UTRECHT UNIVERSITY

INSTITUTE FOR THEORETICAL PHYSICS

MASTER THESIS

An Invitation to the Principal Series

Author
Jim Skulte

Supervisors
Dr. Umut Gürsoy
Dr. Thomas Grimm

Daily Supervisor
Dr. Tarek Anous

June 23, 2020

Abstract

The unitary irreducible representations of the isometry group of d-dimensional de Sitter space SO(1,d) can be distinguished by their conformal dimensions Δ, the eigenvalue of the dilatation operator near the origin. Scalar fields with sufficiently large mass compared to the de Sitter scale $1/L$ have complex conformal weights and physical modes of these fields fall into the continuous principal series representation of $SO(1,d)$. In $d = 2$ and in global coordinates, we show that the generators of the isometry group of dS_2 acting on a massive scalar field reduces exactly to the quantum mechanical model introduced by de Alfaro, Fubini and Furlan (DFF) in the early/late time limit. In its original presentation, the DFF model describes a single degree of freedom on the positive semi-axis subject to a repulsive potential that diverges at the origin. The Hilbert space of this model furnishes the discrete highest weight representation of $SO(1,2)$. To accommodate the principal series representation, the potential must be made attractive, but this comes at the expense of the failure of self-adjointness of the operators, leading to the speculation that DFF can not accommodate a unitary principal series representation. Motivated by the ambient dS_2 construction, we explain in detail how this model must be completed in order to allow for the principal series representation and verify that all operators remain Hermitian and self-adjoint. While the conformal dimensions are complex, the representations are nevertheless completely unitary. By studying this model in detail, we explore some features one must face in the search of a dual quantum field theory that contains states in the principal series.

Contents

1 Introduction

1.1 Motivation

There are many reasons to study de Sitter spacetime. In recent years it was discovered that our universe is expanding [1,2] and hence, the cosmological constant is positive. De Sitter space is the most simple spacetime with a positive cosmological constant and is therefore an important theoretical playground for understanding our universe. Furthermore, the inflationary epoch of our early universe was approximately de Sitter [3,4]. However, despite obvious interest and necessity, quantum gravity in de Sitter remains an unsolved theoretical problem [5–14].
De Sitter space and its features are interesting to study on their own. Bekenstein and Hawking were the first to relate thermodynamic entropy to the area of an event horizon and obtained their macroscopic entropy-area law [15,16]

$$S = \frac{A}{4G} \ , \tag{1.1}$$

where A is the area of the event horizon and G Newton's constant. This should be understood just as any other thermodynamic relation from the 19th century. De Sitter spacetime inherits a cosmological horizon and this entropy formula should also apply here [17]. In [18] the black hole entropy was calculated microscopically for certain black holes using super symmetry. The microscopical origin of the entropy of Schwarzschildt black holes or cosmological event horizons are certainly not understood yet. Conceptually one can imagine that a black hole is some localized quantum object with some quantum microstates. If one finds a correct description of these objects, one could go on and count the microstates and compare these results to the Bekenstein-Hawking formula (1.1). The entropy for cosmological horizons is even more puzzling. The event horizon in de Sitter spacetime is observer dependent, so one needs to identify the location of the possible microstates first. The question what the possible microstates might be stays an open question.
Another reason to study de Sitter space is due to the great success of the anti de Sitter-conformal field theory correspondence introduced by Maldacena [19]. The idea is to relate a gauge theory living in d-dimensions to a $d + 1$-dimensional gravitational bulk theory in anti de Sitter spactime. One might asks the question whether this holographic approach is only applicable to anti de Sitter spacetime or holds true for a generic spacetime. Presumably the simplest spacetime to consider is de Sitter space since it only differs by a sign in front of the cosmological constant from anti de Sitter spacetime. There have been many attempts to understand this correspondence [8, 9, 20–22], but a satisfactory map is still missing and the correspondents is not established yet. In this we attempt to understand

some of the necessary features any theory in de Sitter must exhibit.
Let us consider various coordinates systems of de Sitter, which all have their benefits and conceptual difficulties as we will see.

1.2 De Sitter Spacetime

We will realise d-dimensional de Sitter spacetime (dS_d) by embedding it into $d+1$-dimensional flat Minkowski spacetime ($\mathcal{M}^{1,d}$) via the condition

$$-X_0^2 + X_1^2 + ... + X_{d-1}^2 = L^2 \ , \tag{1.2}$$

where L is the de Sitter radius in units of length. This hypersurface in flat Minkowski space is simply a hyperboloid, as illustrated in figure 1.1. Each horizontal slice represents the extremal volume of a S^{d-1} sphere and the timelike coordinate X^0 flows upwards.

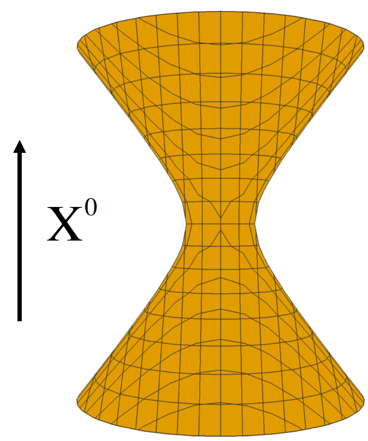

Figure 1.1: Hyperboloid illustrating d-dimensional de Sitter spacetime embedded in a higher dimensional Minkowski spacetime. Each point corresponds to a S^{d-2} sphere.

Due to the embedding of dS_d into $\mathcal{M}^{1,d}$ we can immediately read off the isometry group of d-dimensional de Sitter spacetime to be $SO(1,d)$ [23].
Furthermore, de Sitter is the maximally symmetric solution to the Einstein equations with

positive cosmological constant [24] satisfying

$$G_{ab} + \Lambda g_{ab} = 0 \ , \tag{1.3}$$

where G_{ab} is the Einstein tensor and g_{ab} the metric. Further, the cosmological constant is defined as

$$\Lambda = \frac{(d-2)(d-1)}{2L^2} \ . \tag{1.4}$$

Global coordinates

Since we will often use a parametrization on the S^{d-1} sphere, we will define one here. A convenient parametrization is given by setting [25]

$$\begin{aligned} \omega^1 &= \cos\theta_1 \\ \omega^2 &= \sin\theta_1 \cos\theta_2 \\ &\vdots \\ \omega^{d-1} &= \sin\theta_1 \dots \sin\theta_{d-2} \cos\theta_{d-1} \\ \omega^d &= \sin\theta_1 \dots \sin\theta_{d-2} \sin\theta_{d-1} \ , \end{aligned} \tag{1.5}$$

where $0 \leq \theta_i < \pi$ for $1 \leq i < d-1$, but $0 \leq \theta_{d-1} < 2\pi$. These coordinates naturally satisfy the condition to be on the sphere $\sum_{i=1}^{d} \left(\omega^i\right)^2 = 1$ and the metric on the S^{d-1} sphere is given, as usual, by

$$d\Omega_{d-1}^2 = \sum_{i=1}^{d} \left(d\omega^i\right)^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \dots + \sin^2\theta_1 \dots \sin^2\theta_{d-2} d\theta_{d-1}^2 \ . \tag{1.6}$$

We obtain the global coordinates that cover the whole hyperboloid by setting

$$X_0 = L\sinh(\tau/L) \ , \qquad X_i = L\cosh(\tau/L)\omega_i \ , \tag{1.7}$$

where $I = 1 \dots d$, $\tau \in \{-\infty, \infty\}$ and the ω_i as defined in (1.5). One immediately sees that these coordinates fulfil the condition (1.2) for every point (τ, ω_i). We can plug this choice of coordinates (1.7) into the standard flat metric on $\mathcal{M}^{1,d}$

$$ds^2 = -dX_0^2 + dX_1^2 + \dots + dX_d^2 \tag{1.8}$$

and obtain the induced metric on dS_d as

$$ds^2 = -d\tau^2 + L^2 \cosh^2\left(\frac{\tau}{L}\right) d\Omega_{d-1}^2 \ . \tag{1.9}$$

In global coordinates one can think of dS_d as a S^{d-1} sphere that changes its size in time. More precisely, the $d-1$ sphere starts out infinitely large at $\tau = -\infty$, then the sphere shrinks to the minimal size at $\tau = 0$ with radius L and afterwards starts to grow again to a infinitely large size as $\tau \to \infty$.

Penrose diagram

As the next step we would like to understand the causal structure of de Sitter spacetime, hence we want to draw a Penrose diagram. In order to do this we will use conformal coordinates (T, θ_i). These are simply related to global coordinates by

$$\cosh\tau = \frac{1}{\cos T} \ , \tag{1.10}$$

which restricts $-\pi/2 < T < \pi/2$. Finally, one arrives at the following metric

$$ds^2 = \frac{1}{\cos^2 T}\left(-dT^2 + d\Omega_{d-1}^2\right) \ . \tag{1.11}$$

We can use the fact that null rays with respect to the metric (1.11) stay null rays after conformal transformations. Hence, the causal structure stays the same and we can investigate the causal structure of de Sitter space by considering the more simple and conformally related metric

$$d\tilde{s}^2 = \left(\cos^2 T\right) ds^2 = -dT^2 + d\Omega_{d-1}^2 \ . \tag{1.12}$$

The Penrose diagram corresponding to the metric (1.11) and the metric (1.12) is given in figure 1.2.

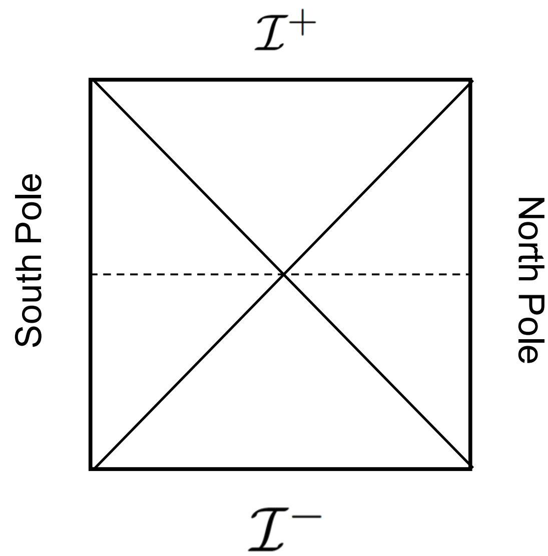

Figure 1.2: Penrose diagram for d-dimensional de Sitter spacetime. Every point corresponds to a S^{d-2} sphere except the points on the left and right, which are the north/south pole of the S^{d-1} that are obtained by taking a complete spacelike slice as denoted with the dashed line. Light rays travel at 45° as denoted with the continuous lines. $\mathcal{I}^+$ and $\mathcal{I}^-$ denote the future and past null infinity.

As usual in a Penrose diagram the light rays or null geodesics travel at 45° angles and spacelike surfaces are more horizontal than vertical. Timelike surfaces are the other way around, namely more vertical than horizontal. All points in the diagram are S^{d-2} spheres except the 'edges' on the left and right, which are actually points. These are the north/south pole of the S^{d-1} spheres that are obtained by taking a complete spacelike slice as denoted with the dashed line. The future and past null infinity are denoted as $\mathcal{I}^+$ and $\mathcal{I}^-$. These are the surfaces, where all light rays terminate on or originate from. One should note that a light ray which is emitted from the south pole at the past infinity will reach the north pole by the time it arrives at $\mathcal{I}^+$ infinitely far in the future. This is one of the rather special features of de Sitter and we should spend some time to briefly discuss what this means for causality in de Sitter space.

Causality

In de Sitter space a local observer is never able to access the entire spacetime and one should be aware what is accessible to an local observer and what is only accessible to some observer outside the spacetime.

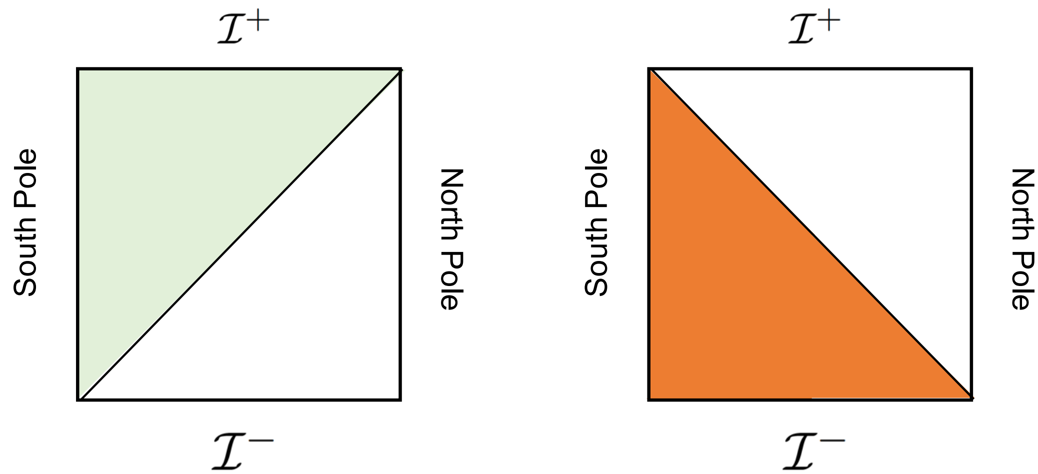

Figure 1.3: The two Penrose diagrams show the causal future and past. On the left in green the causal future $\mathcal{O}^+$ and on the right in orange the causal past $\mathcal{O}^-$. Both regions correspond to an observer sitting at the south pole.

Let us assume a local observer is sitting at past null infinity at the south pole. If she sends out a light ray, it will reach the north pole at future null infinity. Hence, she can only send messages to half of the spacetime. This region is denoted as $\mathcal{O}^+$ and is depicted as the green area on the left of figure 1.3. This differs qualitatively from the causal structure in Minkowski space. If she would send out a light ray at any time in Minkowski space, the light ray will eventually reach any region in a finite time.
On the right of figure 1.3 the region $\mathcal{O}^-$ is shown in orange. This is the region from where one can receive messages from sitting at past null infinity at the south pole. The intersection of these two regions $\mathcal{O}^- \cap \mathcal{O}^+$ is the fully accessible region for a local observer sitting at the south pole and is called the southern diamond. One can send queries to any point in this region and receive an answer before reaching future infinity $\mathcal{I}^+$. In contrast to this the northern diamond is completely inaccessible to an observer sitting at the south pole.

Planar coordinates

Next, let us discuss a chart that covers only half of the de Sitter space, namely the causal past for an observer sitting at the south pole. This chart is usually called planar coordinates, flat slicing or Poincare coordinates. To obtain this chart we choose our coordinates to be

$$X_0 = L\sinh(t/L) + \frac{x_i x^i}{2} e^{-t/L}$$
$$X_1 = L\cosh(t/L) - \frac{x_i x^i}{2} e^{-t/L}$$
$$X_i = e^{-t/L} y_i \ , \tag{1.13}$$

where $i = 2 \dots d$. The induced metric takes the form

$$ds^2 = -dt^2 + e^{-2t/L} d\vec{y}^2 \ . \tag{1.14}$$

The slices of constant t are depicted in figure 1.4.

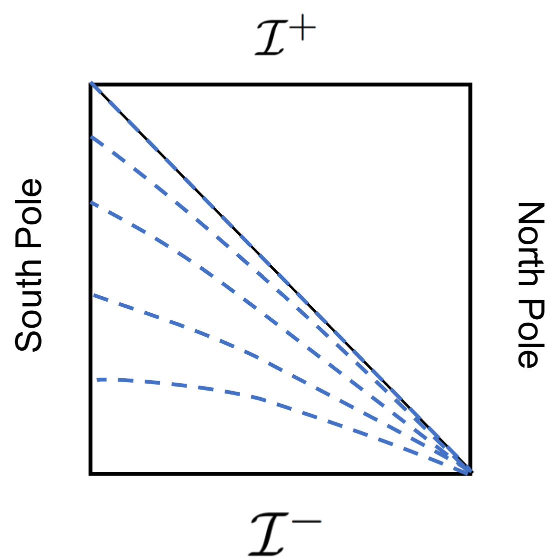

Figure 1.4: The blue dashed lines are slices of constant t in planar coordinates. Each slice is a infinite, flat $d-1$-dimensional plane, hence the name flat slicing. Each plane extends all the way down to $\mathcal{I}^-$.

In these coordinates the rotation and translation symmetries in the x_i coordinates are manifest and the time t is not a Killing vector. This patch is often used in Cosmology since it allows to use Fourier transformation for fixed t, which is often a very useful tool.

Static patch

Finally, we will discuss the static patch. The static patch covers the southern diamond and is constructed to have an explicit timelike Killing vector. To obtain this chart we take

$$X_0 = \sqrt{L^2 - r^2}\,\sinh(t/L)$$
$$X_1 = \sqrt{L^2 - r^2}\,\cosh(t/L)$$
$$X_i = r\omega_i \ , \tag{1.15}$$

where $i = 2 \dots d$, ω^i parameterize the $d-2$-sphere as defined in (1.5) with $\sum_{i=1}^{d}\left(\omega^i\right)^2 = 1$ and $r \in (0, L)$. In figure 1.5 the area that is covered by the static patch is shown in green. The induced metric reads

$$ds^2 = -\left(1 - r^2/L^2\right)dt^2 + \frac{dr^2}{1 - r^2/L^2} + r^2 d\Omega_{d-2}^2 \ . \tag{1.16}$$

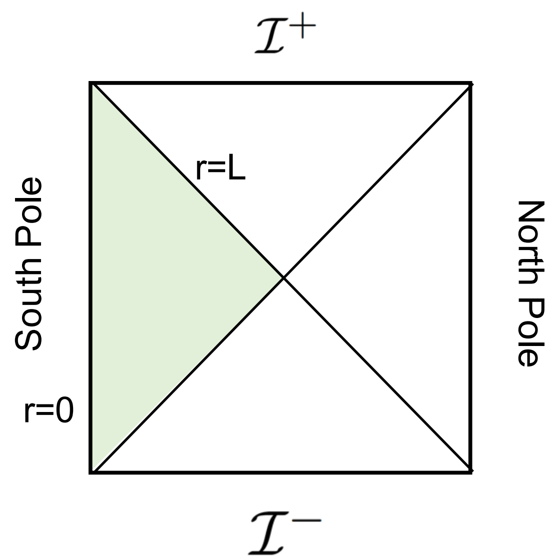

Figure 1.5: The static patch is shown in green on the Penrose diagram. This is the causal connected patch for an observer sitting at the south pole, i.e. $r = 0$ in static coordinates. The left 'edge' of the diagram is $r_{\text{static}} = 0$ and the Killing horizon is at $r_{\text{static}} = L$.

One can also describe the region called the future triangle by taking $r \in (L, \infty)$. From the metric (1.16) one can immediately see that ∂_t is a Killing vector and one can define a Hamiltonian in this patch. It is also obvious that the norm of the Killing vector vanishes at $r = L$. Indeed, $r = L$ is a null surface. This event horizon is known as the cosmological horizon in de Sitter spacetime and is evident in the static patch coordinates. One can see that the direction of the Killing vector changes sign as one passes through the cosmological horizon from the southern diamond into the future triangle. Hence, the Hamiltonian one can possible define changes sign, as illustrated in figure 1.6. We will briefly comment on this observation later.

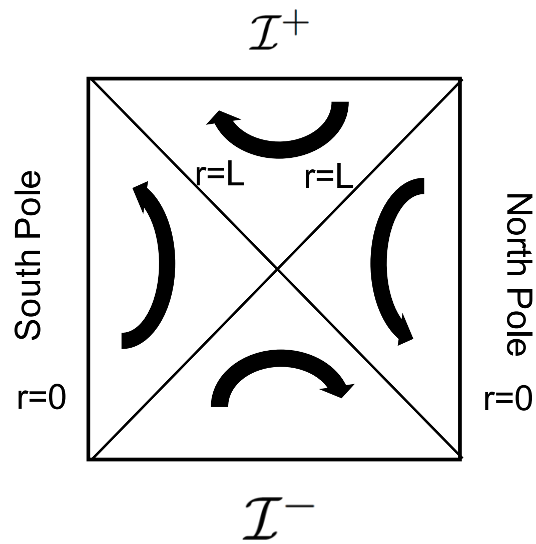

Figure 1.6: The Penrose diagram of de Sitter spacetime includes the direction of the flow generated by the Killing vector ∂_t in static coordinates. The horizons at $r^2 = L^2$ are shown as the continuous black lines. As before the static patch is the triangle on the left side with $r \in (0, L)$.

One can conclude that there is no globally defined timelike Killing vector in de Sitter space. This has important features in the quantum theory of de Sitter, to which we will move on now.

Holographic boundary

We want to briefly mention some confusions about the possible holographic principle for de Sitter spacetime.
The idea of holography [26, 27] relates a d-dimensional quantum field theory living on the boundary of a $d + 1$-dimensional gravitational theory. One of the first examples was es-

tablished by Maldacena [19] and is now a well established general approach to quantum gravity [28, 29]. For a review see [30, 31]. In the AdS/CFT correspondence the conformal boundary is a spatial boundary and is fully accessible for a local observer. In contrast to that, de Sitter spacetime has no spatial boundary as can be seen in global coordinates. The only boundary in de Sitter spacetime are the future and past null infinity. The issue is that this boundary is not spatial and hence, cannot be fully accessed by a local observer. This means using global coordinates and defining the QFT on the boundary the observable computed can never be tested by a local observer. Now, one could go on and say one uses static coordinates instead, which comes at the cost that the boundary is hidden by a horizon and one needs to use an appropriate statistical tool to incorporate this. The interpretation of this approach is also far from obvious and the precise definition of the observable, one should use in de Sitter space to make meaningful prediction, stays unknown.

Quantum field theory in (Anti) de Sitter spacetime

There are three maximally symmetric solutions to Einstein's equations with constant curvature, namely Minkowski spacetime, de Sitter spacetime and anti de Sitter spacetime, where the curvature is non-vanishing in anti de Sitter and de Sitter spacetime. As with de Sitter space, anti de Sitter spacetime can be embedded into a higher dimensional flat space through the equation

$$-X_0^2 + X_1^2 + ... + X_{d-1}^2 = -L^2. \tag{1.17}$$

Unlike to de Sitter space, which has positive constant curvature, it has constant negative curvature.

For a generic d-dimensional spacetime we can write down the action for a non-interacting scalar field with minimal coupling as

$$S = -\frac{1}{2}\int d^d x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \ , \tag{1.18}$$

where $g = \det(g_{\mu\nu})$. One obtains the Klein-Gordon equation by varying the above action with respect to ϕ

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi - m^2\right)\phi = 0 \ . \tag{1.19}$$

As the next step one can go on and calculate the quadratic Casimir C^2 of the isometries as done in [9, 30] or as calculated for the 2-dimensional case in section 4, to show that

$$\mp \frac{C^2}{L^2} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \ , \tag{1.20}$$

where the minus corresponds to de Sitter spacetime and the plus to anti de Sitter spacetime. Furthermore, unitary irreducible representations are labeled by real eigenvalues of the quadratic Casimir

$$C^2 = \Delta(\Delta - d + 1) \ . \tag{1.21}$$

Thus, based on representation theory we expect to identify

$$\Delta(\Delta - d + 1) = \mp m^2 L^2 \ . \tag{1.22}$$

It seems that the results for de Sitter space and anti de Sitter space are essentially the same since they only differ by a sign, but as we will see this is not the case.
From (1.22) the weights Δ in anti de Sitter spacetime follow as

$$\Delta_\pm = \frac{1}{2} \left((d-1) \pm \sqrt{(d-1)^2 + 4m^2L^2} \right) \ . \tag{1.23}$$

In the language of the AdS/CFT correspondence these weights label the two possible spatial falloffs, which correspond to the highest weight representation. It is important to note that for all masses m the weights Δ are real.
Let us discuss the de Sitter case next. As before we can compute the weights from (1.22) as

$$\Delta_\pm = \frac{1}{2} \left((d-1) \pm \sqrt{(d-1)^2 - 4m^2L^2} \right) \ . \tag{1.24}$$

Here, we need to distinguish qualitatively different cases. The first case corresponds to $m^2L^2 < \frac{(d-1)^2}{4}$, where $\Delta_\pm$ is a real number and the falloff (now in time) is the same as in the AdS case and the highest weight representation can be used. Actually one can show that in de Sitter the weights Δ need to be an integer for the highest weight. These corresponds to the complementary series.
The more interesting case is now where the weights $\Delta_\pm$ become complex for $m^2L^2 > \frac{(d-1)^2}{4}$ and we find

$$\Delta_\pm = \frac{1}{2} \left((d-1) \pm i\nu \right) \tag{1.25}$$

with $\nu \in \mathbb{R}$. These correspond precisely to the unitary principal series representation, which do not appear in the anti de Sitter space. Here, the two weights $\Delta_\pm$ are complex conjugates and actually represent the same state in the Hilbert space, as we will see later.

In order to see how this representation can be unitary, even though the weights are complex, we will discuss in section 2 the possible unitary irreducible representations (UIRs) of $SO(1,d)$, which is the isometry group of d-dimensional de Sitter space. Following this, in section 3, we will investigate how the simple conformal quantum mechanical toy model, namely the DFF model, furnishes both the highest weight and the principal series representation. After that, in section 4, we will connect these results from the DFF model to 2-dimensional de Sitter space. In section 5 we will finish with a conclusion. We will further try review some open questions one needs to raise to link our results to the $\mathrm{dS}_3/\mathrm{CFT}_2$ correspondence and the Virasoro algebra.

2 Review of UIRs in de Sitter

Our goal is to investigate the possible UIRs of the isometry group $SO(1,d)$ of de Sitter space.
We start with the standard representation of the $so(1,d)$ algebra with $L_{ab} = -L_{ba}$

$$[L_{ab}, L_{cd}] = i\left(\eta_{bc}L_{ad} + \eta_{db}L_{ca} + \eta_{ad}L_{bc} + \eta_{ca}L_{db}\right) \ , \tag{2.1}$$

and $\eta = \mathrm{diag}(-1,1,\ldots,1)$. We are free to choose a different set of generators, which is more convenient if one thinks about a possible dS/CFT application [32]. We use the following identification

$$\begin{aligned} L_{ij} &= M_{ij} \\ L_{0,d} &= D \\ L_{d,i} &= \frac{1}{2}\left(P_i + K_i\right) \\ L_{0,i} &= \frac{1}{2}\left(P_i - K_i\right) \end{aligned} \tag{2.2}$$

with $L_{ab} = -L_{ba}$, $i,j \in \{1,2,\ldots,d-1\}$ and $a,b \in \{0,1,\ldots,d\}$. This corresponds to choosing a set of generators consisting of translations P_i, special conformal transformations K_i, rotations M_{ij} and a dilatation D. We end up with the algebra

$$\begin{aligned} [D, P_i] &= iP_i \\ [D, K_i] &= -iK_i \\ [K_i, P_j] &= 2i\left(\delta_{ij}D - M_{ij}\right) \\ [M_{ij}, P_k] &= i\left(\delta_{jk}P_i - \delta_{ik}P_j\right) \\ [M_{ij}, K_k] &= i\left(\delta_{jk}P_i - \delta_{ik}K_j\right) \\ [M_{ij}, M_{kl}] &= i\left(\delta_{jk}M_{il} + \delta_{lj}M_{ki} + \delta_{il}M_{jk} + \delta_{ki}M_{lj}\right) \ . \end{aligned} \tag{2.3}$$

In order to obtain a unitary representations of $so(1,d)$, we need our operators to be hermitian operators acting on the Hilbert space

$$\begin{aligned} D^\dagger &= D \\ P_i^\dagger &= P_i \\ K_i^\dagger &= K_i \\ M_{ij}^\dagger &= M_{ji} \ . \end{aligned} \tag{2.4}$$

For the $so(1,d)$ algebra we can define a quadratic Casimir, which commutes with all other generators as

$$\begin{aligned} C^2 \equiv \frac{1}{2} L_{ab} L^{ab} &= -L_{0,d}^2 - L_{0,i}^2 + L_{d,i}^2 + \frac{1}{2} L_{ij}^2 \\ &= -D^2 - \frac{1}{4} (P_i - K_i)^2 + \frac{1}{4} (P_i + K_i)^2 + \frac{1}{2} M_{ij}^2 \\ &= -D^2 + \frac{1}{2} (P_i K_i + K_i P_i) + \frac{1}{2} M_{ij}^2 \\ &= D(d-1-D) + P_i K_i + \frac{1}{2} M_{ij}^2 \ . \end{aligned} \tag{2.5}$$

After all of this layed out we can start with explicit examples for $d = 2, 3$. We will see that there are two distinct cases, namely the principal series representation with complex weights and the highest weight representation with real weights.

2.1 Unitary Irreducible Representations of $SO(1,2)$

Let us start simple by considering the group $SO(1,2)$, which is the isometry group of 2-dimensional de Sitter spacetime. We will investigate how unitarity will put constraints on the possible irreducible representations. To do this we will first write down the action of the generators in position space and for completeness in momentum space as well. The $so(1,2)$ algebra, which consists of one translation P, one special conformal transformation K and one dilatation D, simply reads

$$\begin{aligned} [D, P] &= iP \\ [K, D] &= iK \\ [K, P] &= 2iD \ . \end{aligned} \tag{2.6}$$

We need to assume that these operators are realized as hermitian operators on a Hilbert space $\mathcal{H}$ with a positive definite inner product. For clarity this means the operators fulfil

$$\begin{aligned} D^\dagger &= D \\ P^\dagger &= P \\ K^\dagger &= K \ . \end{aligned} \tag{2.7}$$

The quadratic Casimir can be computed via

$$C^2 = D(1-D) + PK \ . \tag{2.8}$$

2.1.1 Position Space Basis

Our story begins with a position state basis picture. Hence, we start with the state $|\Delta, 0\rangle$ obeying

$$\begin{aligned} K|\Delta, 0\rangle &= 0 \\ D|\Delta, 0\rangle &= i\Delta|\Delta, 0\rangle \ . \end{aligned} \tag{2.9}$$

This state is chosen such that D is diagonalized and preserves the subspace of states annihilated by K. At this point Δ can be any complex number and different values of Δ will correspond to different representations. Of course, we will see later that due to unitarity the possible values are constraint.
We can produce a family of states via acting on these states with translations

$$|\Delta, x\rangle \equiv e^{xP}|\Delta, 0\rangle \ . \tag{2.10}$$

In the following we will drop the label Δ and define $|\Delta, x\rangle \equiv |x\rangle$. From the definition above and the algebra we obtain

$$\begin{aligned} P|x\rangle &= i\partial_x|x\rangle \\ D|x\rangle &= i(x\partial_x + \Delta)|x\rangle \\ K|x\rangle &= i(2xD - x^2\partial_x)|x\rangle = i(x^2\partial_x + 2x\Delta)|x\rangle \ . \end{aligned} \tag{2.11}$$

The first line follows from the definition and the following can be computed using the commutation relation and the Baker–Campbell–Hausdorff formula. We can construct arbitrary states via linear combinations of wavefunctions using the position space basis

$$|\Psi\rangle = \int dx \ \psi(x)|x\rangle \ . \tag{2.12}$$

Eventually we are interested in the action of the generators on those wavefunctions. So let us derive those using integration by parts

$$\begin{aligned} P|\psi\rangle &= \int dx\psi(x)i\partial_x|x\rangle \\ &= \int dx \left(-i\partial_x\psi(x)\right)|x\rangle \implies P\psi(x) = -i\partial_x\psi(x) \ . \end{aligned} \tag{2.13}$$

It follows

$$\begin{aligned} P\psi(x) &= -i\partial_x\psi(x) \\ D\psi(x) &= -i(x\partial_x + \bar{\Delta})\psi(x) \\ K\psi(x) &= -i(x^2\partial_x + 2x\bar{\Delta})\psi(x) \ , \end{aligned} \tag{2.14}$$

where we defined $\bar{\Delta} \equiv 1-\Delta$ as the dual dimension. A quick check reveals that these indeed fulfil the algebra (2.6).

By construction, all possible states $|\Psi\rangle$ are eigenstates of the quadratic Casimir as well

$$C^2|\Psi\rangle = \left(\Delta(1-\Delta)\right)|\Psi\rangle \ . \tag{2.15}$$

We can only obtain a unitary irreducible representation if the quadratic Casimir has only real eigenvalues. This implies two possible cases

$$\textbf{CASE 1}: \ \Delta = \frac{1}{2} + i\nu \ , \ \nu \in \mathbb{R} \tag{2.16}$$

$$\textbf{CASE 2}: \ \Delta \in \mathbb{R} \ . \tag{2.17}$$

2.1.2 Momentum Space Basis

We can go to the momentum space basis $|p\rangle$ by taking the Fourier transform of the position space basis $|x\rangle$ as usual

$$|p\rangle = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx}|x\rangle \ , \tag{2.18}$$

which is simply (2.12) with $\psi(x) = e^{-ipx}$ plugged in. As in the position space case, we can go on and use the algebra to obtain

$$
\begin{aligned}
P|p\rangle &= -p|p\rangle \\
D|p\rangle &= -i(p\partial_p + \bar{\Delta})|p\rangle \\
K|p\rangle &= (p^2\partial_p + 2\bar{\Delta}\partial_p)|p\rangle = (\partial_p^2 p - 2\Delta\partial_p)|p\rangle \ .
\end{aligned}
\tag{2.19}
$$

Again, we can write an arbitrary state as a linear superposition, now with momentum eigenstates

$$
|\Psi\rangle = \int dp \ \psi(p)|p\rangle \ . \tag{2.20}
$$

The action of the generators on the wavefunctions $\psi(p)$ in momentum space is again obtained from (2.19) and integration by parts

$$
\begin{aligned}
P\psi(p) &= -p\psi(p) \\
D\psi(p) &= i(p\partial_p + \Delta)\psi(p) \\
K\psi(p) &= (p\partial_p^2 + 2\Delta\partial_p)\psi(p) \ .
\end{aligned}
\tag{2.21}
$$

2.1.3 Constraints due to Unitarity

Next, we are finally ready to see what restrictions we can put on Δ by demanding unitarity. First, before diving into consequences from the hermiticity of D, K, P, we want to use the already known fact that the quadratic Casimir must be real and hence, as mentioned in (2.16) either $\Delta^* = 1 - \Delta$ or $\Delta^* = \Delta$. This leads to two distinct cases

$$
\textbf{CASE 1}: \ \Delta = \frac{1}{2} + i\nu \ , \ \nu \in \mathbb{R} \tag{2.22}
$$

$$
\textbf{CASE 2}: \ \Delta \in \mathbb{R} \ . \tag{2.23}
$$

We note that in case **1**, $\bar{\Delta} = \Delta^*$. Finally, we want to use the unitarity constraints to obtain the corresponding inner product in the momentum and position space basis for the two qualitatively different cases.
We start by demanding that P is hermitian, $P^\dagger = P$, and our results from (2.19). This leads to $\langle p|P|q\rangle = p\langle p|q\rangle = q\langle p|q\rangle$, which implies

$$
(p - q)\langle p|q\rangle = 0 \implies \langle p|q\rangle = f(p)\delta(p - q) \tag{2.24}
$$

with $f(p)$ some function of the momentum p. Next we can use the hermiticity of the dilatation operator $D^\dagger = D$. It follows that

$$(q\partial_q + p\partial_p)\langle p|q\rangle = -(\bar{\Delta} + \bar{\Delta}^*)\langle p|q\rangle \ , \tag{2.25}$$

where we can now use our previous result $\langle p|q\rangle = f(p)\delta(p-q)$ to obtain

$$f(p) = c|p|^\alpha, \quad c > 0, \quad \alpha = 1 - \bar{\Delta} - \bar{\Delta}^* = \Delta + \Delta^* - 1 \ . \tag{2.26}$$

Thus, we arrive at two different inner products

$$\textbf{CASE 1}: \ \Delta = \frac{1}{2} + i\nu, \ \nu \in \mathbb{R}: \quad \langle p|q\rangle = c\delta(p-q) \tag{2.27}$$

$$\textbf{CASE 2}: \ \Delta \in \mathbb{R}: \quad \langle p|q\rangle = c|p|^\alpha \delta(p-q), \quad \alpha = 2\Delta - 1 \ , \tag{2.28}$$

where again c is some positive number. The case **1** corresponds to the principal series representation and case **2** to the discrete highest/lowest weight representation. The inner product for the states from (2.55) is given by

$$\textbf{CASE 1}: \langle \Psi|\phi\rangle = c\int dp \ \psi(p)^*\phi(p) \tag{2.29}$$

$$\textbf{CASE 2}: \langle \Psi|\phi\rangle = c\int dp \ |p|^{2\Delta-1}\psi(p)^*\phi(p), \quad \Delta \in \mathbb{R} \ . \tag{2.30}$$

By Fourier transformation of the momentum space result we find the position space inner product as

$$\textbf{CASE 1}: \langle \Psi|\phi\rangle = c\int dx \ \psi(x)^*\phi(x) \tag{2.31}$$

$$\textbf{CASE 2}: \langle \Psi|\phi\rangle = c'\int dxdy \ \frac{1}{|x-y|^{2\Delta}}\psi(x)^*\phi(y) \ , \quad \Delta \in \mathbb{R} \ , \tag{2.32}$$

with the latter expression up to contact terms can c' some constant. Furthermore, we deduce that

$$\textbf{CASE 1}: \langle x|y\rangle = c\delta(x-y) \tag{2.33}$$

$$\textbf{CASE 2}: \langle x|y\rangle = c'\frac{1}{|x-y|^{2\Delta}} \tag{2.34}$$

with c' as before and $\Delta > 0$ in the second case to obtain a converging Fourier transformation.

We have seen that there are two possible representations. One is the well known highest weight representation and the second is the less well known principal series representation [33, 34], which we will study in more detail in 3.2.

2.1.4 Inner Products of States belonging to different UIRs

Finally, one can go on and redo the above analysis for states not necessary belonging to the same representation to see how this effects the discussion. The steps are completely analogous to those before and we will omit those. We will label our states by $|\lambda, p\rangle$, $|\lambda', p\rangle$ with λ and λ' labelling different representations. The result we obtain is

$$\langle \lambda, p|\lambda', p'\rangle = c_{\lambda,\lambda'}|p|^{\Delta^*_\lambda+\Delta_{\lambda'}-1}\delta(p-p') \tag{2.35}$$

and to ensure the the inner product to be non zero we find

$$c_{\lambda,\lambda'} \neq 0 \quad \Longrightarrow \quad \Delta_{\lambda'} = 1-\Delta^*_\lambda \quad \text{or} \quad \Delta_{\lambda'} = \Delta^*_\lambda \ . \tag{2.36}$$

Taking these two constraints into account we find

$$\langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\Delta_{\lambda'} = 1-\Delta^*_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\Delta_{\lambda'}-1}\delta(p-p') & (\Delta_{\lambda'} = \Delta^*_\lambda) \ . \end{cases} \tag{2.37}$$

Next, we define $\nu \equiv \Delta - \frac{1}{2}$ to simplify the results from above as

$$\langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = -\nu^*_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = \nu^*_\lambda) \end{cases} \tag{2.38}$$

This means, specialized to those irreducible representations satisfying the unitarity conditions $\nu = -\nu^*$ (**CASE 1**) or $\nu = \nu^*$ (**CASE 2**), we find for the three different possible combinations

$$\langle \mathbf{1}|\mathbf{1}\rangle : \langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = \nu_\lambda, \ \text{i.e.} \ \Delta_{\lambda'} = \Delta_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = -\nu_\lambda, \ \text{i.e.} \ \Delta_{\lambda'} = \bar{\Delta}_\lambda) \end{cases} \tag{2.39}$$

$$\langle \mathbf{2}|\mathbf{2}\rangle : \langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = -\nu_\lambda, \ \text{i.e.} \ \Delta_{\lambda'} = \bar{\Delta}_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = \nu_\lambda, \ \text{i.e.} \ \Delta_{\lambda'} = \Delta_\lambda) \end{cases} \tag{2.40}$$

$$\langle \mathbf{1}|\mathbf{2}\rangle : \langle \lambda, p|\lambda', p'\rangle = 0 \ . \tag{2.41}$$

This corresponds to the fact that the inner product between inequivalent irreducible representations is zero, but this is something one would generally expect.

2.1.5 Summary

We can label our representations by the eigenvalues $\Delta(1-\Delta)$ of the quadratic Casimir. Representations labeled by Δ and by $\bar{\Delta} = 1-\Delta$ are identical. We saw that the generators of $SO(1,2)$ act on states as $|\psi\rangle = \int dx\psi(x)|x\rangle = \int dp\psi(p)|p\rangle$ as follows

Position space

$$\begin{aligned} P\psi(x) &= -i\partial_x\psi(x) \\ D\psi(x) &= -i(x\partial_x + \bar{\Delta})\psi(x) \\ K\psi(x) &= -i(x^2\partial_x + 2x\bar{\Delta})\psi(x) \ , \end{aligned} \tag{2.42}$$

Momentum space

$$\begin{aligned} P\psi(p) &= -p\psi(p) \\ D\psi(p) &= i(p\partial_p + \Delta)\psi(p) \\ K\psi(p) &= (p\partial_p^2 + 2\Delta\partial_p)\psi(p) \ . \end{aligned} \tag{2.43}$$

We saw that there are two distinct representations for $SO(1,2)$. These are the following

- **CASE 1** : $\Delta = \frac{1}{2} + i\nu, \ \nu \in \mathbb{R}$ with an inner product given by $\int dp|\psi(p)|^2$. This corresponds to the *principal series representation.*
- **CASE 2** : $\Delta \in \mathbb{R}$ with an inner product given by $\int dp|p|^{2\Delta-1}|\psi(p)|^2$. This corresponds to the *highest weight representation.*

Interestingly, the principal series representation can be used to realize a unitary irreducible representation with complex weights. This is interesting since we have seen before that in de Sitter space complex weights are important, which are often assumed to signal non-unitarity, but these correspond exactly to the shape of weights in the unitary principal series representation.

2.2 Unitary Irreducible Representations of $SO(1,3)$

Next, we consider the isometry group of 3-dimensional de Sitter spacetime, namely SO(1,3). The steps are going to be the same as for the $SO(1,2)$ case.

In total we have six generators for SO(1,3). The algebra consists of one dilatation D, two translations P_i, two special conformal transformations K_i and one rotations (due to the constraint $M_{ij} = -M_{ji}$), where $i, j \in \{1, 2\}$. We define $M_{ij} = -\epsilon_{ij} J$, where J can be thought of as the usual J_3. The algebra (2.3) simplifies to

$$
\begin{aligned}
[D, P_i] &= iP_i \\
[D, K_i] &= -iK_i \\
[K_i, P_j] &= 2i\left(\delta_{ij} D - \epsilon_{ij} J\right) \\
[J, P_i] &= -i\epsilon_{ij} P_j \\
[J, K_i] &= -i\epsilon_{ij} K_j
\end{aligned}
\tag{2.44}
$$

and the quadratic Casimir is given by

$$
C^2 = D(2 - D) + \vec{P} \cdot \vec{K} + J^2. \tag{2.45}
$$

2.2.1 Position Space Basis

We start with a state $|\Delta, m, 0\rangle$ of dimension Δ and spin m such that

$$
\begin{aligned}
K|\Delta, m, 0\rangle &= 0 \\
J|\Delta, m, 0\rangle &= m|\Delta, m, 0\rangle \\
D|\Delta, m, 0\rangle &= \Delta|\Delta, m, 0\rangle,
\end{aligned}
\tag{2.46}
$$

where $\Delta \in \mathbb{C}$ and $m \in \mathbb{Z}$. Again, we have chosen the states such that they diagonalize the generators J and D in the subspace that is annihilated by K. The other basis elements in the irreducible representation are obtained by translating the primary states:

$$
|\Delta, m, x\rangle \;\equiv\; e^{x_i P_i}\;|\Delta, m, 0\rangle \tag{2.47}
$$

From this definition and the conformal algebra we find (dropping the label Δ, m, meaning $|\Delta, m, 0\rangle \;\equiv\; |0\rangle$)

$$
\begin{aligned}
P_i|x\rangle &= i\partial_i|x\rangle \\
D|x\rangle &= i\left(x_i\partial_i + \Delta\right)|x\rangle \\
J|x\rangle &= i\left(\epsilon_{ij} x_i \partial_j + m\right)|x\rangle \\
K_i|x\rangle &= i\left(2x_i\left(x_k\partial_k + \Delta\right) - x^2\partial_i + 2\epsilon_{ij} x_j m\right)|x\rangle\ .
\end{aligned}
\tag{2.48}
$$

We can construct arbitrary states in the representation via superposition

$$|\Psi\rangle = \int d^2x \sum_m \psi_m(x)|m,x\rangle \ . \tag{2.49}$$

The $so(1,3)$ algebra acts on the wavefunctions $\psi_m(x)$ as follows

$$\begin{aligned} P_i\ \psi_m(x) &= -i\partial_i\ \psi_m(x) \\ D\ \psi_m(x) &= -i\left(x_i\partial_i + \bar{\Delta}\right)\ \psi_m(x) \\ J\ \psi_m(x) &= i\left(-\epsilon_{ij}x_i\partial_j + m\right)\ \psi_m(x) \\ K_i\ \psi_m(x) &= i\left(-2x_i\left(x_k\partial_k + \bar{\Delta}\right) + x^2\partial_i + 2\epsilon_{ij}x_j m\right)\ \psi_m(x)\ , \end{aligned} \tag{2.50}$$

where we have defined $\bar{\Delta} = 2 - \Delta$. All states $|\Psi\rangle$ in the representation are constructed that they also diagonalize the quadratic Casimir

$$C^2|\Psi\rangle = \left(\Delta(2-\Delta) + m^2\right)|\Psi\rangle \ . \tag{2.51}$$

In order to obtain a unitary representation we must require that the Casimir must be real from which follows that

$$\textbf{CASE 1}:\ \Delta = 1 + i\nu,\ \nu \in \mathbb{R} \tag{2.52}$$

$$\textbf{CASE 2}:\ \Delta \in \mathbb{R}\ . \tag{2.53}$$

Again, we are left with two qualitatively cases for the weights, one with complex weights that fall into the principal series representation and one with real weights associate to the highest weight representation. We note that for the weights in the principal series representation $\bar{\Delta} = \Delta^*$.

2.2.2 Momentum Space Basis

As usual, we can go to momentum space by Fourier transforming the position basis

$$|m,p\rangle \equiv \frac{1}{2\pi}\int d^2x\ e^{-ip\cdot x}\psi(x)\ , \tag{2.54}$$

which is again taking $\psi(x) = e^{-ip\cdot x}$. We can construct arbitrary states via linear superpositions

$$|\Psi\rangle = \int d^2p\ \sum_m \psi_m|m,p\rangle \tag{2.55}$$

and can go from the wavefunction in the position space basis to the momentum space basis via

$$\psi(p) = \frac{1}{2\pi} \int d^2x e^{ip\cdot x} \psi(x) \ . \tag{2.56}$$

The action of the so(1,3) generators on the momentum basis states $|p, m\rangle$ is

$$\begin{aligned} P_i|p,m\rangle &= -p|p,m\rangle \\ D|p,m\rangle &= -i(p_i\partial_{p_i} + \bar{\Delta})|p,m\rangle \\ J|p,m\rangle &= i(-\epsilon_{ij}p_j\partial_{p_i} + m)|p,m\rangle \\ K_i|p,m\rangle &= (2(p_k\partial_{p_k} + \bar{\Delta})\partial_{p_i} - p_i\partial^2 + 2m\epsilon_{ij}\partial_{p_j})|p,m\rangle \ . \end{aligned} \tag{2.57}$$

Now, using integration by parts, we obtain the action of the operators on the wavefunctions as

$$\begin{aligned} P_i\ \psi_m(p) &= -p\ \psi_m(p) \\ D\ \psi_m(p) &= i(p_i\partial_{p_i}\Delta)\ \psi_m(p) \\ J\ \psi_m(p) &= i(\epsilon_{ij}p_j\partial_{p_i} + m)\ \psi_m(p) \\ K_i\ \psi_m(p) &= (2(p_k\partial_{p_k} + \Delta)\partial_{p_i} - p_i\partial^2 + 2m\epsilon_{ij}\partial_{p_j})\ \psi_m(p) \ . \end{aligned} \tag{2.58}$$

2.2.3 Constraints due to Unitarity

In this section we want to explore the consequences of unitary and especially the corresponding constraints on Δ. We have already seen, due to the constraint that the Casimir is real $\Delta(2-\Delta)$, that we have two distinct possibilities for Δ

$$\textbf{CASE 1}: \ \Delta = 1 + i\nu,\ \nu \in \mathbb{R} \tag{2.59}$$

$$\textbf{CASE 2}: \ \Delta \in \mathbb{R} \ . \tag{2.60}$$

The steps we need to do here are essentially the same as in (2.1.3), but slightly more involved. Finally we want to find the inner product for the position and momentum basis. Using the fact that $P_i^\dagger = P_i$ and $\langle p|P_i|q\rangle = p_i\langle p|q\rangle = q_i\langle p|q\rangle$ leads to

$$(p_i - q_i)\langle p|q\rangle = 0 \implies \langle p|q\rangle = f(p)\delta(p-q) \ , \tag{2.61}$$

where $f(p)$ is some function. Next we are going to use the constraint that the dilatation is giving us. We again note that $D^\dagger = D$ from which follows that

$$(q_i\partial_{q_i} + p_i\partial_{p_i})\langle p|q\rangle = -(\bar{\Delta} + \bar{\Delta}^*)\langle p|q\rangle \ . \tag{2.62}$$

Using the result from the translation constraint we find that this implies that

$$f(p) = c|p|^{\alpha}, \quad c > 0, \quad \alpha = -\bar{\Delta} - \bar{\Delta}^* + 2 = \Delta + \Delta^* - 2 \ . \tag{2.63}$$

Thus, we arrive that due two unitarity we have two different cases with the following different inner products

$$\textbf{CASE 1}: \ \Delta = 1 + i\nu, \ \nu \in \mathbb{R}: \quad \langle p|q\rangle = c\delta(p-q) \tag{2.64}$$

$$\textbf{CASE 2}: \ \Delta \in \mathbb{R}: \quad \langle p|q\rangle = c|p|^{\alpha}\delta(p-q), \quad \alpha = 2(\Delta - 1) \ , \tag{2.65}$$

where again c is some positive number. The inner product for the states from (2.55) is given by

$$\textbf{CASE 1}: \langle\Psi|\phi\rangle = c\int \mathrm{d}^2p \ \psi(p)^*\phi(p) \tag{2.66}$$

$$\textbf{CASE 2}: \langle\Psi|\phi\rangle = c\int \mathrm{d}^2p \ |p|^{2\Delta-2}\psi(p)^*\phi(p), \quad \Delta \in \mathbb{R} \ . \tag{2.67}$$

By Fourier transformation of the momentum space result we find the position space inner product to be

$$\textbf{CASE 1}: \langle\Psi|\phi\rangle = c\int \mathrm{d}^2x \ \psi(x)^*\phi(x) \tag{2.68}$$

$$\textbf{CASE 2}: \langle\Psi|\phi\rangle = c'\int \mathrm{d}^2x\mathrm{d}^2y \ \frac{1}{|x-y|^{2\Delta}}\psi(x)^*\phi(y), \quad \Delta \in \mathbb{R} \tag{2.69}$$

with the latter expression up to contact terms. From this we deduce

$$\textbf{CASE 1}: \langle x|y\rangle = c\delta(x-y) \tag{2.70}$$

$$\textbf{CASE 2}: \langle x|y\rangle = c'\frac{1}{|x-y|^{2\Delta}} \tag{2.71}$$

with c' as before and $\Delta > 0$ in the second case to obtain a converging Fourier transformation.

We see that there is again the discrete highest/lowest weight representation. We further see that there is as a second option to construct a unitary irreducible representation, the principal series representation with complex weights. This representation is prominent for the de Sitter spacetime isometry group in any dimension d and needs to be better understood to get a better handling of quantum gravity in de Sitter.

2.2.4 Inner Products of States belonging to different UIRs

For completeness we will redo our discussion on states not necessary belonging to the same representation. Again, we are going to label the states as $|\lambda, p\rangle$, where λ denotes the irreducible representations to which the state belongs. Again, we find

$$\langle \lambda, p|\lambda', p'\rangle = c_{\lambda,\lambda'}|p|^{\Delta^*_\lambda+\Delta_{\lambda'}-2}\delta(p-p') \ . \tag{2.72}$$

To ensure the inner product to be non zero we find

$$c_{\lambda,\lambda'} \neq 0 \quad \Longrightarrow \quad \Delta_{\lambda'} = 2-\Delta^*_\lambda \quad \text{or} \quad \Delta_{\lambda'} = \Delta^*_\lambda \ . \tag{2.73}$$

Taking these two constraints into account we find

$$\langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\Delta_{\lambda'} = 2-\Delta^*_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\Delta_{\lambda'}-2}\delta(p-p') & (\Delta_{\lambda'} = \Delta^*_\lambda) \ . \end{cases} \tag{2.74}$$

Next, we define $\nu \equiv \Delta - 1$ and we can write the result from above as

$$\langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = -\nu^*_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = \nu^*_\lambda) \ . \end{cases} \tag{2.75}$$

This means specialized to those irreducible representations satisfying the unitarity conditions $\nu = -\nu^*$ (**CASE 1**) or $\nu = \nu^*$ (**CASE 2**) we find for the three different possible combinations

$$\langle \mathbf{1}|\mathbf{1}\rangle : \langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = \nu_\lambda, \ \ \text{i.e.} \ \ \Delta_{\lambda'} = \Delta_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = -\nu_\lambda, \ \ \text{i.e.} \ \ \Delta_{\lambda'} = \bar{\Delta}_\lambda) \end{cases} \tag{2.76}$$

$$\langle \mathbf{2}|\mathbf{2}\rangle : \langle \lambda, p|\lambda', p'\rangle = \begin{cases} c_{\lambda,\lambda'}\delta(p-p') & (\nu_{\lambda'} = -\nu_\lambda, \ \ \text{i.e.} \ \ \Delta_{\lambda'} = \bar{\Delta}_\lambda) \\ c_{\lambda,\lambda'}|p|^{2\nu_{\lambda'}}\delta(p-p') & (\nu_{\lambda'} = \nu_\lambda, \ \ \text{i.e.} \ \ \Delta_{\lambda'} = \Delta_\lambda) \end{cases} \tag{2.77}$$

$$\langle \mathbf{1}|\mathbf{2}\rangle : \langle \lambda, p|\lambda', p'\rangle = 0 \ . \tag{2.78}$$

This is as expected and completely analogous to our result for $SO(1,2)$.

2.2.5 Summary

We can label our representations by the eigenvalues $\Delta(2-\Delta)$ of the quadratic Casimir. Representations labeled by Δ and by $\bar{\Delta} = 2-\Delta$ are identical. We saw that the generators of $SO(1,3)$ act on states as $|\psi\rangle = \int d^2x\psi(x)|x\rangle = \int d^2p\psi(p)|p\rangle$ as follows

Position space

$$
\begin{aligned}
P_i\ \psi_m(x) &= -i\partial_i\ \psi_m(x) \\
D\ \psi_m(x) &= -i\left(x_i\partial_i+\bar{\Delta}\right)\ \psi_m(x) \\
J\ \psi_m(x) &= i\left(-\epsilon_{ij}x_i\partial_j+m\right)\ \psi_m(x) \\
K_i\psi_m(x) &= i\left(-2x_i\left(x_k\partial_k+\bar{\Delta}\right)+x^2\partial_i+2\epsilon_{ij}x_jm\right)\psi_m(x)\ ,
\end{aligned} \tag{2.79}
$$

Momentum space

$$
\begin{aligned}
P_i\ \psi_m(p) &= -p\ \psi_m(p) \\
D\ \psi_m(p) &= i(p_i\partial_{p_i}\Delta)\ \psi_m(p) \\
J\ \psi_m(p) &= i(\epsilon_{ij}p_j\partial_{p_i}+m)\ \psi_m(p) \\
K_i\ \psi_m(p) &= (2(p_k\partial_{p_k}+\Delta)\partial_{p_i}-p_i\partial^2+2m\epsilon_{ij}\partial_{p_j})\ \psi_m(p)\ .
\end{aligned} \tag{2.80}
$$

We saw that there are two distinct representations for $SO(1,3)$. These are the following

- **CASE 1** : $\Delta = 1+i\nu,\ \nu\in\mathbb{R}$ with an inner product given by $\int dp|\psi(p)|^2$. This corresponds to the *principal series representation.*
- **CASE 2** : $\Delta\in\mathbb{R}$ with an inner product given by $\int dp|p|^{2\Delta-2}|\psi(p)|^2$. This corresponds to the *highest weight representation.*

2.3 Summary

We have seen that there are two qualitatively distinct representations popping up in d-dimensional de Sitter space [9]. We have explicitly shown this for the two and three dimensional case. The principal series representation is always there, which differs from the anti de Sitter case. However, the principal series is not well understood yet in the context of physics and we will try to shed some light on it in the context of the DFF model. To summarize we have seen the following two cases:

- **CASE 1** : $\Delta = \frac{d-1}{2}+i\nu,\ \nu\in\mathbb{R}$ with an inner product given by $\int dp|\psi(p)|^2$. This corresponds to the *principal series representation.*
- **CASE 2** : $\Delta\in\mathbb{R}$ with an inner product given by $\int dp|p|^{2\Delta-d}|\psi(p)|^2$. This corresponds to the *highest weight representation.*

In the following we will go on and study a simple toy model for conformal quantum mechanics, namely the DFF model. We will see what sort of features a principal series representation will bring.

3 DFF Model

In the paper [35] by D'Alfaro, Fubini and Furlan (DFF) introduced a $SL(2,\mathbb{R})$ invariant quantum mechanical model. The model consists of three operators, namely one Hamiltonian H, one special conformal transformation operator K and one dilatation operator D

$$
\begin{aligned}
H &= \frac{1}{2}\left\{-\partial_r^2 + \frac{(4\Delta-1)(4\Delta-3)}{4r^2}\right\} \\
K &= \frac{r^2}{2} \\
D &= -\frac{i}{2}\left\{r\partial_r + \frac{1}{2}\right\} \ .
\end{aligned}
\tag{3.1}
$$

A quick calculation shows that these operators fulfil the $SO(1,2)$ algebra

$$
\begin{aligned}
[D,H] &= iH \\
[K,D] &= iK \\
[K,H] &= 2iD \ .
\end{aligned}
\tag{3.2}
$$

We can also choose a linear combination of those to obtain the usual raising/lowering operators

$$
\begin{aligned}
L_0 &\equiv \frac{1}{2}\left(\frac{K}{a} + aH\right) \\
L_\pm &\equiv \frac{1}{2}\left(\frac{K}{a} - aH\right) \mp iD \ ,
\end{aligned}
\tag{3.3}
$$

where a is a dimensional parameter, which we will set for simplicity to one in the following. These operators obey the $SL(2,\mathbb{R})$ algebra

$$
\begin{aligned}
[L_\pm, L_0] &= \pm L_\pm \\
[L_+, L_-] &= 2L_0 \ .
\end{aligned}
\tag{3.4}
$$

We can use the definition of the operator H,K and D to explicitly write these as

$$\begin{aligned} L_0 &= \frac{1}{4}\left\{-\partial_r^2 + \frac{(4\Delta-1)(4\Delta-3)}{4r^2} + r^2\right\} \\ L_\pm &= \frac{1}{4}\left\{-\partial_r^2 + \frac{(4\Delta-1)(4\Delta-3)}{4r^2} - r^2 \pm 2ir\partial_r + \frac{1}{2}\right\} \ . \end{aligned} \tag{3.5}$$

The algebra admits a quadratic Casimir

$$C^2 = L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) \ . \tag{3.6}$$

The model can be described by two different unitary representations relying on the value of the quadratic Casimir

$$C^2 = \Delta(\Delta - 1) \ . \tag{3.7}$$

The first representation is the highest weight representation for $\Delta(\Delta-1) \geq -1/4$ with $\Delta \in \mathbb{R}$ and describes the radial dynamics of a charged particle interacting with a magnetic monopole at the origin [36]. In this case the operators are self-adjoint with the standard norm on the positive semi-axis

$$\int_0^\infty dr f^*(r) g(r) \ . \tag{3.8}$$

We will briefly review the most important aspects in 3.1. This representation is well known and we will not focus on it.

The second case corresponds to the choice of $\Delta(\Delta-1) < -1/4$ and $\Delta = \frac{1-i\nu}{2}$ where $\nu \in \mathbb{R}$. This correpsonds to an attractive potential at the origin. These states are usually neglected since it is said that the particle will simply fall into the origin [37] or that the states are not normalizable [38]. However, this is the case we need to understand to get a better understanding of their role in de Sitter spacetime. As we will see, the issue is that the operators in the principal series are not self-adjoint as defined on the positive half-line and we need to include the negative half-line as well. This issue will be resolved in detail in 3.2.

3.1 The Highest Weight Representation

It is well known that the DFF model allows a highest weight description for $\Delta(\Delta-1) \geq -\frac{1}{4}$. In the highest weight case we can construct the discrete spectrum for the group $SO(1,2)$ in the following way [39]

$$
\begin{aligned}
L_0|n\rangle &= r_n|n\rangle \\
r_n &= r_0 + n, \qquad n = 0, 1, \ldots, \qquad r_0 > 0 \\
\langle n'|n\rangle &= \delta_{n',n}
\end{aligned}
\tag{3.9}
$$

with Ladder operators $L_\pm$

$$
L_\pm|n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)}|n \pm 1\rangle \ . \tag{3.10}
$$

We note that we can get non-integer eigenvalues of L_0 since we are considering the universal cover of SO(1,2) [40]. From (3.10) one can deduce that

$$
|n\rangle = \sqrt{\frac{\Gamma(2r_0)}{n!\Gamma(2r_0 + n)}} \left(L_+\right)^n |0\rangle \ . \tag{3.11}
$$

We see that the eigenvalue r_0 is the lowest and we can call this state the *vacuum*. The quadratic Casimir is connected to this state due to

$$
C^2|n\rangle = r_0(r_0 - 1)|n\rangle \ . \tag{3.12}
$$

This structure can be translated into the DFF model as defined in (3.1) with

$$
\begin{aligned}
C^2 &= \Delta(\Delta - 1) \\
r_0 &= \frac{1}{2}\left(\sqrt{(2\Delta - 1)^2} + 1\right) \ ,
\end{aligned}
\tag{3.13}
$$

where r has scale dimension $-1/2$ and is a conformal primary.

3.2 The Principal Series Representation

Next, we want to discuss the DFF model in the principal series representation, where we will need the observations laid out in [41, 42]. We will start out by working on a circle with a compact degree of freedom $\psi \in [0, 2\pi)$ as it is the most convenient and appropriate place for the principal series to start. Later we will use a coordinate transformation to make touch to the usual DFF model on the positive semi-axis as discussed before. This irreducible representation is labeled by $\Delta = \frac{1-i\nu}{2}$ with $\nu \in \mathbb{R}$. Δ is chosen such that we obtain a real quadratic Casimir and hence, a unitary representation.
Let us start by deriving the Hamiltonian H, the special conformal transformation operator

K and the dilatation operator D appropriate for the principal series, which obey the algebra (2.6)

$$
\begin{aligned}
[D, H] &= iH \\
[K, D] &= iK \\
[K, H] &= 2iD \ .
\end{aligned}
\tag{3.14}
$$

We are going to use the action of an group element of SU(1,1) acting on an element of S^1 as

$$
(D^\nu(g)f)e^{i\psi} = |\bar{\alpha} + e^{i\psi}\beta|^{i\nu-1} f\left(\frac{\alpha e^{i\psi} + \bar{\beta}}{\bar{\alpha} + \beta e^{i\psi}}\right) \ , \tag{3.15}
$$

where we define for an element of $SU(1,1)$

$$
\text{SU(1,1)} \in \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \to \begin{pmatrix} Re(\alpha+\beta) & -Im(\alpha-\beta) \\ Im(\alpha+\beta) & Re(\alpha-\beta) \end{pmatrix} \in \text{SL}(2,\mathbb{R}) \ . \tag{3.16}
$$

This defines an isomorphism between SU(1,1) and SL(2,$\mathbb{R}$). As an representation of the algebra (3.14) we use

$$
\begin{aligned}
H &= i\sigma_+ \\
K &= -i\sigma_- \\
D &= \frac{i}{2}\sigma_3
\end{aligned}
\tag{3.17}
$$

with $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and σ_i the Pauli matrices.
First we will work out K. For this, we need to determine α and β. In order to do this we will first need to find the group element of K as an element of SL(2,$\mathbb{R}$) and then use the isomorphism to determine α and β. We find

$$
e^{i\lambda K} = e^{i(-i\lambda)\sigma_-} = \mathbb{1}\cos(i\lambda) - i\sin(i\lambda)\sigma_- \underset{\lambda\ll 1}{\approx} \mathbb{1} + \lambda\sigma_- = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \ . \tag{3.18}
$$

Using the isomorphism we can work out that $\alpha = 1 + \frac{i\lambda}{2}$ and $\beta = +\frac{i\lambda}{2}$. From (3.15) we know

$$
\left(e^{i\lambda K} f\right)\left(e^{i\psi}\right) = \left|1 - \frac{i\lambda}{2} + \frac{i\lambda}{2}e^{i\psi}\right|^{i\nu-1} f\left(\frac{(1+\frac{i\lambda}{2})e^{i\psi} - \frac{i\lambda}{2}}{(1-\frac{i\lambda}{2}) + \frac{i\lambda}{2}e^{i\psi}}\right) \ . \tag{3.19}
$$

Expanding both sides in λ and only keeping terms up to first order we find

$$K = -\frac{\nu + i}{2}\sin(\psi) + i(\cos(\psi) - 1)\frac{\partial}{\partial\psi} = -2i\sin\left(\frac{\psi}{2}\right)\left\{\Delta\cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\frac{\partial}{\partial\psi}\right\}\ , \tag{3.20}$$

where we used $\Delta = \frac{1-i\nu}{2}$.
Next, let us consider H. Using the same steps as before for K, we find

$$e^{i\lambda H} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}\ . \tag{3.21}$$

From which we deduce that $\alpha = 1 + \frac{i\lambda}{2}$ and $\beta = -\frac{i\lambda}{2}$. Again, we use (3.15) and expand both sides in λ to find

$$H = \frac{\nu + i}{2}\sin(\psi) - i(1 + \cos(\psi))\frac{\partial}{\partial\psi} = 2i\cos\left(\frac{\psi}{2}\right)\left\{\Delta\sin\left(\frac{\psi}{2}\right) - \cos\left(\frac{\psi}{2}\right)\frac{\partial}{\partial\psi}\right\}\ . \tag{3.22}$$

Finally, we do the same procedure for D. We find

$$e^{i\lambda D} = \begin{pmatrix} 1 + \frac{\lambda}{2} & 0 \\ 0 & 1 - \frac{\lambda}{2} \end{pmatrix}\ , \tag{3.23}$$

corresponding to $\alpha = 1$ and $\beta = \frac{\lambda}{2}$. Using (3.15) and expand both sides in λ we find

$$D = -\frac{\nu + i}{2}\cos(\psi) - i\sin(\psi)\frac{\partial}{\partial\psi} = -i\left\{\Delta\cos(\psi) + \sin(\psi)\right\}\ . \tag{3.24}$$

In total the three operator of the algebra (3.14) are

$$H = \ \ 2i\cos\left(\frac{\psi}{2}\right)\left\{\Delta\sin\left(\frac{\psi}{2}\right) - \cos\left(\frac{\psi}{2}\right)\frac{\partial}{\partial\psi}\right\} \tag{3.25}$$

$$K = -2i\sin\left(\frac{\psi}{2}\right)\left\{\Delta\cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\frac{\partial}{\partial\psi}\right\} \tag{3.26}$$

$$D = -i\left\{\Delta\cos(\psi) + \sin(\psi)\right\}\ . \tag{3.27}$$

We note that these states are all self-adjoint with respect to the norm on the circle

$$(\xi, \chi) = \int_0^{2\pi} d\psi \xi^*(\psi)\chi(\psi)\ . \tag{3.28}$$

3.2.1 L_0 Eigenstates

In this part we want to calculate the eigenstates for the operator L_0. Our raising/lowering operators can be computed via $L_0 = \frac{H+K}{2}$ and $L_\pm = \frac{1}{2}(K-H) \mp iD$ or explicitly these are given by

$$L_0 = -i\partial_\psi \tag{3.29}$$
$$L_+ = e^{-i\psi}\left(-\Delta - i\partial_\psi\right) \tag{3.30}$$
$$L_- = e^{i\psi}\left(+\Delta - i\partial_\psi\right) \tag{3.31}$$

obeying the algebra (3.4). The Hilbert space is now spanned by eigenstates of the operator $L_0 l_n = -n l_n$. These are easily found to be

$$l_n(\psi) = c e^{-in\psi} \ . \tag{3.32}$$

Since we demand that our eigenfunctions are single valued and $\psi \in [0, 2\pi)$ we observe that the eigenvalue n needs to be an integer. We further observe that these functions are orthogonal with the inner product (3.28) and are normalized by setting $c = \frac{1}{\sqrt{2\pi}}$. Hence, we obtain our normalized states as

$$l_n(\psi) = \frac{1}{\sqrt{2\pi}} e^{-in\psi} \ . \tag{3.33}$$

A quick check reveals that the operators $L_\pm$ act on these functions as

$$L_\pm l_n(\psi) = -(n \pm \Delta) l_{n\pm 1}(\psi) \ . \tag{3.34}$$

This action is precisely what one would expect for the principal series.

3.2.2 H and K Eigenstates

Working with the discrete eigenstates of the compact operator L_0 will be eventually most convenient, but one could also work with the eigenstates of the energy operator H or the special conformal transformations operator K. Both operators H and K are non-compact so we will obtain a continuous spectrum and a modified orthogonality relation. We further note that from the point of view of the $SL(2,\mathbb{R})$ algebra H and K are interchangeable.
Let us start by finding the eigenstates of the energy operator H. We need to solve the differential equation

$$H h_E = E h_E \ , \tag{3.35}$$

which is solved by

$$h_E(\psi) = \frac{1}{2\sqrt{\pi}} e^{iE\tan\left(\frac{\psi}{2}\right)} \cos\left(\frac{\psi}{2}\right)^{-2\Delta} \ . \tag{3.36}$$

The solution we wrote is already normalized to the inner product (3.28). One can also check that these functions obey the completeness relation

$$\int_{-\infty}^{\infty} h_E^*(\psi) h_E(\psi') = \delta(\psi - \psi') \ . \tag{3.37}$$

In the same manner we can find eigenfunctions for the special conformal transformations operator K. We begin with the differential equation

$$K\rho\kappa(\psi) = \kappa\rho_\kappa(\psi) \ , \tag{3.38}$$

which is solved by

$$\rho_\kappa(\psi) = \frac{1}{2\sqrt{\pi}} e^{-i\kappa\cot\left(\frac{\psi}{2}\right)} \sin\left(\frac{\psi}{2}\right)^{-2\Delta} \ . \tag{3.39}$$

These states are normalized with respect to (3.28) and obey the completeness relation

$$\int_{-\infty}^{\infty} d\kappa \rho_\kappa(\psi) \rho_\kappa^*(\psi') = \delta(\psi - \psi') \ . \tag{3.40}$$

Since these functions define a complete orthonormal set of functions we can define the following transform and its inverse

$$\begin{aligned} \theta(\kappa) &= \int_0^{2\pi} d\psi \rho_\kappa^*(\psi) \theta(\psi) \\ \theta(\psi) &= \int_{-\infty}^{\infty} d\kappa \rho_\kappa(\psi) \theta(\kappa) \ . \end{aligned} \tag{3.41}$$

3.2.3 L_0 States in the r-basis

As the next step we want to translate our results into the standard picture of the DFF model using the coordinate r. We further use the fact that the eigenfunctions of K build a orthonormal set to define a modified transform as

$$\tilde{\theta}(\kappa) = |2\kappa|^{\frac{3}{4}-\Delta} \int_0^{2\pi} d\psi \rho_\kappa^*(\psi) \theta(\psi) \ , \tag{3.42}$$

which implies a modified norm to be

$$(\chi,\xi)' = \int_{-\infty}^{\infty} \frac{dk}{|2\kappa|^{1/2}} \chi^*(\kappa)\xi(\kappa) \ . \tag{3.43}$$

The prefactor is added to ensure the standard norm in the r-coordinate after the transformation. This norm preserves the overlap, meaning

$$\left(\tilde{\chi}(\kappa), \tilde{\xi}(\kappa)\right)' = (\chi(\psi), \xi(\psi)) \ . \tag{3.44}$$

The operators act on the function space with parameter κ as

$$\begin{aligned} H &= \frac{1}{2}\left\{-2\kappa\partial_\kappa^2 - \partial_\kappa + \frac{(4\Delta-1)(4\Delta-3)}{8\kappa}\right\} \\ K &= \kappa \\ D &= -i\left\{\kappa\partial_\kappa + \frac{1}{4}\right\} \ . \end{aligned} \tag{3.45}$$

Eventually, we want to make touch with the standard description of the DFF model in the r-coordinate.

	DFF with r	DFF with κ
K	$\frac{r^2}{2}$	κ
Range	$r \in \{0, \infty\}$	$\kappa \in \{-\infty, \infty\}$

In table 3.2.3 it is shown that we need to extend the r-coordinate to $r \in \{-\infty, \infty\}$ and to identify

$$\kappa = \frac{r^2}{2} \ , \ r > 0 \qquad \text{and} \qquad \kappa = -\frac{r^2}{2} \ , \ r < 0 \tag{3.46}$$

or in short $\kappa = \text{sign}(r)\frac{r^2}{2}$ in order to allow negative values of K in the r-coordinate as well and to arrive at the standard DFF model. We are now ready to compute our eigenfunctions of L_0 in the κ-basis via

$$\tilde{l}_n(\kappa) = |2\kappa|^{\frac{3}{4}-\Delta} \int_0^{2\pi} \frac{d\psi \sin^{-2\Delta}\left(\frac{\psi}{2}\right) e^{i\kappa\cot\left(\frac{\psi}{2}\right) - in\psi}}{2\pi} \ . \tag{3.47}$$

We find

$$\tilde{l}_n(\kappa) = |2\kappa|^{\frac{3}{4}-\Delta} \int_0^{2\pi} \frac{d\psi \sin^{-2\Delta}\left(\frac{\psi}{2}\right) e^{i\kappa\cot\left(\frac{\psi}{2}\right)-in\psi}}{2\pi} \tag{3.48}$$

$$= -|2\kappa|^{\frac{3}{4}-\Delta} \int_{-\infty}^{\infty} \frac{du\ e^{-i\kappa u}(1-iu)^{-n+\frac{1}{2}i(\nu+i)}(1+iu)^{n+\frac{1}{2}i(\nu+i)}}{\pi} \ , \tag{3.49}$$

where we did the coordinate transformation $\psi \to -2\cot^{-1}(u)$. Next, we can use the Fourier transform relation from [43]

$$\int_{-\infty}^{\infty} (1+iu)^{-2\mu}(1-iu)^{-2\nu} e^{-iyu} du = -2\pi 2^{-\mu-\nu}\Gamma(2\nu)^{-1} y^{\nu+\mu-1} W_{\nu-\mu,1/2-\mu-\nu}(2y) \ , \ y > 0 \tag{3.50}$$

$$\int_{-\infty}^{\infty} (1+iu)^{-2\mu}(1-iu)^{-2\nu} e^{-iyu} du = 2\pi 2^{-\mu-\nu}\Gamma(2\mu)^{-1} (-y)^{\nu+\mu-1} W_{-\nu+\mu,1/2-\mu-\nu}(-2y) \ , \ y < 0 \ , \tag{3.51}$$

where W stands for the Whittaker function, to solve the integral. Hence, we find by making use of our identification $\kappa = \text{sign}(r)\frac{r^2}{2}$ as explained before

$$\tilde{l}_n(r) = \pm \frac{2^{1-\Delta} W_{\pm n, \frac{1}{2}-\Delta}\left(|r|^2\right)}{\sqrt{|r|}\Gamma(n \pm \Delta)} \tag{3.52}$$

for $\pm r > 0$. These states fulfil the relations

$$L_0\ \tilde{l}_n = n\tilde{l}_n$$
$$L_\pm\ \tilde{l}_n = -(n \pm \Delta)\tilde{l}_{n\pm1} \ . \tag{3.53}$$

Next we need to find the orthogonality relation for our states. The induced norm (3.43) implies that all states must be normalized to the standard L_2 norm

$$\int_{-\infty}^{\infty} f^*(r) g(r) dr \ . \tag{3.54}$$

Normalization

We normalize our states according to (3.54), which reads for our states (3.52)

$$(\tilde{l}_n(r), \tilde{l}_m(r)) = \int_0^\infty \left(\frac{W_{n,\frac{1}{2}-\Delta}\left(|r|^2\right) W_{m,\frac{1}{2}-\Delta}\left(|r|^2\right)}{\Gamma(n+\bar{\Delta})\Gamma(m+\Delta)r} + \frac{W_{-n,\frac{1}{2}-\Delta}\left(|r|^2\right) W_{-m,\frac{1}{2}-\Delta}\left(|r|^2\right)}{\Gamma(-n+\bar{\Delta})\Gamma(-m+\Delta)r} \right) . \tag{3.55}$$

Using the result for an integral over two Whittaker functions [44]

$$\int_0^\infty \frac{W_{n,\mu}(x)W_{m,\mu}(x)}{x} dx = \frac{1}{(n-m)\sin(2\mu\pi)} \times \left(\frac{1}{\Gamma(1/2-n+\mu)\Gamma(1/2-m-\mu)} - \frac{1}{\Gamma(1/2-n-\mu)\Gamma(1/2-m+\mu)} \right) \tag{3.56}$$

and a transformation $x \to \sqrt{y}$, we see that our states indeed fulfil

$$\left(\tilde{l}_n(r), \tilde{l}_m(r)\right) = \delta_{n,m} \tag{3.57}$$

and that our states are still normalized as expected.

After the identification with $\kappa = \text{sign}(r)\frac{r^2}{2}$ we find that our operators acting on functions depending now on r as

$$\begin{aligned} H &= \frac{\text{sign}(r)}{2} \left\{ -\partial_r^2 + \frac{(4\Delta-1)(4\Delta-3)}{4r^2} \right\} \\ K &= \text{sign}(r)\frac{r^2}{2} \\ D &= -\frac{i}{2} \left\{ r\partial_r + \frac{1}{2} \right\} . \end{aligned} \tag{3.58}$$

In conclusion, we see that in order to describe the Hilbert space of the principal series with the standard DFF model approach using r as the degree of freedom, we have to extend r to negative values. We further need to *flip the sign of the Hamiltonian* at $r = 0$. Despite this unusual behaviour at $r = 0$ the description of the system is completely unitary. We note that the behaviour at $r = 0$ reminds one of the behaviour of the Hamiltonian at the cosmological horizon in de Sitter spacetime.

Shape of the wavefunction

To get a bit of intuition, we will plot some wavefunctions for different values of n at fixed value $\nu = 2$. We first note the discrete spectrum of the operator L_0 is unbounded from above and below and there is no *ground state*. There is though one state that is symmetric if one reflects at $r = 0$ namely the wavefunction with $n = 0$ as shown in figure 3.1

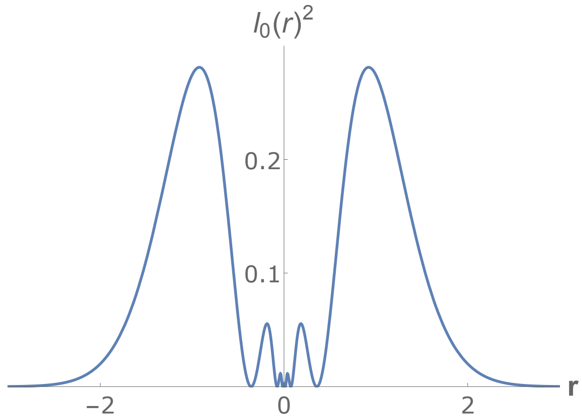

Figure 3.1: Plot of the wavefunction with $n = 0$ and $\nu = 2$. This state is symmetric if one reflects at $r = 0$.

Next, let us look at the first *excited* state with $n = \pm 1$. This state is not symmetric along $r = 0$, but the state $n = 1$ is reflected into the state $n = -1$ by reflecting at the origin. This is shown in figure 3.2.

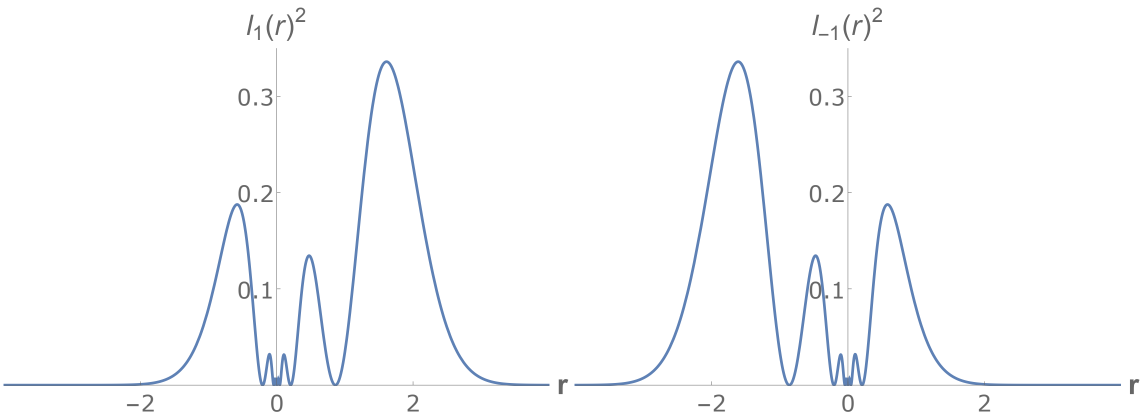

Figure 3.2: Plot of the wavefunction with $n = \pm 1$ and $\nu = 2$. These states are reflected into each other if one reflects at $r = 0$.

We also see that going from state $n \to n + 1$ we add one hill, which can be thought of adding one *excitation* with the raising operator L_+. This can also be seen in figure 3.3 and is shown in figure 3.4 to show that this holds true for larger values of n as well.

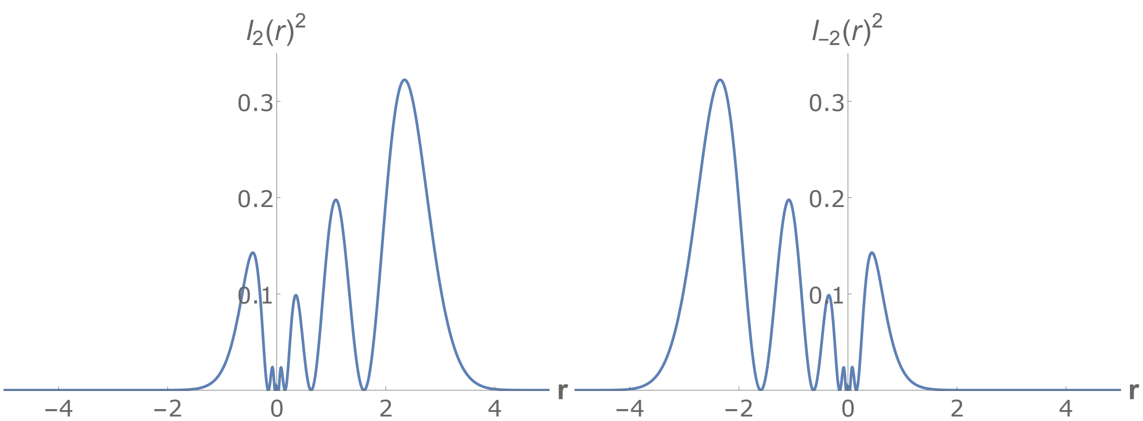

Figure 3.3: Plot of the wavefunction with $n = \pm 2$ and $\nu = 2$. These states are reflected into each other if one reflects at $r = 0$.

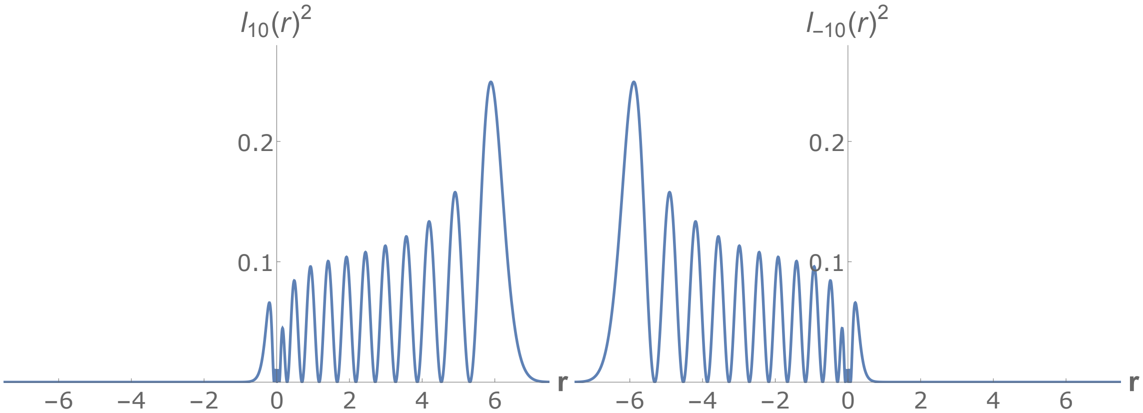

Figure 3.4: Plot of the wavefunction with $n = \pm 10$ and $\nu = 2$. These states are reflected into each other if one reflects at $r = 0$.

3.2.4 Classical and Path Integral Descriptions

Let us step back and try to think, which classical dynamical system can lead to the DFF model in the quantum case. One can consider the symplectic structures on the group manifold of $SO(1, 2)$ [41] to obtain the following functions on phase space

$$\begin{aligned} H &= 2\cos\left(\frac{\theta}{2}\right)\left\{-\nu\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)p\right\} \\ K &= 2\sin\left(\frac{\theta}{2}\right)\left\{\nu\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)p\right\} \\ D &= \nu\cos(\theta) + \sin(\theta) \ . \end{aligned} \tag{3.59}$$

We can define a Poisson bracket with the canonical pair p and θ obeying $\{\theta, p\} = 1$ as

$$\{f, g\} = \partial_\theta f \partial_p g - \partial_p f \partial_\theta g \ . \tag{3.60}$$

The functions fulfil the algebra

$$\begin{aligned} \{D, H\} &= H \\ \{K, D\} &= K \\ \{K, H\} &= 2D \ . \end{aligned} \tag{3.61}$$

This system defines a classical dynamical system incorporating a $SL(2, \mathbb{R})$ symmetry. We note that the functions are linear in p and not as usual quadratic in p. In standard classical mechanics the Lagrange formalism and the Hamiltonian formalism are related by a Legendre transformation. In this case a Legendre transformation is not possible since

$$\frac{\partial H(\theta, p)}{\partial p} = 2\cos\left(\frac{\theta}{2}\right)^2 \equiv \dot{\theta} \tag{3.62}$$

is not invertible. It is puzzling that our model does not seem to admit a local Lagrangian, which can be calculated from the Hamiltonian. We further note that the combination

$$HK - D^2 = -\nu^2 \tag{3.63}$$

is a constant that commutes with H, K, D and therefore is conserved for any dynamics generated by any linear combination of H,K and D. We can recover the quantum operators by replacing $p \to i\partial_\theta$ and $\nu \to -i\Delta$, while ensuring normal ordering. We further need to replace the Poisson brackets with commutators and multiplying the left side of the commutation relation with i.

We note that all of our operators in (3.59) are unbounded from above and from below. This means it is not quite clear, which to use as a time evolution operator. This differs from the highest weight case, where the operators are bounded from below. Furthermore, in the highest weight case one can use the operator $L_0 = \frac{1}{2}(H + K)$, whose spectrum is discrete and bounded. In [35, 36] it was shown that any dynamics is equally valid and are related by time reparametrization. In the following we will first investigate the dynamics generated by $L_0 = \frac{1}{2}(H + K)$. After that we will define and use the operator $K^2 = \frac{1}{2}(H - K) = \frac{1}{2}(L_+ + L_-)$ and investigate the dynamics generated by this operator. This operator is interesting since it corresponds to the generator of time translation in the static patch. This is the boost operator K^2 we obtain by embedding the static patch of dS_2 into 3-dimensional Minkowski space.

Dynamics generated by $L_0 = \frac{1}{2}(H + K)$

Let us first investigate the dynamics generated by L_0. In order to do this we first change our set of functions from H,K and D to L_0 and $L_\pm$, which are given by

$$\begin{aligned} L_0 &= \frac{1}{2}(H + K) = p \\ L_\pm &= \frac{1}{2}(H - K) \mp iD = e^{\mp i\theta}(p \mp i\nu) \ . \end{aligned} \tag{3.64}$$

These satisfy the Poisson bracket algebra

$$\begin{aligned}\{L_\pm, L_0\} &= \mp i L_\pm \\ \{L_+, L_-\} &= -2iL_0\end{aligned} \tag{3.65}$$

and we can find the classical equations by

$$\frac{d\bullet}{dt} = \{\bullet, L_0\} \ . \tag{3.66}$$

We find for our equations

$$\begin{aligned}\frac{dL_0}{dt} &= 0 \\ \frac{d\theta}{dt} &= 1 \\ \frac{dL_\pm}{dt} &= \mp i L_\pm \ .\end{aligned} \tag{3.67}$$

These are solved by

$$\begin{aligned}L_0(t) &= l_0 \\ \theta(t) &= \theta_0 + t \\ L_\pm(t) &= (l_0 \mp i\nu) e^{\mp\theta(t)} \ .\end{aligned} \tag{3.68}$$

We can go on and use the path integral to investigate the dynamics of the system

$$\begin{aligned}\langle\theta_f| e^{-iL_0 T} |\theta_i\rangle &= \int_{\theta(0)=\theta_i}^{\theta(T)=\theta_f} \mathcal{D}p\mathcal{D}\theta \ \exp\left(i\int_0^T dt \ p\left(\dot{\theta}-1\right)\right) \\ &= \int_{\theta(0)=\theta_i}^{\theta(T)=\theta_f} \mathcal{D}\theta\delta\left(\dot{\theta}-1\right) \\ &= \delta\left(\theta_f - \theta_i - T\right) \ ,\end{aligned} \tag{3.69}$$

where in the first step p was integrated out.
It is quite remarkable, how simple the dynamics of this system are. We further see, that due to the linearity in p, there is no simple, local Lagrangian that can give rise to these dynamics.

Dynamics generated by $K^2 = \frac{1}{2}(H - K)$

Another possible choice is to investigate the dynamics generated by $K^2 = \frac{1}{2}(H - K)$. This is hinted by the bulk since this operator coincides with the static patch time. We are going to use the three operators K^2, L_0 and D. These are given by

$$\begin{aligned} K^2 &= \frac{1}{2}(H - K) = -\nu \sin(\theta) + \cos(\theta)\, p \\ L_0 &= \frac{1}{2}(H + K) = p \\ D &= \nu \cos(\theta) + \sin(\theta)\, p \ . \end{aligned} \tag{3.70}$$

They satisfy

$$\begin{aligned} \{L_0, K^2\} &= D \\ \{D, K^2\} &= L_0 \\ \{L_0, D\} &= -K^2 \ . \end{aligned} \tag{3.71}$$

Again, we find the equations that govern the dynamics generated by K^2 by considering

$$\frac{d\bullet}{d\tau} = \{\bullet, K^2\} \ . \tag{3.72}$$

This leads to the equations

$$\begin{aligned} \frac{dL_0}{d\tau} &= D \\ \frac{dD}{d\tau} &= L_0 \ , \end{aligned} \tag{3.73}$$

which are solved by

$$\begin{aligned} L_0(\tau) &= c_1 \cosh(\tau) + c_2 \sinh(\tau) \\ D(\tau) &= c_1 \sinh(\tau) + c_2 \cosh(\tau) \ . \end{aligned} \tag{3.74}$$

We can use the quantum path integral to calculate the dynamics as

$$\begin{aligned}\langle\theta_f|e^{-iK^2T}|\theta_i\rangle &= \int_{\theta(0)=\theta_i}^{\theta(T)=\theta_f} \mathcal{D}p\mathcal{D}\theta\ \exp\left(i\int_0^T d\tau\ \left\{p\left(\dot{\theta}-\cos(\theta)\right)+\nu\sin(\theta)\right\}\right)\\ &= \int_{\theta(0)=\theta_i}^{\theta(T)=\theta_f} \mathcal{D}\theta\ \delta\left(\dot{\theta}-\cos(\theta)\right)\ \exp\left(i\int_0^T d\tau\ \nu\sin(\theta)\right)\ . \end{aligned} \tag{3.75}$$

This expression is explicitly solved by

$$\langle\theta_f|e^{-iK^2T}|\theta_i\rangle = \left(\cosh(T)+\sin(\theta_i)\sinh(T)\right)^{\frac{1}{2}(1+i\nu)}\delta\left(\theta_f-2\tan^{-1}\left[\frac{\sinh\left(\frac{T}{2}\right)+\cosh\left(\frac{T}{2}\right)\tan\left(\frac{\theta_i}{2}\right)}{\cosh\left(\frac{T}{2}\right)+\sinh\left(\frac{T}{2}\right)\tan\left(\frac{\theta_i}{2}\right)}\right]\right)\ . \tag{3.76}$$

We can take the late time limit $T\to\infty$ to obtain

$$\langle\theta_f|e^{-iK^2T}|\theta_i\rangle \propto e^{-(1-\Delta)T}\delta\left(\theta_f-\pi/2\right) \tag{3.77}$$

with $\Delta=\frac{1}{2}(1-i\nu)$ as before. This result is suggesting that localized wavepackets tend towards the ‘horizon’ at $\theta=\pi/2$.

4 DFF Model in dS_2

In the following we want to match the Hilbert space of the DFF model with the 2-dimensional de Sitter spacetime in global coordinates. We will see that the isometries in the early and late time limit become exactly those of the DFF model and from a representation theoretical point of view we can classify the states of the de Sitter bulk with those lying in the Hilbert space of the DFF model.

4.1 Global Coordinates

We want to derive the isometries of 2-dimensional de Sitter space in global coordinates. In order to do so we are going to use the method of embedding dS_2 in 3-dimensional Minkowski space. The line element in Minkowski space in the canonical basis is given by

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 \ . \tag{4.1}$$

The de Sitter space is now the manifold fulfilling the constraint

$$- X_0^2 + X_1^2 + X_2^2 = L^2 \ . \tag{4.2}$$

We can change to global coordinates (τ,θ)

$$\begin{aligned} X_0 &= L \sinh\left(\frac{\tau}{L}\right) \\ X_1 &= L \cos(\theta) \cosh\left(\frac{\tau}{L}\right) \\ X_2 &= L \sin(\theta) \cosh\left(\frac{\tau}{L}\right) \end{aligned} \tag{4.3}$$

with the inverse

$$\begin{aligned} \tau &= L \sinh^{-1}\left(\frac{X_0}{L}\right) \\ \theta &= \tan^{-1}\left(\frac{X_2}{X_1}\right) \ . \end{aligned} \tag{4.4}$$

The global time τ ranges from $\{-\infty, \infty\}$ and the angle θ from $[0, 2\pi)$. This coordinate system induces the metric

$$ds^2 = -d\tau^2 + L^2 \cosh^2\left(\frac{\tau}{L}\right) d\theta^2 \ . \tag{4.5}$$

There are three isometries from the ambient of 3-dimensional Minkowski space that leave the constraint invariant, namely two boosts

$$\begin{aligned} K^1 &= -i(X_0\partial_{X_1} + X_1\partial_{X_0}) \\ K^2 &= -i(X_0\partial_{X_2} + X_2\partial_{X_0}) \end{aligned} \tag{4.6}$$

and one rotation

$$J^3 = -i(X_1\partial_{X_2} - X_2\partial_{X_1}) \ . \tag{4.7}$$

In global coordinates these read

$$\begin{aligned} J^3 &= -i\partial_\theta \\ K^1 &= -i\left(L\cos(\theta)\partial_\tau - \sin(\theta)\tanh\left(\frac{\tau}{L}\right)\partial_\theta\right) \\ K^2 &= -i\left(L\sin(\theta)\partial_\tau + \cos(\theta)\tanh\left(\frac{\tau}{L}\right)\partial_\theta\right) \ . \end{aligned} \tag{4.8}$$

These obey the following commutation relations

$$\begin{aligned} [J^3, K^1] &= iK^2 \\ [K^2, J^3] &= iK^1 \\ [K^2, K^1] &= iJ^3 \ . \end{aligned} \tag{4.9}$$

It is not difficult to see that the following linear combination of these operators is leading to the lowering/raising operators of $SL(2,\mathbb{R})$

$$\begin{aligned} L_0 &= J^3 \\ L_\pm &= K^2 \pm iK^1 \end{aligned} \tag{4.10}$$

fulfilling the $SL(2,\mathbb{R})$ algebra

$$\begin{aligned} [L_\pm, L_0] &= \pm L_\pm \\ [L_+, L_-] &= 2L_0 \ . \end{aligned} \tag{4.11}$$

These read explicitly by using our results before

$$L_0 = -i\partial_\theta$$
$$L_\pm = e^{\mp i\theta}\left(-i\tanh\left(\frac{\tau}{L}\right)\partial_\theta \pm L\partial_\tau\right) . \tag{4.12}$$

As we have discussed earlier the algebra admits a quadratic Casimir

$$C^2 = L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) = -\frac{1}{\cosh(\frac{\tau}{L})^2}\partial_\theta^2 + L\left(\tanh\left(\frac{\tau}{L}\right)\partial_\tau + L\partial_\tau^2\right) . \tag{4.13}$$

As a unitary irreducible representation these are labeled by

$$C^2 = \Delta(\Delta - 1) . \tag{4.14}$$

To map these generators to those of the DFF model we need to take the limit $\tau \to \pm\infty$ to the future/past boundary. In order to do this, we need to solve the wave equations first to see how ∂_τ acts on these solutions in the late/early time limit.

4.2 The Wave Equation

The action for a massive non-interacting scalar field in dS_2 is given by

$$S_0 = -\frac{1}{2}\int d^2x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) . \tag{4.15}$$

One can now go on and vary the action with respect to ϕ to obtain the equations of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi = m^2\phi . \tag{4.16}$$

Comparing this with the quadratic Casimir, one obtains

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi = -\frac{1}{L^2}C^2\phi . \tag{4.17}$$

Thus, we expect on the representation theory of $SL(2,\mathbb{R})$ to label our states as $\Delta(\Delta - 1)$

$$\Delta(\Delta - 1) = -m^2L^2 , \tag{4.18}$$

which implies

$$\Delta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4m^2L^2}\right) . \tag{4.19}$$

Assuming $m^2L^2 > \frac{1}{4}$ our two possible falloffs in time become

$$\Delta_\pm = \frac{1}{2}(1 \pm i\nu) \tag{4.20}$$

with $\nu \in \mathbb{R}$ as expected for the principal series.
The differential equation is separable [45] and the solution can be written in the form

$$\phi = y_l(\tau)Y_n(\theta) \ , \tag{4.21}$$

where Y_n are the spherical harmonics on S^1 obeying

$$\nabla^2_{S^1} Y_n = -n^2 Y_n \ . \tag{4.22}$$

The spherical harmonics on S^1 are given by

$$Y_n(\theta) = e^{-in\theta} \tag{4.23}$$

with $n \in \mathbb{Z}$. In the following we set $L = 1$ and we will recover the factor for our solutions by dimensional analysis later. From this, the differential for the τ dependence reads

$$\partial_\tau^2 y_n + \tanh(\tau)\partial_\tau y_n + \left(m^2 + \frac{n^2}{\cosh^2(\tau)}\right) y_l = 0 \ . \tag{4.24}$$

We can write this equation in terms of the coordinate $\sigma = -e^{2\tau}$ to obtain

$$\sigma(1-\sigma)y_n'' - 2\sigma y_n' + \left\{\frac{m^2}{4}\frac{1-\sigma}{\sigma} - \frac{n^2}{1-\sigma}\right\} y_n = 0 \ . \tag{4.25}$$

Next, let us further substitute

$$y_n^{\text{in}} = \cosh^n(\tau)e^{(n+\frac{1}{2}(1-i\mu))\tau}x = \cosh^n(\tau)e^{(n+\Delta)\tau}x \ , \tag{4.26}$$

where we have defined $\mu = \sqrt{4m^2-1}$ and $\Delta = \frac{1}{2}(1-i\mu)$ as usual. The differential equation simplifies to

$$\sigma(1-\sigma)x'' + [2\Delta - 2(1+n\Delta)\sigma]x' - (n+1)(n+2\Delta)x = 0 \ , \tag{4.27}$$

which is simply the hypergeometric equation. This is solved by

$$y_n^{\text{in}} = \frac{2^n}{\sqrt{\mu}}\cosh^n(\tau)e^{(n+\Delta)\tau/L}{}_2F_1\left(n+\frac{1}{2}, n+\Delta; 2\Delta, -e^{-2\tau/L}\right) \ , \tag{4.28}$$

where F is the hypergeometric function. We can translate our results from incoming states into outgoing states via the map [45]

$$y_n^{\text{out}}(\tau) = y_n^{\text{in}^*}(-\tau) \ . \tag{4.29}$$

The τ-dependence of the outgoing states read explicitly

$$y_n^{\text{out}} = \frac{2^n}{\sqrt{\mu}} \cosh^n(\tau) e^{(-n-\bar{\Delta})\tau/L} F\left(n + \frac{1}{2}, n + \bar{\Delta}; 2\bar{\Delta}, -e^{-2\tau/L}\right) \ . \tag{4.30}$$

4.3 Late and Early Time Limit

Past boundary $\tau \to -\infty$

At the past boundary $\tau \to -\infty$ we observe $F \to 1$. From this we can read off the state as

$$y_n^{\text{in}} \to \frac{1}{\sqrt{\mu}} e^{\Delta\tau/L} \ . \tag{4.31}$$

We can deduce that in this limit

$$\partial_\tau \to \Delta/L \tag{4.32}$$

and our generator become

$$\begin{aligned} L_0 &= -i\partial_\theta \\ L_\pm &= e^{\mp i\theta}\left(i\partial_\theta \pm \Delta\right) \ . \end{aligned} \tag{4.33}$$

We see that our early time wave functions fulfil

$$L_0\phi_n = -n\phi_n \tag{4.34}$$

$$L_\pm\phi_n = (n \pm \Delta)\phi_{n\pm 1} \ . \tag{4.35}$$

Future boundary $\tau \to \infty$

In the late time limit $\tau \to \infty$ we again observe $F \to 1$ and we find

$$y_n^{\text{out}} \to \frac{1}{\sqrt{\mu}} e^{-\bar{\Delta}\tau/L} \ . \tag{4.36}$$

From this we can read off

$$\partial_\tau \to -\bar{\Delta}/L \ . \tag{4.37}$$

With this our generators read in the limit $\tau \to \infty$

$$\begin{aligned} L_0 &= -i\partial_\theta \\ L_\pm &= e^{\mp i\theta}\left(\mp\bar{\Delta} - i\partial_\theta\right) \ , \end{aligned} \tag{4.38}$$

which are the same as for the DFF model. This means one can understand the DFF model with the principal series as the quantum mechanical system, which is defined by the late time generators of 2-dimensional de Sitter space in global coordinates. We see that our late time wave functions fulfil

$$L_0\phi_n = -n\phi_n \tag{4.39}$$

$$L_\pm\phi_n = -(n \pm \bar{\Delta})\phi_{n\pm 1} \ . \tag{4.40}$$

We further note that the shape of the generators and their action on the wavefunction are precisely those of the DFF model with the principal series representation.

5 Conclusion and outlook

Conclusion

We have extensively reviewed the geometry of de Sitter space, where we have shown some of the puzzling features of it. Further, we have seen the need of understanding the principal series representation as it is one of the unitary irreducible representations that pops up at the future and past boundary of de Sitter space. Following this, we systematically investigated the unitary irreducible representations of $SO(1,d)$ for the $d=2,3$ case, which is the isometry group of de Sitter spacetime. It turns out that the principal series is one of the two qualitatively possible UIRs of the group $SO(1,d)$.

After a brief review of the conformal quantum mechanical model DFF, where we have presented the standard description using the highest weight representation, we have derived the generators appropriate for the principal series representation with complex weights. We considered both the classical case as well as the quantum case. The degree of freedom natural for the principal series is the compact angle $\psi \in [0,2\pi)$. In the classical case, we observed that the Hamiltonian is linear in the momentum p and that there is no local simple Lagrangian giving rise to these dynamics. In the quantum case, we saw that all states can be normalized and the description is unitary. We further use a complete set of functions to translate our results into the r coordinate often used to define the DFF model. This required to extend the coordinate, which is defined on the positive semi-axis, to the negative semi-axis as well. Hence, the degree of freedom is $r \in (-\infty,\infty)$. As expected, after the transformation the states still behave well and are still normalized to the standard $\mathbb{R}$-Lebesque norm. We find that in these coordinates the wavefunctions are proportional to the Whittaker functions. Interestingly, one observes that the sign of the Hamiltonian switches as one crosses $r=0$. This behaviour is reminiscent of the sign switch of the Hamiltonian in static coordinates in de Sitter spacetime as one crosses the cosmological horizon.

Finally, we have investigated the isometries of de Sitter space at the future and past boundary. Therefore we used heavy massive scalars in global coordinates. By considering the isometries at the future and past boundary, we were able to show that the algebra can be exactly mapped to those of the principal series description of the DFF model. Thus, we can deduce that we can match the Hilbert spaces of these two theories.

Outlook

It is known that the DFF model in the highest weight representation can be used to describe the radial motion of an electron interacting with a magnetic monopole. It is now natural to ask the question, which physical model, that might be realized in a lab, can be described with the DFF model using the principal series representation. This remains an open question for now, but providing a physical realization would certainly help to understand a few of those bizarre features.

Moreover, it would be interesting to have a closer look at the link between the DFF model in the principal series representation and 2-dimensional de Sitter space. In both cases one can observe that the sign of the Hamiltonian switches. In the DFF model at $r = 0$ and in de Sitter space static patch coordinates at the cosmological horizon $r = L$. One can imagine that there is still a lot to understand with possible applications to the dS/CFT correspondence.

Furthermore, it is interesting to see how our results can be generalized to the higher dimensional case, especially to the dS_3/CFT_2 case. As far as we know, there is no quantum field theory in the literature naturally admitting or at all admitting a unitary description with the principal series representation yet. Our results for the quantum mechanical case might shed some light on the structure of such a field theory. In addition, the principal series can be used to describe the Virasoro algebra with a vanishing central charge. Building a theory containing these features would certainly be a great achievement and is left for future work.

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