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BACHELOR THESIS

The classifying space of the (1+1)-dimensional cobordism category

and an application to topological quantum field theory

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Contents

Chapter 1

Introduction

The main goal of this thesis is the construction and analysis of the classifying space of the (1+1)-cobordism category following a paper by Ulrike Tillmann [1].

We will begin by giving a short introduction of the concept of categories in chapter 2. In chapter 3 the classifying space of a category will be defined, this is a topological space corresponding to the category that can be studied to determine properties of the category. We will see that it is often not clear what the topological properties of a classifying space are from its construction. Thus we will take a better look at the homotopy type of certain classifying spaces in chapter 4.

In chapter 5 the framework of classifying spaces will be used to analyze the (1+1)-dimensional cobordism category, the homotopy type of the classifying space of two of its subcategories will be determined as well as the fundamental group of its classifying space. The last chapter makes a connection between the (1+1)-dimensional cobordism category and topological quantum field theories, ending with a classification of morphism inverting (1+1)-dimensional topological quantum field theories using the results from chapter 5.

Chapter 2

Categories

One of the main, and perhaps the most versatile, mathematical tools used in this thesis is category theory. It allows one to study many properties of mathematical systems in more general terms and transfer ideas between different areas of mathematical study. Within category theory mathematical systems are abstracted to objects with some sort of underlying structure and structure preserving maps between those objects.

In the following chapter some concepts and results from category theory that are of interest to us will be discussed. This chapter is based on [2], [3] and [4].

2.1 Categories

Definition 2.1.1. A category $\mathscr{C}$ consists of a collection of objects and a collection of morphisms, also called arrows, such that:

- Every morphism has specified domain and codomain objects, the notation $f : X \to Y$ means that f is a morphism with domain X and codomain Y.
- For every object X there is a designated identity morphism $id_X : X \to X$.
- For any pair of morphism $f : X \to Y$, $g : Y \to Z$ there is a specified composition $gf : X \to Z$.

Also the following two axioms must hold:

- For any morphism $f : X \to Y$ we have $id_Y f = f = f id_X$.
- The composition of morphisms is associative, meaning that for every triple

  $$f : X \to Y \qquad g : Y \to Z \qquad h : Z \to W$$

  we have that $h(gf) = (hg)f$.

Example 2.1.2. Many classes of mathematical objects form a category, some examples are:

- **Set**, the category with sets as its objects and functions with specified domain and codomain as its morphisms.
- $\mathbf{Top}_*$, the category with topological spaces with a selected basepoint as objects and continuous basepoint preserving functions as its morphisms.
- $\mathbf{Vect}_{\mathbb{K}}$, the category with vector spaces over field $\mathbb{K}$ as objects and $\mathbb{K}$-linear maps as morphisms.
- **Group**, the category with groups as objects and group homomorphisms as morphisms.

A subcategory $\mathscr{C}'$ of a category $\mathscr{C}$ is a category with the property that all its objects are objects in $\mathscr{C}$ and all its morphisms are morphisms in $\mathscr{C}$ with the same identity and composition rules. A subcategory $\mathscr{C}'$ is called full if for any two objects it has all morphisms between those objects in $\mathscr{C}$.

Remark 2.1.3. When discussing certain categories there can be set-theoretical problems that arise, it is for instance not possible to define the set of all sets. For this reason the term collection is used in the definition of categories. In category theory a distinction is made between small and large categories, the first having objects and morphisms that form a set and the second being to "large" to do this. Some large categories are locally small, meaning that the morphisms between any two objects form a set.

The distinction between small and large categories is of little importance to the goals of this thesis and will not be discussed in any more details, interested readers are refered to [3] and [4]. From this point onward all categories used will either be "sufficiently small " such that no set-theoretical problems arise or only a small subcategory of a large category will be considered.

Definition 2.1.4. An invertible morphism, or isomorphism, in a category $\mathscr{C}$ is a morphism $f : X \to Y$ for which there is a morphism $g : Y \to X$ such that $gf = id_X$ and $fg = id_Y$.

A category is called a groupoid if all its morphisms are isomorphisms.

Groupoids with one element play a central role in this thesis. These categories correspond to groups; any given group G defines an one-object groupoid. The morphisms are given by the elements of the group, the composition of morphisms is given by the composition laws of the group and the identity morphism is given by the neutral element of the group. The one-object groupoid corresponding to a group G will also be denoted G.

For a given category it is possible to define a category with the same objects but all the arrows going in the opposite direction, this resulting category is called the opposite category. A more formal definition is:

Definition 2.1.5. Let $\mathscr{C}$ be a category, the opposite category $\mathscr{C}^{op}$ has the same objects as $\mathscr{C}$ and a morphism f^{op} in $\mathscr{C}^{op}$ for each morphism f in $\mathscr{C}$ such that the domain of f^{op} is the codomain of f and the codomain of f^{op} is the domain of f.

For each object X in $\mathscr{C}$ the identity arrow 1_X induces an identity arrow 1_X^{op} for the object X in $\mathscr{C}^{op}$. The composition of two morphisms f^{op} and g^{op} in $\mathscr{C}^{op}$ is defined using composition in $\mathscr{C}$; we define $g^{op}f^{op}$ to be $(fg)^{op}$.

Note that the opposite category $\mathscr{C}^{op}$ is indeed a category if $\mathscr{C}$ is. An important property of the opposite category is that it contains precisely the same information as the original category, as a result every proof in category theory could be called a double proof. Not only the original statement is proven but also a dual statement for the opposite category, however, this dual statement is not always different from the original in any meaningful way.

2.2 Functors

In category theory we consider mathematical objects and the structure preserving maps between them. As categories are also mathematical objects this raises a question about what the structure preserving maps between categories are.

Definition 2.2.1. A functor $F : \mathscr{C} \to \mathscr{D}$ between two categories assigns an object FX in $\mathscr{D}$ to every object X in $\mathscr{C}$ and a morphism $Ff : FX \to FY \in \mathscr{D}$ to every morphism $f : X \to Y$ in $\mathscr{C}$.

The assignments have to follow two functorial axioms, being that for every composable pair of morphisms f and g in $\mathscr{C}$ we have $Fg \circ Ff = F(g \circ f)$ and for every object X in $\mathscr{C}$ we have $F(id_X) = id_{FX}$.

Thus a functor is a mapping between categories that preserves structure. When taking all categories as objects and the functors between them as morphisms one can easily check that the result is a category. Every functor indeed has specified domain and codomain objects, for any category $\mathscr{C}$ there is the identity functor $id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ that maps all objects and morphisms to them self and functors can be composed. The axioms of categories also clearly hold.

Functors arise naturally in many branches of mathematics, for instance in algebraic topology.

Example 2.2.2. The fundamental group defines a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$. Every object in $\mathbf{Top}_*$ gets mapped to its fundamental group and the morphisms, continuous basepoint preserving maps, $f : (X,x) \to (Y,y)$ get mapped to their induced group homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,y)$. Two basic and well known properties of these induced group homomorphisms are that the identity map induces the identity homomorphism and that for a pair of basepoint preserving maps $f : (X,x) \to (Y,y)$ and $g : (Y,y) \to (Z,z)$ we have $(gf)_* = g_* f_*$, meaning that this functor does indeed follow the two functorial axioms.

2.3 Natural transformations

In the same vein as the reasoning above one might wonder what structure preserving maps between functors are, since functors can also be regarded as mathematical objects with some structure.

Definition 2.3.1. Given two categories $\mathscr{C}$ and $\mathscr{D}$ and two functors $F, G : \mathscr{C} \to \mathscr{D}$. A natural transformation $\alpha : F \implies G$ consists of a morphism $\alpha_X : FX \to GX$ in $\mathscr{D}$ for each object X in $\mathscr{C}$ such that for any morphism $f : X \to Y$ in $\mathscr{C}$ the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\alpha_Y} & GY \end{array}$$

The collection of these α_X define the components of the natural transformation.

Per lemma 1.5.1 of [3], a natural transformation $\alpha : F \implies G$ can be seen as a functor

$$H : (0 \to 1) \times \mathscr{C} \to \mathscr{D}.$$

Here $(0 \to 1)$ is the discrete category with two elements and one non identity morphism between them. This alternate point of view will prove to be valuable later.

Similar to how a category can be formed with categories as objects and functors as morphisms, for two fixed categories $\mathscr{C}$ and $\mathscr{D}$ we can form a category with as objects all functors $\mathscr{C} \to \mathscr{D}$ and as morphisms natural transformations between these functors.

The identity natural transformation between a functor and itself has identity arrows for every component. Natural transformations can be composed through composing their components, leading to a composition law that is associative and has the identity natural transformations as neutral morphisms. Thus this indeed forms a category.

Chapter 3

Simplicial sets and the classifying space

In this chapter we discuss a method of relating a topological space to a category. The resulting topological space is called the classifying space and contains much of the information of the category. The classifying space is constructed using the theory of simplicial sets and much of this chapter will be devoted to giving an introduction to this theory.

The benefit of considering the classifying space of a category as opposed to the category itself is that it allows the usage of the vast toolbox that algebraic topology has to offer.

The following chapter is mostly based on [5], the section on the classifying space of a category is also based on [6].

3.1 The simplex category

Definition 3.1.1. Define the simplex category Δ with as objects the natural numbers, that are denoted as $[n]$ and seen as linear orders $[n] = \{0 \leq 1 \leq 2 \leq ... \leq n\}$. The morphisms in Δ are the non-decreasing maps $\alpha : [n] \to [m]$.

Note that this definition is equivalent to defining Δ as the category with as objects free categories

$$[n] = (0 \to 1 \to 2 \to ... \to n)$$

and as morphisms functors $[n] \to [m]$.

Both definitions of the simplex category will be used as the difference between them is purely their notation, depending on the context one notation is more convenient than the other.

There are two types of morphisms in Δ that are of particular interest. The first type are called the elementary faces, which for each $0 \leq i \leq n$ are monotone functions

$$\delta_i : [n-1] \to [n]$$

that skip the value i.

The second type are called the elementary degeneracies, which for each $0 \leq j \leq n-1$ are surjective functions

$$\sigma_j : [n] \to [n-1]$$

that map to j twice and every other value once.

Note that every non identity injective function $[n] \to [m]$ can be written as a composition of elementary faces and every non identity surjective function can be written as a composition of elementary degeneracies. Furthermore note that every morphism $[n] \to [m]$ can be written as a surjection followed by an injection. Meaning that every identity morphism can be written as the composition of elementary faces and degeneracies.

This composition, however, is not unique since the elementary faces and degeneracies satisfy the following relations:

$$\begin{aligned}
\delta_j \delta_i &= \delta_i \delta_{j-1} \text{ for } i < j \\
\sigma_i \sigma_j &= \sigma_{j-1} \sigma_i \text{ for } i < j \\
\sigma_i \delta_j &= \begin{cases} \delta_{j-1} \sigma_i & \text{if } i < j-1 \\ id & \text{if } i = j-1 \text{ or } i = j \\ \delta_j \sigma_{i-1} & \text{if } i > j \end{cases}
\end{aligned}$$

These identities can easily be checked and are called the cosimplicial identities.

Because every morphism in Δ can be written as a composition of elementary faces and degeneracies, it is possible to specify a functor F from Δ into another category by giving the values $F([n])$ for all $n \geq 0$ and the maps $F(\delta_i)$ and $F(\sigma_j)$.

This is sufficient since for every morphism $\alpha : [n] \to [m]$ we can write $\alpha = \delta_{i_k} ... \delta_{i_1} \sigma_{j_l} ... \sigma_{j_1}$ with suitable indices $i_1, .., i_k, j_1, ..j_l$. Using the functorial axioms we get

$$F(\alpha) = F(\delta_{i_k}) ... F(\delta_{i_1}) F(\sigma_{j_l}) ... F(\sigma_{j_1})$$

However, for this $F(\alpha)$ to be well defined, the maps $F(\delta_i)$ and $F(\sigma_j)$ must satisfy the cosimplicial identities.

3.2 Simplicial sets

Definition 3.2.1. A simplicial set is a functor

$$X : \Delta^{op} \to \mathbf{Sets}.$$

As seen before, a category can be formed with as objects functors between fixed categories and natural transformations between these functors as morphisms. The category of simplicial sets formed in this way will be denoted by **sSets**.

Simplicial sets are an example of the more general notion of simplicial objects in an arbitrary category $\mathscr{C}$, being functors $\Delta^{op} \to \mathscr{C}$. However, in this thesis only simplicial sets are of interest to us so we will not look into general simplicial objects.

More explicitly a simplicial set X is given by a sequence of objects $X_n := X([n])$ in **Sets** with $n \geq 0$ together with induced maps $\alpha^* : X_n \to X_m$ for morphisms $\alpha : [m] \to [n]$ in Δ. These induced maps are functorial, with which we mean

$$id^* = id : X_n \to X_n$$
$$(\alpha\beta)^* = \beta^*\alpha^* : X_n \to X_k \quad \text{for} \quad [k] \xrightarrow{\beta} [m] \xrightarrow{\alpha} [n]$$

We will refer to the elements of the set X_n as the n-simplices of X.

As discussed above, a functor $\Delta \to \mathbf{Sets}$ can be specified by giving all sets X_n together with the maps that all (δ_i) and (σ_j) are send to. Similarly, one can describe a simplicial set by giving all n-simplices together with the maps $(\delta_i)^*$ and $(\sigma_j)^*$, as every α^* is a composition of these maps. These maps are called the face maps and degeneracy maps respectively and we will write

$$\begin{aligned} d_i &= (\delta_i)^* : X_n \to X_{n-1} & \qquad & i = 0, 1, ..., n \\ s_j &= (\sigma_j)^* : X_{n-1} \to X_n & & i = 0, 1, ..., n-1. \end{aligned}$$

The requirement that the maps α^* are functorial is equivalent to requiring the following identities that are dual to the cosimplicial identities and are called the simplical identities.

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ for } i < j \\ s_j s_i &= s_i s_{j-1} \text{ for } i < j \\ d_j s_i &= \begin{cases} s_i d_{j-1} & \text{if } i < j-1 \\ id & \text{if } i = j-1 \text{ or } i = j \\ s_{i-1} d_j & \text{if } i > j. \end{cases} \end{aligned}$$

We can use the discussion above to associate a simplical set to a given category $\mathscr{C}$ which we will call the nerve of $\mathscr{C}$.

Definition 3.2.2. Given a category $\mathscr{C}$, its nerve $N\mathscr{C}$ is a simplicial set where every n-simplex X_n is the set of functors from the free category $[n]$ to $\mathscr{C}$. The face maps d_i are given by composition (or omission in the case of d_0 and d_n) and the degeneracy maps s_j are given by inserting identity arrows.

This definition of the nerve of a category is equivalent to defining $N\mathscr{C}$ as a simplicial set with the objects of $\mathscr{C}$ as its 0-simplices, morphisms as 1-simplices and strings of n composable morphisms as its n-simplices.

To see this consider an element of X_n, being a functor $F_n : [n] \to \mathscr{C}$. We can now identity F_n to a string of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_n} c_n.$$

Here c_i is the object in $\mathscr{C}$ that i as object in $[n]$ is mapped to by F_n and f_j is the morphism in $\mathscr{C}$ that the morphism from $j-1$ to j gets mapped to.

The face and degeneracy maps can be written down explicitly. For every $0 < i < n$, $0 < j < n-1$ we have:

$$d_0(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_n} c_n) = c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} ... \xrightarrow{f_n} c_n$$

$$d_n(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_n} c_n) = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} c_{n-1}$$

$$d_i(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_n} c_n) = c_0 \xrightarrow{f_0} ... \; c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \; ... \xrightarrow{f_n} c_n$$

$$s_j(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} c_{n-1}) = c_0 \xrightarrow{f_0} ...c_j \xrightarrow{id_{c_j}} c_j \; ... \xrightarrow{f_{n-1}} c_{n-1}.$$

Since these face and degeneracy maps obey the simplical identities, the induced maps are factorial and thus we see that the nerve of a category as defined above is indeed a simplicial set.

As discussed above the morphisms in **sSets** are natural transformations between functors $\Delta^{op} \to$ **Sets**. Recall that a natural transformation between two functors $X, Y : \Delta^{op} \to$ **Sets** consists of a sequence of morphisms $f_n : X_n \to Y_n$ in **Sets** with the property that $f_m\alpha^* = \alpha^* f_n$ for $\alpha : [m] \to [n]$.

Now, using the simplical identities, we can deduce that a collection of maps $f_n : X_n \to Y_n$ determines a morphism of simplicial sets if and only if it is compatible with the face and degeneracy maps, meaning

$$f_{n-1}d_i = d_i f_n \text{ for } n \geq 0, i = 0, ..., n$$
$$f_n s_j = s_j f_{n+1} \text{ for } n \geq 0, j = 0, ..., n-1.$$

3.3 The standard simplex

We now make a connection between simplical sets and topological spaces.

Definition 3.3.1. For each $n \geq 0$ the standard topological n-simplex is defined as

$$\Delta^n := \left\{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | t_0 + ... + t_n = 1, t_i \geq 0 \text{ for all } i\right\}$$

This standard simplex has $n+1$ vertices $v_0, ..., v_n$ where

$$v_i = (0, ..., 0, 1, 0, ..., 0)$$

with the 1 at the ith position.

It is clear that any function of sets $f : \{0, ..., m\} \to \{0, ..., n\}$ defines an affine map

$$f_* : \Delta^m \to \Delta^n$$

that is uniquely determined by the requirement $f(v_i) = v_{f(i)}$.

As a result the family of standard simplices gives us a functor

$$\Delta^\bullet : \Delta \to \mathbf{Top}.$$

This functor takes the object $[n]$ to the standard n-simplex and a morphism α in Δ to a map Δ^α between topological spaces. Recall that the morphisms in Δ are non decreasing maps $\alpha : [m] \to [n]$, which are in particular functions of sets $\{0, ..., m\} \to \{0, ..., n\}$. Meaning that every morphism α defines a map between standard topological n-simplices in the same way as for f above, we write Δ^α as α_*.

For a morphism $\alpha : [m] \to [n]$, an explicit expression for α_* can be given:

$$\alpha_*(t_0, ..., t_m) = (s_0, ..., s_n) \qquad \text{with } s_i = \sum_{\alpha(j)=i} t_j$$

Using this explicit expression on the elementary face map $\delta_i : [n-1] \to [n]$ we get

$$(\delta_i)_* : \Delta^{n-1} \to \Delta^n, \qquad (\delta_i)_*(t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}).$$

Meaning that $(\delta_i)_*$ embeds Δ^{n-1} as the face opposite the vertex v_i.

This can be generalized, for any injective map $\alpha : [m] \to [n]$ the corresponding map α_* embeds the m-simplex Δ^m as a face of Δ^n of some (possibly high) codimension.

For the elementary degeneracy map $\sigma_j : [n] \to [n-1]$ we see

$$(\sigma_j)_* : \Delta^n \to \Delta^{n-1}, \qquad (\sigma_j)_*(t_0, ..., t_n) = (t_0, ..., t_{j-1}, t_{j+1}, ..., t_n).$$

Meaning that $(\sigma_j)_*$ projects Δ^n onto Δ^{n-1} by a projection parallel to the line connecting vertices v_j and v_{j+1}

3.4 The geometric realization

We will now use these standard topological n-simplices to identify a topological space to each simplicial set X, which we call its geometric realization.

Definition 3.4.1. Given a simplicial set X its geometric realization $|X|$ is a topological space defined as a quotient of the large disjoint sum of simplices

$$\coprod_{n \geq 0} X_n \times \Delta^n = \coprod_{n \geq 0} \coprod_{x \in X_n} \Delta^n$$

the points of which we denote by (x, t) for $x \in X_n$ and $t \in \Delta^n$. The quotient is formed by making the identification

$$(x, \alpha_*) \sim (a^* x, t)$$

for each morphism $\alpha : [m] \to [n]$ of Δ and each $x \in X_m$, $t \in \Delta^n$.

We will write $x \otimes t$ for the equivalence class of a pair $(x, t) \in X_n \times \Delta^n$. The reason for this notation is that there is a sense in which $|X|$ can be interpreted as a tensor product, but we will not elaborate on it as it is not important for the contents of this thesis.

Recall that a map $f : X \to Y$ between simplicial sets consists of a sequence of morphisms $f_n : X_n \to Y_n$, such a map induces a continuous map

$$|f| : |X| \to |Y|, \;\; x \otimes t \mapsto f(x) \otimes t.$$

Here we have not included the subscript n on f in the expression $f(x)$ for $x \in X_n$.

Thus the geometric realization can be regarded as a functor

$$|.| : \mathbf{sSets} \to \mathbf{Top}.$$

3.5 The cellular structure of the geometric realization

In this section we will examine the cellular structure of the geometric realization $|X|$.

Note that in a simplical set, every n-simplex $x \in X_n$ defines a map

$$\hat{x} : \Delta^n \to |X|, t \mapsto x \otimes t$$

with the property that the images of all these maps cover he geometric realization $|X|$. Furthermore these maps respect the simplical structure of X, meaning that for any $\alpha : [m] \to [n]$ and $y \in X_m$ such that $\alpha^* x = y$ the diagram

$$\begin{tikzcd} \Delta^m \arrow[rr, "\hat{y}"] \arrow[dr, "\alpha_*"] & & {|X|} \\ & \Delta^n \arrow[ur, "\hat{x}"] & \end{tikzcd}$$

commutes.

Definition 3.5.1. An n-simplex $x \in X_n$ is called degenerate if it lies in the image of one of the degeneracy operators $s_i : X_{n-1} \to X_n$ for $0 \leq i \leq n-1$.

Note that this definition is equivalent to saying x is degenerate if there exists a surjection $\alpha : [n] \to [m]$ and $y \in X_m$ such that $x = \alpha_* y$.

As a result every point in $|X|$ can be represented in the form $y \otimes s$ with y a non-degenerate simplex of X. Let $x \otimes t \in |X|$. If x is degenerate there is a surjection $\alpha : [n] \to [m]$ and $\in X_m$ such that $x = \alpha_* z$, if in turn that z is degenerate it is possible to chose a further surjection. Clearly there is a surjection $\beta : [n] \to [k]$ such that $x = \beta^* y$ with y a non degenerate k-simplex of X. Then also $\beta_* : \Delta^n \to \Delta^k$ is a surjection and thus there is an $s \in \Delta^k$ such that $\beta_* s = t$. Thus $x \otimes t = y \otimes s$ with y non degenerate.

Now we conclude this section by stating theorem 2.3 from [5] without elaborating on the proof.

Theorem 3.5.2. *Let X be a simplical set. Its geometric realization $|X|$ naturally has the structure of a CW-complex with precisely one closed n-cell $\hat{x} : \Delta^n \to |X|$ for every non-degenerate n-simplex X_n.*

3.6 The classifying space

Definition 3.6.1. The geometric realization $|N\mathscr{C}|$ of the nerve of a category $\mathscr{C}$ is called the classifying space $B\mathscr{C}$.

Recall that the 0-simplices of the nerve $N\mathscr{C}$ are the objects of $\mathscr{C}$ and the n-simplices for $n \geq 1$ are strings of n composable morphisms. Using Theorem 3.5.2 we see that the classifying space of a category $\mathscr{C}$ is a CW-complex with the objects of $\mathscr{C}$ as 0-cells and strings of n composable non degenerate morphisms as n-cells.

A functor $F : \mathscr{C} \to \mathscr{D}$ induces a continuous map of classifying spaces $BF : B\mathscr{C} \to B\mathscr{D}$. Thus a structure preserving map between categories induces a structure preserving map between their classifying spaces. As seen in the previous chapter the structure preserving maps between functors are natural transformations. Natural transformations induce structure preserving maps between the continuous functions induced by functors.

Theorem 3.6.2. *Consider two functors $F, G : \mathscr{C} \to \mathscr{D}$ between categories and a natural transformation $\alpha : F \implies G$ between them. This natural transformation induces a homotopy $\Delta^1 \times B\mathscr{C} \to B\mathscr{D}$ between BF and BG.*

Proof. Recall that the natural transformation between F and G can be seen as a functor

$$(0 \to 1) \times \mathscr{C} \to \mathscr{D}.$$

Thus the natural transformation induces a continuous map between classifying spaces

$$B((0 \to 1) \times \mathscr{C}) \to B\mathscr{D}.$$

As shown with theorem 2 in [7] the canonical map $B(\mathscr{C} \times \mathscr{C}') \to B\mathscr{C} \times B\mathscr{C}'$ is a homeomorphism if every vertex of either $B\mathscr{C}$ or $B\mathscr{C}'$ belongs to finitely many simplices. Clearly every vertex of $B(0 \to 1)$ belongs to finitely many simplices, as it is the unit interval. Thus we can conclude that the natural transformation induces a continuous map $B(0 \to 1) \times B\mathscr{C} \to B\mathscr{D}$ that is a homotopy between BF and BG. □

The classifying space $B\mathscr{C}$ can often be more "complicated" than one might expect. For instance, the classifying space of a category with just one object and one non-identity morphism has a n-cell for every dimension n, as every string of n non identify morphisms

$$X \xrightarrow{\tau} X \xrightarrow{\tau} \dots \xrightarrow{\tau} X$$

is non degenerate.

Note that the category described above is the one-object groupoid corresponding to the group $\mathbb{Z}/2$, meaning that the classifying space of the smallest non-trivial group regarded as category is infinite dimensional. More generally, any category with a non-identity morphism from an object to itself has at least one n-cell for every $n \geq 0$.

Nonetheless, since the homotopy theory of CW-complexes is well understood there is much to gain from considering the classifying space. In the next chapter the homotopy type of certain categories will be discussed.

Chapter 4

The homotopy type of classifying spaces

In this chapter the homotopy type of classifying spaces of certain categories with one object will be explored. First we will introduced the concept of Eilenberg-MacLane spaces of type $K(G, 1)$, being path-connected spaces with a contractible universal cover with fundamental group G. We will state a homotopy uniqueness property for Eilenberg-MacLane spaces of type $K(G, 1)$ and give a construction of such spaces as a group quotient of a space EG called the total space. The space obtained from this group quotient is not only an Eilenberg-MacLane spaces of type $K(G, 1)$ but also the classifying space BG of the one-object groupoid G.

Finally the group completion theorem will be stated. This theorem gives that a category with one object and whose composition of morphisms is commutative has a classifying space homotopic to the classifying space of an one-object groupoid induced by that category.

The following material concerning Eilenberg-MacLane spaces and the construction of BG as a group quotient is adapted from [8].

4.1 Eilenberg-MacLane spaces of type $K(G, 1)$

Recall the definition and some properties of covering spaces.

Definition 4.1.1. A covering space of a space X is a space Y together with a map

$$p : Y \to X$$

with the property that for every point $x \in X$ there is an open neighborhood U in X such that $p^{-1}(U)$ is the union of disjoint open sets in Y that are all mapped homeomorphically to U by p.

If a covering space of X is path-connected and has a trivial fundamental group it is called a universal cover. Recall that universal covers of a space X are unique up to isomorphisms, thus a universal cover is often called the universal cover.

Example 4.1.2. A covering space of the circle S^1 is given by the disjoint union of $n \geq 1$ circles together with a map p that essentially the identity when restricted to a single circle.

Another covering space of the circle S^1 is $\mathbb{R}$ together with the map

$$p : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}.$$

Here S^1 is seen as the unit circle in $\mathbb{C}$. Since $\mathbb{R}$ is path-connected and its fundamental group is trivial, this is the universal cover of S^1.

Definition 4.1.3. For a group G, an Eilenberg-MacLane space of type $K(G, 1)$ is a path-connected space with a fundamental group isomorphic to G and a contractible universal covering space. For convenience such a space will simply be called a $K(G, 1)$ space.

Note that every path-connected space X with a contractible universal covering space is an Eilenberg-MacLane space of type $K(\pi_1(X), 1)$.

A $K(G, 1)$ space is not uniquely determined by the group G. For instance, S^1 is path-connected and $\mathbb{R}$ is a contractible universal covering space of S^1. As the fundamental group of S^1 is $\mathbb{Z}$, it is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 1)$. The Möbius strip is also path-connected, has $\mathbb{R} \times [0, 1]$ as contractible universal covering space and also has fundamental group $\mathbb{Z}$. Thus the Möbius strip is also an Eilenberg-MacLane space of type $K(\mathbb{Z}, 1)$.

There is, however, a homotopical uniqueness property for $K(G, 1)$ spaces that are CW-complexes. The following theorem is theorem 1B.8 in [8] and will be used without elaborating on the proof.

Proposition 4.1.4. *The homotopy type of a CW-complex $K(G, 1)$ space is uniquely determined by G.*

4.2 The classifying space of a group

Recall the definitions of an action of a group and the quotient space.

Definition 4.2.1. Given a group G and a space X, an action of G on X is a group homomorphism from G to the group of homeomorphisms from X to itself. Thus there is a homeomorphism $X \to X$ associated to every $g \in G$, which will be denoted simply as $g : X \to X$.

Definition 4.2.2. Given a group action of G on X, the quotient space X/G is obtained by identifying each point $x \in X$ with its images $g(x)$ for all $g \in G$. The points in x/G are thus given by the orbits $Gx := \{g(x) | x \in X\}$.

Proposition 1.40 from [8] concerning group actions and covering spaces will be used without proof.

Proposition 4.2.3. *Consider an action of a group G on a space X satisfying the condition that for each $x \in X$ there is a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. Then the quotient map $p : X \to X/G, x \mapsto Gx$ is a covering space.*

If furthermore X is path-connected, locally path-connected and has a trivial fundamental group, then $\pi_1(X/G)$ is isomorphic to G.

In example 1B.7 of [8] it is noted that if a group acts on a the geometric realization of a simplicial set by taking the realization of every simplex linearly into the realization of another simplex, the condition of every point having a neighborhood whose images under different elements in G are disjoint is met when only the identity takes any simplex to itself.

Definition 4.2.4. For a group G define the total space, EG, to be the classifying space of the category $\mathscr{G}$, with as objects the elements of G and the morphisms from object $g \in G$ to $h \in G$ given by hg^{-1}.

Thus EG is the geometry realization of the simplicial set $N\mathscr{G}$, the nerve of $\mathscr{G}$. The n-simplices of $N\mathscr{G}$ are ordered $(n+1)$-tuples $[g_0, .., g_n]$ with $g_0, .., g_n \in G$, the face maps are given by composition and the degeneracy maps are given by inserting the identity $e \in G$.

Note that for any group G the total space EG is contractible. Any point $x \in EG$ lies in the geometric realization of some non-degenerate n-simplex $[g_0, ..., g_n]$. Either one of $g_0, .., g_n \in G$ is the identity e or the $(n+1)$-simplex $[e, g_0, ..., g_n]$ is non-degenerate and thus its geometric realization is a subset of EG. In either case a homotopy h_t can be defined, it moves x along the line segment in the geometric realization of $[g_0, ..., g_n]$ or $[e, g_0, ..., g_n]$ respectively.

This homotopy is well defined; if we restrict to a face $[g_0, ..., \hat{g}_i, ..., g_n]$ there is a linear deformation to $[e]$ in $[e, g_0, ..., \hat{g}_i, ..., g_n]$. Here $\hat{g}_i$ indicates that this vertex is deleted.

We will now show that EG/G is the classifying space of the one-object groupoid given by G.

Proposition 4.2.5. *For any group G acting on EG by $g \in G$ taking a simplex $[g_0, .., g_n]$ linearly onto $[gg_0, ..., gg_n]$, the quotient space EG/G is the classifying space of the one-object groupoid defined by G.*

Proof. First note that the group action identifies all the vertices of EG to each other, meaning that EG/G indeed only has one 0-simplex. Every n-simplex of EG can be written uniquely as

$$[g_0, g_0g_1, g_0g_1g_2, ..., g_0g_1 \cdots g_n] = g_0[e, g_1, g_1g_2, ..., g_1g_2 \cdots g_n].$$

Thus the image of such a n-simplex under the quotient map can unambiguously be denoted by the ordered n-tuple $[g_1|g_2|...|g_n]$. Meaning that the n-simplices in EG/G correspond to the composition of n morphisms in the one-object groupoid defined by G. This correspondence is one-to-one, the composition of n morphisms in the groupoid gives an unique n-simplex of EG. As the n-simplices in the nerve of the groupoid are the compositions of n morphisms, we can conclude that EG/G and and the nerve of the groupoid are the same simplicial set. Thus they have the same geometric realization and are the same space. □

This classifying space is called the classifying space of G and denoted BG.

Now we show that the classifying space BG is an Eilenberg-MacLane space of type $K(G, 1)$.

Proposition 4.2.6. *The quotient EG/G is a $K(G, 1)$ space.*

Proof. It is clear that only the identity takes a simplex to itself. Thus, using Proposition 4.2.3 together with example 1B.7 from [8] we see that the quotient is a covering map. Recall that EG has a CW structure and thus is locally path-connected. Since EG is also contractible, Proposition 4.2.3 gives that EG/G has a fundamental group isomorphic to G and thus we can conclude that EG/G is a $K(G, 1)$ space. □

As the classifying space of a category is the geometric realization of a simplicial set, it naturally has the structure of a CW-complex. Using Proposition 4.1.4 this gives the following important result.

Corollary 4.2.7. *The homotopy type of the classifying space BG of a group G is uniquely determined by G and BG is homotopic to every Eilenberg-MacLane space of type $K(G, 1)$.*

4.3 Group completion theorem

Now we relate the classifying space of any category with one object and whose composition of morphisms is commutative to the classifying space of a group.

Categories with one object correspond to the algebraic structure of monoids, the following definition of monoids and their group completion is based on [9].

Definition 4.3.1. A monoid is a set M with an operation $+$ which is associative and has a identity element e in M. A monoid is said to be commutative if its operation is commutative.

A map $\mu : M \to N$ between monoids is said to be a monoid map if for all $m, n \in M$ $\mu(m+n) = \mu(m) + \mu(n)$ and it maps the identity in M to the identity in N.

Monoids are a generalization of groups. Every group is a monoid but a monoid is in general not a group, the difference being that every element in a group has an inverse and this is not a requirement for a monoid.

In the same way that every group corresponds to the category that is an one-object groupoid, every monoid corresponds to a category with one object and a morphism for every element in the monoid with composition of morphisms given by the operation on the monoid. Note that if a monoid is commutative, the composition of the morphisms in its corresponding category is commutative. For monoids that are groups the corresponding category is exactly the one-object groupoid as seen earlier. The classifying space of the category corresponding to a monoid M will be called the classifying space of M and denoted BM.

It is possible turn a monoid M into a group by adding an inverse for every non-invertible element. This process is called the group completion and can be made precise for commutative monoids in the following way.

Definition 4.3.2. A group completion of a commutative monoid M is an commutative group MM^{-1} together with a monoid map $\mu : M \to MM^{-1}$ such that for every commutative group A and every monoid map $\alpha : M \to A$ there is an unique abelian group homomorphism $\tilde{\alpha} : MM^{-1} \to A$ such that $\tilde{\alpha}(\mu(m)) = \alpha(m)$ for all $m \in M$.

Example 4.3.3. The natural numbers $\mathbb{N}$ form a commutative monoid with addition as operation and zero as identity element. As there are no two strictly positive numbers that together add up to zero $\mathbb{N}$ is not a group, many elements miss an inverse.

The group completion of $\mathbb{N}$ is $\mathbb{Z}$, the monoid map $\mu : \mathbb{N} \to \mathbb{Z}$ is given by the canonical inclusion. For every commutative group A and monoid map $\alpha : \mathbb{N} \to A$ the map $\tilde{\alpha} : \mathbb{Z} \to A$ defined as $\tilde{\alpha}(m) = \alpha(m)$ is an uniquely defined group homomorphism.

The classifying spaces of a commutative monoid and its group completion can be related, in fact Theorem 4.4.1 in [10] states that the two classifying spaces are homotopic:

Theorem 4.3.4. *If M is a commutative monoid and MM^{-1} is its group completion, then the classifying space of M and MM^{-1} are homotopic.*

Chapter 5

The classifying space of the (1+1)-dimensional cobordism category

In this chapter we introduce the concept of (n+1)-cobordisms and the (n+1)-dimensional cobordism category. We will focus on the (1+1)-dimensional cobordism category and determine the homotopy type of two of its subcategories. Lastly we will determine the fundamental group of the classifying space of the (1+1)-dimensional cobordism category.

Both the structure of and the proofs in this chapter are based on Tillmann's paper [1].

5.1 The (n+1)-dimensional cobordism category

Definition 5.1.1. A (n+1)-cobordism M from a compact oriented n-manifold Σ_0 to another compact oriented n-manifold Σ_1, is a compact oriented (n+1)-manifold M with boundary such that its boundary is the disjoint union of Σ_0 and Σ_1. The induced orientation of the boundary δM must agree with the orientation on Σ_0 and be opposite to the orientation on Σ_1.

For convenience simply the term cobordism between Σ_0 and Σ_1 will be used to mean an oriented cobordism with the properties of the definition above. The (n+1)-dimensional cobordisms give rise to a category.

Definition 5.1.2. The (n+1)-dimensional cobordism category **(n+1)-Cob** is the category with an object for ever homeomorphism class of compact oriented n-manifolds and the morphisms given by homeomorphism classes of cobordisms between these manifolds. The empty set is also regarded as a n-manifold and the identity morphism on an object Σ is given by $\Sigma \times [0, 1]$. The composition of morphisms is given by the composition of representations from their homeomorphism classes.

Remark 5.1.3. The composition of two cobordisms $M : \Sigma_0 \to \Sigma_1$ and $M' : \Sigma_1 \to \Sigma_2$ is obtained by gluing the two manifolds along Σ_1 and the resulting space is a cobordism from Σ_0 to Σ_1. The homeomorphism class of the resulting cobordism is determined only by the homeomorphism classes of M and M', meaning that the composition of morphisms is well-defined. A detailed proof of this is given in [2].

The main focus of this thesis from this point onward will be the (1+1)-dimensional cobordism category which we will denote by $\mathscr{S}$. The (1+1)-cobordisms can be build from the composition of six "building blocks" as shown in the Figure 5.1, details of this can be found in [2].

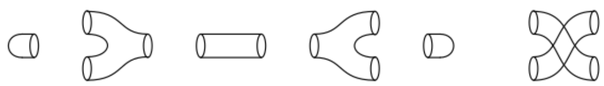

Figure 5.1: The building blocks of (1+1)-cobordism, found in [2].

Note that every compact, connected and oriented manifold of dimension 1 is homeomorphic to the circle and every compact, oriented manifold of dimension 1 is homeomorphic to the disjoint union of circles. Thus there is a bijection between the objects in $\mathscr{S}$ and the natural numbers where $n \in \mathbb{N}$ corresponds to the homeomorphism class of the disjoint union of n circles. Note that the disjoint union of zero circles corresponds to the empty 1-manifold.

Recall that every connected, compact oriented surface is determined up to homeomorphisms by its genus and the number of boundary components via the Euler characteristic. For a connected, compact oriented surface M with genus g and n discs removed the Euler characteristic is given by $\chi(M) = 2 - 2g - n$. The composition of (1+1)-cobordisms corresponds to the addition of their Euler characteristics.

Since the Euler characteristic of the circle S^1 or the disjoint union of any number of circles is zero, the Euler characteristic of every object in $\mathscr{S}$ is zero. Thus we see that the Euler characteristic induces a functor $\chi : \mathscr{S} \to \mathbb{Z}$ from $\mathscr{S}$ to the one-object monoid $\mathbb{Z}$.

5.2 Analysis of subcategories

In this section we determine the homotopy type of two subcategories of $\mathscr{S}$. We begin by defining $\mathscr{S}_0$ to be the full subcategory of $\mathscr{S}$ with the empty 1-manifold as its only object. Note that all morphisms in $\mathscr{S}_0$ are compact surfaces without boundary.

Theorem 5.2.1. *The classifying space $B\mathscr{S}_0$ is homotopic to $\mathbb{R}^\infty/\mathbb{Z}^\infty$.*

Proof. As the morphisms of $\mathscr{S}_0$ are compact surfaces without boundary every morphism corresponds to a sequence $(n_0, n_1, n_2, ...)$ of finitely many non-zero natural numbers, standing for the disjoint union of n_0 spheres, n_1 tori, n_2 surfaces of genus 2, etc. The composition of morphisms corresponds to addition in the components of the sequences.

Sequences $(n_0, n_1, n_2, ...)$ of finitely many integers together with componentwise addition form a monoid we will denote $\mathbb{N}^\infty$. As $\mathscr{S}_0$ is a category with one object, its morphisms correspond to the elements of monoid $\mathbb{N}^\infty$ and composition of morphisms $\mathscr{S}_0$ corresponds to addition in $\mathbb{N}^\infty$ we see that $B\mathscr{S}_0 = B\mathbb{N}^\infty$.

Since addition of natural numbers is commutative, we can use the group completion Theorem 4.3.4. Thus we know that $B\mathbb{N}^\infty$ is homotopic to the classifying space of the group completion of $\mathbb{N}^\infty$. We will denote the group completion of $\mathbb{N}^\infty$ by $\mathbb{Z}^\infty$, being the group with sequences $(n_0, n_1, n_2, ...)$ of finitely many non-zero (possibly negative) integers as elements and componentwise addition as operation.

Now we construct an Eilenberg-MacLane space of type $K(\mathbb{Z}^\infty, 1)$. Define $\mathbb{R}^\infty$ to be the space consisting of all sequences $(x_0, x_1, x_2, ...)$ of finitely many non-zero real numbers with the direct limit topology induced from the finite dimensional subspaces $\mathbb{R}^n$. Note that $\mathbb{R}^\infty$ is locally path-connected and contractible

Consider the group action of $\mathbb{Z}^\infty$ acting on $\mathbb{R}^\infty$ by $(n_0, n_1, n_2, ...) \in \mathbb{Z}^\infty$ taking $(x_0, x_1, x_2, ...) \in \mathbb{R}^\infty$ to $(x_0 + n_0, x_1 + n_1, x_2 + n_2, ...)$. We can now use Proposition 4.2.3 regarding group actions, as $\mathbb{R}^\infty$ is contractible we see that $\mathbb{R}^\infty/\mathbb{Z}^\infty$ is an Eilenberg-MacLane space of type $K(Z^\infty, 1)$.

Thus, using Corollary 4.2.7 we can conclude that $B\mathscr{S}_0$ is homotopic to $\mathbb{R}^\infty/\mathbb{Z}^\infty$. □

Remark 5.2.2. To conclude that the classifying space of $B\mathbb{Z}^\infty$ is isomorphic to $\mathbb{R}^\infty/\mathbb{Z}^\infty$ using Proposition 4.1.4 it is necessary for $\mathbb{R}^\infty/\mathbb{Z}^\infty$ to be a CW-complex. To see that $\mathbb{R}^\infty/\mathbb{Z}^\infty$ is indeed a CW-complex consider the embedding of $\mathbb{R}^n$ into $\mathbb{R}^\infty$ that maps a point $(x_1, ..., x_n)$ to $(x_1, ..., x_n, 0, ...)$. Every point in $\mathbb{R}^\infty$ lies in the image of such an embedding for a n, in this way $\mathbb{R}^\infty$ inherits a CW-structure. As $\mathbb{R}^\infty/\mathbb{Z}^\infty$ is the quotient of a free action on a CW-complex it is again a CW-complex and thus we can conclude that $B\mathscr{S}_0$ is isomorphic to $\mathbb{R}^\infty/\mathbb{Z}^\infty$.

Now consider the subcategory of $\mathscr{S}$ that does not contain the empty 1-manifold as object nor any morphisms that have a component that is a morphism to or from the empty 1-manifold. We denote this subcategory by $\mathscr{S}_{>0}$.

Theorem 5.2.3. *The classifying space of $\mathscr{S}_{>0}$ is homotopic to the circle S^1.*

Proof. We first consider the full subcategory $\mathscr{S}_1$ of $\mathscr{S}_{>0}$ with as only object the circle. As the morphisms in $\mathscr{S}_{>0}$ do not have any components that are morphisms to or from the empty 1-manifold, all the morphisms in $\mathscr{S}_1$ are connected cobordisms with one incoming and one outgoing boundary circle. These morphisms are in one-to-one correspondence with the natural numbers, as they are determined by their genus and composition of morphisms corresponds to the addition of their genera. Thus we see that $B\mathscr{S}_1 = B\mathbb{N}$.

Recall from Example 4.3.3 that $\mathbb{N}$ is a commutative monoid and its group completion is $\mathbb{Z}$. Using the group completion Theorem 4.3.4 we can conclude that $B\mathscr{S}_1$ is homotopic to $B\mathbb{Z}$. From Example 4.1.2 we know that $B\mathbb{Z}$ is homotopic to the circle, as S^1 is a $K(\mathbb{Z}, 1)$ space. Thus the classifying space $B\mathscr{S}_1$ is homotopic to the circle S^1.

Now we will construct a functor $\Phi : \mathscr{S}_{>0} \to \mathscr{S}_1$ such that there is a natural transformation between Φi and the identify functor on $\mathscr{S}_1$ as well as a natural transformation between $i\Phi$ and the identify functor on $\mathscr{S}_{>0}$, with i the inclusion $\mathscr{S}_1 \to \mathscr{S}_{>0}$. Using Theorem 3.6.2, stating that a natural transformation between functors induces a homotopy between induced functions of classifying spaces, we can then conclude that the classifying spaces of $\mathscr{S}_0$ and $\mathscr{S}_{>0}$ are homotopic.

Define the functor $\Phi : \mathscr{S}_{>0} \to \mathscr{S}_1$ to be the constant map on objects, as $\mathscr{S}_1$ only has the circle as object. Define the image of a morphism M in $\mathscr{S}_{>0}$ with n source circles, c connected components, total genus g and m target circles to be the unique morphism in $\mathscr{S}_1$ with genus $\frac{1}{2}(m - n - \chi(M)) = g + m - c$. As no morphism in $\mathscr{S}_{>0}$ has a component that is a morphism to or from the empty 1-manifold, the number of connected components of a morphism in $\mathscr{S}_{>0}$ can not be greater than the number of target circles. Thus $m \geq c$, meaning that $g + m - c \geq 0$ and therefore Φ is well-defined.

We now show that Φ is a functor. Let $M_{n,m}$ be a morphism from n to m circles and $M_{m,k}$ be a morphism from m to k circles. Then, using that composition of (1+1)-cobordisms corresponds to addition of the Euler characteristics,

$$\begin{aligned}
\Phi(M_{n,m} \circ M_{m,k}) &= \frac{1}{2}(k - n - \chi(M_{n,m} \circ M_{m,k})) \\
&= \frac{1}{2}(m - n - \chi(M_{n,m}) + k - m - \chi(M_{m,k})) \\
&= \Phi(M_{n,m}) + \Phi(M_{m,k}).
\end{aligned}$$

Thus we see that Φ is a functor.

Note that $\Phi i : \mathscr{S}_1 \to \mathscr{S}_1$ is the identify functor on $\mathscr{S}_1$. The only object in $\mathscr{S}_1$ gets mapped to itself and since every morphism M in $\mathscr{S}_1$ with genus g has one target circle and one connected component we see $\Phi(M) = g + 1 - 1 = g$, thus every morphism also gets mapped to itself. In particular we can conclude that there is a natural transformation between Φi and the identity on $\mathscr{S}_1$.

Now we show that the function of classifying spaces induced by $i\Phi : \mathscr{S}_{>0} \to \mathscr{S}_{>0}$ is homotopic to the function induced by the identity $Id_{\mathscr{S}_{>0}} : \mathscr{S}_{>0} \to \mathscr{S}_{>0}$. We define a natural transformation $\alpha : Id_{\mathscr{S}_{>0}} \implies \Phi$. For the object n being n copies of the circle, define the morphism $\alpha_n : n \to 1$ to be the connected morphism joining n circles to 1 with genus $1-n$. We need to check that for arbitrary morphisms M in $\mathscr{S}_{>0}$ with genus g and c connected components the following diagram commutes.

$$\begin{tikzcd} n \arrow[r, "\alpha_n"] \arrow[d, "M"] & 1 \arrow[d, "\Phi(M)"] \\ m \arrow[r, "\alpha_m"] & 1 \end{tikzcd}$$

First note that both $\Phi(M) \circ \alpha_n$ and $\alpha_m \circ M$ are connected surfaces with n source and one target circle. Furthermore the genus of $\Phi(M) \circ \alpha_n$ is $g + m - c$ and the genus of M increases with $m - c$ when composed with α_m, thus both $\Phi(M) \circ \alpha_n$ and $\alpha_m \circ M$ have the same genus. Therefore we can conclude that the diagram commutes and α is a natural transformation. Meaning the function of classifying spaces induced by $i\Phi$ is homotopic to the identity map.

Thus we see that the induced maps $Bi\Phi = Bi \circ B\Phi$ and $B\Phi i = B\Phi \circ Bi$ are homotopic to the the identity on $B\mathscr{S}_{>0}$ and $B\mathscr{S}_1$ respectively, meaning that $B\mathscr{S}_{>0}$ and $B\mathscr{S}_1$ are homotopic.

We conclude that $B\mathscr{S}_{>0}$ is homotopic to the circle S^1. □

5.3 The fundamental group of $B\mathscr{S}$

The goal of this section is to prove the following proposition.

Proposition 5.3.1. *The fundamental group of $B\mathscr{S}$ is isomorphic to $\mathbb{Z}$.*

To show this we define a category $\mathscr{G}$ as the groupoid obtained from $\mathscr{S}$ by adding an inverse for every morphism and taking the quotient of certain equivalence relations, to assure that every morphism has an unique inverse.

Note that that the morphisms in this category $\mathscr{G}$ do not correspond to cobordisms anymore. There is for instance no surface that can be composed with a sphere to get the an empty 2-manifold, which is the identity morphism on the empty 1-manifold.

We will use the following result of Proposition 1 from [6] without giving a formal proof.

Lemma 5.3.2. *The fundamental group of $B\mathscr{G}$ is also the fundamental group of $B\mathscr{S}$.*

It is not true that $B\mathscr{S}$ and $B\mathscr{G}$ are homotopic, only their fundamental groups are necessarily equal. Intuitively this is a result of the fact that the fundamental group of a CW-complex is only determined by its 0-,1- and 2-cells and not by its higher dimensional cells. Adding an inverse gives an extra 1-cell for every morphism, but the composition of a morphism with its inverse gives a 2-cell through which these two 1-cells are homotopic.

Denote by $\mathscr{G}_n$ the full subcategory of $\mathscr{G}$ with as only object n. This category is an one-object groupoid and as such corresponds to a group which we will also denote by $\mathscr{G}_n$. We now prove a lemma regarding $\mathscr{G}_n$ as a group and the fundamental group of $\mathscr{G}$.

Lemma 5.3.3. *All $\mathscr{G}_n$ are isomorphic as groups and $\pi_1(B\mathscr{G}) = \mathscr{G}_n$.*

Proof. We begin by proving that all $\mathscr{G}_n$ are isomorphic by constructing a group isomorphism between $\mathscr{G}_1$ and an arbitrary $\mathscr{G}_n$. As every morphism in $\mathscr{G}$ is invertible, any morphism $\alpha : n \to 1$ gives a conjugation $c_\alpha : \mathscr{G}_1 \to \mathscr{G}_n, \beta \mapsto \alpha^{-1}\beta\alpha$ that is a group isomorphism. Since there is a morphism between every two objects in $\mathscr{S}$, there is an isomorphism between $\mathscr{G}_1$ and $\mathscr{G}_n$ for every $n \in \mathbb{N}$. Thus all $\mathscr{G}_n$ are isomorphic.

Now consider the inclusion functor $i : \mathscr{G}_n \to \mathscr{G}$. As $\mathscr{G}_n$ is the full subcategory of $\mathscr{G}$ with only n as object and there is an invertible morphism between n and any object in $\mathscr{G}$, Theorem 1.5.9 of [3] gives that there is a functor $F : \mathscr{G} \to \mathscr{G}_n$ such that there are natural transformations $iF \implies id_{\mathscr{G}}$ and $Fi \implies id_{\mathscr{G}}$, assuming the axiom of choice. Thus, using Theorem 3.6.2 regarding homotopic maps induced by natural transformations between functors, we see that $B\mathscr{G}$ and $B\mathscr{G}_n$ are homotopic. In particular this means that the fundamental group of $B\mathscr{G}$ is isomorphic to the fundamental group of $B\mathscr{G}_n$. As $B\mathscr{G}_n$ is a $K(\mathscr{G}_n, 1)$, we see $\pi_1(B\mathscr{G}_n) = B\mathscr{G}_n$. We conclude that $\pi_1(B\mathscr{G}) = \mathscr{G}_n$. □

In particular this gives that $\pi_1(\mathscr{G}) = \mathscr{G}_1$. We now prove that $\mathscr{G}_1 = \mathbb{Z}$, in combination with Lemma 5.3.3 and Lemma 5.3.2 this concludes the proof of Proposition 5.3.1.

Proof of Proposition 5.3.1. We first take a closer look at morphisms from 1 to 1 in $\mathscr{S}$ and give a monoid with which these morphisms are in one-to-one correspondence. Using this monoid and group isomorphisms $\mathscr{G}_0 \to \mathscr{G}_1$ and $\mathscr{G}_1 \to \mathscr{G}_2$ we will deduce certain relations on $\mathscr{G}_1$ that give that $\mathscr{G}_1 = \mathbb{Z}$.

Consider all morphisms from 1 to 1 in $\mathscr{S}$. Apart from components that are a disjoint union of closed surfaces and thus morphisms from the empty 1-manifold to the empty 1-manifold, every morphism from 1 to 1 in $\mathscr{S}$ is either a connected surface with two boundary circles and genus g or two surfaces with both one boundary circle and genus a and b. These two types of morphisms are shown in Figure 5.2.

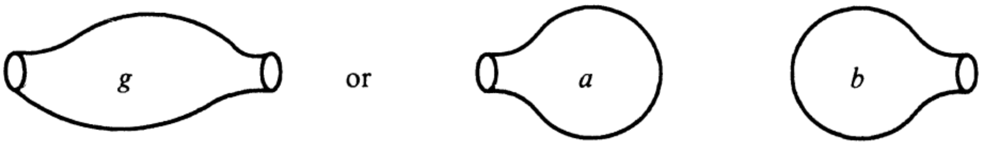

Figure 5.2: Two types of morphisms from 1 to 1, obtained from [1].

Morphisms from 1 to 1 in $\mathscr{S}$ are in one-to-one correspondence with the non-commutative monoid $(\mathbb{N} \cup (\mathbb{N} \times \mathbb{N})) \times \mathbb{N}^{\infty}$ with addition given by the geometry. With this notation we mean that elements of this monoid are either of the form $(g, (n_0, n_1, ...))$ or of the form $((a, b), (n_0, n_1, ...))$. The first corresponds to the morphism that is a connected surface with two boundary circles and genus g together with n_i closed surfaces with genus i. The second corresponds to the morphisms consisting of a connected morphisms from 1 to 0 with genus a, a connected morphism from 0 to 1 with genus b and n_i closed surfaces with genus i. The addition given by the geometry can be explicitly expressed:

$$\begin{aligned}
(g, (n_0, n_1, ...)) \circ (g', (n'_0, n'_1, ...)) &= (g + g', (n_0 + n'_0, n_1 + n'_1, ...)), \\
(g, (n_0, n_1, ...)) \circ ((a, b), (n'_0, n'_1, ...)) &= ((g + a, b), (n_0 + n'_0, n_1 + n'_1, ...)), \\
((a, b), (n_0, n_1, ...)) \circ (g, (n'_0, n'_1, ...)) &= ((a, b + g), (n_0 + n'_0, n_1 + n'_1, ...)), \\
((a, b), (n_0, n_1, ...)) \circ ((a', b'), (n'_0, n'_1, ...)) &= ((a, b'), (n_0 + n'_0, n_1 + n'_1, ..., n_{b+a'} + 1, ...)).
\end{aligned}$$

Now define a group isomorphism $c_\alpha : \mathscr{G}_0 \to \mathscr{G}_1,\ \beta \mapsto \alpha^{-1}\beta\alpha$ with $\alpha : 0 \to 1$ a sphere with one boundary circle. To construct the inverse of α, note that the result of composing α with the morphism $1 \to 0$ that is a sphere with one boundary circle is the sphere as a morphism $0 \to 0$. By identifying the morphisms from $0 \to 0$ in $\mathscr{S}$ with $\mathbb{N}^{\infty}$ as before, the inverse of the sphere can be denoted $(-1, 0, ...)$. Thus the inverse of this $\alpha : 0 \to 1$ is the union of a sphere with one boundary circle and the inverse of the sphere $(-1, 0, ...)$. Now we see that c_α maps

$$\begin{aligned}
(g, (n_0, n_1, ...)) &\mapsto (n_0 - 1, n_1, ..., n_g + 1, ...), \\
((a, b), (n_0, n_1, ...)) &\mapsto (n_0 - 1, n_1, ..., n_a + 1, ..., n_b + 1, ...).
\end{aligned}$$

However, as c_α is a group isomorphism it must obey the group structure and be injective. For any two elements $\beta, \gamma \in \mathscr{G}_1$ we must have that $c_\alpha(\beta \circ \gamma) = c_\alpha(\beta) \circ c_\alpha(\gamma)$. This forces the following identifications on $\mathscr{G}_1$:

$$\begin{aligned}
(g, (n_0, n_1, ...)) &\sim (g + \Sigma i\ n_i, (\Sigma n_i, 0, 0, ...)), \\
((a, b), (n_0, n_1, ...)) &\sim (a + b + \Sigma i\ n_i, (1\Sigma n_i, 0, 0, ...)).
\end{aligned}$$

Note that this means that an equivalence class of a morphism in $\mathscr{G}_1$ is uniquely determined by its genera and the number of connected components.

Now we consider a group isomorphisms $C_\beta : \mathscr{G}_2 \to \mathscr{G}_1, \gamma \mapsto \beta^{-1}\gamma\beta$ with $\beta : 1 \to 2$ being the union of a cylinder and a disk. There are multiple representations of the inverse of β, we consider two of them. Let $\beta_1 : 2 \to 1$ be the connected morphism with genus 0 from two circles to one and $\beta_1 : 2 \to 1$ be the union of a disk, a cylinder and the inverse of a sphere. Now let $\gamma \in \mathscr{G}_2$ be the union of the connected morphism with genus 0 from one circle to two with a disk. We compare the two images of $C_\beta(\gamma)$ using the two different representations of the inverse of β, this is shown in Figure 5.3.

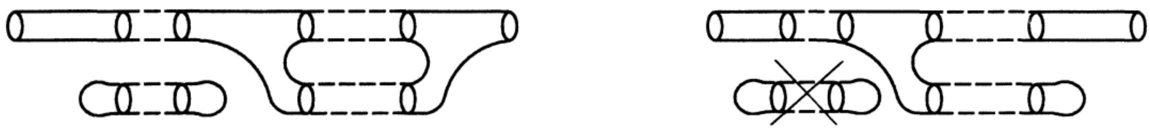

Figure 5.3: $\beta_1\gamma\beta$ and $\beta_2\gamma\beta$, obtained from [1].

These two morphisms must be equivalent in $\mathscr{G}_1$, meaning that

$$(g, (n_0, 0, 0, ...)) \sim (g - n_0, (0, 0, ...)).$$

Thus we have

$$\begin{aligned} (g, (n_0, n_1, ...)) &\sim (g + \Sigma(i-1)n_i, (0, 0, ...)), \\ ((a, b), (n_0, n_1, ...)) &\sim (a + b - 1 + \Sigma(i-1)n_i, (0, 0, ...)). \end{aligned}$$

This means that the equivalence classes of morphisms in $\mathscr{G}_1$ depend only on the difference of the sum of the genera and the number of connected components. This is half of their Euler characteristic and as noted at the beginning of this chapter, the Euler characteristic induces a functor $\chi : \mathscr{S} \to \mathbb{Z}$. For this functor to be well-defined, different Euler characteristics must correspond to different equivalence classes. So there is a one-to-one correspondence of $\mathbb{G}_1$ with $\mathbb{Z}$.

We conclude that $\pi_1(B\mathscr{S}) = \mathscr{G}_1 = \mathbb{Z}$. $\square$

Chapter 6

Topological quantum field theories

Now that we have studied the (1+1)-dimensional cobordism category and its classifying space, we will give an application regarding (1+1)-dimensional topological quantum field theories.

In this chapter we will first formally introduce the concept of topological quantum field theories in a manner based on [2]. Then we will use our knowledge of the classifying space of the (1+1)-dimensional cobordism category to classify morphism inverting (1+1)-dimensional topological quantum field theories, this will be shown using a proof based on [1].

6.1 Topological quantum field theories

There are many different variations on the definition of topological quantum field theories. The definition we use in this thesis is very similar to the definition as originally put forward by Atiyah in [11] and [12]. It is however slightly rewritten following [2], giving an intuitive idea of the interpretation of topological quantum field theories.

Definition 6.1.1. A (n+1)-dimensional topological quantum field theory (TQFT) is a rule $\mathscr{A}$ which to each compact oriented n-manifold Σ associates a vector space $\mathscr{A}\Sigma$ over $\mathbb{C}$, and to each oriented cobordism $M : \Sigma_0 \to \Sigma_1$ associates a linear map $\mathscr{A}M$ from $\mathscr{A}\Sigma_0$ to $\mathscr{A}\Sigma_1$. This rule must satisfy the following five axioms:

1. Two equivalent homeomorphic cobordisms must have the same image:

$$M \cong M' \implies \mathscr{A}M = \mathscr{A}M'$$

2. The cylinder $\Sigma \times I$, thought of as a cobordism form Σ to itself, must be sent to the identity map of $\mathscr{A}\Sigma$.

3. Given a decomposition $M = M'M''$ then

$$\mathscr{A}M = (\mathscr{A}M')(\mathscr{A}M'')$$

4. Disjoint unions are associated with tensor products: if $\Sigma = \Sigma' \amalg \Sigma''$ then $\mathscr{A}\Sigma = \mathscr{A}\Sigma' \otimes \mathscr{A}\Sigma''$.

   This must also hold for cobordisms: if a cobordism M is the disjoint union of two cobordisms M' and M'' then $\mathscr{A}M = \mathscr{A}M' \otimes \mathscr{A}M''$.

5. The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $\mathbb{C}$ and the empty cobordism, being the cylinder over $\Sigma = \emptyset$, must be sent to the identity map of $\mathbb{C}$.

The n-manifolds are meant to represent n-dimensional physical space and the cobordisms between these manifolds represent a time evolution of this space. The first two axioms give TQFT their topological nature, it depends purely on the topology of the manifolds and the cobordisms and not on properties such as metric.

The vector space $\mathscr{A}\Sigma$ is the Hilbert space of the quantum theory while the linear map $\mathscr{A}M$ given by a cobordism M gives the time evolution between Hilbert spaces. This is the quantum field theory nature of TQFT. This is also reflected in the last two axioms; in quantum mechanics the state space of two independent systems is the tensor product of the two state spaces.

This definition of TQFT makes both the topological and the quantum field theory aspects clear, but is rather unwieldy. As we will see this definition can be restated as $\mathscr{A}$ being a functor between the category of n-cobordisms and a category of vector spaces. The first three axioms give that $\mathscr{A}$ is a functor while the fourth and fifth axiom give that this functor has certain properties, namely that it is a non-trivial monoidal functor.

6.2 Monoidal categories

To be able to define monoidal functors we first have to define monoidal categories, to do that we have to define the Cartesian product of categories.

Definition 6.2.1. For a pair of categories $\mathscr{C}$ and $\mathscr{C}'$, define their Cartesian product $\mathscr{C} \times \mathscr{C}'$ to be the category with as objects pairs (X, Y), where X is an object in $\mathscr{C}$ and Y is an object in $\mathscr{C}'$. The set of morphisms from (X, Y) to (X', Y') is the Cartesian product of morphisms from X to X' and morphisms from Y to Y'.

The Cartesian product of two functors $F : \mathscr{C} \to \mathscr{D}$ and $F' : \mathscr{C}' \to \mathscr{D}'$ is the canonical functor $F \times F' : \mathscr{C} \times \mathscr{C}' \to \mathscr{D} \times \mathscr{D}'$.

Now we can define monoidal categories.

Definition 6.2.2. A strict monoidal category is a category $\mathscr{M}$ together with two functors

$$\mu : \mathscr{M} \times \mathscr{M} \to \mathscr{M} \qquad\qquad \eta : \mathscr{I} \to \mathscr{M}$$

with the property that the following three diagrams commute:

$$\begin{tikzcd}
 & \mathscr{M} \times \mathscr{M} \times \mathscr{M} \arrow[dl, "\mu \times id_{\mathscr{M}}"'] \arrow[dr, "id_{\mathscr{M}} \times \mu"] & \\
\mathscr{M} \times \mathscr{M} \arrow[dr, "\mu"'] & & \mathscr{M} \times \mathscr{M} \arrow[dl, "\mu"] \\
 & \mathscr{M} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\eta \times id_{\mathscr{M}}"] \arrow[dr] & \mathscr{M} \times \mathscr{M} \arrow[d, "\mu"] \\
 & \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M} \times \mathscr{M} \arrow[d, "\mu"] & \mathscr{M} \arrow[l, "id_{\mathscr{M}} \times \eta"'] \arrow[dl] \\
\mathscr{M} &
\end{tikzcd}$$

Here $\mathscr{I}$ is the category with one object and only the identity arrow and the diagonal functors in the commuting triangles are the canonical projections.

To ease the notation for general monoidal categories we will write

$$\mu(X, Y) = X \Box Y \qquad \text{and} \qquad \mu(f, g) = f \Box g$$

for the image of objects X and Y and morphisms f and g in $\mathscr{M}$. Here $\Box$ acts more or less as a placeholder for some bifunctor such as a tensorproduct $\otimes$.

We will denote the object that is the image of $\eta : \mathscr{I} \to \mathscr{M}$ by I. The statement of the two commuting triangular diagrams can also be written as

$$I \Box X = X = X \Box I, \qquad\qquad id_I \Box f = f = f \Box id_I.$$

We will refer to a monoidal category by giving the triple $(\mathscr{M}, \Box, I)$.

Remark 6.2.3. Strict monoidal categories are a special case of the more general notion of monoidal categories. Where for strict monoidal categories it is needed for $\mu(\mu \times id_{\mathscr{M}})$ and $\mu(id_{\mathscr{M}} \times \mu)$ to be equal, for general monoidal categories the requirement is that there is an invertible natural transformation between these two compositions.

As explained in chapter 3 of [2] every monoidal category is equivalent to a strict monoidal category and all monoidal categories can be assumed to be strict for the purposes of this thesis.

Example 6.2.4. The category of 2-cobordisms $(\mathscr{S}, \amalg, \emptyset)$ is a monoidal category. For any three objects $n,\ n',\ n''$ and morphisms $M,\ M',\ M''$ in $\mathscr{S}$ we see that

$$(n \amalg n') \amalg n'' = n \amalg (n' \amalg n''), \qquad (M \amalg M') \amalg M'' = M \amalg (M' \amalg M'').$$

And clearly for every object n in $\mathscr{S}$ it holds that $\emptyset \amalg n = n = n \amalg \emptyset$ and for any cobordism M it holds that $id_\emptyset \amalg M = M = M \amalg id_\emptyset$, since the identity morphism of the empty 1-manifold is the empty cobordism $\emptyset \times [0,1]$.

Example 6.2.5. The category $(\mathbf{Vect}_\mathbb{C}, \otimes, \mathbb{C})$, having vector spaces over field $\mathbb{C}$ as objects, $\mathbb{C}$-linear maps as morphisms and the tensor product $\otimes$ as bifunctor, is a monoidal category. For any three vector spaces $V,\ V',\ V''$ over $\mathbb{C}$ and any three $\mathbb{C}$-linear maps $L,\ L',\ L''$ we have

$$(V \otimes V') \otimes V'' \cong V \otimes (V' \otimes V''), \qquad (L \otimes L') \otimes L'' \cong L \otimes (L' \otimes L''),$$

and also

$$\mathbb{C} \otimes V \cong V \cong V \otimes \mathbb{C}, \qquad id_\mathbb{C} \otimes L \cong L \cong L \otimes id_\mathbb{C}.$$

A functor between monoidal categories that preserves the monoidal structure is called a monoidal functor.

Definition 6.2.6. A monoidal functor is a functor $F : \mathscr{M} \to \mathscr{M}'$ between two monoidal categories $(\mathscr{M}, \Box, I)$ and $(\mathscr{M}', \Box', I')$ such that the following diagrams commute:

$$\begin{tikzcd} \mathscr{M} \times \mathscr{M} \arrow[r, "F \times F"] \arrow[d, "\mu"] & \mathscr{M}' \times \mathscr{M}' \arrow[d, "\mu"] \\ \mathscr{M} \arrow[r, "F"] & \mathscr{M} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{M} \arrow[r, "F"] & \mathscr{M}' \\ \mathscr{I} \arrow[u, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

Meaning that for every pair of objects X and Y in $\mathscr{M}$ we have $F(X)\Box' F(Y) = F(X \Box Y)$ and for every pair of morphisms f and g in $\mathscr{M}$ we have $F(f)\Box' F(g) = F(f \Box g)$. The commuting triangle gives that $F(I) = I'$.

Now we can use the notion of monoidal functors to give a definition of a TQFT as a functor.

Definition 6.2.7. An $n+1$-dimensional topological quantum field theory (TQFT) is a non-trivial monoidal functor $\mathscr{A}$ from the category (**(n+1)-Cob**,$\amalg$,$\emptyset$) to category $(\mathbf{Vect}_\mathbb{C}, \otimes, \mathbb{C})$, here non-trivial means that the image of the empty n-manifold $\mathscr{A}(\emptyset) = \mathbb{C}$ and the image of the empty cobordism is the identiy map of $\mathbb{C}$. Recall that **(n+1)-Cob** is the category as objects homeomorphism classes of compact oriented n-manifolds and as morphisms homeomorphism classes of cobordisms between them.

If we compare this definition of a TQFT in terms of categories with the definition at the beginning of the chapter, we see that these two definitions do indeed agree. The first axiom of Definition 6.1.1 amounts to assuring that the functor is well defined and the second and third axiom are equivalent to the functorial axioms. The fourth axiom is the same as $\mathscr{A}$ being monoidal. Lastly the fifth axiom gives that the functor must be non-trivial.

6.3 Classification of morphism inverting (1+1)-dimensional TQFT's

We will now use the functorial definition to give a classification of morphism inverting (1+1)-dimensional topological quantum field theories with morphism inverting meaning that every morphism gets mapped to an invertible morphism.

Theorem 6.3.1. *The (1+1)-dimensional morphism inverting topological quantum field theories are in one-to-one correspondence with pairs $\mu, \lambda \in \mathbb{C}^*$.*

Proof. Consider a (1+1)-dimensional morphism inverting topological quantum field theory $\mathscr{A} : \mathscr{S} \to \mathbf{Vect}_{\mathbb{C}}$. In [6] it is shown that if $\mathscr{A}$ is morphism inverting, it must factor through $\mathscr{G}$. Meaning that $\mathscr{A}$ is a composition of a functor from $\mathscr{S}$ to $\mathscr{G}$ and a functor from $\mathscr{G}$ to $\mathbf{Vect}_{\mathbb{C}}$. Recall that $\mathscr{G}$ is the category obtained from $\mathscr{S}$ by adding inverses for every morphism and that $\mathscr{G}_n$ is its full subcategory with only the object n.

Recall from the proof of Proposition 5.3.1 that $\mathscr{G}_1$ is isomorphic to $\mathbb{Z}$ via half the Euler characteristic of its morphisms. As the Euler characteristic of the sphere S^2 is 2, half of which is 1, it is a generator for the morphisms in $\mathscr{G}_1$. All morphisms in $\mathscr{G}_1$ are the union of either $n \in \mathbb{N}$ spheres or the union of $n \in \mathbb{N}$ anti-spheres, the inverse of the union of n spheres. Thus giving the image of of the sphere $\mathscr{A}(S^2)$, determines all morphisms $\mathscr{G}_1 \to \mathbf{Vect}_{\mathbb{C}}$.

Since every morphism $n \to m$ in $\mathscr{G}$ is an isomorphism between $\mathscr{G}_n$ and $\mathscr{G}_m$, the functor $\mathscr{A}$ can now be determined by giving the images of morphisms $p_n : 0 \to n$ for $n \geq 1$. Define p_n to be the union of n disks as morphisms $0 \to 1$. Because $\mathscr{A}$ is non-trivial, we know that $\mathscr{A}(0) = \mathbb{C}$. Using that every $p_n : 0 \to n$ is an isomorphism we see that every $\mathscr{A}(n)$ is isomorphic to $\mathbb{C}$ as a vector space. As $\mathscr{A}$ is monoidal and $p_n = \coprod_{1 \leq i \leq n} p_1$, the image of every p_n is determined by giving the image of p_1 by taking the n-fold tensor product.

Thus $\mathscr{A}$ is determined by giving the image of S^2 and p_1.

Because all $\mathscr{A}(n)$ are isomorphic to $\mathbb{C}$ we see that $\mathscr{A}$ can be seen as a functor from $\mathscr{S}$ to $GL_1(\mathbb{C})$, mapping all objects to one object and all morphisms to elements of the group $GL_1(\mathbb{C})$. Recall that $GL_1(\mathbb{C}) = \mathbb{C}^*$, the complex numbers without 0.

If we define $\mu = \mathscr{A}(S^2)$ and $\lambda = \mathscr{A}(p_1)$ we see that $\mathscr{A}$ is completely determined by the pair $\mu, \lambda \in \mathbb{C}^*$. $\square$

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