equation in a TNC background, which we found in Equation (96). The following expansion is a solution to that geodesic equation and satisfies $x^m(0) = X_0^m$, $x^m(1) = X_0^m + l_s \tilde{Y}^m$ and $\dot{x}^m(0) = l_s Y^m$ [14],

$$x^m(\lambda) = X_0^m + \lambda l_s Y^m + \frac{\lambda^2}{2} l_s^2 Y_2^m + \mathcal{O}(l_s^3), \tag{123}$$

where Y_2^m is a function yet to be determined. Inserting the solution (123) into the geodesic equation (96) we find the relation,

$$Y_2^t + \Gamma^t_{mn} Y^m Y^n = \frac{Y^t Y_2^m \tau_m + Y^t \partial_r \tau_s Y^s Y^r - \frac{1}{2} \bar{h}_{mn} F_{sr} h^{st} Y^m Y^n Y^r}{Y^p \tau_p}. \tag{124}$$

Following [14] we find that this is solved by

$$Y_2^t = -\Gamma^t_{mn} Y^m T^n - G^t_{mn} Y^m Y^n, \tag{125}$$

where the tensor G^t_{mn} satisfies

$$\tau_{(r} G^t_{mn)} = \tau_s G^s_{(mn} \delta^t_{r)} - \frac{1}{2} \bar{h}_{(mn} F_{r)s} h^{st}. \tag{126}$$

Some interesting TNC properties can be derived from Equation (126). We can first contract the r and the t index and then contract with $h^{nq} h^{mp}$ to find $\tau_s G^s_{mn} h^{nq} h^{mp} = 0$. Alternatively we can first contract Equation (126) with $h^{nq} h^{mp}$ and then contract the r-index with the t-index to find

$$F_{mn} h^{np} h^{mq} = 0. \tag{127}$$

This is the twistless torsion condition. It enforces causality in torsional Newton-Cartan geometry [12] and we see that it arises naturally when demanding that the bosonic string follows a geodesic. Using the twistless torsion condition (127) one can show that

$$G^t_{mn} = \frac{1}{2} \bar{h}_{mn} a_s h^{st} \tag{128}$$

is a solution to Equation (126) if we take the torsion to be twistless, in other words, if Equation (127) holds. We have defined the acceleration $a_s = \hat{v}^t F_{ts}$. Inserting the solution for Y_2^m (125) into (123) with $\lambda = 1$ we can express the quantum field $\tilde{Y}^m$ in terms of the covariant vector Y^m,

$$\tilde{Y}^m = Y^m - \frac{1}{2} l_s \big(\Gamma^m_{rs} + G^m_{rs}\big) Y^r Y^s + \dots \tag{129}$$

This suggests that it is useful to consider a new connection,

$$\mathring{\Gamma}^t_{mn} = \Gamma^t_{mn} + \frac{1}{2} \hat{v}^t F_{mn} + G^t_{mn}, \tag{130}$$

which is symmetric in its lower indices, and thus torsion free. It is also invariant under the $U(1)_m$ transformation of Equation (121). This new connection is no longer compatible with the Galilean metric data h^{mn} and τ_m, but has the following action on them:

$$\mathring{D}_m \tau_n = \frac{1}{2} F_{mn} \qquad , \qquad \mathring{D}_r h^{mn} = a_t h^{t(m} \delta^{n)}_r. \tag{131}$$

With this new connection(130) and the twistless torsion condition(127) we can rewrite the geodesic equation (96) into

$$\ddot{x}^t + \mathring{\Gamma}^t_{mn} \dot{x}^m \dot{x}^n = 0. \tag{132}$$

This looks very much the same as the standard geodesic equation on a Riemannian manifold (22). One would again be tempted to look at equation (129) and note that, when we are in the Riemann normal coordinate system, $\mathring{\Gamma}^m_{rs}$ should be zero, but this is not the case. The new connection now has a tensorial part, such that it cannot be set to zero in one coordinate system.

The geometric data of the action (108) can be expanded to fourth-order in l_s since we only go up to two spacetime derivatives on the geometric fields most of these expansions quickly terminate. Because the geodesic equation (132) has the same form as in the Riemannian case, we can express the quantum field as

$$\tilde{Y}^m = Y^m - \frac{1}{2} l_s \mathring{\Gamma}^m_{rs} Y^r Y^s - \frac{1}{6} l_s^2 \mathring{\nabla}'_t \mathring{\Gamma}^m_{rs} Y^t Y^r Y^s + \dots, \tag{133}$$

where again $\mathring{\nabla}'$ denotes the covariant derivative on lower indices only.

5.2 Expanding the TNC fields

With the new covariant coordinate Y^m (133) in hand we can start a covariant expansion of the action (114) in the quantum fields (122). In Section 3 we saw that we need up to expand this action up to fourth-order to calculate the first-order quantum correction of the dilaton action. Here we will expand the action to second-order in spacetime derivatives. It turns out that this only gives terms up to fourth-order in the quantum fields, as any terms that are fifth order in the quantum fields would have to contain at least three spacetime derivatives.

First of, the expansion for the derivative of the string embedding is easily found by first taking the derivative of (133) and then replacing partial derivatives by covariant ones.

$$\begin{aligned}\partial_\alpha X^m = \partial_\alpha X_0^m + l_s(\mathring{\nabla}_\alpha Y^m - \mathring{\Gamma}^m_{ts} Y^s \partial_a X_0^t) - l_s^2 \Big[\mathring{\Gamma}^m_{rs} \mathring{\nabla}_\alpha Y^r Y^s + \frac{1}{2}(\partial_t \mathring{\Gamma}^m_{rs} - 2\mathring{\Gamma}^m_{ls}\mathring{\Gamma}^l_{tr}) Y^r Y^s \partial_\alpha X_0^t\Big] \\ - \frac{1}{6} l_s^3 \big(\partial_r \mathring{\Gamma}^m_{ts} - \mathring{\Gamma}^l_{rt}\mathring{\Gamma}^m_{ls} - \mathring{\Gamma}^l_{rs}\mathring{\Gamma}^m_{tl}\big) \mathring{\nabla}_\alpha (Y^r Y^t Y^s)\end{aligned} \tag{134}$$

to second-order in spacetime derivatives. We then want to expand all spacetime dependent fields of the action (117), these are all pullbacks of either vectors or rank (0,2) tensors. A rank (0,2) tensor can be expanded as

$$
\begin{aligned}
W_{mn}(X_0+\tilde{Y}) &= W_{mn}(X_0) + l_s\partial_r W_{mn}Y^r + \frac{1}{2}l_s^2\big(\partial_r\partial_s W_{mn} - \partial_l W_{mn}\mathring{\Gamma}^l_{rs}\big)Y^rY^s \\
&= W_{mn}(X_0) + l_sY^i\big(\mathring{\nabla}_i W_{mn} + \mathring{\Gamma}^l_{im}W_{ln} + \mathring{\Gamma}^l_{in}W_{ml}\big) + \frac{1}{2}l_s^2\big(\mathring{\nabla}_i\mathring{\nabla}_j W_{mn} + 2\mathring{\Gamma}^l_{im}\mathring{\nabla}_j W_{ln} \\
&\quad + 2\mathring{\Gamma}^l_{jn}\mathring{\nabla}_i W_{ml} + W_{ml}\mathring{R}^l{}_{jin} + W_{ln}\mathring{R}^l{}_{jim} + \partial_m\mathring{\Gamma}^l_{ji}W_{ln} + \partial_n\mathring{\Gamma}^l_{ij}W_{ml} + 2\mathring{\Gamma}^l_{jm}\mathring{\Gamma}^s_{in}W_{ls}\big)Y^iY^j.
\end{aligned}
\tag{135}
$$

to second-order in spacetime derivatives. We then take the expansions (134) and (135) together to find the expansion for the pullback of a rank (0-2) tensor. All connection terms either cancel each other or combine to form curvature tensors, such that the end result is covariant,

$$
\begin{aligned}
W_{\alpha\beta} &= W_{\alpha\beta}(X_0) + l_s\Big[W_{mn}(X_0)\big(\mathring{\nabla}_\alpha Y^m\partial_\beta X_0^n + \partial_\alpha X_0^m\mathring{\nabla}_\beta Y^n\big) + \partial_\alpha X_0^m\partial_\beta X_0^n\mathring{\nabla}_i W_{mn}Y^i\Big] \\
&\quad + l_s^2\Big[\frac{1}{2}\big(\mathring{\nabla}_i\mathring{\nabla}_j W_{mn} + W_{ml}\mathring{R}^l{}_{jin} + W_{ln}\mathring{R}^l{}_{jim}\big)Y^iY^j\partial_\alpha X_0^m\partial_\beta X_0^n + \partial_\alpha X_0^m\mathring{\nabla}_\beta Y^n\mathring{\nabla}_i W_{mn}Y^i \\
&\qquad + \partial_\beta X_0^n\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_i W_{mn}Y^i + W_{mn}\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^n\Big] \\
&\quad + l_s^3\Big[\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^nY^i\mathring{\nabla}_i W_{mn} + \frac{1}{2}\partial_\alpha X_0^m\mathring{\nabla}_\beta Y^n\big(\mathring{\nabla}_i\mathring{\nabla}_j W_{mn} + \frac{1}{3}W_{ml}\mathring{R}^l{}_{jin} + W_{ln}\mathring{R}^l{}_{jim}\big)Y^iY^j \\
&\qquad + \frac{1}{2}\partial_\beta X_0^n\mathring{\nabla}_\alpha Y^m\big(\mathring{\nabla}_i\mathring{\nabla}_j W_{mn} + W_{ml}\mathring{R}^l{}_{jin} + \frac{1}{3}W_{ln}\mathring{R}^l{}_{jim}\big)Y^iY^j\Big] \\
&\quad + \frac{1}{2}l_s^4\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^nY^iY^j\Big[\mathring{\nabla}_i\mathring{\nabla}_j W_{mn} + \frac{1}{3}W_{ml}\mathring{R}^l{}_{jin} + \frac{1}{3}W_{ln}\mathring{R}^l{}_{jim}\Big]
\end{aligned}
\tag{136}
$$

There are no higher-order terms in l_s^5 as these have at least three spacetime derivatives. One consistency check on this expansion is to see if it matches the expansion of the Riemannian Polyakov action (30). When we plug in the pullback of the spacetime metric $G_{\alpha\beta}$ in (136), these two expansions give the same result.

We also need the expansion for a vector, following the same steps as above we first find

$$
V_m(X_0+\tilde{Y}^m) = V_m(X_0)+l_s\big(\mathring{\nabla}_n V_m+\mathring{\Gamma}^l_{nm}V_l\big)Y^n+\frac{1}{2}l_s^2\Big(\mathring{\nabla}_r\mathring{\nabla}_s V_m+V_l\mathring{R}^l{}_{srm}+\partial_m\mathring{\Gamma}^l_{sr}V_l+2\mathring{\Gamma}^l_{sm}\mathring{\nabla}_r V_l\Big)Y^rY^s
\tag{137}
$$

to second-order in spacetime derivatives. The pullback of this vector to worldsheet can then be found by multiplying (137) with (134). To second-order in worldsheet derivatives we find

$$
\begin{aligned}
V_\alpha(X_0+\tilde{Y}) &= V_\alpha(X_0) + l_s\big(\mathring{\nabla}_n V_m Y^n\partial_\alpha X_0^m + V_m\mathring{\nabla}_\alpha Y^m\big) \\
&\quad + l_s^2\Big[\mathring{\nabla}_n V_m Y^n\mathring{\nabla}_\alpha Y^m + \frac{1}{2}\big(\mathring{\nabla}_r\mathring{\nabla}_s V_m + V_l R^l{}_{srm}\big)Y^rY^s\partial_\alpha X_0^m\Big] \\
&\quad + \frac{1}{2}l_s^3\big(\mathring{\nabla}_r\mathring{\nabla}_s V_m + \frac{1}{3}R^l{}_{srm}V_l\big)Y^rY^s\mathring{\nabla}_\alpha Y^m.
\end{aligned}
\tag{138}
$$

The expansions (136) and (138) agree with what was found in [14]. There the expansion was performed up to second-order in l_s, here we supplement this with results up to fourth-order in l_s.

We can now find explicit expressions for the expansions of the couplings in action (114). The couplings that need to be worked out are $-\gamma^{\alpha\beta}\bar{h}_{\alpha\beta}$, $-\epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}$, $\gamma^{ab}\lambda_a e_b^\beta\partial_\beta\eta$, $\gamma^{ab}\lambda_a e_b^\beta\aleph_\beta$ and $\epsilon^{ab}\lambda_a e_b^\beta\tau_\beta$.

To this end we use the background field quantization for the X^m, λ_a and η fields as given in (122) and the expansions for the pullback of an arbitrary rank (0,2) tensor (136) and that of the pullback of an arbitrary vector (138). The $\eta - \lambda_a$ term can be expanded as

$$-\frac{1}{2}\gamma^{ab}\lambda_a e_b^\alpha \partial_\alpha \eta = -\frac{1}{2}\gamma^{ab}\Big[\lambda_a^0 e_b^\alpha \partial_\alpha \eta_0 + l_s\big(\lambda_a^0 e_b^\alpha \mathring{\nabla}_\alpha \tilde{E} + \tilde{\Lambda}_a e_b^\alpha \partial_\alpha \eta_0\big) + l_s^2 \tilde{\Lambda}_a e_b^\alpha \mathring{\nabla}_\alpha \tilde{E}\Big]. \tag{139}$$

The $\mathcal{O}(l_s)$ can be set to zero by demanding that the fields η and λ_a obey their equations of motion. We now turn to the $\bar{h}_{\alpha\beta}$ coupling.

$$\begin{aligned}
&-\gamma^{\alpha\beta}\bar{h}_{\alpha\beta}(X_0+\tilde{Y}) = -\gamma^{\alpha\beta}\bar{h}_{\alpha\beta}(X_0) - l_s\gamma^{\alpha\beta}\big[2\bar{h}_{mn}\mathring{\nabla}_\alpha Y^m \partial_\beta X_0^n + \partial_\alpha X_0^m \partial_\beta X_0^n \mathring{\nabla}_i \bar{h}_{mn} Y^i\big] \\
&- l_s^2\gamma^{\alpha\beta}\Big[\big(\frac{1}{2}\mathring{\nabla}_i\mathring{\nabla}_j \bar{h}_{mn} + \mathring{R}^l{}_{jin}\bar{h}_{ml}\big)Y^i Y^j \partial_\alpha X_0^m \partial_\beta X_0^n + 2\partial_\alpha X_0^m \mathring{\nabla}_\beta Y^n \mathring{\nabla}_i \bar{h}_{mn} Y^i + \bar{h}_{mn}\mathring{\nabla}_\alpha Y^m \mathring{\nabla}_\beta Y^n\Big] \\
&\quad - l_s^3\gamma^{\alpha\beta}\Big[\mathring{\nabla}_\alpha Y^m \mathring{\nabla}_\beta Y^n Y^i \mathring{\nabla}_i \bar{h}_{mn} + \partial_\alpha X_0^m \mathring{\nabla}_\beta Y^n \big(\mathring{\nabla}_i\mathring{\nabla}_j \bar{h}_{mn} + \frac{4}{3}\bar{h}_{ml}\mathring{R}^l{}_{jin}\big)Y^i Y^j\Big] \\
&\qquad - \frac{1}{2}l_s^4\gamma^{\alpha\beta}\mathring{\nabla}_\alpha Y^m \mathring{\nabla}_\beta Y^n Y^i Y^j \big(\mathring{\nabla}_i\mathring{\nabla}_j \bar{h}_{mn} + \frac{2}{3}\mathring{R}^l{}_{jim}\bar{h}_{ln}\big).
\end{aligned} \tag{140}$$

Furthermore, we find the following expansions for the vector couplings

$$\begin{aligned}
&-\frac{1}{2}\epsilon^{ab}\lambda_a e_b^\alpha \tau_\alpha(X_0+\tilde{Y}) = -\frac{1}{2}\epsilon^{ab}e_b^\alpha\Big\{\lambda_a^0\tau_\alpha(X_0) + l_s\big[\lambda_a^0\big(\frac{1}{2}F_{mn}Y^m\partial_\alpha X_0^n + \tau_m\mathring{\nabla}_\alpha Y^m\big) + \tilde{\Lambda}_a\tau_\alpha(X_0)\Big] \\
&\quad + l_s^2\Big[\frac{1}{2}\lambda_a^0\big(F_{mn}Y^m\mathring{\nabla}_\alpha Y^n + \mathring{\nabla}_r F_{sm}Y^r Y^s \partial_\alpha X_0^m\big) + \tilde{\Lambda}_a\big(\frac{1}{2}F_{mn}Y^m\partial_\alpha X_0^n + \tau_m\mathring{\nabla}_\alpha Y^m\big)\Big] \\
&\qquad + l_s^3\Big[\frac{1}{3}\lambda_a^0\mathring{\nabla}_r F_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m + \frac{1}{2}\tilde{\Lambda}_a\big(F_{mn}Y^m\mathring{\nabla}_\alpha Y^n + \mathring{\nabla}_r F_{sm}Y^r Y^s\partial_\alpha X_0^m\big)\Big] \\
&\qquad\qquad + \frac{1}{3}l_s^4\tilde{\Lambda}_a\mathring{\nabla}_r F_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m\Big\}.
\end{aligned} \tag{141}$$

Here we have used Equation (131) and the relation $\mathring{R}^l{}_{(sr)m}\tau_l = [\mathring{\nabla}_m, \mathring{\nabla}_{(r}]\tau_{s)} = -\frac{1}{2}\mathring{\nabla}_{r)}F_{m(s}$. We have also introduced the field strength $F \equiv d\tau$. Similarly for the $\lambda - \aleph$ coupling we find

$$\begin{aligned}
&-\frac{1}{2}\gamma^{ab}\lambda_a e_b^\alpha\aleph_\alpha(X_0+\tilde{Y}) = -\frac{1}{2}\gamma^{ab}\Big\{\lambda_a^0 e_b^\alpha\aleph_\alpha(X_0) + l_s\Big[\lambda_a^0 e_b^\alpha\mathfrak{h}_{mn}Y^m\partial_\alpha X_0^n - \mathring{\nabla}_\alpha(\lambda_a^0 e_b^\alpha)\aleph_m Y^m + \tilde{\Lambda}_a e_b^\alpha\aleph_\alpha(X_0)\Big] \\
&+ l_s^2\Big[\frac{1}{2}\lambda_a^0 e_b^\alpha\big(\mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\partial_\alpha X_0^m + \mathfrak{h}_{mn}Y^m\mathring{\nabla}_\alpha Y^n\big) - \frac{1}{2}\mathring{\nabla}_r\aleph_s Y^r Y^s\mathring{\nabla}_\alpha(\lambda_a^0 e_b^\alpha) + \tilde{\Lambda}_a e_b^\alpha\mathfrak{h}_{mn}Y^m\partial_\alpha X_0^n \\
&\quad - \aleph_m Y^m\mathring{\nabla}_\alpha(\tilde{\Lambda}_a e_b^\alpha)\Big] + l_s^3\Big[\frac{1}{3}\lambda_a^0 e_b^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m - \frac{1}{6}Y^r Y^s Y^m\mathring{\nabla}_r\mathring{\nabla}_s\aleph_m\mathring{\nabla}_\alpha(\lambda_a^0 e_b^\alpha) \\
&\qquad + \frac{1}{2}\tilde{\Lambda}_a e_b^\alpha\big(\mathfrak{h}_{mn}Y^m\mathring{\nabla}_\alpha Y^n + \mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\partial_\alpha X_0^m\big) - \frac{1}{2}\mathring{\nabla}_r\aleph_s Y^r Y^s\mathring{\nabla}_\alpha(\tilde{\Lambda}_a e_b^\alpha)\Big] \\
&\qquad\qquad + l_s^4\Big[\frac{1}{3}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m - \frac{1}{6}Y^r Y^s Y^m\mathring{\nabla}_r\mathring{\nabla}_s\aleph_m\mathring{\nabla}_\alpha(\tilde{\Lambda}_a e_b^\alpha)\Big]\Big\},
\end{aligned} \tag{142}$$

where partial differentiation has been used to write the $\aleph$ field in its field strength $\mathfrak{h} \equiv d\aleph$. The total derivatives resulting from this partial differentiation are dropped since (142) appears in the action, such that any boundary terms are zero as the string is periodic. Note that in the action (114) we have the combination $-\frac{1}{2}\gamma^{ab}\lambda_a e_b^\beta\aleph_\beta \equiv \Sigma\lambda^\beta\aleph_\beta$. Terms involving $\mathring{\nabla}_\beta\Sigma\lambda^\beta$ are then zero by the equation of motion for λ (342).

Finally we have the coupling

$$-\epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}(X_0+\tilde{Y}) = -\epsilon^{\alpha\beta}l_s^2\Big[H_{mkl}\partial_\alpha X_0^m\mathring{\nabla}_\beta Y^k Y^l + \frac{1}{2}\mathring{\nabla}_k H_{mnl}\partial_\alpha X_0^m\partial_\beta X_0^n Y^k Y^l\Big] - \frac{1}{3}\epsilon^{\alpha\beta}l_s^3 H_{kmn}Y^k\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^n, \tag{143}$$

where we have only written down the $\mathcal{O}(> l_s^2)$ terms that will contribute to the dilaton beta function. The expansion (143) is invariant under the $\mathrm{U}(1)_B$ symmetry of the B field (119). In section 4.3 we saw that the fields $\aleph_m$ and η also transform under these U(1) transformations through (120). Most of the $\aleph$ terms in (142) combined into its field strength $\mathfrak{h}$, which is invariant under (120). The only part of (142) that is not manifestly U(1) invariant is

$$-\frac{1}{2}l_s^2\gamma^{ab}\tilde{\Lambda}_a e_b^\alpha + \tilde{\Lambda}_a e_b^\alpha)\mathring{\nabla}_\alpha\big(\aleph_m Y^m + \frac{1}{2}l_s\mathring{\nabla}_r\aleph_s Y^r Y^s + \frac{1}{6}l_s^2\mathring{\nabla}_r\mathring{\nabla}_s\aleph_m Y^r Y^s Y^m\big). \tag{144}$$

The $\eta - \lambda$ term expansion (139) also has a part that transforms under U(1) transformations. As we noted in section 4.3, the η field has to transform as $\delta_\Lambda \eta = -\Lambda_u$ under U(1) transformations. It then follows that its quantum field transforms as

$$\delta_\Lambda \tilde{E} = -\mathring{\nabla}_r\Lambda_u - \frac{1}{2}l_s\mathring{\nabla}_r\mathring{\nabla}_s\Lambda_u Y^r Y^s - \frac{1}{6}l_s^2\mathring{\nabla}_t\mathring{\nabla}_r\mathring{\nabla}_s\Lambda_u Y^r Y^s Y^t + \mathcal{O}(l_s^3). \tag{145}$$

This precisely cancels the part coming from (144)! Therefore it makes sense to define the $\mathrm{U}(1)_B$ invariant field

$$\hat{E} = \tilde{E} + \aleph_m Y^m + \frac{1}{2}l_s\mathring{\nabla}_r\aleph_s Y^r Y^s + \frac{1}{6}l_s^2\mathring{\nabla}_r\mathring{\nabla}_s\aleph_m Y^r Y^s Y^m. \tag{146}$$

The expanded action can now be written completely in the $\mathrm{U}(1)_B$ invariant fields H_{mnr}, $\mathfrak{h}_{mn}$ and $\hat{E}$. By making the substitution (146) the quantum coordinates are changed from $\{Y^m, \tilde{\Lambda}_\pm, E\}$ to $\{Y^m, \tilde{\Lambda}_\pm, \hat{E}\}$. The original coordinates are independent of each other and only E is redefined, consequently the Jacobian of this transformation is one. With this new field we can write the action in a manifestly U(1) invariant form. Taking all the couplings together we find the TNC action with matter, expanded to second-order in spacetime derivatives,

$$\begin{aligned}
S_{\text{matter}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, e\Big\{ & l_s^2\Big[-\gamma^{\alpha\beta}\bar{h}_{mn}\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^n - \frac{1}{2}\gamma^{ab}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_\alpha\hat{E} - \frac{1}{2}\epsilon^{ab}\tilde{\Lambda}_a e_b^\alpha\tau_m\mathring{\nabla}_\alpha Y^m \\
& - \frac{1}{2}\tilde{\Lambda}_a e_b^\alpha\big(\frac{1}{2}\epsilon^{ab}F_{mn} + \gamma^{ab}\mathfrak{h}_{mn}\big)Y^m\partial_\alpha X_0^n - \big(2\gamma^{\alpha\beta}\mathring{\nabla}_i\bar{h}_{mn} + \epsilon^{\alpha\beta}H_{mni}\big)Y^i\mathring{\nabla}_\alpha Y^m\partial_\beta X_0^n \\
& + \frac{1}{2}\big(-\Delta\lambda^\alpha F_{mn} + \Sigma\lambda^\alpha\mathfrak{h}_{mn}\big)Y^m\mathring{\nabla}_\alpha Y^n - \big(\gamma^{\alpha\beta}C_{klmn} + \epsilon^{\alpha\beta}\frac{1}{2}\mathring{\nabla}_k H_{mnl}\big)Y^k Y^l\partial_\alpha X_0^m\partial_\beta X_0^n \\
& + \frac{1}{2}\big(-\Delta\lambda^\alpha\mathring{\nabla}_r F_{sm} + \Sigma\lambda^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}\big)Y^r Y^s\partial_\alpha X_0^m\Big] \\
& + l_s^3\Big[-\frac{1}{4}\gamma^{ab}\tilde{\Lambda}_a e_b^\alpha\mathfrak{h}_{mn}Y^m\mathring{\nabla}_\alpha Y^n - \frac{1}{4}\gamma^{ab}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\partial_\alpha X_0^m - \frac{1}{4}\epsilon^{ab}\tilde{\Lambda}_a e_b^\alpha Y^m F_{mn}\mathring{\nabla}_\alpha Y^n \\
& - \frac{1}{4}\epsilon^{ab}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_r F_{sm}Y^r Y^s\partial_\alpha X_0^m - \gamma^{\alpha\beta}D_{klmn}Y^k Y^l\mathring{\nabla}_\beta Y^n\partial_\alpha X_0^m \\
& + \frac{1}{3}(-\Delta\lambda^\alpha\mathring{\nabla}_r F_{sm} + \Sigma\lambda^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm})Y^r Y^s\mathring{\nabla}_\alpha Y^m - \big(\gamma^{\alpha\beta}\mathring{\nabla}_k\bar{h}_{mn} + \frac{1}{3}\epsilon^{\alpha\beta}H_{kmn}\big)\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^n Y^k\Big] \\
& + l_s^4\Big[-\frac{1}{6}\epsilon^{ab}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_r F_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m - \frac{1}{6}\gamma^{ab}\tilde{\Lambda}_a e_b^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}Y^r Y^s\mathring{\nabla}_\alpha Y^m - \gamma^{\alpha\beta}E_{rsmn}\mathring{\nabla}_\alpha Y^m\mathring{\nabla}_\beta Y^n Y^r Y^s\Big]\Big\},
\end{aligned} \tag{147}$$

where

$$C_{klmn} = \frac{1}{2}\mathring{\nabla}_k\mathring{\nabla}_l\bar{h}_{mn} + \mathring{R}^s{}_{(kl)(m}\bar{h}_{n)s}, \tag{148}$$

$$D_{klmn} = \mathring{\nabla}_k\mathring{\nabla}_l\bar{h}_{mn} + \frac{4}{3}\mathring{R}^s{}_{(kl)m}\bar{h}_{ns}, \tag{149}$$

$$E_{rsmn} = \frac{1}{2}\mathring{\nabla}_r\mathring{\nabla}_s\bar{h}_{mn} + \frac{1}{3}\mathring{R}^l{}_{(rs)(m}\bar{h}_{n)l}. \tag{150}$$

6 Flat index expansion

The action in Equation 147 contains the kinetic terms of the theory, but for general $\bar{h}_{mn}$ these are not diagonal, such that constructing a propagator is non-trivial. By going to a local Lorentz frame through the vielbein formalism, $Y^m = e^m_I Y^I$, these kinetic terms become diagonal. We can then write the spatial metric as

$$\bar{h}_{mn} = \eta_{IJ} e^I_m e^J_n \tag{151}$$

and its inverse as

$$h^{mn} = \eta^{IJ} e^m_I e^n_J. \tag{152}$$

In the Riemannian case, this came down to swapping curved indices with flat ones, as the vielbein was compatible with the covariant derivative and contracting indices resulted in contracting a vielbein with its inverse. Because our covariant derivative is not metric compatible, the derivative of a vielbein is also non-zero. This results in extra terms. To organize these expansions we consider the different orders of l_s in Equation 147 separately by writing

$$S_{\text{matter}} = S_2\big(=\mathcal{O}(l_s^2)\big) + S_3(=\mathcal{O}(l_s^3)\big) + S_4(=\mathcal{O}(l_s^4)\big). \tag{153}$$

Additionally, we further divide this by the amount of spacetime derivatives in each term, such that $S_a = S_a^{(0)} + S_a^{(1)} + S_a^{(2)}$, with the number in the brackets denoting the amount of spacetime derivatives. One can also consider the flat index expansion separately for spatial indices i and time index 0 by writing,

$$Y^m = -\hat{v}^m(\tau_s Y^s) + e^m_i(\delta^{ij} e^r_j \bar{h}_{rs} Y^s) \equiv -\hat{v}^m \frac{Y^0}{\sqrt{2\Phi}} + e^m_i Y^i, \tag{154}$$

where we have defined $Y^0 \equiv \sqrt{2\Phi}\tau_s Y^s$ and $Y^i \equiv \delta^{ij} e^r_j \bar{h}_{rs} Y^s$. Note that Latin letters i and j are flat indices ranging from 1 to d, whereas the Latin letters m, n, r, s denoted curved indices ranging from 0 to d. Here Φ plays the role of Newtons potential, it is defined by

$$\Phi = -v^s m_s + \frac{1}{2} h^{rs} m_r m_s \tag{155}$$

and it is invariant under local Galilean boost and rotations [14]. The vielbein over all indices I can then be written as $e^m_I = \{\frac{-\hat{v}^m}{\sqrt{2\Phi}}, e^m_i\}$. The zero spatial derivative part of the action is

$$S^{(0)} = S_2^{(0)} = \frac{1}{4\pi\alpha'} l_s^2 \int d^2\sigma\, e\Big\{ -\gamma^{\alpha\beta}\eta_{IJ}\mathring{\nabla}_\alpha Y^I \mathring{\nabla}_\beta Y^J - \frac{1}{2}\gamma^{ab}\tilde{\Lambda}_a e^\alpha_b \mathring{\nabla}_\alpha \hat{E} - \frac{1}{2}\epsilon^{ab}\tilde{\Lambda}_a e^\alpha_b \tau_m e^m_I \mathring{\nabla}_\alpha Y^I\Big\}. \tag{156}$$

The last term has the contraction $\tau_m e^m_I$, by using Equation (154) this can be written as $1/\sqrt{2\Phi}\delta^0_I$. The last term then describes a propagator between the fields $\tilde{\Lambda}_a$ and Y^0, by redefining $\tilde{\Lambda}_\pm = \sqrt{2\Phi}\Lambda$ this propagator becomes canonical. We then also have to rescale $\hat{E} = \frac{1}{\sqrt{2\Phi}}E$ to make the $\Lambda_a - E$ propagator canonical. The covariant derivative in the first term acts on Y^I by $\mathring{\nabla}_\alpha Y^I = \partial_\alpha Y^I + \omega^I_{J\alpha}$, with $\omega^I_{J\alpha}$ the SO$(D-1{,}1)$ spin connection. As there are no terms involving derivatives of the spin connection, no gauge invariant curvature tensor can be formed. Consequently we can ignore any terms involving the spin connection because the end result should be gauge invariant. By using the new fields and dropping the spin connection terms the zero derivative action becomes

$$S^{(0)} = \frac{1}{4\pi\alpha'} l_s^2 \int d^2\sigma\, e\Big\{ -\gamma^{\alpha\beta}\eta_{IJ}\partial_\alpha Y^I \partial_\beta Y^J - \frac{1}{2}\gamma^{ab}\Lambda_a e^\alpha_b \mathring{\nabla}_\alpha E - \frac{1}{2}\epsilon^{ab}\Lambda_a e^\alpha_b \partial_\alpha Y^0\Big\}. \tag{157}$$

This action now features diagonal, canonical kinetic terms and from we can determine the propagators of the theory, we will do this in Section 6.1. We also have to rewrite the other actions in terms of the fields Y^I, $\Lambda_\pm$ and E. The $\mathcal{O}(l_s^2)$, first-order in derivatives action is given by

$$\begin{aligned}S_2^{(1)} =&\frac{1}{4\pi\alpha'}l_s^2\int d^2\sigma\, e\Big\{ -2\partial_\alpha Y^I Y^J e_J^n \bar{h}_{mn}\mathring{\nabla}_n e_I^m \partial^\alpha X_0^n + \frac{1}{4}\gamma^{ab}\Lambda_a e_b^\alpha E\mathring{\nabla}_m(\log\Phi)\partial_\alpha X_0^m \\ &-\frac{1}{2}\epsilon^{ab}\Lambda_a e_b^\alpha \tau_m Y^I \mathring{\nabla}_n e_I^m \partial_\alpha X_0^n - \frac{1}{2}\Lambda_a e_b^\alpha\sqrt{2\Phi}\big(\frac{1}{2}\epsilon^{ab}F_{mn} + \gamma^{ab}\mathfrak{h}_{mn}\big)Y^m\partial_\alpha X_0^n \\ &-\big(2\gamma^{\alpha\beta}\mathring{\nabla}_k\bar{h}_{mn} - \epsilon^{\alpha\beta}H_{mnk}\big)e_I^k Y^I e_J^m \partial_\alpha Y^J \partial_\beta X_0^n - \frac{1}{2}\big(\Delta\lambda^\alpha F_{mn} - \Sigma\lambda^\alpha\mathfrak{h}_{mn}\big)e_I^m Y^I e_J^n \partial_\alpha Y^J\Big\}.\end{aligned} \tag{158}$$

Note that there are also derivatives on the vielbeins coming from terms that were originally of lower order in spacetime derivatives. For the $\mathcal{O}(l_s^2)$, second-order in spacetime derivatives action in terms of the new fields we find,

$$\begin{aligned}S_2^{(2)} =&\frac{l_s^2}{4\pi\alpha'}\int d^2\sigma\, e\Big\{ -\bar{h}_{mn}Y^I Y^J \mathring{\nabla}_r e_I^m \mathring{\nabla}_s e_J^n \partial_\alpha X_0^r \partial^\alpha X_0^s + \big(\epsilon^{\alpha\beta}H_{mnk} - 2\gamma^{\alpha\beta}\mathring{\nabla}_k\bar{h}_{mn}\big)e_I^k Y^I Y^J \mathring{\nabla}_r e_J^m \partial_\alpha X_0^r \partial_\beta X_0^n \\ &+\frac{1}{2}\big(\Sigma\lambda^\alpha\mathfrak{h}_{mn} - \Delta\lambda^\alpha F_{mn}\big)e_I^m Y^I Y^J \mathring{\nabla}_r e_J^n \partial_\alpha X_0^r - \big(\gamma^{\alpha\beta}C_{klmn} + \frac{1}{2}\epsilon^{\alpha\beta}\mathring{\nabla}_k H_{mnl}\big)e_I^k e_J^l Y^I Y^J \partial_\alpha X_0^m \partial_\beta X_0^n \\ &+\frac{1}{2}\big(-\Delta\lambda^\alpha\mathring{\nabla}_r F_{sm} + \Sigma\lambda^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}\big)e_I^r e_J^s Y^I Y^J \partial_\alpha X_0^m\Big\}.\end{aligned} \tag{159}$$

For the $\mathcal{O}(l_3^3)$ action some terms involving the covariant derivative of a vielbein become third-order in spacetime derivatives and can thus be dropped. The first-order in spacetime derivatives terms are

$$\begin{aligned}S_3^{(1)} = \frac{l_s^3}{4\pi\alpha'}\int d^2\sigma\, e\Big\{ &-\frac{1}{4}\sqrt{2\Phi}\Lambda_a e_b^\alpha\big(\gamma^{ab}\mathfrak{h}_{mn} + \epsilon^{ab}F_{mn}\big)e_I^m e_J^n Y^I\partial_\alpha Y^J \\ &- \big(\gamma^{\alpha\beta}\mathring{\nabla}_k\bar{h}_{mn} + \frac{1}{3}\epsilon^{\alpha\beta}H_{kmn}\big)e_I^k e_J^m e_N^n Y^I \partial_\alpha Y^J \partial_\beta Y^N\Big\}.\end{aligned} \tag{160}$$

And the $\mathcal{O}(l_s^3)$ terms that are second-order in spacetime derivatives are

$$\begin{aligned}S_3^{(2)} =&\frac{l_s^3}{4\pi\alpha'}\int d^2\sigma\, e\Big\{ -\frac{1}{4}\sqrt{2\Phi}\gamma^{ab}\Lambda_a e_b^\alpha\Big[\mathfrak{h}_{mn}e_I^m Y^I Y^J \mathring{\nabla}_r e_J^n \partial_\alpha X_0^r + \mathring{\nabla}_r\mathfrak{h}_{sm}e_I^r e_J^s Y^I Y^J \partial_\alpha X_0^m\Big] \\ &-\frac{1}{4}\sqrt{2\Phi}\epsilon^{ab}\Lambda_a e_b^\alpha\Big[F_{mn}e_I^m Y^I Y^J \mathring{\nabla}_r e_J^n \partial_\alpha X_0^r + \mathring{\nabla}_r F_{sm} e_I^r e_J^s Y^I Y^J \partial_\alpha X_0^m\Big] \\ &-\gamma^{\alpha\beta}D_{klmn}e_I^k e_J^l e_N^n \partial_\alpha Y^N Y^I Y^J \partial_\beta X_0^m + \frac{1}{3}\big(-\Delta\lambda^\alpha\mathring{\nabla}_r F_{sm} + \Sigma\lambda^\alpha\mathring{\nabla}_r\mathfrak{h}_{sm}\big)e_I^r e_J^s e_M^m Y^I Y^J \partial_\alpha Y^M \\ &-\big(\gamma^{\alpha\beta}\mathring{\nabla}_k\bar{h}_{mn} + \frac{1}{3}\epsilon^{\alpha\beta}H_{kmn}\big)e_I^k Y^I\big(e_J^m\mathring{\nabla}_r e_N^n \partial_\alpha Y^J Y^N \partial_\beta X_0^r + e_N^n \mathring{\nabla}_r e_J^m Y^J \partial_\beta Y^N \partial_\alpha X_0^r\big)\Big\}.\end{aligned} \tag{161}$$

Finally for the $\mathcal{O}(l_s^4)$ part of the action in (147) there are only terms that are second-order in spacetime derivatives or higher, as we drop any higher-order terms we are left with

$$S_4 = S_4^{(2)} = \frac{l_s^4}{4\pi\alpha'}\int d^2\sigma\, e\bigg\{-\frac{1}{6}\sqrt{2\Phi}\Lambda_a e_b^\alpha\big(\gamma^{ab}\mathring{\nabla}_r \mathfrak{h}_{sm} + \epsilon^{ab}\mathring{\nabla}_r F_{sm}\big)e_I^r e_J^s e_M^m Y^I Y^J \partial_\alpha Y^M$$
$$- \gamma^{\alpha\beta}E_{rsmn}e_I^r e_J^s e_M^m e_N^n Y^I Y^J \partial_\alpha Y^M \partial_\beta Y^N\bigg\}. \quad (162)$$

6.1 TNC Feynman rules

The propagators of the theory can be found by considering the Fourier transform of the zero spatial derivative action (157). By taking the following Fourier transforms for the quantum fields,

$$Y^I(\sigma) = \frac{1}{2\pi}\int d^2k e^{ik\sigma}Y^I(k) \quad , \quad \Lambda_a(\sigma) = \frac{1}{2\pi}\int d^2k e^{ik\sigma}\Lambda_a(k) \quad , \quad E(\sigma) = \frac{1}{2\pi}\int d^2k e^{ik\sigma}E(k), \quad (163)$$

the action (157) can be written as

$$S^{(0)} = \frac{1}{4\pi}\int d^2k\Big\{-\eta_{IJ}k^2 Y^I(k)Y^J(-k) + \frac{1}{2}i\gamma^{ab}\Lambda_a(k)e_b^\alpha k_\alpha E(-k) + \frac{1}{2}i\epsilon^{ab}\Lambda_a(k)e_b^\alpha k_\alpha Y^0(-k)\Big\}. \quad (164)$$

To determine the propagators these fields can be combined into one field $\phi^A = \{Y^0, Y^i, \Lambda_a, E\}$ and we write the action as

$$S^{(0)} = \int d^2k \phi_A(-k) G^{AB}\phi_B(k), \quad (165)$$

where the indices A and B run over the labels of all fields: $0, i, a, -$ and η. The matrix G^{AB} is given by

$$G^{AB} = \frac{1}{4\pi}\begin{pmatrix} k^2 & 0 & -\frac{1}{4}i\epsilon^{ab}k_a & 0 \\ 0 & -k^2\delta_{ij} & 0 & 0 \\ -\frac{1}{4}i\epsilon^{ab}k_b & 0 & 0 & -\frac{1}{4}i\gamma^{ab}k_b \\ 0 & 0 & \frac{1}{4}i\gamma^{ab}k_a & 0 \end{pmatrix}. \quad (166)$$

To determine the propagators we need the inverse of this matrix,

$$G_{AB} = 4\pi\begin{pmatrix} 0 & 0 & \frac{4i}{k^2}\epsilon_{cb}k^c & 0 \\ 0 & -\frac{1}{k^2}\delta^{ij} & 0 & 0 \\ \frac{4i}{k^2}\epsilon_{ac}k^c & 0 & 16(\gamma_{ab} - \frac{k_a k_b}{k^2}) & -\frac{4i}{k^2}k_a \\ 0 & 0 & \frac{4i}{k^2}k_b & 0 \end{pmatrix}. \quad (167)$$

The propagators of the fields involving Y are given by the elements of the first two rows of G^{AB} multiplied by $-\frac{1}{2}$.[5]

$$Y^i \longrightarrow Y^j \;= 2\pi\,\delta^{ij}\frac{1}{k^2}, \quad (168)$$

$$Y^0 \longrightarrow\!\!\leadsto \Lambda_a \;= 8\pi i\frac{1}{k^2}\epsilon_{ab}k^b. \quad (169)$$

[5] These Feynman rules for constructing propagators are taken from Volume 1 of the lecture notes by de Wit, Laenen and Smith [21].

Note that the actions (158), (159), (160) and (162) all have interaction terms involving the fields Y^I instead of Y^i and Y^0. We thus want to write these propagators in terms of the fields Y^I as

$$Y^I \longrightarrow Y^J \;= 2\pi\,(\eta^{IJ} + \delta^I_0\delta^J_0)\frac{1}{k^2} \tag{170}$$

and

$$Y^I \longrightarrow\!\!\rightsquigarrow \Lambda_a \;= 8\pi i \delta^I_0 \frac{1}{k^2}\epsilon_{ab}k^b. \tag{171}$$

The Λ_a propagator is given by

$$\Lambda_a \rightsquigarrow \Lambda_b \;= -32\pi\big(\gamma_{ab} - \frac{k_a k_b}{k^2}\big). \tag{172}$$

The E field only couples to Λ_a,

$$\Lambda_a \rightsquigarrow\!\dashrightarrow E \;= 8\pi i \frac{1}{k^2}k_a. \tag{173}$$

7 Locating the Weyl anomaly

Just as in Equation (37) we can make an ansatz for the Weyl anomaly by considering symmetry and dimensional arguments. As we now have more background fields (λ^0_a and η_0) there are also more possibilities for the Weyl anomaly [14],

$$\langle\frac{\delta S}{\delta f}\rangle = \big(\gamma^{\alpha\beta}\beta^G_{\mu\nu} + \epsilon^{\alpha\beta}\beta^B_{\mu\nu}\big)\partial_\alpha X^\mu_0 \partial_\beta X^\nu_0 + \big(\beta_m\gamma^{ab} + \bar{\beta}_m\epsilon^{ab}\big)\partial_a X^m_0 \lambda^0_b + \beta\lambda^0_a\lambda^0_b\gamma^{ab} + \sqrt{-\gamma}R_\gamma\beta^\phi. \tag{174}$$

All of these β-functions have been computed by Gallegos, Gürsoy and Zinnato in [14] except for β^ϕ, which was only computed at the tree level. The goal of this thesis is to find the last remaining piece of this expression: β^ϕ. As can be seen from Equation (174) it is a scalar function and that there are no background fields attached to it. To find it we should examine diagrams with vertices that do not involve the background fields X^m_0 λ^0_a and η_0.

In section 3 we saw that we could get to the Weyl anomaly through the trace of the energy-momentum tensor. Here we will see that for the TNC string this is a little bit more complicated. Under a Weyl transformation the worldsheet zweibein and the Lagrange multipliers transform as

$$e^\alpha_\pm \to f e^\alpha_\pm \quad , \quad \lambda_\pm \to f\lambda_\pm, \tag{175}$$

for any worldsheet function $f(\xi)$. The action (112) should be invariant under this gauge transformation, we can decompose the variation of the action as [14]

$$\frac{\delta S}{\delta f} = -\frac{2\pi\alpha'}{e}\Big(e^\gamma_c\frac{\delta S}{\delta e^\gamma_c} + \frac{\delta S}{\delta\lambda_+}\lambda_+ + \frac{\delta S}{\delta\lambda_-}\lambda_-\Big) = 0. \tag{176}$$

The first term of (176) is the energy-momentum one-form [14],

$$\begin{aligned}\tau^c_\gamma &\equiv -\frac{2\pi\alpha'}{e}\frac{\delta S}{\delta e^\gamma_c}\\ &= \frac{2\pi\alpha'\mathcal{L}}{e}e^c_\gamma + \bar{h}_{\gamma\beta}e^\beta_b\eta^{cb} + \bar{B}_{\gamma\beta}e^\beta_b\epsilon^{cb} - \frac{1}{2}\delta^c_-\lambda_+\big(\partial_\gamma\eta + \aleph_\gamma + \tau_\gamma\big) - \frac{1}{2}\delta^c_+\lambda_-\big(\partial_\gamma\eta + \aleph_\gamma - \tau_\gamma\big),\end{aligned} \tag{177}$$

where we have used that the variation of the determinant of the zweibein is

$$\frac{\delta e}{\delta e_c^\gamma} = -e e_\gamma^c. \tag{178}$$

The second and third variations are given by

$$C^\pm \equiv -\frac{2\pi\alpha'}{e}\frac{\delta S}{\delta\lambda_\pm} = -\frac{1}{2}e_\mp^\beta\big(\partial_\beta\eta + \aleph_\beta \pm \tau_\beta\big). \tag{179}$$

As we can expect Equation (176) vanishes classically, but, as in the Riemannian case, we do not expect this to hold at the quantum level. To be able to investigate the behavior of $\frac{\delta S}{\delta f}$ at the quantum level we want to parametrize it in physical functions. In the Riemannian case, this was just the trace of the energy-momentum tensor, but here we also have the functions $C^\pm$. By making use of the definition of the energy-momentum tensor from the energy-momentum one-form,

$$T_{\alpha\gamma} \equiv \eta_{cd}e_\alpha^d\tau_\gamma^c, \tag{180}$$

we see that the term $e_c^\gamma\tau_\gamma^c$ is again the trace of the energy-momentum tensor,

$$e_c^\beta\tau_\beta^c = \eta_{dc}\gamma^{\beta\alpha}e_\alpha^d\tau_\beta^c = \gamma^{\alpha\beta}T_{\alpha\beta}. \tag{181}$$

The strategy now is to determine $T_{\alpha\beta}$ and $C^\pm$ in terms of the quantum fields of Equation (122) and to insert them in diagrams constructed from the expanded actions of Section 6 to get their expectation values. To express the energy-momentum tensor and $C^\pm$ in the quantum fields we have to repeat the steps of Section 5 and 6.

Let us first calculate the energy-momentum tensor explicitly by

$$\begin{aligned} T_{\alpha\gamma} &= \eta_{cd}e_\alpha^d\tau_\gamma^c \\ &= \frac{2\pi\alpha'\mathcal{L}}{e}\gamma_{\gamma\alpha} + \big(\bar{h}_{\gamma\beta}\eta^{cb} + \bar{B}_{\gamma\beta}\epsilon^{cb}\big)e_b^\beta e_\alpha^d\eta_{cd} - \frac{1}{2}\delta_-^c\lambda_+\eta_{cd}e_\alpha^d\big(\partial_\gamma\eta + \aleph_\gamma + \tau_\gamma\big) - \frac{1}{2}\delta_+^c\lambda_-\eta_{cd}e_\alpha^d\big(\partial_\gamma\eta + \aleph_\gamma - \tau_\gamma\big). \end{aligned} \tag{182}$$

The $\bar{h}$ term reduces to $\bar{h}_{\alpha\gamma}$ when performing the contractions. We expect the $\bar{B}$ term to drop out altogether to match the Riemannian bosonic string, there the energy-momentum tensor is given by variating the action to the spacetime metric, such that the Kalb-Ramond field never appears as its proportional to the epsilon tensor and not the metric. Here we wrote that same action in the vielbein formalism, but the end result should be the same.

We now want to expand the energy-momentum tensor to second-order in l_s to be able to find its expectation value. The diagrams we will need here already have two spacetime derivatives contained in their vertices, so we can drop any terms that contain spacetime derivatives in this expansion of the energy-momentum tensor[6]. For now we omit the $\bar{B}$ term as it should not be present. We also ignore the term containing the Lagrangian as it is proportional to the worldsheet metric. We will examine the Weyl anomaly through insertions of T_{++} and T_{--} in lightcone coordinates. There

[6]We are only interested in the dilaton two-loop diagrams in this thesis. These either contain a four quantum field vertex which is at least of second-order in spacetime derivatives, or two vertices with three quantum fields, which are both of order one in spacetime derivatives. Then there can be no more derivatives in either the propagators or the energy-momentum insertion. The question is if this is also the case for one-loop diagrams. One could construct diagrams which have one spacetime derivative in the energy-momentum tensor and one in a vertex. For the Riemannian case, this was not relevant as the energy-momentum tensor would contain either zero spacetime derivatives or two.

the diagonal terms of the metric are zero and thus the term containing the Lagrangian does not contribute. The relevant terms are

$$T_{\alpha\beta} = \mathcal{O}(\neq l_s^2) + l_s^2\Big(\bar{h}_{mn}\mathring{\nabla}_\alpha Y^m \mathring{\nabla}_\beta Y^n - \frac{1}{2}\delta^c_- \eta_{cd} e^d_\alpha \big(\tilde{\Lambda}_+ \mathring{\nabla}_\beta \tilde{E} + \tilde{\Lambda}_+ \aleph_m \mathring{\nabla}_\beta Y^m + \tilde{\Lambda}_+ \tau_m \mathring{\nabla}_\beta Y^m\big) \\ - \frac{1}{2}\delta^c_+ \eta_{cd} e^d_\alpha \big(\tilde{\Lambda}_- \mathring{\nabla}_\beta \tilde{E} + \tilde{\Lambda}_- \aleph_m \mathring{\nabla}_\beta Y^m - \tilde{\Lambda}_- \tau_m \mathring{\nabla}_\beta Y^m\big)\Big) + \cdots \times \gamma_{\alpha\beta}. \quad (183)$$

By writing $\aleph_m \mathring{\nabla}_\beta Y^m = \mathring{\nabla}_\beta(\aleph_m Y^m) + \mathcal{O}(\mathring{\nabla}_m)$ we can absorb the $\aleph$ term into the E field by using the redefinition of Equation (146) to zeroth order in spacetime derivatives. We then again perform the flat index expansion through $Y^m = e^m_I Y^I$ and Equation (154),

$$T_{\alpha\beta} = l_s^2\Big(\eta_{IJ}\partial_\alpha Y^I \partial_\beta Y^J - \frac{1}{2}\delta^c_- \eta_{cd} e^d_\alpha \big(\Lambda_+ \partial_\beta E + \Lambda_+ \partial_\beta Y^0\big) - \frac{1}{2}\delta^c_+ \eta_{cd} e^d_\alpha \big(\Lambda_- \partial_\beta E - \Lambda_- \partial_\beta Y^0\big)\Big) + \cdots \times \gamma_{\alpha\beta} + \mathcal{O}(\neq l_s^2). \quad (184)$$

Note that any derivatives on the vielbeins and Φ that emerge from this are dropped as we work only to zeroth order in spacetime derivatives. In flat space the curved Greek indices and the flat Latin indices belong to the same coordinate system and this can be written as

$$T_{ab} = l_s^2\Big(\eta_{IJ}\partial_a Y^I \partial_b Y^J - \frac{1}{2}\eta_{-a}\big(\Lambda_+ \partial_b E + \Lambda_+ \partial_b Y^0\big) - \frac{1}{2}\eta_{+a}\big(\Lambda_- \partial_b E - \Lambda_- \partial_b Y^0\big)\Big) + \cdots \times \eta_{ab} + \mathcal{O}(\neq l_s^2). \quad (185)$$

We will again go to lightcone coordinates to find a simple expression for the trace of the energy-momentum tensor, note that the energy-momentum tensor is not necessarily symmetric here. The flat metric in lightcone coordinates is given by

$$\eta_{ab} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}. \quad (186)$$

Such that the trace of the energy-momentum tensor in lightcone coordinates can be expressed as

$$\gamma^{ab} T_{ab} = -\frac{1}{2} T_{-+} - \frac{1}{2} T_{+-}. \quad (187)$$

We will determine T_{+-} and T_{-+} through energy-momentum conservation as in Equation (39), for this we will need expressions for T_{++} and T_{--},

$$T^{\text{ins}}_{++} = l_s^2\Big(\eta_{IJ}\partial_+ Y^I \partial_+ Y^J + \frac{1}{4}\big(\Lambda_+ \partial_+ E + \Lambda_+ \partial_+ Y^0\big)\Big), \quad (188)$$

$$T^{\text{ins}}_{--} = l_s^2\Big(\eta_{IJ}\partial_- Y^I \partial_- Y^J + \frac{1}{4}\big(\Lambda_- \partial_- E - \Lambda_- \partial_- Y^0\big).\Big) \quad (189)$$

We now perform the same steps to get insertions for $C^\pm$. From Equation (179) we have

$$\lambda_\pm C^\pm = -\frac{1}{2} e^\beta_\mp \lambda_\pm \big(\partial_\beta \eta + \aleph_\beta \pm \tau_\beta\big). \quad (190)$$

This term also appeared in Equation (182), so it is easy to see that

$$\lambda_\pm C^\pm = -\frac{1}{2} l_s^2 \big(\Lambda_\pm \partial_\mp E \pm \Lambda_\pm \partial_\mp Y^0\big) + \mathcal{O}(\neq l_s^2) \quad (191)$$

to zeroth order in spacetime derivatives.

7.1 Energy-momentum conservation

In Section 2 we found a quantum analog for energy-momentum conservation in Equation (20) by examining worldsheet reparametrizations (5). The TNC action (112) is also reparametrization invariant. In the derivation leading to Equation (20) the generating functional W was taken to be general, such that Equation (20) also holds for the TNC case. We can then repeat the steps that lead to Equation (39), but we should be careful that the TNC energy-momentum tensor is not symmetric. We will thus arrive at two equations, one connecting $\langle T_{+-}\rangle$ to $\langle T_{--}\rangle$ and one connecting $\langle T_{-+}\rangle$ to $\langle T_{++}\rangle$.

$$k_-\langle T_{+-}\rangle + k_+\langle T_{--}\rangle = 0, \tag{192}$$

$$k_+\langle T_{-+}\rangle + k_-\langle T_{++}\rangle = 0. \tag{193}$$

7.2 Differential equation to get to the dilaton β-function

To get to the dilaton beta function we need to consider $\langle \gamma^{\alpha\beta}T_{\alpha\beta}\rangle$ and $\langle C^{\pm}\rangle$ on a curved worldsheet. This brings in complications so instead we use the following trick: We write the curved metric as a scale factor times the flat metric, $\gamma_{\alpha\beta} = e^f\delta_{\alpha\beta}$. This can be done because we have reparametrization invariance on the worldsheet. For the theory to be Weyl invariant we want the functions characterizing the Weyl anomaly (T^a_a and $C^{\pm}$) to disappear regardless of what the scale factor f is. As a minimal requirement its first variation to f should be zero. Using the path integral definition (17) we can write this variation on the curved worldsheet as

$$\begin{aligned}\frac{\delta}{\delta f}\langle \gamma^{\alpha\beta}T_{\alpha\beta}+C^+\lambda_+ + C^-\lambda_-\rangle_{e^f\delta_{ab}}\Big|_{f=0} = -\Big(&\langle\gamma^{\alpha\beta}T_{\alpha\beta}\gamma^{\gamma\delta}T_{\gamma\delta}\rangle + 2\langle\gamma^{\alpha\beta}T_{\alpha\beta}C^+\lambda_+\rangle + 2\langle\gamma^{\alpha\beta}T_{\alpha\beta}C^-\lambda_-\rangle\\ &+\langle(C^+\lambda_+)^2\rangle + 2\langle C^+\lambda_+C^-\lambda_-\rangle + \langle(C^-\lambda_-)^2\rangle\Big)_{\delta_{ab}}.\end{aligned} \tag{194}$$

Here any derivatives of the partition function are multiplied by the first-order Weyl anomaly on a flat worldsheet, which can be set to zero using the first-order β-functions found in [14].

Let us start with constructing the first term of Equation (194), two insertions of the trace of the energy-momentum tensor. In lightcone coordinates this is

$$\begin{aligned}\langle \eta^{ab}T_{ab}\eta^{cd}T_{cd}\rangle &= \frac{1}{4}\left(\langle T_{+-}T_{+-}\rangle + 2\left\langle T_{+-}T_{-+}\right\rangle + \langle T_{-+}T_{-+}\rangle\right)\\ &= \frac{1}{4}\left(\frac{k_+^2}{k_-^2}\left\langle T_{--}T_{--}\right\rangle + 2\left\langle T_{++}T_{--}\right\rangle + \frac{k_-^2}{k_+^2}\left\langle T_{++}T_{++}\right\rangle\right),\end{aligned} \tag{195}$$

where we made use of the energy-momentum conservation equations (192). $\langle T_{++}\rangle$ and $\langle T_{--}\rangle$ are given in Equation (188) and Equation (189). Note that the $++$ and the $--$ insertions are very similar, the same is true for the insertions of C_+ and C_-. This brings us to the idea to introduce a tensor M_*^{abcd} which has the following action,

$$\left\langle T_{ab}T_{cd}\right\rangle M_*^{abcd} \equiv \frac{k_+^2}{k_-^2}\left\langle T_{--}T_{--}\right\rangle + 2\left\langle T_{--}T_{++}\right\rangle + \frac{k_-^2}{k_+^2}\left\langle T_{++}T_{++}\right\rangle. \tag{196}$$

Similarly, we can combine the second and third term of Equation (194) in

$$\left\langle T_{ab}C_c\lambda_d\right\rangle M_{**}^{abcd} \equiv \left(\frac{k_-}{k_+}\left\langle T_{++}C^+\lambda_+\right\rangle + \frac{k_+}{k_-}\left\langle T_{--}C^+\lambda_+\right\rangle + \frac{k_-}{k_+}\left\langle T_{++}C^-\lambda_-\right\rangle + \frac{k_+}{k_-}\left\langle T_{--}C^-\lambda_-\right\rangle\right). \tag{197}$$

And finally the last three terms in Equation (194) can be written as

$$\langle C_a\lambda_b C_c\lambda_d\rangle\, M^{abcd}_{***} \equiv \langle (C^+\lambda_+)^2\rangle + 2\langle C^+\lambda_+ C^-\lambda_-\rangle + \langle (C^-\lambda_-)^2\rangle. \tag{198}$$

We can then write Equation (194) as

$$\frac{\delta}{\delta f}\left\langle \frac{\delta S}{\delta f}\right\rangle_{e^f\delta_{ab}}\Bigg|_{f=0} = -\frac{1}{4}M^{abcd}_{*}\langle T_{ab}T_{cd}\rangle - M^{abcd}_{**}\langle T_{ab}C_c\lambda_d\rangle - M^{abcd}_{***}\langle C_a\lambda_b C_c\lambda_d\rangle\,. \tag{199}$$

Using Equations (188), (189) and (191), the expectation values in Equation (199) can be expressed in the quantum fields as

$$\langle T_{a'b'}T_{c'd'}\rangle = l_s^4 K^{ab}_{a'b'}K^{cd}_{c'd'}\Big[\left\langle \eta_{IJ}\eta_{MN}\partial_a Y^I\partial_b Y^J\partial_c Y^M\partial_d Y^N\right\rangle + \frac{1}{2}\left\langle \eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d E\right\rangle$$
$$-\frac{1}{2}P^e_d\left\langle \eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_e Y^0\right\rangle + \frac{1}{16}\left\langle \Lambda_a\partial_b E\Lambda_c\partial_d E\right\rangle + P^e_b P^f_d\frac{1}{16}\left\langle \Lambda_a\partial_e Y^0\Lambda_c\partial_f Y^0\right\rangle - P^e_d\frac{1}{8}\left\langle \Lambda_a\partial_b E\Lambda_c\partial_e Y^0\right\rangle\Big]. \tag{200}$$

Here we introduced the K tensor to produce the momentum factors that appeared because of energy-momentum conservation (192), its non-zero entries are $K^{++}_{++} = \frac{k_-}{k_+}$ and $K^{--}_{--} = \frac{k_+}{k_-}$. Notice that the expressions for T_{++} (188) and T_{--} (189) have the same factors in front of every term, except for a minus sign difference at the $\partial_\pm Y^0$ term. To capture this behavior the tensor P^a_b is added, which has non-zero entries $P^+_+ = 1$ and $P^-_- = -1$. The second term in Equation (199) can be written out as

$$\langle T_{a'b'}C_c\lambda_d\rangle = -\frac{1}{2}l_s^4 K^{ab}_{a'b'}R^e_d\Big[\left\langle \delta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_e E\right\rangle + P^f_c\left\langle \delta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_f\partial_e Y^0\right\rangle + \frac{1}{4}\left\langle \Lambda_a\partial_b E\Lambda_c\partial_e E\right\rangle$$
$$+\frac{1}{4}P^f_c\left\langle \Lambda_a\partial_b E\Lambda_f\partial_e Y^0\right\rangle + \frac{1}{4}P^g_b\left\langle \Lambda_a\partial_g Y^0\Lambda_c\partial_e E\right\rangle + \frac{1}{4}P^g_b P^f_c\left\langle \Lambda_a\partial_g Y^0\Lambda_f\partial_e Y^0\right\rangle\Big]. \tag{201}$$

Here the R tensor is added to reverse the coordinate of the C insertion. Its non-zero entries are $R^+_- = R^-_+ = 1$. Finally for the last term in Equation (199) we find

$$\langle C_a\lambda_b C_c\lambda_d\rangle = \frac{1}{4}l_s^4 R^e_b R^g_d\Big[\left\langle \Lambda_a\partial_e E\Lambda_c\partial_g E\right\rangle + P^h_c\left\langle \Lambda_a\partial_e E\Lambda_h\partial_g Y^0\right\rangle + P^f_a\left\langle \Lambda_f\partial_e Y^0\Lambda_c\partial_g E\right\rangle + P^f_a P^h_c\left\langle \Lambda_f\partial_e Y^0\Lambda_h\partial_g Y^0\right\rangle\Big]. \tag{202}$$

We have now covered any special behavior for the $+$ and $-$ coordinates in the tensors K^{ab}_{cd}, P^a_b and R^a_b. The M tensors of Equations (196), (197) and (198) can then be written as one simple tensor M which gives the available coordinate combinations. It has non-zero entries $M^{++++} = M^{++--} = M^{--++} = M^{----} = 1$, note that this is exactly the behavior of $\delta^{ab}\delta^{cd}$. Taking all of this together we find for the differential equation of Equation (199) that

$$\frac{\delta}{\delta f}\left\langle \frac{\delta S}{\delta f}\right\rangle_{e^f\delta_{ab}}\Bigg|_{f=0} = l_s^4\Big[S^{abcd}\left\langle \delta_{IJ}\delta_{MN}\partial_a Y^I\partial_b Y^J\partial_c Y^M\partial_d Y^N\right\rangle$$
$$+U^{abcd}\left\langle \delta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d E\right\rangle + V^{abce}\left\langle \delta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_e Y^0\right\rangle + W^{abcd}\left\langle \Lambda_a\partial_e E\Lambda_c\partial_d E\right\rangle \tag{203}$$
$$+X^{aecf}\left\langle \Lambda_a\partial_e Y^0\Lambda_c\partial_f Y^0\right\rangle + Z^{abce}\left\langle \Lambda_a\partial_b E\Lambda_c\partial_e Y^0\right\rangle\Big],$$

where

$$S^{abcd} = -\frac{1}{4}\delta^{a'b'}\delta^{c'd'}K^{ab}_{a'b'}K^{cd}_{c'd'}, \tag{204}$$

$$U^{abcd} = \delta^{a'b'}\delta^{c'd'}\left(-\frac{1}{8}K^{ab}_{a'b'}K^{cd}_{c'd'} + \frac{1}{2}K^{ab}_{a'b'}\delta^{c}_{c'}R^{d}_{d'}\right), \tag{205}$$

$$V^{abce} = \delta^{a'b'}\delta^{c'd'}\left(\frac{1}{8}K^{ab}_{a'b'}P^{e}_{d}K^{cd}_{c'd'} + \frac{1}{2}K^{ab}_{a'b'}P^{c}_{c'}R^{e}_{d'}\right), \tag{206}$$

$$W^{abcd} = \delta^{a'b'}\delta^{c'd'}\left(-\frac{1}{64}K^{ab}_{a'b'}K^{cd}_{c'd'} + \frac{1}{8}R^{d}_{d'}K^{ab}_{a'b'}\delta^{c}_{c'} - \frac{1}{4}R^{b}_{b'}R^{d}_{d'}\delta^{a}_{a'}\delta^{c}_{c'}\right), \tag{207}$$

$$X^{aecf} = \delta^{a'b'}\delta^{c'd'}\left(-\frac{1}{64}P^{e}_{b}P^{f}_{d}K^{ab}_{a'b'}K^{cd}_{c'd'} + \frac{1}{8}P^{e}_{b}P^{c}_{c'}R^{f}_{d'}K^{ab}_{a'b'} - \frac{1}{4}P^{a}_{a''}P^{c}_{c'}R^{e}_{b}R^{f}_{d'}K^{a''b}_{a'b'}\right), \tag{208}$$

$$Z^{abce} = \frac{1}{4}\delta^{a'b'}\delta^{c'd'}\left(\frac{1}{8}P^{e}_{d}K^{ab}_{a'b'}K^{cd}_{c'd'} + \frac{1}{2}P^{c}_{c'}R^{e}_{d'}K^{ab}_{a'b'} + \frac{1}{2}P^{e}_{f}R^{b}_{d'}K^{cf}_{a'b'}\delta^{a}_{c'} - R^{b}_{b'}R^{e}_{d'}P^{c}_{c'}\delta^{a}_{a'} - R^{b}_{d'}R^{e}_{b'}P^{c}_{a'}\delta^{a}_{c'}\right). \tag{209}$$

We have now reduced the problem to determining the six expectation values that appear in Equation (203). These can be found by inserting the terms inside the expectation value brackets in diagrams generated by the actions of Section 6. As we saw for the bosonic string, the dilaton beta function contains the contributions that are proportional to the spacetime Ricci scalar. These result from terms that have no background fields associated with them. Examining the actions in the flat index expansion we see that all terms in (161) are proportional to the background field X^{μ}_{0} or $\lambda_{\pm}$. Consequently, we only have to consider the actions (162) and (160).

8 Dilaton β-function Feynman diagrams

We now have all the ingredients to start calculating the dilaton β-function. In Equation (203) we saw that it consists of six different combinations of insertions. We will see that a $\Lambda_a\partial_b E$ insertion is much the same as a $\Lambda_a\partial_b Y^0$ insertion. For this reason it is easier to consider some of these insertions simultaneously. We have subdivided this section into a subsection covering $\left\langle\eta_{IJ}\eta_{MN}\partial_a Y^I\partial_b Y^J\partial_c Y^M\partial_d Y^M\right\rangle$, a subsection covering diagrams with insertions of the form of $\left\langle\eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d A\right\rangle$ and lastly a subsection covering diagrams having insertions of the form $\langle\Lambda_a\partial_b A\Lambda_c\partial_d B\rangle$. Here A and B are either E or Y^0. As there are a lot of complicated integrals, they are listed in Appendix C

8.1 $\left\langle\eta_{IJ}\eta_{MN}\partial_a Y^I\partial_b Y^J\partial_c Y^M\partial_d Y^M\right\rangle$ diagrams

Let us start with the quadruple Y expectation value $\left\langle\eta_{IJ}\eta_{MN}\partial_a Y^I\partial_b Y^J\partial_c Y^M\partial_d Y^M\right\rangle$. We will consider the three topologies of Figure 7, 9 and 10. We expect that for any of these three diagram forms there will be multiple diagrams in the TNC case. This is because we are dealing with two extra fields here (Λ_a and E). As we saw in Section 6.1, a Y field can propagate to another Y-field or to a Λ-field, furthermore the vertices in the actions of Equation (160) and (162) contain terms where a Λ-field is coupled to multiple Y-fields. We then get additional diagrams where we swap certain Y-lines of the diagrams of Figures 7, 9 and 10 by a Λ. This can either happen in an internal line, or in an external line connecting a vertex to the insertion. Note that Y-Y propagator is only

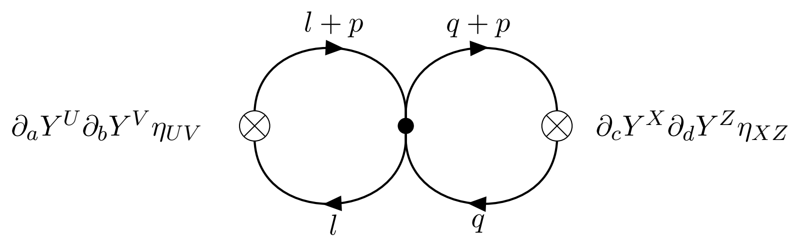

Figure 14: Diagram contributing to the expectation value $\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \rangle$. The Y-fields are inserted at the crosses. The solid lines stand for a Y-Y propagator, whose form is found in Equation (170). The dot in the middle represents a vertex with vertex factor $-E_{rsmn} e^r_I e^s_J e^m_M e^n_N Y^I Y^J \partial_a Y^M \partial^a Y^N$.

between spatial Y^i fields, whereas the Y-Λ propagator connects a Λ_a to a Y^0. The Y-Y insertion that we want to compute here does not mix the components of the Y-fields, as it contracts the Y fields with the flat metric η_{UV}. Therefore it is impossible to have one of the two inserted Y-fields on one insertion propagate into a Λ-field while the other remains a Y-field. Either both Y-fields propagate into a Λ-field and the inserted Y-fields are only the Y^0 component or none of the Y-fields change and the inserted Y-fields are only the spatial Y^i component. We are now ready to examine the three diagram shapes of Figure 7, 9 and 10 and see which diagrams we can form with the TNC fields.

The infinity shaped diagram of Figure 7 has one vertex involving four quantum fields. In the fourth order action (162) there are two terms that can form such a vertex. One involves four Y fields and the other couple a Λ field to three Y fields, but since we cannot have only one inserted Y-field propagate to a Λ, this last vertex will not appear in diagrams. The only diagram of this shape that can be formed is given in Figure 14. This diagram is the same as in the Riemannian case (Figure 7) except that the vertex now contains an extra term involving the derivative of the spatial metric, which can be traced back to the fact that the covariant derivative is no longer metric compatible.

Next, we turn to diagrams with the shape of Figure 9. These diagrams have two third order vertices. For these we look at $S_3^{(1)}$ in Equation (160). This action features one vertex combining three Y fields and one vertex combining two Y fields with a Λ-field. Consider first the case where none of the inserted Y fields flip to a Λ. We could then have two triple Y vertices with an internal Y-Y propagator, see Figure 15. Another option is one $YY\Lambda$-vertex, one YYY-vertex and a $Y-\Lambda$ propagator connecting the two. This diagram is shown in Figure 16. Note that a symmetry factor of 2 appears here as both vertices can be interchanged. Finally we have the case where both vertices have two Y-fields and one Λ-field, shown in Figure 17. The internal line connecting the vertices is then a Λ-Λ propagator.

Because Λ_a can only propagate to Y^0 and Y^0 can only propagate back to a Λ_a the only diagram where the inserted Y-fields flip to Λ-fields directly is the one in Figure 18

We now turn to diagrams with the shape of the diagram in Figure 10, just as the previous diagrams these feature two 3-point vertices of either three Y-fields or two Y-fields and a Λ-field. Here it is not possible to let any of the inserted Y-fields flip to a Λ-field as its partner on the lower leg is directed connected to another YY-insertion. The only place where we can add Λ-fields is on the internal lines connecting the two vertices. Let us start with a diagram where all of the fields are Y-fields,

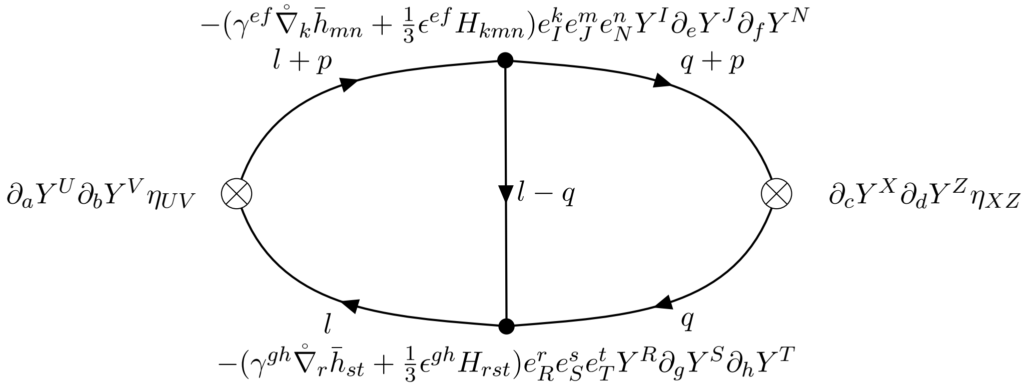

Figure 15: Contribution to the expectation value $\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \rangle$ with only Y propagators. To find the expectation value we insert $\partial_a Y^I \partial_b Y^J \eta_{IJ}$ twice into a second order diagrams composed of the vertices and propagators determined in Section 6.

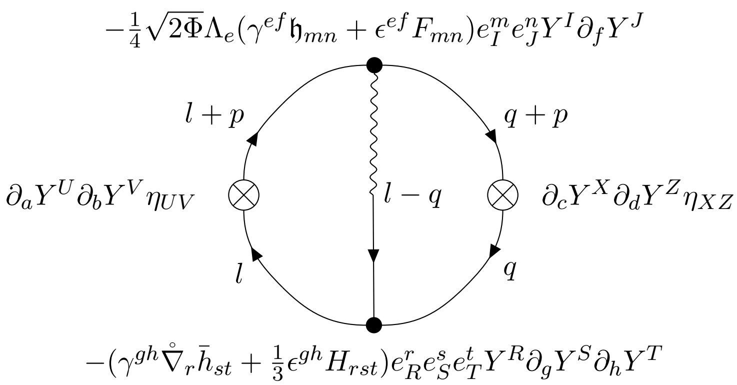

Figure 16: One of the vertices now features a Λ, which then flips to a Y-field through the Y-Λ propagator (171). The diagram should be multiplied by 2 for symmetry because the Λ could also be on the lower half of the inner propagator..

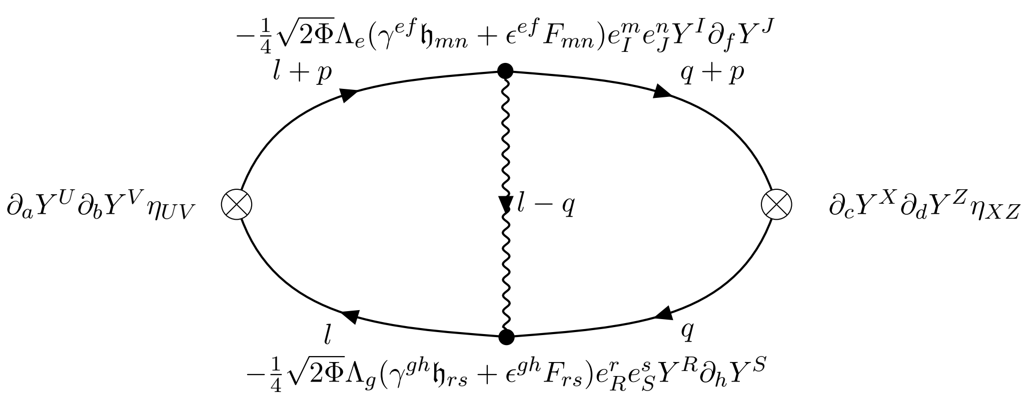

Figure 17: Now both of the vertices have a Λ-field.

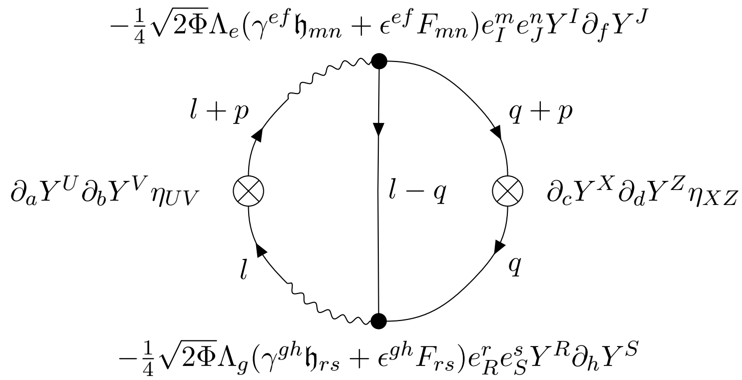

Figure 18: The Y^0 part of the inserted Y-fields can flip to a Λ_a (171). Because the YY insertion does not mix the Y fields the other inserted Y-field also only has the Y^0 component. This Y^0 can then only flip to Λ_a (171). This double flip can be on any of the two YY insertions, but not on both as the vertices in action (160) have at most one Λ-field. The diagram should be multiplied by two to account for this symmetry.

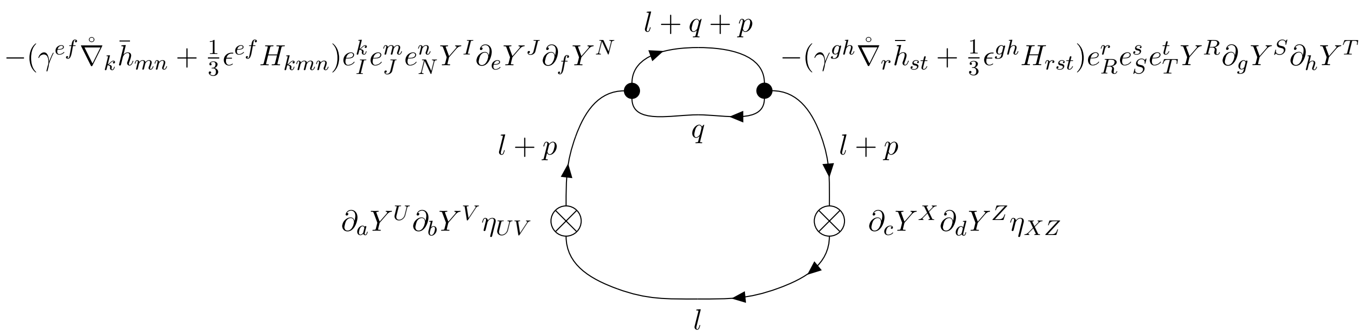

Figure 19: Diagram contributing to the expectation value $\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \rangle$ with only Y fields. This diagram is similar to the diagram of Figure 10 of the Riemannian bosonic string, but here the vertices also feature a derivative of the spatial metric, which was not present in the Riemannian case because there the covariant derivative was metric compatible.

this is shown in Figure 19. Next we consider one Λ-field appearing in the small loop, as shown in Figure 20. The Λ-field can be on any of the four internal legs of the loop, thus a symmetry factor of 4 appears. If we then add a second Λ-field to the small loop there are two options. It should be on the other vertex, but it can go to the line connecting to the first Λ-field or to the opposite side. These diagrams are shown in Figures 21 and 22 respectively.

Having constructed all the diagrams contributing to $\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \rangle$ we can now start computing the corresponding Feynman integrals. The strategy will mostly be the same as for the Riemannian bosonic string; the derivatives put momentum factors in the numerator of an integral. The propagators provide the denominators. In general for two-loop diagrams these integrals will be too difficult to solve right away because there are too many denominators. By rewriting the momenta in the numerator in terms of the terms of the denominator we try to simplify the integral. Separately there are contractions between the TNC fields, as these diagrams contribute to the dilaton β-function these contractions form spacetime scalars. There are not too many scalar terms that can be formed this way, for example in the Riemannian case there were only four:

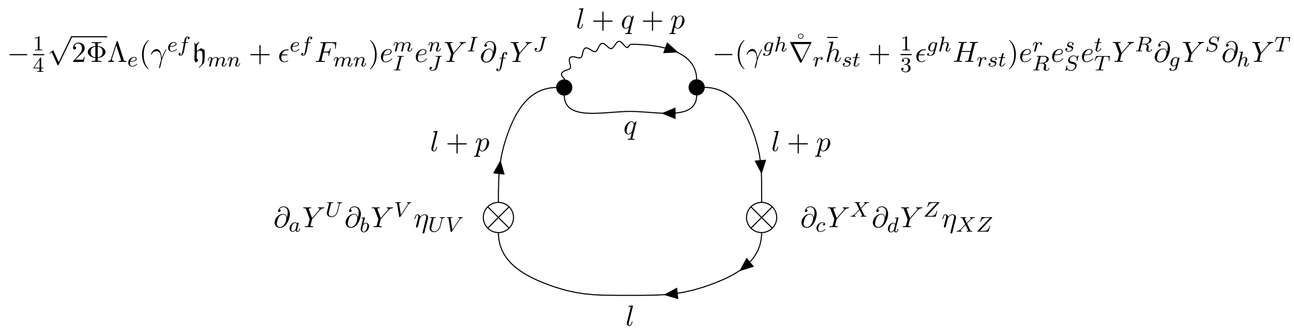

Figure 20: Now we put a Λ-field on one of the legs of the small loop. There are four options, hence a symmetry factor of 4.

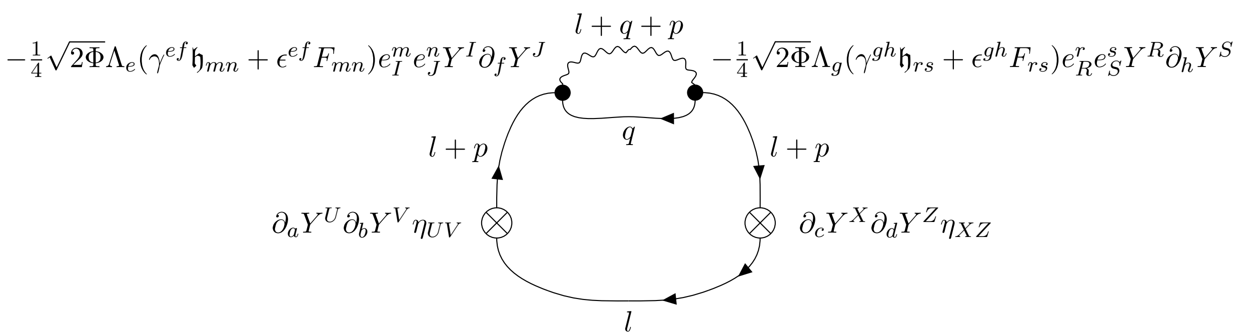

Figure 21: Now we put a second Λ-field on one of the legs of the small loop. The option shown here has the two Λ-fields connecting with a Λ-Λ propagator (172). There is a symmetry factor of 2.

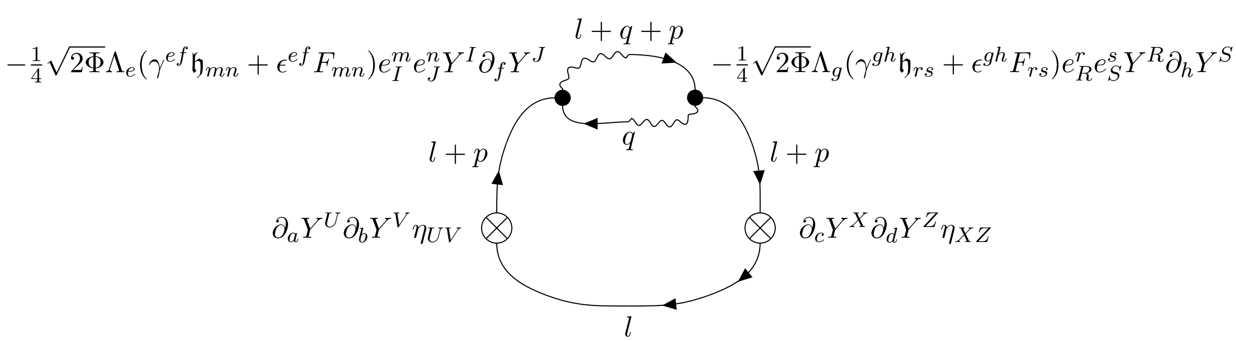

Figure 22: Lastly there is the option to put the two Λ-fields on opposite sides in the small loop, such that they connect with a Y-field of the other vertex. There is a symmetry factor of 2.

The spacetime Ricci scalar R, the square of the Kalb-Ramond field H^2 and two ways to put two spacetime derivatives on the dilaton: $(\nabla \cdot \phi)$ and $\nabla^2\phi$.Schematically Figure 14 can be represented by the following integral,

$$-i^6(2\pi)^4\frac{l_s^2}{4\pi}\int d^nl\int d^nq\frac{l_a(l+p)_b(q+p)_cp_d}{l^2q^2(l+p)^2(q+p)^2}\delta_{UV}\delta_{XZ}\phi^{UR}\phi^{VS}\phi^{XQ}\phi^{ZP}E_{rsmn}e_I^re_J^se_M^me_N^n\Big[-\delta_R^I\delta_S^J\delta_Q^M\delta_P^N q\cdot(q+p)-\delta_R^I\delta_S^J\delta_Q^N\delta_P^M q\cdot(q+p)+\delta_R^I\delta_S^M\delta_Q^J\delta_P^N q\cdot(l+p)+\dots\Big], \quad (210)$$

where the dots represent the other 21 ways to place the vertex Y-fields. The E-tensor was defined in Equation (150). Each insertion has a delta tensor connecting two legs. The Y-Y propagator was just a delta tensor in the Riemannian case, but as we found in Equation (170), in the TNC case it has a different tensorial structure: $\phi^{IJ}\equiv\eta^{IJ}+\delta_0^I\delta_0^J=\delta^{IJ}-\delta_0^I\delta_0^J$. Each set of derivatives brings down at most two factors of the loop momenta l or q, so at most we have one-loop integrals with four momentum factors in the numerator. This is still covered by the integrals of Appendix B.1, the only problem being that there are 12 different momentum configurations and thus 12 different integrals to compute. We will first simplify the tensor contractions a bit to see if we can reduce this. After some algebra we find that an insertion with two connecting propagators combines into an easier result; $\eta_{UV}\phi^{UI}\phi^{VJ}=\phi^{IJ}$. The vertex involves four spatial vielbeins, which contract with the ϕ^{IJ} of the insertion plus propagator contraction. This contraction happens in the 24 possibilities of taking all vertex leg configurations into account. In general $\phi^{IJ}e_I^re_J^s=h^{rs}+e_0^re_0^s\equiv\tilde{h}^{rs}$. This $\tilde{h}^{rs}$ then contracts with the E-tensor appearing in the vertex. From Equation (150) we see that E_{rsmn} is symmetric in its last two indices. When it is contracted with two symmetric rank-2 tensors there are only two distinct terms that can be formed. We can then group the 24 terms appearing in Equation (210) in terms that have the contraction $E_{klmn}\tilde{h}^{kl}\tilde{h}^{mn}\equiv E_1$ or the contraction $E_{klmn}\tilde{h}^{km}\tilde{h}^{ln}\equiv E_2$. This gives two different integrals, one that is proportional to E_1 and another that is proportional to E_2,

$$-16\pi^3l_s^2\int d^nl\int d^nq\frac{l_a(l-p)_b(q-p)_cq_d}{l^2q^2(p-l)^2(q-p)^2}\left[E_1\big(q^2+l^2-q\cdot p-l\cdot p\big)+p^2E_2\right]. \quad (211)$$

These integrals are fourth order in their tensor loop momenta at most, we can thus use the formulas of Appendix B.1 to calculate them.[7] To make reporting the Weyl anomaly clear we will write these diagrams in terms of integrals J, which we then move to Appendix C to be calculated there. The diagram of Figure 14 can then be represented by

$$-16\pi^3l_s^2\Big(E_1J_{14}^{E_1}+E_2J_{14}^{E_2}\Big). \quad (212)$$

Next, we move to the second diagram, Figure 15. Again we have to take all vertex leg configurations into account: We get different momentum factors depending on which fields the derivatives in the vertices act and contracting both vertices with each other gives different results depending on how the indices are permuted. Note that we do not get any terms combining H_{ijk} with $\mathring{\nabla}_i\bar{h}_{jk}$ as there will always be a contraction between a symmetric part and an antisymmetric part. This diagram is much the same as that of Figure 9. There we first used Equation (66) to rewrite the epsilon tensors. Secondly, we wrote contractions among the momenta coming from the vertices in terms of propagators to be able to cancel some of the propagators and to be able to calculate the integral

[7] In the Riemannian case we were careful with keeping track of the tadpole integral $S\equiv\int d^nll^{-2}$. In Appendix B.1 we note that this integral is either infinite or zero. Any term proportional to S will thus either be zero, or stick around until the end and be removed by a counterterm. Because of this, no finite results will ever arise from terms proportional to S. For this reason we will no longer take terms proportional to S into account; effectively we put S to zero.

using expressions from Appendix B. By going through these same steps for the diagram of Figure 15 we can express it as

$$
\begin{aligned}
&(2\pi)^5\left(\frac{l_s}{4\pi}\right)^2\tilde{h}^{uv}\tilde{h}^{wx}\tilde{h}^{yz}\int d^nl\int d^nq\frac{l_a(l+p)_b(q+p)_cq_d}{l^2(l+p)^2(l-q)^2(q+p)^2q^2}\Big\{-2\mathring{\nabla}_z\bar{h}_{vw}\mathring{\nabla}_u\bar{h}_{yx}\Big[(l-q)^4+\big((l+p)^2\\
&\quad-(q+p)^2\big)\big(l^2-q^2\big)-(l-q)^2\big((l+p)^2+l^2+(q+p)^2+q^2\big)\Big]-\mathring{\nabla}_u\bar{h}_{zw}\mathring{\nabla}_v\bar{h}_{yx}\Big[-3(l-q)^4\\
&\quad-3(l+p)^2l^2+l^2(q+p)^2+(l+p)^2q^2-3(q+p)^2q^2+(l-q)^2\big((l+p)^2+l^2+(q+p)^2+q^2\big)\Big]\\
&\quad+f(n)H_{uwy}H_{vxz}\Big[(l-q)^4+\big((l+p)^2-(q+p)^2\big)(l^2-q^2)-(l-q)^2\big((l+p)^2+l^2-2p^2+(q+p)^2+q^2\big)\Big]\Big\}.
\end{aligned}
\tag{213}
$$

This looks complicated but any terms where only one denominator of either l or q remains can be dropped straight away by the arguments of footnote 7. We are then left with only two distinct integrals, one where $(l-q)^2$ gets canceled and we are left with two separate one-loop integrals and one where either $q^2(l+p)^2$ or $l^2(q+p)^2$ is canceled. Note that these two give the same result as the rest of the integral is symmetric under exchange of l and q.[8] The integral then simplifies to

$$2\pi^3l_s^2\tilde{h}^{uv}\tilde{h}^{wx}\tilde{h}^{yz}\Big(\mathring{\nabla}_u\bar{h}_{zw}\mathring{\nabla}_v\bar{h}_{yx}J_{15}^{\nabla h}+\mathring{\nabla}_z\bar{h}_{vw}\mathring{\nabla}_u\bar{h}_{yx}J_{15}^{\nabla h*}+H_{uwy}H_{vxz}J_{15}^{H^2}\Big), \tag{214}$$

with the J integrals written out in Appendix C

After taking all vertex configurations into account we find the following integral expression for the diagram of Figure 16,

$$8\pi^3l_s^2\frac{1}{\Phi}(\mathring{\nabla}_s\bar{h}_{tu}-\mathring{\nabla}_u\bar{h}_{st})h^{ru}\hat{v}^s\hat{v}^t\Big(a_rJ_{16}^{\nabla h\cdot a}-\mathfrak{e}_rJ_{16}^{\nabla h\cdot\mathfrak{e}}\Big). \tag{215}$$

Here we have written out $\tilde{h}^{rs}=h^{rs}+e_0^re_0^s$ and subsequently used $e_0^r=-\frac{\hat{v}^r}{\sqrt{2\Phi}}$, which can be seen from Equation (154). We then made some simplifications by using the twistless torsion condition $F_{mn}h^{mr}h^{ns}=0$ as well as twistless torsion of the $\mathfrak{h}$ field, $\mathfrak{h}_{rm}h^{rs}h^{mn}=0$. This last condition is a result from a different β-function, found in [14]. We then defined the acceleration $a_s\equiv\hat{v}^tF_{ts}$ and the electric field $\mathfrak{e}_s\equiv\hat{v}^t\mathfrak{h}_{ts}$.

Next we turn to the diagram of Figure 17. There two vertices involve a Λ-field, which have a Λ-Λ propagator connecting them (172). The diagram can be represented by the following integral expression,

$$4l_s^2\pi^3h^{uv}\Big[a_ua_vJ_{17}^{a^2}+\mathfrak{e}_u\mathfrak{e}_vJ_{17}^{e^2}+2a_u\mathfrak{e}_vJ_{17}^{a\cdot e}\Big]. \tag{216}$$

The last diagram of this form is given in Figure 18. After rearranging the momentum terms we find the following integral expression for it,

$$\pi^3l_s^2h^{uv}\Big[4\mathfrak{e}_u\mathfrak{e}_vJ_{18}^{\mathfrak{e}^2}+a_ua_vJ_{18}^{a^2}-4a_ue_vJ_{18}^{a\cdot\mathfrak{e}}\Big]. \tag{217}$$

The next diagram to consider is in Figure 19. After taking all possible momentum configurations and rearranging the momentum contractions we find the following integral expression for this diagram,

$$2\pi^3l_s^2\tilde{h}^{wx}\tilde{h}^{uv}\tilde{h}^{yz}\big(3\mathring{\nabla}_u\bar{h}_{zw}\mathring{\nabla}_v\bar{h}_{yx}-2\mathring{\nabla}_z\bar{h}_{vw}\mathring{\nabla}_u\bar{h}_{yx}+f(n)H_{uwy}H_{vxz}\big)J_{19}. \tag{218}$$

[8]For this symmetry to hold one needs the diagram to be symmetric under exchange of $abcd\leftrightarrow dcba$. This is the case when the diagram is contracted S^{abcd} (204), which is mirror symmetric.

The next diagram is in Figure 20, after reordering all the contractions we have

$$\pi^3 \frac{8}{\Phi} l_s^2 h^{sw} \hat{\upsilon}^t \hat{\upsilon}^v \left(\mathring{\nabla}_t \bar{h}_{vw} - \mathring{\nabla}_w \bar{h}_{tv}\right) \left[a_s J_{20}^{a\cdot\mathring{\nabla}h} - 2\mathfrak{e}_s J_{20}^{\mathfrak{e}\cdot\mathring{\nabla}h}\right]. \tag{219}$$

Moving on, the diagram in Figure 21 has the following integral expression,

$$-8\pi^3 l_s^2 h^{rs} \left[a_u a_v J_{21}^{a^2} + \mathfrak{e}_u \mathfrak{e}_v J_{21}^{\mathfrak{e}^2} - 4 a_u \mathfrak{e}_v J_{21}^{a\cdot\mathfrak{e}}\right]. \tag{220}$$

The final diagram contributing to $\left\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \right\rangle$ is given in Figure 22. After rearranging we find the following integral expression for it,

$$\pi^3 l_s^2 h^{xy} \left[a_x a_y J_{22}^{a^2} - \mathfrak{e}_x \mathfrak{e}_y J_{22}^{\mathfrak{e}^2}\right]. \tag{221}$$

We can combine all nine of these diagrams into a final result

$$\begin{aligned}\left\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \right\rangle = \pi^3 l_s^2 \Big\{ &-16E_1 J_{14}^{E_1} - 16E_2 J_{14}^{E_2} + \tilde{h}^{uv}\tilde{h}^{wx}\tilde{h}^{yz}\mathring{\nabla}_u \bar{h}_{zw}\mathring{\nabla}_v \bar{h}_{yx}\left(2J_{15}^{\nabla h} + 6J_{19}\right) \\ &+ \tilde{h}^{uv}\tilde{h}^{wx}\tilde{h}^{yz}\mathring{\nabla}_z \bar{h}_{vw}\mathring{\nabla}_u \bar{h}_{yx}\left(2J_{15}^{\nabla h*} - 4J_{19}\right) + \tilde{h}^{uv}\tilde{h}^{wx}\tilde{h}^{yz} H_{uwy} H_{vxz}\left(2J_{15}^{H^2} + 2f(n)J_{19}\right) \\ &+ \frac{1}{\Phi}(\mathring{\nabla}_s \bar{h}_{tu} - \mathring{\nabla}_u \bar{h}_{st}) h^{ru}\hat{\upsilon}^s\hat{\upsilon}^t a_r \left(8J_{16}^{\nabla h\cdot a} + 8J_{20}^{a\cdot\mathring{\nabla}h}\right) + \frac{1}{\Phi}(\mathring{\nabla}_s \bar{h}_{tu} - \mathring{\nabla}_u \bar{h}_{st}) h^{ru}\hat{\upsilon}^s\hat{\upsilon}^t \mathfrak{e}_r \left(-8J_{16}^{\nabla h\cdot\mathfrak{e}} - 16J_{20}^{\mathfrak{e}\cdot\mathring{\nabla}h}\right) \\ &+ h^{uv} a_u a_v \left(4J_{17}^{a^2} + J_{18}^{a^2} - 8J_{21}^{a^2} + J_{22}^{a^2}\right) + h^{uv}\mathfrak{e}_u\mathfrak{e}_v\left(4J_{17}^{\mathfrak{e}^2} + 4J_{18}^{\mathfrak{e}^2} - 8J_{21}^{a^2} - J_{22}^{\mathfrak{e}^2}\right) + h^{uv}\mathfrak{e}_u a_v \left(8J_{17}^{a\cdot e} - 4J_{18}^{a\cdot\mathfrak{e}} + 32J_{21}^{a\cdot\mathfrak{e}}\right)\Big\}.\end{aligned} \tag{222}$$

8.2 $\left\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d E\right\rangle$ and $\left\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d Y^0\right\rangle$ diagrams

We now turn to the second expectation value in Equation (203): $\left\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d E\right\rangle$. The E-field can only propagate to a Λ-field (173), so this limits the possibilities. Before we go into the possible diagrams note that diagrammatically there is a very small difference between an inserted E-field and an inserted Y^0-field. Both can only propagate to a Λ-field, this means that atleast the layout of the diagrams will be exactly the same, regardless of whether we insert an E-field or a Y^0-field. The only difference between the two is their propagator with Λ, the Y^0-Λ propagator (169) is $8\pi i \frac{1}{k^2}\epsilon_{ab}k^b$, whereas the E-Λ (173) propagator is $-8\pi i \frac{1}{k^2}\gamma_{ab}k^b$. To save some time we will draw general diagrams for $\left\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d A\right\rangle$, where the A field is either E or Y^0. In the Feynman integrals we use a new worldsheet tensor ω_{ab} for the E-Λ propagator, where $\omega_{ab} = \epsilon_{ab}$ for the Y^0-Λ propagator and $\omega_{ab} = -\gamma_{ab}$ for the E-Λ propagator.

We will again consider the three diagram shapes of Figure 7, 9 and 10 one by one. The only diagram contributing to $\left\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d A\right\rangle$ in the shape of Figure 7 is given in Figure 23. For diagrams in the shape of Figure 9 there are two options. The inserted A-field has to propagate to a Λ-field, this fixes the possibilities for the lower vertex. The upper vertex is either all Y-fields (Figure 24), or it can have one Λ-field, which can be either connected to the inserted Λ (Figure 25), or it can be on the inner line (Figure 26). It cannot propagate to the YY-insertion because then the other Y-field would also have to turn into a Λ-field, which is not possible here.

Next we move to diagrams of the shape of Figure 10. Because there is one line connecting the two insertions, the options are very limited. We cannot have the A-field on this line because A can only

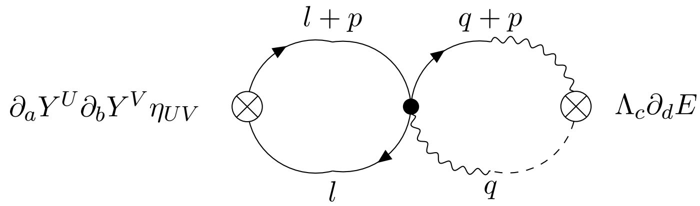

Figure 23: Diagram contributing to $\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d E \rangle$. The vertex in the middle has vertex factor $-\frac{l_s^2}{4\pi}\frac{1}{6}\sqrt{2\Phi}\Lambda_a \left(\gamma^{ab}\mathring{\nabla}_r \mathfrak{h}_{sm} + \epsilon^{ab}\mathring{\nabla}_r F_{sm}\right) e^r_I e^s_J e^m_M Y^I Y^J \partial_b Y^M$. A straight line is a propagating Y-field, a wiggly line a Λ_a-field and a dashed line is an A-field, which is either E or Y^0.

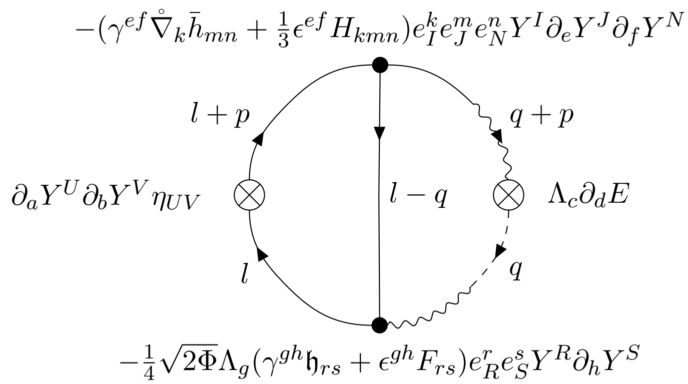

Figure 24

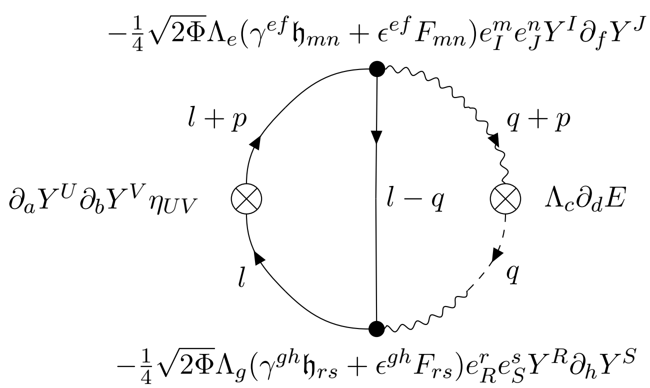

Figure 25

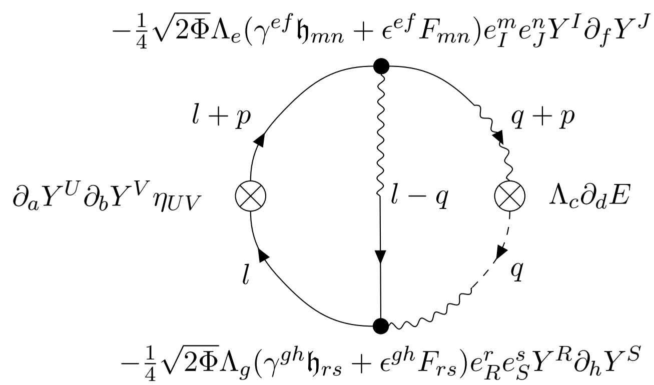

Figure 26

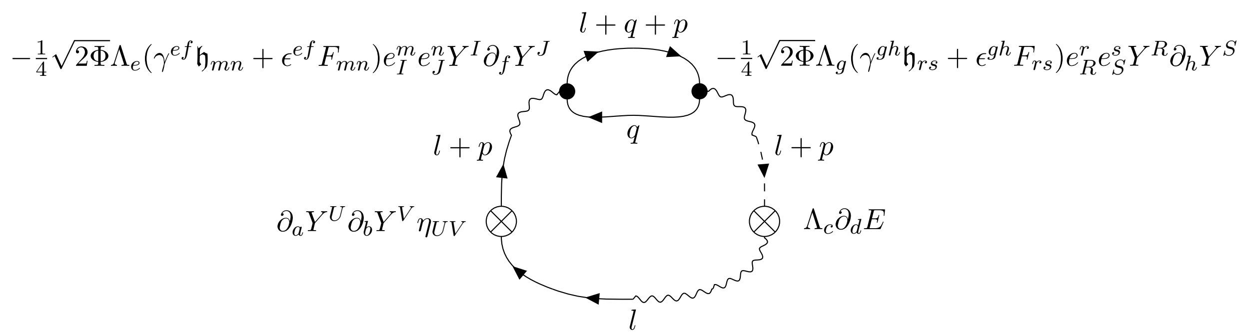

Figure 27: Note that because we want the E-field to go on the leg connecting to the vertex, we have to flip the diagram. So in this the d and a parts connect to the small loop and b and c are on the lower leg.

propagate to Λ. The only option is to place the inserted Λ on the line connecting the two insertions. Because Λ only propagates to Y^0 and the YY-insertion does not mix up the components of Y, the other Y-field of this insertion also only has its 0 component, such that it can only propagate to a Λ. The inserted A-field also has to propagate to a Λ-field and this fixes the internal lines, as the vertices already have the maximum amount of Λ terms. The resulting diagram is given in Figure 27. We have now drawn all diagrams contributing to $\langle \eta_{IJ}\partial_a Y^I \partial_b Y^J \Lambda_c \partial_d A\rangle$, we will express them in Feynman integrals below.

The diagram in Figure 23 can be represented by the integral

$$\pi^3 l_s^2 \frac{32}{3}\hat{\upsilon}^m \tilde{h}^{rs}\Big[\mathring{\nabla}_r \mathfrak{h}_{sm} J_{23}^{\nabla\mathfrak{h}}(\omega) + \mathring{\nabla}_r F_{sm} J_{23}^{\nabla F}(\omega)\Big], \tag{223}$$

with the J integrals are given in Appendix C.

Figure 24 can be represented by the integral expression

$$\frac{4\pi^3}{\Phi}(\mathring{\nabla}_v \bar{h}_{xy} - \mathring{\nabla}_y \bar{h}_{vx}) l_s^2 h^{uy}\hat{\upsilon}^v \hat{\upsilon}^x\Big[a_u J_{24}^{\nabla h\cdot a}(\omega) + 2\mathfrak{e}_u J_{24}^{\nabla h\cdot\mathfrak{e}}(\omega)\Big]. \tag{224}$$

We then turn to Figure 25, which can be represented by the following expression

$$-8\pi^3 l_s^2 h^{vx}\Big[2\mathfrak{e}_v \mathfrak{e}_x J_{25}^{\mathfrak{e}^2}(\omega) + a_v a_x J_{25}^{a^2}(\omega) - 2a_v \mathfrak{e}_x J_{25}^{a\cdot\mathfrak{e}}(\omega)\Big]. \tag{225}$$

The diagram of Figure 26 can be represented as

$$8\pi^3 l_s^2 h^{xy}\Big[\mathfrak{e}_x\mathfrak{e}_y J_{26}^{e^2}(\omega)+a_x a_y J_{26}^{a^2}(\omega)+a_x\mathfrak{e}_y J_{26}^{a\cdot\mathfrak{e}}(\omega)\Big]. \tag{226}$$

The last diagram to contribute to $\langle\eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d E\rangle$ and $\langle\eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d Y^0\rangle$ is given in Figure 27, its contribution is

$$8\pi^3 l_s^2 h^{xy}\Big(\mathfrak{e}_x\mathfrak{e}_y J_{27}^{\mathfrak{e}^2}(\omega)+a_x a_y J_{27}^{a^2}(\omega)+J_{27}^{\mathfrak{e}\cdot a}(\omega)\mathfrak{e}_x a_y\Big). \tag{227}$$

In total we have

$$\begin{aligned}\big\langle\eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d E\big\rangle &= \pi^3 l_s^2\Big\{\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r\mathfrak{h}_{sm}J_{23}^{\nabla\mathfrak{h}}(-\gamma)+\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r F_{sm}J_{23}^{\nabla F}(-\gamma)\\ &+\frac{4}{\Phi}(\mathring{\nabla}_v\bar{h}_{xy}-\mathring{\nabla}_y\bar{h}_{vx})h^{uy}\hat{\upsilon}^v\hat{\upsilon}^x a_u J_{24}^{\nabla h\cdot a}(-\gamma)+\frac{8}{\Phi}(\mathring{\nabla}_v\bar{h}_{xy}-\mathring{\nabla}_y\bar{h}_{vx})h^{uy}\hat{\upsilon}^v\hat{\upsilon}^x\mathfrak{e}_u J_{24}^{\nabla h\cdot\mathfrak{e}}(-\gamma)\\ &+h^{vx}\mathfrak{e}_v\mathfrak{e}_x\Big(-16J_{25}^{\mathfrak{e}^2}(-\gamma)+8J_{26}^{e^2}(-\gamma)+8J_{27}^{\mathfrak{e}^2}(-\gamma)\Big)+h^{vx}a_v a_x\Big(-8J_{25}^{a^2}(-\gamma)+8J_{26}^{a^2}(-\gamma)+8J_{27}^{a^2}(-\gamma)\Big)\\ &+h^{vx}a_v\mathfrak{e}_x\Big(16J_{25}^{a\cdot\mathfrak{e}}(-\gamma)+8J_{26}^{a\cdot\mathfrak{e}}(-\gamma)++J_{27}^{\mathfrak{e}\cdot a}(-\gamma)\Big)\Big\}\end{aligned} \tag{228}$$

and

$$\begin{aligned}\big\langle\eta_{IJ}\partial_a Y^I\partial_b Y^J\Lambda_c\partial_d Y^0\big\rangle &= \pi^3 l_s^2\Big\{\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r\mathfrak{h}_{sm}J_{23}^{\nabla\mathfrak{h}}(\epsilon)+\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r F_{sm}J_{23}^{\nabla F}(\epsilon)\\ &+\frac{4}{\Phi}(\mathring{\nabla}_v\bar{h}_{xy}-\mathring{\nabla}_y\bar{h}_{vx})h^{uy}\hat{\upsilon}^v\hat{\upsilon}^x a_u J_{24}^{\nabla h\cdot a}(\epsilon)+\frac{8}{\Phi}(\mathring{\nabla}_v\bar{h}_{xy}-\mathring{\nabla}_y\bar{h}_{vx})h^{uy}\hat{\upsilon}^v\hat{\upsilon}^x\mathfrak{e}_u J_{24}^{\nabla h\cdot\mathfrak{e}}(\epsilon)\\ &+h^{vx}\mathfrak{e}_v\mathfrak{e}_x\Big(-16J_{25}^{\mathfrak{e}^2}(\epsilon)+8J_{26}^{e^2}(\epsilon)+8J_{27}^{\mathfrak{e}^2}(\epsilon)\Big)+h^{vx}a_v a_x\Big(-8J_{25}^{a^2}(\epsilon)+8J_{26}^{a^2}(\epsilon)+8J_{27}^{a^2}(\epsilon)\Big)\\ &+h^{vx}a_v\mathfrak{e}_x\Big(16J_{25}^{a\cdot\mathfrak{e}}(\epsilon)+8J_{26}^{a\cdot\mathfrak{e}}(\epsilon)++J_{27}^{\mathfrak{e}\cdot a}(\epsilon)\Big)\Big\}.\end{aligned} \tag{229}$$

9 $\langle\Lambda_a\partial_b E\Lambda_c\partial_d E\rangle$, $\langle\Lambda_a\partial_b E\Lambda_c\partial_d Y^0\rangle$ and $\langle\Lambda_a\partial_b Y^0\Lambda_c\partial_d Y^0\rangle$ diagrams

As was noted in the previous section, E and Y^0 behave very much the same in a Feynman diagram. The only difference is in the $E-\Lambda/Y^0-\Lambda$ propagator, this difference be contained in a tensor ω_{ab} which is $-\gamma_{ab}$ for the E-field and ϵ_{ab} for the Y^0-field. We will draw general $\langle\Lambda_a\partial_b A\Lambda_c\partial_d B\rangle$ diagrams and compute all three of the expectation values from those. There are four diagrams that contribute to $\langle\Lambda_a\partial_b A\Lambda_c\partial_d B\rangle$. These are given in Figures 28, 29, 30, 31 and 32.

Figure 28 can be represented as

$$-32\pi^3 l_s^2 h^{xy}\Big(\mathfrak{e}_x\mathfrak{e}_y J_{28}^{\mathfrak{e}^2}(\omega_A,\omega_B)+a_x a_y J_{28}^{a^2}(\omega_A,\omega_B)+ac_x\mathfrak{e}_y J_{28}^{a\cdot\mathfrak{e}}(\omega_A,\omega_B)\Big). \tag{230}$$

After taking all momentum contractions into account, it turns out that the diagram of Figure 29 is always zero by footnote 7. Regardless of whether the A and B field are E or Y^0; there is always a q propagator cancelling, meaning we get a tadpole diagram with no finite contribution.

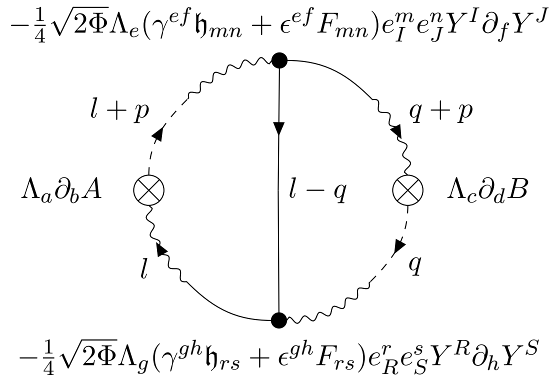

Figure 28: Diagram contributing to $\langle \Lambda_a \partial_b E \Lambda_c \partial_d E \rangle$ for $\{A, B\} = \{E, E\}$, to $\langle \Lambda_a \partial_b E \Lambda_c \partial_d Y^0 \rangle$ for $\{A, B\} = \{E, Y^0\}$ and to *lyly* for $\{A, B\} = \{Y^0, Y^0\}$. The only difference between an E-field and a Y^0-field is in the propagator, which has a $-\gamma_{ab}$ for E and a ϵ_{ab} for Y^0.

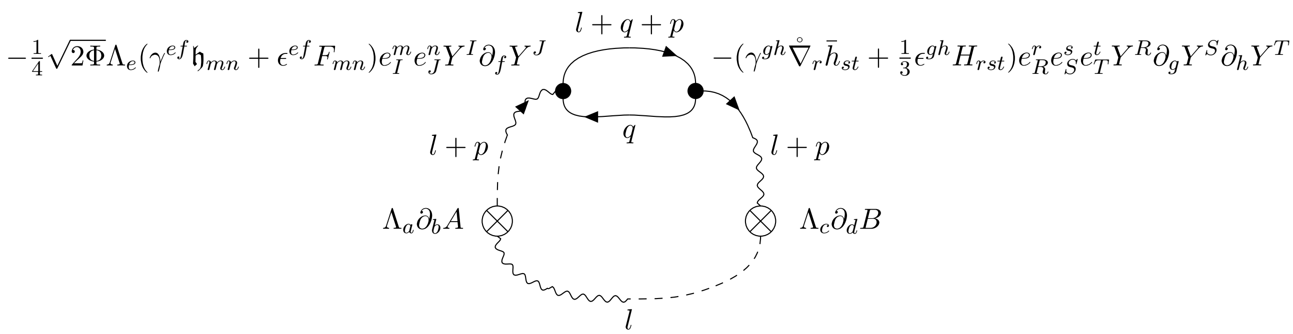

Figure 29

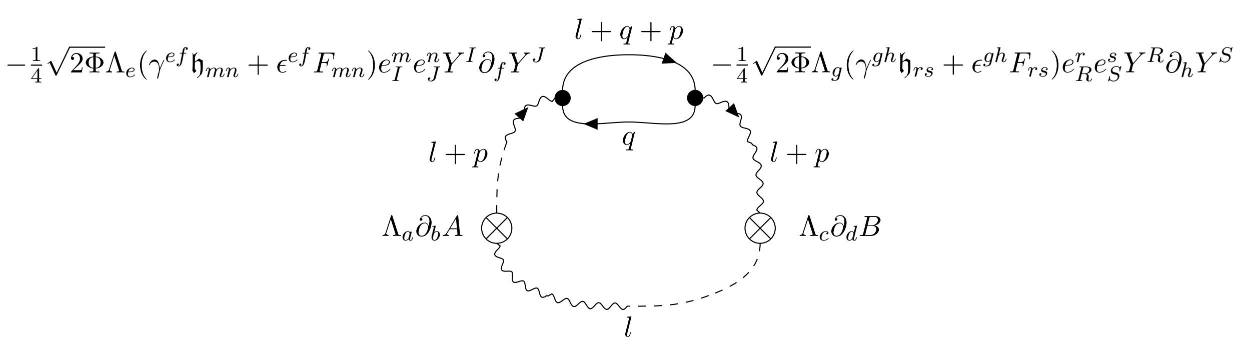

Figure 30

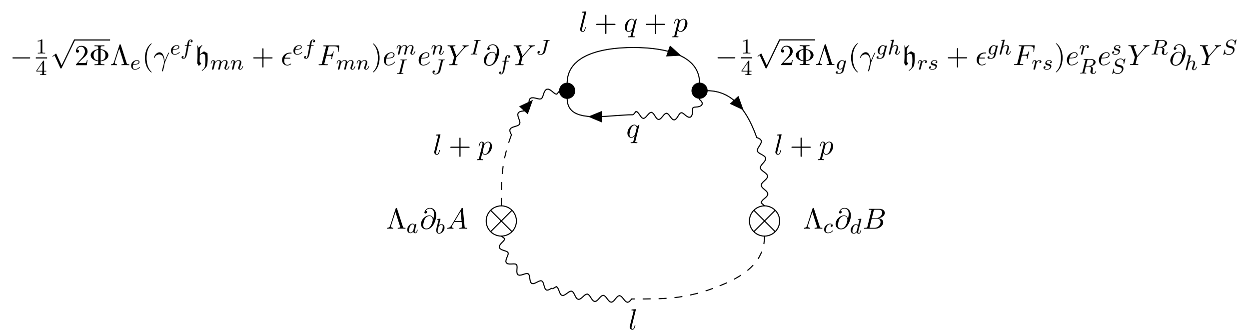

Figure 31: This diagram has a symmetry factor of 2.

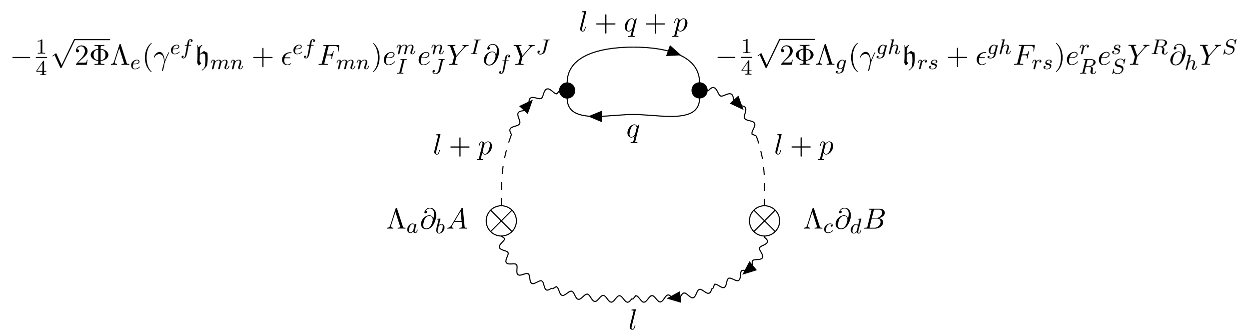

Figure 32

The diagram of Figure 30 can be expressed by

$$-64\pi^3 l_s^2 h^{xy}\Big(\mathfrak{e}_x\mathfrak{e}_y J_{30}^{\mathfrak{e}^2}(\omega_A,\omega_B)+a_x a_y J_{30}^{a^2}(\omega_A,\omega_B)+a_x\mathfrak{e}_y J_{30}^{a\cdot\mathfrak{e}}(\omega_A,\omega_B). \tag{231}$$

We can express the diagram of Figure 31 as

$$-64\pi^3 l_s^2 h^{xy}\Big(\mathfrak{e}_x\mathfrak{e}_y J_{31}^{\mathfrak{e}^2}(\omega_A,\omega_B)+a_x a_y J_{31}^{a^2}(\omega_A,\omega_B)+a_x\mathfrak{e}_y J_{31}^{a\cdot\mathfrak{e}}(\omega_A,\omega_B)\Big). \tag{232}$$

Figure 32 has the following integral expression

$$64\pi^3 l_s^2 h^{uv}\Big(\mathfrak{e}_u\mathfrak{e}_v J_{32}^{\mathfrak{e}^2}(\omega_A,\omega_B)+a_u a_v J_{32}^{a^2}(\omega_A,\omega_B)+\mathfrak{e}_u a_v J_{32}^{a\cdot\mathfrak{e}}(\omega_A,\omega_B)\Big). \tag{233}$$

These integrals simplify considerably when taking ω to be γ or ϵ. For this reason they are reported separately in Appendix C.

In total we have

$$\begin{aligned}\langle \Lambda_a\partial_b E\Lambda_c\partial_d E\rangle = 32\pi^3 l_s^2\Big\{ & h^{xy}\mathfrak{e}_x\mathfrak{e}_y\Big(-J_{28}^{\mathfrak{e}^2}(-\gamma,-\gamma)-2J_{31}^{\mathfrak{e}^2}(-\gamma,-\gamma)\Big)\\ &+a_x a_y h^{xy}\Big(-J_{28}^{a^2}(-\gamma,-\gamma)-2J_{31}^{a^2}(-\gamma,-\gamma)+2J_{32}^{a^2}(-\gamma,-\gamma)\Big)\\ &+a_x\mathfrak{e}_y h^{xy}\Big(-J_{28}^{a\cdot\mathfrak{e}}(-\gamma,-\gamma)-2J_{30}^{a\cdot\mathfrak{e}}(-\gamma,-\gamma)-2J_{31}^{a\cdot\mathfrak{e}}(-\gamma,-\gamma)\Big)\Big\},\end{aligned} \tag{234}$$

$$\big\langle \Lambda_a \partial_b E \Lambda_c \partial_d Y^0 \big\rangle = 32\pi^3 l_s^2 \Big\{ h^{xy} \mathfrak{e}_x \mathfrak{e}_y \Big(- J_{28}^{\mathfrak{e}^2}(-\gamma,\epsilon) - 2J_{30}^{\mathfrak{e}^2}(-\gamma,\epsilon) - 2J_{31}^{\mathfrak{e}^2}(-\gamma,\epsilon)\Big) \\ + a_x a_y h^{xy} \Big(- J_{28}^{a^2}(-\gamma,\epsilon) - 2J_{30}^{a^2}(-\gamma,\epsilon) - 2J_{31}^{a^2}(-\gamma,\epsilon) + 2J_{32}^{a^2}(-\gamma,\epsilon)\Big) \\ + a_x \mathfrak{e}_y h^{xy} \Big(- J_{28}^{a\cdot\mathfrak{e}}(-\gamma,\epsilon) - 2J_{30}^{a\cdot\mathfrak{e}}(-\gamma,\epsilon) - 2J_{31}^{a\cdot\mathfrak{e}}(-\gamma,\epsilon) + 2J_{32}^{a\cdot\mathfrak{e}}(-\gamma,\epsilon)\Big) \Big\} \tag{235}$$

and

$$\big\langle \Lambda_a \partial_b Y^0 \Lambda_c \partial_d Y^0 \big\rangle = 32\pi^3 l_s^2 \Big\{ h^{xy} \mathfrak{e}_x \mathfrak{e}_y \Big(- J_{28}^{\mathfrak{e}^2}(\epsilon,\epsilon) - 2J_{30}^{\mathfrak{e}^2}(\epsilon,\epsilon) - 2J_{31}^{\mathfrak{e}^2}(\epsilon,\epsilon) + 2J_{32}^{\mathfrak{e}^2}(\epsilon,\epsilon)\Big) \\ + a_x a_y h^{xy} \Big(- J_{28}^{a^2}(\epsilon,\epsilon) - 2J_{30}^{a^2}(\epsilon,\epsilon) - 2J_{31}^{a^2}(\epsilon,\epsilon) + 2J_{32}^{a^2}(\epsilon,\epsilon)\Big) \\ + a_x \mathfrak{e}_y h^{xy} \Big(- J_{28}^{a\cdot\mathfrak{e}}(\epsilon,\epsilon) - 2J_{30}^{a\cdot\mathfrak{e}}(\epsilon,\epsilon) - 2J_{31}^{a\cdot\mathfrak{e}}(\epsilon,\epsilon)\Big) \Big\}. \tag{236}$$

10 Results

We can now take Equation (222), (228), (229), (234), (235) and (236) together to find the right-hand side of Equation (203). To this end we have to contract all the tensorial integrals with their corresponding tensor S, U, V, W, X or Z. We will denote an integral J contracted with a tensor T as J_T.

$$\frac{\delta}{\delta f} \left\langle \frac{\delta S}{\delta f} \right\rangle_{e^f \delta_{ab}} \bigg|_{f=0} = \pi^3 l_s^2 \Big\{ - 16 E_1 J_{14\,S}^{E_1} - 16 E_2 J_{14\,S}^{E_2} + \tilde{h}^{uv} \tilde{h}^{wx} \tilde{h}^{yz} \mathring{\nabla}_u \bar{h}_{zw} \mathring{\nabla}_v \bar{h}_{yx} \Big(2J_{15\,S}^{\nabla h} + 6J_{19\,S}\Big) \\ + \tilde{h}^{uv} \tilde{h}^{wx} \tilde{h}^{yz} \mathring{\nabla}_z \bar{h}_{vw} \mathring{\nabla}_u \bar{h}_{yx} \Big(2J_{15\,S}^{\nabla h *} - 4J_{19\,S}\Big) + \tilde{h}^{uv} \tilde{h}^{wx} \tilde{h}^{yz} H_{uwy} H_{vxz} \Big(2J_{15\,S}^{H^2} + 2f(n) J_{19\,S}\Big) \\ + \frac{1}{\Phi} (\mathring{\nabla}_s \bar{h}_{tu} - \mathring{\nabla}_u \bar{h}_{st}) h^{ru} \hat{\upsilon}^s \hat{\upsilon}^t a_r \Big(8J_{16\,S}^{\nabla h\cdot a} + 8J_{20\,S}^{a\cdot\mathring{\nabla} h} + 4J_{24\,U}^{\nabla h\cdot a}(-\gamma) + 4J_{24\,U}^{\nabla h\cdot a}(\epsilon)\Big) \\ + \frac{1}{\Phi} (\mathring{\nabla}_s \bar{h}_{tu} - \mathring{\nabla}_u \bar{h}_{st}) h^{ru} \hat{\upsilon}^s \hat{\upsilon}^t \mathfrak{e}_r \Big(- 8J_{16\,S}^{\nabla h\cdot\mathfrak{e}} - 16J_{20\,S}^{\mathfrak{e}\cdot\mathring{\nabla} h} + 8J_{24\,U}^{\nabla h\cdot\mathfrak{e}}(-\gamma) + 8_{24\,V}^{\nabla h\cdot\mathfrak{e}}(\epsilon)\Big) \\ + h^{uv} a_u a_v \Big(4J_{17\,S}^{a^2} + J_{18\,S}^{a^2} - 8J_{21\,S}^{a^2} + J_{22\,S}^{a^2} - 8J_{25\,U}^{a^2}(-\gamma) + 8J_{26\,U}^{a^2}(-\gamma) + 8J_{27\,U}^{a^2}(-\gamma) - 8J_{25\,V}^{a^2}(\epsilon) + 8J_{26\,V}^{a^2}(\epsilon) \\ + 8J_{27\,V}^{a^2}(\epsilon) - 32J_{28\,W}^{a^2}(-\gamma,-\gamma) - 64J_{31\,W}^{a^2}(-\gamma,-\gamma) + 64J_{32\,W}^{a^2}(-\gamma,-\gamma) - 32J_{28\,Z}^{a^2}(-\gamma,\epsilon) - 64J_{30\,Z}^{a^2}(-\gamma,\epsilon) \\ - 64J_{31\,Z}^{a^2}(-\gamma,\epsilon) + 64J_{32\,Z}^{a^2}(-\gamma,\epsilon) - 32J_{28\,X}^{a^2}(\epsilon,\epsilon) - 64J_{30\,X}^{a^2}(\epsilon,\epsilon) - 64J_{31\,X}^{a^2}(\epsilon,\epsilon) + 64J_{32\,X}^{a^2}(\epsilon,\epsilon)\Big) \\ + h^{uv} \mathfrak{e}_u \mathfrak{e}_v \Big(4J_{17\,S}^{\mathfrak{e}^2} + 4J_{18\,S}^{\mathfrak{e}^2} - 8J_{21\,S}^{a^2} - J_{22\,S}^{\mathfrak{e}^2} - 16J_{25\,U}^{\mathfrak{e}^2}(-\gamma) + 8J_{26\,U}^{e^2}(-\gamma) + 8J_{27\,U}^{\mathfrak{e}^2}(-\gamma) - 16J_{25\,V}^{\mathfrak{e}^2}(\epsilon) \\ + 8J_{26\,V}^{e^2}(\epsilon) + 8J_{27\,V}^{\mathfrak{e}^2}(\epsilon) - 32J_{28\,W}^{\mathfrak{e}^2}(-\gamma,-\gamma) - 64J_{31\,W}^{\mathfrak{e}^2}(-\gamma,-\gamma) - 32J_{28\,Z}^{\mathfrak{e}^2}(-\gamma,\epsilon) - 64J_{30\,Z}^{\mathfrak{e}^2}(-\gamma,\epsilon) \\ - 64J_{31\,Z}^{\mathfrak{e}^2}(-\gamma,\epsilon) - 32J_{28\,X}^{\mathfrak{e}^2}(\epsilon,\epsilon) - 64J_{30\,X}^{\mathfrak{e}^2}(\epsilon,\epsilon) - 64J_{31\,X}^{\mathfrak{e}^2}(\epsilon,\epsilon) + 64J_{32\,X}^{\mathfrak{e}^2}(\epsilon,\epsilon)\Big)$$

$$
\begin{aligned}
&+h^{uv}\mathfrak{e}_u a_v\Big(8J^{a\cdot e}_{17\,S}-4J^{a\cdot\mathfrak{e}}_{18\,S}+32J^{a\cdot\mathfrak{e}}_{21\,S}+16J^{a\cdot\mathfrak{e}}_{25\,U}(-\gamma)+8J^{a\cdot\mathfrak{e}}_{26\,U}(-\gamma)++J^{\mathfrak{e}\cdot a}_{27\,U}(-\gamma)+16J^{a\cdot\mathfrak{e}}_{25\,V}(\epsilon)+8J^{a\cdot\mathfrak{e}}_{26\,V}(\epsilon)\\
&+J^{\mathfrak{e}\cdot a}_{27\,V}(\epsilon)-32J^{a\cdot\mathfrak{e}}_{28\,W}(-\gamma,-\gamma)-64J^{a\cdot\mathfrak{e}}_{30\,W}(-\gamma,-\gamma)-64J^{a\cdot\mathfrak{e}}_{31\,W}(-\gamma,-\gamma)-32J^{a\cdot\mathfrak{e}}_{28\,Z}(-\gamma,\epsilon)-64J^{a\cdot\mathfrak{e}}_{30\,Z}(-\gamma,\epsilon)\\
&\qquad-64J^{a\cdot\mathfrak{e}}_{31\,Z}(-\gamma,\epsilon)+64J^{a\cdot\mathfrak{e}}_{32\,Z}(-\gamma,\epsilon)-32J^{a\cdot\mathfrak{e}}_{28\,X}(\epsilon,\epsilon)-64J^{a\cdot\mathfrak{e}}_{30\,X}(\epsilon,\epsilon)-64J^{a\cdot\mathfrak{e}}_{31\,X}(\epsilon,\epsilon)\Big)\\
&\quad+\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r\mathfrak{h}_{sm}\Big(J^{\nabla\mathfrak{h}}_{23\,U}(-\gamma)+J^{\nabla\mathfrak{h}}_{23\,V}(\epsilon)\Big)+\frac{32}{3}\hat{\upsilon}^m\tilde{h}^{rs}\mathring{\nabla}_r F_{sm}\Big(J^{\nabla F}_{23\,U}(-\gamma)+J^{\nabla F}_{23\,V}(\epsilon)\Big)\Big\}. \qquad (237)
\end{aligned}
$$

Now we certainly do not want to write this down a second time, so let us write

$$
\frac{\delta}{\delta f}\left\langle\frac{\delta S}{\delta f}\right\rangle_{e^f\delta_{ab}}\Bigg|_{f=0}=\mathcal{M} \qquad (238)
$$

for now. We may not be able to solve all of the integrals in $\mathcal{M}$, but we know that they are worldsheet scalars and have momentum dimension 2. The only option is that they are proportional to p^2. In section 3 we saw that in position space p^2 becomes $-\Box\delta(\xi)$. We then have an equation of the form

$$
\frac{\delta}{\delta f}\left\langle\frac{\delta S}{\delta f}\right\rangle=-\mathcal{M}\Box\delta(\xi). \qquad (239)
$$

We did not write this down explicitly, but all of the TNC spacetime scalars appearing in $\mathcal{M}$ are in Fourier space. We never bothered to compute these TNC fields in Fourier space as we will just just transform them back into momentum space here. By performing the same steps as we did in Equation (54) and (60) to solve this differential equation we get

$$
\left\langle\frac{\delta S}{\delta f}\right\rangle=-\mathcal{M}\Box f. \qquad (240)
$$

We then use equation (15) to write the D'Alembertian of the scale factor as the worldsheet Ricci scalar,

$$
\left\langle\frac{\delta S}{\delta f}\right\rangle=\frac{1}{2}\mathcal{M}\sqrt{\gamma}R_\gamma. \qquad (241)
$$

Looking back at Equation (174) we find that the two-loop contribution of the action (117) to the dilaton β-function is given by

$$
\beta^\phi=\frac{1}{2}\mathcal{M}, \qquad (242)
$$

now to get the full form of β^ϕ one needs to consider term coming from the dilaton action itself, as well as possible terms involving higher order energy-momentum insertions. In section 11 we discuss how these two kinds of contributions can be found.

Putting β^ϕ to zero along with the other β-functions in Equation (174), ensures that the TNC string is Weyl invariant at the quantum level. Putting each β function to zero gives six different equations of motion. The first five of these were found in [14]

$$
\boxed{
\begin{aligned}
h^{rs}\nabla_r a_s+h^{rs}a_r a_s&=2h^{rs}\mathfrak{e}_r\mathfrak{e}_s+2h^{rs}a_r\nabla_s\phi,\\
R_{(mn)}-\frac{1}{4}H_{rs(m}H_{n)tw}h^{rt}h^{sw}+2\nabla_{(m}\nabla_{n)}\phi&=h^{tq}\bar{h}_{q(m}\nabla_{n)}a_t+\frac{1}{2}a_m a_n+\mathfrak{e}_r h^{rs}\hat{\upsilon}^t\tau_{(m}H_{n)ts}\\
&\quad+\frac{1}{2}\big(\mathfrak{e}^2(2\Phi\tau_m\tau_n-\bar{h}_{mn})-\mathfrak{e}_m\mathfrak{e}_n\big)-a^2\Phi\tau_m\tau_n,\\
\frac{1}{2}h^{rs}\nabla_r H_{smn}-h^{rp}H_{pmn}\nabla_r\phi&=h^{tq}\bar{h}_{q[m}\nabla_{n]}\mathfrak{e}_t+\hat{\upsilon}^r\tau_{[m}\nabla_{n]}\mathfrak{e}_r-a_{[m}\mathfrak{e}_{n]}-\frac{1}{2}a_r h^{rs}H_{smn}\\
&\quad+\big(\hat{\upsilon}^t\nabla_t\phi-\frac{1}{2}\nabla_t\hat{\upsilon}^t\big)\mathfrak{h}_{mn}.
\end{aligned}
}
$$

Here we have reintroduced the covariant derivative ∇ and Ricci tensor of the standard TNC connection (92). The condition $\mathcal{M} = 0$ should not be independent of the above equations. This can be seen from the fact that we can only β^{ϕ} for a curved worldsheet. The physics, however, should be independent of the worldsheet. After calculating all the integrals comprising $\mathcal{M}$ we thus expect that the boxed equations already give the information coming from the condition $\mathcal{M} = 0$. Calculating the precise form of the equation of motion ensuing from setting β^{ϕ} to zero, is thus a good way to check the validity of the results found in [14].

11 Discussion and future work

We have found all of the diagrams that contribute to the dilaton β-function and we have expressed them in two-loop integrals. About one-third of these integrals can be calculated using the standard integrals of Appendix B.1. Now the other integrals have terms where only one propagator power gets canceled. To compute these we need to extend the list of Equation 269 through 273 with the standard integrals $I_2[l_a l_b l_c l_d l_e, \alpha\beta]$ and $I_2[l_a l_b l_c l_d l_e l_f, \alpha, \beta]$, where this notation means

$$I_2[l_a l_b l_c l_d l_e l_f, \alpha, \beta] = \int d^n l \frac{l_a l_b l_c l_d l_e l_f}{l^{2\alpha}(p-l)^{2\beta}}. \tag{243}$$

We discuss the feasibility of finding these standard integrals at the start of Appendix C, where we conclude that in principle it should not be too difficult. We make the estimate that one should diagonalize a 72 dimensional matrix to find $I_2[l_a l_b l_c l_d l_e l_f, \alpha, \beta]$. On a first look there do appear to be some integrals in Appendix C where not even one denominator power can be canceled, an example is integral $J_{25}^{a\text{-}e}$ in Equation (315). One would then have to construct a (0,8) tensorial standard integral. There are however not that many terms where not even one denominator gets canceled. It could be that on a more careful examination of these few terms they turn out to vanish by, for example, symmetry arguments.

A different approach to solving the integrals of Appendix C is to construct standard three-point integrals

$$I_3[T(l), \alpha, \beta, \gamma] = \int d^n l \frac{T(l)}{l^{2\alpha}(p-l)^{2\beta}(r-l)^{2\gamma}}, \tag{244}$$

where $T(l)$ is some tensorial structure of the loop momentum l, and p and r are external momenta. This has the advantage that rewriting the numerator to cancel denominator terms is not needed. Any two-loop integral that we encounter in Appendix C can be expressed in these standard 3-point integrals right away. However, we have not been able to solve even the most basic integral of this form, where we have $T(l) = 1$ and $r = \frac{1}{2}p$. Links to the literature on massless 2-dimensional three-point functions and attempts to calculate this scalar integral are discussed in Appendix B.2. Even if one would find the scalar integral, then finding tensorial integrals from that is more complicated then for the two-point function. This is because one now has two external momenta, so more ways to distribute the Lorentz indices of a tensorial integral over combinations of the metric γ_{ab} and the external momenta p_a and r_b. Even if one were to compute all this it is still preferred to cancel numerators terms against denominators so as to avoid having to use the standard (0,8) tensorial integral $I_3[l_a l_b l_c l_d l_e l_f l_g l_h, \alpha, \beta, \gamma]$. For the above reasons, we do not recommend using three-point functions to calculate the two-loop integrals of Appendix C. One does need three-point integrals for calculating diagrams that have three external lines, such as the diagram of Figure 4. These diagrams also show up in some of the other TNC β-functions, but (as has been done in [14]) these β-functions can also be calculated by using operator product expansions instead of Feynman diagrams. One would then only need three-point Feynman diagrams when calculating higher-order corrections to these β-functions.

Once all of the integrals of Appendix C have been calculated on has to remove any divergences in them. On a first look only diagrams of the form of Figure 14 are convergent. The others can have $\mathcal{O}(\frac{1}{\epsilon^2})$ divergences. To remove these divergences one needs to construct counterterm diagrams. For the Riemannian bosonic string, there was only one counterterm diagram (Figure 11), but for the TNC case, we expect that there are many more.

There are two more kinds of terms that contribute to the dilaton β-function that we have not examined in this thesis. First of all, there are the first-order corrections coming from the dilaton action, such as the diagram in Figure 13 for the Riemannian bosonic string. These should be relatively straightforward to calculate as the dilaton action has the same form for the TNC string as for the Riemannian bosonic string.

The second class of diagrams that should be considered are terms that involve higher-order spacetime derivatives in the background field expansion of the Weyl anomaly, such as the diagram in Figure 12 for the Riemannian bosonic string. In our present approach we dropped any terms in the background field expansions of the energy-momentum tensor (183) and $C_\pm$ (191). For the Riemannian bosonic string, the only term in the background field expansion of T_{++} that is higher in spacetime derivatives is the one involving the dilaton appearing in the diagram of Figure 12. However, here we could also have terms showing up with spacetime derivatives on the vielbeins, so this could generate some extra diagrams.

When the entire form of the dilaton β-function is found it can be compared to the other β-functions which were found in [14]. As shown in Equation (174), the dilaton β-function is proportional to the worldsheet Ricci scalar R_γ, which vanishes for the flat worldsheet. For this reason, we expect that dilaton β-function is not independent of the other β-functions, as the spacetime equations of motion should be independent of the worldsheet. A different approach to get the $\mathcal{O}(\alpha')$ contributions to the dilaton β-function was undertaken at the same time as this thesis. There the TNC bosonic string was described in the framework of double field theory and an expression for the dilaton β-function was found. Once all integrals have been found it would be a good check on both methods to compare the results.

A slightly different version of Newton-Cartan strings is also studied, known as string Newton-Cartan theory. In [24] the β-functions for the Weyl anomaly of this theory are found. It would be interesting to compare the ensuing equations of motion with the ones that can be found from computing the full form of β^ϕ (242) combined with the equations of motion found in [14].

Once the TNC spacetime equations of motion have been found and checked by the various methods described above, one can start working with these equations, for example, in the context of holography.

A Conventions

For index conventions we use

- α, β (first letters of greek alphabet) for curved worldsheet indices
- a, b (first letters of roman alphabet) for flat worldsheet indices
- μ, ν etc. for curved spacetime indices
- k, m, n etc. for spatial spacetime indices
- I, J for flat spacetime indices
- g, h, i, j for flat spatial spacetime indices

The Riemann tensor is defined in the usual way, one can contract two of its indices to find the Ricci tensor,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{245}$$

When using zweibeins, the following convention is used

$$e_+^{(\alpha} e_-^{\beta)} = -\gamma^{\alpha\beta}, e_+^{[\alpha} e_-^{\beta]} = -\epsilon^{\alpha\beta}. \tag{246}$$

We use the following convention for the Levi-Civita symbol: $\epsilon^{01} = 1$, from which it follows that in lightcone coordinates: $\epsilon^{-+} = 2$ and $\epsilon^{+-} = -2$.
We use the metric signature: -+++

The worldsheet has time coordinate τ and spatial coordinate σ. Lightcone coordinates are defined by

$$\sigma^{\pm} = \tau \pm \sigma. \tag{247}$$

Conformal coordinates on the worldsheet are defined by

$$z = -\tau + \sigma \quad , \quad \bar{z} = \tau + \sigma. \tag{248}$$

In Feynman diagrams a derivative of an outgoing momentum gives a minus sign.

B Dimensional regularization integrals

B.1 One-loop, two-point integrals

A one-loop diagram means we have one internal loop where we integrate over all values of its momenta. Two-point diagram means we are looking at a diagram with two external lines and thus, via momentum conservation, only one external momentum. Consider the following one-loop

Feynman integral with two external lines and some arbitrary numerator $f(l)$ dependent on the loop momentum l, we denote this integral by

$$I_2[f(l), \alpha, \beta] \equiv \int d^2 l \frac{f[l]}{l^{2\alpha}(p-l)^{2\beta}}. \tag{249}$$

Let us start with the scalar integral

$$I_2[1, \alpha, \beta] = \int d^n l \frac{1}{l^{2\alpha}(p-l)^{2\beta}}. \tag{250}$$

By dividing all terms in the numerator and the integral measure by the external momentum p and scaling the loop momentum by $l/p \to l$ the integral can be made dimensionless, which can then be solved in terms of hypergeometric functions[25],

$$I_2[1, \alpha, \beta] = (p^2)^{n/2-\alpha-\beta}\Pi(n)G(n, \alpha, \beta), \tag{251}$$

where

$$G(n, \alpha, \beta) = \frac{\Gamma(\frac{n}{2}-\alpha)\Gamma(\frac{n}{2}-\beta)\Gamma(\alpha+\beta-\frac{n}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)} \qquad \text{and} \qquad \Pi(n) = \frac{(2\pi)^2}{(4\pi)^{n/2}}. \tag{252}$$

In the following we will simplify notation by writing $I_2[f(l), 1, 1] = I_2[f(l)]$ as we will not need general denominator powers until we get to two-loop calculations.
The tensor integrals can be written in terms of this scalar integral using Veltman-Passarino reduction [26]. The idea is to note that by Lorentz invariance, a rank $(0, k)$ tensorial integral $I_2[l_{a_1}...l_{a_k}]$ should be proportional to all possible distinct combinations of the two Lorentz tensors we have at our disposal, namely the external momentum p_a and the worldsheet metric γ_{ab}. Each combination gets an unknown variable, we can then contract both sides with the same tensorial combinations of the external momentum p^a and the inverse metric γ^{ab}. This gives m equations for the m unknown variables, which we can then solve for the unknown variables. By contracting $I_2[l_{a_1}...l_{a_k}]$ with combinations of p^a and γ^{ab} we encounter integrals like $I_2[l \cdot p]$ and $I_2[l^2]$. Integrals of the first type can be rewritten using $(l \cdot p) = \frac{1}{2}(-(p-l)^2 + l^2 + p^2)$,

$$I_2[l \cdot p] = \frac{1}{2}\Big(-\int d^n l \frac{1}{l^2} + \int d^n l \frac{1}{(p-l)^2} + p^2 I_2[1]\Big). \tag{253}$$

The first two integrals are the same upon shifting the loop momentum, such that they cancel each other. The last term is the scalar integral of Equation (251). Integrals with just one denominator like $I_2[l^2]$ can always be shifted in the loop momentum to obtain something proportional to

$$I_2[l^2] = \int d^n l \frac{1}{l^2}, \tag{254}$$

which obeys the following identity in dimensional regularization[25, 27]

$$I_2[l^2] = \int \frac{1}{l^2} = 2i\frac{\pi^{n/2+1}}{\Gamma(n/2)}\delta(\epsilon/2) \equiv S(\epsilon/2). \tag{255}$$

The δ divergence here seems to a problem, but as we will see below, we will only encounter this in the combination $\epsilon\delta(\epsilon)$ which is zero as $x\delta(x) = 0$ for all x. To simplify notation we write $S(\epsilon/2) = S$ below, in the two-loop case we will encounter an $S(\epsilon)$ term.

To illustrate the above described method of Veltman-Passarino reduction consider the integral $I_2[l_a l_b]$. By Lorentz invariance it has to be proportional to γ_{ab} and $p_a p_b$,

$$I_2[l_a l_b] = \int d^2 l \frac{l_a l_b}{l^2 (p-l)^2} = a\gamma_{ab} + b p_a p_b. \tag{256}$$

We know contract once with γ^{ab} and once with $p^a p^b$. This gives the following two relations,

$$an + bp^2 = I_2[l^2] = S, \tag{257}$$
$$ap^2 + bp^4 = I_2[(l\cdot p)^2]. \tag{258}$$

The integral involving $(l\cdot p)^2$ can be found by writing expressing $l\cdot p$ as denominator powers as in Equation (253),

$$\begin{aligned} I_2[(l\cdot p)^2] &= \frac{1}{2}\Big(- I_2[(l\cdot p)(p-l)^2] + I_2[(l\cdot p)l^2] + p^2 I_2[l\cdot p]\Big) \\ &= \frac{1}{2}p^2 S + \frac{1}{4}p^4 I_2[1]. \end{aligned} \tag{259}$$

To get to the second line Equations (253) and (255) were used. We can now determine a and b in terms of the known integrals S and $I_2[1]$. This method has been repeated up to fourth order in numerator powers, the results are

$$I_2[l_a] = \frac{1}{2}p_a I_2[1], \tag{260}$$
$$I_2[l\cdot p] = \frac{1}{2}p^2 I_2[1], \tag{261}$$
$$I_2[(l\cdot p)^2] = \frac{1}{2}p^2 S + \frac{1}{4}p^4 I_2[1], \tag{262}$$
$$I_2[l_a l_b] = \frac{nI_2[1]p^2 + 2S(n-2)}{4(n-1)p^2} p_a p_b + \frac{2S - I_2[1]p^2}{4(n-1)}\gamma_{ab}, \tag{263}$$
$$I_2[(p\cdot l)^3] = \frac{3}{4}p^4 S + \frac{1}{8}p^6 I_2[1], \tag{264}$$
$$I_2[l_a l_b l_c] = \Big(3\frac{n-2}{4p^2(n-1)}S + \frac{n+2}{8(n-1)}I_2[1]\Big)p_a p_b p_c + \frac{2S - p^2 I_2[1]}{8(n-1)} p_{\{a}\gamma_{bc\}}, \tag{265}$$
$$I_2[(l\cdot p)^4] = \frac{1}{16}p^8 I_2[1] + \frac{7}{8}p^6 S, \tag{266}$$
$$\begin{aligned} I_2[l_a l_b l_c l_d] = \frac{1}{16(n^2-1)}\bigg(&\frac{(2+n)(4+n)p^2 I_2[1] + 2(n-2)(8+7n)S}{p^2} p_a p_b p_c p_d \\ &+ p^2(p^2 I_2[1] - 2S)\gamma_{\{ab}\gamma_{cd\}} - (n+2)(p^2 I_2[1] - 2S)p_{\{a}p_b\gamma_{cd\}}\bigg). \end{aligned} \tag{267}$$

Here $A_{\{a_1 \dots a_m\}}$ denotes the sum of all symmetrically different combinations of the indices $a_1 \dots a_n$.

The above are only valid for $\alpha = 1$ and $\beta = 1$. For general α and β we have the following results

$$I_2[l\cdot p,\alpha,\beta] = \frac{1}{2}p^2 I_2[1,\alpha,\beta] + \frac{1}{2}I_2[1,\alpha-1,\beta] - \frac{1}{2}I_2[1,\alpha,\beta-1], \tag{268}$$

$$I_2[l_a] = \frac{1}{2}p_a\Big(I_2[1,\alpha,\beta] + \frac{1}{p^2}I_2[1,\alpha-1,\beta] - \frac{1}{p^2}I_2[1,\alpha,\beta-1], \tag{269}$$

$$I_2[(l\cdot p)^m,\alpha,\beta] = \frac{1}{2}\Big(-I_2[(l\cdot p)^{m-1},\alpha,\beta-1] + I_2[(l\cdot p)^{m-1},\alpha-1,\beta] + p^2 I_2[(l\cdot p)^{m-1},\alpha,\beta]\Big), \tag{270}$$

$$I_2[l_a l_b,\alpha,\beta] = \frac{1}{(n-1)p^2}\Big\{\frac{nI_2[(l\cdot p)^2,\alpha,\beta] - p^2 I_2[1,\alpha-1,\beta]}{p^2}p_a p_b + \big(p^2 I_2[1,\alpha-1,\beta] - I_2[(l\cdot p)^2,\alpha,\beta]\big)\gamma_{ab}\Big\}, \tag{271}$$

$$\begin{aligned} I_2[l_a l_b l_c,\alpha,\beta] = \frac{1}{(n-1)p^4}\Big\{&\big((n+2)I_2[(l\cdot p)^3,\alpha,\beta] - 3I_2[l\cdot p,\alpha-1,\beta]p^2\big)\frac{p_a p_b p_c}{p^2} \\ &+\big(I_2[l\cdot p,\alpha-1,\beta]p^2 - I_2[(l\cdot p)^3,\alpha,\beta]\big)p_{\{a}\gamma_{bc\}}\Big\}, \end{aligned} \tag{272}$$

$$\begin{aligned} I_2[l_a l_b l_c l_d,\alpha,\beta] = &\frac{1}{(n^2-1)p^8}\Big\{p^2\Big((n+3)I_2[(l\cdot p)^2,\alpha-1,\beta]p^2 - (n+2)I_2[(l\cdot p)^4,\alpha,\beta] \\ &-I_2[1,\alpha-2,\beta]p^4\Big)p_{\{a}p_b\gamma_{cd\}} + p^4\Big(I_2[1,\alpha-2,\beta]p^4 - 2I_2[(l\cdot p)^2,\alpha-1,\beta]p^2 + I_2[(l\cdot p)^4,\alpha,\beta]\Big)\gamma_{\{ab}\gamma_{cd\}} \\ &+\Big(-6(n+2)I_2[(l\cdot p)^2,\alpha-1,\beta]p^2 + (n+2)(n+4)I_2[(l\cdot p)^4,\alpha,\beta] + 3I_2[1,\alpha-2,\beta]p^4\Big)p_a p_b p_c p_d\Big\}. \end{aligned} \tag{273}$$

When considering two-loop integrals we will often encounter the following integral

$$\int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{(l+p)^2(l-q)^2 q^2}. \tag{274}$$

We can perform the q integral first: we then consider l as the external momentum and use Equation (260) and (263) to write this integral as

$$I_2[1]p^{2(1+\epsilon/2)}\int d^n l \frac{l_a(l-p)_b}{(p-l)^2 l^{2(1+\epsilon/2)}}\Big(\frac{n}{4(n-1)}l_c l_d - \frac{l^2}{4(n-1)}\gamma_{cd} + \frac{1}{2}p_c l_d\Big). \tag{275}$$

These are all just one-loop integrals with different values for the α power of the l^2 denominator of Equation (250). We thus find

$$\begin{aligned} \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{(l+p)^2(l-q)^2 q^2} = &I_2[1]p^{2(1+\epsilon/2)}\Big(\frac{n}{4(n-1)}I_2[l_a(l-p)_b l_c l_d, 1+\frac{\epsilon}{2},1] \\ &-\frac{1}{2}p_c I_2[l_a(l-p)_b l_d, 1+\frac{\epsilon}{2},1] - \frac{1}{4(n-1)}I_2[l_a(l-p)_b,\frac{\epsilon}{2},1]\gamma_{cd}\Big). \end{aligned} \tag{276}$$

B.2 three-point integral

In Figure 4 we have a one-loop integral with three external lines. In general such a diagram would have as scalar integral

$$I_3[1] \equiv \int d^n l \frac{1}{l^2(l+p_1)^2(l+p_2)^2}. \tag{277}$$

However, this scalar integral is more complicated to solve than the 2-point scalar integral (251). In [28] an analytic expression for this integral is given in terms of the Appell F4 function. Such a function is hard to implement in Mathematica because it is not a basic function. One then has to work with infinite sums of which it is a question if they even converge. For these reasons, we have chosen to not treat 3-point diagrams in this thesis. Luckily, for the dilaton β-function they are not needed.

B.3 Sunset integrals

The integrals of chapter 3 are solved using these integrals. Later on, for the treatment of TNC diagrams, we will take a more direct approach to solve two-loop integrals, as done for example in Equation (276). The only nontrivial two-loop diagram that we encounter in Section 3 is the so-called sunset diagram,

$$I_s[f(l,q)] = \int d^2l \int d^2q \frac{f(l,q)}{l^2 q^2 (p-l-q)^2}. \tag{278}$$

We now have two-loops, and thus two independent loop momenta, l and q. The external momentum is given by p. We again start with the scalar version of this, this can be solved by performing the integrals separately

$$I_s[1] = \int d^2l \frac{1}{l^2} \int d^2q \frac{1}{q^2(p-l-q)^2}. \tag{279}$$

Note that the integral over q is just the one-loop scalar integral, but with external momentum $p-l$. We can solve it using Equation (251) to find

$$I_s[1] = \Pi(n)G(n,1,1) \int d^2l \frac{1}{l^2(p-l)^{2(1+\epsilon/2)}} = \Pi(n)^2 G(n,1,1)G(n,1,1+\epsilon/2)\frac{1}{(p^2)^{2-n/2}}. \tag{280}$$

Here the second denominator has a power $2(1+\epsilon/2)$ instead of the usual 2. Luckily Equation (251) already covers general denominator powers, such that we can easily find the answer in terms of gamma functions. Integrals with tensorial loop momenta in the numerator can be found using the same method, they all reduce to the one-loop integrals of the previous section. The integrals that we will encounter are

$$I_s[l_a q_b] = \Pi G \left(\frac{1}{p^{2(1+\epsilon)}} \Pi G' \frac{n-2}{8(n-1)} - \frac{n-2}{4(n-1)p^2} S(\epsilon) \right) p_a p_b + \gamma \text{ terms}, \tag{281}$$

$$I_s[l_a l_b q_c] = \Pi G \left[\frac{n-2}{16(n-1)} \frac{\Pi G'}{p^{2(1+\epsilon)}} - \frac{n-2}{8(n-1)} \frac{S(\epsilon)}{p^2} \right] p_a p_b p_c + \gamma \text{ terms}, \tag{282}$$

$$\begin{aligned} I_s[l_a l_b q_c q_d] = \frac{n^2(n-2)}{32(n-1)(n^2-1)} &\left[\frac{n}{2} \frac{\Pi^2 G G'}{p^{2(1+\epsilon)}} + (n-2)\frac{S(\epsilon/2)\Pi G}{p^{2(1+\epsilon/2)}} + n \frac{\Pi G S(\epsilon)}{p^2} \right. \\ &\left. + 2(n-2)\frac{S(\epsilon/2)^2}{p^2} \right] p_a p_b p_c p_d + \gamma \text{ terms}. \end{aligned} \tag{283}$$

Where $G = G(n,1,1)$ and $G' = G(n,1,1+\epsilon/2)$. The metric dependent terms are omitted as we will only encounter integrals with a $++++$ signature.

$$I_s[l_a q_b] = p^{2(1+\epsilon/2)} I_2[1] I_2[1,1,1+\epsilon/2]\Big(\frac{n-2}{8(n-1)}p_a p_b - \frac{p^2}{4(n-1)}\gamma_{ab}\Big), \tag{284}$$

$$I_s[l_a l_b q_c] = I[1]I[1,1,1+\epsilon/2]p^{2(1+\epsilon/2)}\Big[\frac{n-2}{16(n-1)}p_a p_b p_c + \frac{p^2}{16(n-1)}\big(p_a\gamma_{bc} + p_b\gamma_{ac} - p_c\gamma_{ab}\big)\Big], \tag{285}$$

$$-I[1]p^{2(1+\epsilon/2)}\frac{S(\epsilon)}{8(n-1)}\Big[\frac{(n-2)}{p^2}p_a p_b p_c + p_a\gamma_{bc} + p_b\gamma_{ac} - p_c\gamma_{ab}\Big]$$

$$I_s[l_a l_b q_c q_d] = I_2[1,1,1+\epsilon/2]p^{2(1+\epsilon/2)}\Big[\frac{n}{4(n-1)}I_2[l_a l_b (p-l)_c (p-l)_d, 1, 1+\epsilon/2] - \frac{1}{4(n-1)}I_2[l_a l_b, 1, \epsilon/2]\gamma_{cd}\Big]$$
$$+\frac{2S}{4(n-1)}I_2[l_a l_b (p-l)_c (p-l)_d]. \tag{286}$$

Here $I_2[1] \equiv I_2[1,1,1]$ as defined in Equation (251). We could simplify $I_s[l_a l_b q_c q_d]$ further by substituting Equations from Appendix B.1, but it is easier to write it like this in Mathematica directly.

C TNC Feynman integrals

This lists all of the integrals needed to compute the TNC dilaton β-function. In this treatment, any integrals that are zero by footnote 7 are already dropped. Apart from the two integrals coming from Figure 14 and Figure 23 (which give two independent one-loop integrals straight away) all integrals have five denominators and eight momentum factors in the numerator. These momentum factors can be the loop momenta l and q, or the external momenta p. It is always a good start to write these momentum factors in terms of the denominators, such that some cancellations occur. Since there are always four momenta in the denominator carrying an external Lorentz index, we can cancel at most two denominators with this method. An integral where two denominators are canceled can almost always easily be solved, either because the denominator coupling the two-loop momenta gets canceled, or because we can write it in the form of Equation (276). When only one denominator gets canceled, one of the integrals can be solved, and the remaining integral can be written in terms of only two denominators to an arbitrary power. As we can handle any general denominator power α and β with the standard integrals of Equations (269), (271), (272) and (273) this can then in theory be solved. However, because we only canceled two of the eight numerator momentum factors the second integral can contain terms that have six factors of the same loop momentum. This is a problem because our standard integrals only go to four numerator loop momentum factors. We could extend these standard integrals, but the amount of terms needed quickly adds up. A rough estimate for the standard integral $I_2[l_a l_b l_c l_d l_e]$ is that there are 26 different combinations of combining p_a and γ_{ab}. For $I_2[l_a l_b l_c l_d l_e l_f]$ this adds up to 76 different combinations. Now, most of these factors have the same coefficient in front of them by symmetry arguments such that it should not be too much work to determine them. Diagonalizing a 72×72 matrix might be the hard part of this.

For the above reasons, we have tried to write part of the numerators of the integrals listed above in terms of denominator powers. For some integrals, this greatly simplifies the expression, but later on when we have more index mixing because of Λ-fields the expressions get very big. The integrals can then be reported more compactly if we do not write out any worldsheet contractions and only perform the spacetime contractions to find the spacetime scalars contributing to the Weyl anomaly.

By lack of flat worldsheet indices (the first eight letters of the Latin alphabet), we started using greek indices in some of the integrals below. This does not mean that these represent curved worldsheet coordinates, all of the worldsheet indices below represent flat worldsheet coordinates.

C.1 $\langle \eta_{IJ}\eta_{MN}\partial_a Y^I \partial_b Y^J \partial_c Y^M \partial_d Y^M \rangle$ integrals

$$J_{14}^{E_1} = \int d^n l \int d^n q \frac{l_a(l-p)_b(q-p)_c q_d}{l^2 q^2 (p-l)^2 (q-p)^2}\big(q^2 + l^2 - q\cdot p - l\cdot p\big), \tag{287}$$

$$J_{14}^{E_2} = \int d^n l \int d^n q \frac{l_a(l-p)_b(q-p)_c q_d}{l^2 q^2 (p-l)^2 (q-p)^2} p^2. \tag{288}$$

$$J_{15}^{\nabla h *} = \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2(q+p)^2 q^2}\Big[-2(l-q)^4 + 2l^2(1+p)^2\Big], \tag{289}$$

$$J_{15}^{\nabla h} = \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2(q+p)^2 q^2}\Big[-3(l-q)^4 + 2l^2(1+p)^2\Big], \tag{290}$$

$$J_{15}^{H^2} = f(n)\int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2(q+p)^2 q^2}\Big[(l-q)^2\big((l-q)^2 - p^2\big) - 2l^2(q+p)^2\Big]. \tag{291}$$

$$\tag{292}$$

$$J_{16}^{\nabla h\cdot a} = f(n)(n-1)\int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2 q^2 (q+p)^2 (l-q)^2} l^2 (q+p)^2, \tag{293}$$

$$J_{16}^{\nabla h\cdot \mathfrak{e}} = \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2 q^2 (q+p)^2 (l-q)^2} \epsilon^{ef}(p_e q_f + l_e p_f + l_e q_f)(l^2 - q^2). \tag{294}$$

$$J_{17}^{a^2} = f(n)\int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2 q^2 (q+p)^2}\Big\{(n-2)(l-q)^2\big((l-q)^2 + 2p^2\big) + l^2(q+p)^2 + q^2(l+p)^2\Big\}, \tag{295}$$

$$J_{17}^{\mathfrak{e}^2} = \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2 q^2 (q+p)^2}\Big\{-(l-q)^2\big((l-q)^2 + 2p^2\big) + l^2(q+p)^2 + q^2(l+p)^2\Big\}, \tag{296}$$

$$J_{17}^{a\cdot \mathfrak{e}} = \int d^n l \int d^n q \frac{l_a(l+p)_b(q+p)_c q_d}{l^2(l+p)^2(l-q)^2 q^2 (q+p)^2}\epsilon^{ef}\Big\{(l^2 - q^2)l_e q_f + \big((l+p)^2 + l^2 - (q+p)^2 - q^2\big)l_e q_f + (l^2 - q^2)p_e q_f\Big\}. \tag{297}$$

$$J_{18}^{\mathfrak{e}^2} = f(n)\int d^n l\int d^n q\frac{l_a(l+p)_b(q+p)_c q_d}{q^2l^2(q+p)^2(l+p)^2(l-q)^2}\Big\{((l-q)^4+2(l-q)^2p^2-l^2(q+p)^2-(l+p)^2q^2\Big\}, \tag{298}$$

$$\begin{aligned}J_{18}^{a^2} = f(n)^2\int d^n l\int d^n q\frac{l_a(l+p)_b(q+p)_c q_d}{q^2l^2(q+p)^2(l+p)^2(l-q)^2}\Big\{&4(l-q)^4-4(n-2)(l-q)^2p^2-(n-2)\Big[(l+p)^4\\ &-2(l+p)^2\big((q+p)^2+q^2\big)+(l^2-q^2)\big(l^2+p^2-2((q+p)^2+q^2)\big)\Big]\Big\},\end{aligned} \tag{299}$$

$$\begin{aligned}J_{18}^{a\cdot\mathfrak{e}} = f(n)\int d^n l\int d^n q\frac{l_a(l+p)_b(q+p)_c q_d}{q^2l^2(q+p)^2(l+p)^2(l-q)^2}\epsilon^{ef}\Big\{&p_eq_f\big(2(l-q)^2-(l+p)^2+l^2+p^2-2q^2\big)\\ &+(l_ep_f+2l_eq_f)\big(2(l-q)^2+p^2-(q+p)^2-q^2\big)\Big\}.\end{aligned} \tag{300}$$

$$J_{19} = \int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2q^2(l+q+p)^2}, \tag{301}$$

$$J_{20}^{a\cdot\mathring{\nabla}h} = f(n)(n-1)\int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2q^2(l+q+p)^2}, \tag{302}$$

$$J_{20}^{\mathfrak{e}\cdot\mathring{\nabla}h} = \int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2q^2(l+p)^2(l+q+p)^2}\epsilon^{fe}q_e(l_f+p_f). \tag{303}$$

$$J_{21}^{\mathfrak{e}^2} = \int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2q^2(l+q+p)^2}, \tag{304}$$

$$J_{21}^{a^2} = f(n)J_{21}^{\mathfrak{e}^2}, \tag{305}$$

$$J_{21}^{a\cdot\mathfrak{e}} = \int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2(l+p)^2q^2(l+q+p)^2}\epsilon^{ef}(l_e+p_e)q_f. \tag{306}$$

$$J_{22}^{\mathfrak{e}^2} = f(n)\int d^n l\int d^n q\frac{l_a(l+p)_b(l+p)_c l_d}{l^2q^2(l+q+p)^2}, \tag{307}$$

$$J_{22}^{a^2} = f(n)(5n-6)J_{22}^{\mathfrak{e}^2}. \tag{308}$$

C.2 $\big\langle\eta_{IJ}\partial_aY^I\partial_bY^J\Lambda_c\partial_dA\big\rangle$ integrals

$$J_{23}^{\nabla\mathfrak{h}}(\omega) = \omega^{fg}\epsilon_{ce}\int d^n l\int d^n q\frac{l_a(l+p)_bq_dq_g}{l^2(l+p)^2q^2(q+p)^2}\Big[2q_fp^e+3p_fp^e+3p_fq^e+2q_fq^e\Big], \tag{309}$$

$$J_{23}^{\nabla F}(\omega) = \omega^{fg}\epsilon_{ce}\epsilon_{hf}\int d^n l\int d^n q\frac{l_a(l+p)_bq_dq_g}{l^2(l+p)^2q^2(q+p)^2}\Big[3p^eq^h+3q^ep^h+2p^eq^h+2q^eq^h\Big]. \tag{310}$$

here ω^{fg} is $-\gamma^{fg}$ for the $\Lambda_c\partial_dE$ insertion and $\omega^{fg}=\epsilon^{fg}$ for the $\Lambda\partial_dY^0$ insertion.

$$
\begin{aligned}
J_{24}^{\nabla h\cdot a}(\omega) = f(n)\int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2(q+p)^2q^2}\Big[2(l-q)^4+(l-q)^2p^2+(l+p)^2\big(2(l+p)^2 \\
-p^2+(q+p)^2+q^2\big)q^e\omega_{ce}+2\big((l+p)^2-(l-q)^2\big)q_cq_eq_g\omega^{ge}-2\big((l-q)^2-(l+p)^2\big)\big(p_eq_cq_f\omega^{ef} \\
-2l_cq_g\big(p_f\omega^{fg}+q_f\omega^{gf}\big)\Big],
\end{aligned} \tag{311}
$$

$$
\begin{aligned}
J_{24}^{\nabla h\cdot \mathfrak{e}}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2(q+p)^2q^2}\Big[\big((l-q)^2-(l+p)^2\big)q^g\Big(2l^e\big(q^f\epsilon_{cg}\omega_{ef}+p^f\epsilon_{cf}\omega_{eg}\big) \\
-q^f\big(q^e\epsilon_{cg}\omega_{fe}+p^e\epsilon_{ce}\omega_{gf}\big)\Big)\Big].
\end{aligned} \tag{312}
$$

$$
\begin{aligned}
J_{25}^{\mathfrak{e}^2}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}q^e(q_c((l-q)^2-(l+p)^2-(q+p)^2)+2l_c(q+p)^2 \\
+p_c((l-q)^2-(l+p)^2+(q+p)^2))(2l^h\omega_{he}-q^h\omega_{eh}),
\end{aligned} \tag{313}
$$

$$
\begin{aligned}
J_{25}^{a^2}(\omega) = \int d^n l\int d^n q\frac{f(n)l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}\Big\{q^h\omega_{ch}(q+p)^4-2(l+p)^2q^h\omega_{ch}(q+p)^2-2l^2q^h\omega_{ch}(q+p)^2 \\
+p^2q^h\omega_{ch}(q+p)^2-2p_cq^hq^e\omega_{eh}(q+p)^2-2q^hq_cq^e\omega_{eh}(q+p)^2+8l_cl^eq^f\omega_{ef}(q+p)^2+4l_cp^eq^f\omega_{ef}(q+p)^2 \\
+2q^hq_cq^f\omega_{fh}(q+p)^2-4l_cq^eq^f\omega_{fe}(q+p)^2+2p^hq^e(p_c(3(l-q)^2+l^2-q^2)+q_c(3(l-q)^2+l^2 \\
-(q+p)^2-q^2))\omega_{he}+2l^hq^e(q_c(2(l-q)^2-2(l+p)^2+p^2-(q+p)^2-q^2)+p_c(2(l-q)^2-2(l+p)^2+p^2 \\
+(q+p)^2-q^2))\omega_{he}+2(l-q)^2p_cq^hq^e\omega_{eh}+4(l+p)^2p_cq^hq^e\omega_{eh}+2l^2p_cq^hq^e\omega_{eh}-2p_cp^2q^hq^e\omega_{eh} \\
+2(l-q)^2q^hq_cq^e\omega_{eh}+4(l+p)^2q^hq_cq^e\omega_{eh}+2l^2q^hq_cq^e\omega_{eh}-2p^2q^hq_cq^e\omega_{eh}\Big\},
\end{aligned} \tag{314}
$$

$$
\begin{aligned}
J_{25}^{a\cdot\mathfrak{e}}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}(q^e(p^hq^f(2(p_c+q_c)q^g\epsilon_{hg}+(q+p)^2\epsilon_{ch})\omega_{fe} \\
+q^h((-2l_c(q+p)^2-p_c((l-q)^2-(l+p)^2+(q+p)^2)+q_c(-(l-q)^2+(l+p)^2+(q+p)^2))\epsilon_{ef}\omega^{f}{}_{h} \\
-q^f(q+p)^2\epsilon_{cf}\omega_{eh}))+2l^h(-2l^eq^f((q+p)^2\epsilon_{ce}+(p_c+q_c)q^g\epsilon_{eg})\omega_{hf}-2l^ep^f(p_c+q_c)q^g\epsilon_{ef}\omega_{hg} \\
+q^e(q^f((q+p)^2\epsilon_{cf}\omega_{he}+((p_c+q_c)q^g\epsilon_{hg}+(q+p)^2\epsilon_{ch})\omega_{fe})+(q_c((l-q)^2-(l+p)^2-(q+p)^2) \\
+2l_c(q+p)^2+p_c((l-q)^2-(l+p)^2+(q+p)^2))\epsilon_{hf}\omega^{f}{}_{e})+p^eq^f((p_c+q_c)q^g(\epsilon_{he}\omega(-g,-f)-2\epsilon_{eg}\omega_{hf}) \\
-(q+p)^2\epsilon_{ce}\omega_{hf}))).
\end{aligned} \tag{315}
$$

$$
J_{26}^{e^2}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}\epsilon_{c\alpha}\epsilon^{e\beta}\omega^{g\gamma}(l-q)_\beta q_\gamma(p^\alpha+q^\alpha)(l+2p+q)_e(2l-q)_g \tag{316}
$$

$$
\begin{aligned}
J_{26}^{a^2}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}\epsilon_{c\alpha}\epsilon^{e\beta}\omega^{g\gamma}(l-q)_\beta q_\gamma(p^\alpha+q^\alpha)\Big[2l^f(2p^h+q^h)\epsilon_{eh}\epsilon_{gf}. \\
+\epsilon_{ef}\epsilon_{gh}\Big(2l^fl^h-q^h(l^f+2p^f+q^f)\Big)\Big],
\end{aligned} \tag{317}
$$

$$
\begin{aligned}
J_{26}^{a\cdot\mathfrak{e}}(\omega) = \int d^n l\int d^n q\frac{l_a(l+p)_b q_d}{l^2(l+p)^2(l-q)^2q^2(q+p)^2}\epsilon_{c\alpha}\epsilon^{e\beta}\omega^{g\gamma}(l-q)_\beta q_\gamma(p^\alpha+q^\alpha)\Big((l^f+q^f \\
+2p^f)(2l-q)_g\epsilon_{ef}+\epsilon_{gf}(l^e+2p^e+q^2(2l^f-q^f).
\end{aligned} \tag{318}
$$

$$J_{27}^{\mathfrak{e}^2} = \int d^n l \int d^n q \frac{(l+p)_a l_b (l+p)_d}{q^2(l+q+p)^2(l+p)^4 l^2} \epsilon_{c\delta}\epsilon^{e\alpha}\omega^{g\beta} l^\delta (l+p)_\alpha (l+p)_\beta (l+q+2p)_e (2l-q)_g, \quad (319)$$

$$J_{27}^{a^2} = \int d^n l \int d^n q \frac{(l+p)_a l_b (l+p)_d}{q^2(l+q+p)^2(l+p)^4 l^2} \epsilon_{c\delta}\epsilon^{e\alpha}\omega^{g\beta} l^\delta (l+p)_\alpha (l+p)_\beta \Big[4l^f p^h + 2l^f q^h + l^h l^f - l^h q^f - 2p^h q^f - q^h q^f\Big]\epsilon_{eh}\epsilon_{gf}, \quad (320)$$

$$J_{27}^{\mathfrak{e}\cdot a} = \int d^n l \int d^n q \frac{(l+p)_a l_b (l+p)_d}{q^2(l+q+p)^2(l+p)^4 l^2} \epsilon_{c\delta}\epsilon^{e\alpha}\omega^{g\beta} l^\delta (l+p)_\alpha (l+p)_\beta \Big[(l^f + q^f + 2p^f)(2l-q)_g \epsilon_{ef} + (l+q+2p)_e (2l^f - q^f)\epsilon_{gf}\Big]. \quad (321)$$

$$(322)$$

C.3 $\langle \Lambda_a \partial_b A \Lambda_c \partial_d B \rangle$ integrals

$$J_{28}^{\mathfrak{e}^2} = \int d^n l \int d^n q \frac{(l+p)_b q_d l^\alpha (l+p)_\gamma (p^\beta + q^\beta) q_\delta}{(l^2(l+p)^2(l-q)^2 q^2 (q+p)^2} \epsilon_{a\alpha}\epsilon_{c\beta}\omega_A^{e\gamma}\omega_B^{g\delta}(l-2q-p)_e(2l-q)_g, \quad (323)$$

$$J_{28}^{a^2} = \int d^n l \int d^n q \frac{(l+p)_b q_d l^\alpha (l+p)_\gamma (p^\beta + q^\beta) q_\delta}{(l^2(l+p)^2(l-q)^2 q^2 (q+p)^2} \epsilon_{a\alpha}\epsilon_{c\beta}\omega_A^{e\gamma}\omega_B^{g\delta}\Big[-2l^f(p^h + 2q^h)\epsilon_{eh}\epsilon_{gf} + \big(l^f(2l^h - q^h) + q^h(p^f + 2q^f)\big)\epsilon_{ef}\epsilon_{gh}\Big], \quad (324)$$

$$J_{28}^{a\cdot\mathfrak{e}} = \int d^n l \int d^n q \frac{(l+p)_b q_d l^\alpha (l+p)_\gamma (p^\beta + q^\beta) q_\delta}{(l^2(l+p)^2(l-q)^2 q^2 (q+p)^2} \epsilon_{a\alpha}\epsilon_{c\beta}\omega_A^{e\gamma}\omega_B^{g\delta}\Big[(l^f - p^f - 2q^f)(2l^g - q^g)\epsilon_{ef} + (l^e - p^e - 2q^e)(2l^f - q^f)\epsilon_{gf}\Big]. \quad (325)$$

$$J_{30}^{\mathfrak{e}^2}(\omega_A,\omega_B) = \int d^n l \int d^n q \frac{(l+p)_b l_d l^\alpha (l^\beta + p^\beta)}{l^2(l+p)^4(l+q+p)^2 q^2}\big(\gamma_{gc}(l+p)^2 - (l+p)_c(l+p)_g\big) \times\omega_{e\beta}^A\omega_{a\alpha}^B(l^e + 2q^e + p^e)(l^g + 2q^g + p^g), \quad (326)$$

$$J_{30}^{a\cdot\mathfrak{e}}(\omega_A,\omega_B) = \int d^n l \int d^n q \frac{(l+p)_b l_d l^\alpha (l^\beta + p^\beta)}{l^2(l+p)^4(l+q+p)^2 q^2}\big(\gamma_{gc}(l+p)^2 - (l+p)_c(l+p)_g\big)\omega_{e\beta}^A\omega_{a\alpha}^B \times(l+2q+p)_f\Big[(l^g + 2q^g + p^g)\epsilon^{ef} + (l^e + 2q^e p^e)\epsilon^{gf}\Big], \quad (327)$$

$$J_{30}^{a^2}(\omega_A,\omega_B) = \int d^n l \int d^n q \frac{(l+p)_b l_d l^\alpha (l^\beta + p^\beta)}{l^2(l+p)^4(l+q+p)^2 q^2}\big(\gamma_{gc}(l+p)^2 - (l+p)_c(l+p)_g\big)\omega_{e\beta}^A\omega_{a\alpha}^B \times\Big[\big(l_f p_h + 2(l+p)_f q_h\big)\epsilon^{eh}\epsilon^{gf} + \Big(l_f(l+p)_h + p_f p_h + 2(l+2q+p)_f q_h\Big)\epsilon^{ef}\epsilon^{gh}, \quad (328)$$

where its good to note that these integrals simplify considerably when the inserted fields are both an E-field. We then have $J_{30}^{a^2}(-\gamma,-\gamma) = 0$, $J_{30}^{\mathfrak{e}^2}(-\gamma,-\gamma) = 0$ and

$$J_{30}^{a\cdot\mathfrak{e}}(-\gamma,-\gamma) = 2\int d^n l \int d^n q \frac{(l+p)_b l_d}{l^2(l+p)^2(l+q+p)^2 q^2}(l+p)_f(l+2q+p)_c q_g \epsilon^{fg}. \quad (329)$$

$$
\begin{aligned}
J_{31}^{\mathfrak{e}^2} &= \int d^n l \int d^n q \frac{(l+)_b l_d l^\gamma (l^\alpha + p^\alpha)(l^\delta + p^\delta) q_\beta}{q^2 (l+q+p)^2 (l+)^4 l^2} \omega_{e\alpha}^A \omega_{a\gamma}^B \epsilon_{c\delta} \epsilon^{g\beta} (l^e + 2q^e + p^e)(2l + 2p + q)_g, && (330)\\
J_{31}^{a\cdot\mathfrak{e}} &= \int d^n l \int d^n q \frac{(l+)_b l_d l^\gamma (l^\alpha + p^\alpha)(l^\delta + p^\delta) q_\beta}{q^2 (l+q+p)^2 (l+)^4 l^2} \omega_{e\alpha}^A \omega_{a\gamma}^B \epsilon_{c\delta} \epsilon^{g\beta} \Big[(l + 2q + p)_f (2l + 2p + q)_g \epsilon^{ef} \\
&\quad + (l^e + p^e + 2q^e)(2l + 2p + q)_f \epsilon^{gf} \Big], && (331)\\
J_{31}^{a^2} &= \int d^n l \int d^n q \frac{(l+)_b l_d l^\gamma (l^\alpha + p^\alpha)(l^\delta + p^\delta) q_\beta}{q^2 (l+q+p)^2 (l+)^4 l^2} \omega_{e\alpha}^A \omega_{a\gamma}^B \epsilon_{c\delta} \epsilon^{g\beta} \Big[2\big(l^f p_h + 2(l^f + p^f) q_h \big) \epsilon^{eh} \epsilon_{gf} \\
&\quad + \big(2 l_f l^h + 2(l+p)_f p^h + (l + p + 2q)_f q^h \big) \epsilon^{ef} \epsilon_{gh} \Big]. && (332)
\end{aligned}
$$

$$
\begin{aligned}
J_{32}^{a^2}(-\gamma, -\gamma) &= \int d^n l \int d^n q \frac{(l+p)_b (l+p)_d}{q^2 (l+q+p)^2 l^2} (l_a l_c + \gamma_{ac} l^2), && (333)\\
J_{32}^{\mathfrak{e}^2}(-\gamma, -\gamma) &= 0, && (334)\\
J_{32}^{a\cdot\mathfrak{e}}(-\gamma, -\gamma) &= 0, && (335)\\
J_{32}^{a^2}(-\gamma, \epsilon) &= \int d^n l \int d^n q \frac{(l+p)_b (l+p)_d}{q^2 (l+p)^4 (l+q+p)^2 l^2} (l_a l_c + \gamma_{ac} l^2) \Big[l_e q_f \epsilon^{ef} \big((l+p)^2 - l^2 + p^2 \big) \\
&\quad - p_e q_f \big((l+p)^2 + l^2 - p^2 \big) \Big], && (336)\\
J_{32}^{\mathfrak{e}^2}(-\gamma, \epsilon) &= 0, && (337)\\
J_{32}^{a\cdot\mathfrak{e}}(-\gamma, \epsilon) &= \int d^n l \int d^n q \frac{(l+p)_b (l+p)_d}{q^2 (l+q+p)^2 l^2} (l_a l_c + \gamma_{ac} l^2), && (338)\\
J_{32}^{a^2}(\epsilon, \epsilon) &= (2-n) f(n) \int d^n l \int d^n q \frac{(l+p)_b (l+p)_d}{q^2 (l+q+p)^2 l^2} (l_a l_c + \gamma_{ac} l^2), && (339)\\
J_{32}^{\mathfrak{e}^2}(\epsilon, \epsilon) &= \int d^n l \int d^n q \frac{(l+p)_b (l+p)_d}{q^2 (l+q+p)^2 l^2} (l_a l_c + \gamma_{ac} l^2), && (340)\\
J_{32}^{a\cdot\mathfrak{e}}(\epsilon, \epsilon) &= 0. && (341)
\end{aligned}
$$

D Equations of motion for TNC fields

We can vary the Lagrangian (114) to its worldsheet fields X^m, Λ_a and η to find the classical equations of motion. Varying with respect to η we find

$$\partial_\alpha \Sigma\lambda^\alpha = 0, \qquad (342)$$

where $\Sigma\lambda^\alpha \equiv -\frac{1}{2}\gamma^{ab}\lambda_a e_b^\alpha$. After varying the other fields we find,

$$
\begin{aligned}
& e_\pm^\alpha (\partial_\alpha \eta + \aleph_\alpha \mp \tau_\alpha) = 0, \\
& \gamma^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n (\partial_r \bar{h}_{mn} - 2\partial_m \bar{h}_{rn}) - 2\bar{h}_{rm} \Box_\sigma X^m + \epsilon^{\alpha\beta} (\partial_r \bar{B}_{mn} - 2\partial_m \bar{B}_{rn}) \partial_\alpha X^m \partial_\beta X^n \qquad (343)\\
& \quad + \Delta\lambda^\alpha F_{rm} \partial_\alpha X^m - (\partial_\alpha \Delta\lambda^\alpha) \tau_r + \Sigma\lambda^\alpha \mathfrak{h}_{rm} \partial_\alpha X^m = 0.
\end{aligned}
$$

References

[1] D. Oriti, "The Bronstein hypercube of quantum gravity," 2018.

[2] C. Kiefer, "Quantum gravity: general introduction and recent developments," *Annalen der Physik*, vol. 15, no. 1-2, pp. 129–148, 2006.

[3] D. Tong, "Lectures on string theory," *arXiv preprint arXiv:0908.0333*, 2009.

[4] E. Cartan, "Sur les variétés à connexion affine, et la théorie de la relativité généralisée," in *Annales scientifiques de l'École Normale Supérieure*, vol. 41, pp. 1–25, 1924.

[5] J. Gomis and H. Ooguri, "Nonrelativistic closed string theory," *Journal of Mathematical Physics*, vol. 42, no. 7, pp. 3127–3151, 2001.

[6] C. M. Will, "The confrontation between general relativity and experiment," *Living Reviews in Relativity*, vol. 9, Mar 2006.

[7] D. Hansen, J. Hartong, and N. A. Obers, "Gravity between Newton and Einstein," *arXiv preprint arXiv:1904.05706*, 2019.

[8] D. Hansen, J. Hartong, and N. A. Obers, "Non-relativistic gravity and its coupling to matter," *arXiv preprint arXiv:2001.10277*, 2020.

[9] D. T. Son, "Newton-cartan geometry and the quantum Hall effect," *arXiv preprint arXiv:1306.0638*, 2013.

[10] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, "Torsional Newton-Cartan geometry and Lifshitz holography," *Physical Review D*, vol. 89, no. 6, p. 061901, 2014.

[11] K. Jensen, "On the coupling of Galilean-invariant field theories to curved spacetime," *arXiv preprint arXiv:1408.6855*, 2014.

[12] E. Bergshoeff, A. Chatzistavrakidis, L. Romano, and J. Rosseel, "Newton-Cartan gravity and torsion," *Journal of High Energy Physics*, vol. 2017, 08 2017.

[13] C. G. Callan, Jr. and L. Thorlacius, "SIGMA MODELS AND STRING THEORY," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particles, Strings and Supernovae (TASI 88) Providence, Rhode Island, June 5-July 1, 1988*, pp. 795–878, 1989.

[14] A. Gallegos, U. Gürsoy, and N. Zinnato, "Torsional Newton Cartan gravity from non-relativistic strings," *arXiv preprint arXiv:1906.01607*, 2019.

[15] P. Deligne, P. I. Etingof, and D. S. Freed, *Quantum fields and strings: a course for mathematicians*, vol. 2. American Mathematical Society Providence, 1999.

[16] M. Peskin, *An introduction to quantum field theory.* CRC press, 2018.

[17] S. M. Carroll, *Spacetime and geometry.* Cambridge University Press, 2019.

[18] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, "The background field method and the ultraviolet structure of the supersymmetric nonlinear σ-model," *Annals of Physics*, vol. 134, no. 1, pp. 85–109, 1981.

[19] J. P. Ellis, “Tikz-feynman: Feynman diagrams with tikz,” *Computer Physics Communications*, vol. 210, pp. 103–123, 2017.

[20] R. Metsaev and A. A. Tseytlin, “Two-loop β-function for the generalized bosonic sigma model,” *Physics Letters B*, vol. 191, no. 4, pp. 354–362, 1987.

[21] B. de Wit, E. Laenen, and J. Smith, “Field theory in particle physics lecture notes.”.

[22] R. Andringa, E. Bergshoeff, J. Gomis, and M. de Roo, “‘Stringy’ Newton–Cartan gravity,” *Classical and Quantum Gravity*, vol. 29, no. 23, p. 235020, 2012.

[23] T. Harmark, J. Hartong, L. Menculini, N. A. Obers, and Z. Yan, “Strings with non-relativistic conformal symmetry and limits of the AdS/CFT correspondence,” *Journal of High Energy Physics*, vol. 2018, no. 11, p. 190, 2018.

[24] J. Gomis, J. Oh, and Z. Yan, “Nonrelativistic string theory in background fields,” *Journal of High Energy Physics*, vol. 2019, no. 10, p. 101, 2019.

[25] A. Kotikov and S. Teber, “Multi-loop techniques for massless Feynman diagram calculations,” *Physics of Particles and Nuclei*, vol. 50, no. 1, pp. 1–41, 2019.

[26] D. Ross, “Veltman-Passarino reduction.” `https://www.southampton.ac.uk/~doug/mhv/vp.pdf`.

[27] S. G. Gorishnii and A. P. Isaev, “An approach to the calculation of many-loop massless Feynman integrals,” *Theoretical and Mathematical Physics*, vol. 62, pp. 232–240, Mar 1985.

[28] A. Suzuki, E. Santos, and A. Schmidt, “Massless and massive one-loop three-point functions in negative dimensional approach,” *The European Physical Journal C-Particles and Fields*, vol. 26, no. 1, pp. 125–137, 2002.