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Bachelor Thesis

Representation Theory and The Eightfold Way

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Abstract

In this thesis we want to understand an organization scheme for subatomic particles proposed in 1961 by the physicist Murray Gell-Mann, called the Eightfold Way. We do this mainly from a mathematical point of view. We combine representation theory and the theory of quarks to obtain an insight in the Eightfold Way. More specifically, the irreducible representations of the symmetry group of the quarks, the Lie group $SU(3)$, will help us to classify the light hadrons into multiples. Since the irreducible representations of $SU(3)$ play such an important role, the first part of this thesis is devoted to classifying those representations. The second part will involve the classification of the light hadrons.

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1 Introduction

In the 1950's, the field of particle physics was very chaotic. New particles were discovered very frequently due to technological developments. Those particles could not be understood or explained by the models available at the time. These turbulent times reminded physicists of the period before the Periodic Table, when chemists were looking for order in the observed elements. In the beginning of the 1960's, the field of particle physics awaited their own 'Periodic Table' ([8, p. 33]).

One of the people who played a big role, in bringing order to the chaos, was the American physicist Murray Gell-Mann. He started grouping the new subatomic particles according to their quantum numbers and found they arranged into peculiar geometric patterns ([8, p. 33]). He published his findings, which he called the *Eightfold Way*, in a paper in 1961. The Eightfold Way gave a way to classify the subatomic particles found in the 1950's and even predicted the existence of the Ω^- particle, which was later found[1]. Moreover, the Eightfold Way led to the postulation of quarks and the quark model. To emphasise the importance of the Eightfold Way we quote David J. Griffiths[2]:

> Classification is the first stage in the development of any science. The Eightfold Way did more than merely classify the hadrons, but its real importance lies in the organizational structure it provided. I think it's fair to say that the Eightfold Way initiated the modern era in particle physics.
> (see [8, p. 37])

Our main goal is to understand the geometric patterns that appear in the Eightfold Way from a mathematical point of view. To accomplish this we will apply representation theory to the quark model. More specifically, the representation theory of the quark symmetry group $SU(3)$ is important. The thesis is divided into two parts: a mathematical part and a physics part. In the mathematical part the main goal is to understand the representation theory of $SU(3)$, which is accomplished in Chapter 3. In Chapter 2 we will discuss the preliminaries needed for Chapter 3, such as Lie groups, Lie algebras and some basic representation theory. In Chapter 4 the application of the representation theory of $SU(3)$ to the quark model is given. Furthermore, an understanding of the geometric patterns occurring in the Eightfold Way is established in Chapter 4.

Finally, this thesis was written for the most part during the COVID-19 pandemic. This caused some practical issues here and there. Because of this, I would like to thank my supervisor prof. dr. Erik van den Ban, since he made sure that our weekly meetings could continue, even during the lockdown and bad internet connection. Moreover, I would like to thank him for his involvement during the process and the time to answer my numerous questions. Furthermore, I would like to thank prof. dr. Eric Laenen for answering my questions concerning the quark model and providing me with useful literature.

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[1] We will elaborate on this topic in Section 4.1.

[2] This quote talks about some concepts that are not yet defined. This will be done in Chapter 4.

2 Lie groups and Lie algebras

In this chapter we introduce the objects that are central in this thesis, such as Lie groups and a Lie algebras. These objects play an important role in many branches of mathematics and physics. After a general discussion we will give some examples of Lie groups and Lie algebras such as $SU(3)$ and $\mathfrak{sl}(3,\mathbb{C})$, which will be important throughout the thesis. Furthermore we will discuss elementary representation theory and how this applies to Lie groups and Lie algebras. We assume that the reader is familiar with group theory and the basics of differential geometry such as the notion of a manifold and tangent space.

2.1 Lie groups

We start by introducing the notion of a Lie group.

Definition 2.1. *A **Lie group** is a smooth manifold G equipped with a group structure so that the group multiplication $\mu : G \times G \to G$, $(x,y) \mapsto xy$ and the inversion $\iota : G \to G$, $x \mapsto x^{-1}$ are smooth maps.*

A very basic example of a Lie group would be $\mathbb{R}^n$ with ordinary addition and 0 as neutral element. Another example of a Lie group, which is important for our purposes, is $SU(3)$: the set of unitary 3×3 matrices with determinant 1. The reason Lie groups are such powerful tools is that the theory of both groups and manifolds can be applied. For Lie groups to be compared we need the right notion of structure preserving mappings, which we define in the following definition.

Definition 2.2. *Let G and H be Lie groups.*

(a) *A **Lie group homomorphism** from G to H is a smooth map $\phi : G \to H$ that is a group homomorphism, so $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$.*

(b) *A **Lie group isomorphism** from G to H is a bijective Lie group homomorphism $\phi : G \to H$ for which the inverse is also a Lie group homomorphism.*

As said above, $SU(3)$ is an important example for us. Further on in this thesis we will intensively study $SU(3)$. Therefore we will give some properties of this Lie group.

Lemma 2.3. *The Lie group $SU(3)$ is connected, compact and simply connected.*

Proof. We start by showing $SU(3)$ is connected. From linear algebra it is known that every $X \in SU(3)$ can be written as $X = U\Lambda U^{-1}$ with $U \in U(3)$ and Λ of the form

$$\Lambda := \begin{pmatrix} e^{i\phi_1} & 0 & \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix},$$

with $\sum_{i=1}^{3}\phi_i = 0$. Now define the map $\gamma : [0,1] \to SU(3)$ by

$$\gamma(t) = U \begin{pmatrix} e^{i\phi_1 t} & 0 & \\ 0 & e^{i\phi_2 t} & 0 \\ 0 & 0 & e^{i\phi_3 t} \end{pmatrix} U^{-1}.$$

Note that γ is a path in $SU(3)$ with endpoints $\gamma(0) = \mathbb{1}$ and $\gamma(1) = X$. This shows that for every element of $SU(3)$ there exists a path with identity, hence $SU(3)$ is connected.

Now we show that $SU(3)$ is compact. Note that $SU(3) = \{X \in U(3) \mid \det(X) = 1\}$. Since the determinant is a continuous function it follows that $SU(3)$ is a closed subset of $U(3)$. We claim that $U(3)$ is compact. Since a closed subset of a compact set is compact we conclude that $SU(3)$ is compact. We will now show that $U(3)$ is compact. Since $\mathrm{M}(3,\mathbb{C})$ is isomorphic to $\mathbb{C}^9$, it is enough to show that $U(3)$ is closed and bounded. Note that, for every $X \in U(3)$ we have $UU^* = \mathbb{1}$. Hence $(UU^*)_{ii} = |u_{i1}|^2 + |u_{i2}|^2 + |u_{i3}|^2 = 1$ for every $i = 1,2,3$. This shows that $|u_{ij}| \leq 1$ for every i,j. Hence $U(3)$ is bounded. To prove that $U(3)$ is closed we define a function $f : \mathrm{M}(3,\mathbb{C}) \to \mathrm{M}(3,\mathbb{C})$ by $f(X) = XX^*$. Note that f is a continuous function and that $U(3) = f^{-1}(\{\mathbb{1}\})$, where $\{\mathbb{1} \subset \mathrm{M}(3,\mathbb{C})\}$. Furthermore, note that $\{\mathbb{1}\} \subset \mathrm{M}(3,\mathbb{C})$ is closed. Hence $U(3)$ is closed and bounded, thus compact.

For the simply connectedness we refer to [9, Prop. 13.11]. ■

Actually, the above theorem generalizes to $SU(n)$ for arbitrary $n \geq 1$.

2.2 Lie algebras

We now move on to the concept of a Lie algebra. We start by giving the abstract definition and then we proceed to the Lie algebra of a Lie group in the next section.

Definition 2.4. *A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ endowed with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following conditions: For all $X, Y, Z \in \mathfrak{g}$ we have*

(a) $[X,Y] = -[Y,X]$;

(b) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

In this thesis we will always work with the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Often the operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the (Lie) bracket. Due to condition (a) in Definition 2.4 we say that the Lie bracket is skew-symmetric or anti-symmetric. Furthermore, condition (b) is known as the Jacobi identity. A classic example of a Lie algebra is the space $M(n,\mathbb{R})$ with the commutator as bracket, thus $[X,Y] = XY - YX$.
Also here we want to specify what are the structure-preserving maps, which is done in the following definition.

Definition 2.5. *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. Then a map $\phi : \mathfrak{g} \to \mathfrak{h}$ is said to be a **Lie algebra homomorphism** if it is linear and $\phi([X,Y]_{\mathfrak{g}}) = [\phi(X),\phi(Y)]_{\mathfrak{h}}$ for all $X, Y \in \mathfrak{g}$, where $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{h}}$ denote the Lie brackets of $\mathfrak{g}$ and $\mathfrak{h}$ respectively.*

Note that Definition 2.5 tells us that a Lie algebra homomorphism preserves the Lie bracket. Using the Lie bracket we can define a linear map by fixing the first entry, which will give rise to an example of a Lie algebra homomorphism. This is done in the following definition.

Definition 2.6. *Let $\mathfrak{g}$ be a Lie algebra. For $X \in \mathfrak{g}$ we define the* ***adjoint map*** $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ *by* $\mathrm{ad}(X)Y := [X, Y]$*. The map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is called the* ***adjoint representation*** *of* $\mathfrak{g}$ *in* $\mathfrak{g}$*.*

Lemma 2.7. *The map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is a Lie algebra homomorphism, where the Lie bracket on* $\mathrm{End}(\mathfrak{g})$ *is the commutator bracket.*

Proof. First we note that $\mathrm{ad}(X)$ is linear for every $X \in \mathfrak{g}$, since the Lie bracket is bilinear. Therefore the claim in Definition 2.6 that ad maps to $\mathrm{End}(\mathfrak{g})$ is justified. Moreover, the bilinearity of the Lie bracket implies that the map ad is linear. Now let $X, Y, Z \in \mathfrak{g}$. Then using the anti-symmetry and Jacobi identity of the Lie bracket we see

$$\begin{aligned}\mathrm{ad}([X,Y])(Z) =&[[X,Y],Z]\\ =&-[[Y,Z],X]-[[Z,X],Y]\\ =&[X,[Y,Z]]-[Y,[X,Z]]\\ =&\mathrm{ad}(X)\circ\mathrm{ad}(Y)(Z)-\mathrm{ad}(Y)\circ\mathrm{ad}(X)(Z)\\ =&(\mathrm{ad}(X)\circ\mathrm{ad}(Y)-\mathrm{ad}(Y)\circ\mathrm{ad}(X))(Z).\end{aligned}$$

The latter expression is precisely the commutator bracket on $\mathrm{End}(\mathfrak{g})$ acting on Z. This shows that ad is a Lie algebra homomorphism. ■

The reason why ad is called the adjoint representation will become clear in Section 2.4 and Lemma 2.7 will play a role in that.

Now we will discuss special types of Lie algebras, namely *simple* and *semisimple* Lie algebras. To define these objects we first need the notion of an ideal. An ideal of a Lie algebra is quite similar to an ideal in ring theory, as can be seen in the following definition.

Definition 2.8. *Let $\mathfrak{g}$ be a Lie algebra. Then we say $\mathfrak{h} \subset \mathfrak{g}$ is an* ***ideal*** *(notation $\mathfrak{h} \lhd \mathfrak{g}$) if $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.*

Definition 2.9. *A* ***simple*** *Lie algebra $\mathfrak{g}$ is a nonabelian Lie algebra such that the only ideals are $\{0\}$ and $\mathfrak{g}$ itself.*

Definition 2.10. *A Lie algebra $\mathfrak{g}$ is said to be* ***semisimple*** *if it is the direct sum of simple Lie algebras.*

The reason to look at semisimple Lie algebras is that they have a very elegant classification. This classification is made using the so called *root system* of the Lie algebra. We will not go into great detail about root systems in this thesis. For more background about root systems we refer to [12, Ch. 9]. We will limit ourself to very specific Lie algebras which will turn out to be semisimple, namely the Lie algebras $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$[1].

[1]See [9, Ex. 7.3] for a proof.

2.3 The Lie algebra of a Lie group

In this section we assume that G is a Lie group and we follow [3, Ch. 4]. Before we start our discussion on the correspondence between Lie groups and Lie algebras we need to define the exponential map. To obtain this definition we need the notion of a left invariant vector field. Since G is a Lie group we can define the left and right translation maps for $x \in G$:

$$l_x : G \to G,\ y \mapsto xy, \quad r_x : G \to G,\ y \mapsto yx. \tag{2.1}$$

Note that, since G is a Lie group, l_x and r_x are diffeomorphisms from G onto itself.

Definition 2.11. *Let M be a manifold and $V \in \mathfrak{X}(M)$ a smooth vector field. We say V is **left invariant** if*

$$V(xy) = T_y(l_x)V(y), \quad (x, y \in G). \tag{2.2}$$

The collection of all left invariant vector fields, denoted by $\mathfrak{X}(M)_L$, is a linear subspace of $\mathfrak{X}(M)$. For $y = e$, Equation (2.2) implies the vector field $V \in \mathfrak{X}(M)_L$ is completely defined by its value $V(e) \in T_eG$ at e. Hence, the map $\epsilon : \mathfrak{X}(M)_L \to T_eG$ given by $V \mapsto V(e)$ is an injective linear map. It turns out this map is surjective as well, as we will see. Conversely, for $X \in T_eG$ we can define a vector field $V \in \mathfrak{X}(M)_L$ by

$$V_X(x) = T_e(l_x)X, \quad (x \in G). \tag{2.3}$$

Lemma 2.12. *The map $X \mapsto V_X$ from T_eG to $\mathfrak{X}(M)_L$ is a linear isomorphism, with inverse the map ϵ from above.*

Proof. See [3, Lemma 3.1]. ■

Now, for $X \in T_eG$ we define α_X to be the maximal integral curve of the corresponding left invariant vector field v_X starting at e. It is known that α_X has domain $\mathbb{R}$, for a proof of this we refer to [3, Lemma 3.2]. Now we can define the exponential map.

Definition 2.13. *Let G be a Lie group. The **exponential map** $\exp = \exp_{\mathrm{G}} : T_eG \to G$ is defined by*

$$\exp(\mathrm{X}) = \alpha_X(1), \quad (X \in T_eG)$$

Remark 2.14. In the case of $\mathrm{GL}(V)$, with V a finite dimensional vector space the exponential map coincides with the ordinary exponential map for endomorphisms $X \mapsto e^X$, $\mathrm{End}(V) \to \mathrm{GL}(V)$.

Proof. We refer to [3, Ex. 3.4]. ■

The exponential map has some very nice properties, some of which we would expect since the name is quite suggestive. These properties are captured in the following lemma.

Lemma 2.15. *For all $s, t \in \mathbb{R}$, $X \in T_eG$ we have*

(a) $\exp(sX) = \alpha_X(s)$.

(b) $\exp((s + t)X) = \exp(sX)\exp(tX)$.

(c) *The tangent map of* $\exp$ *in the origin is given by* $T_0\exp = \mathbb{1}_{T_eG}$.

(d) *The map* $\exp : T_eG \to G$ *is smooth and a local diffeomorphism at 0.*

Proof. We omit the proof here and refer to Lemma 3.6 in [3]. ■

The following lemma will be fundamental in our discussion this section and we will use it a few times throughout this thesis.

Lemma 2.16. *Let* $\phi : G \to H$ *be a homomorphism of Lie groups. Then the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ T_eG \arrow[u, "\exp_G"] \arrow[r, "T_e\phi"] & T_eH \arrow[u, "\exp_H"'] \end{tikzcd}$$

Proof. We omit the proof here and refer to [3, Lemma 3.9]. ■

Recall the left and right translation maps:

$$l_x : G \to G,\ y \mapsto xy, \quad r_x : G \to G,\ y \mapsto yx. \tag{2.4}$$

Furthermore, recall that l_x and r_x are diffeomorphisms from G onto itself. Then the map $C_x := l_x \circ r_x : G \to G$, $y \mapsto xyx^{-1}$ is also a diffeomorphism. This means that for $y \in G$ its tangent map $T_yC_x : T_yG \to T_{C_x(y)}G$ is a linear isomorphism. Note that $C_x(e) = e$, therefore T_eC_x is a linear automorphism of T_eG, thus $T_eC_x \in \mathrm{GL}(T_eG)$. This leads to the following definition.

Definition 2.17. *Let* $x \in G$*. Then we define* $\mathrm{Ad}(x) \in \mathrm{GL}(T_eG)$ *by* $\mathrm{Ad}(x) := T_eC_x$*. The map* $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ *is called the* ***adjoint representation*** *of* G *in* T_eG*.*

The adjoint has some neat properties, such as the following lemmas.

Lemma 2.18. *Let* $x \in G$*. For every* $X \in T_eG$ *we have*

$$x \exp(X) x^{-1} = \exp\big(\mathrm{Ad}(x)X\big).$$

Proof. It is easily seen that the conjugation map $C_x : G \to G$ is a Lie group homomorphism. Hence we may apply 2.16. Thus we find the following commuting diagram:

$$\begin{tikzcd} G \arrow[r, "C_x"] & G \\ T_eG \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}(x)"] & T_eG \arrow[u, "\exp"'] \end{tikzcd}$$

Which gives us the desired result. ■

Lemma 2.19. *The map* $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ *is a Lie group homomorphism.*

Proof. Since G is a Lie group the map $(x, y) \mapsto xyx^{-1}$ from $G \times G$ to G is smooth. Differentiating with respect to y at $y = 0$, it follows that $x \mapsto \mathrm{Ad}(x)$ is smooth from T_eG to $\mathrm{End}(T_eG)$. Since $\mathrm{GL}(T_eG)$ is an open subset of $\mathrm{End}(T_eG)$, which will be shown after the proof, it follows that $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ is smooth.

We note that $C_{xy} = C_x \circ C_y$ and $C_x(e) = e$ for all $x, y \in G$. By differentiating the former equation at e on both sides and applying the chain rule, we see $\mathrm{Ad}(xy) = \mathrm{Ad}(x) \circ \mathrm{Ad}(y)$. Furthermore, we know $C_e = \mathbb{1}_G$ and thus $\mathrm{Ad}(e) = \mathbb{1}_{T_eG}$ ■

After a choice of basis we can identify elements of $\mathrm{GL}(T_eG)$ and $\mathrm{End}(T_eG)$ with real matrices. From this, the fact the determinant is continuous, $\mathbb{R}\setminus\{0\}$ is open in $\mathbb{R}$ and $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$ if follows that $\mathrm{GL}(T_eG)$ is an open subset of $\mathrm{End}(T_eG)$. Therefore $T_{\mathbb{1}}\mathrm{GL}(T_eG) = T_{\mathbb{1}}\mathrm{End}(T_eG) = \mathrm{End}(T_eG)$. We also note that $\mathrm{Ad}(e) = \mathbb{1}$, hence the tangent map of Ad at e is a linear map from T_eG to $\mathrm{End}(T_eG)$.

Definition 2.20. *We define the linear map* $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ *by* $\mathrm{ad} := T_e\mathrm{Ad}$. *With the chain rule we see, for* $X \in T_eG$

$$\mathrm{ad}(X) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tX)).$$

Here we used the same notation as in Definition 2.6. Later on in this section we will see that this is justified.

Lemma 2.21. *For all* $X \in T_eG$ *we have*

$$\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$$

Proof. From Lemma 2.19 we know that Ad is a Lie group homomorphism, hence we can apply Lemma 2.16 with $\phi = \mathrm{Ad}$ and $H = \mathrm{GL}(T_eG)$. Note that in this case $T_eH = T_{\mathbb{1}}\mathrm{GL}(T_eH) = \mathrm{End}(T_eG)$ and $\exp_H$ is given by $e^{\cdot} : \mathrm{End}(T_eG) \to \mathrm{GL}(T_eG)$ by Remark 2.14. Thus we have the following commuting diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(T_eG) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle e^{\cdot}} \\ T_eG & \xrightarrow{\mathrm{ad}} & \mathrm{End}(T_eG) \end{array}$$

This diagram yields $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$. ∎

The following example will give some motivation for why we used similar notation for ad in this section in comparison with the previous section.

Example 2.22. Let V be a finite dimensional vector space. Then we consider the Lie group $\mathrm{GL}(V)$. Note that for $x \in G$ the conjugation map $C_x : \mathrm{GL}(V) \to \mathrm{GL}(V)$ extends to a linear map $C_x : \mathrm{End}(V) \to \mathrm{End}(V)$, hence $\mathrm{Ad}(x) : \mathrm{End}(V) \to \mathrm{End}(V)$ is given by $\mathrm{Ad}(x) = T_eC_x = C_x$. Thus for $Y \in \mathrm{End}(V)$ we have $\mathrm{Ad}(x)Y = xYx^{-1}$. If we take $x = e^{tX}$ for some $X \in \mathrm{End}(V)$ and $t \in \mathbb{R}$, we see by differentiating at $t = 0$

$$\mathrm{ad}(X)Y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(e^{tX})Y = \frac{d}{dt}\bigg|_{t=0} \left(e^{tX}Ye^{-tX}\right) = XY - YX.$$

We thus see that in this case ad coincides with the commutator bracket.

The next definition is strongly motivated by our findings in the previous example. It also gives the justification for the similar notation in this section and the previous section.

Definition 2.23. *For* $X, Y \in T_eG$ *we define a map* $[\cdot,\cdot] : T_eG \times T_eG \to T_eG$ *by*

$$[X,Y] := \mathrm{ad}(X)Y. \tag{2.5}$$

We call this map the Lie bracket.

This name of the map in Definition 2.23 is very suggestive and we will show that it satisfies the conditions in Definition 2.4, thus making T_eG into a Lie algebra. We start by proving the bilinearity and anti-symmetry.

Lemma 2.24. *The map $[\cdot,\cdot] : T_eG \times T_eG \to T_eG$ is bilinear and anti-symmetric.*

Proof. Since $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ is the tangent map of Ad, it is linear. This gives linearity in the first component. The linearity in the second component follows from the fact that $\mathrm{ad}(X) \in \mathrm{End}(T_eG)$ for all $X \in T_eG$. Hence $[\cdot,\cdot]$ is bilinear. Now we will show the anti-symmetry. Let $t, s \in \mathbb{R}$ and $Z \in T_eG$, then by Lemma 2.15 and Lemma 2.18 we see

$$\exp(tZ) = \exp(sZ)\exp(tZ)\exp(-sZ) = \exp\big(\mathrm{Ad}(\exp(sZ))tZ\big). \tag{2.6}$$

Then by differentiating at $t = 0$ we find

$$Z = \mathrm{Ad}(\exp(sZ))Z.$$

Differentiating this expression another time at $s = 0$ we get

$$0 = \frac{d}{dt}\bigg|_{s=0} \mathrm{Ad}(\exp(sZ))Z = \mathrm{ad}(Z)Z.$$

Hence we showed that $[Z, Z] = 0$ for all $Z \in T_eG$. Now let $X, Y \in T_eG$ and set $Z = X+Y$, then by using the bilinearity of the bracket

$$0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] = [X, Y] + [Y, X].$$

Therefore $[X, Y] = -[Y, X]$, proving the anti-symmetry. ∎

To be able to prove the Jacobi identity for the Lie bracket of Definition 2.23 we need the following lemma.

Lemma 2.25. *Let $\phi : G \to H$ be a Lie group homomorphism. Then for all $X, Y \in T_eG$ we have*

$$T_e\phi([X, Y]_G) = [T_e\phi(X), T_e\phi(Y)]_H$$

Proof. We refer to Lemma 4.10 in [3]. ∎

The above lemma is crucial for showing that the tangent map of a Lie group homomorphism is a Lie algebra homomorphism, as we will see later on. We now return to showing the Jacobi identity for the Lie bracket of Definition 2.23.

Lemma 2.26. *For all $X, Y, Z \in T_eG$ we have*

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

Proof. Let $X, Y, Z \in T_eG$. Note that $\phi := \mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ is a Lie group homomorphism. Furthermore, by Example 2.22 we know that $[A, B]_{\mathrm{GL}(T_eG)} = AB - BA$ for all $A, B \in \mathrm{End}(T_eG)$. Then by Lemma 2.25 we find

$$T_e\phi([X, Y]) = T_e\phi X \circ T_e\phi Y - T_e\phi Y \circ T_e\phi X.$$

Note that $T_e\phi = \mathrm{ad}$, thus we have

$$\mathrm{ad}([X,Y]) = \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X).$$

Applying both sides of the above equation to Z gives us the desired conclusion. ■

The conclusion of Lemma 2.26 is actually equivalent to the Jacobi identity. This can easily be seen by grouping the terms on one side of the equal sign and using the anti-symmetry of the bracket multiple times. We leave this as a small exercise for the reader.

Now we have shown that T_eG applied with the bracket of Definition 2.23 is a Lie algebra in the sense of Definition 2.4. This implies that the tangent map of a Lie group homomorphism $\phi : G \to H$, namely $T_e\phi : T_eG \to T_eH$ is a map between Lie algebras. Hence, by Lemma 2.25 $T_e\phi$ is a Lie algebra homomorphism. From now on we will denote the Lie algebra T_eG with the Gothic letter $\mathfrak{g}$, in parallel with Section 2.2.

2.4 Basic representation theory

We have now defined the concepts of Lie group and Lie algebras and we have given a connection between the two. The next idea we want to discuss is the notion of a representation for a Lie group and a Lie algebra. In this section we will restrict ourselves to **finite dimensional** representations.

2.4.1 Definitions and examples

Definition 2.27. *Let G be a Lie group and let V be a finite dimensional vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A* ***representation*** *of G is a pair (π, V), where V is finite dimensional vector space over the field $\mathbb{K}$ and $\pi : G \to GL(V)$ is a Lie group homomorphism. If the field is not mentioned, it is assumed to be $\mathbb{K} = \mathbb{C}$.*

Often, if (π, V) is a representation of G, it is said that π is a representation of G *in* V or that V is a G-module[2].

Example 2.28. The representation $(\pi, \mathbb{C}^3)$ of $SU(3)$ given by $\pi(A)v = Av$, where $A \in SU(3)$ acts on $v \in \mathbb{C}^3$ as a matrix. We readily see that this is a representation in the sense of the above definition. We call this representation the **standard representation**.

Example 2.29. Another example is the so called **trivial representation**. Which is given, for an arbitrary Lie group G, by the pair $(\pi, \mathbb{C})$, where $\pi : G \to \mathrm{GL}(\mathbb{C})$, $x \mapsto \mathbb{1}_{\mathbb{C}}$.

Definition 2.30. *Let $\mathfrak{g}$ be a Lie algebra and let V be a vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A representation of $\mathfrak{g}$ is a pair (π, V), where V is a finite dimensional vector space of a field $\mathbb{K}$ and $\pi : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism, where* $\mathrm{End}(V)$ *is equipped with the commutator bracket. Again, if the field is not mentioned it is assumed to be $\mathbb{C}$.*

[2]The reason V is called a G-module has to do with the group algebra. For more background see [6, Sec. 3.4]

An example of this is the adjoint representation ad. By Lemma 2.7 we know that ad is a Lie algebra homomorphism. Since $\mathfrak{g}$ is a linear space we conclude that $(\mathrm{ad}, \mathfrak{g})$ is a representation, hence the name. Another example of a representation of a Lie algebra $\mathfrak{g}$, which is parallel to the one in Example 2.29, is the pair $(\pi, \mathbb{C})$ such that $\pi(X) = 0 \in \mathrm{End}(\mathbb{C})$ for all $X \in \mathfrak{g}$. This representation is also called the trivial representation.

Of course we can have multiple representations of the same Lie group. We would like to be able to compare those representations. For example, when would we want to say two representations are essentially 'the same'. This is captured in the following definition.

Definition 2.31. *Suppose that (π_1, V_1), (π_2, V_2) are finite dimensional representations of a Lie group G. A linear map $T : V_1 \to V_2$ is said to be* ***equivariant****, or* ***intertwining*** *if for all $x \in G$ the following diagram commutes:*

$$\begin{array}{ccc} V_1 & \xrightarrow{T} & V_2 \\ {\scriptstyle \pi_1(x)}\big\uparrow & & \big\uparrow{\scriptstyle \pi_2(x)} \\ V_1 & \xrightarrow{T} & V_2 \end{array}$$

We say (π_1, V_1) and (π_2, V_2) are ***equivalent*** *if there exists a linear isomorphism $T : V_1 \to V_2$, which is equivariant. One also could say π_1 and π_2 are equivalent or V_1 and V_2 are isomorphic.*

For a Lie algebra $\mathfrak{g}$ we have a definition analogous to Definition 2.31, where we replace $x \in G$ by $X \in \mathfrak{g}$.

Definition 2.32. *Let (π, V) be a representation of G. We say a linear subspace $W \subset V$ is* ***invariant*** *if $\pi(x)W \subset W$ for all $x \in G$. The representation is said to be* ***irreducible*** *if the only invariant subspaces of V are $\{0\}$ and V itself.*

Note that the trivial representation for both Lie groups and Lie algebras is irreducible, since $\mathbb{C}$ has no non-trivial subspaces. An elementary result, yet truly useful, about irreducible representations is Schur's lemma.

Lemma 2.33 (Schur's Lemma). *Let (π, V) and (ρ, W) be irreducible representations over $\mathbb{C}$ of a Lie group G or Lie algebra $\mathfrak{g}$ and let $T : V \to W$ be an intertwining map, then*

(a) *Either T is an isomorphism, or $T = 0$.*

(b) *If $V = W$ and $\pi = \rho$, then $T = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.*

Proof. We start by proving the first claim. To do this we first show that $\ker(T) \subset V$ and $\mathrm{im}(T) \subset W$ are invariant under the actions of π and ρ, respectively. Let $x \in G$ and $v \in \ker(T)$. Then, since T intertwines the representations π and ρ we have

$$T(\pi(x)v) = \rho(x)(Tv) = \rho(x)0 = 0.$$

Hence $\pi(x)v \in \ker(T)$, since this holds for all $x \in G$ we conclude that $\ker(T)$ is an invariant subspace of V. Now let $x \in G$ and $w \in \mathrm{im}(T)$, then there exists a $v \in V$ such that $w = Tv$. We see

$$\rho(x)w = \rho(x)(Tv) = T(\pi(x)v).$$

We note that $\pi(x)v \in V$, thus $T(\pi(x)v) \in \text{im}(T)$. Which shows that $\text{im}(T)$ is an invariant subspace of W. Since (π, V) is an irreducible representation, we know that $\ker(T) = 0$ or $\ker(T) = V$. If $\ker(T) = V$, then $T = 0$. If $\ker(T) = 0$ it means that $\text{im}(T) \neq 0$. Yet, since (ρ, W) is also an irreducible representation, we conclude that $\text{im}(T) = W$. Showing that T is an isomorphism.

Now suppose that $V = W$ and $\pi = \rho$. We know that T has a complex eigenvalue λ, since $\mathbb{C}$ is algebraically closed. Associated with this eigenvalue, there exists an eigenvector $v \in V$. This means that $(T - \lambda\mathbb{1})v = 0$, hence $\ker(T - \lambda\mathbb{1})$ is not trivial. We readily see that, since $T : V \to V$ is an intertwining map, $T - \lambda\mathbb{1} : V \to V$ is an intertwining map as well. Since its kernel is not trivial it follows from our discussion above that $T - \lambda\mathbb{1} = 0$, which shows our claim. The proof for a Lie algebra $\mathfrak{g}$ is completely analogous as above, but with the replacement of $x \in G$ by $X \in \mathfrak{g}$. ∎

2.4.2 Complete reducibility

For certain Lie groups, irreducible representations can be viewed as the 'atoms' of their representations, since every representation decomposes into irreducibles. This property is captured in the following definition.

Definition 2.34. *Let G be a Lie group and let (π, V) be a representation of G. Then (π, V) is said to be* ***completely reducible*** *if there exists a direct sum decomposition*

$$V = \bigoplus_{i=1}^{n} V_i,$$

where V_i is an invariant subspace for all $1 \leq i \leq n$ such that the representation $(\pi|_{V_i}, V_i)$ is irreducible.

It turns out, not every representation is completely reducible. Actually, it is quite a specific property of a representation. Yet, in this chapter we will see that specific properties of the Lie group implies complete reducibility for its representations. Note that, if every representation of a Lie group G is completely reducible, the study of its representations reduces to the irreducible ones. Which drastically simplifies the discussion. In this thesis we focus on the Lie group $SU(3)$ and we will see that this Lie group has the desirable properties for its representations to be completely reducible.

Definition 2.35. *We say a representation (π, V) of a Lie group G is* ***unitarizable*** *when there exists a Hermitian inner product on V for which $\pi(x) \in \text{GL}(V)$ is unitary for every $x \in G$.*

Furthermore, we say a representation (π, V) is **unitary** if for every $x \in G$ the operator $\pi(x)$ is unitary. There is a useful characterization for a representation to be unitary. To give this characterization we need the following lemma.

Lemma 2.36. *Let G be a connected Lie group. Then the subgroup G_e generated by $\exp(X)$, for $X \in \mathfrak{g}$, equals G. Actually, the converse is also true.*

Proof. We refer to [3, Lemma 5.8]. ∎

For the following lemma, we follow the proof of [3, Lemma 29.2].

Lemma 2.37. *Let G be a connected compact Lie group and (π, V) be a representation of G. Then, π is unitary if and only if*

$$\pi_*(X)^* = -\pi_*(X) \tag{2.7}$$

for $X \in \mathfrak{g}$. In the above equation π_* denotes the tangent map $T_e\pi : \mathfrak{g} \to \mathrm{End}(V)$.

Proof. Since $\pi : G \to \mathrm{GL}(V)$ is a Lie group homomorphism, Lemma 2.16 tells us

$$\pi(\exp(tX)) = e^{t\pi_*(X)} \tag{2.8}$$

for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. If π is unitary, we see $\pi(\exp(tX))^* = \pi(\exp(-tX))$. Hence, for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$

$$e^{t\pi_*(X)^*} = \pi(\exp(-tX)) = e^{-t\pi_*(X)}. \tag{2.9}$$

Differentiating this equation at $t = 0$, yields

$$\pi_*(X)^* = -\pi(X).$$

Now assume Equation (2.7) holds. Then we see Equation (2.9) holds for every $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Hence, $\pi(x)$ is unitary for every $x \in \exp(\mathfrak{g})$ and thus for every $x \in G_e$. By Lemma 2.36 it follows $G_e = G$, which completes the proof. ■

For unitarizable representations we will see that they are completely reducible. To prove that we need the following lemma.

Lemma 2.38. *Let (π, V) be a unitarizable representation of a Lie group G. If U is an invariant subspace of V, then $U^\perp$ is also an invariant subspace and we have $V = U \oplus U^\perp$.*

Proof. Let $U \subset V$ be an invariant subspace and let us denote the Hermitian inner product on V, for which $\pi(x)$ is unitary for every $x \in G$, by $\langle \cdot, \cdot \rangle$. Since V is finite dimensional we know $V = U \oplus U^\perp$ from linear algebra. Now let $x \in G$, $v \in U$ and $w \in U^\perp$. Then, using the fact that π is a unitarizable representation, we find

$$\begin{aligned}
\langle v, \pi(x)w \rangle =& \langle \pi(x^{-1})v, \pi(x^{-1})\pi(x)w \rangle \\
=& \langle \pi(x^{-1})v, w \rangle \\
=& 0,
\end{aligned}$$

because U is an invariant subspace, meaning that $\pi(x^{-1})v \in U$, and $w \in U^\perp$. Since $v \in U$ was arbitrary we conclude that $\pi(x)w \in U^\perp$. This argument holds for every $x \in G$, which implies that $U^\perp$ is an invariant subspace. ■

Corollary 2.39. *Let (π, V) be a unitarizable representation of a Lie group G. Then (π, V) is completely reducible.*

Proof. We consider a Hermitian inner product for which $\pi(x)$ is unitary for all $x \in G$ and repeatedly apply Lemma 2.38. ■

Now we state the main theorem of this section.

Theorem 2.40. *Let G be a compact Lie group. Then every representation (π, V) is completely reducible.*

Corollary 2.41. *Let (π, V) be a representation of a compact Lie group G. Then every invariant subspace is completely reducible.*

Proof. Let $U \subset V$ be an invariant subspace. Then apply Theorem 2.40 to the representation $(\pi|_U, U)$. ■

Since $SU(3)$ is a compact Lie group, this theorem shows that $SU(3)$ is completely reducible. To prove Theorem 2.40 we need a result about compact Lie groups and unitarizable representations. To prove this result we need some object from measure theory, namely the *left Haar measure.*

Definition 2.42. *Let G be a locally compact Lie group and μ be a nonzero Borel measure*[3] *on G. We say μ is a* ***left Haar measure*** *if it has the following properties:*

(a) *The measure μ is invariant under left translations: $\mu(xA) = \mu(A)$ for every $x \in G$ and Borel set $A \subset G$.*

(b) *The measure μ if finite on compact subsets of G: $\mu(K) < \infty$ for all compact $K \subset G$.*

(c) *the measure μ is* inner regular *on open subsets $U \subset G$: $\mu(U) = \sup\{\mu(K) \mid K \subset U \text{ compact}\}$.*

(d) *the measure μ is* outer regular *on borel sets $A \subset G$: $\mu(A) = \inf\{\mu(U) \mid A \subset U,\ U \text{ open}\}$.*

It can be proven that there exists a left Haar measure on every locally compact Lie group G. This left Haar measure is unique up to a real positive factor. For a proof of these statements we refer to [5, Thm. 9.2.2, 9.2.6]. Furthermore, it can be shown μ is finite if and only if G is compact (see [5, Prop. 9.3.3]). Hence, for each compact Lie group G there exists a unique left Haar measure, denoted by dx, such that

$$\int_G dx = 1.$$

Another property for the (normalized) left Haar measure of a compact Lie group G is that it is bi-invariant (see [3, Remark 19.15]). As a consequence of this bi-invariance, one can show

$$\int_G f(yx)dx = \int_G f(x)dx, \qquad \int_G f(xy)dx = \int_G f(x)dx.$$

Where the first equation is a consequence of the left-invariance and the second of the right-invariance.

Now we have enough background to prove Theorem 2.40 using the following lemma.

Lemma 2.43. *Let G be a compact Lie group and (π, V) a representation of G. Then the representation (π, V) is unitarizable.*

[3] A Borel measure is any measure defined on the σ-algebra of Borel sets, see [5, Ch.1] for more background

Proof. Let $\langle\cdot,\cdot\rangle_1$ be any Hermitian inner product on V. Then we define a new Hermitian inner product by 'averaging' over G: for $v, w \in V$ we set

$$\langle v, w\rangle := \int_G \langle \pi(x)v, \pi(x)w\rangle dx.$$

Note that $f : x \mapsto \langle \pi(x)v, \pi(x)w\rangle$ is a continuous function, hence the integral is well-defined. It can easily be shown that this indeed defines another Hermitian inner product on V. Now let $y \in G$, then we see

$$\begin{aligned}\langle \pi(y)v, \pi(y)w\rangle &= \int_G \langle \pi(x)\pi(y)v, \pi(x)\pi(y)w\rangle dx\\ &= \int_G \langle \pi(xy)v, \pi(xy)w\rangle dx\\ &= \int_G f(xy)dx\\ &= \int_G f(x)dx\\ &= \int_G \langle \pi(x)v, \pi(x)w\rangle dx\\ &=\langle v, w\rangle,\end{aligned}$$

using the right invariance of the Haar measure. This shows $\pi(y)$ is unitary with respect to this new Hermitian inner product. Since this argument holds for all $y \in G$ we conclude that (π, V) is unitarizable. ■

Now Theorem 2.40 follows immediately from Lemma 2.43 and Corollary 2.39.

2.4.3 Connection between Lie group representations and Lie algebra representations

Let (π, V) be a representation of a Lie group G. Then by definition $\pi : G \to \mathrm{GL}(V)$ is a Lie group homomorphism. The from Lemma 2.25 we know that $\pi_* := T_e\pi : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism. In view of Definition 2.30, we conclude that (π_*, V) is a representation of $\mathfrak{g}$. Using the chain rule we find for $X \in \mathfrak{g}$ and $v \in V$

$$\pi_*(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tX))v. \tag{2.10}$$

Furthermore, from Lemma 2.16 we see

$$\pi(\exp(X)) = e^{\pi_*(X)}. \tag{2.11}$$

An interesting question one could ask at this point is: does (π_*, V) inherit properties from (π, V). Especially, a property like irreducibility. To get an answer we will use Lemma 2.36.

Theorem 2.44. *Let G be a connected Lie group and let (π, V), (ρ, W) be representations of G. Then the following assertions are valid*

(a) *A linear subspace $U \subset V$ is invariant under G if and only if it is invariant under $\mathfrak{g}$.*

(b) *(π, V) is an irreducible representation of G if and only if (π_*, V) is an irreducible represenation of $\mathfrak{g}$.*

(c) *Let $T : V \to W$ be a linear map. Then T is G-equivariant if and only if it is $\mathfrak{g}$-equivariant.*

(d) *(π, V), (ρ, W) are isomorphic as representations of G if and only if they are isomorphic as representations of $\mathfrak{g}$.*

Proof. Assume that $U \subset V$ is invariant under π. Let $X \in \mathfrak{g}$ and $v \in U$. Then, because U is invariant under π, we have $\pi(\exp(tX))v \in U$ for all t. Since U is a linear subspace, it follows by differentiating at $t = 0$ from Equation (2.10) that $\pi_*(X)v \in U$.

Now assume that $U \subset V$ is invariant under π_*. Let $X \in \mathfrak{g}$ and $v \in U$. By Equation (2.11) we see $\pi(\exp(X))v = e^{\pi_*(X)}v$. Since U is invariant under π_* we have $\pi_*(X)v \in U$, for all $X \in \mathfrak{g}$. Then by the power series of the exponential map for endomorphisms it follows that $e^{\pi_*(X)}v \in U$. Hence $\pi(\exp(X))v \in U$, for all $X \in \mathfrak{g}$. Therefore U is invariant under G_e. Since G is connected it follows from Lemma 2.36 that $G = G_e$. Assertion (a) follows.

Assertion (b) is a direct consequence of (a).

Assume that T is G-equivariant. Then for all $x \in G$ we have $T \circ \pi(x) = \rho(x) \circ T$. In particular, $T \circ \pi(\exp(tX)) = \rho(\exp(tX)) \circ T$ holds for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Then by straightforward differentiating at $t = 0$ and Equation (2.10) we get $T \circ \pi_*(X) = \rho_*(X) \circ T$. Hence T is $\mathfrak{g}$-equivariant.

Now assume T is $\mathfrak{g}$-equivariant, then $T \circ \pi_*(X) = \rho_*(X) \circ T$ for all $X \in \mathfrak{g}$. Using the last equation repeatedly we see

$$T \circ \pi_*(X)^n = \rho_*(X)^n \circ T$$

for all $n \in \mathbb{N}$. Using the power series of the exponential map of Remark 2.14 it follows that $T \circ e^{\pi_*(X)} = e^{\rho_*(X)} \circ T$. Using Equation (2.11) the last equation becomes $T \circ \pi(\exp(X)) = \rho(\exp(X)) \circ T$. Hence T is G_e-equivariant. Again, because G is connected it follows from Lemma 2.36 that $G_e = G$. The result follows.

Assertion (d) follows directly from assertion (c). ∎

Another useful tool to find representations of a Lie algebra $\mathfrak{g}$ is the notion of its complexification $\mathfrak{g}_{\mathbb{C}}$. It turns out that determining representations of the complexification of a Lie algebra is usually easier, since $\mathbb{C}$ is algebraically complete.

Definition 2.45. *Let W be a real vector space. The* ***complexification***[4] *of W is defined by*

$$W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C},$$

as real linear space. $W_{\mathbb{C}}$ is equipped with a complex scalar multiplication defined by $\lambda(v \otimes z) = v \otimes \lambda z$.

Note that we can embed W as a real linear subspace of $W_{\mathbb{C}}$ by the map $v \mapsto v \otimes 1$. Furthermore, every vector $v \in W_{\mathbb{C}}$ can be uniquely written as $v = v_1 \otimes 1 + v_2 \otimes i$, by the

[4]For some background about the tensor product see [1].

nature of the tensor product. Therefore we see $W_{\mathbb{C}} \cong W \oplus iW$, with the usual complex multiplication. From now on we will always make this identification. If $\mathfrak{g}$ is a Lie algebra, then by a complex bilinear extension of the Lie bracket, its complexification $\mathfrak{g}_{\mathbb{C}}$ is made into a Lie algebra.

If ρ is a representation of $\mathfrak{g}$ in a complex linear space V, there is a unique extension of ρ to a representation of $\mathfrak{g}_{\mathbb{C}}$ in V. We denote this representation by $\rho_{\mathbb{C}}$ and it is given by

$$\rho_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(V), \quad \rho_{\mathbb{C}}(X + iY) = \rho(X) + i\rho(Y), \tag{2.12}$$

for $X, Y \in \mathfrak{g}$. Also here we are interested in which properties are preserved after complexifying. This captured in the following theorem.

Theorem 2.46. *Let $\mathfrak{g}$ be a Lie algebra and let (ρ, V), (ρ', V') be representations of $\mathfrak{g}$ over $\mathbb{C}$. Then*

(a) *A linear subspace $U \subset V$ is invariant under $\mathfrak{g}$ if and only if it is invariant under $\mathfrak{g}_{\mathbb{C}}$.*

(b) *(ρ, V) is an irreducible representation of $\mathfrak{g}$ if and only if $(\rho_{\mathbb{C}}, V)$ is an irreducible represenation of $\mathfrak{g}_{\mathbb{C}}$.*

(c) *Let $T : V \to V'$ be a (complex) linear map. Then T is $\mathfrak{g}$-equivariant if and only if it is $\mathfrak{g}_{\mathbb{C}}$-equivariant.*

(d) *(ρ, V), (ρ', V') are isomorphic as representations of $\mathfrak{g}$ if and only if (ρ, V),(ρ', V') are isomorphic as representations of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. We start by showing assertion (a). Let $U \subset V$ be invariant under ρ. This means $\rho(X) \in U$ for all $X \in \mathfrak{g}$. Since U is a (complex) subset of V we have $\rho_{\mathbb{C}}(X + iY) = \rho(X) + i\rho(Y) \in U$, because $\rho(X), i\rho(Y) \in U$ for all $X, Y \in U$. Since every element $Z \in \mathfrak{g}_{\mathbb{C}}$ can be uniquely written as $Z = X + iY$, for $X, Y \in \mathfrak{g}$, we conclude that U is invariant under $\rho_{\mathbb{C}}$.

Now suppose U is invariant under $\rho_{\mathbb{C}}$, then $\rho_{\mathbb{C}}(Z)W \subset W$ for all $Z \in \mathfrak{g}_{\mathbb{C}}$. Note $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}_{\mathbb{C}}$. Hence by restricting to $\mathfrak{g}$ we see that U is invariant under ρ.

Now we will show assertion (c). Suppose T is $\mathfrak{g}$-equivariant. Then by straightforward calculation we see, for $X, Y \in \mathfrak{g}$ and $v \in V$

$$\begin{aligned} T \circ \rho_{\mathbb{C}}(X + iY)v &= T(\rho(X)v + i\rho(Y)v) \\ &= T(\rho(X)v) + iT(\rho(Y)v) \\ &= \rho'(X)(Tv) + i\rho'(Y)(Tv) \\ &= \rho'_{\mathbb{C}}(X + iY) \circ Tv. \end{aligned}$$

Since every element of $\mathfrak{g}_{\mathbb{C}}$ is of the form $X + iY$ with $X, Y \in \mathfrak{g}$, the above calculation shows that T is $\mathfrak{g}_{\mathbb{C}}$-equivariant. The other implication is immediate after restricting to $\mathfrak{g}$ as we did before.

Assertions (b) and (d) are direct consequences of (a) and (c), respectively. ∎

We now know that every representation of a Lie algebra $\mathfrak{g}$ has a unique extension to a representation of $\mathfrak{g}_{\mathbb{C}}$. The other way around is also true; if we start with a representation of $\mathfrak{g}_{\mathbb{C}}$ we can produce a representation of $\mathfrak{g}$ by restricting the representation. Then from

Theorem 2.46 it follows that there is a one-to-one correspondence between irreducible representations of $\mathfrak{g}$ and its complexification $\mathfrak{g}_{\mathbb{C}}$.

Example 2.47. The Lie algebra $\mathfrak{su}(n)$ consists of complex $n \times n$ matrices $X \in \mathrm{M}(n, \mathbb{C})$ such that $\mathrm{tr}X = 0$ and $X^* = -X$. For a proof of this we refer to [9, Prop. 3.24]. Note that $i\mathfrak{su}(n)$ then consists of the matrices $X \in \mathrm{M}(n, \mathbb{C})$ such that $\mathrm{tr}X = 0$ and $X^* = X$. In particular, we see $\mathfrak{su}(n) \cap i\mathfrak{su}(n) = 0$. Hence $\mathfrak{su}_{\mathbb{C}}(n) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$. This implies that the embedding $\mathfrak{su}(n) \hookrightarrow \mathrm{M}(n, \mathbb{C})$ extends to a complex linear embedding

$$j : \mathfrak{su}_{\mathbb{C}}(n) \hookrightarrow \mathrm{M}(n, \mathbb{C}).$$

Note that j maps into $\mathfrak{sl}(n, \mathbb{C})$, since $\mathfrak{sl}(n, \mathbb{C})$ consists of the traceless matrices[5]. On the other hand if $X \in \mathfrak{sl}(n, \mathbb{C})$, we can write $X = \frac{1}{2}(X - X^*) + \frac{1}{2}(X + X^*)$. Note that $\frac{1}{2}(X - X^*) \in \mathfrak{su}(n)$ and $\frac{1}{2}(X + X^*) \in i\mathfrak{su}(n)$, hence $X \in \mathfrak{su}_{\mathbb{C}}(n)$. This shows that j is an isomorphism and we will identify $\mathfrak{su}_{\mathbb{C}}(n)$ with $\mathfrak{sl}(n, \mathbb{C})$ from now on.

This example will be quite extensively used in the next chapter. There we will classify the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ and use the correspondence of Theorem 2.46 to link them with irreducible representations of $\mathfrak{su}(3)$. Yet, our main goal is classifying the irreducible representations of $SU(3)$, since those play a big role in the description of the Eightfold Way. We now ask ourselves if there is a way to *lift* the representation of the Lie algebra of a Lie group to a representation of the Lie group. It turns out this depends on the topological structure of the Lie group, as is seen in the following theorem.

Theorem 2.48. *Let G, H be Lie groups and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Suppose that G is connected and simply connected. Then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that its tangent map at e equals ϕ.*

Proof. We refer to [2, Thm. A.1] or [16, Prop. 1.20]. ■

Corollary 2.49. *Let G be a connected simply connected Lie group and let (ρ, V) be a representation of $\mathfrak{g}$. Then there exists a unique representation (π, V) of G such that $\pi_* = \rho$.*

Proof. Direct consequence of Theorem 2.48. ■

Since $SU(3)$ is a connected simply connected Lie group, combining Theorem 2.48 and Theorem 2.46 tells us there is a one-to-one correspondence between the irreducible representations of $SU(3)$ and the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$.

2.4.4 Operations on representations

Given representations of a Lie group or Lie algebra we can generate new representations through several operations. Here we utilize the definitions in [9, Sec. 4.3]. Suppose (π_1, V_1), (π_2, V_2) are representations of the a Lie group G. Then we can apply several operations on the spaces V_1 and V_2, such as the direct sum and tensor product[6]. We will now give the definitions of the direct sum representations, the tensor product of representations and the dual of a representation.

[5] For a proof we refer to [9, Prop. 3.23].
[6] The definition of the tensor product and some useful theorems can be found in [1].

Definition 2.50. *Let G be a Lie group and $\mathfrak{g}$ a Lie algebra and let (π_1, V_1), (π_2, V_2) be representations of G or $\mathfrak{g}$.*

(a) *The* ***direct sum*** *of π_1 and π_2 is a representation of G in $V_1 \oplus V_2$ defined by*

$$(\pi_1 \oplus \pi_2)(x)(v_1, v_2) := (\pi_1(x)v_1, \pi_2(x)v_2),$$

for $x \in G$. Similarly, the direct sum representation $\mathfrak{g}$ in $V_1 \oplus V_2$ is defined by

$$(\pi_1 \oplus \pi_2)(X)(v_1, v_2) := (\pi_1(X)v_1, \pi_2(X)v_2),$$

for $X \in G$.

(b) *The* ***tensor product*** *of π_1 and π_2 is a representation of G in $V_1 \otimes V_2$ defined by*

$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) = \pi_1(x)v_1 \otimes \pi_2(x)v_2,$$

for $x \in G$. Similarly, the tensor product representation of $\mathfrak{g}$ in $V_1 \otimes V_2$ is defined by

$$(\pi_1 \otimes \pi_2)(X)(v_1 \otimes v_2) = \pi_1(X)v_1 \otimes \mathbb{1}v_2 + \mathbb{1}v_1 \otimes \pi_2(X)v_2,$$

for $X \in \mathfrak{g}$.

One can easily check that the above definitions actually define true representations in the sense of Definition 2.27.
For the dual of a representation we need the notion of a dual space. Let V be a linear space over a field $\mathbb{K}$. The dual space of V is a linear space (with respect to pointwise addition and scalar multiplication), denoted by V^*, given by all linear maps $\phi : V \to \mathbb{K}$. If $T : V \to V$ is a linear map then we can define the dual or adjoint of T by $T^* : V^* \to V^*$, $\phi \mapsto \phi \circ T$. We use this in the following definition.

Definition 2.51. *Let G be a Lie group and let (π, V) be a representation of G. Then the* ***dual representation*** *of π is the representation of G in V^* defined by*

$$\pi^\vee(x) = \pi(x^{-1})^*,$$

for $x \in G$. Similarly, the dual representation of $\mathfrak{g}$ in V^* is defined by

$$\pi^\vee(X) = -\pi(X)^*,$$

for $X \in \mathfrak{g}$.

Note that the inverse of the element x in Definition 2.51 is necessary for $\pi^\vee$ to be group homomorphism. Furthermore, it is readily seen that $\pi^\vee$ is smooth and thus $\pi^\vee$ is a genuine representation. The same is true for the dual representation of a Lie algebra.

3 Irreducible representations of $SU(3)$

As said before, our main focus is understanding the representations of $SU(3)$. From Theorem 2.40 we know it suffices to understand the irreducible representations, since $SU(3)$ is compact. From our discussion in Section 2.4.3 it follows that there is a one-to-one correspondence between the irreducible representations of $SU(3)$ and the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$. In this chapter we will classify the irreducible representations of $SU(3)$ through the one-to-one correspondence with $\mathfrak{sl}(3,\mathbb{C})$. In the description of $\mathfrak{sl}(3,\mathbb{C})$ we will use some general machinery from representation theory called *weights* and *roots*. Also, it will turn out that a description of the representations of $\mathfrak{sl}(2,\mathbb{C})$ is very useful for the classification of irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$. Therefore, this description is included in this chapter as well.

3.1 Weights

We will begin with some general theory. In this section we will discuss the notion of a *weight* and use Chapter 31 in [3] for reference. Furthermore, this chapter we assume $\mathfrak{g}$ is a semisimple Lie algebra coming from a compact Lie group. Let $\mathfrak{t} \subset \mathfrak{g}$ be a finite dimensional commutative subalgebra of $\mathfrak{g}$ and (ρ, V) a representation of $\mathfrak{t}$ in V. We denote the space of complex-valued linear functions on $\mathfrak{t}_\mathbb{C}$ by $\mathfrak{t}_\mathbb{C}^*$. Note that the space of real linear functionals, denoted by $\mathfrak{t}^*$, can be realized as a real linear subspace of $\mathfrak{t}_\mathbb{C}^*$. Since we can identity $\mathfrak{t}^*$ with those $\lambda \in \mathfrak{t}_\mathbb{C}^*$ that take values in $\mathbb{R}$ restricted to $\mathfrak{t}$. Similarly, the space $i\mathfrak{t}^*$ can be identified with the $\lambda \in \mathfrak{t}_\mathbb{C}^*$ such that $\lambda\big|_\mathfrak{t}$ maps to $i\mathbb{R}$.

Let $\lambda \in \mathfrak{t}_\mathbb{C}^*$. Then we define the subspace $V_\lambda \subset V$ by

$$V_\lambda = \bigcap_{H \in \mathfrak{t}} \ker(\rho(H) - \lambda(H)\mathbb{1}). \tag{3.1}$$

If we take a closer look at Equation (3.1), we see that V_λ consists of those vectors that satisfy $\rho(H)v = \lambda(H)v$ for all $H \in \mathfrak{t}$. One could think of these vectors as a generalization of eigenvectors with $\lambda \in \mathfrak{t}_\mathbb{C}^*$ as a generalized eigenvalue.

Definition 3.1. *We say $\lambda \in \mathfrak{t}_\mathbb{C}^*$ is a **weight** if $V_\lambda \neq 0$. In that case the space V_λ is called the associated **weight space**. The dimension of the weight space is called the **multiplicity**. We denote the set of weights of $\mathfrak{t}$ in V by $\Lambda(\rho)$ (or sometimes by $\Lambda(V)$),* [3, p.105].

We now define a subspace $V' \subset V$ by

$$V' := \sum_{\lambda \in \Lambda(\rho)} V_\lambda, \tag{3.2}$$

by which we mean that we take the vector sum of the spaces V_λ

Lemma 3.2. *The vector sum in equation* (3.2) *is actually a direct sum.*

Proof. We refer to [3, Lemmma 31.2]. ■

Lemma 3.3. *If $\rho(X)$ is diagonalizable for every $X \in \mathfrak{t}$, then $V = V'$.*

Proof. We also refer to [3, Lemma 31.2]. ■

Lemma 3.4. *The set $\Lambda(\rho)$ is non-empty finite subset of $\mathfrak{t}_{\mathbb{C}}^*$.*

Proof. From Lemma 3.2 we deduce that $\Lambda(\rho)$ has at most $\dim V$ elements. So it is left to show that $\Lambda(\rho)$ is non-empty. Let $X_1, X_2, \ldots, X_n$ be a basis for $\mathfrak{t}$. The endomorphism $\rho(X_1)$ has at least one eigenvalue λ_1 (maybe complex). Denote the corresponding eigenspace by E_{λ_1}. Note that $E_{\lambda_1} \subset V$ is a nontrivial invariant subspace for the endomorphism $\rho(X_2)$. This easily follows from the commutativity of $\mathfrak{t}$. Due to this invariance we can restrict $\rho(X_2)$ to E_1 and get an endomorphism of E_1. Again, this endomorphism has at least one eigenvalue, say λ_2. We denote the corresponding eigenspace by E_2, as we did before. Note that $E_2 \subset E_1$ by construction. Continuing in this way we find after a finite number of steps a sequence of non-trivial subspaces $E_n \subset E_{n-1} \subset \cdots \subset E_1 \subset V$ such that $\rho(X_i)v = \lambda_i v$ for $v \in E_i$. Now define $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ by $\lambda(X_i) = \lambda_i$. Note $E_n \subset V_\lambda$, thus V_λ is not empty. As a consequence, it follows that $\lambda \in \Lambda(\rho)$. Proving the assertion. ■

Definition 3.5. *We say $\mathfrak{t} \subset \mathfrak{g}$ is a* ***torus*** *if it is a commutative subalgebra of $\mathfrak{g}$. We say a torus $\mathfrak{t} \subset \mathfrak{g}$ is called* ***maximal*** *if there exists no torus of $\mathfrak{g}$ that properly contains $\mathfrak{t}$.*

This definition is typically used for compact Lie algebras[1]. A basic result can be deduced from the maximality of a torus, as is shown in the following lemma.

Lemma 3.6. *The centralizer of a maximal torus $\mathfrak{t}$ in $\mathfrak{g}$ equals $\mathfrak{t}$.*

Proof. Let $\mathfrak{t}$ be a maximal torus. Since a torus $\mathfrak{t}$ is commutative we know $t \subset Z(\mathfrak{t})$. On the other hand, suppose $X \in Z(\mathfrak{t})$. Then $\mathfrak{t}' := \mathfrak{t} + \mathbb{R}X$ is a commutative subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. From the maximality of $\mathfrak{t}$ it follows that $\mathfrak{t} = \mathfrak{t}'$, hence $X \in \mathfrak{t}$. Completing the proof. ■

Until this point we have only discussed representations of $\mathfrak{t}$. Yet, the objects we want to consider are representations of $\mathfrak{g}$ or equivalently $\mathfrak{g}_{\mathbb{C}}$. Let (π, V) be a representation of $\mathfrak{g}_{\mathbb{C}}$. We want to extend our notion of weights to the representation (π, V). Since weights are only defined for representations of a torus $\mathfrak{t}$, we need to construct a representation of $\mathfrak{t}$ from (π, V). The most natural way to do this is by restriction. So, $\Lambda(\pi) = \Lambda(\pi, \mathfrak{t})$ denotes set of the weights of the representation $\rho := \pi|_{\mathfrak{t}}$ of $\mathfrak{t}$ in V. To use the previous definitions ρ is substituted by $\pi|_{\mathfrak{t}}$. As a direct consequence of Lemma 3.4 it follows that $\Lambda(\pi)$ is a finite subset of $\mathfrak{t}_{\mathbb{C}}^*$.

For every representation (π, V) of a Lie group G we know its tangent map π_* is a representation of $\mathfrak{g}$ in V. Furthermore, we have seen that there exists a unique extension of π_* to a representation of $\mathfrak{g}_{\mathbb{C}}$ in V. Using this we can formulate the following lemma.

Lemma 3.7. *Let (π, V) be a representation of a compact Lie group G. Then $\Lambda(\pi_*)$ is a finite subset of $i\mathfrak{t}^*$. Furthermore,*

$$V = \bigoplus_{\lambda \in \Lambda(\pi_*)} V_\lambda. \tag{3.3}$$

[1] By a compact Lie algebra we mean a Lie algebra coming from a compact Lie group.

Proof. Since G is a compact Lie group, there exists a Hermitian inner product on V such that $\pi(x)$ is unitary for every $x \in G$ (see Lemma 2.43). We equip V with this inner product and denote it by $\langle \cdot, \cdot \rangle$. This means that π is now a function from G to the unitary operators on V, denoted by $U(V)$. Hence its tangent map, π_*, maps $\mathfrak{g}$ to $\mathfrak{u}(V)$. Where $\mathfrak{u}(V) \subset \mathrm{End}(V)$ is the subalgebra of anti-Hermitian operators on V (see [9, Prop. 3.24]). From linear algebra we know that anti-Hermitian operators on a finite dimensional vector space are diagonalizable with imaginary eigenvalues. Equation 3.3 now follows after applying Lemma 3.2. ∎

3.2 Roots

At this point, we have developed some basic theory about weights for an arbitrary representation of $\mathfrak{g}_{\mathbb{C}}$. In this section we will talk about the weights of an important representation, namely the adjoint representation ad. In this section we assume that $\mathfrak{t}$ is a maximal torus of $\mathfrak{g}$.

Note for every $A \in \mathrm{End}(\mathfrak{g})$ there exists a unique complex linear extension to $\mathfrak{g}_{\mathbb{C}}$, denoted by $A_{\mathbb{C}}$. Furthermore, $\mathrm{End}(\mathfrak{g})$ can be viewed as a real linear subspace of $\mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ by the map $\iota : \mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g}_{\mathbb{C}}), A \mapsto A_{\mathbb{C}}$. Hence the adjoint representation Ad can be viewed as a representation of a Lie group G in $\mathfrak{g}_{\mathbb{C}}$. And after extending its tangent map, we may view the adjoint representation ad as a representation of $\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Note that the weight space $\mathfrak{g}_{\mathbb{C}0}$ is non-empty, since $\mathfrak{t} \subset \mathfrak{g}_{\mathbb{C}0}$. Hence $0 \in \Lambda(\mathrm{ad})$. Actually, using the definition of a weight space one readily sees that $\mathfrak{g}_{\mathbb{C}0} = Z(\mathfrak{t})_{\mathbb{C}}$, where $Z(\mathfrak{t})$ denotes the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. From Lemma 3.6 it follows that $Z(\mathfrak{t}) = \mathfrak{t}$, hence $\mathfrak{g}_{\mathbb{C}0} = \mathfrak{t}_{\mathbb{C}}$.

Definition 3.8. *The nonzero weights of* ad *are called* ***roots*** *of* $\mathfrak{t}$ *in* $\mathfrak{g}_{\mathbb{C}}$. *We denote the set of roots by* $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ *and for* $\alpha \in R$ *the associated weight space, denoted by* $\mathfrak{g}_{\mathbb{C}\alpha}$, *is called the* ***root space*** *associated with* α.

From the construction above it follows that we can apply Lemma 3.7 to obtain the so called *root space decomposition* of $\mathfrak{g}_{\mathbb{C}}$.

Corollary 3.9. *The set of roots* $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ *is a finite subset of* $i\mathfrak{t}^*$. *Moreover, we have the decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\mathbb{C}\alpha}.$$

It turns out there is a connection between roots and the weights of an arbitrary representation. This connection, although easy to prove, will turn out to be very useful for the classification of representations of $\mathfrak{sl}(3, \mathbb{C})$, which we will discuss later on this chapter.

Lemma 3.10. *Let* (π, V) *be a representation of* $\mathfrak{g}_{\mathbb{C}}$. *Then for every* $\lambda \in \Lambda(\pi)$ *and* $\alpha \in R$ *we have*

$$\pi(\mathfrak{g}_{\mathbb{C}\alpha})V_\lambda \subset V_{\lambda+\alpha}.$$

In particular, if $\lambda + \alpha \notin \Lambda(\pi)$ *then* $\pi(\mathfrak{g}_{\mathbb{C}\alpha})$ *annihilates* V_λ.

Proof. Let $\lambda \in \Lambda(\pi)$ be a weight and $\alpha \in R$ be a root. Furthermore, let $v \in V_\lambda$ and $X \in \mathfrak{g}_{\mathbb{C}\alpha}$. Then for $H \in \mathfrak{t}$ we have

$$\begin{aligned} \pi(H)\pi(X)v &= \pi(X)\pi(H)v + [\pi(H), \pi(X)]v \\ &= \lambda(H)\pi(X)v + \pi([H, X])v \\ &= \lambda(H)\pi(X)v + \pi(\alpha(H)X)v \\ &= (\lambda(H) + \alpha(H))\pi(X)v. \end{aligned}$$

The above calculation shows that $\pi(X)v \in V_{\lambda+\alpha}$. If $\lambda + \alpha \notin \Lambda(\pi)$ it follows by definition that $V_{\lambda+\alpha} = 0$, hence $\pi(\mathfrak{g}_{\mathbb{C}\alpha})$ annihilates V_λ. ∎

The above lemma has some neat consequence that we will discuss now.

Corollary 3.11. *Let $\alpha, \beta \in R$, then*

$$[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \subset \mathfrak{g}_{\mathbb{C}(\alpha+\beta)}.$$

Specifically, if $\alpha + \beta \notin R \cup \{0\}$ then $\mathfrak{g}_{\mathbb{C}\alpha}$ and $\mathfrak{g}_{\mathbb{C}\beta}$ commute.

Proof. Apply Lemma 3.10 to the adjoint representation ad of $\mathfrak{g}_{\mathbb{C}}$ and note $\Lambda(\mathrm{ad}) = R \cup \{0\}$. ∎

Corollary 3.12. *Let (π, V) be a representation of $\mathfrak{g}_{\mathbb{C}}$. Then, the space*

$$W := \bigoplus_{\lambda \in \Lambda(\pi)} V_\lambda \tag{3.4}$$

is a non-trivial invariant subspace for the representation (π, V). In particular, if (π, V) is irreducible then $W = V$.

Proof. From Lemma 3.4 it follows that $\Lambda(\pi)$ is non-empty, hence W is non-trivial. Let $w \in W$ and $X \in \mathfrak{g}_{\mathbb{C}}$. Then there exists a weight $\lambda \in \Lambda(\pi)$ such that $w \in V_\lambda$. Moreover, either $X \in \mathfrak{t}_{\mathbb{C}}$ or there exists a root $\alpha \in R$ such that $X \in \mathfrak{g}_{\mathbb{C}\alpha}$, by Corollary 3.9. If $X \in \mathfrak{t}_{\mathbb{C}}$ it follows that $\pi(X)w \in V_\lambda$, by the definition of a weight space. On the other hand if $X \in \mathfrak{g}_{\mathbb{C}\alpha}$ for some $\alpha \in R$, Lemma 3.10 tells us that either $\pi(X)w = 0$ or $\pi(X)w \in V_{\lambda+\alpha}$ where $\lambda + \alpha \in \Lambda(\pi)$. In both cases we have $\pi(X)w \in W$. Hence W is an invariant subspace for the representation (π, V). If (π, V) is irreducible it follows, since $W \subset V$ is an invariant subspace, that $W = V$. ∎

3.3 The representations of $\mathfrak{sl}(2, \mathbb{C})$

So far, this chapter, we have developed the general notions of weights and roots and showed some of their properties. At this point we are ready to apply this general theory to a specific Lie algebra, namely the complexification of $\mathfrak{su}(2)$: $\mathfrak{sl}(2, \mathbb{C})$. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ consists of traceless complex 2×2 matrices[2]. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has a very

[2]To see this we again refer to [9, Prop. 3.23].

natural basis given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In this section we will assume that (π, V) is a representation of $\mathfrak{sl}(2,\mathbb{C})$. Since $\mathfrak{sl}(2,\mathbb{C})$ is equipped with the commutator bracket we can calculate the commutation relations. After a simple calculation we see

$$[H, X] = 2X, \; [H, Y] = -2Y, \; [X, Y] = H. \tag{3.5}$$

Recall that $\mathfrak{su}(2)$ consists of the complex 2×2 matrices X such that $\mathrm{tr}X = 0$ and $X^* = -X$. Then $\mathfrak{t} = i\mathbb{R}H$ is a maximal torus in $\mathfrak{su}(2)$. Let us define the linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ by $\alpha(H) = 2$. For $H' \in \mathfrak{t}$, given by $H' = irH$ with $r \in \mathbb{R}$, we see by using the commutation relations

$$[H', X] = ir[H, X] = 2irX = ir\alpha(H)X = \alpha(H')X. \tag{3.6}$$

Since the adjoint representation $(\mathrm{ad}, \mathfrak{sl}(2,\mathbb{C}))$ of $\mathfrak{sl}(2,\mathbb{C})$ coincides with the commutation bracket (see Example 2.22), Equation (3.6) shows that $\alpha \in R(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{t})$. In a similar way one can show that $-\alpha \in R(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{t})$, with $Y \in \mathfrak{sl}(2,\mathbb{C})_{-\alpha}$. One can actually show, through straightforward calculation, that $\mathfrak{sl}(2,\mathbb{C})_\alpha = \mathbb{C}X$ and $\mathfrak{sl}(2,\mathbb{C})_{-\alpha} = \mathbb{C}Y$. Also, note that $\mathfrak{t}_{\mathbb{C}} = \mathbb{C}H$. Since H, X, Y form a basis of $\mathfrak{sl}(2,\mathbb{C})$ it follows that $R(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{t}) = \{\alpha, -\alpha\}$. Now we can apply our theory of roots.

Lemma 3.13. *Let v be an eigenvector of $\pi(H)$ with eigenvalue $\lambda \in \mathbb{C}$. Then*

$$\pi(H)\pi(X)v = (\lambda + 2)\pi(X)v, \qquad \pi(H)\pi(Y)v = (\lambda - 2)\pi(Y)v.$$

Showing, either $\pi(X)v = 0$ or $\pi(X)v$ is again an eigenvector of $\pi(H)$ with eigenvalue $\lambda + 2$. Similarly for $\pi(Y)v$ but with eigenvalue $\lambda - 2$.

Proof. Since our torus $\mathfrak{t}$ is one dimensional the functional $\tilde{\lambda} \in \mathfrak{t}_{\mathbb{C}}^*$ defined by $\tilde{\lambda}(H) = \lambda$ is actually a weight, so $\tilde{\lambda} \in \Lambda(\pi)$. Since $\alpha, -\alpha$ are roots it follows from Lemma 3.10 that $\pi(X)V_{\tilde{\lambda}} \subset V_{\lambda+\alpha}$. In particular, this means $\pi(H)\pi(X)v = (\tilde{\lambda}(H) + \alpha(H))\pi(X)v = (\lambda + 2)\pi(X)v$. In a similar way we get $\pi(H)\pi(Y)v = (\lambda - 2)\pi(Y)v$. ■

Note that $\pi(H)$ always has at least one eigenvalue, since $\mathbb{C}$ is algebraically complete. Since the operator $\pi(X)$ raises the eigenvalue of $\pi(H)$, it is called, fittingly, the *raising operator*. Similary, since $\pi(Y)$ lowers the eigenvalue of $\pi(H)$, it is called the *lowering operator*. If we repeatedly apply Lemma 3.13 we find

$$\pi(H)\pi(X)^n v = (\lambda + 2n)\pi(X)^n v \tag{3.7}$$

and a similar result for $\pi(Y)$. Yet, since V is finite dimensional $\pi(H)$ only has finitely many eigenvalues. Therefore, $\pi(X)^n v$ cannot be nonzero for every $n \in \mathbb{N}$. Hence there exists an $N \in \mathbb{N}$ such that $\pi(X)^N v \neq 0$, but $\pi(X)^{N+1} v = 0$. This brings us to the following definition.

Definition 3.14. *We say $v \in V \setminus \{0\}$ is a* ***primitive vector*** *if $\pi(X)v = 0$.*

By our discussion above Definition 3.14 it follows that every representation of $\mathfrak{sl}(2,\mathbb{C})$ has a primitive vector that is simultaneously an eigenvector of $\pi(H)$, namely $\pi(X)^N v$. Let us capture that property in a lemma.

Lemma 3.15. *Let (π, V) be a representation of $\mathfrak{sl}(2,\mathbb{C})$, then there exists a primitive vector that is simultaneously an eigenvector of $\pi(H)$.*

Let us denote $w := \pi(X)^N v$ and its eigenvalue for $\pi(H)$ by μ, which is equal to $\lambda+2N$. Now we define the collection of vectors v_k inductively by $v_0 = w$ and $v_{k+1} = \pi(Y)v_k$. So $v_k = \pi(Y)^k w$. By a similar reasoning as above there exists an $n \in N$ such that $v_n \neq 0$, but $v_{n+1} = 0$.

Lemma 3.16. *Assume the representation (π, V) of $\mathfrak{sl}(2,\mathbb{C})$ is irreducible.*

(a) *For every $0 \leq k \leq n$, the following holds*

$$\pi(H)v_k = (\mu - 2k)v_k, \qquad \pi(X)v_k = k(\mu - k + 1)v_{k-1}.$$

(b) *The vectors v_k for $0 \leq k \leq n$ form a basis for V.*

(c) *The eigenvalue μ equals* $\dim(V) - 1$

Proof. We start by proving (a). By repeated usage of Lemma 3.13,

$$\begin{aligned} \pi(H)v_k &= \pi(H)\pi(Y)^k w \\ &= (\mu - 2k)\pi(Y)^k w \\ &= (\mu - 2k)v_k. \end{aligned}$$

Proving the first part of (a). We will prove the second part of (a) by induction. For $k = 0$ we have $v_k = w$. Since w is a primitive vector we know $\pi(X)w = 0 = k(\mu - k + 1)$, so the assertion is true for $k = 0$. Now assume the equality holds for some $0 \leq k < n$. Then for $k + 1$,

$$\begin{aligned} \pi(X)v_{k+1} &= \pi(X)\pi(Y)v_k \\ &= \pi(Y)\pi(X)v_k + [\pi(X), \pi(Y)]v_k \\ &= \pi(Y)k(\mu - k + 1)v_{k-1} + \pi([X, Y])v_k \\ &= k(\mu - k + 1)v_k + \pi(H)v_k \\ &= k(\mu - k + 1)v_k + (\mu - 2k)v_k \\ &= (k\mu + \mu - k^2 - k)v_k \\ &= (k + 1)(\mu - (k + 1) + 1)v_k. \end{aligned}$$

Hence the second assertion of (a) is true for every $0 \leq k \leq n$.

Now we prove part (b). We define $W \subset V$ to be the linear span of the collection of vectors v_k. Note that, by the definition of the vectors v_k it is clear that W is invariant under the action of $\pi(Y)$. Furthermore from part (a) it follows that W is invariant under the actions of $\pi(H)$ and $\pi(X)$, as well. Since H, X, Y form a basis of $\mathfrak{sl}(2,\mathbb{C})$, it follows that W is an invariant subspace (and non-trivial by definition) of V. Since the representation (π, V) is irreducible we conclude $W = V$, proving (b).

For (c) we note that $v_{n+1} = 0$. Using the second part of (a) we see[3]

$$0 = (n+1)(\mu - n)v_n.$$

Since $v_n \neq 0$ by construction and $n + 1 > 0$, we conclude $\mu = n$. Note that, from (b) we see that $\dim(V) = n + 1$. Hence $\mu = \dim(V) - 1$, completing the proof. ■

Corollary 3.17. *Let (π, V) be a (*not necessarily irreducible*) representation of $\mathfrak{sl}(2, \mathbb{C})$. Then, every eigenvalue of $\pi(H)$ is an integer.*

Proof. First we note that in the proof of part (c) we only used the irreducibility of (π, V) to show that μ quals $\dim(V) - 1$. Yet, for a general representation the argument shows that μ equals some natural number n. We argued before that $\mu = \lambda + 2N$, where N is a natural number. Hence the original eigenvalue λ of $\pi(H)$, from Lemma 3.13, is an integer. Since this eigenvalue was arbitrary, the result follows. ■

Note that it is not clear if a primitive vector is unique. We made an explicit choice for our primitive vector, yet it is not excluded that other primitive vectors exist. It turns out we need irreducibility to guarantee a uniqueness property of a primitive vector.

Corollary 3.18. *Let (π, V) be an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. A vector $v \in V$ is a primitive vector if and only if it is a nonzero multiple of v_0.*

Proof. Due to the linearity of $\pi(X)$, it is clear that a nonzero multiple of v_0 is a primitive vector. Now let us assume that $v \in V$ is a primitive vector, so $\pi(X)v = 0$. By Lemma 3.16(b) we know v can be represented as a linear combination of the v_k's. Hence

$$v = \sum_{j=0}^{n} \lambda_i v_i.$$

Yet, by the second equation of Lemma 3.16(a) it follows that λ_i must be zero for $i > 0$. Otherwise, $\pi(X)v = 0$ cannot hold. Consequently, we are left with $v = \lambda_0 v_0$. ■

Remark 3.19. From Corollary 3.18 and Lemma 3.15 it follows that *every* primitive vector is an eigenvector of $\pi(H)$ with the same eigenvalue. Hence, the eigenvalue μ in Lemma 3.16 is unique.

Now we have all the tools to classify the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 3.20. *Let (π, V) and (π', V') be irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. Then V and V' are isomorphic if and only if $\dim(V) = \dim(V')$. Moreover, if $v \in V$ and $v' \in V'$ an primitive vectors then there exists a unique equivariant isomorphism $T : V \to V'$, mapping v to v'.*

Proof. Clearly, if V and V' are isomorphic their dimensions must be equal. Suppose $\dim(V) = \dim(V') = n+1$. By our discussion this section we know there exists primitive vectors v, v' in V, V', respectively. Let λ,λ' denote the eigenvalue for π By Lemma 3.16(b) we know $v, \pi(Y)v, \dots, \pi(Y)^n v$ forms a basis of V and $v', \pi'(Y)v', \dots, \pi'(Y)^n v'$ forms a

[3]Note that, even though k should be between 0 and n for (a) the calculation in the proof of (a) still holds for $k = n + 1$.

basis of V'. Let us adopt the notation from before: $v_k := \pi(Y)^k v$ and $v'_k := \pi'(Y)^k v'$ for $0 \leq k \leq n$. Now define the linear map $T : V \to V'$ by $v_k \mapsto v'_k$, for every $0 \leq k \leq n$. Clearly, T is an isomorphism of vector spaces. Now we show T is an intertwiner. Since H, X, Y is a basis for $\mathfrak{sl}(2, \mathbb{C})$, it suffices to show T intertwines the actions of H, X, Y. By the definitions of T and the bases of V and V', for $0 \leq k \leq n$

$$\begin{aligned} T(\pi(Y)v_k) &= T(v_{k+1}) \\ &= v'_{k+1} \\ &= \pi'(Y)v'_k \\ &= \pi'(Y)T(v_k). \end{aligned}$$

The above computation shows that T intertwines the action of Y. Using Lemma 3.16, for $0 \leq k \leq n$

$$\begin{aligned} T(\pi(H)v_k) &= (n-2k)T(v_k) \\ &= (n-2k)v'_k \\ &= \pi'(H)v'_k \\ &= \pi'(H)T(v_k). \end{aligned}$$

Hence T intertwines the action of H. Finally, again by Lemma 3.16

$$\begin{aligned} T(\pi(X)v_k) &= k(n-k+1)T(v_{k-1}) \\ &= k(n-k+1)v'_{k-1} \\ &= \pi'(H)v'_k \\ &= \pi'(H)T(v_k), \end{aligned}$$

for all $0 \leq k \leq n$. Hence T is an intertwiner and thus V and V' are isomorphic.

Note that, if $T : V \to V'$ is an equivariant isomorphism mapping v to v' we must have

$$T(v_k) = T(\pi(Y)^k v) = \pi'(Y)^k T(v) = \pi'(Y)^k v' = v'_k.$$

Hence T is uniquely defined. ■

Due to the one-to-one correspondence between irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ and $SU(2)$, the above theorem also classifies the irreducible representations of $SU(2)$.

3.4 More on roots and weights

In our description of $\mathfrak{sl}(2, \mathbb{C})$ we saw that the notion of a primitive vector was crucial in the classification of irreducible representations. Yet, $\mathfrak{sl}(2, \mathbb{C})$ has a very simple basis which simplifies the discussion al lot. In this section we want to generalize the notion of a primitive vector for more complicated Lie algebras. To obtain this generalization we return to some general theory about *roots* and *weights*.

In this section we use [3, Ch. 31] as our main reference. Furthermore, we assume $\mathfrak{g}$ is a semisimple Lie algebra coming from a compact Lie group and $\mathfrak{t} \subset \mathfrak{g}$ is a maximal torus. We recall that the collection of roots $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ is a finite subset of $i\mathfrak{t}^*$. Moreover, $i\mathfrak{t}^*$ was identified with the real linear subspace of $\mathfrak{t}_{\mathbb{C}}$ consisting of λ such that $\lambda|_{\mathfrak{t}}$ has values in

$i\mathbb{R}$. Note that, this is equivalent to requiring $\lambda|_{i\mathfrak{t}}$ takes on real values. Hence, $i\mathfrak{t}^*$ and $(i\mathfrak{t})^*$ are isomorphic. If we use this isomorphism to identify $i\mathfrak{t}^*$ with $(i\mathfrak{t})^*$, then we can view R as a finite subset of $(i\mathfrak{t})^* \setminus \{0\}$. Since roots map to $\mathbb{R}$, are linear and not identically zero, they must be surjective. Hence, by the rank-nullity theorem it follows that $\ker \alpha \subset i\mathfrak{t}$ is a hyperplane[4], for all $\alpha \in R$. Let us define $H := \bigcup_{\alpha \in R} \ker \alpha \subset i\mathfrak{t}$. Then, the complement of H in $i\mathfrak{t}$ is a finite union of convex connected subsets of $i\mathfrak{t}$. These subsets are referred to by a special term, namely such a subset is called a **Weyl chamber** associated with R. Let $\mathcal{C}$ be a fixed Weyl chamber. Since roots are linear maps between finite dimensional vector spaces, they are continuous. Hence, by the intermediate value theorem and the convexity of the Weyl chambers, every root is either positive or negative by construction of $\mathcal{C}$. Now we can define the notion of a *system of positive roots*, denoted by $R^+ := R^+(\mathcal{C})$, associated with $\mathcal{C}$ by

$$R^+ = \{\alpha \in R \mid \alpha > 0 \text{ on } \mathcal{C}\}.$$

A system of positive roots turns out to give a useful decomposition of $\mathfrak{g}_\mathbb{C}$. To show this we need the following lemma.

Lemma 3.21. *Let $\alpha \in R$ be a root, then $-\alpha$ is also a root.*

Proof. We start by defining a conjugate linear automorphism of $\mathfrak{g}_\mathbb{C}$ regarded as a *real* Lie algebra, which we will refer to as the conjugation map,

$$\tau : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}, \ \tau(X + iY) = X - iY.$$

Let $Z = X + iY \in \mathfrak{g}_\mathbb{C}$ and $H \in \mathfrak{t}$. Then, by straightforward calculation

$$\begin{aligned} [H, \tau(Z)] &= [H, \tau(X + iY)] \\ &= [H, X - iY] \\ &= [H, X] - i[H, Y] \\ &= \tau([H, Z]). \end{aligned}$$

In particular, for $Z = X + iY \in \mathfrak{g}_{\mathbb{C}\alpha}$ the above calculation shows

$$[H, \tau(Z)] = \tau(\alpha(H)Z) = \overline{\alpha(H)}\tau(Z).$$

To complete the proof we note that R is a finite subset of $i\mathfrak{t}^*$. Therefore, $\alpha(H)$ is a pure imaginary number for all $H \in \mathfrak{t}$. Hence $\overline{\alpha(H)} = -\alpha(H)$, for $H \in \mathfrak{t}$. Using this we see $[H, \tau(Z)] = -\alpha(H)\tau(Z)$, thus $\tau(Z) \in \mathfrak{g}_{\mathbb{C}(-\alpha)}$. ∎

Remark 3.22. The proof of Lemma 3.21 shows that τ maps $\mathfrak{g}_{\mathbb{C}\alpha}$ to $\mathfrak{g}_{\mathbb{C}(-\alpha)}$. Note that, it is easy to see τ is its own inverse. And due to symmetry in α and $-\alpha$ one could also argue τ maps $\mathfrak{g}_{\mathbb{C}(-\alpha)}$ to $\mathfrak{g}_{\mathbb{C}\alpha}$, hence $\tau^2 = \mathrm{id}$.

As said before, every root is either positive or negative on a Weyl chamber $\mathcal{C}$. Lemma 3.21 shows, for every root $\alpha \in R$, both α and $-\alpha$ are roots. Yet, only one of them is contained in R^+. Hence, we can write

$$R = R^+ \cup (-R^+). \tag{3.8}$$

[4] A hyperplane is a linear subspace whose dimension is one less than that of the total space.

The above union is actually disjoint. This splitting of the collection of roots gives rise to a decomposition of $\mathfrak{g}_{\mathbb{C}}$.

Lemma 3.23. *Define the subspaces*

$$\mathfrak{g}_{\mathbb{C}}^{+} := \sum_{\alpha \in R^{+}} \mathfrak{g}_{\mathbb{C}\alpha}, \quad \mathfrak{g}_{\mathbb{C}}^{-} := \sum_{\beta \in (-R^{+})} \mathfrak{g}_{\mathbb{C}\beta}$$

of $\mathfrak{g}_{\mathbb{C}}$. *Then, the spaces* $\mathfrak{g}_{\mathbb{C}}^{+}$ *and* $\mathfrak{g}_{\mathbb{C}}^{-}$ *are* $\mathrm{ad}(\mathfrak{t})$*-invariant subalgebras of* $\mathfrak{g}_{\mathbb{C}}$. *Moreover,*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{+} \oplus \mathfrak{g}_{\mathbb{C}}^{-}.$$

Proof. Let $\alpha, \beta \in R^{+}$. Then $\alpha + \beta$ must be positive on $\mathcal{C}$. In particular, $\alpha + \beta \neq 0$. Suppose $[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \neq 0$, then by Corollary 3.11 and the fact $\alpha + \beta \neq 0$ we see $\alpha + \beta \in R$. Note that, $\alpha + \beta > 0$ on $\mathcal{C}$ as said before. Hence, the root $\alpha + \beta$ belongs to R^{+}. This implies that $\mathfrak{g}_{\mathbb{C}(\alpha+\beta)} \subset \mathfrak{g}_{\mathbb{C}}^{+}$, thus $[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \subset \mathfrak{g}_{\mathbb{C}}^{+}$ by Corollary 3.11. If $\mathfrak{g}_{\mathbb{C}\alpha}$ and $\mathfrak{g}_{\mathbb{C}\beta}$ do commute it is clear that $[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \subset \mathfrak{g}_{\mathbb{C}}^{+}$. It follows that $\mathfrak{g}_{\mathbb{C}}^{+}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. A similar argument shows that $\mathfrak{g}_{\mathbb{C}}^{-}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

The spaces $\mathfrak{g}_{\mathbb{C}}^{+}$ and $\mathfrak{g}_{\mathbb{C}}^{-}$ are $\mathrm{ad}(\mathfrak{t})$-invariant, because root spaces are. Furthermore, the direct sum decomposition is a direct consequence of Corollary 3.9 and Equation (3.8). ■

Now we have developed enough theory to generalize the notion of a primitive vector from Section 3.3.

Definition 3.24. *Let* (π, V) *be a representation of* $\mathfrak{g}_{\mathbb{C}}$. *Then a* ***highest weight vector*** *of* V, *relative to the system of positive roots* R^{+}, *is a non-trivial vector* $v \in V$ *satisfying*

(a) $\pi(\mathfrak{t}_{\mathbb{C}})v \subset \mathbb{C}v$

(b) $\pi(X)v = 0$ *for all* $X \in \mathfrak{g}_{\mathbb{C}}^{+}$.

Note that the highest weight vector depends on the choice of Weyl chamber.

Lemma 3.25. *Let* (π, V) *be a representation of* $\mathfrak{g}_{\mathbb{C}}$. *Then,* V *has a highest weight vector.*

Proof. Let $\mathcal{C}$ be the Weyl chamber corresponding to the system of positive roots R^{+} and $X \in \mathcal{C}$. Then $\alpha(X) > 0$ for every $\alpha \in R^{+}$. On the other hand, let $\lambda_0 \in \Lambda(\pi)$ denote the weight such that the real part of $\lambda_0(X)$ is maximal. Such a weight exists, because $\Lambda(\pi)$ is finite. Then $\lambda_0 + \alpha \notin \Lambda(\pi)$ for all $\alpha \in R^{+}$. Otherwise, because $\alpha(X) > 0$ for $\alpha \in R^{+}$, the real part of $\lambda_0(X) + \alpha(X)$ would be bigger than the real part of $\lambda_0(X)$, contradicting the definition of λ_0. Then, by Lemma 3.10 we see $\pi(\mathfrak{g}_{\mathbb{C}\alpha})V_{\lambda_0} = 0$ for every $\alpha \in R^{+}$. Hence $\pi(\mathfrak{g}_{\mathbb{C}}^{+})V_{\lambda_0} = 0$. Consequently, every nonzero vector $v \in V_{\lambda_0}$ is a highest weight vector. ■

Definition 3.26. *Let* (π, V) *be a representation of* $\mathfrak{g}_{\mathbb{C}}$. *We say* $v \in V$ *is* ***cyclic****, if the smallest invariant subspace containing* v *is* V *itself.*

The following lemma will be important for our understanding about highest weight vectors and their corresponding weights. We closely follow the proof of [3, Prop. 31.20]. But first we introduce some notation. By $\mathbb{N}R^{+}$ we mean the collection of linear combinations of positive roots with natural numbers as coefficients[5].

[5] We use the convention that $0 \in \mathbb{N}$.

Lemma 3.27. *Let (π, V) be a representation of $\mathfrak{g}_{\mathbb{C}}$ and $v \in V$ be a cyclic highest weight vector. Then, the following assertions hold*

(a) *There exists a unique weight $\lambda \in \Lambda(\pi)$ such that $v \in V_\lambda$. Moreover, $V_\lambda = \mathbb{C}v$.*

(b) *The space V is equal to the span of the vectors $\pi(Y_1)\pi(Y_2)\dots\pi(Y_n)v$, where $n \in \mathbb{N}$ and $Y_i \in \mathfrak{g}_{\mathbb{C}}^-$ for every $0 \leq j \leq n$.*

(c) *Every weight $\mu \in \Lambda(\pi)$ is of the form $\lambda - \nu$, where $\nu \in \mathbb{N}R^+$.*

Proof. Since $v \in V$ is a highest weight vector, by Definition 3.24 we know $\pi(\mathfrak{t}_{\mathbb{C}})v \subset \mathbb{C}v$. In particular, $\pi(H)v = \lambda(H)v$ for every $H \in \mathfrak{t}$, where $\lambda(H) \in \mathbb{C}$. Using this we can define a weight $\lambda \in \Lambda(\pi)$ such that it maps H to the corresponding complex number $\lambda(H)$ on $\mathfrak{t}$. Then, $v \in V_\lambda$ by construction and λ is uniquely defined. We will inductively define linear subspaces of V by $V_0 = \mathbb{C}v$ and $V_{n+1} = V_n + \pi(\mathfrak{g}_{\mathbb{C}}^-)$ for $n \geq 1$. Let W denote the union of the spaces V_n. We claim that W is an invariant subspace of V. First we note that, by definition,

$$\pi(\mathfrak{g}_{\mathbb{C}}^-)V_n \subset V_{n+1}.$$

Hence W is $\mathfrak{g}_{\mathbb{C}}^-$-invariant. We will show the $\mathfrak{t}_{\mathbb{C}}$- and $\mathfrak{g}_{\mathbb{C}}^+$-invariance of W by induction.

Note that $\mathfrak{t}_{\mathbb{C}}$-invariance is equivalent to being invariant under $\mathfrak{t}$. Since v is a highest weight vector it follows that V_0 is $\mathfrak{t}$-invariant. Now suppose that V_n is $\mathfrak{t}$-invariant for some $n \geq 1$. Furthermore, let $u \in V_n$ and $Y \in \mathfrak{g}_{\mathbb{C}}^-$. Then, for $H \in \mathfrak{t}$

$$\pi(H)\pi(Y)u = \pi(Y)\pi(H)u + \pi([H, Y])u.$$

By our assumption we see $\pi(H)u \in V_n$, hence $\pi(Y)\pi(H)u \in V_{n+1}$. On the other hand, due to Lemma 3.23 it follows $[H, Y] \in \mathfrak{g}_{\mathbb{C}}^-$, thus $\pi([H, Y])u \in V_{n+1}$. Consequently, $\pi(H)\pi(Y)u \in V_{n+1}$. So we have shown

$$\pi(\mathfrak{t})\pi(\mathfrak{g}_{\mathbb{C}}^-)V_n \subset V_{n+1}.$$

Since V_n is $\mathfrak{t}$-invariant by assumption and $V_{n+1} = V_n + \pi(\mathfrak{g}_{\mathbb{C}}^-)V_n$ we conclude V_{n+1} is $\mathfrak{t}$-invariant.

Again, since v is a highest weight vector it follows that the space V_0 is $\mathfrak{g}_{\mathbb{C}}^+$-invariant. Now assume V_n is $\mathfrak{g}_{\mathbb{C}}^+$-invariant for some $n \geq 1$. Note that, by the above discussion and this assumption we can deduce

$$\pi(\mathfrak{g}_{\mathbb{C}})V_n \subset V_{n+1}. \tag{3.9}$$

Let $u \in V$ and $Y \in \mathfrak{g}_{\mathbb{C}}^-$. Then, for $X \in \mathfrak{g}_{\mathbb{C}}^+$,

$$\pi(X)\pi(Y)u = \pi(Y)\pi(X)u + \pi([X, Y])u.$$

By the induction hypothesis it follows $\pi(X)u \in V_n$, hence $\pi(Y)\pi(X)u \in V_{n+1}$. Furthermore, since $[X, Y] \in \mathfrak{g}_{\mathbb{C}}$ we see $\pi([X, Y])u \in V_{n+1}$, by Equation (3.9). Thus, $\pi(X)\pi(Y)u \in V_{n+1}$. By a similar reasoning as before, we conclude V_{n+1} is $\mathfrak{g}_{\mathbb{C}}^+$-invariant. Hence, W is an invariant subspace of V. Note that, $v \in W$ by construction. Since v is cyclic we conclude $W = V$. This proves assertion (b).

Let $w \in W = V$, then by the above we can write $w = \pi(Y_1)\dots\pi(Y_n)v$ with $n \in \mathbb{N}$, $Y_i \in \mathfrak{g}_{\mathbb{C}(-\alpha)}$ and $\alpha \in R^+$. Then, by Lemma 3.10 it follows that $w \in V_{\lambda-(\alpha_1+\dots+\alpha_n)}$. Since $W = V$ is spanned by v and elements like w, we conclude every weight $\mu \in \Lambda(\pi)$ is of the form $\lambda - \nu$, where $\nu = \sum_{\alpha\in R^+} n_\alpha \alpha \in \mathbb{N}R^+$. This concludes the proof of part (c).

Finally, the discussion for part (c) shows that any vector $w = \pi(Y_1)\dots\pi(Y_n)v \in V$ with $n \geq 1$ cannot have weight λ. Hence the only vectors with weight λ are multiples of v, concluding the proof of part (a). ∎

The weight of a highest weight vector is fittingly called the **highest weight**. We note that if (π, V) is an irreducible representation of $\mathfrak{g}_\mathbb{C}$, every vector $v \in V$ is cyclic. Using this and the previous lemma we obtain the follow corollary.

Corollary 3.28. *Let (π, V) be an irreducible representation of $\mathfrak{g}_\mathbb{C}$. Then V has a highest weight vector, which is unique up to a scalar.*

Proof. By Lemma 3.25 it follows that V has a highest weight v. Since (π, V) is irreducible, the vector v is cyclic. Hence, all assertions of Lemma 3.27 hold. Suppose $w \in V$ is another highest weight vector, with weight $\mu \in \Lambda(\pi)$. Then, again all assertions of Lemma 3.27 are valid for w and μ. By Lemma 3.27(c) we see $\mu = \lambda - \nu_1$, where $\nu_1 \in \mathbb{N}R^+$. Similarly, we can write $\lambda = \mu - \nu_2$ with $\nu_2 \in \mathbb{N}R^+$. Combining the two equations shows $\nu_1 = -\nu_2$. Since $\nu_2 \in \mathbb{N}R^+$, this means $\nu_1 \in (-\mathbb{N}R^+)$. Which implies that $\nu_1 \leq 0$ on $\mathcal{C}$. Yet, we also had $\nu_1 \in \mathbb{N}R^+$ so $\nu_1 \geq 0$ on $\mathcal{C}$. Therefore $\nu_1 = 0$ on $\mathcal{C}$. Since $\mathcal{C}$ is a non-empty open subset of $i\mathfrak{t}^*$ it follows that $\nu_1 = 0$. This is because of the general fact, in a non-empty open subset U of a vector space V, there exists a collection of vectors in U that form a basis for V. Hence, if a linear map is zero on U, it is zero on a basis of V and thus zero on V. Since $\nu_1 = 0$ we also have $\nu_2 = 0$, implying that $\lambda = \mu$. Lemma 3.27(a) now implies $\mathbb{C}v = V_\lambda = V_\mu = \mathbb{C}w$. Consequently, there exists a complex number $\gamma \in \mathbb{C}$ such that $v = \gamma w$, completing the proof. ∎

Note the above proof also shows that the highest weight is unique. From now on we assume the Lie group G, for which $\mathfrak{g}$ is the associated Lie algebra, is furthermore connected and simply connected. Then we can use the one-to-one correspondence between irreducibles we discussed in Section 2.4.3. Combining this with the results of Section 2.4.2 we deduce that every representation (π, V) of $\mathfrak{g}_\mathbb{C}$ is completely reducible. Actually, this is true for a general semisimple Lie algebra (see [9, Thm. 10.9]). Yet, the extra assumptions make the discussion less involved. Moreover, we will apply the theory of this section to the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, which is the complexification of a Lie algebra coming from a connected compact simply-connected Lie group. So, the assumptions suffices for our purposes. The following lemma is a specific case of a more general proposition, namely [9, Prop. A.17].

Lemma 3.29. *Let (π, V) be a representation of $\mathfrak{g}_\mathbb{C}$. Let $\lambda_1, \dots, \lambda_n \in \Lambda(\pi)$ be distinct weights, with $v_1, \dots, v_n \in V$ vectors in the associated weight spaces. If $v_1 + \dots + v_n = 0$, then $v_i = 0$ for all i. Furthermore, if $v_1 + \dots + v_n$ is a weight vector with weight λ, then $\lambda = \lambda_i$ for some $1 \leq i \leq n$ and $v_j = 0$ for all $j \neq i$.*

Proof. We will prove the first part by induction. If $n = 1$, then $v_1 = 0$ thus the claim is true. Now let $n > 1$ and suppose the if sum of $m < n$ vectors is zero it follows that each vector is zero. Then, choose $H \in \mathfrak{t}$ such that $\lambda_1(H) \neq \lambda_2(H)$. By applying the endomorphism $\pi(H) - \lambda_2(H)\mathbb{1}$ to $v_1 + \cdots + v_n$ we see

$$\begin{aligned} 0 &= (\pi(H) - \lambda_2(H)\mathbb{1})v_1 + \cdots + v_n \\ &= \sum_{i=1}^{n} (\lambda_i(H) - \lambda_2(H))v_i. \end{aligned}$$

Note that, the second term in the sum above is zero. Hence, there are at most $n-1$ terms in the sum. By the induction hypothesis we conclude that every term is zero. In particular,

$$(\lambda_1(H) - \lambda_2(H))v_1 = 0.$$

Since $\lambda_1(H) \neq \lambda_2(H)$, we conclude $v_1 = 0$. Therefore, the sum $v_1 + \cdots + v_n$ only contains $n-1$ terms. Again, by the induction hypothesis we conclude $v_i = 0$ for all i.

Now suppose $v := v_1 + \cdots + v_n$ is a weight vector with weight λ. Then, there exists an index j such that $v_j \neq 0$ (since a weight vector is nonzero). Furthermore, for every $H \in \mathfrak{t}$ we see

$$\begin{aligned} 0 &= \pi(H)v - \lambda(H)v \\ &= \sum_{i=1}^{n} (\lambda_i(H) - \lambda(H))v_i \end{aligned}$$

By the first part it follows that every term in the sum above must be zero. In particular,

$$(\lambda_j(H) - \lambda(H))v_j = 0.$$

Since $v_j \neq 0$, we conclude $\lambda_j(H) = \lambda(H)$. Since this holds for every $H \in \mathfrak{t}$ we see $\lambda_j = \lambda$. But then the other terms become

$$(\lambda_i(H) - \lambda_j(H))v_i = 0, \tag{3.10}$$

for $i \neq j$. Since all the weights are distinct, for every $i \neq j$ there exists an $H \in \mathfrak{t}$ such that $\lambda_i(H) \neq \lambda_j(H)$. For this specific H, Equation (3.10) shows that $v_i = 0$ for every $i \neq j$. Hence, $v = v_j$. ■

Lemma 3.30. *Suppose (π, V) is a (completely reducible) representation of $\mathfrak{g}_{\mathbb{C}}$ that has a highest weight vector, which is cyclic. Then, (π, V) is irreducible.*

Proof. By the discussion directly above the lemma we know (π, V) is completely reducible[6]. Hence we can write

$$V = \bigoplus_{i=1}^{n} V_i, \tag{3.11}$$

with V_i an irreducible subspace of V for every $1 \leq i \leq n$. By Corollary 3.12, every V_i is the direct sum of its weight spaces. Let $v \in V$ be a highest weight vector that is also cyclic.

[6]The reason for requiring the complete reducibility in brackets is to emphasize the importance of the property for the lemma to be true.

Let us denote its weight by λ. From Equation (3.11) it follows that $v = v_1 + \cdots + v_n$, where $v_i \in V_i$ for $1 \leq i \leq n$. Since every V_i is the direct sum of its weight spaces, v_i is a weight vector for every i. Hence by Lemma 3.29 it follows that λ must be a weight in some V_i and it follows that $v \in V_i$. Therefore, V_i is an invariant subspace of V containing v. Since v is cyclic we must have $V = V_i$. This shows that V is irreducible. ■

At this point, we have enough background to prove the main theorem of this section and perhaps of this chapter. This theorem will classify irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ using the highest weight. Which we will use next section in our discussion of $\mathfrak{sl}(3, \mathbb{C})$.

Theorem 3.31. *Let (π, V) and (ρ, W) be two irreducible representations of $\mathfrak{g}_{\mathbb{C}}$. If (π, V) and (ρ, W) have the same highest weight, then (π, V) and (ρ, W) are equivalent.*

Proof. Let $v \in V$ and $w \in W$ be highest weight vectors with highest weight λ. Consider the direct sum of the representations (π, V) and (ρ, W), namely the representation $(\pi \oplus \rho, V \oplus W)$ defined in Definition 2.50. Let U denote the smallest invariant subspace of $V \oplus W$ that contains (v, w). By definition of the direct sum of representations it follows that (v, w) is a highest weight vector with weight λ. Hence U contains a highest weight vector that is also cyclic. Note that, the representation $(\pi \oplus \rho, V \oplus W)$ is completely reducible by construction. Hence, by Corollary 2.41 and the one-to-one correspondence between irreducibles of G and $\mathfrak{g}_{\mathbb{C}}$ it follows that U is completely reducible. Therefore, by Lemma 3.30 the representation $(\left.(\pi \oplus \rho)\right|_U, U)$ is irreducible.

Consider the projection maps $p_V : V \oplus W \to V$ and $p_W : V \oplus W \to W$. Then for $v' \in V, w' \in W$ and $x \in G$ we see

$$\begin{aligned} p_V((\pi \oplus \rho)(x)(v', w')) &= p_V(\pi(x)v', \rho(x)w') \\ &= \pi(x)v' \\ &= \pi(x)p_V(v', w'). \end{aligned}$$

Hence p_V intertwines $(\pi \oplus \rho, V \oplus W)$ with (π, V). In a similar way we see that p_W intertwines $(\pi \oplus \rho, V \oplus W)$ with (ρ, W). Therefore, $\left.p_V\right|_U$ and $\left.p_W\right|_U$ intertwine $(\left.(\pi \oplus \rho)\right|_U, U)$ with (π, V) and (ρ, W), respectively. Note that $\left.p_V\right|_U$ and $\left.p_W\right|_U$ are not trivial since $\left.p_V\right|_U(v, w) = v$ and $\left.p_W\right|_U(v, w) = w$. Then by Schur's lemma, $\left.p_V\right|_U$ is an isomorphism between U and V and $\left.p_W\right|_U$ is an isomorphism between U and W. Hence V and W are isomorphic. Note that, this isomorphism is also equivariant. Therefore (π, V) and (ρ, W) are equivalent. ■

3.5 The representations of $\mathfrak{sl}(3, \mathbb{C})$

In his section we will be looking at the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. More specifically, we want to classify its irreducible representations. To accomplish this we will extensively use the theory from the previous section. Recall, the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ are relevant for the Eightfold Way since they correspond to the irreducible representations of $SU(3)$. Therefore, this section will be the main mathematical background for the next chapter. In this section we use [9, Ch. 6] and [6, Ch. 12] as main reference.

Similarly to $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ consists of traceless complex 3×3 matrices. For which we choose the following basis

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Again, similar to $\mathfrak{sl}(2,\mathbb{C})$, we are interested in the commutation relations between the basis elements. Before one starts computing commutators, it is worth to take a closer look at some specific basis elements. Note that, by forgetting the last row and last column we see that the subalgebra generated by H_1, X_1 and Y_1 is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. We denote this subalgebra by $\mathfrak{s}_1$. Similarly, the subalgebra generated by H_2, X_2 and Y_2 is also isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and we denote it by $\mathfrak{s}_2$. Hence, the commutation relations between these elements is know

$$\begin{aligned} &[H_1, X_1] = 2X_1, \quad [H_1, Y_1] = 2Y_1, \quad [X_1, Y_1] = H_1, \\ &[H_2, X_2] = 2X_2, \quad [H_2, Y_2] = 2Y_2, \quad [X_2, Y_2] = H_2. \end{aligned}$$

After straightforward calculation, the other commutation relations are given by

$$[H_1, H_2] = 0;$$

$$\begin{aligned} &[H_1, X_2] = -X_2, \quad [H_1, Y_2] = Y_2, \quad [H_1, X_3] = X_3, \quad [H_1, Y_3] = -Y_3, \\ &[H_2, X_1] = -X_1, \quad [H_2, Y_1] = Y_1, \quad [H_2, X_3] = X_3, \quad [H_2, Y_3] = -Y_3; \end{aligned}$$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = 0, \quad [X_1, Y_2] = 0, \quad [X_1, Y_3] = -Y_2;$$

$$[X_2, X_3] = 0, \quad [X_2, Y_1] = 0, \quad [X_2, Y_3] = Y_1;$$

$$[X_3, Y_1] = -X_2, \quad [X_3, Y_2] = X_1, \quad [X_3, Y_3] = H_1 + H_2;$$

$$[Y_1, Y_2] = -Y_3, \quad [Y_1, Y_3] = 0, \quad [Y_2, Y_3] = 0.$$

Since H_1 and H_2 commute we have found candidates for a basis of our maximal torus $\mathfrak{t}$. Just as in the case of $\mathfrak{sl}(2,\mathbb{C})$ we define $\mathfrak{t} \subset \mathfrak{su}(3)$ to be the real Lie algebra generated by

iH_1 and iH_2. Then, $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is the complex Lie generated by H_1 and H_2. Hence,

$$\mathfrak{t}_{\mathbb{C}} = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \mid d_1, d_2, d_3 \in \mathbb{C},\ d_1 + d_2 + d_3 = 0 \right\}. \tag{3.12}$$

Now that we have defined the maximal torus, we can talk about weights. Since $\mathfrak{sl}(3, \mathbb{C})$ contains subalgebras isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ we can use the theory of $\mathfrak{sl}(2, \mathbb{C})$.

Lemma 3.32. *Let (π, V) be a representation of $\mathfrak{sl}(3, \mathbb{C})$ and $\lambda \in \Lambda(\pi)$. Then $\lambda(H_1)$ and $\lambda(H_2)$ are integers.*

Proof. First note that, since $\lambda \in \Lambda(\pi)$ is a weight, $\lambda(H_1)$ and $\lambda(H_2)$ are eigenvalues of $\pi(H_1)$ and $\pi(H_2)$, respectively. Consider the restriction of π to the subalgebra generated by H_1, X_1, Y_1. By Corollary 3.17 we see $\lambda(H_1)$ is an integer. Similarly, by restricting to the subalgebra generated by H_2, X_2, Y_2 it follows that $\lambda(H_2)$ is an integer. ■

We want to apply the theory of the previous section to $\mathfrak{sl}(3, \mathbb{C})$. Before we can achieve that, we need to understand the root system of $\mathfrak{sl}(3, \mathbb{C})$. First we try to find roots by straightforward computation, let us make an observation. Let M be a matrix and D a diagonal matrix with entries d_i on the diagonal. Then, left multiplication of M by D multiplies the ith row of M by d_i. On the other hand, right multiplication of M by D multiplies by the ith column of M by d_i. Let the entries of M be m_{ij}, then the entries of the commutator $[D, M]$ are

$$(d_i - d_j)m_{ij}. \tag{3.13}$$

Observe that the commutator $[D, M]$ can only be a nonzero multiple of M if every entry m_{ij} is zero but one.

Applying this observation to $\mathfrak{sl}(3, \mathbb{C})$ we see, using the commutation relations, there are six roots. The corresponding root spaces are $\mathbb{C}X_i$ and $\mathbb{C}Y_i$ for $i = 1, 2, 3$, also by the above discussion. To obtain neat expressions for these roots, let us define $L_i \in \mathfrak{t}_{\mathbb{C}}^*$ for $i = 1, 2, 3$ by

$$L_i(\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}) = d_i.$$

Since $\mathfrak{t}_{\mathbb{C}}$ has a very explicit form given in Equation (3.12), we can express $\mathfrak{t}_{\mathbb{C}}^*$ in terms of the L_i's

$$\mathfrak{t}_{\mathbb{C}}^* = \mathbb{C}L_1 \oplus \mathbb{C}L_2 \oplus \mathbb{C}L_3 / \mathbb{C}(L_1 + L_2 + L_3).$$

The quotient in the above equation is needed to account for the tracelessness of elements of $\mathfrak{t}_{\mathbb{C}}$. Then by (3.13) the roots of $\mathfrak{sl}(3, \mathbb{C})$ are $L_i - L_j$ for $i \neq j$. Let us give some more explicit names to the roots. Define

$$\alpha_1 := L_1 - L_2, \quad \alpha_2 := L_2 - L_3, \quad \alpha_3 := \alpha_1 + \alpha_2$$

The other roots are then given by $-\alpha_1, -\alpha_2$ and $-\alpha_3$.

One could verify, there exists a Weyl chamber such that $\alpha_1, \alpha_2, \alpha_3 \in R^+$. This verification is done by explicit calculations of the kernels of the roots. We denote this Weyl chamber by $\mathcal{C}$ and keep it fixed throughout this section. Since $\mathfrak{g}_{\mathbb{C}\alpha_i} = \mathbb{C}X_i$ for $i = 1, 2, 3$, as said before, we conclude $\mathfrak{g}_{\mathbb{C}}^+$ is the subalgebra generated by X_1,X_2 and X_3. Similarly, $\mathfrak{g}_{\mathbb{C}}^-$ is the subalgebra generated by Y_1,Y_2 and Y_3. We call α_1 and α_2 are **positive simple roots**, for some background about this naming we refer to [9, p. 206].

Now we are ready to apply the theory of the previous sections. One observation we can make directly is that the difference of two weights, for an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$, is a linear combination of roots with integer coefficients. Since every vector is cyclic in an irreducible representation and by Lemma 3.25, this observation follows from Lemma 3.27(c). By Λ_R we denote the lattice spanned by the α_i's and call it the **root lattice**. Note that the root lattice is made of special points on a finer lattice, namely the lattice spanned by L_1,L_2 and L_3. This lattice is called the **weight lattice** and denoted by Λ_W. It turns out these lattices give a very neat visual depiction of the representation theory of $\mathfrak{sl}(3,\mathbb{C})$. Furthermore, these lattices are very symmetrical if an appropriate Hermitian inner product is applied.

A beneficial way to construct this appropriate inner product is by starting with a Hermitian inner product on $\mathfrak{t}_{\mathbb{C}}$ and then lift it to $\mathfrak{t}_{\mathbb{C}}$. We choose the following Hermitian inner product on $\mathfrak{t}_{\mathbb{C}}$, for $H, H' \in \mathfrak{t}_{\mathbb{C}}$

$$\langle H, H' \rangle = \operatorname{tr}(H^* H'). \tag{3.14}$$

Note that, the Hermitian inner product defined above is linear in the second argument, which is not standard in mathematics. Since $\mathfrak{t}_{\mathbb{C}}$ is a finite dimensional vector space, it follows by the Riesz representation theorem[7] that for every functional $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ there exists a unique element $L \in \mathfrak{t}_{\mathbb{C}}$ such that

$$\lambda(H) = \langle L, H \rangle \tag{3.15}$$

for all $H \in \mathfrak{t}_{\mathbb{C}}$. This gives us a way to identify elements of $\mathfrak{t}_{\mathbb{C}}$ with elements of $\mathfrak{t}_{\mathbb{C}}^*$. After some thought one sees that the corresponding elements of α_1 and α_2 are H_1 and H_2, respectively. From now on we will use the same letter for the functional and the corresponding element in $\mathfrak{t}_{\mathbb{C}}$. In this setting our notion of a weight changes a bit. If (π, V) is a representation of $\mathfrak{sl}(3,\mathbb{C})$ (can also be done more general), then we say $\lambda \in \mathfrak{t}_{\mathbb{C}}$ is a weight when there exists a nonzero vector $v \in V$ such that

$$\pi(H)v = \langle \lambda, H \rangle v$$

for all $H \in \mathfrak{t}_{\mathbb{C}}$.

We would like to introduce some terminology. Let λ be a weight, then we say λ is **integral** if both $\langle \lambda, H_1 \rangle$ and $\langle \lambda, H_2 \rangle$ are integers. Furthermore, λ is called **dominant** if $\langle \lambda, H_1 \rangle \geq 0$ and $\langle \lambda, H_2 \rangle \geq 0$. We are interested in the weights, let us denote them by ω_1

[7]For more background about this theorem we refer to [14, Thm. 5.2].

and ω_2, such that

$$\langle \omega_1, \alpha_1 \rangle = 1, \quad \langle \omega_1, \alpha_2 \rangle = 0$$
$$\langle \omega_2, \alpha_1 \rangle = 0, \quad \langle \omega_2, \alpha_2 \rangle = 1.$$

These weights are called the **fundamental weights**. This name comes from the identification of α_i with H_i, for $i = 1, 2$, and Equation (3.15). It follows that ω_1 is the weight such that $\omega(H_1) = \langle \omega_1, H_1 \rangle = 1$ and $\omega_1(H_2) = \langle \omega_1, H_2 \rangle = 0$. Similarly for ω_2, it is the weight such that $\omega(H_2) = \langle \omega_2, H_1 \rangle = 0$ and $\omega_2(H_2) = \langle \omega_2, H_2 \rangle = 1$. Hence ω_1 and ω_2 are the most basic nonzero dominant integral elements. Consequently, every dominant integral element must be of the form $n_1\omega_1 + n_2\omega_2$ with $n_1, n_2 \in \mathbb{N}$. By Lemma 3.7 we know $\omega_1, \omega_2 \in i\mathfrak{t}^*$, which was isomorphic to $(i\mathfrak{t})^*$. Therefore, for the corresponding element of ω_1 in $\mathfrak{t}_\mathbb{C}$ we can write $\omega_1 = aH_1 + bH_2$, with $a, b \in \mathbb{R}$. Then, the equations $\langle \omega_1, H_1 \rangle = 1$ and $\langle \omega_1, H_2 \rangle = 0$ yield a system of two equations in a and b. Solving this system gives $a = \frac{2}{3}$ and $b = \frac{1}{3}$. A similar computation can be done for ω_2. After this computation, one finds the corresponding elements in $\mathfrak{t}_\mathbb{C}$ are given by

$$\omega_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}.$$

Then we can write $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -\omega_1 + 2\omega_2$. Note α_1 and α_2 have length $\sqrt{2}$ and $\langle \alpha_1, \alpha_2 \rangle = -1$. Hence, the angle θ between α_1 and α_2 equals $\frac{2}{3}\pi$. Using this we can create an elegant and symmetrical depiction of the roots within the weight lattice (see Figure 3.1).

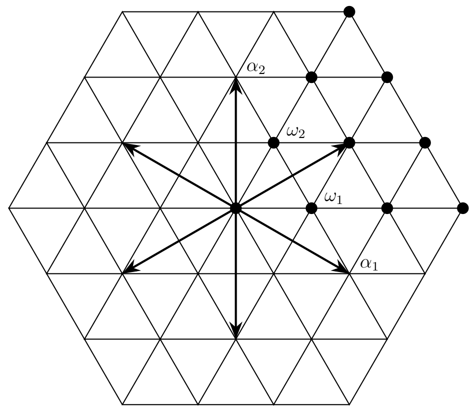

FIGURE 3.1: The weight lattice for $\mathfrak{sl}(3, \mathbb{C})$ relative to the Hermitain inner product defined in Equation (3.14). The arrows denotes the roots and the black dots are the dominant integral elements. Furthermore, the fundamental weights are depicted.

Now we have enough specific background to state the main theorem about irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. We follow the proof of [9, Prop. 6.17].

Theorem 3.33. (a) *Two irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$, with the same highest weight, are isomorphic.*

(b) *If λ is the highest weight of an irreducible representation (π, V) of $\mathfrak{sl}(3,\mathbb{C})$, then there exists $n_1, n_2 \in \mathbb{N}$ such that $\lambda = n_1\omega_1 + n_2\omega_2$.*

(c) *For every pair $n_1, n_2 \in \mathbb{N}$, there exists an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight $\lambda := n_1\omega_1 + n_2\omega_2$.*

Proof. Part (a) is a direct consequence of Theorem 3.31.

Now we will prove part (b). Suppose (π, V) is an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight λ. By Lemma 3.32 it follows that $\lambda(H_i) = \langle\lambda, H_i\rangle \in \mathbb{Z}$ for $i = 1, 2$. Let v be a highest weight vector corresponding to λ. Then, by definition we know $\pi(X_1)v = 0$ and $\pi(X_2)v = 0$. If we restrict the representation to $\mathfrak{s}_1$ and $\mathfrak{s}_2$ we can apply our description of $\mathfrak{sl}(2,\mathbb{C})$. In the proof of Lemma 3.27 we discussed that $\mu := \lambda(H_1) + 2N$ equals a natural number[8]. Recall that $N \in \mathbb{N}$ was the natural number such that $\pi(X_1)^N v \neq 0$, but $\pi(X)^{N+1}v = 0$. In this particular case we have $N = 0$, since v is a highest weight vector. Therefore $\langle\lambda, H_1\rangle = \lambda(H_1) = \mu - 2N = \mu \in \mathbb{N}$. A similar argument shows that $\langle\lambda, H_2\rangle = \lambda(H_2) \in \mathbb{N}$. So, λ is a dominant integral element and we can write $\lambda = \langle\lambda, H_1\rangle\omega_1 + \langle\lambda, H_2\rangle\omega_2$. This completes the proof of part (b).

Finally, we will prove part (c). This will be a very constructive proof. We will start by constructing irreducible representations with highest weights ω_1 and ω_2 and then using tensor products to construct the rest. But first we note that, the trivial representation has highest weight $0 \cdot \omega_1 + 0 \cdot \omega_2$ and the trivial representation is irreducible. Consider the standard representation of $\mathfrak{sl}(3,\mathbb{C})$ acting on $\mathbb{C}^3$. It is easily seen that e_1 is a highest weight vector with highest weight ω_1. Now it is left to show that this representation is irreducible. Since $\mathfrak{sl}(3,\mathbb{C})$ is the complexification of a Lie algebra coming from a compact connected simply-connected Lie group, the standard representation is completely reducible. Hence we can write

$$\mathbb{C}^3 = V_1 \oplus \cdots \oplus V_k,$$

where $V_i \subset \mathbb{C}^3$ is an irreducible subspace for every i. Define a subalgebra[9] of $\mathfrak{sl}(3,\mathbb{C})$ by

$$\mathfrak{n} := \mathbb{C}X_1 \oplus \mathbb{C}X_2 \oplus \mathbb{C}X_3.$$

Furtheremore, we define a subspace of $\mathbb{C}^3$ by

$$(\mathbb{C}^3)^{\mathfrak{n}} = \{v \in \mathbb{C}^3 \mid Xv = 0, \text{ for all } X \in \mathfrak{n}\}.$$

Note that

$$(\mathbb{C}^3)^{\mathfrak{n}} = V_1^{\mathfrak{n}} \oplus \cdots \oplus V_k^{\mathfrak{n}}$$

By Lemma 3.28 it follows that $V_i^{\mathfrak{n}}$ is one dimensional for every i. Therefore, if $\dim(\mathbb{C}^3)^{\mathfrak{n}} = 1$ then the standard representation is irreducible. By explicit calculation we see

$$(\mathbb{C})^{\mathfrak{n}} = \mathbb{C}e_1.$$

[8]Note that, we do not use irreducibility here. The natural number we refer to is the one described underneath Lemma 3.15.

[9]This subalgebra is actually $\mathfrak{g}_{\mathbb{C}}^{+}$, but for simplicity we denote it by $\mathfrak{n}$.

Now we consider the dual of the standard representation, given by

$$\pi^\vee(X) = -X^*$$

for $X \in \mathfrak{sl}(3,\mathbb{C})$. It is easily shown that $X^* = X^T$, where T denotes the usual transpose *not* the conjugate transpose. By a similar argument we see that this representation is irreducible. One can easily check that e_3 is a highest weight vector with highest weight ω_2. Let (π_1, V_1) and (π_2, V_2) denote the standard representation and its dual, respectively. Furthermore, let $v_1 = e_1$ and $v_2 = e_3$ be the respective highest weight vectors. Then define the representation π_{n_1,n_2} of $\mathfrak{sl}(3,\mathbb{C})$ in

$$V := V_1 \otimes \cdots \otimes V_1 \otimes V_2 \otimes \cdots \otimes V_2,$$

by

$$\pi_{n_1,n_2}(X) := (\pi_1 \otimes \cdots \otimes \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_2)(X),$$

where π_1 and V_1 occur n_1 times and π_2 and V_2 occur n_2 times. One readily sees that

$$v := \underbrace{v_1 \otimes \cdots \otimes v_1}_{n_1 \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{n_2 \text{ times}}$$

is a highest weight vector with highest weight $n_1\omega_1 + n_2\omega_2$. Let $W \subset V$ be the smallest invariant subspace that contains v. If we know (π_{n_1,n_2}, W) is completely reducible, Lemma 3.30 implies that the representation is irreducible. Which would conclude our proof of part (c).

So, it remains to show that (π_{n_1,n_2}, W) is completely reducible. As we have seen before, for $X \in \mathfrak{su}(3)$ we have $X^* = -X$. Hence, by Lemma 2.37 it follows that the lift of the standard representation to a representation of $SU(3)$ is unitary (which is the standard representation of G). On the other hand, for the dual representation we see

$$\begin{aligned}(\pi^\vee(X))^* &= (-X^T)^* \\ &= -\overline{X} \\ &= X^T \\ &= -\pi^\vee(X)\end{aligned}$$

for $X \in \mathfrak{su}(3)$. Here we used that $-X = X^* = \overline{X}^T$ for $X \in \mathfrak{su}(3)$. Again, by Lemma 2.37 we see the lift of (π_2, V) to a representation of $SU(3)$ is unitary. Using the properties of the tensor product, if V and W are two inner product spaces there exists a unique inner product[10] $\langle\cdot,\cdot\rangle$ on $V \otimes W$ such that

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \langle v_1, v_2\rangle_V \langle w_1, w_2\rangle_W.$$

If we start with the inner products on V_1 and V_2 for which the lifts of π_1 and π_2 are unitary, this property gives an inner product on V. One readily sees that the lift of π_{n_1,n_2}

[10] This follows from the universal property of the tensor product ([1, Def. 2.8]).

is unitary with respect to this inner product[11]. Let us denote this lift by $\tilde{\pi}_{n_1,n_2}$. Then, by Theorem 2.40 we see $(\tilde{\pi}_{n_1,n_2}, V)$ is completely reducible, since $SU(3)$ is compact. Hence, by Corollary 2.41 and the one-to-one correspondence between irreducibles of $SU(3)$ and $\mathfrak{sl}(3,\mathbb{C})$ it follows that (π_{n_1,n_2}, W) is completely reducible. ∎

The proof of Theorem 3.33 was quite involved. But it gives a way to construct an irreducible representation with a particular highest weight. An example of such a construction can be found in [9, Sec. 6.5]. Let us introduce some notation. Let (π, V) be the irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight $\lambda = n_1\omega_1 + n_2\omega_2$ constructed in Theorem 3.33. Then we denote V by $R(n_1, n_2)$.

The last part of this section is devoted to understanding the structure of these irreducible representations, such as which other weights does the representation have and what are the multiplicities. An important tool to accomplish is the so-called the *Weyl group.* Let $\alpha \in R$ be a root. Then by $s_\alpha : i\mathfrak{t} \to i\mathfrak{t}$ we denote the orthogonal reflection, with respect to the inner product defined in Equation (3.14), in $\ker(\alpha)$ in $i\mathfrak{t}$. Since $s_\alpha(\alpha) = -\alpha$ and $s_\alpha = \mathbb{1}$ on $\ker(\alpha)$ we deduce

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha.$$

Definition 3.34. *The **Weyl group**, denoted by W, of the pair $(i\mathfrak{t}, R)$ is the group generated by the s_α's for $\alpha \in R$.*

Since we know all the roots of $\mathfrak{sl}(3,\mathbb{C})$, we easily see W is the group generated by s_{α_1}, s_{α_2} and s_{α_3}. Note that, this group corresponds to the symmetry group of an equilateral triangle, which is S_3. Hence, W is isomorphic to S_3. We note that, for a representation (π, V) of $\mathfrak{sl}(3,\mathbb{C})$, W defines an action on $\Lambda(\pi)$ by applying s_{α_i} to $\lambda \in \Lambda(\pi)$, with $i = 1, 2, 3$.

Remark 3.35. The inner product defined in (3.14) is Weyl group invariant.

Proof. Note that s_α, for $\alpha \in R$, is defined as the orthogonal reflection with respect to the inner product defined in (3.14). Therefore, the inner product is s_α-invariant. Since W is generated by s_α we conclude the inner product is Weyl group invariant. ∎

The following theorem is a special case of [9, Thm. 9.3] and we adapted the proof to the case of $\mathfrak{sl}(3,\mathbb{C})$.

Theorem 3.36. *Let (π, V) be a representation of $\mathfrak{sl}(3,\mathbb{C})$. If $\lambda \in \Lambda(\pi)$ is a weight, then $w \cdot \lambda$ is also a weight with the same multiplicity for all $w \in W$.*

Proof. Since W is generated by s_i, for $i = 1, 2, 3$, it suffices to prove the claim for those elements. Let α_i be fixed, with $i = 1, 2, 3$. Then we define the operator

$$S_{\alpha_i} := e^{\pi(X_i)}e^{-\pi(Y_i)}e^{\pi(X_i)}.$$

One can show that

$$S_{\alpha_i}\pi(H_i)S_{\alpha_i}^{-1} = -\pi(H_i).$$

[11] Here we use that, for $SU(3)$, the lift of a tensor product representation is the tensor product representation of the lifts. This follows from the unique lifting property of $SU(3)$.

For a proof of this claim we refer to [9, Thm. 4.34(3)]. Note that, $-\pi(H_i) = \pi(s_{\alpha_i} \cdot H_i)$ since H_i is the element corresponding to α_i. On the other hand, if $\langle \alpha_i, H \rangle = 0$ for $H \in \mathfrak{t}_{\mathbb{C}}$ then $\pi(H)$ commutes with $\pi(X_i)$ and $\pi(Y_i)$. Note that, such an element H is contained in $\ker \alpha_i$. Since s_{α_i} is defined by its action on $\ker \alpha_i$ and $(\ker \alpha_i)^{\perp}$ we conclude

$$S_{\alpha_i} \pi(H) S_{\alpha_i}^{-1} = \pi(s_{\alpha_i} \cdot H), \tag{3.16}$$

for every $H \in \mathfrak{t}_{\mathbb{C}}$. Suppose v is a weight vector with weight λ, then for $H \in \mathfrak{t}_{\mathbb{C}}$ Equation 3.16 tells us

$$\pi(H) S_{\alpha_i}^{-1} v = S_{\alpha_i}^{-1} \pi(s_{\alpha_i} \cdot H) v.$$

In the case of $\mathfrak{sl}(3, \mathbb{C})$ one can easily check, through explicit calculations, that $s_{\alpha_i} \cdot H_1$ and $s_{\alpha_i} \cdot H_2$ always equal a linear combination of H_1 and H_2 (for $i = 1, 2, 3$). Since H_1, H_2 form a basis for $\mathfrak{t}_{\mathbb{C}}$ we conclude $s_{\alpha_i} \cdot \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{t}_{\mathbb{C}}$, for $i = 1, 2, 3$. Therefore, since λ is a weight, it follows that

$$\begin{aligned} S_{\alpha_i}^{-1} \pi(s_{\alpha_i} \cdot H) v &= S_{\alpha_i}^{-1} \langle \lambda, s_{\alpha_i} \cdot H \rangle v \\ &= \langle s_{\alpha_i}^{-1} \cdot \lambda, H \rangle S_{\alpha_i}^{-1} v. \end{aligned}$$

Note that, the Weyl group invariance of the inner product is used in the last step of the above computation. The calculation shows that $S_{\alpha_i}^{-1} v$ is a weight vector for $s_{\alpha_i}^{-1} \cdot \lambda$. Hence, $s_{\alpha_i}^{-1} \cdot \lambda$ is a weight. The argument shows that $S_{\alpha_i}^{-1}$ maps V_λ to $V_{s_{\alpha_i}^{-1} \cdot \lambda}$. An almost identical argument show shows that S_{α_i} maps $V_{s_{\alpha_i}^{-1} \cdot \lambda}$ to V_λ, hence the two spaces are isomorphic. This means that λ and $s_{\alpha_i}^{-1} \cdot \lambda$ have the same multiplicity. Note that, $s_{\alpha_i}^{-1} = s_{\alpha_i}$. Therefore, we have deduced that weights are invariant under s_{α_i} and thus invariant under the action of W. ■

Definition 3.37. *For $v_1, \ldots, v_n$ vectors in a real or complex vector space, we define the* ***convex hull*** *of $v_1, \ldots, v_n$ as the collection of vectors*

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where the coefficients satisfy $c_i \geq 0$ for every i and $\sum_i c_i = 1$.

In other words, the convex hull is the smallest convex set containing the vectors v_i. The proof of the following theorem is from [9, Thm. 6.24].

Theorem 3.38. *Let λ be a dominant integral element and (π, V) the irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight λ. If $\mu \in \Lambda(\pi)$ is a weight, then μ belongs to the convex hull of $W \cdot \lambda$.*

Proof. Let $\mu \in \Lambda(\pi)$ be a weight and denote the convex hull of $W \cdot \mu$ by $\mathrm{Conv}(W \cdot \lambda)$. Then by Theorem 3.36 it follows $w \cdot \mu \in \Lambda(\pi)$ for all $w \in W$. Note that, if $\mu \in W \cdot \lambda$ the claim is evident, by the definition of the convex hull. So suppose $\mu \notin W \cdot \lambda$. In particular, one readily sees there exists an element $w \in W$ such that $w \cdot \mu$ is dominant. Hence, for this particular element $w \in W$ the weight $w \cdot \mu$ lies in the closure of the positive Weyl chamber C (which is the top-right pie slice in Figure 3.2). Since λ is the highest weight, $w \cdot \mu$ must lie in the intersection of $\mathrm{Conv}(W \cdot \mu)$ and $\overline{\mathcal{C}}$ (this is the dark shaded area depicted in Figure 3.2). Otherwise, if $w \cdot \mu$ was in the complement of $\mathrm{Conv} \cap \overline{\mathcal{C}}$ in $\overline{\mathcal{C}}$ (the light shaded area in Figure 3.2) we would have that $\mu - \lambda \in \mathbb{N}R^+$. Since $w \cdot \mu \neq \lambda$ by assumption, this would contradict Lemma 3.27(c). Let us define this $Q_\lambda := \mathrm{Conv} \cap \overline{\mathcal{C}}$, so $w \cdot \mu \in Q_\lambda$. We want to show that Q_μ is contained in $\mathrm{Conv}(W \cdot \lambda)$. From Figure 3.2 we see Q_μ is convex, therefore it suffices to show the vertices are contained in $\mathrm{Conv}(W \cdot \lambda)$.

Define,

$$\nu := \sum_{w \in W} w \cdot \mu.$$

Note that, $w \cdot \nu = \nu$ for all $w \in W$, by construction. In particular, $s_{\alpha_i} \cdot \nu = \nu$ for all $i = 1, 2, 3$. By the definition of s_{α_i}, it follows that $\nu \in \ker \alpha_i$ for every $i = 1, 2, 3$. One could verify, by explicit computation,

$$\bigcap_{i=1}^{3} \ker \alpha_i = 0.$$

Hence, $\nu = 0$. Consequently, $0 \in \mathrm{Conv}(W \cdot \lambda)$. We define the point p_1 as the intersection of $\ker \alpha_1$ and the line segment connecting λ and $s_{\alpha_1}(\lambda)$. Similarly, the point p_2 is defined as the intersection of $\ker \alpha_2$ and the line segment connecting λ and $s_{\alpha_2}(\lambda)$ (see Figure 3.2). Explicitly, the points are given by

$$p_1 = \frac{1}{2}(\lambda + s_{\alpha_2}(\lambda)), \quad p_2 = \frac{1}{2}(\lambda + s_{\alpha_1}(\lambda)).$$

From these expressions we immediately see $p_1, p_2 \in \mathrm{Conv}(W \cdot \lambda)$, by definition. Obviously, $\lambda \in \mathrm{Conv}(W \cdot \lambda)$ and thus all vertices of Q_μ are contained in $\mathrm{Conv}(W \cdot \lambda)$. Hence, $w \cdot \mu \in Q_\mu \subset \mathrm{Conv}(W \cdot \lambda)$. Clearly, $W \cdot \lambda$ is invariant under the action of W. Therefore, $\mathrm{Conv}(W \cdot \lambda)$ is invariant under W. Consequently, $\mu = w^{-1} \cdot (w \cdot \mu) \in \mathrm{Conv}(W \cdot \lambda)$ since $w \cdot \mu$ is contained in $\mathrm{Conv}(W \cdot \lambda)$. ■

For the proof of the following lemma we use [9, Lemma 6.26].

Lemma 3.39. *Let (π, V) be an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight λ. Let μ be a weight, α a root, and suppose ν is a point on the line segment joining μ and $s_\alpha \cdot \mu$ such that $\mu - \nu$ is an integer multiple of α. Then ν is a weight of (π, V).*

Proof. Since the orthogonal reflections s_{α_i} and $s_{-\alpha_i}$ are equal, suffices to only consider s_{α_i} for $i = 1, 2, 3$. First, let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be as before and let $\mathfrak{s}_3$ denote the subalgebra generated by $H_3 := H_1 + H_2$, X_3 and Y_3. Furthermore, let us denote the line segment connecting μ and $s_\alpha \cdot \mu$ by $[\mu, s_\alpha \cdot \mu]$. Then the claim of the lemma translates to

$$[\mu, s_\alpha \cdot \mu] \cap (\mu + \mathbb{Z}\alpha) \subset \Lambda(\pi). \tag{3.17}$$

Let $i = 1, 2, 3$ be a fixed index. Note that,

$$\begin{aligned} \langle s_{\alpha_i} \cdot \mu, H_i \rangle &= \langle \mu, s_{\alpha_i} \cdot H_i \rangle \\ &= -\langle \mu, H_i \rangle. \end{aligned}$$

In the above calculation we used a couple properties. Firstly, we used the Weyl group invariance of the inner product. Secondly, we applied the fact that H_i is the element that

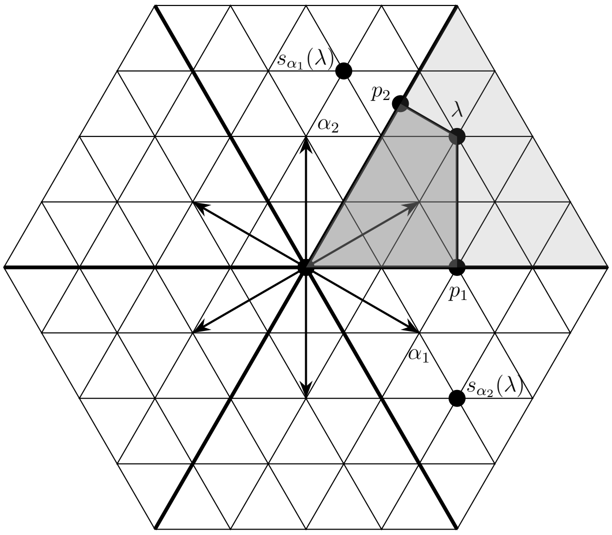

FIGURE 3.2: Weight diagram of (π, V) with highest weight λ. The light shaded area depicts the complement of $\mathrm{Conv}(W\cdot\lambda)\cap\overline{\mathcal{C}}$ in $\overline{\mathcal{C}}$. These dominant elements are said to be 'higher' than λ. And the dark shaded area depicts $\mathrm{Conv}(W\cdot\lambda)\cap\overline{\mathcal{C}}$, which are the dominant elements 'lower' than λ.

corresponds to α_i, hence $s_{\alpha_i}\cdot H_i = -H_i$. Finally, we used that $s_{\alpha_i}^2 = \mathbb{1}$. By the above computation we can assume, without loss of generality, that $k := \langle \mu, H_i\rangle \geq 0$. By explicit calculation, we see

$$\begin{aligned} s_{\alpha_i}\cdot\mu &= \mu - \langle \mu, \alpha_i\rangle\alpha_i \\ &= \mu - \langle \mu, H_i\rangle\alpha_i \\ &= \mu - k\alpha_i. \end{aligned}$$

Hence,

$$[\mu, s_\alpha\cdot\mu]\cap(\mu+\mathbb{Z}\alpha) = \{\mu, \mu-\alpha_i, \ldots, \mu-k\alpha_i\}. \tag{3.18}$$

Let $v \in V\setminus\{0\}$ be a weight vector with weight μ. In particular, v is an eigenvector of $\pi(H_i)$ with eigenvalue $\mu(H_i) = \langle\mu, H_i\rangle = k$. Let $U \subset V$ be the smallest $\mathfrak{s}_i$-invariant subspace containing v. Then, $(\pi|_{\mathfrak{s}_i}, U)$ induces a representation of $\mathfrak{sl}(2,\mathbb{C})$ (not necessarily irreducible). By Corollary 3.17 it follows that $k = \mu(H_i) \in \mathbb{Z}$. Yet, $k \geq 0$ so $k \in \mathbb{N}$. Moreover, by repeated use of Lemma 3.13 we see $\pi(Y_i)^l v$ is a nonzero eigenvector of $\pi(H_i)$ with eigenvalue $k-2l$, for every $l = 0, \ldots, k$. This shows that all the values $-k, -k+2, \ldots, k-2, k$ are eigenvalues of $\pi(H_i)$. Recall, Y_i is a root vector for α_i. Therefore $\pi(Y_i)v \in \pi(\mathfrak{g}_{\mathbb{C}\alpha_i})^l v$. By Lemma 3.10 it follows

$$\pi(\mathfrak{g}_{\mathbb{C}\alpha_i})^l v \subset V_{\mu - l\alpha_i}.$$

Consequently, $\pi(Y_i)v \in V_{\mu-l\alpha_i}$. Since $\pi(Y_i)v$ is nonzero we conclude $\mu - l\alpha_i$ is a weight, for every $l = 0, \ldots, k$. This shows

$$\{\mu, \mu-\alpha_i, \ldots, \mu-k\alpha_i\} \subset \Lambda(\pi),$$

which was the desired result stated in Equation (3.17). ■

We will use this lemma to characterize the weights of an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$. This characterization is captured in the following theorem. For the proof we use [9, Thm. 6.25].

Theorem 3.40. *Let* (π, V) *be an irreducible representation of* $\mathfrak{sl}(3, \mathbb{C})$ *with highest weight* λ. *If* μ *is an integral element satisfying*

(a) $\lambda - \mu$ *can be expressed as a linear combination of roots;*

(b) μ *is contained in* $\mathrm{Conv}(W \cdot \lambda)$,

then μ *is a weight for* (π, V).

Proof. Suppose μ is an integral element satisfying the above conditions. Then, we can write $\mu = \lambda - n_1\alpha_1 - n_2\alpha_2$. Since μ must lie in the convex hull of $W \cdot \lambda$, we conclude $n_1, n_2 \geq 0$. Let us assume $n_1 \geq n_2$. We can rewrite the previous expression into

$$\begin{aligned} \mu &= \lambda - (n_1 - n_2)\alpha_1 - n_2(\alpha_1 + \alpha_2) \\ &= \lambda - (n_1 - n_2)\alpha_1 - n_2\alpha_3. \end{aligned}$$

Now, if we travel in the direction of α_3 we will hit the boundary of $\mathrm{Conv}(W \cdot \lambda)$ at a given moment. Denote this point by ν. Since $\mu \in \mathrm{Conv}(W \cdot \lambda)$ and the assumption that $n_1 \geq n_2$, we know ν must lie on $[\lambda, s_{\alpha_1} \cdot \lambda]$. Note that, every point on $[\lambda, s_{\alpha_1} \cdot \lambda]$ is of the form $\lambda - l\alpha_1$, with $l \in \mathbb{N}$ (see Equation (3.18)). Hence, the point where we hit the boundary is given by

$$\nu = \mu - (n_1 - n_2)\alpha_1,$$

since we only moved in the direction of α_3. Note that, ν is on the line segment connecting λ and $s_{\alpha_1} \cdot \lambda$. Furthermore, $\lambda - \nu$ is an integer multiple of α_1, namely $(n_1 - n_2)\alpha_1$. Therefore, by Lemma 3.39 we see ν is a weight of (π, V). By construction μ is on the line segment connecting ν and $s_{\alpha_3} \cdot \nu$ and the difference $\nu - \mu$ an integer multiple of α_3, namely $n_2\alpha_3$. Hence, again by Lemma 3.39 it follows that μ is a weight of (π, V). If $n_2 \geq n_1$, a similar argument would show μ is a weight of (π, V) but with switched roles for α_1 and α_2. ■

At this point we have characterized all irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ and understand their weight structure. To conclude our discussion of the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$, we will give a formula for the dimension of the irreducible representations.

Theorem 3.41. *Let* (π, V) *be an irreducible representation of* $\mathfrak{sl}(3, \mathbb{C})$ *with highest weight* $\lambda = n_1\omega_1 + n_2\omega_2$. *Then,*

$$\dim V = \frac{1}{2}(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2).$$

Proof. It is a special case of the Weyl dimension fomula, see [9, Thm. 10.18]. ■

At this point we have a very good understanding of the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ and thus of $SU(3)$. We will use this knowledge in the next chapter, where will apply the theory of this chapter to the theory of quarks.

4 The group $SU(3)$ and Quarks

4.1 Historical background

In this section our main references are [8, Sec. 1.6,1.7] and [10, Sec. 1.1]. In the middle of last century, in 1947, people thought the job of a particle physicist was done. Physicists understood that a force called the *strong force* bound protons and neutrons together to form nuclei. All the particles that undergo this strong force were called *baryons* ([10, p. 2]). Within this group of particles there were two distinctions, *mesons* (meaning 'middle-weight') and *baryons* (meaning 'heavy-weight')[1]. The smallest building blocks of ordinary matter were believed to be protons, neutrons, electrons and photons. Furthermore, some more exotic particles were needed to explain observations made in cosmic ray experiments, such as the pion, muon and neutrino. These latter particles caused some difficulties, especially their role was a bit unclear ([8, p. 29]). Yet, they were understood fairly well. Even the idea of antiparticles by Dirac and the discovery of the positron did not cause much disturbance.

Yet, this prosperity took a turn in December of 1947, when Rochester and Butler published a cloud chamber photo[2]. The photograph showed the production of a neutral particle that eventually decayed into two charged particles. Thorough analysis showed the charged decay products were π^+ and π^-. Rochester and Butler had found a new particle, that was eventually called the kaon, which was denoted by K^0 (See [8, p. 29]). In the years that followed lots of new mesons and baryons were found. These discoveries were very unexpected and not well understood, therefore these new particles were named *'strange'*. It turned out not only the unexpectedness of these particles made them 'strange'. Experiments showed that these particles were produced on a very small time scale, yet decayed relatively slowly (see [8, p. 32]). This was an indication that there was a different mechanism at work, as suggested by Abraham Pais in 1952. Today we know these 'strange' particles are produced by the *strong force*, the same one as before, but they decay through the *weak force* ([8, p. 32]).

In 1953 Murray Gell-Mann and Kazuhiko Nishijima introduced a new property, postulated to be conserved in strong interactions but *not* in weak interactions. Gell-Mann called this property *strangeness* ([8, p. 32]). The 'old' particles, such as the proton, were assigned a strangeness of zero and the new 'strange' got strangeness ± 1. In just over a decade the field over particles physics went from a closed chapter to downright chaos. Around 1960 a whole zoo of new 'elementary' particles (mesons and baryons) were found. These particles were characterized by strangeness, charge and mass. Yet, there was no underlying structure to explain it all ([8, p. 33]). This time period is now known as the 'particle zoo' era.

[1] See [8, Sec. 1.3].

[2] We refer to [8, p. 29]

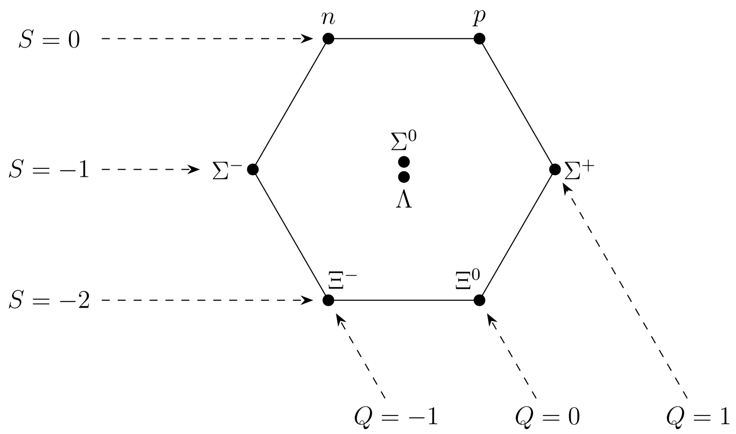

FIGURE 4.1: Particles put in a geometric pattern according to their strangeness (S) and charge (Q), called the baryon octet.

Yet, in 1961, there was some light at the end of the tunnel. In that year Gell-Mann introduced, what he called the **Eightfold Way** (a similar description was given by Yuval Ne'eman, independently). The Eightfold Way placed baryons and mesons in peculiar geometric patterns based on their strangeness and charge (see Figure 4.1)[3].

Many of these diagrams were made and different geometric patterns were seen. In particular, there is an arrangement containing ten baryons and it is called the *decuplet*. Gell-Mann found something astounding: only nine of these particles were experimentally observed. A particle with strangeness -3 and charge -1 was missing. Gell-Mann calculated the mass of the particle and explained how the particle could be found experimentally. And, sure enough, the particle was found and named Ω^-. This showed the Eightfold Way was more than just bookkeeping ([8, p. 36]).

But the success of the Eightfold Way raises a lot of questions, such as 'Why do these patterns occur?'. Trying to answer this question lead Gell-Mann (and also George Zweig, independently) to the theory of even more fundamental particles, which he called **quarks** ([8, Sec. 1.8]). In the next section we will provide the mathematical description of the quark model proposed by Gell-Mann.

4.2 Quarks

The quark model proposed by Gell-Mann in 1964 states that every hadron consists of quarks. There are three types[4] of quarks, often called *flavours*, namely the up quark (u), down quark (d) and strange quark (s). The model suggested that interchanging the flavour of the quarks within a hadron would not change the physics since the strong force interacts in the same way for every quark, regardless of their flavour[5]. Mathematically, these replacements are given by elements of $SU(3)$. Yet, it turns out the three quarks all have different masses. Even though this difference in mass is very small, the result is that $SU(3)$ is only an *approximate* symmetry ([8, p. 121-122]). This symmetry is called the *flavour $SU(3)$ symmetry.*

[3] We refer to the beginning of [8, Sec. 1.7]

[4] Today, the standard model contains six types of quarks, but in this thesis we will restrict ourself to the up, down and strange quark.

[5] This is called *flavour independence* and we refer to [13, Sec. 3.3.1].

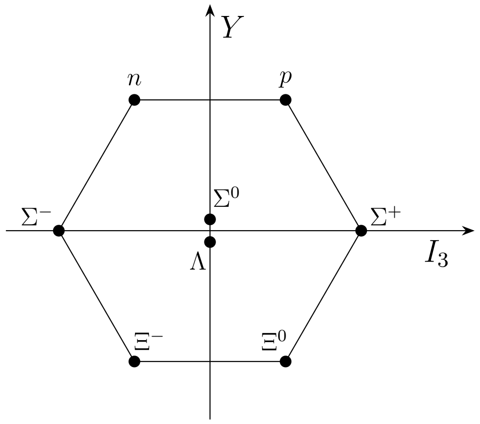

FIGURE 4.2: The baryon octet put in a geometric pattern according to their third component of isospin (I_3) and hypercharge (Y).

Each quark is assigned spin $\pm\frac{1}{2}$ and baryon number of $B = \frac{1}{3}$ ([10, p. 46]). The other quantum numbers of the quarks can be found in Table 4.1. We refer to [10, Table 2.1] for the content of this table. Furthermore, baryons consist of three quarks (qqq) and mesons consist of a quark-antiquark pair ($q\overline{q}$). Note that the axes in Figure 4.1 are a bit unnatural. Therefore, we define two new additive quantum numbers, namely the **hypercharge**

$$Y := B + S$$

and the third component of **isospin**

$$I_3 = Q - \frac{1}{2}Y.$$

A thorough discussion about isospin is given in [8, Sec. 4.5] and relies on the representation theory of $SU(2)$, which we described[6] in Section 3.3. Figure 4.1 now has natural axes (see Figure 4.2). Now we will use the representation theory of $SU(3)$. The previous chapter tells us that every irreducible representation of $SU(3)$ is constructed out of the standard representation and its dual. Therefore, we expect the quarks to fit the standard representation of $SU(3)$, which is reflected in the isospin and hypercharge quantum numbers in Table 4.1.

Flavour	Spin	B	Q	I_3	S	Y
u	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{3}$
d	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	$\frac{1}{3}$
s	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	-1	$-\frac{2}{3}$

TABLE 4.1: The quantum numbers of the different flavours of quarks

[6]Similar to $SU(3)$, the Lie group $SU(2)$ is also simply-connected. Therefore, there is a one-to-one correspondence between the irreducible representations of $SU(2)$ and $\mathfrak{sl}(2, \mathbb{C})$, which we utilize here.

4.2.1 Baryons

At this point we want to give a quantitative description of the subjects we discussed before. In this section we will restrict ourselves to baryons, so particles consisting of three quarks. The main idea is to decompose tensor products of the quark Hilbert space into irreducible representations of the quark symmetry group $SU(3)$ and link a specific basis of these representation spaces to subatomic particles ([15, p. 225]).

We denote the quark Hilbert space by $\mathcal{Q}$ and identify

$$e_1 := |u\rangle\,, \quad e_2 := |d\rangle\,, \quad e_3 := |s\rangle\,. \tag{4.1}$$

Then we see $\mathcal{Q} = \mathbb{C}^3$. Consider the standard representation of $SU(3)$. The corresponding representation of $\mathfrak{sl}(3,\mathbb{C})$ is also the standard one. Let us denote the standard representation of $\mathfrak{sl}(3,\mathbb{C})$ by $(\pi, \mathcal{Q})$. In this representation we see e_1, e_2 and e_3 are weight vectors, relative to $\mathfrak{t}_{\mathbb{C}}$ (as defined in Equation (3.12)). Their corresponding weights are L_1, L_2 and L_3 (from Section 3.5). Note that, in the standard representation e_1 is a highest weight vector with highest weight L_1.

Yet, the basis we chose in Section 3.5 (H_1,H_2) has no physical meaning. Therefore, we introduce two operators on $\mathcal{Q}$

$$\hat{I}_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{Y} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \tag{4.2}$$

Note that, $\hat{I}_3 = \frac{1}{2}H_1$ and $\hat{Y} = \frac{1}{3}(H_1 + 2H_2)$. Therefore $\hat{I}_3$ and $\hat{Y}$ form a basis of $\mathfrak{t}_{\mathbb{C}}$. Moreover, the operators are self-adjoint, hence they represent observables. A close look at Table 4.1 shows that $\hat{I}_3$ and $\hat{Y}$ represent the observables: (third component of) isospin and hypercharge, respectively.

Now we move on to composed quark states and in particular to baryons. Baryons consist of three quarks, hence their Hilbert space is given by

$$\mathcal{B} := \mathcal{Q} \otimes \mathcal{Q} \otimes \mathcal{Q} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3.$$

The $\mathfrak{sl}(3,\mathbb{C})$ representation we want to consider, for baryons, is the pair $(\tilde{\pi}, \mathcal{B})$, where

$$\tilde{\pi} := \pi \otimes \pi \otimes \pi.$$

Note that, the elements $e_i \otimes e_j \otimes e_k$, for $i, j, k = 1, 2, 3$, form a basis of $\mathcal{B}$. Furthermore, every element $e_i \otimes e_j \otimes e_k$ is a weight vector with weight $L_i + L_j + L_k$ (for $i, j, k = 1, 2, 3$), by Definition 2.50. By these two properties of the elements $e_i \otimes e_j \otimes e_k$, it follows that every weight of $(\tilde{\pi}, \mathcal{B})$ is of the form $L_i + L_j + L_k$. Using this property, we can decompose $\mathcal{B}$ into irreducible subspaces (this decomposition is unique up to isomorphism). The existence of such a decomposition is guaranteed by the fact that $SU(3)$ is compact and Theorem 2.40. As we have seen in Section 3.5, an irreducible representation corresponds to a dominant integral element (see Theorem 3.33). By the described property, we deduce that the representation $(\tilde{\pi}, \mathcal{B})$ has three dominant integral weights (see Figure 4.3), namely

$$\lambda_1 := L_1 + L_1 + L_1 = 3L_1, \qquad \lambda_2 := 2L_1 + L_2, \qquad \lambda_3 := L_1 + L_2 + L_3 = 0.$$

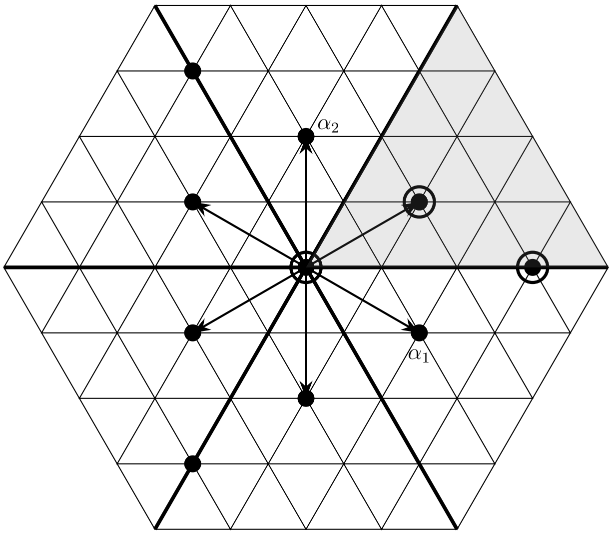

FIGURE 4.3: The weight lattice Λ_W of the representation $(\tilde{\pi}, \mathcal{B})$. The gray shaded area depicts the positive Weyl chamber $\mathcal{C}$. Furthermore, the rings indicate which integral elements are dominant.

We easily see that

$$V_{\lambda_1} = \mathbb{C}(e_1 \otimes e_1 \otimes e_1).$$

Furthermore, one easily verifies that the space V_{λ_2} is spanned by $e_1 \otimes e_1 \otimes e_2$, $e_1 \otimes e_2 \otimes e_1$ and $e_2 \otimes e_1 \otimes e_1$. Similarly, the space V_{λ_3} is spanned by $e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$, for $\sigma \in S_3$ (here S_3 denotes the permutation group of three elements).

We will verify that λ_1 is a highest weight. Note that, the element $e_1 \otimes e_1 \otimes e_1 \in \mathcal{B}$ is a weight vector for λ_1, such that

$$\begin{aligned}\tilde{\pi}(X_i)(e_1 \otimes e_1 \otimes e_1) &= (\pi(X_i)e_1) \otimes e_1 \otimes e_1 + e_1 \otimes (\pi(X_i)e_1) \otimes e_1 + e_1 \otimes e_1 \otimes (\pi(X_i)e_1) \\ &= 0\end{aligned}$$

for $i = 1, 2, 3$. Here we used that e_1 is a highest weight vector of $(\pi, \mathcal{Q})$. Hence, the vector $e_1 \otimes e_1 \otimes e_1$ is a highest weight vector the corresponding irreducible representation, with highest weight $3L_1 = 3\omega_1$, is a submodule of $\mathcal{B}$. Note that, by Theorem 3.33 it follows this submodule is isomorphic to $R(3, 0)$[7]. By Theorem 3.41 it follows that $\dim R(3, 0) = 10$. Let us denote the irreducible submodule of $\mathfrak{sl}(3, \mathbb{C})$, with highest weight $3L_1$, by V_{10}.

To find the other irreducible submodules of $\mathcal{B}$, we are looking for other highest weight vectors[8]. Since $2L_1 + L_2$ is a dominant integral element, we want to find a highest weight vector for $2L_1 + L_2$. Since $\lambda_2 = 2L_1 + L_2 = \omega_1 + \omega_2$, we see the corresponding irreducible submodule of $\mathcal{B}$ is isomorphic to $R(1, 1)$. Note that, every vector $v \in V_{\lambda_2}$ is of the form

$$v = c_1(e_1 \otimes e_1 \otimes e_2) + c_2(e_1 \otimes e_2 \otimes e_1) + c_3(e_2 \otimes e_1 \otimes e_1),$$

[7]The numbers inside the parantheses correspond to the coefficients $n_1, n_2 \in \mathbb{N}$ in Theorem 3.33.
[8]Note that, since $(\tilde{\pi}, \mathcal{B})$ is not irreducible, a highest weight is not necessarily unique.

with $c_1, c_2, c_3 \in \mathbb{C}$. Let us write e_{112} for the element $e_1 \otimes e_1 \otimes e_2$. Then, $v = c_1 e_{112} + c_2 e_{121} + c_3 e_{211}$. The vector v is a highest weight vector of a submodule, when

$$\tilde{\pi}(X_i)v = 0,$$

for every $i = 1, 2, 3$. Note that, e_1 is a highest weight vector of $(\pi, \mathcal{Q})$, so $\pi(X_i)e_1 = 0$ for every i. Moreover, one easily sees that $\pi(X_i)e_2 = 0$ for $i = 2, 3$ and $\pi(X_1)e_2 = e_1$. This shows that $\tilde{\pi}(X_i)v = 0$ for $i = 2, 3$. For $i = 1$ we have

$$\tilde{\pi}(X_1)v = (c_1 + c_2 + c_3)e_{111}.$$

Requiring that v is a highest weight vector implies that $c_1 + c_2 + c_3 = 0$. Hence, the subspace V_{λ_2} contains two linearly independent highest weight vectors. Both of these highest weight vectors correspond to an irreducible submodule, isomorphic to $R(1,1)$. Consequently, two irreducible submodules of $\mathcal{B}$ are isomorphic to $R(1,1)$. By Theorem 3.41 it follows that $\dim R(1,1) = 8$. Let us denote these two irreducible submodules by V_8 and V_8', respectively.

Finally, The last dominant integral element is $\lambda_3 = L_1 + L_2 + L_3 = 0$. Note that, the space V_{λ_3} has dimension six. This is because every permutation of indices of $e_1 \otimes e_2 \otimes e_3$ is an element of V_{λ_3}. Therefore, λ_3 has multiplicity six. One readily sees that λ_3 is a weight for $R(3,0)$ with multiplicity one and a weight for $R(1,1)$ with multiplicity two. Recall, $\mathcal{B}$ contains one submodule isomorphic to $R(3,0)$ and two submodules isomorphic to $R(1,1)$. Therefore, if we strip away these submodules, we are left with the weight λ_3 with multiplicity one. One can verify, in a similar way as we did for the weight λ_2, that V_{λ_3} contains a highest weight vector. Actually, through this method it follows that this vector v_3 is given by

$$v_3 := e_{123} - e_{132} + e_{231} - e_{213} + e_{312} - e_{321}.$$

Note that, this vector cannot be contained in V_{10}, V_8 or V_8', since a highest weight vector is unique (up to a factor) in an irreducible representation. The irreducible submodule of $\mathcal{B}$ corresponding to λ_3 is isomorphic to $R(0,0)$, by Theorem 3.33. Let us denote this irreducible submodule by V_1. Note $\dim V_1 = \dim R(0,0) = 1$, by Theorem 3.41.

We note that,

$$\dim V_{10} + \dim V_8 + \dim V_8' + \dim V_1 = 10 + 8 + 8 + 1 = 27.$$

Since $\dim \mathcal{B} = 3^3 = 27$ we conclude that the decomposition of $\mathcal{B}$ is given by

$$\mathcal{B} = V_{10} \oplus V_8 \oplus V_8' \oplus V_1.$$

The next step in classifying the baryons is linking particle states to basis vectors of the irreducible submodules. One way of doing this is through calculations of Clebsch-Gordan coefficients, as is done in [7, Sec. 8.10] for mesons. For a calculation considering baryons we refer to [7, Ex. 8.14, p. 270]. Yet, these calculations are quite involved. Therefore we will not give these calculations of the Clebsch-Gordan coefficients in this thesis. We will use the theory we have developed in Chapter 3.

We start with the irreducible submodule V_{10}. Recall, the highest weight vector of the module was $e_{111} := e_1 \otimes e_1 \otimes e_1$. Since V_{10} is irreducible, every vector in V_{10} is cyclic and

Baryon	State	Y	I_3
Δ^{++}	uuu	1	$\frac{3}{2}$
Δ^{+}	$\frac{1}{\sqrt{3}}(uud + udu + duu)$	1	$\frac{1}{2}$
Δ^{0}	$\frac{1}{\sqrt{3}}(udd + dud + ddu)$	1	$-\frac{1}{2}$
Δ^{-}	$\frac{1}{\sqrt{3}}(uud + udu + duu)$	1	$-\frac{3}{2}$
Σ^{*+}	$\frac{1}{\sqrt{3}}(uus + usu + suu)$	0	1
Σ^{*0}	$\frac{1}{\sqrt{6}}(uds + usd + sud + sdu + dsu + dus)$	0	0
Σ^{*-}	$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$	0	-1
Ξ^{*+}	$\frac{1}{\sqrt{3}}(uss + sus + ssu)$	-1	$\frac{1}{2}$
Ξ^{*-}	$\frac{1}{\sqrt{3}}(dss + sds + ssd)$	-1	$-\frac{1}{2}$
Ω^{-}	sss	-2	0

TABLE 4.2: The states of the baryon decuplet with corresponding quantum numbers.

thus all the assertions of Lemma 3.27 hold. In particular, we can apply Lemma 3.27(b). Recall, the action of Y_i lowers a weight by α_i. One can easily check that the following vectors

$$e_{111}, \quad \tilde{\pi}(Y_1)^k e_{111}, \quad \tilde{\pi}(Y_3)^k e_{111}, \quad \tilde{\pi}(Y_2)^l \tilde{\pi}(Y_1)^3 e_{111}, \quad \tilde{\pi}(Y_3)\tilde{\pi}(Y_1)e_{111}, \tag{4.3}$$

for $k = 1, 2, 3$ and $l = 1, 2$ form a basis of V_{10}. After explicit calculation we get the vectors

$$\begin{gathered} e_{111}, \\ e_{112} + e_{121} + e_{211}, \quad 2(e_{122} + e_{212} + e_{221}), \quad 6e_{222}, \\ e_{113} + e_{131} + e_{311}, \quad 2(e_{133} + e_{313} + e_{331}), \quad 6e_{333}, \\ e_{223} + e_{232} + e_{322}, \quad 2(e_{233} + e_{323} + e_{332}), \quad 6e_{333}, \\ e_{123} + e_{132} + e_{213} + e_{231} + e_{312} + e_{321}. \end{gathered}$$

These basis elements, after normalization, correspond to particle states. To see this, recall that we chose the operators $\hat{I}_3$ and $\hat{Y}$ as basis for $\mathfrak{t}_{\mathbb{C}}$. Hence, the isospin and hypercharge of the basis elements are known. Linking those values to the same quantum numbers for isospin and hypercharge of baryons we know experimentally, gives us the correspondence. In Table 4.2 we have depicted every baryon in the so-called *baryon decuplet* with their corresponding state. If we plot the baryons of Table 4.2 according to their isospin and hypercharge we find Figure 4.4. This figure is one of the geometric patterns described in the Eightfold Way by Murray Gell-Mann.

We are left to find a basis for V_8 and V_8'. Recall, the highest weight vectors of V_8 and V_8' are of the form

$$c_1 e_{112} + c_2 e_{121} + c_3 e_{211},$$

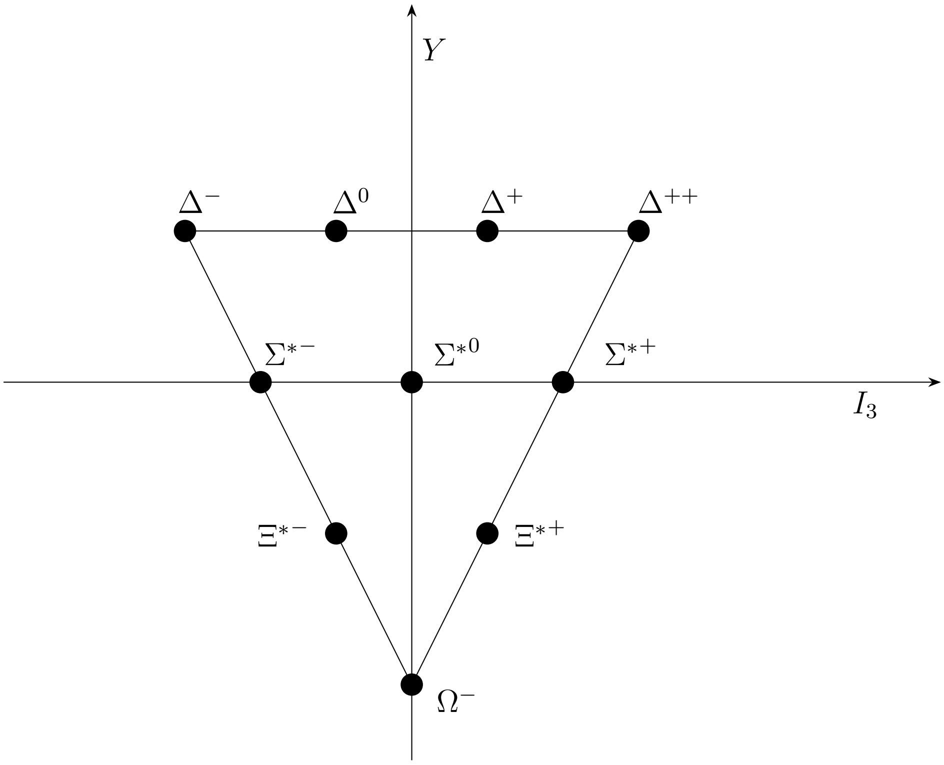

FIGURE 4.4: The baryon decuplet.

with $c_1 + c_2 + c_3 = 0$. Let us choose[9], $c_1 = 0$ and $c_2 = -c_3$ for the highest weight vector of V_8' (as is done in [10, p. 51]). Then we have,

$$v_A = e_{121} - e_{211}.$$

Note that, the weight vector of V_{10} associated with the weight $2L_1 + L_2$ is given by $e_{112}+e_{121}+e_{211}$. Thus, we see this vector and v_A are orthogonal. If we require orthogonality for the highest weight vector v_S of V_8 we find

$$v_S = e_{121} + e_{211} - 2e_{112}.$$

In a similar way to the baryon decuplet, a basis of V_8' can be found through computations similar to Equation (4.3). Furthermore, by requiring orthogonality, as we did for v_S, we find a basis for V_8 ([10, p. 51]).

Note that, the vectors v_S and v_A have mixed symmetry. The subscripts tell us that v_S is symmetric under interchange of the first two quarks. Similarly, v_A is anti-symmetric under such an interchange. Actually, every vector in V_8 is symmetric under interchange of the first quarks and every vector in V_8' is anti-symmetric under such an interchange. Therefore, we give V_8 and V_8' the labels M_S and M_A, respectively. One may think both V_8 and V_8' correspond to baryon octet, but it turns out there is only one baryon octet in observed nature for the lowest-mass baryons (or ground state baryons). This has to do with some properties of baryons.

It turns out that the baryon flavour states must be fully symmetric. This follows from an internal quantum number of the quarks, namely *colour*. For more background about

[9]This choice is made due to symmetry considerations and will become clear later on.

this topic we refer to [4, Sec. 4.5]. The crucial part for our discussion is that observed baryons are postulated to be a colour singlet representation. This implies the wave function of baryons is completely anti-symmetric under interchange of colour indices[10]. Due to the fact that quarks are *fermions*, the Pauli exclusion principle implies that the baryon wave functions must be anti-symmetric with respect to interchange of all their characteristics ([4, p. 62]). Since the wave functions are fully anti-symmetric with respect to colour, the baryon wave functions must be fully symmetric with respect to space, spin and flavour ([10, p. 53]). So, the *baryon octet* depicted in Figure 4.2 corresponds to fully symmetric states. To fully understand why there is only one baryon octet, we need to also consider spin. Since quarks are fermions, they have spin $\pm\frac{1}{2}$. The symmetry group of spin is $SU(2)$. The irreducible representations of $SU(2)$ are classified by the discussion in Section 3.3. Therefore, we could use this discussion to decompose tensor products of spin spaces, but we will mainly refer to literature. Since baryons consist of three quarks, physicists[11] write their spin space as $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$. In [10, Sec. 2.4] it is shown this spin space decomposes as

$$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \underbrace{\mathbf{4}}_{S}\oplus\underbrace{\mathbf{2}}_{M_S}\oplus\underbrace{\mathbf{2}}_{M_A}. \tag{4.4}$$

Here the labels indicate the symmetry within the representations, similar to the labels we introduced for the vectors v_S and v_A. For example, the representation $\mathbf{4}$ is fully symmetric under interchange of two spins. Moreover, the label M_S means elements of the representation are symmetric under interchange of the first two spins. One can verify, the compositions of the spin 'up' state in the three spin representations of Equation (4.4) are given by ([10, p. 52])

$$\begin{aligned}\chi(S) &= \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow),\\ \chi(M_S) &= \frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow),\\ \chi(M_A) &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow).\end{aligned}$$

Now we want to combine the $SU(3)$ flavour symmetry with the $SU(2)$ spin symmetry, so we are considering $(SU(3), SU(2))$ multiplets. It follows that, one of the fully symmetric multiplet is an octet (See [10, Eq. 2.67]). This fully symmetric baryon octet is obtained by

$$\frac{1}{\sqrt{2}}\Big(\underbrace{(\mathbf{8},\mathbf{2})}_{(M_S,M_S)} + \underbrace{(\mathbf{8},\mathbf{2})}_{(M_A,M_A)} \Big). \tag{4.5}$$

Here we used the notation of [10, Eq. 2.68]. The octet described in Equation (4.5) is the one corresponding to the baryon octet depicted in Figure 4.2. The wave functions of the particles in the baryon octet are obtained by considering combination according to Equation (4.5). We will give the wave function of the spin-up proton. Denote $p_S = \frac{1}{\sqrt{6}}v_S$

[10]We refer to [4, p. 63] and [11, p. 59]

[11]In this notation we use the dimension to label the irreducible representations ([10, p. 40]).

and $p_A = \frac{1}{2}v_A$. Then, using equation (4.1) we have

$$p_S = \frac{1}{\sqrt{6}}(udu + duu - 2uud), \qquad p_A = \frac{1}{2}(udu - duu).$$

The wave function of the spin-up proton is then given by ([10, Eq. 2.71])

$$\begin{aligned}|p\uparrow\rangle =& \frac{1}{2}(p_S\chi(M_S) + p_A\chi(M_A))\\ =& \frac{1}{\sqrt{18}}(uud(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow -2\uparrow\uparrow\downarrow) + udu(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow -2\uparrow\uparrow\downarrow)+\\ & duu(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow -2\uparrow\uparrow\downarrow)).\end{aligned}$$

The other wave function can be calculated in a similar way or by using the lowering operators $\pi(Y_i)$, for $i = 1, 2, 3$. The latter is done in [7, Ex. 8.15].

This concludes the discussion on the baryon octet and although we did not need the spin consideration for the baryon decuplet, this can be incorporated in the discussion. We omit the discussion here, but for more background and for the calculation of the spin-flavour wave functions of the baryon decuplet we refer to [7, Sec. 8.11], [7, Ex. 8.14].

To conclude this section we take a closer look at our discussion of the baryon octet. We considered $(SU(3), SU(2))$ multiplets. In fact, by incorporating spin we went from three quark states to six quark states. Mathematically, we took the tensor product of the quark Hilbert space ($\mathcal{Q} = \mathbb{C}^3$) and the spin space ($\mathbb{C}^2$), which is isomorphic to $\mathbb{C}^6$. On $\mathbb{C}^3 \otimes \mathbb{C}^2 \cong \mathbb{C}^6$ we can consider an action of $SU(6)$ and we postulated that physics is invariant under $SU(6)$-transformations of the six quark states into one another. We call this $SU(6)$ *spin-flavour* symmetry ([4, p. 58]). It turns out, the threefold tensor product of the standard representation of $SU(6)$ (denoted by physicists as **6**) decomposes as[12]

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \underbrace{\mathbf{56}}_{S} \otimes \underbrace{\mathbf{70}}_{M_S} \otimes \underbrace{\mathbf{70}}_{M_A} \otimes \underbrace{\mathbf{20}}_{A}.$$

As said before, the spin-flavour wave functions of baryons must be fully symmetric. Therefore, we will focus on the representation **56**. Instead of the group $SU(6)$, we can consider the group $SU(3) \times SU(2)$ acting on $\mathbb{C}^3 \otimes \mathbb{C}^2 \cong \mathbb{C}^6$. This action is defined by

$$(T, S)(v \otimes w) = Tv \otimes Sw,$$

for $T \in SU(3),\ S \in SU(2)$ and $v \otimes w \in \mathbb{C}^3 \otimes \mathbb{C}^2$. Since both $SU(3)$ and $SU(2)$ are compact, the group $SU(3) \times SU(2)$ is also compact. Hence, the space $\bigotimes^3(\mathbb{C}^3 \otimes \mathbb{C}^2) \cong \bigotimes^3(\mathbb{C}^6)$ decomposes into irreducible representations of $SU(3) \times SU(2)$, by Theorem 2.40. The decomposition of **56** under this group is given by ([4, Eq. 4.3])

$$\mathbf{56} = (\mathbf{10}, \mathbf{4}) \oplus (\mathbf{8}, \mathbf{2}). \tag{4.6}$$

In the notation of Equation 4.6, **10** and **8** denote irreducible representations of $SU(3)$. Similarly, **4** and **2** denote irreducible representations of $SU(2)$. By the notation $(\mathbf{10}, \mathbf{4})$ we mean the tensor product of the irreducible representation **10** of $SU(3)$ and the irreducible representation **4** of $SU(2)$, similarly for $(\mathbf{8}, \mathbf{2})$. In the decomposition of Equation 4.6,

[12]We refer to [4, Eq. 4.4]

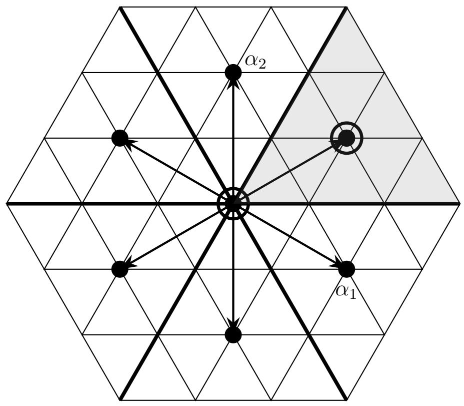

FIGURE 4.5: The weight lattice Λ_W of the representation $(\rho, \mathcal{M})$. The gray shaded are depicts the positive Weyl chamber $\mathcal{C}$. Furthermore, the rings indicate the dominant integral elements.

$(\mathbf{10}, \mathbf{4})$ corresponds to the baryon decuplet and $(\mathbf{8}, \mathbf{2})$ to the baryon octet. The $SU(6)$ spin-flavour symmetry thus explains why only one baryon octet is observed in nature. We only gave a general discussion on spin-flavour symmetry and for further reading we refer to [7, Sec. 8.11], [4, Sec. 4.2] and [11, Ch. 10].

4.2.2 Mesons

Finally, we want to consider the other type of hadrons: mesons. This will be a bit less involved, since we do not have to consider spin-flavour symmetry[13]. Recall, mesons consist of a quark and an antiquark bound together ([10, p. 47]). We denote the antiquarks by $\overline{u}, \overline{d}$ and $\overline{s}$. Mathematically, these antiquarks correspond to the weight vectors of the dual of the standard representation (see Definition 2.51). Recall, the corresponding representation of $\mathfrak{sl}(3, \mathbb{C})$ is given by

$$\pi^\vee(X) = -X^T,$$

for $X \in \mathfrak{sl}(3, \mathbb{C})$. The Hilbert space of the antiquarks is given by $\mathcal{A} := \mathcal{Q}^*$ (the dual space of $\mathcal{Q}$). Let us denote the dual of the standard representation by $(\pi^\vee, \mathcal{A})$ and we identify

$$f^1 := |\overline{u}\rangle\,, \quad f^2 := |\overline{d}\rangle, \quad f^3 := |\overline{s}\rangle\,.$$

Then, we see $\mathcal{A}$ is isomorphic to $(\mathbb{C}^3)^*$, which is in turn isomorphic to $\mathbb{C}^3$. Furthermore, the vectors f^1, f^2 and f^3 are weight vectors of $(\pi^\vee, \mathcal{A})$ with weights $K_1 := -L_1$, $K_2 := -L_2$ and $K_3 := -L_3$, respectively. Hence, the quantum numbers of the antiquarks are precisely opposite of those of the quarks. This is what we would expect of anti-particles. It can be verified through explicit calculations that f^3 is a highest weight vector of $(\pi^\vee, \mathcal{A})$.

Now, the Hilbert space for mesons is given by

$$\mathcal{M} := \mathcal{Q} \otimes \mathcal{A}.$$

[13] One could apply this to mesons and for further reading we refer to [7, Sec. 8.11].

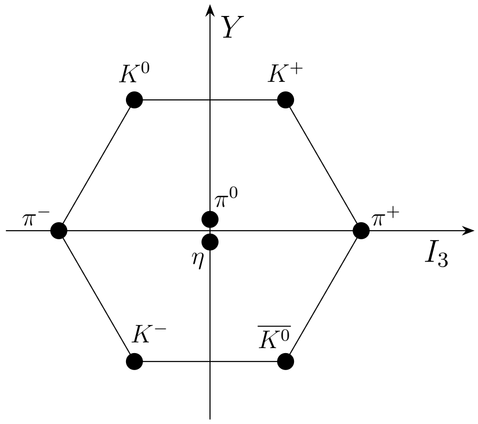

FIGURE 4.6: The meson octet.

The representation of $\mathfrak{sl}(3,\mathbb{C})$ we want to consider is the pair $(\rho,\mathcal{M})$, where

$$\rho := \pi \otimes \pi^\vee.$$

Note that, $\mathcal{M}$ has dimension $3^2 = 9$. Similar to the baryon case, we want to decompose $\mathcal{M}$ into irreducible submodules. To obtain this decomposition we will analyze highest weights. Again, by the definition of the tensor product representation it follows that every weight of $(\rho,\mathcal{M})$ can be written as the sum of individual weights: $L_i + K_j$, with $i,j = 1,2,3$. The weight diagram of $(\rho,\mathcal{M})$ is given in Figure 4.5. Note that, $(\rho,\mathcal{M})$ has two integral elements:

$$\lambda_1 := L_1 + K_3 = L_1 - L_3, \quad \lambda_2 := 0.$$

We readily see

$$V_{\lambda_1} = \mathbb{C}(e_1 \otimes f^3).$$

Since e_1 and f^3 are both highest weight vectors for the standard representation and the dual of the standard representation, respectively, it follows that $e_1 \otimes f^3$ is a highest weight vector. Note that, $L_1 - L_3 = \omega_1 + \omega_2$. Hence, $\mathcal{M}$ contains an irreducible submodule isomorphic to $R(1,1)$. Let us denote this submodule by V_8. Since $\dim R(1,1) = 8$ and $\dim \mathcal{M} = 9$ we expect the last irreducible submodule to be isomorphic to $R(0,0)$. By explicit computation one can verify that

$$(V_{\lambda_2})^{\mathfrak{n}} = \mathbb{C}(e_1 \otimes f^1 + e_2 \otimes f^2 + e_3 \otimes f^3).$$

So, indeed $\mathcal{M}$ contains an irreducible submodule isomorphic to $R(0,0)$, denoted by V_1. Then we see

$$\mathcal{M} = V_8 \oplus V_1.$$

A basis for V_8 can be found in a similar way we did for the baryon decuplet. Only the basis vectors of V_8 corresponding to the weight λ_2 are a bit tricky. Here we will use an orthogonality argument to find the vectors. One of the basis vectors is part of an isospin

Meson	State	Y	I_3
K^+	$u\overline{s}$	1	$\frac{1}{2}$
K^0	$d\overline{s}$	1	$-\frac{1}{2}$
π^+	$u\overline{d}$	0	1
π^0	$\frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d})$	0	0
π^-	$d\overline{u}$	0	-1
$\overline{K^0}$	$s\overline{d}$	-1	$\frac{1}{2}$
K^-	$s\overline{u}$	-1	$-\frac{1}{2}$
η	$\frac{1}{\sqrt{6}}(u\overline{u} + d\overline{d} - 2s\overline{s})$	0	0
η'	$\frac{1}{\sqrt{3}}(u\overline{u} + d\overline{d} + s\overline{s})$	0	0

Table 4.3: The states of the meson octet and singlet with corresponding quantum numbers. The first eight mesons correspond to the octet and the last meson to the singlet. We refer to [7, p. 233] and [10, p.47].

triplet and is computed by

$$\rho(Y_1)\rho(Y_2)(e_1 \otimes f^3) = e_2 \otimes f^2 - e_1 \otimes f^1.$$

The other basis vector follows from requiring it must orthogonal to $e_2 \otimes f^2 - e_1 \otimes f^1$ and $e_1 \otimes f^1 + e_2 \otimes f^2 + e_3 \otimes f^3$ (see [10, p. 47]). By explicit computation it follows this basis vector is given by

$$e_1 \otimes f^1 + e_2 \otimes f^2 - 2e_3 \otimes f^3.$$

It follows that, the basis of V_8 is given by

$$\begin{gathered} e_1 \otimes f^3, \quad e_2 \otimes f^3, \\ e_1 \otimes f^2, \quad e_2 \otimes f^2 - e_1 \otimes f^1, \quad e_1 \otimes f^1 + e_2 \otimes f^2 - 2e_3 \otimes f^3, \quad e_2 \otimes f^1, \\ e_3 \otimes f^1, \quad e_1 \otimes f^2. \end{gathered}$$

Again, after normalizing, we can identify these basis vectors to meson states by matching the quantum numbers. This is shown in Table 4.3. Graphically, this corresponds to the *meson octet* depicted in Figure 4.6. This also corresponds to one of the diagrams proposed by Gell-Mann in the Eightfold Way. It turns out that many other mesons fit into the expected quark-antiquark multiplets and at the time this was at the core for accepting the quark model ([10, p. 49]).

Bibliography

[1] E.P. van den Ban. *Aantekingen over tensoren.* 2000. URL: https://www.staff.science.uu.nl/~ban00101/lecnotes/tensor.pdf.

[2] E.P. van den Ban. *Appendix A: Lifting representations.* 2016. URL: https://www.staff.science.uu.nl/~ban00101/lie2016/lie2010_App_A.pdf.

[3] E.P. van den Ban. *Lecture Notes on Lie groups.* 2010. URL: https://www.staff.science.uu.nl/~ban00101/lecnotes/lie2010.pdf.

[4] J. Chýla. *Quarks, partons and Quantum Chromodynamics.* 2009. URL: https://www.fzu.cz/~chyla/lectures/text.pdf.

[5] D.L. Cohn. *Measure Theory.* (Second Edition). Birkhäuser Basel, 2013.

[6] W. Fulton and J. Harris. *Representation Theory: A First Course.* Springer Graduate Texts in Mathemtics, 2004.

[7] W. Greiner and B. Müller. *Quantum Mechanics - Symmetries.* (Second Edition). Springer-Verlag Berlin Heidelberg, 1994.

[8] D.J. Griffiths. *Introduction to Elementary Particles.* John Wiley & Sons, Inc., 1987.

[9] B.C. Hall. *Lie Groups, Lie Algebras and Representations.* (Second Edition). Springer-Verlag, 2015.

[10] F. Halzen and A.D. Martin. *Quarks & Leptons: An Introductary Course in Modern Particle Physics.* John Wiley & Sons, Inc., 1984.

[11] G. 't Hooft. *Lie Group in Physics.* 2007. URL: https://www.staff.science.uu.nl/~hooft101/lectures/lieg07.pdf.

[12] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag New York Inc, 1972.

[13] B.R. Martin. *Nuclear and Particle Physics.* John Wiley & Sons, Ltd., 2006.

[14] B.P. Rynne and M.A. Youngson. *Linear Functional Analysis.* (Second Edition). Springer-Verlag London Limited, 2008.

[15] E. Zeidler. *Quantum Field Theory III: Gauge Theory.* Springer-Verlag Berlin Heidelberg, 2011.

[16] W. Ziller. *Lie Groups. Representation Theory and Symmetric Spaces.* 2010. URL: https://www.math.upenn.edu/~wziller/math650/LieGroupsReps.pdf.