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1:3 RESONANT DOUBLE HOPF BIFURCATION

MACHIEL KRUGER, 6224059

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Universiteit Utrecht

Contents

Introduction.

Non-linear dynamical systems are often used to model physical problems. One of the most famous examples is the predator-prey model, more specifically the Lotka–Volterra equations. This system is used, as the name implies, to study the relations between predators and prey in an isolated system, where the growth of the predators and prey are dependent on each other.

While this is an interesting and physically relevant example, it is easier for us to start with one of the simpler non-linear systems. Consider the system defined by the equation

$$\frac{d}{dt}x = r + x^2, \quad x \in \mathbb{R}, \tag{1.1}$$

with a parameter $r \in \mathbb{R}$. In this system, changing the r parameter results in profound changes in the solutions. As for $r < 0$ the system has two fixed points and for a $r > 0$ none. The change in r can be very small, while the difference between no fixed point and two fixed points is much more substantial.

These sudden qualitative or topological changes in the dynamics of a system are caused by small smooth changes in the parameters of the system. These changes are known as bifurcations, and due to their importance for the behavior of non-linear systems their study is more than worthwhile.

Fortunately these changes are not completely chaotic, bifurcations can be classified by the changes they induce. The change that the previous example system (1.1) experiences at $r = 0$ is known as the saddle-node bifurcation.

More abstractly, the bifurcations themselves can be classified by the amount of parameters required to fully reveal all different shifts in behavior near the equilibrium point they effect. This number is also known as the co-dimension of the bifurcation and while this statement gives a rough idea of what the co-dimension is, the formal definitions are of course much more rigorous.

Usually a higher co-dimension implies a higher complexity of behaviors. This is backed by the fact that all local bifurcations of co-dimension 1 are completely understood, while this is not the case for some co-dimension 2 bifurcations.

For a complete explanation regarding bifurcations and their co-dimension see the books by Guckenheimer and Holmes [6] or Kuznetsov [8].

The study of bifurcations is usually performed by applying normal form theory, where one tries to find a simplified system that is locally topologically equivalent to the local neighborhood around the equilibrium which exhibits the bifurcation in question. This equivalence preserves the changes in the dynamics near the equilibrium, including the bifurcation.

The advantage is obvious, as usually the simplified system is much easier to analyze. More importantly most instances of the bifurcations that occur in the wild can be proven to be locally topologically equivalent to the same simplified system. This more or less allows us to study the behavior of an identified bifurcation in any system by studying their respective normal form. This is a major advantage, and consequently the concept of the normal form has become central in the study of bifurcations.

This thesis does not differ from this standard, as it makes extensive use of the normal form. In fact the first step in the study of the double Hopf bifurcation is the derivation of the normal form. Again for the complete theory regarding normal forms and their derivation see the book by Guckenheimer and Holmes [6].

Co-dimension 1 bifurcations.

We continue the introduction with a few examples of co-dimension 1 bifurcations.

- Saddle-node (fold) bifurcation, where a node and a saddle points collide and cancel each other out.
- Trans-critical bifurcation, where two fixed points collide and switch stability in the collision.
- Pitchfork bifurcation, where an equilibrium switches stability and forks off two additional equilibria.
- Period-doubling (flip) bifurcation, this is a cycle bifurcation that, as the name implies, doubles the period of a cycle.
- Hopf bifurcation, where an equilibrium switches stability and splits off a cycle.

Due to their relevance to later analysis we examine the trans-critical, pitchfork and the Hopf bifurcation in a bit more detail.

Trans-critical bifurcation.

The normal form of the trans-critical bifurcation is given by

$$\frac{d}{dt}x = rx - x^2, \quad x \in \mathbb{R}. \tag{1.2}$$

This bifurcation is characterized by the eigenvalue r of the equilibrium at the origin and this system, for $r \neq 0$, has two fixed points given by $x = 0$ and $x = r$.

When the parameter r is negative, the fixed point at $x = 0$ is stable and the fixed point $x = r$ is unstable.

At $r = 0$ both fixed points have collided and only one fixed point remains at $x = 0$. Solutions starting at $x > 0$ diverge, while solutions starting at $x < 0$ converge to the fixed point. This point, while displaying some degree of stability, is not stable and therefore unstable.

When the parameter r becomes positive, the equilibria switch stability. The fixed point at $x = 0$ becomes unstable and the fixed point at $x = r$ becomes stable.

Lastly we should notice that during the bifurcation the second point moves across the $x = 0$-axis.

Pitchfork bifurcation.

The normal form of the pitchfork bifurcation is given by

$$\frac{d}{dt}x = rx \pm x^3, \quad x \in \mathbb{R}. \tag{1.3}$$

The eigenvalue of the linearization of this system at the origin is identical to that of the trans-critical bifurcation. Therefore to distinguish the pitchfork and trans-critical bifurcations the higher order terms also come into play. The actual difference is that the x^2 term in the trans-critical normal form system is replaced by an x^3 term. Furthermore the pitchfork bifurcation comes in 2 variations, the super- and sub-critical case.

- If x^3 is subtracted, the pitchfork bifurcation is super-critical. In this case 2 stable equilibria split off from the original equilibrium, when r becomes postive. The original equilibrium is not destroyed, instead the bifurcation switches the stability of the original equilibrium from stable to unstable.
- If x^3 is added to rx, the bifurcation is sub-critical. Contrary to the super-critical case, when r becomes negative, 2 unstable equilibria split off the original equilibrium. The central equilibrium is still maintained and displays the same stability behavior. Therefore, when r becomes negative, the stability of the central equilibrium goes from unstable to stable.

Hopf bifurcation.

The Hopf bifurcation has the normal form

$$\frac{d}{dt}z = (\lambda + i\omega)z + (\alpha + i\beta)\bar{z}z^2 + O(|z|^4), \quad z \in \mathbb{C}. \tag{1.4}$$

Notice that, due to the complex normal form, this bifurcation requires two dimensions. The eigenvalue is given by $\lambda + i\omega$. The parameter is λ and the other important quantity is α, which is known as the first Lyapunov coefficient.

Just as the pitchfork bifurcation the Hopf bifurcation has two distinct cases, the super- and sub-critical case, depending on the sign of the Lyapunov coefficient.

- Super-critical, when $\alpha < 0$. In this case a stable limit cycle bifurcates off a stable equilibrium point as the parameter λ goes from negative to positive. At $\lambda = 0$ the equilibrium switches from stable to unstable.
- Sub-critical, when $\alpha > 0$. In this case an unstable limit cycle bifurcates off a unstable equilibrium point as the parameter λ goes from positive to negative. At $\lambda = 0$ the equilibrium switches from unstable to stable.

If $\alpha = 0$, the bifurcation is degenerate and the equilibrium at $\lambda = 0$ is stable or unstable depending on the higher order term of the normal form. This requires the generalized form of the bifurcation also known as the Bautin bifurcation. As this bifurcation merits it own analysis it is omitted here, but more information about this special case can be found in the book by Kuznetsov [8].

Double Hopf bifurcation.

An example of a co-dimension two bifurcation that brings us closer to the object of study is the regular double Hopf bifurcation. The Poincaré normal form of this bifurcation is also well-known and given by

$$\dot{z}_1 = (\mu_1 + \omega_1(\mu))\, z_1 + G_{2100}(\mu)z_1|z_1|^2 + G_{1011}(\mu)z_1|z_2|^2 + O(4), \tag{1.5a}$$

$$\dot{z}_2 = (\mu_2 + \omega_2(\mu))\, z_2 + H_{1110}(\mu)z_2|z_1|^2 + H_{0021}(\mu)z_2|z_2|^2 + O(4). \tag{1.5b}$$

Just as in the single Hopf bifurcation the variables are complex. But in this case we have two variables instead of one variable, it follows that the system requires 4 dimensions. The coefficients in the system are usually dependent on the value of the parameters μ_1 and μ_2. To simplify the notation of this dependence, the variable μ is introduced as the vector $\mu := (\mu_1, \mu_2)$. The eigenvalues that characterize the equilibrium, which undergoes the double Hopf bifurcation, are given by

$$\lambda_{1,2} = \mu_1 \pm i\omega_1 \quad \text{and} \quad \lambda_{3,4} = \mu_2 \pm i\omega_2. \tag{1.6}$$

It is easy to see where the double Hopf bifurcation got its name, as the bifurcation essentially entails two Hopf bifurcations that occur at the same equilibrium. Where each parameter μ_j induces a Hopf bifurcation on their respective z_j variable. The system can be further simplified if the non-degeneracy conditions

$$\begin{array}{ll} \operatorname{Re} G_{2100}(0) \neq 0, & \operatorname{Re} G_{1011}(0) \neq 0, \\ \operatorname{Re} H_{1110}(0) \neq 0, & \operatorname{Re} H_{0021}(0) \neq 0, \end{array} \tag{1.7}$$

apply, as then (1.5) is locally smoothly orbitally equivalent to the system

$$\dot{z}_1 = (\mu_1 + \omega_1(\mu))\, z_1 + P_{11}(\mu)z_1|z_1|^2 + P_{12}(\mu)z_1|z_2|^2 + O(4), \tag{1.8a}$$

$$\dot{z}_2 = (\mu_2 + \omega_2(\mu))\, z_2 + P_{21}(\mu)z_2|z_1|^2 + P_{22}(\mu)z_2|z_2|^2 + O(4). \tag{1.8b}$$

For a proof of this lemma and the definition of the new coefficients, which is omitted from this introduction, consult the book by Kuznetsov [8]. If the non-degeneracy conditions are not satisfied the analysis requires the higher order terms, this makes the analysis identical to that of the system (1.5). This analysis is much more difficult and as such only the non-degenerate double Hopf bifurcations are considered in this thesis.

The system (1.8) can be analyzed by considering the system of only the amplitudes, which reads as

$$\dot{r}_1 = \left(\mu_1 + \mathrm{Re}(P_{11}(\mu))r_1^2 + \mathrm{Re}(P_{12}(\mu))r_2^2\right) r_1 + O\left(5\right), \tag{1.9a}$$

$$\dot{r}_2 = \left(\mu_2 + \mathrm{Re}(P_{21}(\mu))r_1^2 + \mathrm{Re}(P_{22}(\mu))r_2^2\right) r_2 + O\left(5\right). \tag{1.9b}$$

We notice the absence of any lower order phase variables in this system, which is also what makes this analysis possible. This is as the terms in the equations for the time evolution of the amplitudes are typically of high order. This means their effect on the time derivatives of the amplitudes, r, is negligible when the amplitudes are close to zero. The amplitudes themselves are small when μ is small, as then the limit cycles bifurcating from the equilibrium are still very close to the equilibrium. Therefore the phases can be omitted, when studying the evolution of the amplitudes near the equilibrium. This independence of the amplitudes relative to the phases is also referred to as the uncoupling of the amplitudes from the phases.

The equilibria of the amplitudes can come on the r_1 or on the r_2 axis, these are the so-called "pure" modes. These modes result in invariant 1-tori in the full system (1.5) where the solution must have one amplitude zero and the other amplitude fixed. Or the amplitudes can have a stable point with $r_1 \neq 0$ and $r_2 \neq 0$, a "mixed" mode. This mode results in invariant 2-tori in the full system (1.5) as both amplitudes are fixed, while the phases can still vary.

In the difficult case of the double Hopf bifurcation, the mixed mode exhibits a Hopf bifurcation in the amplitude system (1.5). This bifurcation results in a limit cycle around this mixed mode in the amplitude system. This corresponds to invariant 3-tori in the full system (1.5).

The frequencies at which the cycles rotate are, in the non-resonant regular case, not in resonance. Furthermore the phases are not required to be close to zero near the equilibrium and this implies that all higher order terms must be included to correctly give the solution. As the higher order terms can be difficult to work with, the solutions for the phases are usually unknown. Therefore the phases can differ widely and behave unpredictably with respect to each other.

This affects on how the solutions evolve on the invariant tori, corresponding to the fixed amplitudes. The pure modes correspond to regular cycles, as there is only one dimension on the 1-torus. The mixed modes can firstly correspond to quasi-periodic solutions on the 2-torus, as the amplitudes remain fixed and when the phases vary in a conditionally periodic way. The mixed modes can also correspond to a family of periodic cycles that foliate the 2 torus, when the phases are locked. The last option, the limit cycles, correspond to quasi-periodic solutions on a familiy of 2-tori that foliate a 3-torus.

For the full derivation of the normal form, its analysis, its unfolding diagrams and their associated phase plots, please refer to the book by Kuznetsov [8].

Resonant double Hopf bifurcation.

That brings us to the final bifurcation we consider, the resonant double Hopf bifurcation. This bifurcation has the same eigenvalues as the double Hopf bifurcation,

$$\lambda_{1,2} = \pm i\omega_1, \quad \text{and} \quad \lambda_{3,4} = \pm i\omega_2. \tag{1.10}$$

But in this case, the frequencies of the oscillations are in resonance at the parameter value $(\mu_1, \mu_2) = (0, 0)$. The resonance corresponds to the condition that $n\omega_1 + m\omega_2 = 0$ for some non-zero pair of integers $(n, m) \neq (0, 0)$.

As the parameters μ change the phases might not be in perfect resonance anymore. How fast or slow the phases lose resonance is based on the system that exhibits the resonant double Hopf bifurcation. This can be quite varied and as such it is convenient to introduce a detuning parameter ν. This parameter determines how much the phases are in resonance and replaces the effect of changing the μ parameters on the base frequency of the phases. This does not mean that the μ parameters are completely replaced, as they still very much effect the amplitudes. This greatly simplifies the study of the resonant case.This addition of the detuning parameter also implies that the resonant double Hopf bifurcation can be interpreted as a co-dimension 3 bifurcation.

Lastly we make a classification among the resonances. If for all pairs of integers

$$n\omega_1 + m\omega_2 \neq 0 \tag{1.11}$$

then the system is non-resonant and is just a regular double Hopf bifurcation with the normal form given by (1.8).

If the bifurcation is resonant, and a non-zero pair of integers exists such that $|n| + |m| \leq 4$ and $n_1\omega_1 + n_2\omega_2 = 0$, then the system is strongly resonant. If the bifurcation is still resonant, but no pair of integers can be found such that $|n| + |m| \leq 4$, then the system is weakly resonant.

Now suppose a double Hopf bifurcation is weakly resonant, then the above amplitude system (1.9) is still satisfactory. The resonance still results in the addition of resonance terms in the normal form, but the orders of these terms are high enough to be neglibile and thus can be safely truncated. The equilibria of the amplitudes are the same as in the non-resonant case, but the phases may be resonant for certain parameters. This case results, instead of the quasi-periodic solutions on the 2-torus, in a family of closed 1-tori orbits that foliate the 2-torus. Outside these rational frequency ratio cases the phases are not resonant and the solutions are the regular quasi-periodic solutions on the 2-torus.

However if the resonance is strong, then it leads to a rather interesting situation where the normal form described above does not suffice anymore. Notably the resonant terms have a low enough order in the normal form that the amplitudes are no longer uncoupled from the phases. This gives rise to a bifurcation that is lot richer in dynamics than the regular double Hopf bifurcation.

Besides this thesis, a large number of studies have already been done on the subject of resonant double Hopf bifurcations. For example two studies using the relatively new technique, multiple timescale analysis, in the first study [5] by A. Luongo, A. Paolone and A. Di Egidio have studied both the 1:2 and the 1:3 resonances and in the second study [9] J. Xu and W. Wang have focused entirely on the 1:2 resonance.

Another pair of more relevant studies are those done by S.H. Davi and P.H. Steen [4] and by A.K. Bajaj and P.R. Sethna [2]. Notably both these studies include the derivation of a normal form of the 1:1 resonance.

For a more complete analysis regarding the stability of the invariant tori for all resonances of the double Hopf bifurcation and a discussion of their differences, consult the study done by H. Broer, H. Hanßmann and F. Wagener [3]. Lastly I mention a numerical study [1] by D.M. Alonso, G. Revel and J.L. Moiola that has been performed on the 1:2 resonance, which sheds light on some of the global dynamics. The non-numerical studies usually do not provide much information on the global situation, which makes the numerical studies a very important part of the understanding of the total dynamics.

The structure of the thesis.

This leads us to the object of study in this thesis, the 1:3 resonant double Hopf bifurcation.

As this is a strong resonance, we require a resonant normal form. We derive this form in the next section 2.1. This form differs from the non-resonant case by the addition of a phase difference variable, this variable is interpreted as how much the frequencies of the oscillations are out of phase. The time evolution of the phase difference tells us if the phases for certain fixed amplitude are resonant. A constant phase difference implies that both phases change in the precise 1:3 ratio and thus are in resonance. In the resonant normal form, the equations for amplitudes also contain resonant terms in addition to the non-resonant ones. These terms vary according to the amplitudes themselves as well as the new phase difference variable.

The derivation of the normal form gives us

$$\dot{X} = \mu_1 X + (a_{1r}X^2 + b_{1r}Y^2)X + Y^3\cos(\varphi), \tag{1.12a}$$

$$\dot{Y} = \mu_2 Y + (a_{2r}X^2 + b_{2r}Y^2)Y + XY^2\cos(\varphi + \Phi), \tag{1.12b}$$

$$\dot{\varphi} = \nu + a_{im}X^2 + b_{im}Y^2 - \frac{Y^3}{X}\sin(\varphi) - 3XY\sin(\varphi + \Phi), \tag{1.12c}$$

where X and Y correspond to the amplitudes, and φ corresponds to the phase difference. Lastly μ_1, μ_2 and ν are the parameters.

After the derivation of (1.12), we discuss its equilibria. These come in two distinct types. The first type are the pure modes, which occur on the X-axis, i.e. where $Y = 0$. The second type are the mixed modes, which occur where both amplitudes are higher than 0. In this discussion we make some general statements on the existence and stability of both modes.

Afterwards we start the study of a particularly interesting set of pure modes on the X-axis. These points have the amplitudes $(X, Y) = (X_0, 0)$, where the value X_0 is determined by the parameter μ_1. The phase differences are yet to be determined, as when $Y = 0$ the evolution of the phase difference is equal to $\nu + a_{im}X_0^2$, which can be zero for certain ν values. So these pure modes may have a running phase or a locked phase depending on this parameter.

The property of interest of these pure modes is that one of them, at a certain set of parameters, is characterized by the eigenvalues in the amplitude directions $(\lambda_X, \lambda_Y) = (-2\mu_1, 0)$. Furthermore the evolution of the phase difference is near constant around this point. This implies that this point experiences a double zero bifurcation and makes this pure mode a degenerate point around which expansions can be made in the parameter space.

Before encountering a fatal mistake, the goal of the thesis was determining the local dynamics of the 1:3 resonant normal form near this point. It was hoped for that this method gives some insight in the behavior of the full resonant system.

Conclusions.

After concluding the study of the local dynamics we compare our 1:3 dynamics to the 1:2 resonant case. This is considered to be the main goal of this thesis, as I was personally very interested in these differences when I came across a similar study.

This was a study done on the 1:2 resonance by E. Knobloch and M.R.E. Procter in 1988 [7]. Wherein they studied the dynamics of the 1:2 resonance, by locating degenerate point in parameter space. Just as in the 1:3 resonant case, this point was a pure mode, which also exhibited a double zero bifurcation. An important difference already is that their pure mode had a specific phase difference by $(X_0, 0, \varphi_0)$.

In addition to explaining the differences between the local dynamics around the degenerate point, we discuss the usefulness of the method of locating a degenerate points in parameter space for the study of resonant dynamics.

Resonant normal form.

In the strongly resonant case of the double Hopf bifurcation, even at the lowest significant order, the amplitude remains coupled to the phase. Therefore instead of the usual non-resonant normal form of the double Hopf bifurcation, we need a resonant normal form that takes this coupling into account.

In this section, we derive the general resonant normal form for the 1:3 resonance. Afterwards we consider the fixed points of the resulting system.

2.1 Derivation.

In our particular case, the resonance is 1:3 at $\mu = (0,0)$. Therefore we assume $\omega_1 = 3\omega_2$, where ω_1 is the frequency of the first Hopf bifurcation and ω_2 the frequency of the second Hopf bifurcation, that make up the double Hopf bifurcation. Then at $(0,0)$ the complex amplitudes (u,v) of the system are symmetric under the operations

$$u \to ue^{i\omega_1}, \quad v \to ve^{i\frac{1}{3}\omega_1}.$$

These operations have the four basic invariants

$$\tau_1 = u\bar{u} \in \mathbb{R}, \quad \tau_2 = v\bar{v} \in \mathbb{R}, \quad \tau_3 = u\bar{v}^3 \in \mathbb{C} \text{ and } \tau_4 = \bar{\tau}_3 \in \mathbb{C}. \tag{2.1}$$

The most general commuting vector field associated with these invariants and operations has the form

$$\dot{u} = f_1(\tau_1,\tau_2,\tau_3,\tau_4)u + f_2(\tau_1,\tau_2,\tau_3,\tau_4)v^3, \tag{2.2a}$$

$$\dot{v} = f_3(\tau_1,\tau_2,\tau_3,\tau_4)v + f_4(\tau_1,\tau_2,\tau_3,\tau_4)u\bar{v}^2. \tag{2.2b}$$

This general vector field must satisfy two additional conditions to be a 1:3 resonant normal form. When $(u,v) = (0,0)$, at the point of the double Hopf bifurcation, the linear part must be resonant with the 1:3 ratio. Therefore, for this vector field to display a 1:3 resonant double Hopf bifurcation, $f_1(0,0,0,0) = i\omega_1$ and $f_3(0,0,0,0) = i\omega_2 = \frac{1}{3}i\omega_1$. Furthermore $f_2(0,0,0,0)$ and $f_4(0,0,0,0)$ are assumed be non-zero, as these correspond to the non-degeneracy conditions (1.7) seen in the introduction. In this thesis we only study the non-degenerate case of the double Hopf bifurcation, satisfying the conditions (1.7).

Now we can take the Taylor expansion of all f_j about $(0,0,0,0)$,

$$f_j = \hat{c}_j + \hat{a}_j\tau_1 + \hat{b}_j\tau_2 + \hat{d}_j\tau_3 + \hat{e}_j\tau_4 + O(2). \tag{2.3}$$

As all f_j are complex-valued functions, their Taylor coefficients are complex numbers. If we substitute the invariants τ_j for their definitions, we see for $j \in \{1,2,3,4\}$

$$f_j = \hat{c}_j + \hat{a}_j u\bar{u} + \hat{b}_j v\bar{v} + O(4). \tag{2.4}$$

Then if we replace the f_j in the system (2.2) with their expansions and collect the higher order terms of u and v in the big O, we obtain

$$\dot{u} = \hat{c}_1 u + (\hat{a}_1 u\bar{u} + \hat{b}_1 v\bar{v})u + \hat{c}_2 v^3 + O(5), \tag{2.5a}$$

$$\dot{v} = \hat{c}_3 v + (\hat{a}_3 u\bar{u} + \hat{b}_3 v\bar{v})v + \hat{c}_4 u\bar{v}^2 + O(5). \tag{2.5b}$$

We know from the restrictions on $f_{1,2}(0,0,0,0)$ that $\hat{c}_1 = i\omega_1$ and $\hat{c}_3 = i\omega_2 = \frac{1}{3}i\omega_1$. This allows us to somewhat simplify the equations by relabeling the coefficients

$$\begin{cases} \bar{c}_1 = \hat{c}_2, & \bar{a}_1 = \hat{a}_1, \quad \bar{b}_1 = \hat{b}_1, \\ \bar{c}_2 = \hat{c}_4, & \bar{a}_2 = \hat{a}_3, \quad \bar{b}_2 = \hat{b}_3. \end{cases} \tag{2.6}$$

Substituting these relabeled coefficients in (2.5) leads us to

$$\dot{u} = i\omega_1 u + (\bar{a}_1 u\bar{u} + \bar{b}_1 v\bar{v})u + \bar{c}_1 v^3 + O(5), \tag{2.7a}$$

$$\dot{v} = \tfrac{1}{3}i\omega_1 v + (\bar{a}_2 u\bar{u} + \bar{b}_2 v\bar{v})v + \bar{c}_2 u\bar{v}^2 + O(5). \tag{2.7b}$$

Now let us take the bifurcation parameters μ_1, μ_2 into account. From the linear dynamics we know that both are to be added to the linear coefficients of $\dot{u}$ and $\dot{v}$ to obtain the correct eigenvalues. Also we notice that all coefficients of the f_j Taylor expansions are dependent on μ, therefore for $j \in \{1,2\}$ let

$$\begin{cases} \tilde{\omega}_j = \omega_j + O(\mu_1,\mu_2), & \tilde{a}_j = \bar{a}_j + O(\mu_1,\mu_2), \\ \tilde{b}_j = \bar{b}_j + O(\mu_1,\mu_2), & \tilde{c}_j = \bar{c}_j + O(\mu_1,\mu_2). \end{cases} \tag{2.8}$$

If we add the bifurcation parameters to the linear part and if we substitute the parameter dependent coefficients (2.8) in system (2.7), we obtain

$$\dot{u} = (\mu_1 + i\tilde{\omega}_1)u + (\tilde{a}_1 u\bar{u} + \tilde{b}_1 v\bar{v})u + \tilde{c}_1 v^3 + O(5), \tag{2.9a}$$

$$\dot{v} = (\mu_2 + i\tilde{\omega}_2)v + (\tilde{a}_2 u\bar{u} + \tilde{b}_2 v\bar{v})v + \tilde{c}_2 u\bar{v}^2 + O(5). \tag{2.9b}$$

The resulting system is the complex resonant normal form of the double Hopf bifurcation corresponding to the 1:3 resonance. To make the study of this system easier, let us perform a switch of variables to transform the complex amplitude system into a system of real amplitudes and phases. Suppose

$$\begin{cases} u = \tilde{X}e^{i\varphi_1}, & v = \tilde{Y}e^{i\varphi_2}, \\ \tilde{c}_1 = |\tilde{c}_1|e^{i\Phi_1}, & \tilde{c}_2 = |\tilde{c}_2|e^{i\Phi_2}, \end{cases} \tag{2.10}$$

then if substitute these new variables in (2.9a) we obtain

$$\dot{\tilde{X}}e^{i\varphi_1} + i\tilde{X}\dot{\varphi}_1 e^{i\varphi_1} = (\mu_1 + i\tilde{\omega}_1)\tilde{X}e^{i\varphi_1} + (\tilde{a}_1\tilde{X}^2 + \tilde{b}_1\tilde{Y}^2)\tilde{X}e^{i\varphi_1} + |\tilde{c}_1|e^{i\Phi_1}\tilde{Y}^3 e^{i3\varphi_2}. \tag{2.11}$$

Here we notice that the real amplitude can be derived using the equality

$$\operatorname{Re}\left(e^{-i\varphi_1}(\dot{\tilde{X}}e^{i\varphi_1} + i\tilde{X}\dot{\varphi}_1 e^{i\varphi_1})\right) = \dot{\tilde{X}}. \tag{2.12}$$

Thus

$$\dot{\tilde{X}} = \operatorname{Re}\left(e^{-i\varphi_1}\left((\mu_1 + i\tilde{\omega}_1)\tilde{X}e^{i\varphi_1} + (\tilde{a}_1\tilde{X}^2 + \tilde{b}_1\tilde{Y}^2)\tilde{X}e^{i\varphi_1} + |\tilde{c}_1|e^{i\Phi_1}\tilde{Y}^3 e^{i3\varphi_2}\right)\right) \tag{2.13}$$

$$= \operatorname{Re}\left(\mu_1\tilde{X} + (\tilde{a}_1\tilde{X}^2 + \tilde{b}_1\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3 e^{i(\Phi_1 - \varphi_1 + 3\varphi_2)}\right). \tag{2.14}$$

Here we introduce a shorthand for the real values of the coefficients, $\tilde{a}_{1r} = \operatorname{Re}\tilde{a}_1$ and $\tilde{b}_{1r} = \operatorname{Re}\tilde{b}_1$. This gives us

$$\dot{\tilde{X}} = \mu_1\tilde{X} + (\tilde{a}_{1r}\tilde{X}^2 + \tilde{b}_{1r}\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3\cos(\Phi_1 - \varphi_1 + 3\varphi_2). \tag{2.15}$$

After finalizing the amplitude equation we continue with the equation for the corresponding phase, which is given by

$$\dot{\varphi}_1 = \frac{1}{\tilde{X}}\operatorname{Im}\left(e^{-i\varphi_1}(\dot{\tilde{X}}e^{i\varphi_1} + i\tilde{X}\dot{\varphi}_1 e^{i\varphi_1})\right). \tag{2.16}$$

This leads us to

$$\dot{\varphi}_1 = \frac{1}{\tilde{X}}\operatorname{Im}\left(\mu_1\tilde{X} + i\tilde{\omega}_1\tilde{X} + (\tilde{a}_1\tilde{X}^2 + \tilde{b}_1\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3 e^{i(\Phi_1 - \varphi_1 + 3\varphi_2)}\right) \tag{2.17}$$

$$= \frac{1}{\tilde{X}}\left(\tilde{\omega}_1\tilde{X} + \operatorname{Im}(\tilde{a}_1\tilde{X}^2 + \tilde{b}_1\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3\sin(\Phi_1 - \varphi_1 + 3\varphi_2)\right). \tag{2.18}$$

We introduce the shorthand for the imaginary parts of $\tilde{a}_1$, $\tilde{b}_1$, $\tilde{a}_{1i} = \operatorname{Im}\tilde{a}_1$ and $\tilde{b}_{1i} = \operatorname{Im}\tilde{b}_1$, and substituting these leads to

$$\dot{\varphi}_1 = \tilde{\omega}_1 + \tilde{a}_{1i}\tilde{X}^2 + \tilde{b}_{1i}\tilde{Y}^2 + |\tilde{c}_1|\tilde{Y}^3\frac{1}{\tilde{X}}\sin(\Phi_1 - \varphi_1 + 3\varphi_2). \tag{2.19}$$

We need to apply the same process to (2.9b) to obtain the other amplitude and phase. To this end let us substitute the variables (2.10) in (2.9b), which leads us to

$$\dot{\tilde{Y}}e^{i\varphi_2} + i\tilde{Y}\dot{\varphi}_2 e^{i\varphi_2} = (\mu_2 + i\tilde{\omega}_2)\tilde{Y}e^{i\varphi_2} + (\tilde{a}_2\tilde{X}^2 + \tilde{b}_2\tilde{Y}^2)\tilde{Y}e^{i\varphi_2} + |\tilde{c}_2|e^{i\Phi_2}\tilde{X}e^{i\varphi_1}\tilde{Y}^2 e^{-i2\varphi_2}. \tag{2.20}$$

We again notice that the amplitude is given by

$$\operatorname{Re}\left(e^{-i\varphi_2}(\dot{\tilde{Y}}e^{i\varphi_2} + i\tilde{Y}\dot{\varphi}_2 e^{i\varphi_2})\right) = \dot{\tilde{Y}}, \tag{2.21}$$

therefore

$$\dot{\tilde{Y}} = \operatorname{Re}\left(e^{-i\varphi_2}\left((\mu_2 + i\tilde{\omega}_2)\tilde{Y}e^{i\varphi_2} + (\tilde{a}_2\tilde{X}^2 + \tilde{b}_2\tilde{Y}^2)\tilde{Y}e^{i\varphi_2} + |\tilde{c}_2|e^{i\Phi_2}\tilde{X}e^{i\varphi_1}\tilde{Y}^2 e^{-i2\varphi_2}\right)\right) \tag{2.22}$$

$$= \operatorname{Re}\left(\mu_2\tilde{Y} + (\tilde{a}_2\tilde{X}^2 + \tilde{b}_2\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2 e^{i(\Phi_2 + \varphi_1 - 3\varphi_2)}\right). \tag{2.23}$$

Again we introduce the shorthand for the real values, $\tilde{a}_{2r} = \operatorname{Re}\tilde{a}_2$ and $\tilde{b}_{2r} = \operatorname{Re}\tilde{b}_2$, which gives us the final amplitude equation

$$\dot{\tilde{Y}} = \mu_2\tilde{Y} + (\tilde{a}_{2r}\tilde{X}^2 + \tilde{b}_{2r}\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2\cos(\Phi_2 + \varphi_1 - 3\varphi_2). \tag{2.24}$$

Then we continue with the equation of the second phase, given by

$$\dot{\varphi}_2 = \frac{1}{\tilde{Y}} \operatorname{Im}\left(e^{-i\varphi_1}(\dot{\tilde{Y}} e^{i\varphi_2} + i\tilde{Y}\dot{\varphi}_2 e^{i\varphi_2})\right). \tag{2.25}$$

We continue as we have done before,

$$\dot{\varphi}_2 = \frac{1}{\tilde{Y}} \operatorname{Im}\left((\mu_2 + i\tilde{\omega}_2)\tilde{Y} + (\tilde{a}_2\tilde{X}^2 + \tilde{b}_2\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2 e^{i(\Phi_2+\varphi_1-3\varphi_2)}\right) \tag{2.26}$$

$$= \frac{1}{\tilde{Y}}\left(\tilde{\omega}_2\tilde{Y} + \operatorname{Im}(\tilde{a}_{2i}\tilde{X}^2 + \tilde{b}_{2i}\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2\sin(\Phi_2 + \varphi_1 - 3\varphi_2)\right), \tag{2.27}$$

until we reach

$$\dot{\varphi}_2 = \tilde{\omega}_2 + \tilde{a}_{2i}\tilde{X}^2 + \tilde{b}_{2i}\tilde{Y}^2 + |\tilde{c}_2|\tilde{X}\tilde{Y}\sin(\Phi_2 + \varphi_1 - 3\varphi_2), \tag{2.28}$$

where we used the shorthand $\tilde{a}_{2i} = \operatorname{Im}\tilde{a}_2$ and $\tilde{b}_{2i} = \operatorname{Im}\tilde{b}_2$.

With the extracted real amplitudes, (2.15) and (2.24), and the real phases, (2.19) and (2.28), we have obtained the real form

$$\dot{\tilde{X}} = \mu_1\tilde{X} + (\tilde{a}_{1r}\tilde{X}^2 + \tilde{b}_{1r}\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3\cos(\Phi_1 - \varphi_1 + 3\varphi_2), \tag{2.29a}$$

$$\dot{\tilde{Y}} = \mu_2\tilde{Y} + (\tilde{a}_{2r}\tilde{X}^2 + \tilde{b}_{2r}\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2\cos(\Phi_2 + \varphi_1 - 3\varphi_2), \tag{2.29b}$$

$$\dot{\varphi}_1 = \tilde{\omega}_1 + \tilde{a}_{1i}\tilde{X}^2 + \tilde{b}_{1i}\tilde{Y}^2 + |\tilde{c}_1|\tilde{Y}^3\frac{1}{\tilde{X}}\sin(\Phi_1 - \varphi_1 + 3\varphi_2), \tag{2.29c}$$

$$\dot{\varphi}_2 = \tilde{\omega}_2 + \tilde{a}_{2i}\tilde{X}^2 + \tilde{b}_{2i}\tilde{Y}^2 + |\tilde{c}_2|\tilde{X}\tilde{Y}\sin(\Phi_2 + \varphi_1 - 3\varphi_2), \tag{2.29d}$$

of the system (2.9). As we are more interested in the phase locking and unlocking of the system than its absolute phases, it is more useful to calculate the evolution of the relative phase difference, than to let the phases remain as they are. This also has the added benefit that it reduces the variables we need to track by 1.

Notice that it is not the case that the 4-dimensional system has truly been reduced in dimension, as the phases do still very much exist. However, because the resonant terms in the amplitudes and the phases depend only on the phase difference and not the absolute value of each phase, we are not required to know their precise values beyond their difference. Also very important is that the phase difference allows for a separate time evolution, uncoupled from the phases individually. Therefore we may treat it as a separate variable and get a system whose right hand side depends on the phase differences and the amplitudes, but not on the absolute phases.

The result is a 3 dimensional system, that is separate from the original system, but still correctly gives the time evolution of the amplitudes and phase difference of said system. This is a trade off however as by deriving only the phase difference, we do not obtain much information about the time evolution of the phases of the oscillations, excluding the difference between them. However, once the reduced system (see (2.32) or (2.36) below) is solved, we can substitute these solutions in (2.29) and compute $\varphi_j(t)$, $j \in \{1, 2\}$.

Let $\tilde{\varphi} = \varphi_1 - 3\varphi_2$, then $\dot{\tilde{\varphi}} = \dot{\varphi}_1 - 3\dot{\varphi}_2$ and let us define

$$\nu = \tilde{\omega}_1 - 3\tilde{\omega}_2, \quad \tilde{a}_{im} = \tilde{a}_{1i} - 3\tilde{a}_{2i} \quad \text{and} \quad \tilde{b}_{im} = \tilde{b}_{1i} - 3\tilde{b}_{2i}. \tag{2.30}$$

This simplifies the phase difference equation to

$$\dot{\tilde{\varphi}} = \nu + \tilde{a}_{im}\tilde{X}^2 + \tilde{b}_{im}\tilde{Y}^2 + |\tilde{c}_1|\tilde{Y}^3\frac{1}{\tilde{X}}\sin(\Phi_1 - \tilde{\varphi}) - 3|\tilde{c}_2|\tilde{X}\tilde{Y}\sin(\Phi_2 + \tilde{\varphi}). \tag{2.31}$$

This is the time evolution of the phase difference, which, as discussed above, 'replaces' the phases in (2.29) to give us the 3-dimensional system

$$\dot{\tilde{X}} = \mu_1\tilde{X} + (\tilde{a}_{1r}\tilde{X}^2 + \tilde{b}_{1r}\tilde{Y}^2)\tilde{X} + |\tilde{c}_1|\tilde{Y}^3\cos(\tilde{\varphi} - \Phi_1), \tag{2.32a}$$

$$\dot{\tilde{Y}} = \mu_2\tilde{Y} + (\tilde{a}_{2r}\tilde{X}^2 + \tilde{b}_{2r}\tilde{Y}^2)\tilde{Y} + |\tilde{c}_2|\tilde{X}\tilde{Y}^2\cos(\tilde{\varphi} + \Phi_2), \tag{2.32b}$$

$$\dot{\tilde{\varphi}} = \nu + \tilde{a}_{im}\tilde{X}^2 + \tilde{b}_{im}\tilde{Y}^2 - |\tilde{c}_1|\tilde{Y}^3\frac{1}{\tilde{X}}\sin(\tilde{\varphi} - \Phi_1) - 3|\tilde{c}_2|\tilde{X}\tilde{Y}\sin(\tilde{\varphi} + \Phi_2). \tag{2.32c}$$

We can simplify this system one more time by eliminating the $|c_i|$ coefficients via a scaling of the amplitudes and offsetting the phase difference. Let

$$\begin{cases} \Phi = \Phi_1 + \Phi_2, \qquad \varphi = \tilde{\varphi} - \Phi_1, \\ X = \dfrac{|\tilde{c}_2|}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{4}}}\tilde{X}, \quad Y = |\tilde{c}_1\tilde{c}_2|^{\frac{1}{4}}\tilde{Y}. \end{cases} \tag{2.33}$$

This results in

$$\frac{\tilde{Y}^3}{\tilde{X}} = \frac{|\tilde{c}_2|}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{4}}}\frac{1}{|\tilde{c}_1\tilde{c}_2|^{\frac{3}{4}}}\frac{Y^3}{X} = \frac{|\tilde{c}_2|}{|\tilde{c}_1\tilde{c}_2|}\frac{Y^3}{X} = \frac{1}{|\tilde{c}_1|}\frac{Y^3}{X}, \tag{2.34}$$

$$\tilde{X}\tilde{Y} = \frac{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{4}}}{|\tilde{c}_2|}\frac{1}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{4}}}XY = \frac{1}{|\tilde{c}_2|}XY. \tag{2.35}$$

If we substitute these in the system (2.32), we obtain the simplest form of the system (2.32).

$$\dot{X} = \mu_1 X + (a_{1r}X^2 + b_{1r}Y^2)X + Y^3\cos(\varphi), \tag{2.36a}$$

$$\dot{Y} = \mu_2 Y + (a_{2r}X^2 + b_{2r}Y^2)Y + XY^2\cos(\varphi + \Phi), \tag{2.36b}$$

$$\dot{\varphi} = \nu + a_{im}X^2 + b_{im}Y^2 - \tfrac{Y^3}{X}\sin(\varphi) - 3XY\sin(\varphi + \Phi), \tag{2.36c}$$

where

$$\begin{cases} a_{1r} = \dfrac{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}{|\tilde{c}_2|^2}\tilde{a}_{1r}, \quad a_{2r} = \dfrac{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}{|\tilde{c}_2|^2}\tilde{a}_{2r}, \quad a_{im} = \dfrac{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}{|\tilde{c}_2|^2}\tilde{a}_{im}, \\ b_{1r} = \dfrac{1}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}\tilde{b}_{1r}, \quad b_{2r} = \dfrac{1}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}\tilde{b}_{2r}, \quad b_{im} = \dfrac{1}{|\tilde{c}_1\tilde{c}_2|^{\frac{1}{2}}}\tilde{b}_{im}. \end{cases} \tag{2.37}$$

2.2 Equilibria of the normal form.

To start, because X and Y both represent amplitudes, the only solutions of interest are those with $X, Y \geq 0$, as negative amplitudes do not exist.The second thing we should deduce from the system (2.36) is that at $Y = 0$, $\dot{Y} = 0$ as every term of (2.36b) contains the Y amplitude. The evolution of the X amplitude (2.36a) when on the X-axis is also much simpler, resulting in

$$\dot{X} = \mu_1 X + a_{1r}X^3. \tag{2.38}$$

We see that when $Y = 0$, the remaining amplitude uncouples from the phase difference and it is possible to have fixed amplitude points, where, regardless of the value of φ, the amplitudes remain fixed. The distinction between true equilibrium points and these fixed amplitude points is that, in the fixed amplitude points the phase difference might still be able to change, while in the true fixed points the phase difference is fixed as well.

As the evolution of the phase difference is an important part of the discussion, we introduce two types of fixed amplitude points.

The first being those which satisfy $Y = 0$ and $X \neq 0$, which we refer to as the pure modes. These pure modes usually have a running phase φ_1 associated with the X amplitude. However, to study the solutions near this pure mode that are nearly in resonance we should consider the phase difference φ of the pure mode. This is slightly difficult as the second phase φ_2 is not defined on the pure mode, because a zero amplitude oscillation has little meaning. Consequently the phase difference φ is also not defined, as it uses the φ_2 phase in its definition.

Therefore we use the limit of the derivative of the phase difference $\dot{\varphi}$, to study how the phase difference evolves in the neighborhood of the pure mode. Depending on the limit of the time derivative phase difference φ, we classify these pure modes in two sub-categories.

- Pure modes, where the limit implies a fixed phase difference φ, we call phase locked pure modes.
- Pure modes, where the limit implies a monotonically increasing or decreasing phase difference φ, we call running phase difference pure modes.

The second fixed amplitude points are those that satisfy $Y \neq 0$ and $X \neq 0$, which we call the mixed modes. The mixed modes, due to the coupled amplitude and phase difference, only occur when the phase difference φ remains constant. Otherwise the cosine term in the derivatives of the amplitudes, (2.36a) and (2.36b), would still vary and make a fixed point impossible. Therefore all mixed modes must have a constant phase difference, which we refer to as phase locked.

In addition to the pure mode and the mixed mode, the system also has a trivial fixed amplitude point at the origin of the normal form system (2.36), $(X, Y) = (0, 0)$.

We start the study of the normal form by discussing the stability and eigenvalues of these points.

Trivial point.

The first fixed amplitude point is at $(X, Y) = (0, 0)$, which are the values around which the double Hopf bifurcation takes place. The linearization of (2.36a) and (2.36b) to this point has the eigenvalues

$$\lambda_1 = \mu_1 \quad \text{and} \quad \lambda_2 = \mu_2 \tag{2.39}$$

The third eigenvalue for the phase difference is not defined, as, when both amplitudes are zero, no phases are defined and in extension no phase difference.

We can however, just as with the pure modes, study the phase difference evolution of the surrounding neighborhood by calculating the limit

$$\lim_{(X,Y)\to(0,0)} \dot{\varphi} = \nu. \tag{2.40}$$

When ν is zero, we see that solutions passing very close to the origin are almost phase locked. And when $\nu \neq 0$, we see that those solutions have a running phase difference.

Pure modes.

The pure modes are situated at the amplitudes $(X_0, 0)$, with X_0 defined by $\mu_1 = -a_{1r}X_0^2$ and μ_1 chosen freely within certain bounds. The property that $Y = 0$ simplifies the equations (2.36a) and (2.36b) to

$$\dot{X} = (\mu_1 + a_{1r}X_0^2)X_0, \tag{2.41a}$$

$$\dot{Y} = 0. \tag{2.41b}$$

Firstly we see that the amplitudes are fixed as $\dot{Y} = 0$ and via the definition of X_0, $\dot{X} = 0$ at $(X, Y) = (X_0, 0)$, regardless of the value of the phase difference φ. The phase difference in the neighborhood of the pure mode is given by the limit

$$\lim_{(X,Y)\to(X_0,0)} \dot{\varphi} = \nu + a_{im}X_0^2. \tag{2.42}$$

This limit is, just as at the trivial point, decoupled from variable φ and only dependent on the constant $\nu + a_{im}X_0^2$.

When this constant is not zero, the phase difference φ monotonically increases or decreases near the pure mode, while the amplitudes in this neighborhood remain largely fixed. This results in a running phase difference region around the pure mode, and as such these pure modes will be referred to as running phase difference pure modes.

In the special case where ν equals $\nu_c = \dfrac{\mu_1 a_{im}}{a_{1r}} = -a_{im}X_0^2$, the phase difference limit, (2.42), is zero. This results in a arbitrarily small phase difference evolution in the neighborhood of the pure mode, regardless of the starting phase difference. In addition, the amplitudes in this neighborhood have an arbitrarily small evolution as well. Therefore, for this particular parameter value, the pure mode is phase locked pure mode for every starting phase difference $\varphi \in S^1$.

As the phase difference is not defined at the pure mode, we only derive the eigenvalues of the amplitudes X and Y. We linearize the amplitudes of the original system (2.36a) and (2.36b) at $(X, Y) = (X_0, 0)$ to obtain

$$\dot{X} = \mu_1 X_0 + \mu_1 X + a_{1r}X_0^3 + 3a_{1r}X_0^2 X, \tag{2.43a}$$

$$\dot{Y} = \mu_2 Y + a_{2r}X_0^2 Y. \tag{2.43b}$$

If we substitute the values of X_0 and the parameter value $\mu_1 = -a_{1r}X_0^2$ then (2.43) simplifies to

$$\dot{X} = -2\mu_1 X, \tag{2.44a}$$

$$\dot{Y} = (\mu_2 + a_{2r}X_0^2)Y. \tag{2.44b}$$

Conveniently this linearization is diagonal, which allows us to easily read the eigenvalues

$$\lambda_1 = -2\mu_1 \quad \text{and} \quad \lambda_2 = \mu_2 + a_{2r}X_0^2. \tag{2.45}$$

All eigenvalues are real and if we set $\mu_2 = \mu_{2c} = -a_{2r}X_0^2$ then we see that the eigenvalue corresponding to the Y amplitude, λ_2, equals 0. This implies a bifurcation that splits a solution off from the pure mode.

If, in addition to this particular μ_2 parameter, we also set ν equal to ν_c then this bifurcation occurs when the neighborhood of the pure mode is arbitrarily close to phase locked. This implies the bifurcation on the phase locked mixed mode.

Therefore at these special parameter values, $(\nu, \mu_1, \mu_2) = (\nu_c, -a_{1r}X_0^2, \mu_{2c})$, the eigenvalues of the pure mode and the limit of the phase difference imply a rather interesting situation. In the rest of the thesis we study this degenerate point at amplitudes $(X, Y) = (X_0, 0)$ and the parameters $(\nu, \mu) = (\nu_c, \mu_c)$. The bifurcation occurs regardless of the phase difference, so we do not specify a particular value and study the entire space $\varphi \in S^1$.

Mixed modes.

The mixed modes are given by the equations

$$(X_0, Y_0, \varphi_0) : \begin{cases} \mu_1 X_0 + (a_{1r}X_0^2 + b_{1r}Y_0^2)X_0 + Y_0^3\cos(\varphi_0) & = 0, \\ \mu_2 + a_{2r}X_0^2 + b_{2r}Y_0^2 + X_0Y_0\cos(\varphi_0 + \Phi) & = 0, \\ \nu + a_{im}X_0^2 + b_{im}Y_0^2 - \frac{Y_0^3}{X_0}\sin(\varphi_0) - 3X_0Y_0\sin(\varphi_0 + \Phi) & = 0. \end{cases} \tag{2.46}$$

The eigenvalues of these points can be calculated by the linearization of the system about the point (X_0, Y_0, φ_0). This however leads to a complicated 3rd order characteristic equation

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0. \tag{2.47}$$

The coefficients A, B and C are given by complicated expressions, which are not further discussed in this thesis. It is possible, despite being difficult, to show that these mixed modes undergo bifurcations via the calculation of the eigenvalues.

However, while this indeed determines the local dynamics around these points, which is worthwhile by itself, it is more advantageous to consider certain points in the parameter space, where multiple bifurcations occur at the same parameter values. At these points the bifurcations combine to form one bifurcation of a higher co-dimension. The study of such a bifurcation gives much more insight how the constituent bifurcation interact with each other.

The small changes of ν around ν_c result in a significant change in the fixed points of the phase difference and can therefore be interpreted as bifurcation. Althrough this does not imply that the changes around this point in the phase difference correspond to actual differences in the dynamics of the normal form, for now we treat this small change as a bifurcation.

Therefore in the following sections we study the simplest of such higher co-dimension bifurcations. The double bifurcation of the pure mode at the degenerate point discussed above, where a cycle or mixed mode splits off from the X-axis.

Preliminaries.

The dynamics of the system near the degenerate bifurcation on the pure mode can be studied in detail. This bifurcation occurs on $(X_0, 0, \varphi)$ when the parameters are given by

$$\nu_c = -a_{im}X_0^2 \quad \text{and} \quad \mu_{2c} = -a_{2r}X_0^2. \tag{3.1}$$

The φ can take any value, but since it is a periodic variable, it is convenient to normalize to a value within $[0, 2\pi]$. Also important to realize is that both of the critical values depend via X_0 on μ_1. As X_0 must be non-zero to define a pure mode, μ_1 should be non-zero and chosen with a sign opposite to a_{1r}. Also we require X_0 to be relatively small, as otherwise all higher order terms of X must have been included in the normal form due to their non-negligible impact on the dynamics. Therefore μ_1 must be small as well and lastly μ_1 remains fixed in the rest of the analysis.

3.1 Localized system.

To study the bifurcation, we focus only on X and Y values near $(X_0, 0)$ and the parameters near (ν_c, μ_{2c}). The localization near $(X_0, 0)$ allows us to truncate the higher order terms of the normal form. Near $(X_0, 0)$ the truncated localized system displays the same dynamics as the normal form, outside this neighborhood the dynamics of the normal form can significantly differ to those of the truncated system. This is beneficial as the truncated system is much simpler than the original normal form.

The focus on (ν_c, μ_{2c}) prevents the mixed mode, which we discover near the degenerate point, to leave the neighborhood of $(X_0, 0)$. If the parameters differ too much, the truncated localized system would have to include the higher order terms to fully capture the dynamics around the mixed mode. This is beyond the scope of this thesis, as we only study the truncated localized form.

You might have noticed that we do not focus on a particular value φ_0 of φ, this is because the bifurcation may occur at any phase difference φ.

We start the localization with the introduction of a small $\delta > 0$ and define the localized coordinates and parameters

$$\begin{cases} X = X_0 + \delta^2\xi, & \mu_2 = \mu_{2c} + \delta^2\bar{\mu}, \\ Y = \delta\eta, & \nu = \nu_c + \delta\bar{\nu}, \\ \varphi = \psi, & t = \frac{1}{\delta}\tau, \end{cases} \tag{3.2}$$

for the system (2.36). Recall the resonant normal form (2.36) derived in section 2.1,

$$\dot{X} = \mu_1 X + (a_{1r}X^2 + b_{1r}Y^2)X + Y^3\cos(\varphi), \tag{3.3a}$$

$$\dot{Y} = \mu_2 Y + (a_{2r}X^2 + b_{2r}Y^2)Y + XY^2\cos(\varphi + \Phi), \tag{3.3b}$$

$$\dot{\varphi} = \nu + a_{im}X^2 + b_{im}Y^2 - \tfrac{Y^3}{X}\sin(\varphi) - 3XY\sin(\varphi + \Phi). \tag{3.3c}$$

Firstly we apply the localization to the left hand side of (3.3a) and get

$$\dot{X} = \frac{d(X_0 + \delta^2\xi)}{dt} = \delta^2\frac{d\xi}{dt}. \tag{3.4}$$

This gives us the derivative of ξ in the normal time t, however we require the slow time τ derivative, which is from now on denoted by the prime, $\frac{d}{d\tau}\xi = \xi'$. To this end we apply the chain rule

$$\xi' = \frac{d}{d\tau}\xi = \frac{d\xi}{dt}\frac{dt}{d\tau} = \frac{1}{\delta}\frac{d\xi}{dt}. \tag{3.5}$$

Together with (3.4) this leads to $\dot{X} = \delta^3\xi'$, and this can be substituted with the rest of the new coordinates in (3.3a) to yield

$$\delta^3\xi' = \mu_1(X_0 + \delta^2\xi) + (a_{1r}(X_0 + \delta^2\xi)^2 + b_{1r}\delta^2\eta^2)(X_0 + \delta^2\xi) + \delta^3\eta^3\cos(\psi) \tag{3.6}$$

$$= \left(\mu_1 + a_{1r}X_0^2 + 2a_{1r}X_0\delta^2\xi + a_{1r}\delta^4\xi^2 + b_{1r}\delta^2\eta^2\right)(X_0 + \delta^2\xi) + \delta^3\eta^3\cos(\psi), \tag{3.7}$$

$$\delta\xi' = \frac{1}{\delta^2}\left(2a_{1r}X_0\delta^2\xi + a_{1r}\delta^4\xi^2 + b_{1r}\delta^2\eta^2\right)(X_0 + \delta^2\xi) + \delta\eta^3\cos(\psi) \tag{3.8}$$

$$= \left(2a_{1r}X_0\xi + a_{1r}\delta^2\xi^2 + b_{1r}\eta^2\right)(X_0 + \delta^2\xi) + \delta\eta^3\cos(\psi) \tag{3.9}$$

$$= 2a_{1r}X_0^2\xi + b_{1r}X_0\eta^2 + \delta\eta^3\cos(\psi) + O(\delta^2). \tag{3.10}$$

Similarly for (3.3b), we first calculate the slow time derivative using the same methods. This gives us $\dot{Y} = \delta^2\eta'$ and once again substituting this in (3.3b) we obtain

$$\begin{aligned}
\delta^2\eta' &= (\mu_{2c} + \delta^2\bar{\mu})\delta\eta + (a_{2r}(X_0 + \delta^2\xi)^2 + b_{2r}\delta^2\eta^2)\delta\eta + (X_0 + \delta^2\xi)\delta^2\eta^2\cos(\psi + \Phi) && (3.11)\\
&= (\mu_{2c} + \delta^2\bar{\mu} + a_{2r}X_0^2 + 2a_{2r}X_0\delta^2\xi + a_{2r}\delta^4\xi^2)\delta\eta + b_{2r}\delta^3\eta^3 + (X_0 + \delta^2\xi)\delta^2\eta^2\cos(\psi + \Phi), && (3.12)\\
\eta' &= \frac{1}{\delta^2}\left(\delta^2\bar{\mu} + 2a_{2r}X_0\delta^2\xi + a_{2r}\delta^4\xi^2\right)\delta\eta + b_{2r}\delta\eta^3 + (X_0 + \delta^2\xi)\eta^2\cos(\psi + \Phi) && (3.13)\\
&= \left(\bar{\mu} + 2a_{2r}X_0\xi + a_{2r}\delta^2\xi^2\right)\delta\eta + b_{2r}\delta\eta^3 + (X_0 + \delta^2\xi)\eta^2\cos(\psi + \Phi) && (3.14)\\
&= \left(\bar{\mu} + 2a_{2r}X_0\xi\right)\delta\eta + b_{2r}\delta\eta^3 + X_0\eta^2\cos(\psi + \Phi) + O(\delta^2). && (3.15)
\end{aligned}$$

We apply the same process to the last equation (3.3c)

$$\begin{aligned}
\delta\psi' &= \nu_c + \delta\bar{\nu} + a_{im}(X_0 + \delta^2\xi)^2 + b_{im}\delta^2\eta^2 - \frac{\delta^3\eta^3}{X_0 + \delta^2\xi}\sin(\psi) - 3(X_0 + \delta^2\xi)\delta\eta\sin(\psi + \Phi), && (3.16)\\
&= \delta\bar{\nu} + 2a_{im}X_0\delta^2\xi + a_{im}\delta^4\xi^2 + b_{im}\delta^2\eta^2 - \frac{\delta^3\eta^3}{X_0 + \delta^2\xi}\sin(\psi) - 3(X_0 + \delta^2\xi)\delta\eta\sin(\psi + \Phi) && (3.17)\\
\psi' &= \bar{\nu} + 2a_{im}X_0\delta\xi + a_{im}\delta^3\xi^2 + b_{im}\delta\eta^2 - \frac{\delta^2\eta^3}{X_0 + \delta^2\xi}\sin(\psi) - 3(X_0 + \delta^2\xi)\eta\sin(\psi + \Phi) && (3.18)\\
&= \bar{\nu} + 2a_{im}X_0\delta\xi + b_{im}\delta\eta^2 - 3X_0\eta\sin(\psi + \Phi) + O(\delta^2). && (3.19)
\end{aligned}$$

After the transformation we have the new system

$$\begin{aligned}
\delta\xi' &= 2a_{1r}X_0^2\xi + b_{1r}X_0\eta^2 + \delta\eta^3\cos(\psi) + O(\delta^2), && (3.20a)\\
\eta' &= (\bar{\mu} + 2a_{2r}X_0\xi)\delta\eta + X_0\eta^2\cos(\psi + \Phi) + b_{2r}\delta\eta^3 + O(\delta^2), && (3.20b)\\
\psi' &= \bar{\nu} + 2a_{im}X_0\delta\xi + b_{im}\delta\eta^2 - 3X_0\eta\sin(\psi + \Phi) + O(\delta^2). && (3.20c)
\end{aligned}$$

It is possible to reduce this system more, by eliminating the ξ variable from (3.20b) and (3.20c). We achieve this by introducing a yet to determine function f, such that δf is equal to the leading part of $\delta\xi'$. This allows us to give ξ as an expression of f and by some clever derivation the system (3.20) allows us to find the leading order terms of f. Lastly we are able to use the leading term of f to eliminate f from the equation for ξ and this gives us a value for ξ in terms of η and ψ.

We equate δf to the leading part of (3.20a) and obtain

$$2a_{1r}X_0^2\xi + b_{1r}X_0\eta^2 = \delta f(\delta, \xi, \eta, \psi), \tag{3.21}$$

where f is O(1), therefore we can isolate the ξ, using $\mu_1 = -a_{1r}X_0^2$, which gives us

$$\xi = \frac{1}{2\mu_1}(b_{1r}X_0\eta^2 - \delta f(\delta, \xi, \eta, \psi)). \tag{3.22}$$

If we multiply this equation for ξ by δ and take the derivative, we can equate it to (3.20a), where we have substituted (3.21), resulting in

$$\frac{\delta}{\mu_1}b_{1r}X_0\eta\eta' = \delta f(\delta, \xi, \eta, \psi) + \delta\eta^3\cos(\psi) + O(\delta^2), \tag{3.23}$$

and after canceling the common factor δ, we have found an equation

$$f(\delta, \xi, \eta, \psi) = \frac{1}{\mu_1}b_{1r}X_0\eta\eta' - \eta^3\cos(\psi) + O(\delta) \tag{3.24}$$

for f. With the equation for f, we can eliminate the function from our formula (3.22) for ξ, after which we obtain

$$\xi = \frac{1}{2\mu_1}\left(b_{1r}X_0\eta^2 - \frac{\delta}{\mu_1}b_{1r}X_0\eta\eta' + \delta\eta^3\cos(\psi)\right) + O(\delta^2). \tag{3.25}$$

This result allows us to cancel the ξ terms in both η' and ψ'.

We see that the equation (3.25) still contains an η' in one of its terms. We could substitute this for (3.20b) and further continue the derivation, but all resulting terms of such a substitution would be $O(\delta)$ in the equation (3.25) for ξ. This is due to the δ in the term containing the derivative η'. At the same time all terms containing ξ in (3.20b) and (3.20c) are already $O(\delta)$. Therefore all of those terms would have fallen under $O(\delta^2)$ and no explicit calculation is required.

We start with the elimination of ξ in η' (3.20b)

$$\eta' = (\bar{\mu} + 2a_{2r}X_0\xi)\,\delta\eta + X_0\eta^2\cos(\psi+\Phi) + b_{2r}\delta\eta^3 + O(\delta^2) \tag{3.26}$$

$$= \delta\bar{\mu}\eta + X_0\eta^2\cos(\psi+\Phi) + \left(b_{2r} + \frac{a_{2r}}{\mu_1}X_0^2 b_{1r}\right)\delta\eta^3 + O(\delta^2). \tag{3.27}$$

We continue and perform the cancellation of ξ in the ψ' equation (3.20c)

$$\psi' = \bar{\nu} + 2a_{im}X_0\delta\xi + b_{im}\delta\eta^2 - 3X_0\eta\sin(\psi+\Phi) + O(\delta^2) \tag{3.28}$$

$$= \bar{\nu} + 2a_{im}X_0\delta\left(\frac{1}{2\mu_1}(b_{1r}X_0\eta^2 - \delta f(\tau,\delta))\right) + b_{im}\delta\eta^2 - 3X_0\eta\sin(\psi+\Phi) + O(\delta^2) \tag{3.29}$$

$$= \bar{\nu} + \frac{a_{im}b_{1r}}{\mu_1}X_0^2\delta\eta^2 + b_{im}\delta\eta^2 - 3X_0\eta\sin(\psi+\Phi) + O(\delta^2) \tag{3.30}$$

$$= \bar{\nu} - 3X_0\eta\sin(\psi+\Phi) + \left(b_{im} + \frac{a_{im}b_{1r}}{\mu_1}X_0^2\right)\delta\eta^2 + O(\delta^2). \tag{3.31}$$

After the cancellation of the ξ in all terms we have the following two dimensional system, which are the desired equations describing the dynamics near the double zero bifurcation.

$$\eta' = \delta\bar{\mu}\eta + X_0\eta^2\cos(\psi+\Phi) + \left(b_{2r} + \frac{a_{2r}b_{1r}}{\mu_1}X_0^2\right)\delta\eta^3 + O(\delta^2), \tag{3.32a}$$

$$\psi' = \bar{\nu} - 3X_0\eta\sin(\psi+\Phi) + \left(b_{im} + \frac{a_{im}b_{1r}}{\mu_1}X_0^2\right)\delta\eta^2 + O(\delta^2). \tag{3.32b}$$

We can still further simplify the system, firstly by eliminating the X_0 parameter via a transformation of the variable η and secondly by introducing the constants K_1 and K_2. Let

$$\begin{cases} y = X_0\eta, & K_1 = \dfrac{b_{2r}}{X_0^2} + \dfrac{a_{1r}b_{1r}}{\mu_1} = \left(b_{im} + \frac{a_{im}b_{1r}}{\mu_1}X_0^2\right)\frac{1}{X_0^2}, \\ x = \psi, & K_2 = \dfrac{b_{im}}{X_0^2} + \dfrac{a_{im}b_{1r}}{\mu_1} = \left(b_{2r} + \frac{a_{1r}b_{1r}}{\mu_1}X_0^2\right)\frac{1}{X_0^2}. \end{cases} \tag{3.33}$$

With these new variables we obtain (3.32) in the form

$$y' = \delta\bar{\mu}y + y^2\cos(\Phi+x) + \delta K_2 y^3 + O(\delta^2), \tag{3.34a}$$

$$x' = \bar{\nu} - 3y\sin(\Phi+x) + \delta K_1 y^2 + O(\delta^2). \tag{3.34b}$$

Lastly we truncate the $O(\delta^2)$ term, as within the non-degenerate cases, its effects are negligible for the dynamics near the x-axis. In addition we introduce four new coefficients to help with notation. We obtain

$$x' = \alpha - 3y\sin(\Phi+x) + K_x y^2, \tag{3.35a}$$

$$y' = \beta y + y^2\cos(\Phi+x) + K_y y^3, \tag{3.35b}$$

where

$$\begin{cases} \alpha = \bar{\nu}, & K_x = \delta K_1, \\ \beta = \delta\bar{\mu}, & K_y = \delta K_2. \end{cases} \tag{3.36}$$

This is the form we use to study the solutions of the localized system.

3.2 Equilibria of the localized system.

Within the localized system around $(X_0, 0)$, we maintain the classification of the pure modes and mixed modes for the equilibria on the x-axis and the equilibria with $y_0 > 0$, respectively.

The pure modes correspond with the equilibria on the x-axis, as the $Y = 0$ condition in the original normal form (2.36) corresponds to $y = 0$ in (3.35). This is because y is the localization of the Y amplitude in (3.35). Consequently it is no surprise that $y' = 0$ for all point on the x-axis.

As discussed in section 2.2, pure modes may or may not have a neighborhood with a arbitrarily small phase difference. This means that near the x-axis in the localized system, the localized phase difference x may or may not be constant. Therefore we make the same distinction between the phase locked and running phase difference cases as in section 2.2.

- If x is monotonically increasing or decreasing near the x-axis, we refer to the entire axis as a single running phase difference pure mode.
- If some point $(x, 0)$ on the x-axis has small neighborhood with a fixed x, we refer to these points as a phase locked pure mode. Notice that when $\alpha = 0$, the entirety of the x-axis consists of an infinite amount of phase locked pure modes.

This gives us the localization of the pure modes of the normal form (2.36) in the localized system (3.35).

The mixed modes are represented by the regular equilibrium points above the x-axis in (3.35). The $Y \neq 0$ and $\dot{Y} = 0$ conditions in the normal from system, are equivalent to $y \neq 0$ and $\dot{y} = 0$ in the local system due to the correspondence between y and Y. Furthermore in section 2.2 it was determined that the mixed modes are necessarily phase locked, i.e. $\dot{\varphi} = 0$. This corresponds to $x' = 0$ in (3.35), which together with the condition $y' = 0$ implies that the mixed mode is an equilibrium in (3.35). The last condition $y \neq 0$ finishes the correspondence, the mixed modes correspond to equilibria above the x-axis in (3.35).

3.3 Mixed mode near $(X_0, 0)$.

Important for the analysis of (3.35) is the existence of a mixed mode that occurs near the pure mode in (3.35). We prove the existence of this point in the following derivations.

We start by calculating the fixed phase difference when given a certain y-value. The equilibrium implies that the derivative of the phase difference, (3.35a), must be equal to 0, therefore we can set

$$\alpha - 3y\sin(\Phi + x) + K_x y^2 = 0, \tag{3.37}$$

$$3y\sin(\Phi + x) = \alpha + K_x y^2, \tag{3.38}$$

$$\sin(\Phi + x) = \frac{\alpha + K_x y^2}{3y}. \tag{3.39}$$

This results in

$$\Phi + x = \arcsin\left(\frac{\alpha + K_x y^2}{3y}\right). \tag{3.40}$$

For convenience we maintain the phase shift Φ, as we need the value of $\Phi + x$ for the next equation. The arcsin is bounded to $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, therefore π might need to be added to obtain the correct $\Phi + x$. However this difference does not matter for the calculation of y. Now from the definition of a fixed point we know that the slow time derivative of y, (3.35b), also must be equal to 0. And after substituting (3.40), our equation for $\Phi + x$, in y', we obtain the equation

$$\beta + y\cos\left(\arcsin\left(\frac{\alpha + K_x y^2}{3y}\right)\right) + K_y y^2 = 0, \tag{3.41}$$

that gives us the correct $y_0 \neq 0$. We can apply the trigonometric identity $\cos(\arcsin(x)) = \sqrt{1 - x^2}$ here, to obtain

$$\beta + y\sqrt{1 - \left(\frac{\alpha + K_x y^2}{3y}\right)^2} + K_y y^2 = 0. \tag{3.42}$$

And bringing the variable y into the square root yields

$$\beta + \sqrt{y^2 - \frac{1}{9}\left(\alpha + K_x y^2\right)^2} + K_y y^2 = 0, \tag{3.43}$$

$$\beta + K_y y^2 = -\sqrt{y^2 - \frac{1}{9}\left(\alpha + K_x y^2\right)^2}. \tag{3.44}$$

We square to eliminate the root

$$\left(\beta + K_y y^2\right)^2 = y^2 - \frac{1}{9}\left(\alpha + K_x y^2\right)^2. \tag{3.45}$$

After expanding and restructuring the equation, we obtain the quadratic formula

$$\left(K_y^2 + \frac{1}{9}K_x^2\right)y^4 + \left(2\beta K_y + \frac{2}{9}\alpha K_x - 1\right)y^2 + \left(\beta^2 + \frac{1}{9}\alpha^2\right) = 0 \tag{3.46}$$

for y^2. To solve this equation we can simply apply the abc-formula, which gives us the discriminant

$$D_1 = \left(2\beta K_y + \frac{2}{9}\alpha K_x - 1\right)^2 - 4\cdot\left(K_y^2 + \frac{1}{9}K_x^2\right)\cdot\left(\beta^2 + \frac{1}{9}\alpha^2\right) \tag{3.47}$$

and the formula

$$y_0^2 = \frac{1 - 2\beta K_y - \frac{2}{9}\alpha K_x \pm \sqrt{D_1}}{2\left(K_y^2 + \frac{1}{9}K_x^2\right)} \tag{3.48}$$

for y_0^2. The fact that we obtain the square of y_0 does not matter, as the interpretation of y as an amplitude in our analysis, means we are only interested in positive real y amplitudes. Therefore the y_0 can be assumed to be positive in (3.49). The real positivity of y_0 also has the additional effect that complex solutions of y_0 that are the result of $D_1 < 0$, do not matter.

We can obtain the corresponding x_0 from y_0^2, via the earlier formula (3.40), which results in

$$x_0 = \arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right) - \Phi. \tag{3.49}$$

We might need to add π to this x_0, due to the bounded range of the arcsin.

These restrictions result in that (3.48) gives us two, one or zero possibilities for y_0, depending on the value of D_1.

Existence of this mixed mode near $y = 0$.

What we first notice is the condition that D_1 must be real positive or zero for the system (3.35) to have any mixed modes.

The second important condition to consider, is if these mixed modes are within the bounds of the localized system. Mixed points that fall far outside the neighborhood of the degenerate point may exhibit secondary bifurcation not captured in our analysis. If the y_0 position of the mixed mode is in the magnitude of $\frac{1}{\delta}$ or higher, the analysis requires the higher order terms of the normal form to be included for a full picture.

We consider the discriminant condition first

$$\left(2\beta K_y + \frac{2}{9}\alpha K_x - 1\right)^2 - 4\cdot\left(K_y^2 + \frac{1}{9}K_x^2\right)\cdot\left(\beta^2 + \frac{1}{9}\alpha^2\right) \geq 0, \tag{3.50}$$

and this only occurs when

$$4\beta^2 K_y^2 + \frac{8}{9}\alpha\beta K_x K_y + \frac{4}{81}\alpha^2 K_x^2 - 4\beta K_y - \frac{4}{9}\alpha K_x + 1 \geq 4\cdot\left(\beta^2 K_y^2 + \frac{1}{9}\beta^2 K_x^2 + \frac{1}{9}\alpha^2 K_y^2 + \frac{1}{81}\alpha^2 K_x^2\right), \tag{3.51}$$

$$\frac{8}{9}\alpha\beta K_x K_y - 4\beta K_y - \frac{4}{9}\alpha K_x + 1 \geq \frac{4}{9}\beta^2 K_x^2 + \frac{4}{9}\alpha^2 K_y^2, \tag{3.52}$$

$$1 \geq 4\beta K_y + \frac{4}{9}\alpha K_x + \frac{4}{9}\beta^2 K_x^2 + \frac{4}{9}\alpha^2 K_y^2 - \frac{8}{9}\alpha\beta K_x K_y. \tag{3.53}$$

Notice that the right hand side is $O(\delta)$. Therefore, as long as the parameters are sufficiently small as to not counteract the $O(\delta)$, which is true for the parameter space used in the analysis, this condition is always satisfied. Consequently two mixed modes exist in the system (3.35) for all considered parameters.

Position of the mixed mode

Now for the second condition, whether the mixed modes are near $y = 0$. We first calculate the discriminant with all terms expanded and sorted according to the implicit order of δ

$$D_1 = \left(2\beta K_y + \frac{2}{9}\alpha K_x - 1\right)^2 - 4 \cdot \left(K_y^2 + \frac{1}{9}K_x^2\right) \cdot \left(\beta^2 + \frac{1}{9}\alpha^2\right) \tag{3.54}$$

$$= 4\beta^2 K_y^2 + \frac{4}{81}\alpha^2 K_x^2 + 1 + \frac{8}{9}\alpha\beta K_x K_y - \frac{4}{9}\alpha K_x - 4\beta K_y \tag{3.55}$$

$$- 4\beta^2 K_y^2 - \frac{4}{9}\beta^2 K_x^2 - \frac{4}{9}\alpha^2 K_y^2 - \frac{4}{81}\alpha^2 K_x^2$$

$$= 1 + \frac{8}{9}\alpha\beta K_x K_y - \frac{4}{9}\alpha K_x - 4\beta K_y - \frac{4}{9}\beta^2 K_x^2 - \frac{4}{9}\alpha^2 K_y^2. \tag{3.56}$$

We collect the higher order terms of δ within a $O(\delta^2)$ term to obtain

$$D_1 = 1 - \frac{4}{9}\alpha K_x + O(\delta^2). \tag{3.57}$$

Now let us calculate the magnitude of y_0 using this discriminant

$$y_0^2 = \frac{1 - 2\beta K_y - \frac{2}{9}\alpha K_x \pm \sqrt{D_1}}{2\left(K_y^2 + \frac{1}{9}K_x^2\right)}, \tag{3.58}$$

$$y_0 = \frac{\sqrt{1 - 2\beta K_y - \frac{2}{9}\alpha K_x \pm \sqrt{D_1}}}{\sqrt{2\left(K_y^2 + \frac{1}{9}K_x^2\right)}}. \tag{3.59}$$

Notice that the denominator is $O(\delta)$. If y_0 is to be of lower magnitude than $\frac{1}{\delta}$, the numerator in (3.59) should be of order $O(\delta)$ and this in turn requires $1 - \frac{2}{9}\alpha K_x \pm \sqrt{D_1}$ to be $O(\delta^2)$.

We can prove that this is only the case when $\sqrt{D_1}$ is subtracted. We use the Taylor expansion of the square root of $1 + x$, which is given by

$$\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3), \tag{3.60}$$

to estimate the magnitude of $\sqrt{D_1}$. This gives us

$$\sqrt{D_1} = \sqrt{1 + \left(-\frac{4}{9}\alpha K_x + O(\delta^2)\right)} = 1 - \frac{2}{9}\alpha K_x + O(\delta^2). \tag{3.61}$$

If we substitute this in the numerator of (3.59) we obtain

$$1 - \frac{2}{9}\alpha K_x \pm \sqrt{D_1} = 1 - \frac{2}{9}\alpha K_x \pm \left(1 - \frac{2}{9}\delta\alpha K_x\right) + O(\delta^2). \tag{3.62}$$

Here we see that the discriminant must be subtracted to counteract the constant and the δ term in the numerator. We notice that this subtraction results in a value y_0 of at most order 1 of δ for all parameters within the considered parameter space and all constants, therefore the lower mixed mode is always near the x-axis.

However the higher mixed mode, where the discriminant is added, has a constant, i.e. $O(\delta^0)$, in the numerator of (3.59). This results in an y_0 value in the magnitude of $\frac{1}{\delta}$ for all parameters and, as stated before, requires the higher terms of the normal form to be included in the analysis and is therefore not considered any further.

3.4 Transition to the non-resonant case.

We briefly discuss what happens in the truncated localized system (3.35) when $\alpha \to \infty$.

Firstly we see that the discriminant D_1 becomes zero and eventually negative when α reaches.

$$D_1 = 1 - \frac{4}{9}\alpha K_x + O(\delta^2). \tag{3.63}$$

This corresponds to a saddle-node bifurcation between the two mixed modes. This is supported by the eigenvalues at this point, whose calculation we omit.

However this bifurcation occurs so far from x-axis that there is no certainty that the same bifurcation is present in the dynamics of the non-localized normal form (2.36). Therefore this bifurcation can only be hypothesized to occur in the normal form system and consequently is not considered any further in this thesis.

Analysis of localized system.

In this chapter we start the analysis of the dynamics near the pure mode by determining the stability and bifurcations of the pure mode(s). Afterwards we consider the stability of the nearby mixed mode. We achieve this by considering the localized system calculated in section 3.1, which is given by

$$x' = \alpha - 3y\sin(\Phi + x) + K_x y^2, \tag{4.1a}$$

$$y' = \beta y + y^2\cos(\Phi + x) + K_y y^3. \tag{4.1b}$$

Recall from section 3.2 that the considered range of the phase space of this localized system is given by $(x, y) \in S^1 \times \mathbb{R}^+$, where $\mathbb{R}^+ := \{y \in \mathbb{R} | y \geq 0\}$.

The x variable is periodic in nature, because it represents the phase difference between the phases of the normal form (2.36). As both phases are in S^1, their difference, given by x, is in S^1 too. The y variable is restricted to the positive or zero real number due to its correspondence with the amplitude Y in the normal form.

Also recall from section 3.2 that the pure mode(s) are located in the manifold $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^+$. This subspace is referred to as the x-axis in the rest of the analysis and the points $(x, 0) \in S^1 \times \{0\}$ are referred to as points on the x-axis.

4.1 Stability of the pure modes.

The stability of the pure modes is easily calculated, as the localized system (4.1) at the points on the x-axis, is given by

$$x' = \alpha - 3y\sin(\Phi + x + x_0) + K_x y^2, \tag{4.2a}$$

$$y' = \beta y + y^2\cos(\Phi + x + x_0) + K_y y^3. \tag{4.2b}$$

We can instantly derive the linearization around $(x, y) = (x_0, 0)$ of these equations which gives us

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.3}$$

which in turn gives us the eigenvalues

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \beta. \tag{4.4}$$

The eigenvalue λ_2 has the eigenvector $(0, 1)$, this implies that it determines the stability of all points on the x-axis in the y direction.

- If $\beta < 0$ then solutions near the x-axis, which refers to solutions within $S^1 \times (0, \epsilon)$ with a non-zero real and suitably small ε, converge to $y = 0$.
- If $\beta > 0$ then the x-axis is unstable, solutions near the x-axis may converge to a bifurcated running phase difference solution. Or when this cycle does not exit, the nearby solutions increase in y until they leave the neighborhood of the x-axis.

More care should be taken when discussing the $(1, 0)$ directional stability of the pure modes; recall from section 3.3 that the phase difference x is not defined at the pure mode. However we can use the limit of the phase difference when approaching $y = 0$ to define the x variable at $y = 0$. Conveniently, due to the continuity of the limit, this definition allows us to treat the x values at $y = 0$ as normal variables. This slight difference does not impact the dynamics of the localized system much, but is does affect the interpretation of the bifurcations of the localized system.

- If $\alpha = 0$ all points on the x-axis are equilibria and no motion takes place along the x coordinate. This confirms the $\lambda_1 = 0$ result.
- If $\alpha \neq 0$ then the pure mode has a running phase difference, as every point on the x-axis satisfies $x' = \alpha$. This implies that the pure mode is a single periodic orbit.

The pure modes do in fact experience a bifurcation at $\beta = 0$. Here we must distinguish the phase locked pure modes from the running phase difference pure modes, as they behave quite differently when β approaches zero.

Running phase difference pure modes.

We first cover the cases when the pure mode has a running phase difference. We consider the points near the x-axis, these points are defined by $(x, y) = (x_0, \varepsilon)$ with $x_0 \in S^1$ and ε positive, real and in the neighborhood of 0. This definition is used for the rest of the analysis. On these points the derivative x' is given by

$$x' = \alpha - 3\varepsilon \sin(\Phi + x_0) + K_x \varepsilon^2 = \alpha + O(\varepsilon). \tag{4.5}$$

When $\alpha \neq 0$, the value of $O(\varepsilon)$ is negligible compared to the parameter term α, due to the smallness of ε. Therefore on the points near the x-axis we can approximate the derivative x' by $\alpha + O(\varepsilon) \approx \alpha$. When starting at some point (x_0, ε) near the x-axis, this approximation leads to

$$x(\tau) \approx \alpha\tau + x_0. \tag{4.6}$$

Substituting this approximation in y' gives us

$$y' = \beta y + y^2 \cos(\Phi + \alpha\tau + x_0) + K_y y^3. \tag{4.7}$$

Now when near the pure mode, and as such the x-axis, the derivative y' itself is very small, due to the y in each term. From this it follows that any solution starting near the pure mode will stay there for a long time and by starting even closer to $y = 0$ we can arbitrarily extend the time spent near $y = 0$. Therefore the term $\cos(\Phi + \alpha\tau + x_0)$ oscillates on a time scale much faster than y' changes. This reduces the term to its median, which equals zero. We conclude that the evolution of y' near the pure mode is determined by

$$y' = \beta y + K_y y^3. \tag{4.8}$$

Recall from the introduction that this is in fact the normal form of a pitchfork bifurcation, where the case is determined by the sign of K_y.

However, due to the correspondence of x and y to the phase difference and the amplitude in the normal form, this bifurcation in fact corresponds to a Hopf bifurcation in the normal form. The periodicity of the x variable and the non-existence of the phase difference at $y = 0$, together with the coupled oscillation y experiences, support this observation.

It is also apparent that the pitchfork normal form (4.8) breaks when $K_y = 0$. For now we exclude this possibility, as this results in a degenerate localized system, we briefly consider that system in section 5.8.

If $K_y < 0$, we know that the pitchfork bifurcation is super-critical. Therefore when $\beta < 0$ the running phase difference pure mode is stable and no other running phase difference solutions exist. When $\beta > 0$ we know that two stable solutions have bifurcated off the original pure mode, while the stability of said pure mode has switched from stable to unstable. One of the bifurcated stable solutions has negative y-values; as these do not describe valid amplitudes, this solution is not considered any further. Within the total normal form (2.29), this bifurcation corresponds to the splitting of a stable invariant 2-torus, which contains the quasi-periodic solutions, off the 1-torus corresponding to the pure mode as argued in section 2.2. Lastly this 1-torus switches stability from stable to unstable.

If $K_y > 0$, we know that the pitchfork bifurcation is sub-critical. Therefore when $\beta > 0$ the running phase difference pure mode is unstable and no other running phase difference solutions exist. When $\beta < 0$ we know that two unstable solutions have bifurcated and the pure mode is stable. Again one of the bifurcated solutions is negative and not considered any further. Within the total normal form (2.29), this bifurcation corresponds to the splitting of a unstable invariant 2-torus, which contains the quasi-periodic solutions, off the 1-torus corresponding to the pure mode as argued in section 2.2. Lastly this 1-torus switches stability from unstable to stable.

Phase locked pure modes.

When the pure modes are phase locked the situation is different. Firstly we notice that, due to the eigenvalue λ_2 with the $(0,1)$ eigenvector, the x-axis still switches stability from stable to unstable when β becomes positive. Secondly the derivative x' is zero at all points on the x-axis, but slightly above the x-axis the $-3y\sin(\Phi+x)$ term becomes important. The term causes the derivative x' to be non-zero except at the x values where $-3y\sin(\Phi+x)$ is zero.

At the points on the x-axis, with a neighborhood wherein all points above the x-axis satisfy $x' \neq 0$, the change in stability of the x-axis does not lead to the bifurcation of fixed points. The switch in x-axis stability does cause nearby solutions to reverse their y direction, but unlike the running phase difference case no periodic running phase difference solution bifurcates off the pure mode. This is due to the points near the x-axis values where x' is zero. These points form a curve, which starts perpendicular to the x-axis and which no running phase difference solution near the x-axis can cross, instead these solution either bend toward or bend away from the x-axis.

At the points on the x-axis, with a neighborhood that contains points above the x-axis which satisfy $x' = 0$, equilibria bifurcate from the x-axis.

At all points near the x-axis, see the start of the running phase difference paragraph for the definition of these points, the derivative x' is given by

$$x' = -3\varepsilon\sin(\Phi+x_0) + K_x\varepsilon^2 = -3\varepsilon\sin(\Phi+x_0) + O(\varepsilon^2). \tag{4.9}$$

As ε is very small, the $O(\varepsilon^2)$ term is negligible and can be truncated to obtain the approximation of the derivative x' near the x-axis

$$x' = -3\varepsilon\sin(\Phi+x_0). \tag{4.10}$$

This right hand side of this equation is only zero at the values $\Phi + x_0 = k\pi$, $k \in \mathbb{Z}$. At all other values of x_0, all points in the neighborhood of $(x_0, 0)$ satisfy $x' \neq 0$. As discussed before, the switch of the stability of the x-axis does not lead to bifurcated equilibria in these points, therefore these points at these x_0 values are not considered any further.

We consider the derivative y' at the considered x_0 values

$$y' = \beta y + y^2\cos(\Phi+x_0) + K_y y^3. \tag{4.11}$$

Close enough to the x-axis the y^3-term is negligible, resulting in the evolution

$$y' = \beta y + y^2\cos(\Phi+x_0), \tag{4.12}$$

and at the considered x_0 values the cosine is either 1 or -1. Therefore the derivative y' at the points $(k\pi, y)$, with $k \in \mathbb{Z}$, equals

$$y' = \beta y \pm y^2. \tag{4.13}$$

Recall from the introduction that this is the normal form of a trans-critical bifurcation, where the direction of the bifurcation is determined by the sign of y^2. This normal form implies a second equilibrium at $y = \mp\beta$ in addition to the one at $y = 0$, this second equilibrium has its stability in the $(0,1)$ direction characterized by the eigenvalue $\lambda = -\beta$. Here we exclude the fixed points below the x-axis, that might occur due the trans-critical bifurcation. This is again because the y variable still corresponds to an amplitude and therefore should not be negative.

We can also calculate the eigenvalue of the x variable at the second equilibrium for both signs of the cosine. If we linearize at $\Phi + x_0 = 0 + 2k\pi$, $y_0 = -\beta > 0$ we obtain

$$\begin{aligned} x' &= -3(y_0+y)\sin(\Phi+x_0+x) + O(y^2) && (4.14)\\ &= -3(y_0+y)\left(\sin(\Phi+x_0)\cos(x) + \cos(\Phi+x_0)\sin(x)\right) + O(y^2) && (4.15)\\ &= -3(y_0+y)\sin(x) + O(y^2) && (4.16)\\ &= 3\beta x + O(2). && (4.17)\end{aligned}$$

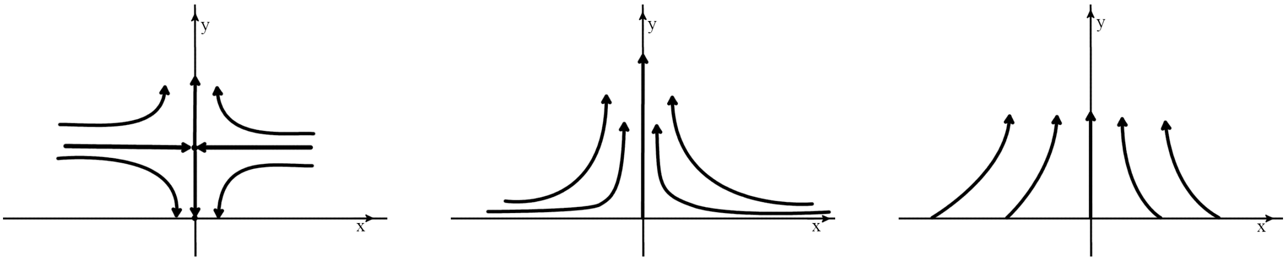
Therefore the eigenvalues of this mixed mode are $\lambda_1 = -\beta$ and $\lambda_2 = 3\beta$, which implies that this mixed mode is a saddle point.

If we linearize at $\Phi + x_0 = \pi + 2k\pi$, $y_0 = \beta > 0$ we obtain

$$x' = -3(y_0 + y)\sin(\Phi + x_0 + x) + O(y^2) \tag{4.18}$$
$$= 3(y_0 + y)\sin(x) + O(y^2) \tag{4.19}$$
$$= 3\beta x + O(2). \tag{4.20}$$

Therefore the eigenvalues of this mixed mode are $\lambda_1 = -\beta$ and $\lambda_2 = 3\beta$, which implies this mixed mode is again a saddle point.

We can summarize these results in phase portraits of the dynamics of the system (4.2) near the points $(x, y) = (k\pi - \Phi, 0)$, $k \in \mathbb{Z}$. The origin in the phase portraits has been set to these points. In all cases the x-axis in these phase portraits consists of equilibria, as both derivatives x' and y' are zero when $y = 0$.

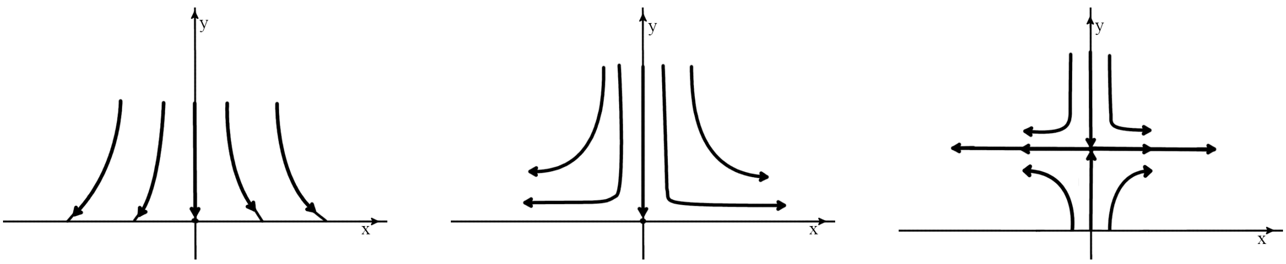


(a) $\beta < 0$ **(b)** $\beta = 0$ **(c)** $\beta > 0$

Figure 4.1: $\Phi + x_0 = 0 + 2k\pi$

Near the phase locked pure modes at $\Phi + x_0 = 0 + 2k\pi$, we see $y' = \beta y + y^2 = y(\beta + y)$. We distinguish three cases depending on the parameter β.

- When $\beta < 0$, the pure mode is stable and the derivative y' is zero at $y = -\beta$, which implies a fixed point near $(x_0, -\beta)$. The eigenvalue calculation determined that this equilibrium is a saddle point.
- When $\beta = 0$, for all points that satisfy $x_0 \neq k\pi$, the derivative y' is tiny compared to the x' derivative. This implies that running phase difference solutions near the x-axis approximately maintain their y-value until they approach one of the considered x_0 values, where $x' = 0$. Only when nearing the line given by $\Phi + x_0 = k\pi$, the y' derivative becomes dominant. In the case $\Phi + x_0 = 0 + 2k\pi$ the running phase difference solutions tend toward the point $(x_0, 0)$ in the x direction and away from the point $(x_0, 0)$ in the y direction, due to the sign of the y^2 term and the linearization of x' at (x_0, ϵ).
- When $\beta > 0$, the pure mode is unstable and the derivative y' is still zero at $y = -\beta$. However due to the sign of β, this now implies a fixed point with a negative y value and therefore the fixed point is not considered further.



(a) $\beta < 0$ **(b)** $\beta = 0$ **(c)** $\beta > 0$

Figure 4.2: $\Phi + x_0 = \pi + 2k\pi$

Near the phase locked pure modes at $\Phi + x_0 = 0 + 2k\pi$, we see $y' = \beta y - y^2 = y(\beta - y)$. We distinguish three cases depending on the parameter β.

- When $\beta < 0$, the pure mode is unstable and the derivative y' is zero at $y = \beta$. Due to the sign of β, this results in a fixed point with a negative y value, which is not considered further.
- When $\beta = 0$, the same argument as in the $\Phi + x_0 = \pi + 2k\pi$ case applies for all the points where $x' \neq 0$. Therefore the running phase difference solutions near the x-axis approximately maintain their y-value until they approach one of the considered x_0 values, where $x' = 0$. When nearing the line given by

$\Phi + x_0 = \pi + k\pi$, the y' derivative becomes dominant. In the case $\Phi + x_0 = 0 + 2k\pi$ the running phase difference solutions tend away from the point $(x_0, 0)$ in the x direction and towards the point $(x_0, 0)$ in the y direction, due to the sign of the y^2 term and the linearization of x' at (x_0, ϵ).

- When $\beta > 0$, the pure mode is unstable and the derivative y' is zero at $y = \beta$, which implies a fixed point near (x_0, β). The eigenvalue calculation determined that this equilibrium is a saddle point.

Let us summarize the situation when β moves from a negative value to a positive one while $\alpha = 0$. We firstly see that the saddle mixed mode, which is unstable in the y direction, collides with the pure mode at the point $(-\Phi, 0)$ at $\beta = 0$. When $\beta > 0$, the mixed mode reappears as a saddle point, which is stable in the y direction, at the point $(\pi - \Phi, 0)$.

Results.

Thus we can confidently state that when β switches sign, all pure modes switch y-stability. This causes, in running phase difference pure modes, a stable or unstable running phase solution to bifurcate depending on the sign of K_2. In phase locked pure modes, the sign switch of β causes saddle node mixed modes to collide and split off the x-axis at the points $(k\pi - \Phi, 0)$, $k \in \mathbb{Z}$.

4.2 Eigenvalues of the mixed mode.

In section 3.3 we had derived the location of the localized phase locked mixed mode. We denote this location as (x_0, y_0), where the x_0 and y_0 values are obtained by (3.49) and (3.48) respectively.

To obtain the eigenvalues that characterize this equilibrium, we linearize the localized system

$$x' = \alpha - 3y\sin(\Phi + x) + K_x y^2, \tag{4.21a}$$
$$y' = \beta y + y^2\cos(\Phi + x) + K_y y^3, \tag{4.21b}$$

at this point and consider the resulting equations. We start with the linearization of (4.21a)

$$x' = \alpha - 3(y + y_0)\sin(\Phi + x + x_0) + K_x(y + y_0)^2. \tag{4.22}$$

Recall that the $\Phi + x_0$ value is given by (3.40), this can be substituted in (4.22) to obtain

$$x' = \alpha - 3(y + y_0)\sin\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right) + x\right) + K_x(y^2 + 2yy_0 + y_0^2). \tag{4.23}$$

We leave the y_0 coordinate as it is, as using the derived equation (3.48) for it would only complicate this linearization. We now use the sinus sum identity to separate the x value from the arcsin in the sinus, this leads us to

$$x' = \alpha - 3(y + y_0)\left(\sin\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right)\right)\cos(x) + \cos\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right)\right)\sin(x)\right) + K_x(y^2 + 2yy_0 + y_0^2). \tag{4.24}$$

And after further derivation we end with

$$x' = \alpha - 3(y + y_0)\left(\frac{\alpha + K_x y_0^2}{3y_0}\cos(x) + \sqrt{1 - \left(\frac{\alpha + K_x y_0^2}{3y_0}\right)^2}\sin(x)\right) + K_x y^2 + 2K_x yy_0 + K_x y_0^2. \tag{4.25}$$

Fortunately it is possible to remove the square root from this equation. For this we need to extract the square root found (3.42), in the derivation of the mixed mode. This gives us

$$\frac{\beta + K_y y_0^2}{y_0} = -\sqrt{1 - \left(\frac{\alpha + K_x y_0^2}{3y_0}\right)^2}, \tag{4.26}$$

which can be used to simplify (4.25) to

$$x' = \alpha - 3(y + y_0)\left(\frac{\alpha + K_x y_0^2}{3y_0}\cos(x) - \frac{3\beta + 3K_y y_0^2}{3y_0}\sin(x)\right) + K_x y^2 + 2K_x yy_0 + K_x y_0^2. \tag{4.27}$$

We replace the $\cos(x)$ and $\sin(x)$ in the equation with their respective Taylor expansion and collect the higher order terms of x and y in $O(2)$. This leads to

$$x' = \alpha - y\frac{\alpha + K_x y_0^2}{y_0} - y_0\left(\frac{\alpha + K_x y_0^2}{y_0} - \frac{3\beta + 3K_y y_0^2}{y_0}x\right) + 2K_x yy_0 + K_x y_0^2 + O(2), \tag{4.28}$$

which after simplification yields

$$x' = \left(2K_x y_0 - \frac{\alpha + K_x y_0^2}{y_0}\right) y + (3\beta + 3K_y y_0^2)x + O(2). \tag{4.29}$$

This is the equation for x' linearized at (x_0, y_0). We continue with the linearization of (4.21b)

$$y' = \beta(y + y_0) + (y + y_0)^2 \cos(\Phi + x + x_0) + K_y(y + y_0)^3. \tag{4.30}$$

We substitute again the $\Phi + x_0$ constant for its equation (3.40) to obtain

$$y' = \beta(y + y_0) + (y + y_0)^2 \cos\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right) + x\right) + K_y(y + y_0)^3. \tag{4.31}$$

This time we use the trigonometric sum identity of the cosine.

$$\begin{aligned} y' &= \beta(y + y_0) + (y + y_0)^2 \cos\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right)\right)\cos(x) \qquad (4.32)\\ &\qquad - (y + y_0)^2 \sin\left(\arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right)\right)\sin(x) + K_y(y + y_0)^3 \\ &= \beta(y + y_0) + (y + y_0)^2\left(\sqrt{1 - \left(\frac{\alpha + K_x y_0^2}{3y_0}\right)^2}\cos(x) - \frac{\alpha + K_x y_0^2}{3y_0}\sin(x)\right) + K_y(y + y_0)^3. \qquad (4.33)\end{aligned}$$

Here the root has reappeared, but we can again use the equality (4.26) to continue to

$$y' = \beta(y + y_0) - (y + y_0)^2\left(\frac{\beta + K_y y_0^2}{y_0}\cos(x) + \frac{\alpha + K_x y_0^2}{3y_0}\sin(x)\right) + K_y(y + y_0)^3. \tag{4.34}$$

And after expanding the cube and square of $(y - y_0)$, we obtain

$$y' = \beta y + \beta y_0 - (y^2 + 2y_0 y + y_0^2)\left(\frac{\beta + K_y y_0^2}{y_0}\cos(x) + \frac{\alpha + K_x y_0^2}{3y_0}\sin(x)\right) + K_y(y^3 + 3y_0 y^2 + 3y_0^2 y + y_0^3). \tag{4.35}$$

We once again replace the $\cos(x)$ and $\sin(x)$ terms with their respective Taylor expansion and collect all of the higher order terms in $O(2)$

$$y' = \beta y + \beta y_0 - 2y\left(\beta + K_y y_0^2\right) - y_0\left(\beta + K_y y_0^2 + \frac{\alpha + K_x y_0^2}{3}x\right) + 3K_y y_0^2 y + K_y y_0^3 + O(2), \tag{4.36}$$

which after simplification yields the equation

$$y' = -\frac{1}{3}(\alpha y_0 + K_x y_0^3)x + \left(K_y y_0^2 - \beta\right) y + O(2). \tag{4.37}$$

for y' linearized at (x_0, y_0). Together with (4.29), we can finally obtain the linearized system in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3\beta + 3K_y y_0^2 & 2K_x y_0 - \dfrac{\alpha + K_x y_0^2}{y_0} \\ -\dfrac{1}{3}(\alpha y_0 + K_x y_0^3) & K_y y_0^2 - \beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.38}$$

The eigenvalues of this matrix characterize the dynamics in the local neighborhood of the mixed mode. As the matrix is 2 by 2 we can easily give the characteristic polynomial

$$c(\lambda) = \left(3\beta + 3K_y y_0^2 - \lambda\right)\left(K_y y_0^2 - \beta - \lambda\right) + \left(2K_x y_0 - \frac{\alpha + K_x y_0^2}{y_0}\right)\left(\frac{1}{3}(\alpha y_0 + K_x y_0^3)\right). \tag{4.39}$$

This polynomial can be simplified to

$$c(\lambda) = \lambda^2 - \left(2\beta + 4K_y y_0^2\right)\lambda - \left(3\beta^2 - 3K_y^2 y_0^4\right) - \frac{1}{3}\left(\alpha^2 - K_x^2 y_0^4\right). \tag{4.40}$$

We are left with another quadratic equation $c(\lambda) = 0$, which we can easily solve using the abc-formula. To summarize, the mixed mode is given by (3.49) and (3.59) with its characterizing eigenvalues defined by

$$D_2 = \left(2\beta + 4K_y y_0^2\right)^2 - 4\left(3K_y^2 y_0^4 + \frac{1}{3}K_x^2 y_0^4 - 3\beta^2 - \frac{1}{3}\alpha^2\right) \quad \text{and} \quad \lambda_{1,2} = \frac{2\beta + 4K_y y_0^2 \pm \sqrt{D_2}}{2}. \tag{4.41}$$

4.3 Mixed mode stability

The mixed mode discovered near the degenerate point is a saddle point for all considered parameters, which is demonstrated in this section.

Firstly a saddle point equilibrium is characterized by real eigenvalues, that have opposite sign. The equation for the eigenvalues of the mixed point, (4.41), allows for this when two conditions are met,

$$C_1 : D_2 > 0 \quad \text{and} \quad C_2 : \left(2\beta + 4K_y y_0^2\right)^2 < D_2. \tag{4.42}$$

The first condition is that the discriminant, D_2 in the eigenvalue equation (4.41), is above zero. We set this condition to assure the root, $\sqrt{D_2}$ in (4.41), is real. An imaginary root is undesirable, as it would result in conjugate complex eigenvalues instead of the desired real eigenvalues.

The second condition ensures the opposite sign of the eigenvalues. If C_2 is true then the two possibilities of $2\beta + 4K_y y_0^2 \pm \sqrt{D_2}$ have the opposite sign.

We start the proof that any considered mixed mode satisfies both conditions by introducing the variables

$$A = \left(K_y^2 + \frac{1}{9}K_x^2\right), \quad B = \left(2\beta K_y + \frac{2}{9}\alpha K_x - 1\right) \quad \text{and} \quad C = \left(\beta^2 + \frac{1}{9}\alpha^2\right), \tag{4.43}$$

to simplify further derivations. The formula for y_0^2, given by (3.48), is simplified by substituting the new variables, resulting in

$$y_0^2 = \frac{-B - \sqrt{D_1}}{2A} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{4.44}$$

The introduction of (4.43) also simplifies the discriminant D_2, used in the calculation of the eigenvalues, to

$$D_2 = \left(2\beta + 4K_y y_0^2\right)^2 - 12\left(Ay_0^4 - C\right). \tag{4.45}$$

We notice that the first term of the D_2 discriminant is always positive due to the square. Now it is possible to prove that, in addition to the first, the second term is always positive. We start by stating

$$-12\left(Ay_0^4 - C\right) > 0, \tag{4.46}$$

which leads us to the equivalent statement

$$Ay_0^4 - C < 0. \tag{4.47}$$

We substitute the y_0 term with (4.44) and obtain

$$\frac{B^2 + 2B\sqrt{D_1} + D_1}{4A} < C, \tag{4.48}$$

which, by using $D_1 = B^2 - 4AC$, can be further simplified to

$$0 > 2D_1 + 2B\sqrt{D_1}. \tag{4.49}$$

Now for any relevant mixed mode, we have assumed that its y-position is real positive, due to its interpretation as amplitude and the definition of the mixed mode. In addition to this we notice that A is always positive due to the squares. Therefore the upper part of (4.44), $-B - \sqrt{D_1}$, is real positive and consequently

$$0 > B + \sqrt{D_1}. \tag{4.50}$$

If we multiply this by $2\sqrt{D_1}$ we see

$$0 > 2D_1 + 2B\sqrt{D_1}. \tag{4.51}$$

Therefore (4.49) is true for every possible mixed mode and in extension the proposition that the second term of the discriminant D_2 is positive.

The guaranteed posititivy of both terms in the discriminant D_2 imply that the discriminant is always positive, and therefore the first condition C_1 is satisfied for all mixed modes.

Therefore we now only consider the second condition

$$\left(\beta + 4K_y y_0^2\right)^2 < \left(2\beta + 4K_y y_0^2\right)^2 - 12\left(Ay_0^4 - C\right). \tag{4.52}$$

This is equivalent to

$$0 < -12\left(Ay_0^4 - C\right), \tag{4.53}$$

which just has been proven true for all considered mixed modes.

Therefore both saddle conditions in (4.42) are satisfied by all considered mixed modes. We conclude that the nearby mixed mode, as long as it exist near the x-axis, is a saddle for all considered parameters.

4.4 Degenerate case $K_2 = 0$.

This special case was already mentioned in section 4.1, where it was noted for breaking the pitchfork normal form that occurs near the x-axis in the running phase difference pure mode case. In this section we consider the entire localized system (4.2) when $K_2 = 0$. Firstly we notice that $K_2 = 0$ reduces the localized system to

$$x' = \alpha - 3y\sin(\Phi + x) + K_x y^2, \tag{4.54a}$$
$$y' = \beta y + y^2\cos(\Phi + x). \tag{4.54b}$$

This system is still non-degenerate, which implies the dynamics still have an accurate correspondence to the dynamics of the total normal form (2.36). However at the parameter value $\beta = 0$, the system reduces even more to

$$x' = \alpha - 3y\sin(\Phi + x) + K_x y^2, \tag{4.55a}$$
$$y' = y^2\cos(\Phi + x). \tag{4.55b}$$

This system has a property that leads to dynamics that are almost certainly not shared by the total normal form, the system is integrable. This is defined as the existence of a function $E(x, y)$ such that

$$\frac{d}{dt}E(x, y) = 0. \tag{4.56}$$

This implies that for all solutions the function E remains constant along their path. This function E is called a first integral and when found it implies the existence of closed orbits along its level curves. The reduced system (4.55) has the first integral

$$E = y^3\sin(x) - \frac{1}{2}y^2\alpha - \frac{1}{4}y^4K_x. \tag{4.57}$$

We can demonstrate this by taking the τ time derivative

$$\frac{d}{d\tau}E = y^3\cos(\Phi + x)x' + 3y^2y'\sin(\Phi + x) - yy'\alpha - y^3y'K_x. \tag{4.58}$$

We calculate each term separately,

$$y^3\cos(\Phi + x)x' = \alpha y^3\cos(\Phi + x) - 3y^4\cos(\Phi + x)\sin(\Phi + x) + K_x y^5\cos(\Phi + x), \tag{4.59}$$
$$3y^2y'\sin(\Phi - x) = 3y^4\cos(\Phi + x)\sin(\Phi + x), \tag{4.60}$$
$$-yy'\alpha - y^3y'K_x = -\alpha y^3\cos(\Phi + x) - K_x y^5\cos(\Phi + x). \tag{4.61}$$

Therefore, when these value are substituted in (4.58), we see

$$y^3\cos(\Phi - x)x' + 3y^2y'\sin(\Phi + x) - yy'\alpha - y^3y'K_x = 0. \tag{4.62}$$

And thus $\frac{d}{d\tau}E(x, y) = 0$, and consequently E is a first integral of the system (4.55).

We can plot the level curves of the first integral and calculate the direction of solutions on these curves. When done for the parameter value $\alpha > 0$, this results in figure 4.3.

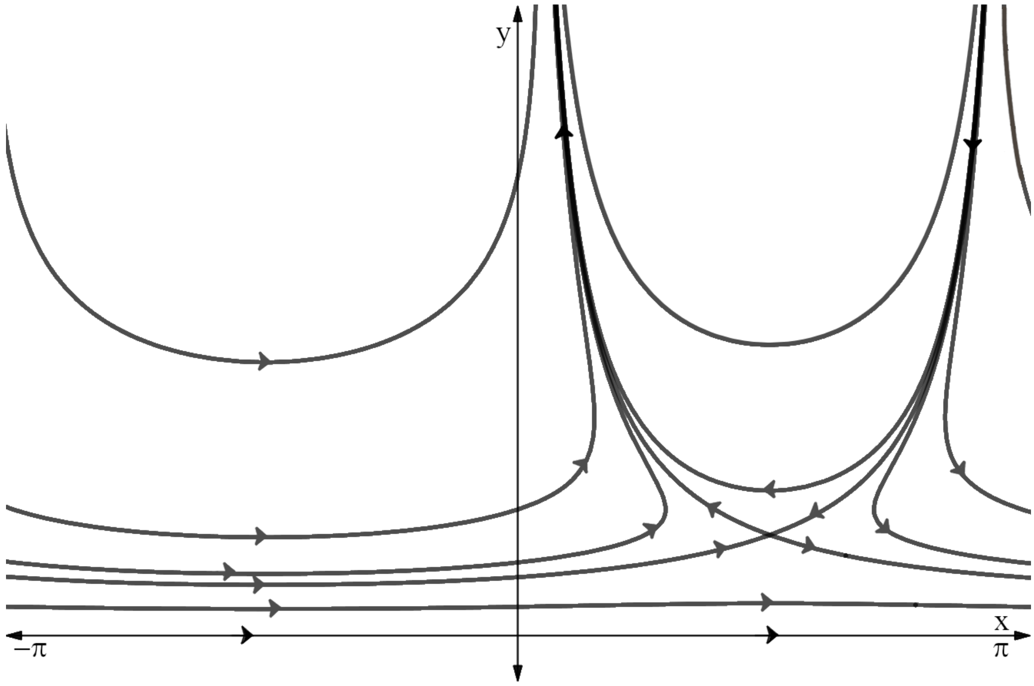


Figure 4.3: Closed cycles on the level curves of E, with the parameter value $\alpha > 0$.

In figure 4.3 the level curves are plotted, including the direction of the solutions on the level curves. When $\alpha < 0$, the dynamics are mirrored in the y-axis.

This figure also gives a good approximation on how the perturbed system behaves, as when δ approaches zero, we expect the solutions of the localized system to approach those of the system (4.55). This results in solutions that are nearly on the level curves of the first integral E.

Changing β to a positive or negative value breaks the first integral invariance of the solutions and results in convergence to the stable pure mode, or divergence until the solution leave the considered space. As this happens for all solutions simultaneously, no region of the parameter space allows the existence of running phase solutions near $y = 0$.

However all of these deductions are made with the absence of higher order terms, which will invariably disturb the closed cycles and other solutions on the level curves present in the dynamics of (4.55). This will lead to a wide array of new bifurcation possibilities and as such these must be included in the study of the degenerate case when β is nearly zero.

As the analysis of the higher order localized system has not been performed in this thesis, the degenerate case is unfortunately not considered any further.

An analysis of a faulty localized system.

5.1 Mistake in the thesis.

From this point on I must admit that I have made a fatal mistake, which resulted in the use of an incorrect normal form. This error then cascaded and worsened throughout all the following sections.

In my studies I forgot to change a minus to a plus sign in the derivation of the normal form, after I used a different definition of φ. This error eventually led to the incorrect resonant normal form

$$\dot{X} = \mu_1 X + (a_{1r}X^2 + b_{1r}Y^2)X + Y^3\cos(\varphi), \tag{5.1a}$$

$$\dot{Y} = \mu_2 Y + (a_{2r}X^2 + b_{2r}Y^2)Y + XY^2\cos(\Phi - \varphi), \tag{5.1b}$$

$$\dot{\varphi} = \nu - a_{im}X^2 - b_{im}Y^2 - \tfrac{Y^3}{X}\sin(\varphi) - 3XY\sin(\Phi - \varphi), \tag{5.1c}$$

and the incorrect localized system

$$x' = \alpha - 3y\sin(\Phi - x) - K_x y^2, \tag{5.2a}$$

$$y' = \beta y + y^2\cos(\Phi - x) + K_y y^3, \tag{5.2b}$$

which I used for the remainder of thesis.

The main difference that within this system (5.2) in the x derivative is that we have the sinus $\sin(\Phi - x)$ with $\Phi - x$, while the correct term is $\sin(\Phi + x)$. The same difference is present in the cosine, however due to the symmetry of the cosine this has considerably less impact than in the sine case.

The second mistake in the normal form was the sign of the $-a_{im}X^2$ and $-b_{im}Y^2$ terms, due to the same error during derivation when redefining φ. However this mistake was not as debilitating as the first and could have been easily solved by redefining a_{im} and b_{im} to have the opposite sign.

These errors do not change the existence of the mixed mode near the degenerate point. They do however lead to the slightly different y_0 position of the nearby mixed mode

$$D_1 = \left(2\beta K_y - \frac{2}{9}\alpha K_x - 1\right)^2 - 4\cdot\left(K_y^2 + \frac{1}{9}K_x^2\right)\cdot\left(\beta^2 + \frac{1}{9}\alpha^2\right) \quad \text{and} \quad y_0^2 = \frac{1 - 2\beta K_y - \frac{2}{9}\alpha K_x - \sqrt{D_1}}{2\left(K_y^2 + \frac{1}{9}K_x^2\right)}. \tag{5.3}$$

And a slightly different x_0 position

$$x_0 = \Phi - \arcsin\left(\frac{\alpha + K_x y_0^2}{3y_0}\right). \tag{5.4}$$

However the errors do have far reaching consequences for the stability of the mixed mode and in extension the dynamics of the system.

For example in the correct localized system the mixed point near the x-axis is a saddle point instead of the source or sink seen in the old analysis. This correct mixed mode does not exhibit any bifurcations, so the Hopf bifurcation found in the old system is not found in the correct system.

These errors have made the remainder of the original thesis completely unsalvageable. The different dynamics required a modified analysis and a completely new result, comparison and conclusion sections. The rewriting of all sections amounted to a sizable workload and would have taken too much time. This is the reason that the second half of this thesis is missing.

However I was able to include a corrected analysis in this thesis in section 4. I have still included the old analysis to exemplify the differences between the two systems. In the last section, section 6, I briefly reflect the impact of the error.

5.2 Original start of chapter.

In this chapter we start the analysis of the dynamics near the pure mode by determining the stability and bifurcations of the pure mode(s). Afterwards we consider the stability of the nearby mixed mode. We achieve this by considering the localized system given by (5.2),

Just as with the correct localized system the considered range of the phase space of this localized system is given by $(x, y) \in S^1 \times \mathbb{R}^+$, where $\mathbb{R}^+ := \{y \in \mathbb{R} | y \geq 0\}$.

The x variable is periodic in nature, because it represents the phase difference between the phases of the normal form (5.1). As both phases are in S^1, their difference, given by x, is in S^1 too. The y variable is restricted to the positive or zero real number due to its correspondence with the amplitude Y in the normal form.

Lastly the pure mode(s) are also located in the manifold $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^+$. This subspace is referred to as the x-axis in the rest of the analysis and the points $(x, 0) \in S^1 \times \{0\}$ are referred to as points on the x-axis.

5.3 The two regions of the phase space.

Before we start with the derivations regarding bifurcations in this system, we denote two regions that will be present the phase space when δ is small, and the parameters, $\bar{\mu}$ and $\bar{\nu}$, are not zero and within the considered parameter space. These regions, while not formally defined, are to help denote the location of solutions.

The first region is near the x-axis and is characterized by a running phase difference. Solutions that stay in this region have a monotonically increasing or decreasing x value. We refer to this region as the running phase region.

The second region is the region near the mixed mode, here the phase difference x and the amplitude y oscillate around the mixed mode and depending on the parameter converge to the mixed mode or a limit cycle. Solutions that stay in this region have a single period oscillating x and y and this region is referred to as the oscillating phase region.

Outside the considered parameter space of the analysis, when β and α are of a similar or greater magnitude than $\frac{1}{\delta}$, the regions become unclear. The stability of the mixed mode can be great enough that no oscillation occurs near the fixed point or the mixed mode could not exist at all. The dynamics at this point are beyond the scope of this thesis, so these cases are not considered further.

5.4 Stability of the pure modes.

The stability of the pure modes is easily calculated, as the localized system (4.1) at the points on the x-axis, is given by

$$x' = \alpha - 3y \sin(\Phi - x - x_0) - K_x y^2, \tag{5.5a}$$
$$y' = \beta y + y^2 \cos(\Phi - x - x_0) + K_y y^3. \tag{5.5b}$$

We can instantly derive the linearization around $(x, y) = (x_0, 0)$ of these equations which gives us

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.6}$$

which in turn gives us the eigenvalues

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \beta. \tag{5.7}$$

The eigenvalue λ_2 has the eigenvector $(0, 1)$, this implies that it determines the stability of all points on the x-axis in the y direction.

- If $\beta < 0$ then solutions near the x-axis, which refers to solutions within $S^1 \times (0, \epsilon)$ with a non-zero real and suitably small ε, converge to $y = 0$
- If $\beta > 0$ then the x-axis is unstable, solutions near the x-axis may converge to a bifurcated running phase difference solution. Or when this cycle does not exit, the nearby solutions increase in y until they leave the neighborhood of the x-axis.

More care should be taken when discussing the $(1, 0)$ directional stability of the pure modes; recall from section 3.3 that the phase difference x is not defined at the pure mode. However we can use the limit of the phase difference when approaching $y = 0$ to define the x variable at $y = 0$. Conveniently, due to the continuity of the limit, this definition allows us to treat the x values at $y = 0$ as normal variables. This slight difference does not impact the dynamics of the localized system much, but is does affect the interpretation of the bifurcations of the localized system.

- If $\alpha = 0$ all points on the x-axis are equilibriaequilibria and no motion takes place along the x coordinate. This confirms the $\lambda_1 = 0$ result.
- If $\alpha \neq 0$ then the pure mode has a running phase difference, as every point on the x-axis satisfies $x' = \alpha$. This implies that the pure mode is a single periodic orbit.

The pure modes do in fact experience a bifurcation at $\beta = 0$. Here we must distinguish the phase locked pure modes from the running phase difference pure modes, as they behave quite differently when β approaches zero.

Running phase pure modes.

We first cover the cases when the pure mode has a running phase difference. We consider the points near the x-axis, these points are defined by $(x, y) = (x_0, \varepsilon)$ with $x_0 \in S^1$ and ε positive, real and in the neighborhood of 0. This definition is used for the rest of the analysis. On these points the derivative x' is given by

$$x' = \alpha - 3\varepsilon \sin(\Phi - x_0) - K_x \varepsilon^2 = \alpha + O(\varepsilon). \tag{5.8}$$

When $\alpha \neq 0$, the value of $O(\varepsilon)$ is negligible compared to the parameter term α, due to the smallness of ε. Therefore on the points near the x-axis we can approximate the derivative x' by $\alpha + O(\varepsilon) \approx \alpha$. When starting at some point (x_0, ε) near the x-axis, this approximation leads to

$$x(\tau) \approx \alpha\tau + x_0. \tag{5.9}$$

Substituting this approximation in y' gives us

$$y' = \beta y + y^2 \cos(\Phi - \alpha\tau - x_0) + K_y y^3. \tag{5.10}$$

Now when near the pure mode, and as such the x-axis, the derivative y' itself is very small, due to the y in each term. From this it follows that any solution starting near the pure mode will stay there for a long time and by starting even closer to $y = 0$ we can arbitrarily extend the time spent near $y = 0$. Therefore the term $\cos(\Phi - \alpha\tau - x_0)$ oscillates on a time scale much faster than y' changes. This reduces the term to its median, which equals zero. We argue that the evolution of y' near the pure mode is determined by

$$y' = \beta y + K_y y^3. \tag{5.11}$$

Recall from the introduction that this is in fact the normal form of a pitchfork bifurcation, where the case is determined by the sign of K_y.

However, due to the correspondence of x and y to the phase difference and the amplitude in the normal form, this bifurcation in fact corresponds to a Hopf bifurcation in the normal form. The periodicity of the x variable and the non-existence of the phase difference at $y = 0$, together with the coupled oscillation y experiences, support this observation.

It is also apparent that the pitchfork normal form (5.11) breaks when $K_y = 0$. For now we exclude this possibility, as this results in a degenerate localized system; we briefly consider that system in section 4.4.

If $K_y < 0$, we know that the pitchfork bifurcation is super-critical. Therefore when $\beta < 0$ the running phase pure mode is stable and no other running phase solutions exist. When $\beta > 0$ we know that two stable solutions have bifurcated off the original pure mode, while the stability of said pure mode has switched from stable to unstable. One of the bifurcated stable solutions has negative y-values, as these do not describe valid amplitudes, this solution is not considered any further. Within the total normal form, this bifurcation corresponds to the splitting of a stable invariant 2-torus, which contains the quasi-periodic solutions, off the 1-torus corresponding to the pure mode. Lastly this 1-torus switches stability from stable to unstable.

If $K_y > 0$, we know that the pitchfork bifurcation is sub-critical. Therefore when $\beta > 0$ the running phase pure mode is unstable and no other running phase solutions exist. When $\beta < 0$ we know that two unstable solutions have bifurcated and the pure mode is stable. Again one of the bifurcated solutions is negative and not considered any further. Within the total normal form, this bifurcation corresponds to the splitting of a unstable invariant 2-torus, which contains the quasi-periodic solutions, off the 1-torus corresponding to the pure mode. Lastly this 1-torus switches stability from unstable to stable.

Phase locked pure modes.

When the pure modes are phase locked the situation is different. Firstly we notice that, due to the eigenvalue λ_2 with the $(0, 1)$ eigenvector, the x-axis still switches stability from stable to unstable when β becomes positive.

Secondly the derivative x' is zero at all points on the x-axis, but slightly above the x-axis the $-3y\sin(\Phi - x)$ term becomes important. The term causes the derivative x' to be non-zero except at the x values where $-3y\sin(\Phi - x)$ is zero.

At the points on the x-axis, with a neighborhood wherein all points above the x-axis satisfy $x' \neq 0$, the change in stability of the x-axis does not lead to the bifurcation of fixed points. The switch in x-axis stability does cause nearby solutions to reverse their y direction, but unlike the running phase difference case no periodic running phase difference solution bifurcates off the pure mode. This is due to the points near the x-axis values where x' is zero. These points form a curve, which starts perpendicular to the x-axis and which no running phase difference solution can cross, instead these solution either bend toward or bend away from the x-axis.

At the points on the x-axis, with a neighborhood that contains points above the x-axis which satisfy $x' = 0$, equilibria bifurcate from the x-axis.

At all points near the x-axis, see the start of the running phase difference paragraph for the definition of these points, the derivative x' is given by

$$x' = -3\varepsilon\sin(\Phi - x_0) - K_x\varepsilon^2 = -3\varepsilon\sin(\Phi + x_0) + O(\varepsilon^2). \tag{5.12}$$

As ε is very small, the $O(\varepsilon^2)$ term is negligible and can be truncated to obtain the approximation of the derivative x' near the x-axis

$$x' = -3\varepsilon\sin(\Phi - x_0). \tag{5.13}$$

This right hand side of this equation is only zero at the values $\Phi - x_0 = k\pi$, $k \in \mathbb{Z}$. At all other values of x_0, all points in the neighborhood of $(x_0, 0)$ satisfy $x' \neq 0$. As discussed before, the switch of the stability of the x-axis does not lead to bifurcated equilibria in these points, therefore these points at these x_0 values are not considered any further.

We consider the derivative y' at the considered x_0 values

$$y' = \beta y + y^2\cos(\Phi - x_0) + K_y y^3. \tag{5.14}$$

Close enough to the x-axis the y^3-term is negligible, resulting in the evolution

$$y' = \beta y + y^2\cos(\Phi - x_0), \tag{5.15}$$

and at the considered x_0 values the cosine is either 1 or -1. Therefore the derivative y' at the points $(\Phi - k\pi, y)$, with $k \in \mathbb{Z}$, equals

$$y' = \beta y \pm y^2. \tag{5.16}$$

Recall from the introduction that this is the normal form of a trans-critical bifurcation, where the direction of the bifurcation is determined by the sign of y^2. This normal form implies a second equilibrium at $y = \mp\beta$ in addition to the one at $y = 0$, this second equilibrium has its stability in the $(0, 1)$ direction characterized by the eigenvalue $\lambda = -\beta$. Here we exclude the fixed points below the x-axis, that might occur due the trans-critical bifurcation. This is again because the y variable still corresponds to an amplitude and therefore should not be negative.

We can also calculate the eigenvalue of the x variable at the second equilibrium for both signs of the cosine. If we linearize at $\Phi - x_0 = 0 + 2k\pi$, $y_0 = -\beta > 0$ we obtain

$$\begin{aligned}
x' &= -3(y_0 + y)\sin(\Phi - x_0 - x) + O(y^2) && (5.17)\\
&= -3(y_0 + y)\left(\sin(\Phi - x_0)\cos(x) - \cos(\Phi - x_0)\sin(x)\right) + O(y^2) && (5.18)\\
&= 3(y_0 + y)\sin(x) + O(y^2) && (5.19)\\
&= -3\beta x + O(2). && (5.20)
\end{aligned}$$

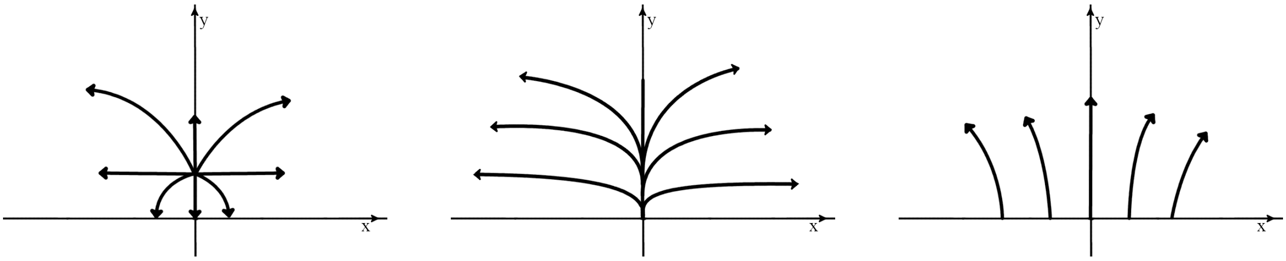
Therefore the eigenvalues of this mixed mode are $\lambda_1 = -\beta$ and $\lambda_2 = -3\beta$. If the fixed point is to exist above the x-axis then we require $\beta < 0$ and we see that both eigenvalues are positive. Therefore the mixed mode is an unstable fixed point.

If we linearize at $\Phi - x_0 = \pi + 2k\pi$, $y_0 = \beta > 0$ we obtain

$$\begin{aligned}
x' &= -3(y_0 + y)\sin(\Phi - x_0 - x) + O(y^2) && (5.21)\\
&= -3(y_0 + y)\sin(x) + O(y^2) && (5.22)\\
&= -3\beta x + O(2). && (5.23)
\end{aligned}$$

Therefore the eigenvalues of this mixed mode are $\lambda_1 = -\beta$ and $\lambda_2 = -3\beta$. If the fixed point is to exist above the x-axis, $\beta > 0$ must be true and we see that both eigenvalues are negative. Therefore the mixed mode is a stable fixed point.

We can summarize these results in phase portraits of the dynamics of the system (5.5) near the points $(x, y) = (\Phi - k\pi, 0)$, $k \in \mathbb{Z}$. The origin in the phase portraits has been set to these points. In all cases the x-axis in these phase portraits consists of equilibria, as both derivatives x' and y' are zero when $y = 0$.

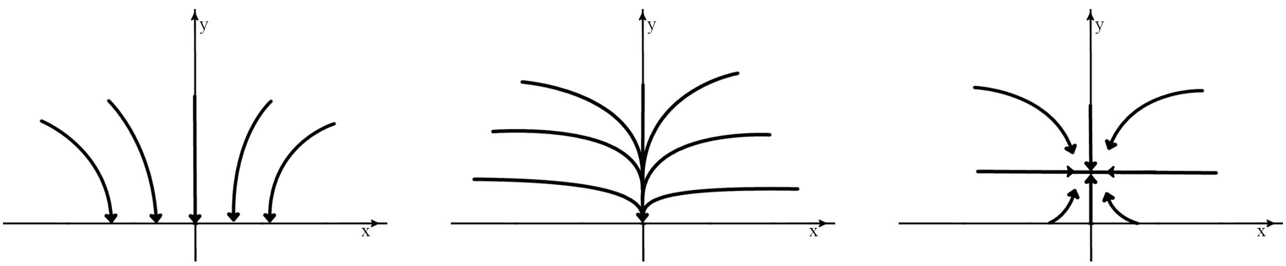


(a) $\beta < 0$ **(b)** $\beta = 0$ **(c)** $\beta > 0$

Figure 5.1: $\Phi - x_0 = 0 + 2k\pi$

Near the phase locked pure modes at $\Phi - x_0 = 0 + 2k\pi$, we see $y' = \beta y + y^2 = y(\beta + y)$. We distinguish three cases depending on the parameter β.

- When $\beta < 0$, the pure mode is stable and the derivative y' is zero at $y = -\beta$, which implies that a fixed point near $(x_0, -\beta)$. The eigenvalue calculation determined that this point is a unstable equilibrium.
- When $\beta = 0$, for all points that satisfy $x_0 \neq k\pi$, the derivative y' is tiny compared to the x' derivative. This implies that running phase difference solutions near the x-axis approximately maintain their y-value until they approach one of the considered x_0 values, where $x' = 0$. Only when nearing the line given by $\Phi + x_0 = k\pi$ zero, the y' derivative becomes dominant. In the case $\Phi + x_0 = 0 + 2k\pi$ the running phase difference solutions tend away from the point $(x_0, 0)$ in both directions, due to the sign of the y^2 term and the linearization of x' at (x_0, ϵ).
- When $\beta > 0$, the pure mode is unstable and the derivative y' is still zero at $y = -\beta$. However due to the sign of β, this now implies a fixed point with a negative y value and therefore the fixed point is not considered further.



(a) $\beta < 0$ **(b)** $\beta = 0$ **(c)** $\beta > 0$

Figure 5.2: $\Phi - x_0 = \pi + 2k\pi$

Near the phase locked pure modes at $\Phi - x_0 = \pi + 2k\pi$, we see $y' = \beta y - y^2 = y(\beta - y)$. We distinguish three cases depending on the parameter β.

- When $\beta < 0$, the pure mode is stable and the derivative y' is zero at $y = \beta$. However due to the sign of β, this now implies a fixed point with a negative y value and therefore the fixed point is not considered further.
- When $\beta = 0$, the same argument as in the $\Phi - x_0 = 0 + 2k\pi$ case applies for all the points where $x' \neq 0$. Therefore the running phase difference solutions near the x-axis approximately maintain their y-value until they approach one of the considered x_0 values, where $x' = 0$. When nearing the line given by $\Phi + x_0 = k\pi$ zero, the y' derivative becomes dominant. In the case $\Phi + x_0 = 0 + 2k\pi$ the running phase difference solutions tend toward the point $(x_0, 0)$ in both directions, due to the sign of the y^2 term and the linearization of x' at (x_0, ϵ).
- When $\beta > 0$, the pure mode is unstable and the derivative y' is still zero at $y = \beta$, which implies a fixed point near (x_0, β). The eigenvalue calculation has shown that this point is a stable equilibrium.

Let us summarize the situation when β moves from a negative value to a positive one while $\alpha = 0$. We firstly see that an unstable mixed mode collides with the pure mode at $(\Phi, 0)$ at $\beta = 0$. When β becomes positive, the mixed mode split off $(\Phi - \pi, 0)$ as a stable equilibrium.

Results.

Thus we can confidently state that when β switches sign, all pure modes switch y-stability. This causes, in running phase difference pure modes, a stable or unstable running phase solution to bifurcate off the pure mode depending on the sign of K_2. These running phase difference solutions are studied in more detail in section 5.7. In phase locked pure modes, the sign switch of β causes a mixed modes to bifurcate at the points $(\Phi - k\pi, 0)$, $k \in \mathbb{Z}$.

5.5 Eigenvalues of the mixed mode.

In section 3.3 we had derived the location of the localized phase locked mixed mode. We denote this location as (x_0, y_0), where the x_0 and y_0 values are obtained by (5.4) and (5.3), respectively.

To obtain the eigenvalues that characterize this equilibrium, we linearize the localized system

$$x' = \alpha - 3y\sin(\Phi - x) - K_x y^2, \tag{5.24a}$$
$$y' = \beta y + y^2\cos(\Phi - x) + K_y y^3, \tag{5.24b}$$

at this point and consider the resulting equations. We start with the linearization of (5.24a)

$$x' = \alpha - 3(y + y_0)\sin(\Phi - x - x_0) - K_x(y + y_0)^2. \tag{5.25}$$

Recall that the $\Phi - x_0$ value is given by (3.40), this can be substituted in (5.25) to obtain

$$x' = \alpha - 3(y + y_0)\sin\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right) - x\right) - K_x(y^2 + 2yy_0 + y_0^2). \tag{5.26}$$

We leave the y_0 coordinate as it is, as using the derived equation (3.48) for it would only complicate this linearization. We now use the sinus sum identity to separate the x value from the arcsin in the sinus, this leads us to

$$x' = \alpha - 3(y + y_0)\left(\sin\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right)\right)\cos(x) - \cos\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right)\right)\sin(x)\right) - K_x(y^2 + 2yy_0 + y_0^2). \tag{5.27}$$

And after further derivation we end with

$$x' = \alpha - 3(y + y_0)\left(\frac{\alpha - K_x y_0^2}{3y_0}\cos(x) - \sqrt{1 - \left(\frac{\alpha - K_x y_0^2}{3y_0}\right)^2}\sin(x)\right) - K_x y^2 - 2K_x yy_0 - K_x y_0^2. \tag{5.28}$$

Fortunately it is possible to remove the square root from this equation. For this we need to extract the square root found in the old derivation of x_0, which is omitted due to being incorrect. This gives us

$$\frac{\beta + K_y y_0^2}{y_0} = -\sqrt{1 - \left(\frac{\alpha - K_x y_0^2}{3y_0}\right)^2}, \tag{5.29}$$

which can be used to simplify (5.28) to

$$x' = \alpha - 3(y + y_0)\left(\frac{\alpha - K_x y_0^2}{3y_0}\cos(x) + \frac{3\beta + 3K_y y_0^2}{3y_0}\sin(x)\right) - K_x y^2 - 2K_x yy_0 - K_x y_0^2. \tag{5.30}$$

We replace the $\cos(x)$ and $\sin(x)$ terms with their respective Taylor expansions and collect all the higher order terms of x and y in $O(2)$, to obtain

$$x' = \alpha - y\frac{\alpha - K_x y_0^2}{y_0} - y_0\left(\frac{\alpha - K_x y_0^2}{y_0} + \frac{\beta + K_y y_0^2}{y_0}x\right) - 2K_x yy_0 - K_x y_0^2 + O(2). \tag{5.31}$$

Which after simplification yields

$$x' = \left(-2K_x y_0 - \frac{\alpha - K_x y_0^2}{y_0}\right)y - (3\beta + 3K_y y_0^2)x + O(2). \tag{5.32}$$

This is the equation for x' linearized at (x_0, y_0). We continue with the linearization of (5.24b)

$$y' = \beta(y + y_0) + (y + y_0)^2 \cos(\Phi - x - x_0) + K_y(y + y_0)^3. \tag{5.33}$$

We substitute again the $\Phi - x_0$ constant for its equation (3.40) to obtain

$$y' = \beta(y + y_0) + (y + y_0)^2 \cos\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right) - x\right) + K_y(y + y_0)^3. \tag{5.34}$$

This time we use the trigonometric sum identity of the cosine.

$$\begin{aligned} y' = \beta(y + y_0) + (y + y_0)^2 \cos\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right)\right) \cos(x) \qquad (5.35)\\ + (y + y_0)^2 \sin\left(\arcsin\left(\frac{\alpha - K_x y_0^2}{3y_0}\right)\right) \sin(x) + K_y(y + y_0)^3 \\ = \beta(y + y_0) + (y + y_0)^2 \left(\sqrt{1 - \left(\frac{\alpha - K_x y_0^2}{3y_0}\right)^2} \cos(x) + \frac{\alpha - K_x y_0^2}{3y_0} \sin(x)\right) + K_y(y + y_0)^3. \qquad (5.36) \end{aligned}$$

Here the root has reappeared, but we can again use the equality (5.29) to continue to

$$y' = \beta(y + y_0) - (y + y_0)^2 \left(\frac{\beta + K_y y_0^2}{y_0} \cos(x) - \frac{\alpha - K_x y_0^2}{3y_0} \sin(x)\right) + K_y(y + y_0)^3. \tag{5.37}$$

And after expanding the cube and square of $(y - y_0)$, we obtain

$$y' = \beta y + \beta y_0 - (y^2 + 2y_0 y + y_0^2) \left(\frac{\beta + K_y y_0^2}{y_0} \cos(x) - \frac{\alpha - K_x y_0^2}{3y_0} \sin(x)\right) + K_y(y^3 + 3y_0 y^2 + 3y_0^2 y + y_0^3). \tag{5.38}$$

We once again replace the $\cos(x)$ and $\sin(x)$ terms with their respective Taylor expansion and collect all of the higher order terms in $O(2)$

$$y' = \beta y + \beta y_0 - 2y\left(\beta + K_y y_0^2\right) - y_0 \left(\beta + K_y y_0^2 - \frac{\alpha - K_x y_0^2}{3} x\right) + 3K_y y_0^2 y + K_y y_0^3 + O(2), \tag{5.39}$$

which after simplification yields the equation

$$y' = \frac{1}{3}(\alpha y_0 - K_x y_0^3)x + \left(K_y y_0^2 - \beta\right) y + O(2) \tag{5.40}$$

for y' linearized at (x_0, y_0). Together with (5.32), we can finally obtain the linearized system in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -(3\beta + 3K_y y_0^2) & -2K_x y_0 - \dfrac{\alpha - K_x y_0^2}{y_0} \\ \dfrac{1}{3}(\alpha y_0 - K_x y_0^3) & K_y y_0^2 - \beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.41}$$

The eigenvalues of this matrix characterize the dynamics in the local neighborhood of the mixed mode. As the matrix is 2 by 2 we can easily give the characteristic polynomial

$$c(\lambda) = \left(-3\beta - 3K_y y_0^2 - \lambda\right)\left(K_y y_0^2 - \beta - \lambda\right) - \left(-2K_x y_0 - \frac{\alpha - K_x y_0^2}{y_0}\right)\left(\frac{1}{3}(\alpha y_0 - K_x y_0^3)\right). \tag{5.42}$$

This polynomial can be simplified to

$$c(\lambda) = \lambda^2 + \left(4\beta + 2K_y y_0^2\right)\lambda + \left(3\beta^2 - 3K_y^2 y_0^4\right) + \frac{1}{3}\left(\alpha^2 - K_x^2 y_0^4\right). \tag{5.43}$$

We are left with another quadratic formula $c(\lambda) = 0$, which we can easily solve using the abc-formula. To summarize, the mixed mode is given by (3.54) and (3.48) and (3.59) with its characterizing eigenvalues defined by

$$D_2 = \left(4\beta + 2K_y y_0^2\right)^2 - 4\left(3\beta^2 - 3K_y^2 y_0^4 + \frac{1}{3}\alpha^2 - \frac{1}{3}K_x^2 y_0^4\right) \quad \text{and} \quad \lambda_{1,2} = \frac{-4\beta - 2K_y y_0^2 \pm \sqrt{D_2}}{2}. \tag{5.44}$$

5.6 Hopf bifurcation of the mixed mode.

An equilibrium point undergoes a Hopf bifurcation when two of its eigenvalues are purely imaginary conjugates, as we have seen in the introduction (5.59).

The equation for the eigenvalues (5.44) allows for this when two conditions are met,

$$C_1: \quad -4\beta - 2K_y y_0^2 = 0 \quad \text{and} \quad C_2: \quad D_2 < 0. \tag{5.45}$$

The first condition to ensure the real part of the eigenvalue is zero. In the equation for the eigenvalues (5.44), we see the term $-4\beta - 2K_y y_0^2$. This term is a real number, as β, K_y and y_0 are real numbers. If $-4\beta - 2K_y y_0^2$ is non-zero, we would have a non-zero real term in the equation for eigenvalues and this would result in eigenvalue with a non-zero real part. This is undesirable at the Hopf bifurcation the eigenvalues of the equilibrium must be purely imaginary. Therefore C_1 must be true if the mixed mode undergoes a Hopf bifurcation.

The second condition is that the discriminant, D_2 in (5.44), is below zero, as otherwise the root, $\sqrt{D_2}$ would be real and this would result in eigenvalues with a non-zero real part.

However if both conditions are met then $\sqrt{D_2}$ is imaginary and the eigenvalues are given by

$$\lambda_{1,2} = \pm\frac{1}{2}\sqrt{D_2}. \tag{5.46}$$

The first condition implies the second.

We now prove that the first condition implies the second. To this end let us introduce the variables

$$A = \left(K_y^2 + \frac{1}{9}K_x^2\right), \quad B = \left(2\beta K_y - \frac{2}{9}\alpha K_x - 1\right) \quad \text{and} \quad C = \left(\beta^2 + \frac{1}{9}\alpha^2\right). \tag{5.47}$$

This simplifies the discriminant D_1, used in the calculation of y_0, to $B^2 - 4AC$, while the second discriminant D_2 simplifies to

$$D_2 = \left(4\beta + 2K_y y_0^2\right)^2 - 12(C - Ay_0^4). \tag{5.48}$$

The first Hopf condition implies $4\beta + 2K_y y_0^2 = 0$, this simplifies the second determinant even more to $D_2 = -12(C - Ay_0^4)$. The second Hopf condition now only requires

$$C > Ay_0^4. \tag{5.49}$$

The formula for y_0^2 stated in (3.48) is also significantly simplified by the introduction of (5.47), resulting in

$$y_0^2 = \frac{-B - \sqrt{D_1}}{2A}. \tag{5.50}$$

Thus the second condition is satisfied when

$$C > A\left(\frac{-B - \sqrt{D_1}}{2A}\right)^2. \tag{5.51}$$

which can be further simplified to

$$4AC > B^2 + 2B\sqrt{D_1} + D_1, \tag{5.52}$$

and finally, using $D_1 = B^2 - 4AC$ to

$$0 > 2D_1 + 2B\sqrt{D_1}. \tag{5.53}$$

Now for any relevant mixed mode, we have assumed that its y-position is real positive, due to its interpretation as amplitude and the definition of the mixed mode. In addition to this we notice that A is always positive due to the squares. Therefore the upper part of (5.50), $-B - \sqrt{D_1}$, is real positive and consequently

$$0 > B + \sqrt{D_1}. \tag{5.54}$$

If we multiply this by $2\sqrt{D_1}$ we see

$$0 > 2D_1 + 2B\sqrt{D_1}. \tag{5.55}$$

Therefore (5.53) is true for every possible mixed mode. We conclude that the first Hopf condition implies the second one.

First condition in term of the parameters.

The first condition in (5.45) states

$$2\beta + K_y y_0^2 = 0, \tag{5.56}$$

which simplifies to

$$4A\beta = K_y B + K_y\sqrt{D_1}. \tag{5.57}$$

For further derivations we replace A, B and C by their original values given in (5.47). This leads to

$$4\beta\left(K_y^2 + \frac{1}{9}K_x^2\right) = K_y\left(2\beta K_y - \frac{2}{9}\alpha K_x - 1\right) + K_y\sqrt{D_1}, \tag{5.58}$$

and

$$4\beta K_y^2 + \frac{4}{9}\beta K_x^2 = 2\beta K_y^2 - \frac{2}{9}\alpha K_x K_y - K_y + K_y\sqrt{D_1}. \tag{5.59}$$

The expansion D_1 is given by

$$D_1 = 1 - \frac{8}{9}\alpha\beta K_x K_y + \frac{4}{9}\alpha K_x - 4\beta K_y - \frac{4}{9}\beta^2 K_x^2 - \frac{4}{9}\alpha^2 K_y^2. \tag{5.60}$$

The derivation of this expansion is omitted in this thesis. If we again using the Taylor expansion of the root (3.60), we obtain

$$\sqrt{D_1} = 1 + \frac{2}{9}\alpha K_x - 2\beta K_y - \frac{2}{9}\alpha^2 K_y^2 - \frac{2}{81}\alpha^2 K_x^2 + O(\delta^3). \tag{5.61}$$

If this is in turn substituted in (5.59), we have

$$4\beta K_y^2 + \frac{4}{9}\beta K_x^2 = 2\beta K_y^2 - \frac{2}{9}\alpha K_x K_y - K_y + K_y + \frac{2}{9}\alpha K_x K_y - 2\beta K_y^2 - \frac{2}{9}\alpha^2 K_y^3 - \frac{2}{81}\alpha^2 K_x^2 K_y + O(\delta^4). \tag{5.62}$$

From which follows

$$4\beta K_y^2 + \frac{4}{9}\beta K_x^2 = -\frac{2}{9}\alpha^2 K_y^3 - \frac{2}{81}\alpha^2 K_x^2 K_y + O(\delta^4), \tag{5.63}$$

and finally

$$\left(K_y^2 + \frac{1}{9}K_x^2\right)\left(4\beta + \frac{2}{9}\alpha^2 K_y\right) + O(\delta^4) = 0. \tag{5.64}$$

Ergo the first condition in (5.45) is satisfied when

$$4\bar{\mu} + \frac{2}{9}\bar{\nu}^2 K_2 = 0. \tag{5.65}$$

The expression on the right hand side of this condition can also be used to determine the stability of the mixed mode, as the value corresponds to the real value of the eigenvalues. Therefore

$$4\bar{\mu} + \frac{2}{9}\bar{\nu}^2 K_2 < 0 \quad \text{implies} \quad \mathrm{Re}(\lambda_{1,2}) > 0, \tag{5.66}$$

and

$$4\bar{\mu} + \frac{2}{9}\bar{\nu}^2 K_2 > 0 \quad \text{implies} \quad \mathrm{Re}(\lambda_{1,2}) < 0. \tag{5.67}$$

Recall that positive eigenvalues imply an unstable equilibrium point and negative ones imply a stable equilibrium point.

5.7 The evolution of the running phase solutions.

In section 5.4, the eigenvalue of the pure mode indicates that it loses its stability at $\beta = 0$. This implies the existence of stable and unstable bifurcated solutions with a running phase. To ascertain how these solutions evolve over time we can interpret the localized system as a perturbation of the system

$$x' = \alpha - 3y\sin(\Phi - x), \tag{5.68a}$$

$$y' = y^2\cos(\Phi - x). \tag{5.68b}$$

When δ approaches 0 we expect the solutions of the complete system (2.36) to approach the cycles of the system (5.68)

The system (5.68) is luckily integrable, this is defined as the existence of a function $E(x, y)$ such that

$$\frac{d}{dt}E(x, y) = 0. \tag{5.69}$$

This implies that for all solutions the function E remains constant along their path. This function E is called a first integral and when found it implies the existence of closed orbits along its level curves. This property allows us to find one explicit solution by considering the level curves of the first integral.

Conveniently, this discovered solution lies exactly on the border between the oscillating phase region and the running phase region. Thus if we calculate the rate of change of a first integral along this solution then we can deduce if the solutions around this cycle tends to the latter or the former region.

We are also able to calculate the time evolution of first integral E_2, defined below in (5.79), for any particular point in the phase space. As we will see this value is $O(\delta)$, this confirms that the perturbed solutions stay close to their original cycles when δ approaches zero.

First integral of leading order system

The leading order system (5.68) has the first integral

$$E_1 = \frac{\sin(\Phi - x)}{y^3} - \frac{\alpha}{4y^4}. \tag{5.70}$$

We can demonstrate this by taking the τ time derivative of (5.70),

$$\frac{d}{d\tau}E_1 = \frac{-y^3\cos(\Phi - x)x' - 3y^2y'\sin(\Phi - x)}{y^6} + \frac{y'\alpha}{y^5}. \tag{5.71}$$

Here we can substitute (5.68) for x' and y' in (5.71), to obtain

$$\frac{d}{d\tau}E_1 = \frac{-\alpha y^3\cos(\Phi - x) + 3y^4\cos(\Phi - x)\sin(\Phi - x) - 3y^4\cos(\Phi - x)\sin(\Phi - x)}{y^6} + \frac{\alpha y^3\cos(\Phi - x)}{y^6} = 0. \tag{5.72}$$

As such E_1 is constant over for all solutions of the leading order system (5.68), thus proving it is a first integral.

We can plot the level curves of the first integral and calculate the direction of solutions on these curves. When done for the parameter value $\alpha > 0$, this results in figure 5.3.

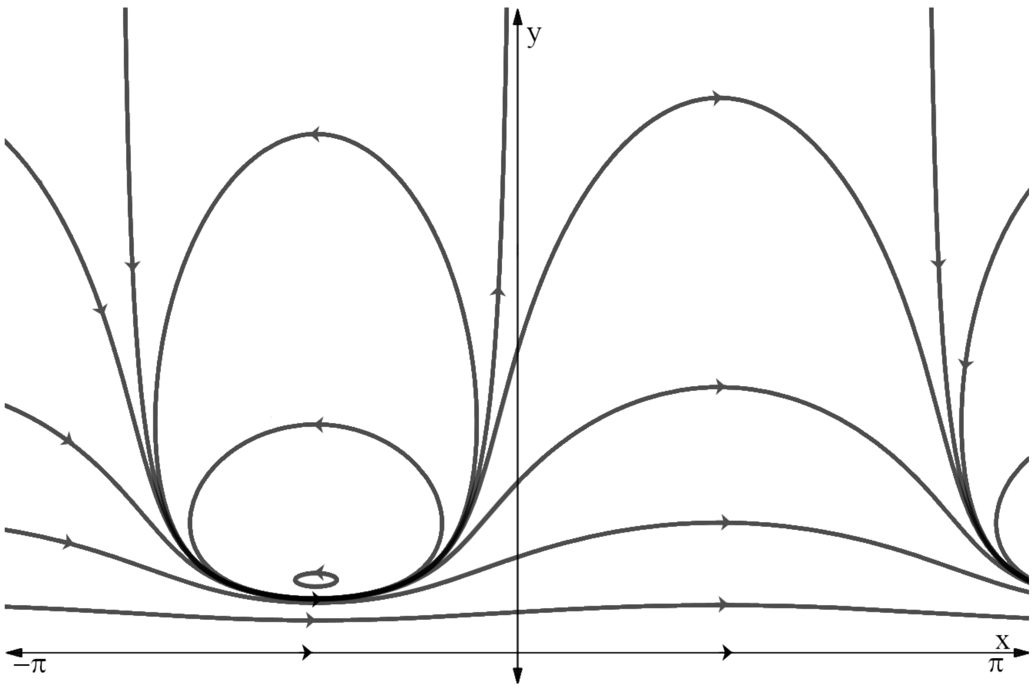


Figure 5.3: Closed cycles on the level curves of E_1, with the parameter value $\alpha > 0$.

In figure 5.3 the level curves are plotted, including the direction of the solutions on the level curves. When $\alpha < 0$, the dynamics are mirrored in the y-axis. The case $\alpha = 0$ is excluded in this section as it concerns the running phase difference solutions and in section 5.4 we determined those only occur when $\alpha \neq 0$.

One parameter solution.

With the first integral, E_1 given by (5.70), we are able to calculate one explicit solution to the unperturbed system (5.68). This solution corresponds to the level curve

$$E_1 = \frac{\sin(\Phi - x)}{y^3} - \frac{\alpha}{4y^4} = 0. \tag{5.73}$$

This equation implies

$$y \sin(\Phi - x) = \frac{1}{4}\alpha, \tag{5.74}$$

and this can directly be used in (5.68a) to obtain

$$x' = \alpha - 3y \sin(\Phi - x) = \alpha - \frac{3}{4}\alpha = \frac{1}{4}\alpha. \tag{5.75}$$

And in turn this leads to us to a one parameter solution for x, given by $x(\tau) = \frac{1}{4}\alpha\tau + x_0$. When starting at $x_0 = \Phi$, which we assume to simplify further derivations, the solution for x is given by $\frac{1}{4}\alpha\tau + \Phi$. This $x(\tau)$ can be substituted in (5.74) to obtain the solution for $y(\tau)$,

$$y(\tau) = \frac{\alpha}{4\sin(\Phi - x(\tau))} = \frac{-\alpha}{4\sin(\frac{1}{4}\alpha\tau)}. \tag{5.76}$$

The results for $x(\tau)$ and $y(\tau)$ give us the following curve

$$C(\tau) = (x(\tau), y(\tau)) = \left(\frac{1}{4}\alpha\tau + \Phi, \frac{-\alpha}{4\sin(\frac{1}{4}\alpha\tau)}\right) \tag{5.77}$$

which is a one parameter solution for the system (5.68). This solution does have gaps, as it switches from positive y-values to negative ones. We do not consider the parts of the curve below the x-axis, as these do not correspond to valid solutions in the localized system, due to the $y \geq 0$ requirement.

Instead we omit these parts in the following derivations, as later is it proven that solutions near C of the localized system experience almost no change in the value of the first integral in the regions where the curve C has a negative y value.

Second integrable system.

The second system we consider is (3.35) with β and K_y equal to 0, while the K_x is not necessarily zero. This system is defined by

$$x' = \alpha - 3y\sin(\Phi - x) - K_x y^2, \tag{5.78a}$$

$$y' = y^2\cos(\Phi - x). \tag{5.78b}$$

This system has the first integral

$$E_2 = \frac{\sin(\Phi - x)}{y^3} - \frac{\alpha}{4y^4} - \frac{K_x}{2y^2}. \tag{5.79}$$

We again can demonstrate this by taking the τ time derivative

$$\frac{d}{d\tau}E_2 = \frac{-y^3\cos(\Phi - x)x' - 3y^2y'\sin(\Phi - x)}{y^6} + \frac{yy'\alpha}{y^6} + \frac{y^3y'K_x}{y^6}. \tag{5.80}$$

We calculate each term separately,

$$-y^3\cos(\Phi - x)x' = -\alpha y^3\cos(\Phi - x) + 3y^4\cos(\Phi - x)\sin(\Phi - x) - K_x y^5\cos(\Phi - x), \tag{5.81}$$

$$-3y^2y'\sin(\Phi - x) = -3y^4\cos(\Phi - x)\sin(\Phi - x), \tag{5.82}$$

$$yy'\alpha + y^3y'K_x = \alpha y^3\cos(\Phi - x) + K_x y^5\cos(\Phi - x). \tag{5.83}$$

Therefore, when these value are substituted in (5.80), we see

$$-y^3\cos(\Phi - x)x' - 3y^2y'\sin(\Phi - x) + yy'\alpha + y^3y'K_x = 0. \tag{5.84}$$

And thus $\frac{d}{d\tau}E_2(x, y) = 0$, and consequently E_2 is a first integral of the system (5.78).

We use this system and its first integral to estimate the behavior of solutions around C. Using the E_2 integral is more convenient. Firstly the second system is closer to the original system (5.2). Secondly and more importantly by using E_2, we remove the lower order terms containing K_x out of the rate of change of E_2 along C.

We have not used this first integral to calculate an explicit solutions, as the required calculation are many times more difficult than the ones performed for C.

E_2 time evolution of the localized system.

Now if we take the integral of the time derivative of the first integral along the curve C while it is positive then we have an estimate of the change E_2 a perturbed solutions near the curve C.

We can prove that, for certain values of K_2 and $\bar{m}u$, this value is zero and in extension that the solutions near C are close to invariant. Let us first calculate the change of E_2 over time for the localized system (5.2). This gives us

$$\frac{d}{d\tau}E_2 = \frac{-y^3\cos(\Phi - x)x' - 3y^2y'\sin(\Phi - x)}{y^6} + \frac{yy'\alpha}{y^6} + \frac{y^3y'K_x}{y^6}. \tag{5.85}$$

This value can be much more simplified, we calculate each term separately,

$$-y^3\cos(\Phi - x)x' = -\alpha y^3\cos(\Phi - x) + 3y^4\cos(\Phi - x)\sin(\Phi - x) - K_x y^5\cos(\Phi - x), \tag{5.86}$$

$$-3y^2y'\sin(\Phi - x) = -3\beta y^3\sin(\Phi - x) - 3y^4\cos(\Phi - x)\sin(\Phi - x) - 3K_y y^5\sin(\Phi - x), \tag{5.87}$$

$$yy'\alpha + y^3y'K_x = \alpha\beta y^2 + \alpha y^3\cos(\Phi - x) + \alpha K_y y^4 + \beta K_x y^4 + K_x y^5\cos(\Phi - x) + K_x K_y y^6. \tag{5.88}$$

Substituting these values into (5.85) results in

$$\frac{d}{d\tau}E_2 = \frac{\beta}{y^6}\left(\alpha y^2 - 3y^3\sin(\Phi - x)\right) + \frac{K_y}{y^6}\left(\alpha y^4 - 3y^5\sin(\Phi - x)\right) + O(\delta^2). \tag{5.89}$$

After one final simplification step we can write (5.89) as

$$\frac{d}{d\tau}E_2 = \left(\frac{\beta}{y^4} + \frac{K_y}{y^2}\right)(\alpha - 3y\sin(\Phi - x)) + O(\delta^2). \tag{5.90}$$

We see that this derivative tends to zero when $y \to \infty$.

It is also important to notice that the derivative of E_2 has no singularities near our known solutions. Near such singularities, no matter how small δ would have been, the change of energy would have exploded and invalidate the estimation.

Recall that the explicit solution C has regions where its y value is negative. Within these regions, the solutions of the localized system near C leave the neighborhood of C. Instead of having a negative y-value, these perturbed solutions cross these regions with an positive y-value around the magnitude $\frac{1}{\delta}$. The conclusion that $\frac{d}{d\tau}E_2$ tends to zero when $y \to \infty$ implies that the rate of change of E_2 of the perturbed solution is very small in these regions. This allows us to exclude these regions in the estimate of the change over the entire period, as the change of E_2 in these regions is negligible when compared to the remaining regions.

Energy evolution along C.

To finally estimate the change of energy along solution C let us calculate equation (5.90) along its path. We only consider the regions where C has a positive y value. Regardless of the value of α, this region is given by the time interval $\tau \in \left[-\frac{4\pi}{|\alpha|}, 0\right]$.

We first consider the change for E_2, (5.90), along the points of the explicit solution C.

$$\frac{d}{d\tau}E_2(x(\tau), y(\tau)) = \left(\frac{\beta}{y^4(\tau)} + \frac{K_y}{y^2(\tau)}\right)(\alpha - 3y(\tau)\sin(\Phi - x(\tau))) + O(\delta^2). \tag{5.91}$$

We can simplify this equation by calculating the $\alpha - 3y\sin(\Phi - x)$ term first,

$$\alpha - 3y(\tau)\sin(\Phi - x(\tau)) = \alpha - \frac{3\alpha}{4\sin(\frac{1}{4}\alpha\tau)}\sin(\frac{1}{4}\alpha\tau) = \frac{1}{4}\alpha. \tag{5.92}$$

When we substitute this in (5.91), we see

$$\frac{d}{d\tau}E_2(\tau) = \frac{1}{4}\alpha\left(\frac{\beta}{y^4(\tau)} + \frac{K_y}{y^2(\tau)}\right) + O(\delta^2) \tag{5.93}$$

$$= \frac{1}{4}\alpha\left(\frac{4^4\beta\sin^4(-\frac{1}{4}\alpha\tau)}{\alpha^4} + \frac{16K_y\sin^2(-\frac{1}{4}\alpha\tau)}{\alpha^2}\right) + O(\delta^2) \tag{5.94}$$

$$= \frac{64\beta\sin^4(-\frac{1}{4}\alpha\tau)}{\alpha^3} + \frac{4K_y\sin^2(-\frac{1}{4}\alpha\tau)}{\alpha} + O(\delta^2). \tag{5.95}$$

If we take the integral of this value (5.95) over the time interval $\left[-\frac{4\pi}{|\alpha|}, 0\right]$, we can calculate the total change E_2 experiences when we follow the solution C in the region where it has a positive y value. This result in

$$\Delta E_2 = \int_{-\frac{4\pi}{|\alpha|}}^{0} \frac{d}{d\tau}E_2(\tau)d\tau = \int_{-\frac{4\pi}{|\alpha|}}^{0} \frac{64\beta\sin^4(-\frac{1}{4}\alpha\tau)}{\alpha^3} + \frac{4K_y\sin^2(-\frac{1}{4}\alpha\tau)}{\alpha}d\tau + O(\delta^2). \tag{5.96}$$

Let us introduce a new time variable $T = \frac{1}{4}|\alpha|\tau$, substituting this new variable in (5.96) results in

$$\Delta E_2 = \int_{-\frac{|\alpha|}{|\alpha|}\pi}^{0} \frac{d}{d\tau}E_2(T)dT = \int_{-\pi}^{0} \frac{64\beta\sin^4(-\mathrm{sgn}(\alpha)T)}{\alpha^3} + \frac{4K_y\sin^2(-\mathrm{sgn}(\alpha)T)}{\alpha}dT + O(\delta^2). \tag{5.97}$$

Due to the squares we have the symmetry $\sin^2(-x) = \sin^2(x)$, and this allows us to simplify the integral to

$$\Delta E_2 = \int_{-\pi}^{0} \frac{64\beta\sin^4(T)}{\alpha^3} + \frac{4K_y\sin^2(T)}{\alpha}dT + O(\delta^2), \tag{5.98}$$

which can be easily solved to obtain

$$\Delta E_2 = \frac{64\beta}{\alpha^3}\int_{-\pi}^{0}\sin^4(T)dT + \frac{4K_y}{\alpha}\int_{-\pi}^{0}\sin^2(T)dT + O(\delta^2) = \frac{24\beta}{\alpha^3}\pi + \frac{2K_y}{\alpha}\pi + O(\delta^2). \tag{5.99}$$

We truncate the $O(\delta^2)$ term, as this value is very small, and obtain the final estimate

$$\Delta E_2 \approx \frac{24\beta}{\alpha^3}\pi + \frac{2K_y}{\alpha}\pi. \tag{5.100}$$

The estimation equals zero when

$$12\beta + \alpha^2 K_y = \delta\left(\bar{\mu} + \bar{\nu}^2 K_2\right) = 0. \tag{5.101}$$

If the estimate for ΔE_2 is zero, all the solutions near C have a near constant first integral E_2 value. This implies that these solutions near C stay near their original position and only slowly move from away.

When the estimate is not zero, we can use the result (5.100) calculate the direction of change of E_2 for the solution near C. This value allows us to determine how the running phase difference solutions evolve over time.

Before we begin with this argument, we first notice that we have two distinct cases relating to the value of E_2 depending on the parameter α.

- When $\alpha < 0$, all points in the oscillating phase region have $E_2 < 0$. The running phase region includes all points where $E_2 > 0$. Lastly the curve C lies on the border between the two regions with $E_2 = 0$.
- When $\alpha > 0$, all points in the oscillating phase region have $E_2 > 0$. The running phase region includes all points where $E_2 < 0$. And again the curve C lies on the border between the two regions with $E_2 = 0$.

Now we can start the argument with the statement $12\bar{\mu} + \bar{\nu}^2 K_2 < 0$. This statement is equivalent to

$$\frac{24\beta}{\alpha^2}\pi + 2K_y\pi < 0. \tag{5.102}$$

The left hand side of (5.102) is a factor of the estimate (5.100),

$$\Delta E_2 \approx \alpha\left(\frac{24\beta}{\alpha^2}\pi + 2K_y\pi\right). \tag{5.103}$$

Therefore, depending on the value of α, we can deduce the sign of ΔE_2.

- When $\alpha < 0$ and $12\bar{\mu} + \bar{\nu}^2 K_2 < 0$, we see $\Delta E_2 > 0$. This implies that all solution near C, over time, have an increasing E_2 value. And as $E_2 > 0$ corresponds to the running phase region, we conclude that these solutions converge towards the running phase region.
- When $\alpha > 0$ and $12\bar{\mu} + \bar{\nu}^2 K_2 < 0$, we see $\Delta E_2 < 0$. This implies that all solution near C, over time, have a decreasing E_2 value. And as in this case $E_2 < 0$ also corresponds to the running phase region, we conclude again that these solutions converge towards the running phase region.

This argument is mirrored by the case $12\bar{\mu} + \bar{\nu}^2 K_2 > 0$. This case leads to the conclusion that, for all α values, the solution near C tend to the oscillating region.

To summarize

- When $12\bar{\mu} + \bar{\nu}^2 K_2 > 0$, the solution near C tend to the oscillating phase region.
- When $12\bar{\mu} + \bar{\nu}^2 K_2 = 0$, the solution near C are almost closed and stay near C.
- When $12\bar{\mu} + \bar{\nu}^2 K_2 < 0$, the solution near C tend to the running phase region.

Let us consider the specific case, with $\beta > 0$, $K_y < 0$ and the value of α small enough that $12\beta + \alpha^2 K_y > 0$ is true, as example.

Recall from section 5.4 that $\beta > 0$ implies that the pure mode is unstable and a stable running phase solution has bifurcated. And from the cycle condition we know that any solution near C will move toward the running phase region and in extension the x-axis.

With this information we can deduce at least one stable running phase difference solution must be present between C and the pure mode.

We can argue that there is only one stable running phase difference solution, as in section 5.4 we have discussed the pitchfork bifurcation of the running phase difference pure mode. This bifurcation splits only one positive solution off from the x-axis. This, in addition to a numerical model, supports that there is only one bifurcated running phase solution.

This same argument applies when $\beta < 0$, $K_y > 0$ and the value of α small enough that $12\beta + \alpha^2 K_y < 0$. In this case, $\beta < 0$ implies that an unstable running phase difference solution has bifurcated off from the x-axis, and the cycle condition implies that this solution keeps existing as long as $12\beta + \alpha^2 K_y < 0$ remains true.

5.8 Degenerate case $K_2 = 0$.

Consider the special form of the system where $K_2 = 0$, this statement reduces the system to

$$x' = \alpha - 3y\sin(\Phi - x) - K_x y^2, \tag{5.104a}$$
$$y' = \beta y + y^2 \cos(\Phi - x). \tag{5.104b}$$

Furthermore it reduces the condition for the Hopf bifurcation (5.45) and the value (5.101) used to argue how the running phase difference solutions near C cycle evolve

$$H : 4\bar{\mu} + \frac{2}{9}\bar{\nu}^2 K_y = 0 \quad \text{and} \quad V : 12\bar{\mu} + \bar{\nu}^2 K_y \tag{5.105}$$

to

$$H : 4\bar{\mu} = 0 \quad \text{and} \quad V : 12\bar{\mu}. \tag{5.106}$$

This radically changes the dynamics as Hopf bifurcation and the first intgral near-invariance of one-parameter solution C occur simultaneously.

When $\bar{\mu} = 0$, the system reduces even more to

$$x' = \alpha - 3y\sin(\Phi - x) - K_x y^2, \tag{5.107a}$$
$$y' = y^2 \cos(\Phi - x). \tag{5.107b}$$

Which we have seen to be integrable in the previous section. The existence of the first integral and its corresponding level curves imply that all solutions of (5.107) remain on the level curve. This implies existence of closed cycles around the equilibrium and closed periodic running phase difference solutions above the pure mode, while the mixed and pure mode neither attract nor repel.

Changing $\bar{\mu}$ from a zero to a positive value changes the stability of the mixed mode from a center to stable equilibrium and the stability of the pure mode to unstable instantly. This act also breaks the first integral invariance of the solutions and results in convergence to a stable pure mode, or to a stable mixed mode. As for all solutions this happens simultaneous, no region of the parameter space allows the existence of closed running phase difference solutions near $y = 0$ and closed bifurcated cycles around the mixed mode.

The same argument applies when changing $\bar{\mu}$ from a zero to a negative value, however in this case the mixed mode becomes unstable and the pure mode stable.

However all of these deductions are made with the absence of higher order terms, which will invariably disturb the closed cycles and other solution on the level curves present in the dynamics of (5.107). This will lead to a wide array of new bifurcation possibilities and as such these must be included in the study of the degenerate case when $\bar{\mu}$ is nearly zero.

As the analysis of the higher order localized system has not been performed in this thesis, the degenerate case is unfortunately not considered any further.

Impact of the mistake.

As seen in both analyses, a simple sign error has had important implications for the dynamics of the systems. The consequence of these differences was that nothing from the original interpretation and conclusion, which both were made with the results obtained with the faulty system, could be saved.

Regrettably I was not able to write a new conclusion within the allowed time-frame, as such the rest of the thesis is missing. I have decided not to include the incorrect conclusion and interpretation of the analysis, as it is of no use for further discussion. None of their statements relate to the true dynamics of the 1:3 resonant Hopf bifurcation, and I have no desire to spread incorrect information.

The unfortunate consequences of a single plus-minus error illustrated in this thesis, should be taken as a lesson to not be underestimate small errors. This was however not the only mistake I made regarding signs (All others were manageable in severity.), and I must admit I very much overestimated my ability to spot these errors and underestimated their importance to truly verify.

Extra care should be placed on the verification of the results and the checking of derivations after changing variables.

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