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TWIN: Mathematics & Physics and Astronomy

Hamiltonian Torus Actions on Symplectic Manifolds

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Introduction

In this thesis we study Lie group actions on symplectic manifolds. The origin of this area, equivariant symplectic geometry, lies in classical mechanics where the phase space of a physical system represents a symplectic manifold and the symmetries of the system are represented by an action of a Lie group on the phase space. The evolution of a physical system in time is governed by a Hamiltonian function on its phase space. A successful generalization of such a Hamiltonian function is the so-called *momentum map*, introduced by Kostant [27] and Souriau [40].

Equivariant symplectic geometry is related to many other areas in mathematics and theoretical physics such as completely integrable systems, Hamiltonian dynamics, Poisson geometry, Morse–Bott theory, representation theory, combinatorics, equivariant cohomology, algebraic geometry, (almost) complex geometry, gauge theory and quantum field theory. The aim of this thesis is to explore some of these relations by studying several spectacular theorems concerning Hamiltonian torus actions on symplectic manifolds. Specifically, we will prove the Marsden–Weinstein–Meyer symplectic reduction theorem, the Atiyah–Guillemin–Sternberg convexity theorem and the Duistermaat–Heckman theorems. Another aim is to provide a text in which these theorems are studied assuming minimal background knowledge.

Structure, Prerequisites and Assumptions

This thesis is structured as follows.

In the first chapter we briefly discuss symplectic geometry to better understand the notion of a symplectic form and other relevant structures, namely Riemannian metrics, almost complex structures and the Poisson bracket. If more background information is desired, we recommend to supplement your knowledge on this subject with, for example, chapters 1, 2, 6-8, 12, 13, 18, 19 from Cannas da Silva's book *Lectures on Symplectic Geometry* [9].

The purpose of the second chapter is two-fold. First, we recall some theory regarding Lie groups and Lie group actions on manifolds. Then we introduce our main object of study, namely the momentum map associated to a Hamiltonian action of a Lie group on a symplectic manifold. We use the momentum map to prove the Marsden–Weinstein–Meyer symplectic reduction theorem, which is a method of taking quotients in the setting of symplectic geometry.

From the third chapter onward, we restrict our attention to the Abelian Lie groups called tori. We first state some facts about tori and discuss how we can use the exponential map of a torus to 'generate subtori'. Furthermore, this chapter provides a fast introduction to representation theory and Morse–Bott theory. Then we incorporate these topics into our study of the momentum map and use it to prove the Atiyah–Guillemin–Sternberg convexity theorem.

The fourth chapter is on the Duistermaat–Heckman theorems. After briefly studying principal bundles and connection one-forms, we prove the Duistermaat–Heckman theorem which compares certain de Rham cohomology classes of reduced spaces (obtained by symplectic reduction). Finally, we study the Cartan model of equivariant differential forms. We use this model to prove the Atiyah–Bott–Berline–Vergne localization theorem in the more general setting of equivariant cohomology. In this way, we obtain the Duistermaat–Heckman localization theorem as a corollary.

We assume that the reader is familiar with differential topology on the level of a first course on smooth manifolds, for example, chapters 1-5, 7-10, 14-17, (20-21) of Lee's *Introduction to Smooth Manifolds* [28].

All manifolds are assumed to be *smooth*, second-countable and Hausdorff, possibly with boundary.

Chapter 1

Symplectic Geometry

1.1 Symplectic Linear Algebra

Definition 1.1.1. Let (V, ω) be a finite-dimensional real vector space V together with a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$. The pair (V, ω) is a **symplectic vector space** if the map ω is nondegenerate, that is: $\omega(v, w) = 0$ for all $w \in V$ implies $v = 0$. We refer to the map ω as the **symplectic structure** of (V, ω).

We define the **rank** of a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$ to be the rank of the natural linear map

$$\omega : V \to V^*, \ v \mapsto \omega(v, \cdot),$$

which we also denote by ω. Similarly, we define the **kernel** of ω to be the kernel of $\omega : V \to V^*$. We see that ω is nondegenerate if and only if ω has full rank if and only if ω has trivial kernel.

The following theorem describes a standard form for nondegenerate skew-symmetric bilinear maps, which may be proven using a skew-symmetric analogue of the Gram-Schmidt procedure.

Theorem 1.1.2. *[9, Theorem 1.1] Let (V, ω) be a symplectic vector space. Then there exists an ordered basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ of V such that the following holds, for all $j, k = 1, \ldots, m$:*

$$\omega(e_j, e_k) = 0 = \omega(f_j, f_k), \qquad \omega(e_j, f_k) = \delta_{j,k}. \tag{1.1}$$

In particular, this theorem implies that any symplectic vector space (V, ω) is even-dimensional. An ordered basis as in Theorem 1.1.2 is called a **symplectic basis**.

Definition 1.1.3. Let (V, ω) and (V', ω') be two symplectic vector spaces of dimension $2m$. A **symplectomorphism** between (V, ω) and (V', ω') is a linear isomorphism $\Phi : V \to V'$ which preserves the symplectic structure, meaning that

$$\Phi^* \omega' = \omega.$$

In this case, we say that (V, ω) and (V', ω') are **symplectomorphic**.

In view of Theorem 1.1.2, we find that two symplectic vector spaces are symplectomorphic if and only if their dimensions agree.

Definition 1.1.4. Let (V, ω) be a symplectic vector space, and let $W \subseteq V$ be a linear subspace. The **symplectic complement**, denoted W^ω, is the linear subspace of V defined by

$$W^\omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$$

Note that the *symplectic* complement does not share all of the properties of the *orthogonal* complement. For example, the intersection of W with its symplectic complement W^ω is not necessarily trivial.

Definition 1.1.5. Let (V, ω) be a symplectic vector space, and $W \subseteq V$ a linear subspace. The subspace W is called

- **isotropic** if $W \subseteq W^\omega$,
- **coisotropic** if $W^\omega \subseteq W$,
- **symplectic** if $W \cap W^\omega = \{0\}$,
- **Lagrangian** if $W = W^\omega$.

However, the symplectic complement does have some properties in common with the notion of orthogonal complement, as the following proposition asserts.

Proposition 1.1.6. [34, Lemma 2.1.1] Let (V, ω) be a symplectic vector space, and let $W \subseteq V$ be a linear subspace. Then

$$\dim W + \dim W^\omega = \dim V, \qquad (W^\omega)^\omega = W.$$

1.2 Symplectic Manifolds

Definition 1.2.1. Let M be a manifold. A two-form $\omega \in \Omega^2(M)$ is a **symplectic form** if it is closed ($d\omega = 0$) and the skew-symmetric bilinear map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is nondegenerate for every $p \in M$. In this case, the pair (M, ω) is called a **symplectic manifold**.

Given any two-form $\eta \in \Omega^2(M)$, we define the **rank** of η at a point $p \in M$ to be the rank of the resulting linear map $\eta_p : T_pM \to T_p^*M$. Thus, a symplectic form is a closed two-form on M which has full rank everywhere. Note that if (M, ω) is a symplectic manifold, each pair (T_pM, ω_p) is a symplectic vector space.

As a consequence of Theorem 1.1.2, we find that a symplectic manifold is necessarily even-dimensional.

Example 1.2.2. We describe the standard symplectic structure on $\mathbb{R}^{2m}$.
Let $x_1, \dots, x_m, y_1, \dots, y_m$ be standard coordinates on $\mathbb{R}^{2m}$. Define $\omega_0 \in \Omega^2(\mathbb{R}^{2m})$ by

$$\omega_0 = \sum_{j=1}^{m} dx_j \wedge dy_j,$$

which is clearly nondegenerate. As the de Rham differential squares to zero, we find that ω_0 is closed. Thus, we conclude that $(\mathbb{R}^{2m}, \omega_0)$ is a symplectic manifold. Note that for each $p \in M$ the ordered basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_m}\right)_p$$

is a symplectic basis of (T_pM, ω_p).

Definition 1.2.3. Let (M, ω) and (M', ω') be $2m$-dimensional symplectic manifolds. A **symplectomorphism** between (M, ω) and (M', ω') is a diffeomorphism $\Phi : M \to M'$ which preserves the symplectic forms in the sense that $\Phi^*\omega' = \omega$. We denote by $\mathrm{Sympl}(M, \omega)$ the group of symplectomorphisms from (M, ω) to itself.

Definition 1.2.4. Let (M, ω) be a symplectic manifold, and let $S \subseteq M$ be a submanifold. The submanifold S is **isotropic (coisotropic, symplectic, Lagrangian)** if for all $p \in S$ the tangent space T_pS is an isotropic (coisotropic, symplectic, Lagrangian) subspace of the symplectic vector space (T_pM, ω_p).

Example 1.2.5. We show that the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ admits a symplectic form. Define the two-form $\omega \in \Omega^2(\mathbb{S}^2)$ pointwise by

$$\omega_p(u, v) = \langle p, u \times v \rangle, \text{ for } u, v \in T_p\mathbb{S}^2,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $\times$ denotes the cross product. Note that ω is trivially closed being a two-form on a two-manifold. Figure 1.1 illustrates that ω is a nondegenerate two-form on $\mathbb{S}^2$.

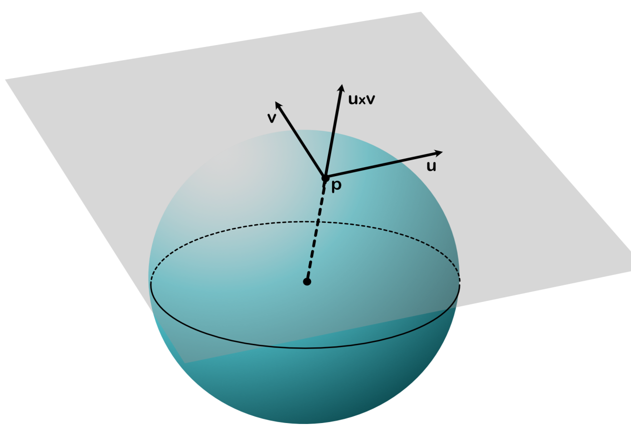


Figure 1.1: By application of the right hand rule we deduce that ω is a nondegenerate form on the sphere $\mathbb{S}^2$.

By checking the values of ω_p on basis tangent vectors of $T_p\mathbb{S}^2$, we find that ω is the standard volume form on the two-sphere, namely:

$$\omega = x_1 \; dx_2 \wedge dx_3 + x_2 \; dx_3 \wedge dx_1 + x_3 \; dx_1 \wedge dx_2 \in \Omega^2(\mathbb{S}^2).$$

By similar arguments, we find that any orientable surface admits a symplectic form.

Which manifolds admit a symplectic structure?

We study necessary properties of a manifold to be a symplectic manifold. The main purpose of this subsection is to show that a symplectic manifold is orientable and to recall some theory regarding de Rham cohomology.

Proposition 1.2.6. Let V be a $2m$-dimensional vector space equipped with a skew-symmetric bilinear form ω. Then ω is symplectic if and only if m-fold wedge product $\omega^m = \omega \wedge \cdots \wedge \omega$ does not vanish.

Proof. In view of Theorem 1.1.2, we may assume without loss of generality that $(V, \omega) = (\mathbb{R}^n, \omega_0)$ (under suitable identifications). We find that the m-fold wedge product of $\omega_0 = \sum_{j=1}^{m} dx_j \wedge dy_j$ is given by

$$\omega_0^m = m!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_m \wedge dy_m. \tag{1.2}$$

It follows that ω_0^m does not vanish.

We prove the converse statement by contraposition. Suppose that $\ker \omega \neq \{0\}$. Then there exists a nonzero vector $u \in V$ such that $\omega(u, \cdot) \equiv 0$. We find that

$$i_u \omega^m = m(\imath_u \omega) \wedge \omega^{m-1} = 0,$$

which implies that ω^m vanishes. □

Let (M, ω) be a $2m$-dimensional symplectic manifold. The above proposition implies that $\omega_p^m \neq 0$ for all $p \in M$, or equivalently, that $\omega^m \in \Omega^{2m}(M)$ is a volume form. Recall that a volume form induces an orientation [28, Proposition 15.5]. Thus, a symplectic manifold (M, ω) is canonically oriented through the symplectic form. The above discussion and the constant appearing in Equation (1.2) leads to the following definition.

Definition 1.2.7. Let (M, ω) be a $2m$-dimensional symplectic manifold. The **Liouville volume form** on M is the volume form $\frac{\omega^m}{m!} \in \Omega^{2m}(M)$.

Whenever we use the orientability of a symplectic manifold (M, ω), we refer to the orientation on M induced by the Liouville volume form.

Proposition 1.2.8. [4, Proposition II.1.6] Let (M, ω) be a $2m$-dimensional compact connected symplectic manifold without boundary. For each $k = 1, \ldots, n$ the de Rham cohomology group $H^{2k}_{\mathrm{dR}}(M)$ is non-trivial.

Proof. By definition, the symplectic form ω is closed, so that ω represents a cohomology class $[\omega] \in H^2_{\mathrm{dR}}(M)$ and similarly we have $[\omega^k] \in H^{2k}_{\mathrm{dR}}(M)$ for $k = 1, \ldots, m$. For the sake of contradiction, suppose that $H^{2k}_{\mathrm{dR}}(M) = \{0\}$ for some k. Then $\omega^k = d\eta$ for some $2k-1$-form $\eta \in \Omega^{2k-1}(M)$. Since ω is closed and the de Rham differential is a derivation, we have $d(\eta \wedge \omega^{m-k}) = \omega^m$. By Stoke's theorem, we have

$$\int_M \omega^m = \int_M d(\eta \wedge \omega^{m-k}) = \int_{\partial M} \eta \wedge \omega^{m-k} = 0,$$

which contradicts the fact that $\omega^m \in \Omega^{2m}(M)$ is a volume form. □

Example 1.2.9. Recall that $H^2_{\mathrm{dR}}(\mathbb{S}^{2m}) = \{0\}$ for $m > 1$ [28, Theorem 17.21]. In view of Proposition 1.2.8, we find that the $2m$-sphere $\mathbb{S}^{2m}$ ($m > 1$) does not admit a symplectic structure.

Thus, the even-dimensional de Rham cohomology groups form a potential obstruction for the existence of a symplectic structure.

1.3 Riemannian Manifolds

We now discuss another important structure on manifolds, namely Riemannian metrics. The main purpose of this subsection is to quickly get to the definition of a geodesic, which is necessary to define the exponential map of a Riemannian metric. In the subsequent sections, we will use the exponential map of a Riemannian metric to construct a local model of a manifold. This subsection is based on Lee's *Introduction to Riemannian Manifolds* [29].

Definition 1.3.1. Let M be a manifold. A **Riemannian metric** on M, denoted by $\langle\cdot,\cdot\rangle$ or m, is a smooth assignment of an inner product $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM to each point $p \in M$. We require a Riemannian metric to be smooth in the sense that, for vector fields $X, Y \in \mathfrak{X}(M)$, the real-valued function $M \ni p \mapsto \langle X_p, Y_p\rangle_p$ is smooth.

A **Riemannian manifold** is a pair $(M, \langle\cdot,\cdot\rangle)$, where M is a manifold and $\langle\cdot,\cdot\rangle$ is a Riemannian metric on M.

A Riemannian metric always exists, as the following proposition asserts. It may be proven using a partition of unity argument.

Proposition 1.3.2. [29, Proposition 2.4] Every manifold admits a Riemannian metric.

Definition 1.3.3. Let (M, m) and (M', m') be Riemannian manifolds. A diffeomorphism $F : M \to M'$ is an **isometry** from (M, m) to (M', m') if the map F preserves the Riemannian metric, meaning that $F^*m' = m$. Explicitly, a diffeomorphism F is an isometry if

$$(F^*m')_p(u, v) = m'_{F(p)}(dF_p(u), dF_p(v)) = m_p(u, v), \text{ for } p \in M,\ u, v \in T_pM.$$

An isometry of (M, m) is an isometry from (M, m) to itself.

Definition 1.3.4. Let $\pi : E \to M$ be a vector bundle over a manifold M, and let $\Gamma(E)$ be the vector space of sections corresponding to $\pi : E \to M$. A **connection** on E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),\ (X, s) \mapsto \nabla_X s,$$

which satisfies the following properties, for all $X \in \mathfrak{X}(M),\ s \in \Gamma(E),\ f \in C^\infty(M)$:

- $\nabla_X s$ is $C^\infty(M)$-linear in X;
- $\nabla_X s$ is $\mathbb{R}$-linear in s;
- $\nabla_X(fs) = \mathcal{L}_X(f)s + \nabla_X s$.

Let M be a manifold, and consider the tangent bundle $TM \to M$. Recall that $\Gamma(TM) = \mathfrak{X}(M)$, so that a connection on the tangent bundle is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying the three properties above. In this setting, we have the following notion.

Definition 1.3.5. Let M be a manifold, and let ∇ be a connection on the tangent bundle $TM \to M$. We define the **torsion** T_∇ of the connection ∇ by

$$T_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

A connection on the tangent bundle TM is called **torsion-free** if $T_\nabla \equiv 0$.

Definition 1.3.6. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. A connection ∇ on the tangent bundle TM is **compatible with the metric** $\langle\cdot,\cdot\rangle$ if for all vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have:

$$\mathcal{L}_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Theorem 1.3.7 (Levi-Civita Connection). *[29, Theorem 5.10] Let (M, m) be a Riemannian manifold. There exists a unique ∇ on the tangent bundle TM which is compatible with the metric m and torsion-free. This connection is called the* ***Levi-Civita connection*** *associated to the metric m.*

Definition 1.3.8. Let M be a manifold, and let $\gamma : I \to M$ be a smooth curve. A **vector field along the curve** γ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$. We denote by $\mathfrak{X}(\gamma)$ the vector space of vector fields along the curve γ. By definition of the pullback bundle, we have $\mathfrak{X}(\gamma) = \Gamma(\gamma^*(TM))$.)

Let M be a manifold, and suppose $\gamma : I \to M$ is a smooth curve. A natural example of a vector field along γ is the map γ' defined by

$$\gamma' : I \to TM,\ s \mapsto \gamma'(s) := \left.\frac{d\gamma}{dt}\right|_{t=s}.$$

We will need the following proposition.

Proposition 1.3.9 (Covariant Derivative Along a Curve). [29, Theorem 4.24] Let M be a manifold and let ∇ be a connection on the tangent bundle TM. Suppose $\gamma : I \to M$ is a smooth curve. Then the connection ∇ determines a unique operator

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

satisfying the following properties, for all $V \in \mathfrak{X}(\gamma)$ and $f \in C^\infty(I)$:

- D_t is linear over $\mathbb{R}$.
- $D_t(fV) = \frac{df}{dt}V + fD_t(V)$.
- If there exists a local vector field $\tilde{V}$ defined on a neighborhood of $\gamma(I)$ such that $V(t) = \tilde{V}_{\gamma(t)}$ for all $t \in I$, then

$$D_t V(t) = \nabla_{\gamma'(t)}\tilde{V}.$$

The map $D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ is called the **covariant derivative along the curve** γ. Finally, we introduce the notion of a geodesic:

Definition 1.3.10. Let M be a manifold, and let ∇ be a connection on the tangent bundle TM. A smooth curve $\gamma : I \to M$ is a **geodesic** if the covariant derivative of the velocity vector field $\gamma' \in \mathfrak{X}(\gamma)$ along γ is identically zero, that is: $D_t(\gamma') \equiv 0$.

Note that we only used a connection on the tangent bundle to define geodesics, and that we did not make use of a Riemannian structure. Whenever we refer to a geodesic on a Riemannian manifold (M, m), we implicitly assume that the curve is a geodesic with respect to the Levi-Civita connection associated with the metric m. Fortunately, geodesics always exist and are unique.

Theorem 1.3.11. *[29, Corollary 4.28, 6.22] Let M be a manifold, and let ∇ be a connection in the tangent bundle TM. For each $p \in M$ and each tangent vector $v \in T_pM$, there exists a unique maximal geodesic $\gamma : I \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$, defined on an open interval I containing 0. If in addition, the Riemannian manifold M is compact, then maximal geodesics exist for all time.*

Geodesics are well-behaved under composition with an isometry:

Proposition 1.3.12. [29, Corollary 5.14] Let (M, m) and (M', m') be Riemannian manifolds. Suppose that $F : M \to M'$ is an isometry. Then F takes geodesics to geodesics. More precisely, if γ is a geodesic in (M, m), then the composition $F \circ \gamma$ is a geodesic in (M', m').

Exponential map of a Riemannian metric

We now introduce the exponential map of a Riemannian metric. There exists also an exponential map of a Lie group, which we will introduce later on. The construction of an exponential map is roughly as follows. Suppose we have a manifold endowed with some structure. We consider the curve on the manifold which reflects properties of the structure and satisfies some specified initial conditions, then the exponential map is defined to be the value of the curve at time $t = 1$. In the case of a Riemannian manifold (M, m), the structure on the manifold is the Riemannian metric, and the relevant curves are the geodesics with respect to the Levi-Civita connection.

Definition 1.3.13. Let (M, m) be a Riemannian manifold, and let $p \in M$. Let U be am open neighborhood of $0 \in T_pM$ such that for each $v \in U$ the maximal geodesic $\gamma_v(t)$ satisfying $\gamma_v(0) = p$, $\gamma_v'(0) = v$ is defined for all $t \in [0, 1]$. (If M is compact, we set $U := T_pM$.)

We define the **exponential map of the Riemannian metric** (at p), denoted by $\exp_p$, to be the map

$$\exp_p : T_pM \supseteq U \to M, \ v \mapsto \gamma_v(1),$$

where $\gamma_v : \mathbb{R} \to M$ is the unique maximal geodesic satisfying $\gamma_v(0) = p$ and $\gamma'(0) = v$. In view of Theorem 1.3.11 this map is well-defined.

We summarize some of its properties.

Proposition 1.3.14. [29, Proposition 5.19] Let (M, m) be a Riemannian manifold, and let $p \in M$. The exponential map $\exp_p : T_pM \supseteq U \to M$ of the Riemannian metric satisfies the following properties.

- The exponential map is smooth.
- For each tangent vector $v \in T_pM$, the geodesic γ_v starting at p with initial velocity v is given by

$$\gamma_v(t) = \exp_p(tv). \tag{1.3}$$

- The differential $d(\exp_p)_0 : T_pM \to T_pM$ at 0 is the identity map on the tangent space T_pM.

By virtue of the inverse function theorem, the latter property implies that $\exp_p$ is a local diffeomorphism from an open neighborhood of $0 \in T_pM$ to an open neighborhood of $p \in M$. In this way, we obtain a local model (a local parametrization compatible with the relevant structures) of each point in the manifold.

1.4 The Darboux Theorem

We have seen that symplectic vector spaces of the same dimension are symplectomorphic. A natural question would be to ask if symplectic manifolds of the same dimension are also symplectomorphic. It turns out that this is true *locally*, a result due to Darboux. We will take the following theorem, due to A. Weinstein and J. Moser, for granted.

Theorem 1.4.1 (**Moser–Weinstein Theorem**). *[4, Theorem II.1.9] Let $S \subseteq M$ be a submanifold of M. Let ω_0 and ω_1 be two symplectic forms on M such that $(\omega_0)_p = (\omega_1)_p$ for all $p \in S$. Then there exists a neighborhood U of S in M and a smooth map $\Phi : U \to M$ such that $i^*\Phi = Id_S$ and $\Phi^*\omega_1 = \omega_0$.*

We will use it to prove the Darboux Theorem. Note the role played by the exponential map of a Riemannian metric to identify a neighborhood of the manifold with a neighborhood of the tangent space.

Corollary 1.4.2 (**Darboux Theorem**). *[4, Corollary II.1.11] Let (M, ω) be a $2m$-dimensional symplectic manifold. For every $p \in M$ there exist local coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ centered at p in which the symplectic form ω on M is the standard symplectic form $\sum_{j=1}^m dx_j \wedge dy_j$.*

Proof. Let $p \in M$. Choose a Riemannian metric on M, and consider the exponential map of the metric. By Proposition 1.3.14, there exists an open neighborhood V of 0 in T_pM and an open neighborhood U of p in M such that $\exp_p$ maps V diffeomorphically onto U, denote $\varphi = \exp_p : V \to U$. Then $(\varphi^{-1})^*(\omega_p), \omega \in \Omega^2(U)$ are two symplectic forms on U, which coincide at the submanifold $Y := \{p\}$.

By application of the Moser–Weinstein theorem (and shrinking U if necessary), there exists an open neighborhood U' of p in M and a diffeomorphism $\Phi : U \to U'$ such that

$$\Phi^*\omega = (\varphi^{-1})^*(\omega_p). \tag{1.4}$$

By Theorem 1.1.2, there exist coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ on $V \subseteq T_pM$ such that $\omega_p = \sum_j dx_j \wedge dy_j$. Now, Equation (1.4) implies that

$$(\Phi \circ \varphi)^*\omega = \omega_p = \sum_{j=1}^{m} dx_j \wedge dy_j,$$

which proves the claim. □

As a consequence of the Darboux Theorem, we conclude that all symplectic manifolds of equal dimension are locally symplectomorphic.

1.5 Almost Complex Structures

We start this subsection with a motivating example.

Example 1.5.1. Consider $\mathbb{C}^m$ with the Hermitian inner product

$$H : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C},\ H(u, v) = \sum_{j=1}^{m} \overline{u_j} v_j.$$

Define a map $J : \mathbb{C}^m \to \mathbb{C}^m$ by $Ju := iu$. Note that $J^2 = -\mathrm{Id}_{\mathbb{C}^m}$. We decompose the Hermitian form into a real part and an imaginary part:

$$H(u, v) = \langle u, v\rangle + i\omega_0(u, v),$$

so that $\langle \cdot, \cdot\rangle = \mathrm{Re}\ H(\cdot, \cdot)$ and $\omega_0(\cdot, \cdot) = \mathrm{Im}\ H(\cdot, \cdot)$. Note that $\langle \cdot, \cdot\rangle$ is the standard Euclidean inner product on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Since $H(u, v) = \overline{H(v, u)}$, we have that $\omega_0 : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{R}$ is a skew-symmetric bilinear map. Note that $H(u, Jv) = iH(u, v)$, this implies that

$$\langle u, v\rangle = \omega_0(u, Jv),\ \text{for } u, v \in \mathbb{C}^m.$$

From this equality, it follows that ω_0 is nondegenerate. Thus, $\omega = \mathrm{Im}\ H$ is a symplectic form on $\mathbb{C}^m$. Writing the complex coordinates on $\mathbb{C}^m$ as $z_j = x_j + iy_j$, it is readily verified that ω is the standard symplectic form $\sum_{j=1}^{m} dx_j \wedge dy_j$. Indeed, let $u \in \mathbb{C}^m$ be an arbitrary nonzero vector, then

$$\omega_0(u, Ju) = \langle u, u\rangle > 0.$$

Since $H(Ju, Jv) = i\bar{i}H(u, v) = H(u, v)$, we find that J preserves the symplectic structure ω_0 in the sense that

$$\omega_0(Ju, Jv) = \omega_0(u, v),\ \text{for all } u, v \in \mathbb{C}^m.$$

These observations motivate the following definition.

Definition 1.5.2. A **linear complex structure** on a real finite-dimensional vector space V is an automorphism $J : V \to V$ satisfying $J^2 = -\mathrm{Id}_V$.

Let (V, ω) be a symplectic vector space. A complex structure J on V is called ω-**compatible** if $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in V$ and $\omega(u, Ju)$ for all nonzero $u \in V$.

Let (V, ω) be a symplectic vector space, and suppose J is a complex structure on V. Note that the condition that J is ω-compatible is equivalent to the condition that the symmetric bilinear form $m_J(u, v) := \omega(u, Jv)$ defines an inner product on V. We generalize this definition to a symplectic manifold, as follows.

Definition 1.5.3. An **almost complex structure** on a manifold M is a section $J \in \Gamma(\mathrm{End}(TM))$ such that at each point $p \in M$ the map $J_p : T_pM \to T_pM$ satisfies $J_p^2 = -\mathrm{Id}_{T_pM}$. The pair (M, J) is called an **almost complex** manifold.

Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called ω-**compatible** if at each $p \in M$ we have:

- $\omega_p(J_p u, J_p v) = \omega_p(u, v)$ for all tangent vectors $u, v \in T_pM$;
- $\omega_p(u, J_p u) > 0$ for each nonzero tangent vector $u \in T_pM$.

Let (M, ω) be a symplectic manifold equipped with an almost complex structure J. The almost complex structure is ω-compatible if and only if the map m_J, defined pointwise by

$$(m_J)_p : T_pM \times T_pM \to \mathbb{R}, \ (m_J)_p(u, v) := \omega_p(u, Jv), \tag{1.5}$$

is a Riemannian metric on M. The almost complex structure J turns each tangent space (T_pM, J_p) into a complex vector space via

$$(a + ib) \cdot v := av + bJ_p(v), \ \text{for } a, b \in \mathbb{R}, \ v \in T_pM.$$

Proposition 1.5.4 (**Polar Decomposition** [37, Theorem 6.59])**.** Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $A : \mathcal{H} \to \mathcal{H}$ be an invertible continuous linear operator. Then there exists a unique decomposition $A = |A|J$ into a symmetric positive-definite operator $|A|$ and an isometry J. The operators A and $|A|$ commute.

Proposition 1.5.5. [4, Lemma II.2.1.] Let (V, ω) be a symplectic vector space. Then there exists an ω-compatible complex structure J on V.

Proof. We equip the vector space V with an inner product $\langle \cdot, \cdot \rangle$. As ω and $\langle \cdot, \cdot \rangle$ are nondegenerate pairings, the equation

$$\langle Au, v \rangle = \omega(u, v), \ \text{for } u, v \in V \tag{1.6}$$

defines a linear isomorphism $A : V \to V$.

Let $u, v \in V$ be arbitary vectors. We find

$$\langle Au, v \rangle = \omega(u, v) = -\omega(v, u) = -\langle Av, u \rangle = -\langle u, Av \rangle,$$

it follows that A is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$, that is, we have $A^T = -A$. Applying the polar decomposition, we write $A = |A|J$. Since $|A|$ is symmetric and A is skew-symmetric, we obtain

$$J^T = (|A|^{-1}A)^T = A^T|A|^{-1} = -A|A|^{-1} = -|A|^{-1}A = -J. \tag{1.7}$$

The map J is an isometry for the inner product $\langle \cdot, \cdot \rangle$, meaning that $\mathrm{Id} = J^TJ$. Combining this with Equation (1.7) we find $J^2 = -\mathrm{Id}$, so that J is a complex structure on V. We compute

$$-|A|J = -A = A^T = (|A|J)^T = J^T|A|^T = -J|A|,$$

which shows that $|A|$ and J commute. Thus, for all $u, v \in V$, we obtain:

$$\omega(Ju, Jv) = \langle AJu, Jv \rangle = \langle JAu, Jv \rangle = \langle J^TJAu, v \rangle = \langle Au, v \rangle = \omega(u, v). \tag{1.8}$$

Using positive-definiteness of $|A|$, we find for all nonzero vectors $u \in V$ that:

$$\omega(u, Ju) = \langle Au, Ju \rangle = \langle -JAu, u \rangle = \langle |A|u, u \rangle > 0. \tag{1.9}$$

By Equation (1.8) and Equation (1.9), we conclude that J is an ω-compatible complex structure on the symplectic vector space (V, ω). ☐

Remark 1.5.6. Note that the inner product defined by $m_J(\cdot,\cdot) := \omega(\cdot, J\cdot) = \langle |A|\cdot, \cdot\rangle$ is generally different from the inner product $\langle\cdot,\cdot\rangle$ that we started with.

Let (M, ω) be a symplectic manifold. By Proposition 1.3.2 there exists a Riemannian metric $\langle\cdot,\cdot\rangle$ on M. For each $p \in M$ we obtain an ω-compatible complex structure J_p on the tangent space (T_pM, ω_p). One can verify that the polar decomposition is smooth (cf. [9, 12.2]), so that we obtain the following corollary:

Corollary 1.5.7. *Let (M, ω) be a symplectic manifold. Then there exists an ω-compatible almost complex structure J on (M, ω).*

The following proposition gives a useful criterion to check that a submanifold is a symplectic submanifold.

Proposition 1.5.8. Let (M, ω) be a symplectic manifold equipped with an ω-compatible almost complex structure J. Suppose $S \subseteq M$ is a submanifold which is J-invariant, that is: $J_p(T_pS) \subseteq T_pS$ for all $p \in S$. Then $(S, i^*\omega)$ is a symplectic submanifold of (M, ω), where $i : S \hookrightarrow M$ denotes the inclusion map.

Proof. Let $p \in S$, and let $v \in T_pS$ be any nonzero tangent vector. Since the submanifold S is J-invariant, we have that $J_p(v) \in T_pS$. As J is ω-compatible, it follows that

$$\omega_p(v, J_p(v)) > 0,$$

which implies that the restricted form $i^*\omega$ is nondegenerate. As the de Rham differential and pullbacks commute, we also have that $i^*\omega$ is closed. We conclude that $(S, i^*\omega)$ is a symplectic submanifold. □

1.6 Hamiltonian Vector Fields

Let (M, ω) be a symplectic manifold. By nondegeneracy of ω, we obtain a bijection between vector fields on M and one-forms on M through the map

$$\omega : \mathfrak{X}(M) \to \Omega^1(M),\ V \mapsto i_V\omega.$$

This leads to the following definition.

Definition 1.6.1. Let (M, ω) be a symplectic manifold, and let $H \in C^\infty(M)$ be a smooth real-valued function on M. The unique vector field $X_H \in \mathfrak{X}(M)$ satisfying $i_{X_H}\omega = dH$ is the **Hamiltonian vector field** of H. In this case, the function H is called the **Hamiltonian** for the vector field $X_H \in \mathfrak{X}(M)$.

Equivalently, a Hamiltonian vector field is a vector field V such that the one-form $i_V\omega$ is exact.

Definition 1.6.2. Let (M, ω) be a symplectic manifold. A vector field $V \in \mathfrak{X}(M)$ is a **symplectic vector field** if the one-form $i_V\omega$ is closed.

Suppose $V \in \mathfrak{X}(M)$ is a symplectic vector field. Since $d\omega = 0$, Cartan's formula

$$\mathcal{L}_V = di_V + i_Vd$$

implies that the vector field V is symplectic if and only if $\mathcal{L}_V\omega = 0$. By definition of the Lie derivative, this is equivalent to the flow ρ_t of V preserving the symplectic structure, meaning that $\rho_t^*\omega = \omega$ for all t.

Remark 1.6.3. By virtue of the Poincaré lemma any closed form is locally exact. This explains why symplectic vector fields are also called locally Hamiltonian vector fields.

Definition 1.6.4. Let (M, ω) be a symplectic manifold. We define the **Poisson bracket**, denoted by $\{\cdot,\cdot\}$, on the algebra of smooth functions $C^\infty(M)$ as the following bilinear map:

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),\ \{f, g\} := \omega(X_f, X_g).$$

The following proposition summarizes some of the properties of the Poisson bracket, an important consequence being that $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra.

Proposition 1.6.5. [9, 18.3] Let M be a manifold. The Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfies the following properties. For smooth real-valued functions $f, g, h \in C^\infty(M)$, we have:

- $\{f, g\} = -\{g, f\}$ (*Skew-symmetry*);
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (*Jacobi-identity*);
- $\{f, gh\} = \{f, g\} \cdot h + g \cdot \{f, h\}$ (*Leibniz rule*);
- $X_{\{f,g\}} = -[X_f, X_g]$ (*Preservation of brackets*).

Theorem 1.6.6. *[9, Theorem 18.9] Let (M, ω) be a symplectic manifold, and let $H, f \in C^\infty(M)$ be smooth functions on M. Then f and H Poisson commute ($\{f, H\} = 0$) if and only if f is constant along the integral curves of the Hamiltonian vector field $X_H \in \mathfrak{X}(M)$.*

Proof. Denote the flow of the Hamiltonian vector field X_H by ρ_t. By definition of a Hamiltonian vector field and Cartan's magic formula, we have

$$\begin{aligned}\mathcal{L}_{X_H} f &= \left.\frac{d}{dt}\right|_{t=0} \rho_t^* f = \left.\frac{d}{dt}\right|_{t=0} f(\rho_t(\cdot)) \\ &= i_{X_H} df = i_{X_H} i_{X_f} \omega = \omega(X_f, X_H) = \{f, H\},\end{aligned}$$

which proves the assertion. □

Let (M, ω) be a symplectic manifold, and $H \in C^\infty(M)$ a smooth function. An important special case of this proposition is that the integral curves of a Hamiltonian vector field X_H are contained in the level set of H, that is,

$$H(\rho_t(p)) = H(p),\ \text{for all } t,$$

where ρ_t denotes the flow of $X_H \in \mathfrak{X}(M)$. If we think of the Hamiltonian H as the total energy, this observation amounts to the conservation of total energy. The following example is in similar spirit.

Example 1.6.7 (**Hamilton's Equations of Motion** [9, 18.2])**.** Consider the phase-space $\mathbb{R}^{2m}$ with coordinates $(q_1, \dots, q_m, p_1, \dots, p_m)$, equipped with the standard symplectic form $\omega_0 = \sum_j dq_j \wedge dp_j$. Here we think of the coordinates q_j as position vectors and the coordinates p_j as momentum vectors of a physical system, which is subject to a Hamiltonian function $H : \mathbb{R}^{2m} \to \mathbb{R}$.

Let $\gamma(t) = (q_1(t), \ldots, q_m(t), p_1(t), \ldots, p_m(t))$ denote an integral curve of the Hamiltonian vector field X_H. Hamilton's equations of motions dictate that the component functions satisfy

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \ \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \qquad j = 1, \ldots, m.$$

It follows that the vector field $X_H \in \mathfrak{X}(\mathbb{R}^{2m})$ can be written as:

$$X_H = \sum_{j=1}^{m} \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}.$$

This explains why the defining relation $i_{X_H}\omega = dH$ of a Hamiltonian vector field is commonly refered to as Hamilton's equations. Note that the Poisson bracket is given by

$$\{f, g\} = \omega(X_f, X_g) = \sum_{j=1}^{m} \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j},$$

so that we can rewrite Hamilton's equations, as follows:

$$\{H, p_j\} = \frac{dp_j}{dt}, \ \{H, q_j\} = \frac{dq_j}{dt} \qquad j = 1, \ldots, m.$$

Summary

In this chapter we studied several structures regarding symplectic geometry. We looked at symplectic structures, Riemannian metrics, almost complex structures and the Poisson bracket. Furthermore, we defined Hamiltonian vector fields using the symplectic structure.

Chapter 2

Momentum Maps and Symplectic Reduction

In the previous chapter we saw that any smooth real-valued function on a symplectic manifold (M, ω) gives rise to a Hamiltonian vector field. We now want to explore this notion in the setting of a Lie group G acting on our manifold (M, ω) in a "Hamiltonian fashion". The main goal of this section is to introduce Lie groups and the momentum map μ, which encodes how the Lie group G generates Hamiltonian vector fields on M.

We start with a recollection of the definitions and basic properties concerning Lie groups. Then we introduce an important tool called the exponential map of a Lie group and apply it in our study of Lie group actions on manifolds. Next we define Hamiltonian actions and their associated momentum maps. Finally, we will use the previously mentioned tools to prove the symplectic reduction theorem, which states that (under some conditions) the quotient of a symplectic manifold by a Lie group is also a symplectic manifold.

2.1 Lie Groups

The following two sections are based on M. Audin's *Torus Actions on Symplectic Manifolds* [4], E. van den Ban's *Lecture Notes - Lie Groups* [5], and J. Lee's *Introduction to Smooth Manifolds* [28].

Definition 2.1.1. A **Lie group** is a manifold G equipped with a compatible group structure, meaning that the group operations multiplication $m : G \times G \to G,\ (g, h) \mapsto gh$ and inversion $j : G \to G,\ g \mapsto g^{-1}$ are smooth maps.

Suppose we have a subgroup H of a Lie group G. We say that H is a **Lie subgroup** of G if it also an embedded submanifold of G. This definition ensures that H is a Lie group with group operations inherited from G by restriction. Indeed, since H is a submanifold, the inclusion map $i : H \hookrightarrow G$ is smooth, which implies that $m|_{H\times H} = m \circ (i, i)$ and $j|_H = j \circ i$ are smooth.

Definition 2.1.2. Let G and H be Lie groups. A **Lie group homomorphism** from G to H is a smooth group homomorphism $\Phi : G \to H$. If it is invertible and $\Phi^{-1} : H \to G$ is also a Lie group homomorphism, we say that Φ is a **Lie group isomorphism** and that G, H are isomorphic Lie groups.

Note that if H is a Lie subgroup of G, then the inclusion map $i : H \hookrightarrow G$ is an injective Lie group homomorphism.

Example 2.1.3.

- The vector space $(\mathbb{R}^n, +)$ is an Abelian Lie group.
- The circle $\mathbb{S}^1 \subseteq \mathbb{C}^*$ is an Abelian Lie group under multiplication by complex numbers. The Cartesian product of Lie groups is a Lie group under componentwise multiplication. For example, the **torus** $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a compact connected Abelian Lie group being the n-fold product of circles, with multiplication given by:

$$\tau \cdot \tau' = (z_1, \ldots, z_n) \cdot (z_1', \ldots, z_n') = (z_1 \cdot z_1', \ldots, z_n \cdot z_n')$$

  for $\tau, \tau' \in \mathbb{T}^n$. Note that the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{T}^n$ given by $z \mapsto (z, z, \ldots, z)$ is indeed a Lie group homomorphism.
- The general linear group $\mathrm{GL}(n, \mathbb{R})$, consisting of invertible $n \times n$ matrices, is a Lie group under matrix multiplication. By continuity of the determinant $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{M}(n, \mathbb{R})$ is open and thus a submanifold of $\mathrm{M}(n, \mathbb{R})$. The multiplication map $(X, Y) \mapsto XY$ is smooth as it is given by polynomials in the entries of X, Y. Similarly, the inversion map is smooth by Cramer's rule.

We introduce the notion of a Lie algebra, which can be interpreted as an infinitesimal version of the object it is associated with.

Definition 2.1.4. A **Lie algebra** is a real vector space $\mathfrak{g}$ equipped with a bilinear map (called a Lie bracket)

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

which is antisymmetric and satisfies the Jacobi identity, for all $X, Y, Z \in \mathfrak{g}$:

- $[X, Y] = -[Y, X]$;
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

A **Lie subalgebra** of $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$, which is closed under the Lie bracket.

Suppose G is a Lie group. For every $g \in G$, define the left translation map $L_g : G \to G$ by $L_g(h) = gh$. Let $X \in \mathfrak{X}(G)$ be a smooth vector field on G, we say that X is a left-invariant vector field if $(L_g)_* X = X$ for all $g \in G$. Denote by $\mathrm{Lie}(G)$ the set of all left-invariant smooth vector fields on G. By naturality of Lie brackets we have for all $X, Y \in \mathrm{Lie}(G)$:

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y].$$

Hence the Lie bracket on $\mathfrak{X}(G)$ induces a Lie bracket on $\mathrm{Lie}(G)$. This is the Lie algebra of the Lie group G. We recall the following result:

Proposition 2.1.5. Let G be a Lie group. Then the evaluation map $\epsilon : \mathrm{Lie}(G) \to T_e G, X \mapsto X_e$ is a vector space isomorphism.

Henceforth, we won't distinguish between the two and will denote by $\mathfrak{g}$ the Lie algebra of the Lie group G.

Definition 2.1.6. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A **Lie algebra homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map $L : \mathfrak{g} \to \mathfrak{h}$ which is compatible with brackets, meaning that:

$$L[X, Y] = [LX, LY]$$

for all $X, Y \in \mathfrak{g}$. Similarly, we define a **Lie algebra antihomomorphism** K by the relation $K[X, Y] = -[KX, KY]$.

Given a Lie group homomorphism, there exists a natural induced Lie algebra homomorphism, as the following proposition shows.

Proposition 2.1.7. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose $\Phi : G \to H$ is a Lie group homomorphism. Then $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ defined by $\Phi_*(X) = d\Phi_e(X)$ is a Lie algebra homomorphism.

Proposition 2.1.8. Let $\Phi : G \to H$ be a Lie group homomorphism. Then Φ has constant rank.

Corollary 2.1.9. *Let $\Phi : G \to H$ be a Lie group homomorphism. If Φ is injective (surjective, bijective), then the induced Lie algebra homomorphism Φ_* is injective (surjective, bijective).*

Let H be a Lie subgroup of G. We have seen that the inclusion map $i : H \hookrightarrow G$ is an injective Lie group homomorphism. By the above, the induced Lie algebra homomorphism $i_* : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is injective. Therefore we identify $\mathfrak{h}$ with the Lie subalgebra $i_*(\mathfrak{h})$ of $\mathfrak{g}$.

The following theorem, due to Cartan, gives a very useful criterion for a subgroup to be a Lie subgroup.

Theorem 2.1.10 (**Cartan's Closed Subgroup Theorem**). *A (topologically) closed subgroup of a Lie group is an embedded Lie subgroup.*

Exponential map of a Lie group

We recall the definition of an integral curve. Let V be a smooth vector field on a manifold M. A smooth curve $\gamma : J \to M$ is an **integral curve** of V if

$$\gamma'(t_0) = V_{\gamma(t_0)} \quad \text{for all } t_0 \in J.$$

We now introduce an important tool, called the exponential map of a Lie group. This map allows us to study the Lie group, by working instead with the Lie algebra, a vector space. In the case of a group action on a manifold, we will use the exponential map of the Lie group to generate vector fields on the manifold.

Definition 2.1.11. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define the **exponential map** by

$$\exp : \mathfrak{g} \to G, \; X \mapsto \gamma_X(1),$$

where $\gamma : \mathbb{R} \to G$ is the (maximal) integral curve of X which starts at the identity.

Remark 2.1.12. Recall that a left-invariant vector field on a Lie group is complete ([28, Proposition 9.17]) so that the previous definition makes sense.

Note that, by definition, the image of the exponential map is contained in the path-component of the identity element of G. We now look at some properties of the exponential map.

Proposition 2.1.13. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ satisfies the following properties:

1. For any $X \in \mathfrak{g}$, the integral curve γ_X of X is given by $\gamma_X(t) = \exp(tX)$.
2. The exponential map $\exp : \mathfrak{g} \to G$ is a smooth map.
3. For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}, \exp(s+t)X = \exp sX \exp tX$.
4. For any $X \in \mathfrak{g}, (\exp X)^{-1} = \exp(-X)$.
5. The exponential map is a local diffeomorphism from $0 \in \mathfrak{g}$ to the identity $e \in G$.
6. Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively and $\Phi : G \to H$ be a Lie group homomorphism. Then the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\Phi} & H. \end{array}$$

7. Suppose H be a Lie subgroup of G, then $\mathfrak{h} \subseteq \mathfrak{g}$ is given by

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}$$

8. If G is Abelian, then for any $X, Y \in \mathfrak{g},\ \exp(X+Y) = \exp X \exp Y$.
9. The exponential map for the matrix Lie groups is the ordinary matrix exponential.

Example 2.1.14.

- Suppose G is an Abelian Lie group, then the Lie bracket is trivial. One way to see this is by noting that the inversion map $i : G \to G, g \mapsto g^{-1}$ is a group homomorphism. By Proposition 2.1.7 we have that $di_e : \mathfrak{g} \to \mathfrak{g}$ preserves Lie brackets. We have that $di_e(X) = -X$. Indeed, $\gamma(t) = \exp tX$ is a smooth curve starting at the identity with initial velocity X. In view of Proposition 2.1.13.4 we find:

$$di_e(X) = (i \circ \gamma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp(-tX) = -X.$$

Thus:

$$-[X, Y] = di_e[X, Y] = [di_e(X), di_e(Y)] = [X, Y],$$

which implies that the Lie bracket is trivial.

- We determine the Lie algebra $\mathfrak{s}$ of the circle $\mathbb{S}^1$. As the circle is one-dimensional its Lie algebra is one-dimensional also. Define

$$\frac{\partial}{\partial \theta} := \frac{d}{dt}\bigg|_{t=0} e^{2\pi i t} \in \mathfrak{s},$$

which is the tangent vector at the identity determined by rotation with period 1.

Consider the torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ with Lie algebra $\mathfrak{t}$. Similarly as before, we can find tangent vectors to the identity of the torus by rotating the j'th $\mathbb{S}^1$-factor:

$$\frac{\partial}{\partial \theta_j} := \frac{d}{dt}\bigg|_{t=0} (1, \dots, 1, e^{2\pi i t}, 1, \dots, 1) \in \mathfrak{t},$$

where j runs from 1 to n. The resulting tangent vectors at the identity span the Lie algebra of the torus. Using this basis we identify the Lie algebra $\mathfrak{t}$ of the torus with $\mathbb{R}^n$. Then the exponential map of the torus is given by:

$$\exp : \mathbb{R}^n \to \mathbb{T}^n, (x_1, \dots, x_n) \mapsto (e^{2\pi i x_1}, \dots, e^{2\pi i x_n}).$$

- We determine the Lie algebra $\mathfrak{u}(k)$ of the unitary group $U(k)$. Recall that

$$U(k) = \{U \in \mathrm{GL}(k, \mathbb{C}) : \; U^{-1} = U^*\},$$

where U^* is the adjoint (or Hermitian) of U, which is obtained by conjugating and transposing the matrix U. Define $g : \mathrm{GL}(k, \mathbb{C}) \to \mathrm{GL}(k, \mathbb{C})$ by $g(X) = XX^* - I$, so that $U(k) = g^{-1}(0)$. Differentiating the expression $g(X) = 0$ at the identity I yields

$$0 = g(X + I) - g(X) = (X + I)(X + I)^* - XX^* = X + X^*.$$

Thus the Lie algebra $\mathfrak{u}(k)$ is given by

$$\mathfrak{u}(k) = \{X \in M(k, \mathbb{C}) : X^* = -X\},$$

that is, the space of skew-adjoint matrices.

2.2 Lie Group Actions on Manifolds

We now consider Lie group actions on manifolds. Let G be a Lie group and M a manifold.

Definition 2.2.1. A **smooth action** of G on M is a smooth map $\psi : G \times M \to M$ satisfying the following properties,

- $\psi_g \circ \psi_g = \psi_{gg'}$;
- $\psi_e = \mathrm{Id}_M$,

for all $g, g' \in G$. The action is either denoted by $\psi_g : M \to M$ or by the shorthand notation $g \cdot p$.

Note that ψ_g is a diffeomorphism, since $\psi_{g^{-1}}$ is its smooth inverse. Therefore, we often think of ψ as a group homomorphism $\psi : G \to \mathrm{Diff}(M)$. We recall some definitions regarding group actions.

Definition 2.2.2. Let $\psi : G \times M \to M$ be a smooth action, and let $p \in M$.

The point p is a **fixed point** of the G-action on M if $g \cdot p = p$ for all $g \in G$. The fixed point set of the action is denoted by M^G.

Define the **orbit of p**, denoted $G \cdot p$, to be the images of p in M under the action ψ. In other words:

$$G \cdot p = \{\psi_g(p) : g \in G\} \subseteq M.$$

Define the **stabilizer of p**, denoted by G_p, by the subgroup of G consisting of all the group elements fixing p, that is:

$$G_p = \{g \in G : \psi_g(p) = p\} \subseteq G.$$

The action ψ is said to be **free** if all the stabilizers are trivial, or equivalently if the identity $e \in G$ is the only element that fixes points on the manifold M.

We describe a common procedure to generate vector fields on our manifold M using elements of the Lie algebra. Let $X \in \mathfrak{g}$ an element of the Lie algebra. Using the exponential map, we find for any $t \in \mathbb{R}$, a group element $\exp(tX) \in G$. By letting t vary we obtain for each point p in M a smooth curve contained in the orbit of p, defined by $\mathbb{R} \to M, t \mapsto \psi_{\exp(tX)}(p)$. Differentiating at $t = 0$ results in a tangent vector at p: $\frac{d}{dt}\big|_{t=0} \psi_{\exp(tX)}(p) \in T_pM$. This leads to the following definition.

Definition 2.2.3. Let $X \in \mathfrak{g}$ be an element of the Lie algebra. We define the **fundamental vector field of** X on M, denoted $\underline{X}$, pointwise by:

$$\underline{X}_p := \frac{d}{dt}\bigg|_{t=0} \exp tX \cdot p \in T_pM.$$

The map $\underline{\psi} : \mathfrak{g} \to \mathfrak{X}(M)$ defined by $\underline{\psi}(X) = \underline{X}$ is called the **infinitesimal generator of the action**.

It turns out that the infinitesimal generator of the action is a Lie algebra antihomomorphism, (Theorem 20.18, [28]).

Proposition 2.2.4. Let $\psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. Then the infinitesimal generator of the action $\underline{\psi} : \mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra antihomomorphism: $\underline{[X, Y]} = -[\underline{X}, \underline{Y}]$ for all $X, Y \in \mathfrak{g}$.

Let us now try to relate the stabilizer of p to the orbit of p in M. For this, define the **orbit map** $\Psi_p : G \to M$ by $\Psi_p(g) = g \cdot p$. Note that the image of Ψ_p is the orbit of p, and the stabilizer of p is the pre-image of p, that is:

$$\Psi_p(M) = G \cdot p, \;\; \Psi_p^{-1}(p) = G_p.$$

This shows that for any $p \in M$ the stabilizer G_p is closed, and thus a Lie subgroup of G. Denote by $\mathfrak{g}_p \subseteq \mathfrak{g}$ its Lie algebra. The following result describes this Lie algebra in more detail.

Lemma 2.2.5. *For any $p \in M$, the Lie algebra $\mathfrak{g}_p$ of the stabilizer of p is given by*

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : \underline{X}_p = 0\}.$$

Proof. Let $p \in M$. By a previous proposition, we have the following characterization of $\mathfrak{g}_p$:

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : \exp tX \in G_p \text{ for all } t \in \mathbb{R}\}.$$

Let $X \in \mathfrak{g}_p$. This characterization implies

$$\underline{X}_p = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot p = 0,$$

which proves the first inclusion.

Now suppose $X \in \mathfrak{g}$ such that $\underline{X}_p = 0$. Let $t_0 \in \mathbb{R}$ arbitrarily. Then by virtue of the chain rule,

$$\left.\frac{d}{dt}\right|_{t=t_0} \psi_{\exp tX}(p) = \left.\frac{d}{ds}\right|_{s=0} \psi_{\exp(t_0+s)X}(p) = \left.\frac{d}{ds}\right|_{s=0} \psi_{\exp t_0 X} \circ \psi_{\exp sX}(p)$$
$$= (d\psi_{\exp t_0 X})_p \left.\frac{d}{ds}\right|_{s=0} \psi_{\exp sX}(p) = (d\psi_{\exp t_0 X})_p(\underline{X}_p) = 0.$$

Hence $t \mapsto \exp tX \cdot p$ is constant and thus $\exp tX \cdot p = p$ for all $t \in \mathbb{R}$. In view of the mentioned characterization, this proves the opposite inclusion. □

We use this lemma in the following important result. We will implicitly use that the left coset space G/G_p is a quotient manifold (with dimension $\dim G - \dim G_p$ and that its tangent space at the identity may be identified with $\mathfrak{g}/\mathfrak{g}_p$ ([28, Theorem 21.17]).

Proposition 2.2.6. The orbit map $\Psi_p : G \to M$ descends to an injective immersion $\tilde{\Psi}_p : G/G_p \to M$. In particular, it follows that $\dim(G \cdot p) = \dim(G) - \dim(G_p)$.

Proof. We argue that it suffices to show that $\tilde{\Psi}_p$ is immersive at the identity. Define the right translation $R_g : G \to G$ by $R_g(h) = hg$, which is readily seen to be a diffeomorphism. Since $\Psi_p \circ R_g = \Psi_{g \cdot p}$, it follows by the chain rule that $d(\Psi_p)_g \circ d(R_g)_e = (d\Psi_{g \cdot p})_e$. Then $d(\Psi_p)_g$ is injective if and only if $d(\Psi_{g \cdot p})_e$ is injective, because $d(R_g)_e$ is an isomorphism.

Consider $d(\Psi_p)_e : \mathfrak{g} \to T_pM$. Let $X \in \mathfrak{g}$ such that $d(\Psi_p)_e(X) = 0$. Consider the smooth curve $\gamma(t) = \exp(tX)$, which satisfies: $\gamma(0) = p$ and $\gamma'(0) = X$. We find that:

$$0 = d(\Psi_p)_e(X) = (\Psi_p \circ \gamma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp tX \cdot p = \underline{X}_p,$$

which implies that $\ker(d\Psi_p)_e = \{X \in \mathfrak{g} : \underline{X}_p = 0\} = \mathfrak{g}_p$. Hence $\tilde{\Psi}_p : G/G_p \to M$ is an injective immersion. The last assertion follows by noting that the image of $\tilde{\Psi}_p$ is the orbit of p. □

Note that the previous proposition implies that any orbit of the action is an immersed submanifold. A crucial fact is that the **fundamental vector fields generate the tangent spaces to the orbits**, meaning that:

$$T_p(G \cdot p) = \{\underline{X}_p : X \in \mathfrak{g}\}.$$

Inspecting the proof again, we see that $d(\Psi_p)_e(X) = \underline{X}_p$ for any $X \in \mathfrak{g}$. This shows that $\{\underline{X}_p : X \in \mathfrak{g}\}$ is a linear subspace of $T_p(G \cdot p)$. The result of the proposition gives that their dimensions are equal: rank $[d(\Psi_p)_e] = \dim T_p(G \cdot p)$, which implies that $T_p(G \cdot p) = \{\underline{X}_p : X \in \mathfrak{g}\}$.

Note that the orbit map Ψ_p is compatible with the group actions of G on G and M in the sense that $\Psi_p(g \cdot g') = g \cdot \Psi_p(g')$. We will see many examples of such equivariant maps in the subsequent sections. This observation leads us to the following definition:

Definition 2.2.7. Let M and N be manifolds equipped with a smooth action of a Lie group G and let $\varphi : M \to N$ be a smooth map. It is said that φ is an **equivariant map** if

$$\varphi(g \cdot p) = g \cdot \varphi(p)$$

for all $g \in G$ and $p \in M$.

Finally, we state the quotient manifold theorem, which we will use in the proof of the symplectic reduction theorem. This remarkable theorem is also an inspiration for equivariant cohomology, which we will explore in section 4.3.

Theorem 2.2.8 (**Quotient Manifold Theorem** [28, Theorem 21.10])**.** *Suppose G is a compact Lie group acting freely on a manifold M. Then the orbit space M/G is a topological manifold of dimension* $\dim M - \dim G$*, and has a unique smooth structure with the property that $\pi : M \to M/G$ is a smooth submersion.*

Adjoint and coadjoint actions

Suppose we have a smooth G-action $\psi : G \times M \to M$ on a manifold M. We describe a natural G-action on the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Recall that G acts on itself by conjugation, let $g \in G$:

$$C_g : G \to G,\ h \mapsto ghg^{-1}.$$

We see that C_g is a diffeomorphism with smooth inverse $C_{g^{-1}}$, thus differentiating C_g at the identity $e \in G$ yields an isomorphism $d(C_g)_e : \mathfrak{g} \to \mathfrak{g}$. Allowing $g \in G$ to vary, we obtain the **adjoint action** of G on the Lie algebra $\mathfrak{g}$:

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}),\ g \mapsto \mathrm{Ad}_g.$$

Let $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R},\ (\xi, X) \mapsto \langle \xi, X \rangle = \xi(X)$ be the pairing between the Lie algebra and its dual. Since $\mathfrak{g}$ is finite-dimensional, any linear map $A : \mathfrak{g} \to \mathfrak{g}$ gives rise to a unique dual map $A^* : \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $A^*\xi = \xi \circ A$ for $\xi \in \mathfrak{g}^*$. As the adjoint action gives an endomorphism of the Lie algebra for each element in the group, we may apply the above procedure to obtain endomorphisms of the dual of the Lie algebra.

Given $\xi \in \mathfrak{g}^*$, define Ad_g^* by the equation $\langle \mathrm{Ad}_g^*\xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle$, where $X \in \mathfrak{g}$. This construction yields the **coadjoint action** of G on the dual of the Lie algebra $\mathfrak{g}^*$:

$$\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*),\ g \mapsto \mathrm{Ad}_g^*.$$

The terms adjoint and coadjoint *action* are not fully substantiated yet: it remains to check that they are indeed group homomorphisms.

Proposition 2.2.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The adjoint action Ad of G on $\mathfrak{g}$ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$ are group homomorphisms.

Proof. Let $g, h \in G$. Note that $C_g \circ C_h = C_{gh}$, we have that:

$$\mathrm{Ad}_g \circ \mathrm{Ad}_h = d(C_g)_e \circ d(C_h)_e = d(C_g \circ C_h)_e = d(C_g h)_e = \mathrm{Ad}_{gh}.$$

Hence, the adjoint action is a group homomorphism.

Let $\xi \in \mathfrak{g}^*, X \in \mathfrak{g}$, we find:

$$\langle \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^* \xi, X\rangle = \langle \mathrm{Ad}_h^* \xi, Ad_{g^{-1}} X\rangle = \langle \xi, \mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_{g^{-1}} X\rangle = \langle \xi, \mathrm{Ad}_{(gh)^{-1}} X\rangle = \langle \mathrm{Ad}_{gh}^* \xi, X\rangle,$$

which shows that the coadjoint action is a group homomorphism. □

Now it is natural to ask how the fundamental vector fields of the adjoint- and coadjoint action look like. The following proposition gives a result in this direction.

Proposition 2.2.10 ([4], [9])**.** Let G be a Lie group. Let $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Denote by $\underline{X}_Y$ the fundamental vector field at Y on the Lie algebra $\mathfrak{g}$ associated to the adjoint action, and by $\underline{X}_\xi$ the fundamental vector field at ξ on $\mathfrak{g}^*$ associated to the coadjoint action. Then:

- $\underline{X}_Y = [X, Y]$;
- $\langle \underline{X}_\xi, Y\rangle = \langle \xi, [Y, X]\rangle$.

Note that the first part of this proposition gives another way of seeing that the Lie bracket of Abelian Lie groups is trivial.

2.3 Momentum Maps

Throughout the following, let G be a Lie group and (M, ω) be a symplectic manifold.

Definition 2.3.1. Let ψ be a smooth action of G on M. We say that $\psi : G \times M \to M$ is a **symplectic action** if it preserves the symplectic structure for all $g \in G$, that is:

$$\psi_g^* \omega = \omega.$$

Thus $\psi_g : M \to M$ is a symplectomorphism for all $g \in G$, consequently we think of a symplectic action as a group homomorphism $\psi : G \to \mathrm{Sympl}(M, \omega)$.

Let $\psi : G \times M \to M$ be a symplectic action of G on M. The above definition is natural in the sense that elements of the Lie algebra give rise to symplectic fundamental vector fields on M.

To see this, let $X \in \mathfrak{g}$ and consider its fundamental vector field $\underline{X}$ on M. The flow ρ_t of $\underline{X}$ is given by $\rho_t = \psi_{\exp tX}$, which implies that

$$\mathcal{L}_{\underline{X}} \omega = \left.\frac{d}{dt}\right|_{t=t_0} (\psi_{\exp tX})^* \omega = 0.$$

Since ω is closed, it follows by Cartan's formula $\mathcal{L}_{\underline{X}} = d \circ i_{\underline{X}} + i_{\underline{X}} \circ d$ that $i_{\underline{X}} \omega$ is closed: the fundamental vector field $\underline{X}$ is a symplectic vector field on M.

One particular class of symplectic actions are those where the one-forms $i_{\underline{X}}\omega$ are exact for all elements $X \in \mathfrak{g}$. We now want to introduce the notion of a Hamiltonian action of G on (M, ω). Let us first think of which properties it should have. Similarly as in the case of a symplectic action, we want each element of the Lie algebra to give rise to a Hamiltonian fundamental vector field on M. As any Hamiltonian vector field is in particular a symplectic vector field, we must require that the action ψ is symplectic.

Now suppose we have a symplectic action $\psi : G \times M \to M$ such that each fundamental vector field is Hamiltonian. Let $X_1, \ldots, X_k$ be a basis for the Lie algebra $\mathfrak{g}$, and let $\mu_1, \ldots, \mu_k \in C^\infty(M)$ be the Hamiltonians corresponding to the fundamental vector fields $\underline{X_1}, \ldots, \underline{X_k}$. Let $X = \sum_{j=1}^k a_j X_k \in \mathfrak{g}$ be an arbitrary element of the Lie algebra expressed in this basis. As the assignment $\mathfrak{g} \to \mathfrak{X}(M), X \mapsto \underline{X}$ is linear, the function $\mu_X := \sum_{j=1}^k a_j \mu_j$ is a Hamiltonian for the fundamental vector field $\underline{X}$:

$$d\mu_X = \sum_{j=1}^k a_j d\mu_j = \sum_{j=1}^k a_j i_{\underline{X_j}}\omega = i_{\underline{X}}\omega.$$

In this way we obtain a linear map $\mu : \mathfrak{g} \to C^\infty(M)$ satisfying

$$d\mu_X = i_{\underline{X}}\omega, \text{ for all } X \in \mathfrak{g}.$$

Let $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ be the (nondegenerate) pairing between the Lie algebra and its dual. By the expression

$$\langle \mu(p), X \rangle = \mu_X(p)$$

we may equivalently view the linear map $\mu : \mathfrak{g} \to C^\infty(M)$ as a smooth map $\mu : M \to \mathfrak{g}^*$ (and the other way around). We will only use such maps $\mu : M \to \mathfrak{g}^*$ which are equivariant with respect to the coadjoint action. The above discussion leads to the following definition:

Definition 2.3.2. A symplectic action $\psi : G \to \mathrm{Sympl}(M, \omega)$ is a **Hamiltonian action** if there exists a smooth map $\mu : M \to \mathfrak{g}^*$, satisfying the following properties:

- For each $X \in \mathfrak{g}$, the smooth function $\mu_X := \langle \mu, X \rangle \in C^\infty(M)$ is a Hamiltonian function for the fundamental vector field $\underline{X}$ on M:

$$d\mu_X = i_{\underline{X}}\omega. \tag{2.1}$$

  Here $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural pairing, so that we read $\mu_X : p \mapsto \langle \mu(p), X \rangle$.

- The map $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$:

$$\mu(g \cdot p) = \mathrm{Ad}_g^* \cdot \mu(p) \tag{2.2}$$

  for all $g \in G$ and $p \in M$.

In this case, it is said that $\mu : M \to \mathfrak{g}^*$ is a **momentum map** for the Hamiltonian action of G on M. The tuple (M, ω, G, μ) is called a **Hamiltonian G-space**.

Remark 2.3.3. A more physics-oriented motivation of the momentum map can be found in Souriau's *Structure des systèmes dynamiques* [40], where it is introduced as a generalization of Noether's theorem.

Before moving on to examples of Hamiltonian actions, let us study some important properties of the momentum map.

Proposition 2.3.4. Let $\mu : M \to \mathfrak{g}^*$ be a momentum map for the Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), and let $p \in M$. Then the following holds.

- For all $X \in \mathfrak{g}, v \in T_pM$, we have:
$$\langle d\mu_p(v), X\rangle = \omega_p(\underline{X}_p, v). \tag{2.3}$$
- For all $X \in \mathfrak{g}$, we have:
$$d\mu_p(\underline{X}_p) = \underline{X}_{\mu(p)}. \tag{2.4}$$
- The linear map
$$\mu : \mathfrak{g} \to C^\infty(M),\ X \mapsto \mu_X \tag{2.5}$$
is a Lie algebra homomorphism: $\mu_{[X,Y]} = \{\mu_X, \mu_Y\}$ for all $X, Y \in \mathfrak{g}$.

Proof. Let $X \in \mathfrak{g}$. By definition of a momentum map, μ_X is the Hamiltonian function for the fundamental vector field $\underline{X}$, that is, $i_{\underline{X}}\omega = d\mu_X$. Define the contraction map $c_X : \mathfrak{g}^* \to \mathbb{R}$ by $c_X(\xi) = \langle \xi, X\rangle$, so that we can write $\mu_X = c_X \circ \mu$. Since $\mathfrak{g}^*$ is a vector space and c_X is linear, it follows that $(dc_X)_{\mu(p)} = c_X$, where we identify the tangent space of $\mathfrak{g}^*$ with $\mathfrak{g}^*$. By application of the chain rule, we find:

$$\omega_p(\underline{X}_p, v) = (d\mu_X)_p(v) = (dc_X)_{\mu(p)} \circ d\mu_p(v) = c_X \circ d\mu_p(v) = \langle d\mu_p(v), X\rangle.$$

This proves the first assertion.

By equivariance of μ, we find:

$$\underline{X}_{\mu(p)} = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp tX} \cdot \mu(p) = \frac{d}{dt}\bigg|_{t=0} \mu(\exp tX \cdot p) = d\mu_p(\underline{X}_p),$$

since $t \mapsto \exp tX \cdot p$ is a smooth curve starting at p with initial velocity $\underline{X}_p$.

Let $X, Y \in \mathfrak{g}$. Note that μ_X, μ_Y are Hamiltonian functions for the fundamental vector fields $\underline{X}, \underline{Y}$ on M, so that their Poisson bracket is given by $\{\mu_X, \mu_Y\}(p) = \omega_p(\underline{X}_p, \underline{Y}_p)$. By the first and second paragraph we obtain $\omega_p(\underline{X}_p, \underline{Y}_p) = \langle d\mu_p(\underline{Y}_p), X\rangle = \langle \underline{Y}_{\mu(p)}, X\rangle$. Recall from Proposition 2.2.10 that $\langle \underline{Y}_{\mu(p)}, X\rangle = \langle \mu(p), [X, Y]\rangle = \mu_{[X,Y]}(p)$. Combining the above equalities yields $\mu_{[X,Y]} = \{\mu_X, \mu_Y\}$, which proves that $\mu : X \mapsto \mu_X$ is a Lie algebra homomorphism. □

Example 2.3.5.

1. We show that the symplectic action of $\mathbb{S}^1$ on $(\mathbb{C}^n, \omega_0)$ given by $s \cdot (z_1, \ldots, z_n) = (s \cdot z_1, \ldots, s \cdot z_n)$ is a Hamiltonian action with momentum map
$$\mu : \mathbb{C}^n \to \mathbb{R},\ \mu(z_1, \ldots, z_n) = -\pi \sum_{j=1}^{n} |z_j|^2.$$
Here we identify the dual of the Lie algebra $\mathfrak{s}^*$ with $\mathbb{R}^*$ using the previously mentioned basis and identify $\mathbb{R}^*$ with $\mathbb{R}$ through the inner product. Write the complex coordinates

as $z_k = x_k + iy_k$ so that the symplectic form on $\mathbb{C}^n$ is given by $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$.

The fundamental vector field of $X := \frac{\partial}{\partial\theta} \in \mathfrak{s}$ is given by:

$$\underline{X} = 2\pi \cdot (\sum_{j=1}^{n} -y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}),$$

which is the vector field on $\mathbb{C}^n$ generated by rotations.
As $\mu = \mu_X$ we find:

$$i_{\underline{X}}\omega_0 = 2\pi \sum_{j=1}^{n}(-y_j\omega_0(\frac{\partial}{\partial x_j}, \ \cdot\ ) - x_j\omega_0(\ \cdot\ , \frac{\partial}{\partial y_j})) = -2\pi \sum_{j=1}^{n}(y_j dy_j + x_j dx_j) = d\mu_X$$

Thus $d\mu_X = i_{\underline{X}}\omega$ holds and μ is $\mathbb{S}^1$-invariant, so that $\mu(z_1, \ldots, z_n) = -\pi \sum_{j=1}^n |z_j|^2$ is a momentum map for the action.

2. The symplectic action of the torus $\mathbb{T}^n$ on $(\mathbb{C}^n, \omega_0)$ given by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 \cdot z_n, \ldots, t_n \cdot z_n)$$

is Hamiltonian with momentum map

$$\mu : \mathbb{C}^n \to \mathbb{R}^n, \mu(z_1, \ldots, z_n) = -\pi(|z_1|^2, \ldots, |z_n|^2).$$

This can be seen by identifying the Lie algebra $\mathfrak{t}$ with $\mathbb{R}^n$, where the basis vectors of $\mathfrak{t}$ are the tangent vectors to the identity obtained by rotating a single $\mathbb{S}^1$-factor, and then following the above procedure for each basis vector of $\mathfrak{t}$.

3. Consider the symplectic action of $\mathbb{S}^1$ on the sphere $(\mathbb{S}^2, \omega)$ by rotation about the z-axis:

$$\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2,\ \theta \cdot (\varphi, h) = (\theta + \varphi, h).$$

Here $(\varphi, h) = (\rho(h)e^{i\varphi}, h)$ are cylindrical coordinates on the sphere, so that the standard volume form on the sphere is given by $\omega = d\varphi \wedge dh$. (Here $\rho(h)$ is the distance of a point p to the z-axis as a function of height.) We show that the action is Hamiltonian with momentum map

$$\mu : \mathbb{S}^2 \to \mathbb{R},\ \mu(\varphi, h) = 2\pi h.$$

The fundamental vector field of $X := \frac{\partial}{\partial\theta} \in \mathfrak{s}$ on the sphere at a point $p = (\varphi, h) = (\rho(h) \cdot e^{i\varphi}, h)$ is given by

$$\underline{X}_p = \frac{d}{dt}\bigg|_{t=0} e^{2\pi it} \cdot p = \frac{d}{dt}\bigg|_{t=0} (e^{2\pi it}\rho(h)e^{i\varphi}, h) = 2\pi \frac{\partial}{\partial\varphi}\bigg|_p \in T_p\mathbb{S}^2.$$

As $\mu = \mu_X$, we find $i_{\underline{X}}\omega = 2\pi dh = d\mu_X$, so that μ_X is a Hamiltonian function for the fundamental vector field $\underline{X}$ on the sphere. The momentum map $\mu(\varphi, h) = 2\pi h$ is $\mathbb{S}^1$-invariant, so that μ is a momentum map for the action.

4. Consider the action of $U(k)$ on $\mathbb{C}^k$ via matrix multiplication:

$$U(k) \times \mathbb{C}^k \to \mathbb{C}^k, (U, z) \mapsto Uz.$$

We show that

$$\mu : \mathfrak{u}(k) \to C^\infty(\mathbb{C}^k), X \mapsto \mu_X.$$

where $\mu_X(z) := \frac{i}{2} z^* X z$, is a momentum map for the action. Here we view the symplectic form ω_0 as the imaginary part of the Hermitian inner product $H(u, v) = v^* u$, that is,

$$\omega_0 : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{R},\ \omega_0(u, v) = \mathrm{Im}(H(u, v)).$$

As unitary matrices act by isometries (w.r.t the Hermitian inner product H and thus ω_0), the action is symplectic.

Let $X \in \mathfrak{u}(k)$, so that X is a skew-adjoint matrix ($X^* = -X$). Using the matrix exponential we find that the fundamental vector field of X at a point $z \in \mathbb{C}^k$ is given by

$$\underline{X}_z = \frac{d}{dt}\bigg|_{t=0} e^{tX} z = Xz.$$

We find:

$$\begin{aligned}(d\mu_X)_z(v) &= \frac{d}{dt}\bigg|_{t=0} \mu_X(z + tv) = \frac{d}{dt}\bigg|_{t=0} \frac{i}{2}[(z + tv)^* X (z + tv)] = \frac{i}{2}[v^* X z + z^* X v] \\ &= \frac{i}{2}[H(Xz, v) + H(Xv, z)] = \frac{i}{2}[H(Xz, v) + H(v, X^* z)] \\ &= \frac{i}{2}[H(Xz, v) - H(v, Xz)] = \mathrm{Im}[H(Xz, v)] = \omega_0(\underline{X}_z, v),\end{aligned}$$

which shows that μ_X is a Hamiltonian function for the fundamental vector field $\underline{X} \in \mathfrak{X}(\mathbb{C}^n)$.

Let $U \in U(k)$, we show that μ is equivariant with respect to the coadjoint action:

$$\mu_X(U \cdot z) = \frac{i}{2}(Uz)^* X (Uz) = \frac{i}{2} z^* U^* X U z = \mathrm{Ad}^*_U \cdot \mu_X(z)$$

for all $X \in \mathfrak{u}(k), z \in \mathbb{C}^n$. We conclude that μ is a momentum map for the action. By identifying the dual $\mathfrak{u}(k)^*$ with the Lie algebra $u(k)$ through the inner product on $M(k, \mathbb{C})$, we can write the momentum map as $\mu : \mathbb{C}^k \to \mathfrak{u}(k), \mu(z) = \frac{i}{2} z z^*$.

Proposition 2.3.6. Let $\mu : M \to \mathfrak{g}^*$ be the momentum map for a Hamiltonian action of a Lie group G on M. Suppose $H \subseteq G$ is a Lie subgroup of G. Denote by $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ the inclusion map with dual map $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$. Then $\mu' = \mathrm{pr} \circ \mu : M \to \mathfrak{h}^*$ is a momentum map for the restricted action of H on M.

Proof. Let $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ be the natural pairing. Recall that $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ being a dual map for $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ precisely means that $\langle \mathrm{pr}(\xi), X \rangle = \langle \xi, i(X) \rangle$ for all $\xi \in \mathfrak{g}^*, X \in \mathfrak{h}$.

Let $X \in \mathfrak{h}$. Then $\mu'_X = \langle \mu', X \rangle = \langle \mathrm{pr} \circ \mu, X \rangle = \langle \mu, i(X) \rangle = \mu_{i(X)}$ implies:

$$d(\mu'_X) = d\mu_{i(X)} = \omega(\underline{i(X)}, \cdot) = \omega(\underline{X}, \cdot),$$

so that μ'_X is a Hamiltonian function for the fundamental vector field of $X \in \mathfrak{h}$. □

Note that this proposition gives another way of finding the momentum map for the circle-action in Example 2.3.5. Let $\mathbb{S}^1 \to \mathbb{T}^n, i(z) = (z, \dots, z)$ denote the inclusion map of $\mathbb{S}^1$ into the torus, with induced Lie algebra homomorphism the inclusion of its Lie algebra $i : \mathbb{R} \hookrightarrow \mathbb{R}^n$, $i(v) = (v, \dots, v)$. Then the projection $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}$ dual to the inclusion is given by $\mathrm{pr}(w_1, \dots, w_n) = w_1 + \cdots + w_n$. Then composing the momentum map of the torus-action with the projection pr indeed gives the momentum map of the circle-action.

As previously mentioned, a crucial property of the momentum map is that it satisfies Noether's theorem, which relates the invariance of a Hamiltonian under transformations (symmetries of the system) to corresponding conservation laws. Classical examples of Noether's theorem include: translational invariance implies conservation of linear *momentum* and rotational invariance implies conservation of angular *momentum*.

Theorem 2.3.7 (Noether's theorem). *Let $\psi : G \to Sympl(M, \omega)$ be a Hamiltonian action of G on M with corresponding momentum map μ. A smooth function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the integral curves of the Hamiltonian vector field X_H of H. (Here G-invariant means that $\psi_g^* H = H$ for all $g \in G$.)*

Proof. Let $X \in \mathfrak{g}$ with fundamental vector field $\underline{X}$ on M. The flow of $\underline{X}$ is given by $\psi_{\exp tX}$. By definition of momentum map and Hamiltonian vector field we have:

$$\mathcal{L}_{X_H} \mu_X = i_{X_H} d\mu_X = i_{X_H} i_{\underline{X}} \omega = -i_{\underline{X}} i_{X_H} \omega = -\mathcal{L}_{\underline{X}} H = -\frac{d}{dt}\bigg|_{t=0} (\psi_{\exp tX})^* H.$$

Thus, μ is constant on the integral curves of X_H if and only if the smooth function H is G-invariant. □

2.4 Symplectic Reduction

The goal of this section is to prove the symplectic reduction theorem, which states that under some conditions the quotient of a symplectic manifold by a Lie group is again a symplectic manifold. In this thesis we will use symplectic reduction to obtain interesting examples of symplectic manifolds. In mechanics, it serves as a method to reduce the number of (redundant) variables in the presence of a symmetry (see, for example, *Introduction to Mechanics and Symmetry* [32]). We follow the proof of symplectic reduction as done in *Torus Actions on Symplectic Manifolds* [4] and *Symplectic geometry and analytical mechanics* [30].

We briefly sketch the procedure of symplectic reduction. Consider a momentum map $\mu : M \to \mathfrak{g}^*$ and suppose ξ is a regular value of μ. Let $i : \mu^{-1}(\xi) \hookrightarrow M$ be the inclusion map and $\pi : \mu^{-1}(\xi) \to M_{\mathrm{red}}$ be the (submersive) projection onto what we call the reduced symplectic manifold $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$. The defining equation for the reduced symplectic form ω_{red} on the reduced symplectic manifold M_{red} is given by:

$$\pi^* \omega_{\mathrm{red}} = i^* \omega.$$

We recall a notion from linear algebra concerning the natural pairing of a vector space with its dual. Note the similarity with the definition of an orthogonal complement.

Definition 2.4.1. Let V be a vector space with a linear subspace $F \subseteq V$, and consider the natural pairing $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. Define the **annihilator** of F, denoted by F^0, as the linear subspace of V^* whose elements are identically zero on F, that is:

$$F^0 = \{\alpha \in V^* : \langle \alpha, v \rangle = 0 \text{ for all } v \in F\}.$$

The following lemma studies the differential of the momentum map $d\mu_p : T_pM \to \mathfrak{g}^*$.

Lemma 2.4.2. *Suppose $\mu : M \to \mathfrak{g}^*$ be the momentum map of a Hamiltonian action of G on the symplectic manifold (M, ω), and let $p \in M$. Then the following hold for $d\mu_p : T_pM \to \mathfrak{g}^*$:*

1. $\ker d\mu_p = (T_p(G \cdot p))^\omega$;
2. $\operatorname{Im} d\mu_p = \mathfrak{g}_p^0$.

Proof. Let $v \in T_pM$. By *Proposition* 2.3.4, we have $\langle d\mu_p(v), \underline{X}_p \rangle = \omega_p(\underline{X}_p, v)$. This implies that $v \in \ker d\mu_p$ if and only if $v \in \{\underline{X}_p : X \in \mathfrak{g}\}^\omega$. As the fundamental vector fields generate the tangent spaces to the orbits, we conclude $\ker d\mu_p = (T_p(G \cdot p))^\omega$.

We determine the dual map $d\mu_p^* : \mathfrak{g} \to T_p^*M$: the unique map which satisfies $\langle d\mu_p(v), X \rangle = \langle v, (d\mu_p)^* X \rangle$ for all $v \in T_pM, X \in \mathfrak{g}$. In view of $\langle d\mu_p(v), X \rangle = \omega_p(\underline{X}_p, v)$, we find that $(d\mu_p)^*$ is given by $(d\mu_p)^*(X) = (i_{\underline{X}}\omega)_p$. for $X \in \mathfrak{g}$. By linear algebraic considerations we see that $\operatorname{Im} d\mu_p = (\ker d\mu_p^*)^0$. If X is in the kernel of the dual map $d\mu_p^*$, it follows by nondegeneracy of ω that $\underline{X}_p = 0$. By Lemma 2.2.5 we have that $\underline{X}_p = 0$ if and only if $X \in \mathfrak{g}_p$, where $\mathfrak{g}_p \subseteq \mathfrak{g}$ is the Lie algebra of the stabilizer of p, so that $\operatorname{Im} d\mu_p = \mathfrak{g}_p^0$. □

Suppose ξ is a regular value of a momentum map $\mu : M \to \mathfrak{g}^*$. Then $\mu^{-1}(\xi) \subseteq M$ is a submanifold of M and thus the inclusion map $i_\xi : \mu^{-1}(\xi) \hookrightarrow M$ is smooth. Note that the above lemma yields that the stabilizer of a point p is discrete, that is, zero-dimensional. We will now study in more detail the geometry of $\mu^{-1}(\xi)$ with the restricted form $i_\xi^*\omega$ (note that this form is generally not a symplectic form). The goal is to smoothly quotient out the kernel of this restricted form to obtain a nondegenerate form. Assuming that the reduced form ω_{red} exists and is nondegenerate, the relation $\pi^*\omega_{\text{red}} = i^*\omega$ shows that the restricted form $i_\xi^*\omega$ must have constant rank. This is the content of the following lemma.

Lemma 2.4.3. *Let $\mu : M \to \mathfrak{g}^*$ be the momentum map of a Hamiltonian action, and let ξ be a regular value of μ. The restricted form $i_\xi^*\omega$ has constant rank, and its kernel at a point $p \in \mu^{-1}(\xi)$ is given by $\ker(i_\xi^*\omega_p) = T_p(G_\xi \cdot p)$.*

Proof. To determine the kernel of the form $(i_\xi^*\omega)_p$, we view it as a linear map

$$(i_\xi^*\omega)_p : T_p(\mu^{-1}(\xi)) \to T_p^*(\mu^{-1}(\xi)).$$

Thus, we have that

$$\ker(i_\xi^*\omega)_p = T_p(\mu^{-1}(\xi)) \cap (T_p(\mu^{-1}(\xi))^\omega.$$

Since ξ is a regular value of μ, we find by application of the regular value theorem that $\ker d\mu_p = T_p(\mu^{-1}(\xi))$. Using Lemma 2.4.2 we can then rewrite the kernel as follows:

$$\ker(i_\xi^*\omega)_p = \ker d\mu_p \cap T_p(G \cdot p).$$

As the fundamental vector fields generate the tangent spaces to the orbits, we have that $v \in \ker d\mu_p \cap T_p(G \cdot p)$ if and only if $v = \underline{X}_p$ for some $X \in \mathfrak{g}$ and $\underline{X}_p \in \ker d\mu_p$. In view of Proposition 2.3.4 and Lemma 2.2.5, the following implications hold:

$$\underline{X}_p \in \ker d\mu_p \Leftrightarrow \underline{X}_\xi = \underline{X}_{\mu(p)} = 0 \Leftrightarrow X \in \mathfrak{g}_\xi.$$

As fundamental vector field generate the tangent spaces to the orbits, we have $X \in \mathfrak{g}_\xi$ if and only if $\underline{X}_p \in T_p(G_\xi \cdot p)$, so we conclude that $\ker(i_\xi^*\omega)_p = T_p(G_\xi \cdot p)$.

As ξ is a regular value of μ and $p \in \mu^{-1}(\xi)$, it follows that the rank of $d\mu_p$ is maximal, and by Lemma 2.4.2 that the stabilizer G_p of p is zero-dimensional. In particular, $(G_\xi)_p$ is zero-dimensional. Recall that $\dim G_\xi - \dim(G_\xi)_p = \dim G_\xi \cdot p$, it follows that:

$$\begin{aligned} \operatorname{rank}(i_\xi^*\omega)_p &= \dim \mu^{-1}(\xi) - \dim G_\xi \cdot p \\ &= \dim \mu^{-1}(\xi) - \dim G_\xi. \end{aligned}$$

We see that $i_\xi^*\omega$ has constant rank on $\mu^{-1}(\xi)$. □

The following theorem, called the Symplectic Reduction Theorem, is due to Marsden,Weinstein [33] and Meyer [35] independently.

Theorem 2.4.4 (**Marsden-Weinstein-Meyer Symplectic Reduction**). *Let (M, ω) be a symplectic manifold equipped with a Hamiltonian action ψ of a compact Lie group G. Let $\mu : M \to \mathfrak{g}^*$ be the momentum map associated to the Hamiltonian action. Suppose $\xi \in \mathfrak{g}^*$ is a regular value of μ and that the stabilizer G_ξ of ξ (w.r.t. the coadjoint action) acts freely on $\mu^{-1}(\xi)$. Then $M_{red} = \mu^{-1}(\xi)/G_\xi$ is a manifold and there exists a symplectic form ω_{red} on M_{red} satisfying*

$$\pi_\xi^*\omega_{red} = i_\xi^*\omega,$$

where $i_\xi : \mu^{-1}(\xi) \hookrightarrow M$ is the inclusion map and $\pi_\xi : \mu^{-1}(\xi) \to M_{red}$ is the projection map.

Proof. As ξ is a regular value of μ, the regular value theorem yields that $\mu^{-1}(\xi)$ is a submanifold of M. Let us now establish that $M_{\text{red}} = \mu^{-1}(\xi)/G_\xi$ is a manifold. For this, let $p \in \mu^{-1}(\xi)$ and $g \in G_\xi$. Since μ is equivariant, we find

$$\mu(g \cdot p) = \operatorname{Ad}_g^* \cdot \mu(p) = \operatorname{Ad}_g^* \cdot \xi = \xi,$$

which implies that $g \cdot p \in \mu^{-1}(\xi)$. As G_ξ is a Lie subgroup of the compact Lie group G, the action ψ thus restricts to a (smooth) free action of G_ξ, denoted by $\hat{\psi}$:

$$\hat{\psi} : G_\xi \times \mu^{-1}(\xi) \to \mu^{-1}(\xi).$$

By virtue of the quotient manifold theorem (Theorem 2.2.8), it follows that $M_{\text{red}} = \mu^{-1}(\xi)/G_\xi$ is a manifold with the property that the projection $\pi : \mu^{-1}(\xi) \to M_{\text{red}}$ is a submersion. Furthermore, the tangent space of M_{red} at $\pi(p) = G_\xi \cdot p$ can be canonically identified with the quotient vector space $T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p)$ through the surjective map $d\pi_p : T_p(\mu^{-1}(\xi)) \to T_{\pi(p)}M_{\text{red}}$, which has kernel $\ker d\pi_p = T_p(G_\xi \cdot p)$.

We make the identification $T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p) \cong T_{\pi(p)}M_{\text{red}}$ as above. Define $\omega_{\text{red}} \in \Omega^2(M_{\text{red}})$ pointwise by:

$$(\omega_{\text{red}})_{\pi(p)}([u], [v]) = \omega_p(u, v)$$

for $[u], [v] \in T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p)$. Note that $[u] = d\pi_p(u)$ and $[v] = d\pi_p(v)$, so that $\pi_\xi^*\omega_{\text{red}} = i_\xi^*\omega$ holds by construction. We show that ω_{red} is well-defined:

- Since the kernel of the restricted form $(i_\xi^*\omega)_p$ is $T_p(G_\xi \cdot p)$ and ω is bilinear, the value of ω_{red} is independent of the representatives picked for the equivalence classes $[u], [v] \in T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p)$.

- It remains to show that $(\omega_{\text{red}})_{\pi(p)}$ is independent of the point in $G_\xi \cdot p$ used for the identification of $T_{\pi(p)}M_{\text{red}}$. To this end, note that for $g \in G_\xi$ the differential $d(\hat{\psi}_g)_p : T_p(\mu^{-1}(\xi)) \to T_{g\cdot p}(\mu^{-1}(\xi))$ is an isomorphism which sends $T_p(G_\xi \cdot p)$ into $T_{g\cdot p}(G_\xi \cdot p)$. Thus $d(\hat{\psi}_g)_p$ descends to an isomorphism

$$d(\hat{\psi}_g)_p : T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p) \xrightarrow{\simeq} T_{g\cdot}(\mu^{-1}(\xi))/T_{g\cdot p}(G_\xi \cdot p).$$

We have $\pi \circ \hat{\psi}_g = \pi$ by definition of the projection π, so the following diagram commutes:

$$\begin{array}{ccc} T_p(\mu^{-1}(\xi))/T_p(G_\xi \cdot p) & \xrightarrow{d(\hat{\psi}_g)_p} & T_{g\cdot p}(\mu^{-1}(\xi))/T_{g\cdot p}(G_\xi \cdot p) \\ & {\scriptstyle d\pi_p} \searrow \quad \swarrow {\scriptstyle d\pi_{g\cdot p}} & \\ & T_{\pi(p)}M_{\text{red}} & \end{array}.$$

As $\hat{\psi}_g^*\omega = \omega$ for all $g \in G_\xi$, we conclude that ω_{red} is well-defined.

In view of the lemma above, the restriction $i_\xi^*\omega$ has constant rank with kernel at points $p \in \mu^{-1}(\xi)$ given by $T_p(G_\xi \cdot p)$ so that ω_{red} is nondegenerate.

Using that ω is closed and the fact that pullbacks and the de Rham differential commute, we find:

$$\pi^* d\omega_{\text{red}} = d\pi^*\omega_{\text{red}} = di_\xi^*\omega = i_\xi^* d\omega = 0.$$

Since π is a submersion, it follows that $\pi^* : \Omega^*(M_{\text{red}}) \to \Omega^*(\mu^{-1}(\xi))$ is an injective linear map. Then the result $d\omega_{\text{red}} \in \ker(\pi^*)$ implies that $d\omega_{\text{red}} = 0$. We conclude that ω_{red} is a symplectic form on M_{red}. ☐

Corollary 2.4.5. *Let $\mu : M \to \mathfrak{g}^*$ be the momentum map associated to a Hamiltonian action. Suppose G acts freely on $\mu^{-1}(0)$. Then $M_{red} = \mu^{-1}(0)/G$ is a manifold and there exists a symplectic form ω_{red} on M_{red} satisfying*

$$\pi^*\omega_{red} = i^*\omega.$$

Proof. As G acts freely on $\mu^{-1}(0)$, it follows that 0 is a regular value of μ. Note that the stabilizer of 0 with respect to the coadjoint action is the group G. The assertion now follows by application of the symplectic reduction theorem. ☐

We end this section with two examples of symplectic quotient manifolds, which can be found as exercises in [4], [9], [34].

Example 2.4.6 (**Complex Projective Space**)**.** Recall that the **complex projective space** of complex dimension n, denoted by $\mathbb{C}P^n$, is defined as the space of all complex lines in $\mathbb{C}^{n+1}$. Thus we can write

$$\mathbb{C}P^n = \{[z_0 : z_1 : \dots : z_n] : (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}\backslash\{0\}\},$$

where $[z_0 : z_1 : \dots : z_n]$ denotes the equivalence class of $(z_0, z_1, \dots, z_n)$ under the equivalence relation

$$(z_0, z_1, \dots, z_n) \sim (z_0', z_1', \dots, z_n') \Leftrightarrow (z_0', z_1', \dots, z_n) = (\lambda z_0, \lambda z_1, \dots, \lambda z_n)$$

for some non-zero complex scalar $\lambda \in \mathbb{C}^*$. We can also view $\mathbb{C}P^n$ as the quotient $\mathbb{S}^{2n}/\mathbb{S}^1$. Consider again the Hamiltonian action of the circle $\mathbb{S}^1$ on $\mathbb{C}^{n+1}$:

$$\mathbb{S}^1 \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, s \cdot (z_1, \dots z_n) = (s \cdot z_1, \dots, s \cdot z_n),$$

with momentum map

$$\mu : \mathbb{C}^n \to \mathbb{R},\ \mu(z_1, \dots, z_n) = -\pi \sum_{j=1}^{n} |z_j|^2.$$

We now show that $\mathbb{C}P^n$ is a symplectic manifold. As $\mu^{-1}(\frac{-1}{\pi}) = \mathbb{S}^{2n}$ and $\mathbb{S}^1$ acts freely on the sphere $\mathbb{S}^{2n}$, we conclude by application of the symplectic reduction theorem that $\mathbb{C}P^n = \mu^{-1}(\frac{-1}{\pi})/\mathbb{S}^1$ is a symplectic quotient manifold. The corresponding reduced symplectic form ω_{red} on $\mathbb{C}P^n$ obtained in this way is called the **Fubini-Study form**.

Example 2.4.7 (**Complex Grassmanian**)**.** We introduce the **Stiefel manifold**

$$V_k(\mathbb{C}^n) = \{A \in \mathbb{C}^{k \times n} : AA^* = I_k\},$$

which is the space of all ordered orthonormal k-frames in $\mathbb{C}^n$, and the **Complex Grassmannian** by

$$G_k(\mathbb{C}^n) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

Here a k-frame is an ordered set which consists of k linearly independent vectors.

The map

$$V_k(\mathbb{C}^n)/U(k) \to G_k(\mathbb{C}^n),\ [A] = U(k) \cdot A \mapsto A^T(\mathbb{C}^k)$$

defines a bijection between the quotient of the Stiefel manifold by the unitary group $U(k)$ and the complex Grassmannian. We think of $A^T(\mathbb{C})$ as the span of the k linearly independent vectors. We now show that the complex Grassmannian is a symplectic quotient manifold.

Let $k \leq n$ be two positive integers. Consider the symplectic action of the unitary group $U(k)$ on $(\mathbb{C}^{k \times n}, \omega_0)$ by matrix multiplication:

$$U(k) \times \mathbb{C}^{k \times n} \to \mathbb{C}^{k \times n},\ (U, A) \mapsto UA.$$

This action is Hamiltonian with a momentum map given by

$$\mu : \mathbb{C}^{k \times n} \to \mathfrak{u}(k),\ \mu(A) = \frac{i}{2} AA^*.$$

To see this, note that if we have n Hamiltonian actions of a Lie group G on symplectic manifolds (M_j, ω_j) ($j \in \{1, \dots, n\}$) with momentum maps $\mu : M_j \to \mathfrak{g}^*$, then

$$\mu : M_1 \times \cdots \times M_n \to \mathfrak{g}^*,\ \mu(p_1, \dots, p_n) = \mu_1(p_1) + \cdots + \mu_n(p_n)$$

is a momentum map for the action

$$G \times (M_1 \times \cdots \times M_n) \to M_1 \times \cdots \times M_n,\ g \cdot (p_1, \dots, p_n) \mapsto (g \cdot p_1, \dots, g \cdot p_n).$$

(The symplectic form ω on $M_1 \times \cdots \times M_n$ is given by $\omega = \mathrm{pr}_1^*\omega_1 + \cdots + \mathrm{pr}_n^*\omega_n$, where $\mathrm{pr}_j : M_1 \times \cdots \times M_n \to M_j$ the projection maps onto the j'th factor.)

Take $M_j = \mathbb{C}^k$ for $j = 1, \ldots, n$ and consider the Hamiltonian $U(k)$-action from Example 2.3.5.4 with momentum maps $\mu_j : \mathbb{C}^k \to \mathfrak{u}(k),\ \mu_j(z_j) = \frac{i}{2} z_j z_j^*$. Write a complex matrix $A \in \mathbb{C}^{k\times n}$ as an n-tuple of its column vectors: $A = (z_1, \cdots, z_n)$ with $z_j \in \mathbb{C}^k$. By the above remark we find that $\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)$ defined by

$$\mu(A) = \mu(z_1, \ldots, z_n) = \frac{i}{2}(z_1 z_1^* + \cdots + z_n z_n^*) = \frac{i}{2} AA^*$$

is a momentum map for the action on $\mathbb{C}^{k\times n}$.

Note that $\mu^{-1}(\frac{i}{2} I_k) = V_k(\mathbb{C}^n)$, and $\frac{i}{2} I_k \in \mathfrak{u}(k)$ is a fixed point of the adjoint action of $U(k)$ on its Lie algebra $\mathfrak{u}(k)$ and $U(k)$ acts freely on $V_k(\mathbb{C}^n)$. By application of the Symplectic Reduction theorem, we conclude that the complex Grassmannian $\mu^{-1}(\frac{i}{2} I_k)/U(k) = V_k(\mathbb{C})/U(k) = G_k(\mathbb{C}^n)$ is a symplectic manifold.

Summary

We first introduced Lie groups and Lie group actions on manifolds. In the case of such a Lie group action on a manifold, we used the exponential map to generate a vector field on the manifold for each element in the Lie algebra, called the fundamental vector fields. We then introduced Hamiltonian actions that are characterized by the existence of a momentum map, that is, an equivariant map $\mu : M \to \mathfrak{g}^*$ which gives a Hamiltonian function for each fundamental vector field. Finally, we used the momentum map to prove the symplectic reduction theorem, which allows us to obtain new symplectic manifolds by taking the quotient of symplectic manifolds with a Hamiltonian action by a Lie group.

Chapter 3

Atiyah–Guillemin–Sternberg Convexity Theorem

In the last chapter we have seen several examples of Hamiltonian torus actions, and that the image $\mu(M)$ of the momentum map is convex in these cases. Atiyah and Guillemin and Sternberg have shown that the convexity of the image holds for any Hamiltonian torus action on a compact symplectic manifold. The main goal of this section is to prove the Atiyah–Guillemin–Sternberg Convexity Theorem. For this, we will first learn some representation theory about compact Abelian Lie groups and apply Morse–Bott theory to the functions μ_X obtained from the momentum map.

3.1 Tori

In this chapter and the subsequent sections we will mainly use torus actions, so we spend some time on several characterizations of tori. Let $\mathbb{T}^n$ be a torus. Throughout, we identify the Lie algebra $\mathfrak{t} \cong \mathbb{R}^n$ using the basis obtained by the tangent vectors obtained by rotating a single $\mathbb{S}^1$-factor with period 1.

In the last section, we have seen the torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ as a compact connected Abelian Lie group being the Cartesian product of circles, with componentwise multiplication. Consider the Lie group homomorphism

$$\Phi : \mathbb{R}^n \to \mathbb{T}^n,\ (x_1, \ldots, x_n) \mapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$$

between Abelian Lie groups. This map has kernel $\mathbb{Z}^n$, so by the first isomorphism theorem for groups, Φ descends to a Lie group isomorphism ([28, Example 21.14]):

$$\tilde{\Phi} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n.$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ denote the quotient map of the quotient manifold $\mathbb{R}^n/\mathbb{Z}^n$. Under the identification $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the exponential map $\exp : \mathfrak{t} \cong \mathbb{R}^n \to \mathbb{T}^n$ and the quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ are identical.

In practice, the latter viewpoint might be more useful when studying the image of a linear subspace under the exponential map.

Example 3.1.1. [28, Example 4.20] Let $\mathbb{T}$ be a torus, and let $X \in \mathbb{R}^n$ be an element in the Lie algebra with rationally independent coefficients. Then the curve $t \mapsto \exp(tX)$ in the torus is dense:

$$\overline{\{\exp(tX) : t \in \mathbb{R}\}} = \mathbb{T}.$$

The following theorem can be viewed as a generalization of the preceding example.

Theorem 3.1.2. *[13, Corollary 1.12.4] Let G be a connected Abelian Lie group. Then G is isomorphic to $(\mathbb{R}^a/\mathbb{Z}^a) \times \mathbb{R}^b$ for some integers a, b. In particular, if in addition G is compact, it is isomorphic (as a Lie group) to $\mathbb{R}^n/\mathbb{Z}^n$, where $n = dim(G)$.*

Let $\mathbb{T}^n$ be a torus, with Lie algebra $\mathfrak{t} \cong \mathbb{R}^n$. The above theorem allows us to **generate subtori**. More explicitly, let $V \subseteq \mathbb{R}^n$ be a linear subspace. Consider the set

$$H := \overline{\exp(V)} \subseteq \mathbb{T}^n$$

obtained by taking the closure of the image of V under the exponential map. Using that $\mathbb{T}^n$ is Abelian, one readily verifies that H is an Abelian subgroup of the torus. It is a closed subgroup, so H is a Lie subgroup by Cartan's Closed Subgroup Theorem. Then H is compact as well, being a closed subset of a compact space. This implies that $H \subseteq \mathbb{T}^n$ is a subtorus by application of the theorem.

For example, if $X \in \mathbb{R}^n$ has rationally dependent coefficients, one finds that $\exp(\mathbb{R} \cdot X) = \pi(\mathbb{R} \cdot X) \subseteq \mathbb{T}^n$ is closed, and thus a one-dimensional subtorus of $\mathbb{T}^n$, which we identify with a circle $\mathbb{S}^1$. We generalize these observations by the following definition.

Definition 3.1.3. Let $\mathbb{T}^n$ be a torus, and suppose $X \in \mathbb{R}^n$ is an element of the Lie algebra. Define the **torus generated by** X, denoted $\mathbb{T}_X$, to be:

$$\mathbb{T}_X := \overline{\{\exp(tX) : t \in \mathbb{R}\}} \subseteq \mathbb{T}^n.$$

The previous discussion shows that $\mathbb{T}_X$ is indeed a subtorus of $\mathbb{T}^n$.

Proposition 3.1.4. Let $\mathbb{T}$ be a torus, with exponential map $\exp : \mathfrak{t} \to \mathbb{T}$. Then the following holds.

- The exponential map is surjective.
- The exponential map is a Lie group homomorphism, where we view the Lie algebra $\mathfrak{t}$ as an Abelian Lie group.

Remark 3.1.5. In the coming chapters we will mainly prove theorems for Hamiltonian actions of a torus. However, these theorems are also of use in the study of Hamiltonian actions of non-Abelian compact Lie groups. The reason being that a compact connected Lie group G contains a **maximal torus** $\mathbb{T} \subseteq G$ ([13, Theorem 3.7.1]). We say that a torus $\mathbb{T} \subseteq G$ is maximal if $\mathbb{T} \subseteq \mathbb{T}'$ for some other subtorus $\mathbb{T}'$ of G implies that $\mathbb{T} = \mathbb{T}'$.

Haar measure

The goal of this section is to show that a symplectic action of a compact Lie group G gives rise to an invariant Riemannian metric and subsequently an invariant almost complex structure J. The main tool to derive such invariant structures is the so-called Haar measure.

The Lebesgue measure λ defined on the Borel measurable subsets of Euclidean space is invariant under translations, that is, for all $a \in \mathbb{R}^n$ and Borel measurable sets $B \subseteq \mathbb{R}^n$ we have:

$$\lambda(a + B) = \lambda(B)$$

One can generalize such Borel measures to locally compact Lie groups:

Definition 3.1.6. Let G be a locally compact Lie group, and let ν be a nonzero regular Borel measure on G. We say that ν is a **left Haar measure** if it is invariant under left translations, meaning that $\nu(L_g(B)) = \nu(B)$ for all $g \in G$ and all Borel measurable sets $B \subseteq G$.

Proposition 3.1.7 (**Haar measure** [10, Theorem 9.2.2.])**.** Let G be a compact Lie group. There exists a unique normalized left Haar measure on G, denoted by dh, satisfying $\int_G dh = 1$. This normalized Haar measure is also a right Haar measure.

Let $f \in C^\infty(G)$ be a smooth function on a compact Lie group, then the left invariance of the Haar measure dh implies that

$$\int_G f(gh)\ dh = \int_G f(h)\ dh, \text{ for all } g \in G.$$

Suppose we have a symplectic action $\psi : G \times M \to M$ of a compact Lie group on a symplectic manifold (M, ω). We construct a G-invariant Riemannian metric m_0 on M as follows. Let m be an arbitrary Riemannian metric on M, and define m_0, with respect to the normalized left Haar measure, pointwise by:

$$(m_0)_p(u, v) := \int_G (\psi_h^* m)_p(u, v)\ dh,$$

for $u, v \in T_pM$. This is well-defined, as the function $h \mapsto (\psi_h^* m)_p(u, v)$ is smooth. It can be checked that m_0 defines a Riemannian metric on M. This Riemannian metric is G-invariant, since the pullback is a contravariant functor and the left Haar measure is G-invariant:

$$\psi_g^* m_0 = \int_G \psi_g^* \circ (\psi_h^* m)\ dh = \int_G (\psi_{hg})^* m\ dh = \int_G \psi_h^* m\ dh = m_0.$$

We say that the G-invariant Riemannian metric m_0 is obtained by averaging a Riemannian metric on M over the group G, and that ψ_g **acts by isometries** with respect to m_0. (This construction only works for compact Lie groups, as a Haar measure for non-compact groups is not finite, that is, $\int_G dh = \infty$ ([10, Proposition 9.3.3.]).

We use this metric to obtain a G-invariant ω-compatible almost complex structure J on M, meaning that $\psi_g^* J = J$ for all $g \in G$. Unpacking this property, we require the almost complex structure $J \in \Gamma(\mathrm{End}(TM))$ to satisfy:

$$d(\psi_g)_p \circ J_p = J_{g \cdot p} \circ d(\psi_g)_p$$

for all $p \in M$ and $g \in G$. In view of the construction of an ω-compatible almost complex structure in section 1.5, we note that it suffices to show that $A \in \Gamma(\mathrm{End}(TM))$ defined by

$$(m_0)_p(A_p(u), v) = \omega_p(u, v) \qquad (u, v \in T_pM)$$

is G-invariant. Recall that we used the polar decomposition to write $A = |A|J$. Let $u, v \in T_pM$. Using the G-invariance of the symplectic structure and the Riemannian metric, we find:

$$\begin{aligned}(m_0)_{g\cdot p}(d(\psi_g)_p \circ A_p(u), v) &= (m_0)_p(A_p(u), d(\psi_{g^{-1}})_{g\cdot p}(v)) \\ &= \omega_p(u, d(\psi_{g^{-1}})_{g\cdot p}(v)) \\ &= \omega_{g\cdot p}(d(\psi_g)_p(u), v) \\ &= (m_0)_{g\cdot p}(A_{g\cdot p} \circ d(\psi_g)_p(u), v).\end{aligned}$$

We conclude that A is G-invariant, so that J is G-invariant. It is readily checked that $m_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is also a G-invariant Riemannian metric. Let us summarize this subsection:

Proposition 3.1.8. Suppose we have a symplectic action $\psi : G \times M \to M$ of a compact Lie group G on a symplectic manifold (M, ω). Then there exists an ω-compatible almost complex structure J on M, which is G-invariant:

$$\psi_g^* J = J, \text{ for all } g \in G.$$

Furthermore, the Riemannian metric m_J defined by $m_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is G-invariant. The form H defined by $H(\cdot,\cdot) := m_J(\cdot,\cdot) + i\omega(\cdot,\cdot)$ is a G-invariant Hermitian form.

In particular, the result holds for a symplectic action of a torus $\mathbb{T}$.

Representation theory and Equivariant Darboux Theorem

Let $\psi : G \times M \to M$ be a symplectic action of a compact Lie group G on a symplectic manifold (M, ω), and suppose $p \in M$ is a fixed point of the action. Then we find a smooth action of the group G on the tangent space T_pM, as follows:

$$\pi : G \to \mathrm{GL}(T_pM), \ \pi(g) := d(\psi_g)_p : T_pM \to T_pM.$$

In this subsection we study the above action of G on the tangent space T_pM of a fixed point p. For this, we will first delve into representation theory, based on Duistermaat and Kolk's book *Lie Groups* [13]. Then, using representation theory and an equivariant version of the Darboux Theorem, we describe the action ψ on M near a fixed point, in the case that $G = \mathbb{T}$ is a torus.

Definition 3.1.9. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let G be a Lie group. A **representation** of G in V is a Lie group homomorphism

$$\pi : G \to \mathrm{GL}(V).$$

We say that π is a finite-dimensional, complex representation of G to reflect that V is a finite-dimensional complex vector space. Furthermore, the dimension of π is defined to be the dimension of V.

Definition 3.1.10. Let π be a finite-dim. representation of G in a complex vector space V.

- A linear subspace $U \subseteq V$ of a representation is said to be $\pi(G)$-invariant if U is $\pi(g)$-invariant for all $g \in G$:
$$\pi(g)(U) \subseteq U.$$

- We say that the representation π of G in V is an **irreducible representation** if V does not have proper and nontrivial $\pi(G)$-invariant linear subspaces. Equivalently, if $U \subseteq V$ is a $\pi(G)$-invariant subspace, then $U = 0$ or $U = V$.
- The representation π of G in V is **completely reducible** if for every $\pi(G)$-invariant linear subspace $U \subseteq V$ there exists another $\pi(G)$-invariant linear subspace $U' \subseteq V$ such that V splits as their direct sum:

$$V = U \oplus U'.$$

- Suppose that $\sigma : G \to \mathrm{GL}(W)$ is another finite-dimensional representation of G in a complex vector space W. We say that π and σ are **equivalent** representations if there exists a linear isomorphism $L : V \to W$ such that:

$$L \circ \pi(g) = \sigma(g) \circ L, \text{ for all } g \in G.$$

- Suppose that $H : V \times V \to \mathbb{C}$ is a Hermitian inner product on V. The representation π is **unitary** (with respect to H) if

$$H(\pi(g)(v), \pi(g)(w)) = H(v, w), \text{ for all } g \in G,\ v, w \in V.$$

Definition 3.1.11. Let π be a finite-dimensional representation of G in V. A **character** of the *representation*, denoted by χ_π, is a map

$$\chi_\pi : G \to \mathbb{C},\ g \mapsto \mathrm{Tr}(\pi(g)).$$

A **multiplicative character** of the *group* G is a Lie group homomorphism

$$\chi : G \to \mathbb{C}^\times.$$

Theorem 3.1.12 (Schur's lemma for Abelian groups [13, Cor. 4.1.2.]**).** *Let G be a compact Abelian Lie group, and suppose that $\pi : G \to GL(V)$ is an irreducible, finite-dimensional, complex representation of G. Then the representation π is one-dimensional. Moreover, the character χ_π of the representation π is a multiplicative character of G:*

$$\chi_\pi : G \to \mathbb{C}^\times,$$

and $\pi(g)$ acts on V via multiplication by the nonzero complex scalar $\chi_\pi(g)$:

$$\pi(g) : v \mapsto \chi_\pi(g) \cdot v.$$

Theorem 3.1.13. *[13, Cor. 4.2.2.] Let π be a unitary representation of a compact Lie group G in a finite-dimensional complex vector space V with a Hermitian inner product. Then π is completely reducible, and V splits as a direct sum of $\pi(G)$-invariant mutually orthogonal linear subspaces V_j, such that $\pi|_{V_j} : g \mapsto \pi(g)|_{V_j}$ is irreducible for each index j.*

We will need the following proposition:

Proposition 3.1.14. [5, Ex. 27.5] Let $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. The multiplicative characters of the torus are given by:

$$\chi_\lambda : \mathbb{T} \to \mathbb{C}^\times,\ \exp(X) \mapsto e^{2\pi i \langle \lambda, X \rangle},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is an n-tuple in the lattice of the torus. (Here we view $X \in \mathbb{R}^n$, and $\exp(X) \in \mathbb{T}$.)

Definition 3.1.15. Denote by $\chi_{\lambda_j} : \mathbb{T}^n \to \mathbb{S}^1$ the character of an irreducible $\mathbb{T}^n$ representation on a complex line V_j. By differentiating the character χ_{λ_j} at the identity, we obtain a linear map

$$\lambda_j : \mathfrak{t} \to \mathbb{R},\ X \mapsto \langle \lambda_j, X \rangle = \sum_{k=1}^{n} (\lambda_j)_k \cdot X_k,$$

which we view as an element in the dual $\mathfrak{t}^*$ of the Lie algebra of the torus. We define the **weight** of the representation of $\mathbb{T}^n$ on V_j to be this dual element.

We now apply representation theory to construct a local model of a Hamiltonian torus action around a fixed point. Let $\psi : \mathbb{T}^n \times M \to M$ be a Hamiltonian torus action on a compact symplectic manifold (M, ω) with momentum map $\mu : M \to \mathfrak{t}^*$, and suppose $p \in M^{\mathbb{T}}$ is a fixed point. Then we have a representation π of the torus on the tangent space T_pM which preserves the symplectic structure ω_p:

$$\pi : \mathbb{T} \to \mathrm{GL}(T_pM),\ g \mapsto d(\psi_g)_p : T_pM \to T_pM.$$

Since $\mathbb{T}$ is compact, we obtain by Proposition 3.1.8 a $\mathbb{T}$-invariant ω-compatible almost complex structure J on M, which makes (T_pM, J_p) into a complex vector space. Since J is $\mathbb{T}^n$-invariant, we find that π is a finite-dimensional *complex* representation. The resulting Riemannian metric $m_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is also $\mathbb{T}$-invariant, so that ψ acts by *isometries* on M with respect to the metric m_J. Similarly, the Hermitian inner product $H(\cdot,\cdot) = m_J(\cdot,\cdot) + i\omega(\cdot,\cdot)$ is $\mathbb{T}$-invariant, so that $\mathbb{T}$ acts by *unitary* transformations on (T_pM, J_p).

By the above theorems, we can split T_pM as follows:

$$T_pM = V_1 \oplus \cdots \oplus V_m, \tag{3.1}$$

where the summands $V_1, \ldots, V_m$ are $\mathbb{T}$-invariant mutually orthogonal complex lines. Thus, we have a vector e_j in each complex line V_j such that $e_1, \ldots, e_m$ is an orthonormal $\mathbb{C}$-basis for the tangent space T_pM. The action of $\mathbb{T}$ on each line V_j is given by:

$$\pi(\exp(X))(e_j) = e^{2\pi i \langle \lambda_j, X \rangle} e_j, \tag{3.2}$$

for some $\lambda_j \in \mathbb{Z}^n$, $j = 1, \ldots, m$. Since $e_1, \ldots, e_m$ is an orthonormal $\mathbb{C}$-basis and J_p is an ω-compatible complex structure, it follows that $e_1, \ldots, e_m, Je_1, \ldots, Je_m$ is a symplectic basis for (T_pM, ω_p). This symplectic basis provides a symplectomorphism between (T_pM, ω_p) and $(\mathbb{C}^n, \omega_0)$, where ω_0 is the standard symplectic structure.

As we have seen before, the exponential map of a Riemannian metric allows us to identify a neighborhood of the tangent space with a neighborhood of the manifold. We show that the exponential map

$$\exp_p : T_pM \to M$$

associated to the $\mathbb{T}$-invariant Riemannian metric m_J is $\mathbb{T}$-equivariant, so that we can make an *equivariant* identification. Let γ_v be the geodesic starting at p such that $\exp_p(v) = \gamma_v(1)$. For $g \in \mathbb{T}$, the initial velocity of the curve $\psi_g \circ \gamma$ is given by:

$$\left.\frac{d}{dt}\right|_{t=0} (\psi_g \circ \gamma_v)(t) = d(\psi_g)_p(v).$$

As ψ acts by isometries, it takes geodesics to geodesics, so that we conclude:

$$\exp_p(\pi(g)(v)) = \exp_p(d(\psi_g)_p(v)) = \psi_g(\exp_p(v)),$$

for $v \in T_pM$ and $g \in \mathbb{T}$. This proves that the exponential map is $\mathbb{T}$-equivariant. In order to construct a local model, we will make use of the following equivariant Darboux theorem:

Theorem 3.1.16 (**Equivariant Darboux Theorem** [19, Theorem 22.1])**.** *Let X be a manifold equipped with an action of a compact Lie group G, and let $x \in X^G$ be a fixed point. Suppose ω_1, ω_2 are G-invariant symplectic forms on X such that $(\omega_1)_x = (\omega_2)_x$. Then there exist G-invariant open neighborhoods V and W of x together with a G-equivariant diffeomorphism $\Phi : V \to W$ such that $\Phi(x) = x$ and $\Phi^*\omega_2 = \omega_1$.*

We take a G-invariant open neighborhood V_0 of 0 in T_pM and an open neighborhood U_p of p in M such that

$$\exp_p : V_0 \to U_p$$

is a $\mathbb{T}$-equivariant diffeomorphism. Then $\exp_p^*\omega$ and ω_p are $\mathbb{T}$-invariant symplectic forms on $V_0 \subseteq T_pM$ which agree at $0 \in V_0^{\mathbb{T}}$. By the equivariant Darboux theorem (and restricting V_0 if necessary), we obtain an open neighborhood V of $0 \in T_pM$ together with a $\mathbb{T}$-equivariant symplectomorphism

$$\Phi : V \to V_0 \tag{3.3}$$

satisfying $\Phi(0) = 0$ and $\Phi^* \exp_p^* \omega = \omega_p$.

Consider the symplectic $\mathbb{T}^n$-action on $(\mathbb{C}^m, \omega_0)$ given by

$$\mathbb{T}^n \times \mathbb{C}^m \to \mathbb{C}^m,\ \exp(X) \cdot (z_1, \dots, z_m) = (e^{2\pi i \langle \lambda_1, X\rangle} z_1, \dots, e^{2\pi i \langle \lambda_m, X\rangle} z_m).$$

This linear action is Hamiltonian with momentum map

$$\mu_0 : \mathbb{C}^m \to \mathfrak{t}^*,\ \mu_0(z_1, \dots, z_m) = \mu(p) - \pi \sum_{j=1}^{m} |z_j|^2 \lambda_j. \tag{3.4}$$

In view of the decomposition in Equation (3.1) and the remarks thereafter, we obtain a $\mathbb{T}$-equivariant symplectomorphism

$$\varphi : (\mathbb{C}^m, \omega_0) \to (T_pM, \omega_p)$$

satisfying $\varphi(0) = 0$ and $\varphi^*\omega_p = \omega_0$. It follows that

$$\Psi := \exp_p \circ \Phi \circ \varphi : (\varphi^{-1}(V), \omega_0) \to (U_p, \omega) \tag{3.5}$$

is a $\mathbb{T}$-equivariant symplectomorphism satisfying $\Psi(0) = p$ and $\Psi^*\omega = \omega_0$. Then

$$\mu \circ \Psi : \varphi^{-1}(V) \to \mathfrak{t}^* \tag{3.6}$$

is another momentum map for the $\mathbb{T}$-action on $(\varphi^{-1}(V), \omega_0) \subseteq (\mathbb{C}^m, \omega_0)$. As momentum maps are unique up to a constant, we find that in local coordinates around the fixed point $p \in M$ the momentum map $\mu : M \to \mathfrak{t}^*$ is given by:

$$\hat{\mu}(z_1, \dots, z_m) = \mu(p) - \pi \sum_{j=1}^{m} |z_j|^2 \ \lambda_j. \tag{3.7}$$

The process of finding such a local model of a Hamiltonian torus action on M around a fixed point is called **linearization of the action near a fixed point**.

Example 3.1.17. The exponential map of a $\mathbb{T}$-invariant Riemannian metric provides a $\mathbb{T}$-equivariant identification between a neighborhood of 0 in the tangent space and a neighborhood of the the manifold containing a fixed point of the action. As this idea played a key role in constructing the local model above, we illustrate it for the usual circle action on the sphere $\mathbb{S}^2$ in Figure 3.1.

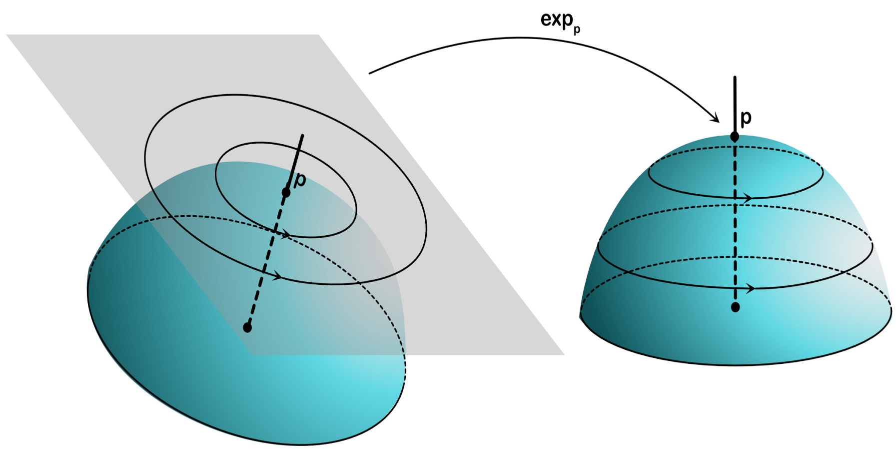


Figure 3.1: The circle acts on $\mathbb{S}^2$ by rotations. The North pole is a fixed point of the action, so that the circle acts on the tangent space. The exponential map intertwines the circle action on the tangent space with the circle action on the sphere.

3.2 Morse–Bott Theory

A successful theory in differential topology is Morse theory, where functions on a manifold with isolated critical points are used to study the topology of the manifold. Suppose we have a Hamiltonian action of a Lie group G on a symplectic manifold M with momentum map $\mu : M \to \mathfrak{g}^*$, this map gives us a smooth function μ_X for each element X in the Lie algebra of G. Thus, we can study the Hamiltonian action by applying Morse theory. However, as the momentum map is equivariant, the critical points of the smooth functions $\mu_X \in C^\infty(M)$ are generally not isolated, but contain orbits of the action. Therefore we will use an extension of Morse theory which allows the critical points of a smooth function to be smooth submanifolds, namely *Morse–Bott* theory.

Definition 3.2.1. Let $N \subseteq M$ be a submanifold of M. Consider the tangent bundle $\pi : TM \to M$. We define the restriction of the tangent bundle of M to N to be

$$TM|_N = \bigsqcup_{p \in N} T_pM,$$

which is a vector bundle over N. The **normal bundle** of a submanifold N in the ambient manifold M is the following quotient bundle over N:

$$\nu_M(N) := (TM)|_N/TN = \bigsqcup_{p \in N} T_pM/T_pN.$$

Another way of thinking about the normal bundle is as follows: the normal bundle of N in M is the vector bundle over N which makes the following sequence of vector bundles over N exact:

$$0 \longrightarrow TN \longrightarrow TM|_N \longrightarrow \nu_M(N) \longrightarrow 0.$$

If $(M, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold with submanifold $N \subseteq M$, then we can alternatively define the normal bundle by:

$$\nu_M(N) := \bigsqcup_{p \in N} T_p N^{\perp},$$

which is isomorphic (as a vector bundle) to the previously defined notionof the normal bundle of N in M.

Let $f \in C^\infty(M)$. Recall that a **critical point** of f is a point $p \in M$ such that $df_p = 0$.

Definition 3.2.2. Let p be a critical point of a smooth real-valued function $f : M \to \mathbb{R}$. Define the **Hessian** of f at p, denoted $H_p(f)$, by the following bilinear form:

$$H_p(f) : T_pM \times T_pM \to \mathbb{R}, \; H_p(f)(v, w) := (VWf)(p) \; (= (\mathcal{L}_V \mathcal{L}_W f)(p)),$$

where V, W are vector fields on M such that $V_p = v, W_p = w$.

The Hessian is well-defined, since a tangent vector can always be extended to a vector field ([28, Proposition 8.7]) and the Hessian is independent of the extension used.

Note that the Hessian is a *symmetric* bilinear form. Indeed, we find:

$$H_p(f)(v, w) - H_p(f)(w, v) = (\mathcal{L}_V \mathcal{L}_W f - \mathcal{L}_W \mathcal{L}_V f)(p) = \mathcal{L}_{[V,W]} f(p) = df_p([V, W]) = 0,$$

because p is a critical point.

Definition 3.2.3. Let $f : M \to \mathbb{R}$ be a smooth function, and let $\mathrm{Crit}(f)$ denote the set of critical points of f. A compact connected submanifold $C \subseteq M$ is a **nondegenerate critical submanifold** of f if $C \subseteq \mathrm{Crit}(f)$ and for all $p \in C$ we have $\ker H_p(f) = T_pC$. The function f is a **Morse–Bott** function if $\mathrm{Crit}(f)$ is a disjoint union of nondegenerate critical submanifolds.

Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on M, and suppose p is a critical point for a Morse–Bott function $f \in C^\infty(M)$. As the Riemannian metric is a nondegenerate pairing, we can define an endomorphism $h_p(f) : T_pM \to T_pM$ by:

$$H_p(f)(v, w) = \langle h_p(f)(v), w \rangle_p, \qquad \text{for all } v, w \in T_pM.$$

Note that h_p is self-adjoint with respect to the metric as the Hessian is a symmetric bilinear form. Using the Riemannian metric we decompose T_pM as follows:

$$T_pM = T_pC \oplus \nu_p(C).$$

Since the Hessian $h_p(f) : T_pM \to T_pM$ is self-adjoint, we can further decompose T_pM into the eigenspaces of $h_p(f)$:

$$T_pM = T_pC \oplus \nu_p^+(C) \oplus \nu_p^-(C). \tag{3.8}$$

We elaborate on this decomposition:

- As the function f is Morse–Bott, the 0-eigenspace of $h_p(f)$ is given by:

$$\ker h_p(f) = \ker H_p(f) = T_pC.$$

- The subspaces $\nu_p^{\pm}(C)$ of the normal space $\nu_p(C)$ are spanned by eigenvectors of $h_p(f)$ corresponding to positive and negative eigenvalues, respectively.

We can find an explicit expression for the Hessian $h_p(f) : T_pM \to T_pM$. Let ∇ be the Levi-Civita connection associated to the metric $\langle \cdot, \cdot \rangle$, and let $\text{grad} f \in \mathfrak{X}(M)$ be the unique vector field satisfying $\langle \text{grad} f, V \rangle = df(V)$ for all $V \in \mathfrak{X}(M)$. Define $h_p(f) \in \text{End}(T_pM)$ by $h_p(f)(v) := \nabla_V \text{grad}(f)$, where V is an extension of v to a vector field on M. We show that $H_p(f)(v, w) = \langle h_p(f)(v), w \rangle$ holds. The Levi-Civita connection is compatible with the metric, so we find:

$$\begin{aligned} \langle h_p(f)(v), w \rangle &= \langle \nabla_V \text{grad}(f), w \rangle \\ &= V \langle \text{grad}(f), W \rangle - \langle \text{grad}(f), \nabla_V W \rangle \\ &= \mathcal{L}_V \mathcal{L}_W f(p) - df_p(\nabla_V W) = H_p(f)(v, w). \end{aligned}$$

We will need the following lemma. Its purpose is to extend the decomposition in eq. (3.8) to a decomposition of the tangent bundle of M to an open subset of M.

Lemma 3.2.4 (**Morse–Bott lemma** [6])**.** *Let $f : M \to \mathbb{R}$ be a Morse–Bott function and $C \subseteq Crit(f)$ a d-dimensional nondegenerate critical submanifold. Let $\dim M = m$. For each $p \in C$ there exists a chart (U, φ) of p in the ambient manifold M adapted to C such that:*

- $\varphi(p) = 0$*;*
- $(f \circ \varphi^{-1})(x_1, \ldots, x_d, y_1, \ldots, y_{m-d}) = f(C) - y_1^2 - \cdots y_k^2 + y_{k+1}^2 + \cdots + y_{m-d}^2.$

In this way we obtain a local splitting of the normal bundle and subsequently the tangent bundle:

$$TM|_U = TC|_U \oplus \nu_M^+(C)|_U \oplus \nu_M^-(C)|_U,$$

where $\nu_q^{\pm}(C)$ $(q \in U)$ are spanned by the eigenvectors of the Hessian $h_q(f)$ corresponding to positive and negative eigenvalues, respectively.

As a consequence of this lemma we find that $C \ni p \mapsto \dim \nu_p^-(C)$ is locally constant on C. Since a nondegenerate critical submanifold is connected, it follows that $p \mapsto \dim \nu_p^-(C)$ is constant on C. Similarly, $p \mapsto \dim \nu_p^+(C)$ is constant on C. This leads to the following definition:

Definition 3.2.5. Let $f \in C^\infty(M)$ be a Morse–Bott function, and C a nondegenerate critical submanifold of f. We define the **index** of C, denoted by $n^-(C)$, to be the common value of $\dim \nu_p^-(C)$ $(p \in C)$. Similarly, the **coindex** of C, denoted $n^+(C)$, is the common value of $\dim \nu_p^+(C)$ $(p \in C)$.

Intuitively, the index of a criticial submanifold is the number of negative directions normal to the submanifold. We now look at some examples of Morse–Bott functions.

Example 3.2.6. The height function on $\mathbb{T}^2$ (see Figure 3.2) is a Morse–Bott function with two circles as critical submanifolds. The green arc indicates that the critical submanifold has index 1, that is, one negative direction normal to the critical submanifold. The red arc indicates that the critical submanifold has coindex 1.

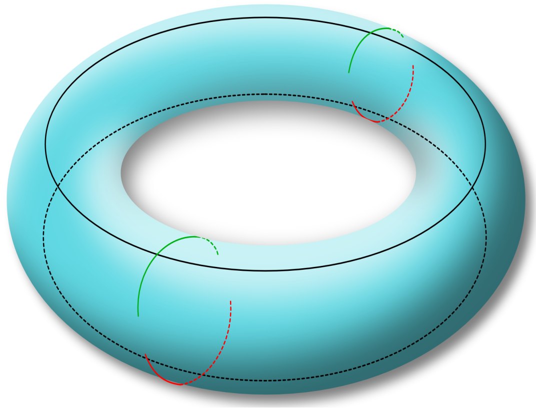


Figure 3.2: The torus $\mathbb{T}^2$.

We will need the following connectivity lemma, due to Atiyah [1], for the proof of the AGS convexity theorem:

Lemma 3.2.7 (**Connectivity Theorem**). *Let $f : M \to \mathbb{R}$ be a Morse–Bott function on a compact connected manifold M. Suppose that all the nondegenerate critical submanifolds have index and coindex $n^{\pm}(C) \neq 1$. Then $f^{-1}(c)$ is connected for every $c \in \mathbb{R}$. Furthermore, the function f has a unique local maximum and a unique local minimum.*

The following example demonstrates that level sets of a Morse–Bott function with index or coindex equal to 1 need not be connected.

Example 3.2.8. Consider the height function f on the torus standing vertically (see Figure 3.3). If the height function crosses a critical value of (co)index 1 ($f(Q)$ and $f(R)$ here), the number of connected components of the fibers changes.

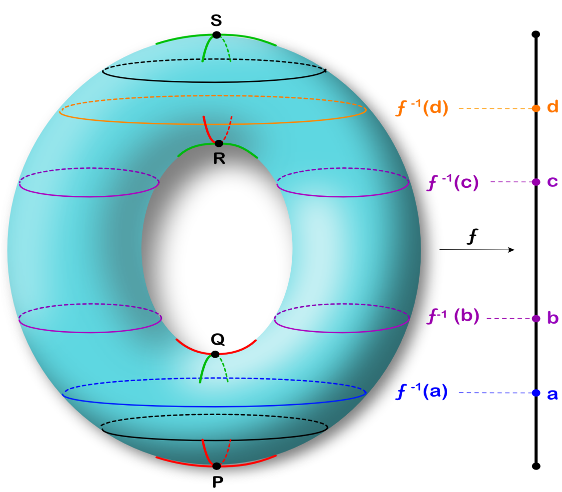


Figure 3.3: The vertical torus $\mathbb{T}^2$ with height function f.

Transversality and normal bundles

Definition 3.2.9. Let S and N be embedded submanifolds of M, and let $p \in S \cap N$. One says that S and N **intersect transversally** at p if

$$T_pM = T_pS + T_pN. \tag{3.9}$$

If eq. (3.9) holds for all $p \in S \cap N$, then we say that S and N intersect transversally, and denote $S \pitchfork N$.

Proposition 3.2.10. [28, Theorem 6.30] Let $S, N \subseteq M$ be submanifolds of M. Suppose S and N intersect transversally. Then the following holds:

- The intersection $S \cap N$ is an embedded submanifold of M with codimension

$$\operatorname{codim}(S \cap N) = \operatorname{codim}(S) + \operatorname{codim}(N).$$

- For all $p \in S \cap N$, we have:

$$T_p(S \cap N) = T_pS \cap T_pN.$$

Proposition 3.2.11. Let $S, N \subseteq M$ be submanifolds of M. Suppose S and N intersect transversally. Then the normal bundle of $S \cap N$ in N is isomorphic to the restriction of the normal bundle of S in M to $S \cap N$. More explicitly, we have:

$$\nu_N(S \cap N) \cong \nu_M(S)|_{S \cap N}.$$

Proof. Let $p \in S \cap N$. Since S and Q are transversal at p, we find:

$$\nu_N(S \cap N)_p = T_pN/T_p(S \cap N) = T_pN/(T_pS \cap T_pN) \cong (T_pS + T_pN)/T_pS = T_pM/T_pS,$$

where the indicated isomorphism is a consequence of the second isomorphism theorem for linear subspaces. As $\nu_M(S)_p = T_pM/T_pS$, we find that:

$$\nu_N(S \cap N)_p \cong \nu_M(S)_p$$

for all $p \in S \cap N$. The assertion now follows by doing the above approach fiberwise. $\square$

We discuss an application in the setting of Morse–Bott functions. Let $f : M \to \mathbb{R}$ be a Morse–Bott function with C as a nondegenerate critical submanifold, which intersects transversally with a submanifold N. Then Proposition 3.2.10 shows that $C \cap N$ is a nondegenerate critical submanifold of $f|_N : N \to \mathbb{R}$, and Proposition 3.2.11 implies that the pair $(f|_N, C \cap N)$ has the same index and coindex as the pair (f, C).

Momentum map

We will now prove that a momentum map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^n$ of a Hamiltonian torus action gives rise to Morse–Bott functions $\mu_X \in C^\infty(M)$ which satisfy the conditions of the Connectivity Theorem. The following lemma, due to Frankel ([15]), demonstrates another reason why the functions obtained from the momentum map are good choices in order to study the action. The critical points of μ_X correspond to the fixed points of the action (of a subtorus). We follow the approach as done in *Torus Actions on Symplectic Manifolds* [4] and *Introduction to Symplectic Topology* [34].

Lemma 3.2.12. *Suppose we have a Hamiltonian action ψ of a torus $\mathbb{T}^n$ on a compact connected symplectic manifold (M,ω) with momentum map $\mu : M \to \mathbb{R}^n$. For each $X \in \mathbb{R}^n$, the smooth function $\mu_X = \langle \mu, X\rangle \in C^\infty(M)$ is a Morse–Bott function with even-dimensional nondegenerate critical submanifolds of even index (and even coindex). Moreover, the critical submanifolds*

$$\mathrm{Crit}(\mu_X) = M^{\mathbb{T}_X}$$

are $\mathbb{T}$-invariant symplectic submanifolds.

Proof. Let J be the $\mathbb{T}$-invariant ω-compatible almost complex structure with $\mathbb{T}^n$-invariant Riemannian metric $m_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$. Let $G \subseteq \mathbb{T}$ be a subgroup, and $p \in M^G$ a fixed point of the G-action.

We use the exponential map of the invariant Riemannian metric m_J to obtain a local model of M^G, proving that M^G is a submanifold of M. Recall that the exponential map $\exp_p : T_pM \to M$is $\mathbb{T}$-equivariant:

$$\exp_p(d(\psi_g)_p(v)) = \psi_g(\exp_p(v)),$$

for all $v \in T_pM$ and all $g \in G$. As the exponential map is a local diffeomorphism, we find for sufficiently small v that v is a fixed point of G on the tangent space if and only if $\exp_p(v)$ is a fixed point of G on the manifold M. Thus, the exponential map defines a local parametrization of the fixed points M^G on M by the vector space $(T_pM)^G$. We conclude that M^G is a submanifold of M with tangent space at p is given by:

$$T_p(M^G) = (T_pM)^G.$$

By G-invariance of J, it follows that $(T_pM)^G$ is J-invariant:

$$d(\psi_g)_p \circ J_p(v) = J_p \circ d(\psi_g)_p(v) = J_p(v),$$

for $v \in (T_pM)^G$ and $g \in G$. Thus, we conclude that M^G is a symplectic submanifold of M. Note that $d(\mu_X)_p = 0$ if and only if $\underline{X}_p = 0$ if and only if $p \in M^{\mathbb{T}_X}$, it follows that:

$$\mathrm{Crit}(\mu_X) = M^{\mathbb{T}_X}.$$

As M is compact, we obtain that $\mathrm{Crit}(\mu_X)$ is a finite disjoint union of compact connected submanifolds.

It remains to show that the critical submanifolds of μ_X are nondegenerate (in the normal direction). We will do this by linearizing the action around the fixed point $p \in M^{\mathbb{T}_X}$. Let C be the critical component of μ_X containing p. Let $\lambda_1, \ldots, \lambda_m \in \mathfrak{t}^*$ denote the weights corresponding to the representation of $\mathbb{T}_X$ on T_pM. By Equation (3.7) and Proposition 2.3.6, we find that μ_X in local coordinates around p is given by:

$$\widehat{\mu_X}(z_1, \ldots, z_m) = \mu_X(p) - \sum_{j=1}^{m} (x_j^2 + y_j^2)\ \langle \lambda_j, X\rangle, \tag{3.10}$$

where we write $z_j = x_j + iy_j$. Since a weight λ_j is zero if and only if the basis vector of the complex line V_j is a fixed point of the torus action on the tangent space, we find that μ_X is normal to the critical submanifold C. Thus, we have that μ_X is Morse–Bott. Furthermore, from Equation (3.10) we read off that the critical submanifold C of μ_X has even index and even coindex. □

3.3 The Atiyah–Guillemin–Sternberg Convexity Theorem

Following Atiyah's proof [1] as explained in *Introduction to Symplectic Topology*[34], we now prove the convexity theorem for Hamiltonian torus actions. The convexity theorem was proved by Atiyah [1], and by Guillemin and Sternberg [20] independently.
In order to state the theorem, we need the definition of a convex hull:

Definition 3.3.1. Let $\eta_1, \ldots, \eta_N \in \mathbb{R}^n$ be a finite number of points in $\mathbb{R}^n$. The **convex hull** of the points $\eta_1, \ldots, \eta_N$, denoted by $[\eta_1, \ldots, \eta_N]$, is defined by:

$$[\eta_1, \ldots, \eta_N] := \{\sum_{j=1}^{N} t_j \eta_j : t_j \geq 0 \text{ and } \sum_{j=1}^{N} t_j = 1\}.$$

Equivalently, the convex hull of $\eta_1, \ldots, \eta_N$ is the smallest convex subset of $\mathbb{R}^n$ containing the points $\eta_1, \ldots, \eta_N$.

Theorem 3.3.2 (Atiyah–Guillemin–Sternberg Convexity Theorem). *Let (M, ω) be a compact connected symplectic manifold. Let $\psi : \mathbb{T}^n \times M \to M$ be a Hamiltonian torus action on M with momentum map $\mu : M \to \mathbb{R}^n$. Then the following holds.*

- *The regular level sets $\mu^{-1}(\eta)$ are connected.*
- *The image $\mu(M)$ of the momentum map is convex.*
- *The fixed point set $M^{\mathbb{T}}$ is a finite disjoint union of compact connected submanifolds $C_1, \ldots, C_N$, on which the momentum map μ is constant. Moreover, the image $\mu(M)$ is the convex hull of the images $\mu(C_j) =: \eta_j$ of the fixed points.*

We briefly outline the proof. Consider the following statements:

- (A_n): The regular level sets $\mu^{-1}(\eta)$ are connected, for every Hamiltonian $\mathbb{T}^n$-action on M.
- (B_n): The image $\mu(M)$ is convex, for every Hamiltonian $\mathbb{T}^n$-action on M.
- (C_n): The image $\mu(M)$ is the convex hull of the images of the fixed points under μ, for every Hamiltonian $\mathbb{T}^n$ action on M.

The proof is done by induction on the dimension n of the torus $\mathbb{T}^n$. We divide the proof into three steps. First, we will prove by induction that (A_n) holds for all n. Here we will mainly use the Morse–Bott theory introduced in the previous section. Secondly, we prove that (A_n) implies (B_{n+1}). The idea behind this statement is roughly as follows. Let $\mu = (\mu_1, \ldots, \mu_{n+1})$ be a momentum map for a torus action, and $\mathrm{pr} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be *any* linear projection. Writing $\mu' = \mathrm{pr} \circ \mu$, we find for any $\eta \in \mathbb{R}^n$ that:

$$\mu(M) \cap \mathrm{pr}^{-1}(\eta) = \mu(\mu'^{-1}(\eta)).$$

If $\mu'^{-1}(\eta)$ is connected, then the image of $\mu(M)$ intersected (or "cut out") by the line $\mathrm{pr}^{-1}(\eta)$ is connected. If the image $\mu(M)$ "cut out" by *any* line is connected, we may conclude that it is convex. The difficulty here lies in the fact that $\mu' = \mathrm{pr} \circ \mu$ is not necessarily a momentum map coming from a Hamiltonian torus action. Finally, we prove that (B_n) implies (C_n).

The first thing is to rule out trivial cases where the induction step follows immediately from the induction hypothesis. This leads to the following definition and lemma.

Definition 3.3.3. Let $\mu : M \to \mathbb{R}^n$ be a momentum map of a Hamiltonian torus action. We say that the momentum map μ is irreducible (or effective) if the differentials $d\mu_1, \ldots, d\mu_n$ are linearly independent, and **reducible** if the differentials are linearly dependent. If $d\mu_1, \ldots, d\mu_n$ are linearly dependent, then there exists an element $X = (X_1, \ldots, X_n)$ in the Lie algebra such that $\mu_X = \sum_j X_j \mu_j$ is constant, because the manifold is connected.

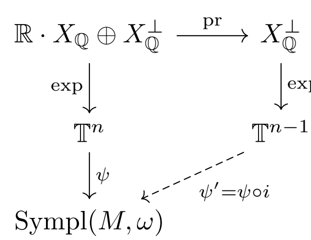
Lemma 3.3.4. *Let $\mu : M \to \mathbb{R}^n$ be the momentum map for a Hamiltonian action ψ of an n-torus $\mathbb{T}^n$ on a symplectic manifold (M, ω). If μ is reducible, then the action can be reduced to the action of an $(n-1)$-torus.*

Proof. Suppose that $\mu : M \to \mathbb{R}^n$ is reducible. Then there exists an element $X \in \mathbb{R}^n$ in the Lie algebra such that $\psi_{\exp tX} = \mathrm{Id}$ for all $t \in \mathbb{R}$. By a continuity argument, we find a vector $X_{\mathbb{Q}}$, with rationally dependent components, satisfying $\psi_{\exp tX_{\mathbb{Q}}} = \mathrm{Id}$ for all $t \in \mathbb{R}$.

We split the Lie algebra of the torus into a direct sum of the line spanned by $X_{\mathbb{Q}}$ and its orthogonal hyperplane:

$$\mathbb{R}^n = \mathbb{R} \cdot X_{\mathbb{Q}} \ \oplus X_{\mathbb{Q}}^{\perp}.$$

Since $X_{\mathbb{Q}}$ has rationally dependent components, it follows that $\exp(X_{\mathbb{Q}}^{\perp}) = \mathbb{T}^{n-1} \subseteq \mathbb{T}^n$ is an $(n-1)$-dimensional subtorus. Consider the following diagram:



Here the projection pr is the transpose of the inclusion of Lie algebras $i : X_{\mathbb{Q}}^{\perp} \hookrightarrow \mathbb{R}^n$ induced by the inclusion $i : \mathbb{T}^{n-1} \hookrightarrow \mathbb{T}$. Therefore the inclusion $i : X_{\mathbb{Q}}^{\perp} \hookrightarrow \mathbb{R}^n$ can be represented by an integer matrix, so that we can also use the projection pr as a Lie group homomorphism between these tori. The above diagram is commutative. For this, let $v \in \mathbb{R}^n$ and write $v = x + y$ with $x \in \mathbb{R} \cdot X_{\mathbb{Q}}$ and $y \in X_{\mathbb{Q}}^{\perp}$. Since the exponential map is a Lie group homomorphism, we find:

$$\begin{aligned}\psi_{\exp(v)} &= \psi_{\exp(x+y)} = \psi_{\exp(x)\cdot\exp(y)} \\ &= \psi_{\exp(x)} \circ \psi_{\exp(y)} = \mathrm{Id} \circ \psi_{\exp(y)} = \psi_{\exp(y)}.\end{aligned}$$

Then $\psi' : \mathbb{T}^{n-1} \to \mathrm{Sympl}(M, \omega)$ given by $\psi' = \psi \circ i$ is a Hamiltonian action, with momentum map μ', satisfying:

$$\psi = \psi' \circ \mathrm{pr}, \ \ \mu = i \circ \mu'.$$

Or equivalently, the action reduces to a Hamiltonian action of an $(n-1)$-dimensional subtorus. $\square$

Proof of the AGS Convexity Theorem.
Part 1: Connectedness of the regular levels.

We prove by induction on $n = \dim \mathbb{T}^n$ that the regular level sets of a momentum map μ of a Hamiltonian torus action are connected. For the base case $n = 1$, we have that $\mu : M \to \mathbb{R}$ is a Morse–Bott function with even indices and coindices by Lemma 3.2.12. By application of the Connectivity Theorem, it follows that the (regular) level sets of μ are connected.

For the induction step, suppose that for any Hamiltonian $\mathbb{T}^n$-action the regular level sets of the corresponding momentum map are connected. We now prove that the regular levels of the momentum map for any Hamiltonian $\mathbb{T}^{n+1}$-action are connected. Let $\mu : M \to \mathbb{R}^{n+1}$ be a momentum map for a $\mathbb{T}^{n+1}$-action. If μ is reducible, then by Lemma 3.3.4 it does not have regular values. Thus, we assume that μ is irreducible. In this case, the function $\mu_X \in C^\infty(M)$ is nonconstant for every nonzero element $X \in \mathbb{R}^{n+1}$ in the Lie algebra. Consider the set

$$Z := \bigcup_{X \neq 0} \mathrm{Crit}(\mu_X) = \bigcup_{||X||=1} \mathrm{Crit}(\mu_X)$$

By Lemma 3.2.12 we have that $\mathrm{Crit}(\mu_X) = M^{\mathbb{T}_X}$ is an even-dimensional submanifold of M, for all X. Since μ is irreducible, these submanifolds must necessarily be proper submanifolds. Note that $\mathbb{T}_X \subseteq \mathbb{T}_Y$ implies that $M^{\mathbb{T}_Y} \subseteq M^{\mathbb{T}_X}(M)$. In any subtorus generated by an element X we can find a circle, which in turn is generated by an element in the Lie algebra with rationally dependent components. There are countably many elements $X \in \mathbb{R}^n$ with rationally dependent components and satisfying $||X|| = 1$. Hence, Z is a countable union of proper even-dimensional submanifolds. We now show that $M - Z$ is open and dense. For density, note that $M - Z$ is a countable intersection of open dense subsets. By virtue of Baire's Category Theorem, which states that a countable intersection of open dense subsets is still dense, we conclude that $M - Z$ is dense. Note that $p \in M - Z$ if and only if the differentials $(d\mu_1)_p, \ldots, (d\mu_{n+1})_p$ are linearly independent. By continuity of the determinant, it follows that linear independence is an open condition, so that $M - Z$ is open.

Denote the set of regular values of the momentum map μ by $\mathfrak{t}^*_{\mathrm{reg}}$ (with some abuse of notation). We show that $\mu(M) \cap \mathfrak{t}^*_{\mathrm{reg}}$ is dense in $\mu(M)$ by showing for each point in the image there is a sequence contained in the intersection which converges to this point. Let $\eta = \mu(x) \in \mu(M)$. Since $M - Z$ is dense, there is a sequence $(x_j)_{j \in \mathbb{N}} \subseteq M - Z$ such that $x_j \to x$ as $j \to \infty$. For each $j \in \mathbb{N}$, there is an open neighborhood $U_j \subseteq M - Z$ containing x_j. As μ is a submersion on these neighborhoods, the image $\mu(M)$ contains an open neighborhood V_j of $\mu(x_j)$, for each $j \in \mathbb{N}$. (Note that V_j is open in $\mathbb{R}^{n+1}$ and thus has nonzero measure.) By application of Sard's Theorem, which states that the critical values have negligible measure, we find a regular value $\eta_j \in V_j \subseteq \mu(M)$ arbitrarily close to $\mu(x_j)$. Thus, we have found a sequence $(\eta_j)_{j \in \mathbb{N}} \subseteq \mu(M) \cap \mathfrak{t}^*_{\mathrm{reg}}$ which converges to η. Denote by $\widetilde{\mathfrak{t}^*_{\mathrm{reg}}}$ the set of values $\eta \in \mathbb{R}^{n+1}$ such that $(\eta_1, \ldots, \eta_n)$ is a regular value for the reduced momentum map $\tilde{\mu} = (\mu_1, \ldots, \mu_n)$. Similarly, we find that that $\mu(M) \cap \widetilde{\mathfrak{t}^*_{\mathrm{reg}}}$ is dense in the image $\mu(M)$.

Suppose that $\eta \in \mathbb{R}^{n+1}$ is a regular value for the momentum map μ and that $\tilde{\eta} := (\eta_1, \ldots, \eta_n)$ is a regular value for the reduced momentum map $\tilde{\mu}$, that is: $\eta \in \mathfrak{t}^*_{\mathrm{reg}} \cap \widetilde{\mathfrak{t}^*_{\mathrm{reg}}}$. By the induction hypothesis (and the regular value theorem), we have that

$$N := \tilde{\mu}^{-1}(\tilde{\eta}) = \mu_1^{-1}(\eta_1) \cap \cdots \cap \mu_n^{-1}(\eta_n)$$

is a connected submanifold. We show that the function $\mu_{n+1}|_N : N \to \mathbb{R}$ is a Morse–Bott function with even index and coindex, so that we may apply the Connectivity Theorem to deduce that the submanifold

$$(\mu_{n+1}|_N)^{-1}(\eta_{n+1}) = N \cap \mu_{n+1}^{-1}(\eta_{n+1}) = \mu^{-1}(\eta)$$

is connected. Subject to the constraints given by N, we use Lagrange multipliers to find that a point $p \in N$ is a critical point for $\mu_{n+1}|_N$ if and only if there exists a vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} \lambda_j (d\mu_j)_p + (d\mu_{n+1})_p = 0.$$

Thus, p is a critical point for the function μ_X, where $X := (\lambda_1, \ldots, \lambda_n, 1) \in \mathbb{R}^{n+1}$. By lemma 3.2.12, it follows that μ_X is Morse–Bott with nondegenerate critical submanifolds of even index and coindex. Let C be the critical submanifold containing p. We show that C and N intersect transversally at p. By linear algebraic considerations, we find:

$$(T_pC + T_pN)^0 = T_pC^0 \cap T_pN^0 = T_pC^0 \cap \operatorname{span}\{(d\mu_1)_p, \cdots, (d\mu_n)_p\}.$$

The last equality can be seen by noting that $\operatorname{span}\{(d\mu_1)_p, \cdots, (d\mu_n)_p\} \subseteq T_pN^0$ and that their dimensions agree. Therefore, to show that C and N intersect transversally at p, it suffices to show that $(d\mu_1)|_{T_pC}, \ldots, (d\mu_n)_p|_{T_pC}$ are linearly independent. Consider the fundamental vector field $\underline{X}$ on M with flow given by $\psi_{\exp(tX)}$. As each μ_j is $\mathbb{T}$-invariant, the coadjoint action being trivial, we find (for all j):

$$\{\mu_j, \mu_X\} = d\mu_j(\underline{X}) = \left.\frac{d}{dt}\right|_{t=0} \mu_j(\psi_{\exp(tX)}) = 0.$$

We find that the Poisson brackets of μ_j and μ_X commute for all $j = 1, \ldots, n$. By Theorem 1.6.6, we find that μ_X is constant along the integral curves of X_{μ_j}. It follows that the integral curves of each vector field X_{μ_j} are contained in the critical submanifold C of μ_X, so that necessarily $X_{\mu_j} \in T_pC$ for all $j = 1, \ldots, n$. By Lemma 3.2.12 the critical submanifold C is a symplectic submanifold: the symplectic form ω_p is nondegenerate on T_pC. Thus, for any tangent vector $Y_p = \sum_j a_j X_{\mu_j} \in T_pC$ there exists some tangent vector $v \in T_pC$ such that:

$$0 \neq \omega_p(Y_p, u) = \sum_{j=1}^{n} a_j (d\mu_j)_p(v).$$

Or equivalently, the 1-forms $(d\mu_1)_p|_{T_pC}, \ldots, (d\mu_n)_p|_{T_pC}$ are linearly independent, which proves that C and N intersect transversally in any point $p \in C \cap N$. As the critical submanifold C of μ_X has even index and coindex, an application of Proposition 3.2.11 yields that $(\mu_X|_N, C \cap N)$ is a Morse–Bott function with critical submanifold $C \cap N$ of even index and coindex. Note that $\mu_X|_N - \mu_{n+1}|_N = \sum_{j=1}^{n} \lambda_j \mu_j|_N$ is constant on N. It follows that $\mu_{n+1}|_N$ is Morse–Bott with critical submanifold $C \cap N$ of even index and coindex.

Applying the Connectivity Theorem to the function $\mu_{n+1}|_N$, we find that

$$(\mu_{n+1}|_N)^{-1}(\eta_{n+1}) = N \cap \mu^{-1}(\eta_{n+1}) = \mu^{-1}(\eta)$$

is connected.

Thus, we have proved that if $\eta \in \mathbb{R}^{n+1}$ is a regular value for μ such that $(\eta_1, \ldots, \eta_n)$ is a regular value for the reduced momentum map $\tilde{\mu} = (\mu_1, \ldots, \mu_n)$, then the level set $\mu^{-1}(\eta)$ is connected. We extend this result to *all* regular values of the momentum map $\mu : M \to \mathbb{R}^{n+1}$. Note that $\mu(M) \cap \mathfrak{t}^*_{\text{reg}}$ is open in $\mu(M)$, since M is compact. Let $\xi \in \mu(M) \cap \mathfrak{t}^*_{\text{reg}}$ be a regular value of μ, and let $U \subseteq \mu(M) \cap \mathfrak{t}^*_{\text{reg}}$ be a connected open neighborhood of ξ contained in the image of μ. Then we have that

$$\mu : \mu^{-1}(U) \to U$$

is a proper surjective submersion. By Ehresmann's fibration theorem [14], which states that a proper surjective submersion is a (locally trivial) fiber bundle, we obtain that the regular level sets over points in U are diffeomorphic to each other. Since $\mu(M) \cap \mathfrak{t}^*_{\text{reg}} \cap \widetilde{\mathfrak{t}^*_{\text{reg}}} \subseteq \mu(M) \cap \mathfrak{t}^*_{\text{reg}}$ is dense (by Sard's theorem) and $\mu^{-1}(\eta)$ connected for $\eta \in \mathfrak{t}^*_{\text{reg}} \cap \widetilde{\mathfrak{t}^*_{\text{reg}}}$, it follows that each regular level set of μ is connected.

Part 2: Convexity of the image.

We prove by induction on $n = \dim \mathbb{T}^n$ that the image $\mu(M)$ is convex. For $n = 1$, note that $\mu(M) \subseteq \mathbb{R}$ is connected as M is connected. Thus, for the base case, $\mu(M)$ is convex. For the induction step, suppose that the image of the momentum map of any Hamiltonian $\mathbb{T}^n$-action on M is convex. Let $\mu : M \to \mathbb{R}^{n+1}$ be a momentum map of a Hamiltonian $\mathbb{T}^{n+1}$-action on M, we show that its image $\mu(M)$ is convex. Suppose that μ is reducible. By Lemma 3.3.4 we may write $\mu = i \circ \mu'$, where μ' is a momentum map of a $\mathbb{T}^n$-action on M and i an inclusion of Lie algebras. By the induction hypothesis, we immediately find that $\mu(M)$ is convex.

Hence, we assume that $\mu : M \to \mathbb{R}^{n+1}$ is irreducible. Let $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be an injective linear map, represented by an integer matrix. Set $\text{pr} := i^T : \mathbb{R}^{n+1} \to \mathbb{R}^n$, which we think of as a projection map. Since $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is represented by an integer matrix, we may use it as a Lie group homomorphism between tori: $i : \mathbb{T}^n \to \mathbb{T}^{n+1}$ is well-defined.

Using the given action ψ, we define a torus action of $\mathbb{T}^n$ on M by:

$$\mathbb{T}^n \times M \to M, \ \tau \cdot p := \psi_{i(\tau)}(p).$$

This action is Hamiltonian with momentum map given by:

$$\mu' = \text{pr} \circ \mu : M \to \mathbb{R}^n.$$

Recall that $\mu(M) \cap \mathfrak{t}^*_{\text{reg}}$ is dense in the image $\mu(M) \subseteq \mathbb{R}^{n+1}$, as μ is irreducible. Then the momentum map $\mu' = \text{pr} \circ \mu$ is also irreducible, so that $\mu'(M) \cap \mathfrak{t}'^*_{\text{reg}}$ is dense in $\mu'(M) \subseteq \mathbb{R}^n$. (Here $\mathfrak{t}'^*_{\text{reg}}$ denotes the set of regular values of μ' in $\mathbb{R}^n$.)

Let $\eta' \in \mu'(M)$ be a regular value of μ', so that $\mu'^{-1}(\eta')$ is a non-empty connected submanifold of M. Let $p_0 \in \mu'^{-1}(\eta')$ arbitrarily. By definition of μ' we may then write the regular level set $\mu'^{-1}(\eta')$ in the following way:

$$\mu'^{-1}(\eta') = \{p \in M : \mu(p) \in \mu(p_0) + \ker(\mathrm{pr})\}.$$

Suppose $p_1 \in \mu'^{-1}(\eta')$ is another point in this regular level set. Since this level set is a connected submanifold, there exists a path $\gamma : I \to \mu'^{-1}(\eta')$ connecting p_0 and p_1. It follows that

$$\mu(\gamma(t)) \in \mu(p_0) + \ker(\mathrm{pr}), \text{ for all } t \in I.$$

Since the kernel of $\mathrm{pr} : \mathbb{R}^{n+1} \to \mathbb{R}$ is one-dimensional and $\mu(\gamma(I)) \subseteq \mu(M)$ is connected, we obtain that

$$(1-t)\mu(p_0) + t\mu(p_1) \in \mu(M), \text{ for all } t \in I.$$

Thus, for a regular value η' of μ' and any two points $p_0, p_1 \in \mu'^{-1}(\eta')$, we have established that the image $\mu(M)$ contains all convex combinations of $\mu(p_0)$ and $\mu(p_1)$.

Let $x, y \in M$ be arbitrary points in the manifold. We can approximate x and y by sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subseteq M$ such that for each k there exists a surjective projection $\mathrm{pr}_k : \mathbb{R}^{n+1} \to \mathbb{R}^n$, represented by an integer matrix, satisfying:

$$\mu(x_k) \in \mu(y_k) + \ker(\mathrm{pr}_k).$$

Since $\mu'(M) \cap \mathfrak{t}'_{\mathrm{reg}} \subseteq \mu'(M)$ is dense, we may assume that $\mu'(y_k)$ is a regular value of $\mu' = \mathrm{pr} \circ \mu$. By the previous paragraph, we know that $\mu(M)$ contains all convex combinations of $(1-t)\mu(x_k) + t\mu(y_k)$. Note that $\mu(M)$ is closed by compactness of M. Now, taking the limit $k \to \infty$, we find that

$$(1-t)\mu(x) + t\mu(y) \in \mu(M), \text{ for all } t \in I.$$

As the points $x, y \in M$ were picked arbitrarily, we conclude that $\mu(M)$ is convex.

Part 3: The image is a convex polytope.

Finally, we show that $\mu(M)$ is the convex hull of the images of the fixed points. By Lemma 3.2.12 and compactness of M, the components of the fixed point set $\mathrm{Fix}_{\mathbb{T}}(M)$ are a finite number of symplectic submanifolds $C_1, \dots, C_N$ of M. Since the critical points of the momentum map correspond to fixed points of the action (of a subtorus), we find that μ is constant on each component C_j. Thus, for each C_j, we have that $\mu(C_j) = \eta_j$ for some $\eta_j \in \mathbb{R}^n$. The image $\mu(M)$ is convex so it contains the convex hull $[\eta_1, \dots, \eta_N]$ of the points η_j.

Let η be a vector which is not contained in the convex hull $[\eta_1, \dots, \eta_N]$. In view of the Cauchy-Schwarz (in)equality, there exists some $X \in \mathbb{R}^n$ in the Lie algebra satisfying:

$$\langle \eta_j, X \rangle < \langle \eta, X \rangle \text{ for } j = 1, \dots, N.$$

By perturbing this element X slightly, we may assume that it has rationally independent coefficients. It follows that $\mathbb{T}_X = \mathbb{T}$. Thus, we have

$$\mathrm{Crit}(\mu_X) = M^{\mathbb{T}} = \bigsqcup_{j=1}^{N} C_j.$$

In view of the Connectivity Theorem, the above implies that μ_X attains a global maximum on one of the components C_j. Taking the supremum of μ_X over M, we find:

$$\sup_{p \in M} \langle \mu(p), X \rangle < \langle \eta, X \rangle,$$

which implies that η is not contained in the image $\mu(M)$. We conclude that the image $\mu(M)$ is the convex hull of the images of the fixed points: $\mu(M) = [\eta_1, \ldots, \eta_N]$. $\square$

Now that we have established that the image of the momentum map of a Hamiltonian torus action on a compact connected symplectic manifold is a convex polytope, we call $\mu(M)$ the **momentum polytope** and denote $\mu(M) = \Delta$.

The following example can be found in [34, Example 5.5.2], [4, IV.4.6].

Example 3.3.5. Consider the following symplectic torus action of $\mathbb{T}^3$ on $(\mathbb{C}P^3, \omega_{\mathrm{FS}})$:

$$(t_1, t_2, t_3) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : -t_1 z_1 : -t_2 z_2 : -t_3 z_3].$$

This action is Hamiltonian with momentum map μ given by:

$$\mu : \mathbb{C}P^3 \to \mathbb{R}^3,\ \mu([z_0 : z_1 : z_2 : z_3]) = \pi\left(\frac{|z_1|^2}{\sum_{j=0}^3 |z_j|^2}, \ldots, \frac{|z_3|^2}{\sum_{j=1}^3 |z_j|^2}\right).$$

One verifies that the action has 4 fixed points, namely:

$$p_1 = [1 : 0 : 0 : 0],\ p_2 = [0 : 1 : 0 : 0],\ p_3 = [0 : 0 : 1 : 0],\ \text{and } p_4 = [0 : 0 : 0 : 1].$$

By virtue of the Atiyah–Guillemin–Sternberg convexity theorem, we find that the image $\mu(M)$ is given by the simplex displayed in Figure 3.4.

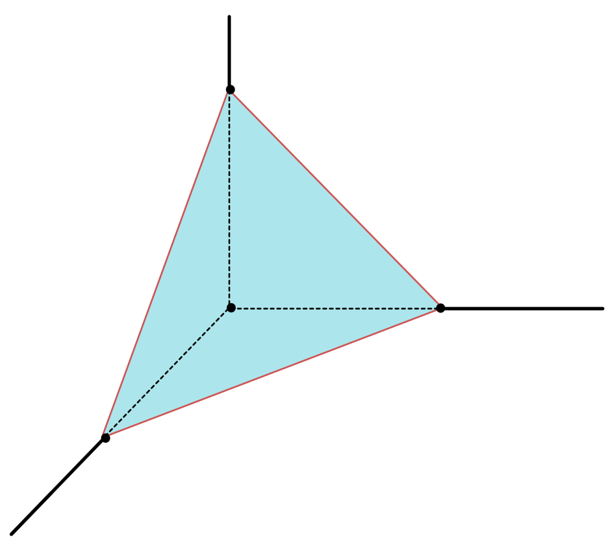


Figure 3.4: The image $\mu(M) = \Delta$ of the momentum map.

The amount of vertices of the resulting polytope Δ is related to the following proposition.

Proposition 3.3.6. Let $\mu : M \to \mathbb{R}^n$ be the momentum map of a Hamiltonian torus action on a compact connected symplectic manifold (M, ω). If the action is effective, that is, each $1 \neq \tau \in \mathbb{T}$ moves at least one point on the manifold, then the resulting polytope $\mu(M) = \Delta$ has at least $n+1$ vertices.

Proof. Since the action is effective, there exists some point $p \in M$, such that $d\mu_p : T_pM \to \mathbb{R}^n$ is a surjection. Thus, μ is a submersion on a sufficiently small neighborhood U around p. Therefore, $\mu|_U$ is an open map, and the polytope $\mu(M) = \Delta$ contains an open (in $\mathbb{R}^n$). This implies that $\mu(M)$ must have at least $n+1$ vertices. □

In particular, an effective Hamiltonian $\mathbb{T}^n$ action has atleast $n+1$ fixed points. In *Hamiltoniens périodiques et images convexes de l'application moment* [11], Delzant has shown that, for effective Hamiltonian torus actions with $\dim \mathbb{T} = \frac{1}{2} \dim M$, the momentum polytope completely determines the torus action.

Summary

In this chapter, we first used the Haar measure of a torus to obtain an invariant almost complex structure and an invariant Riemannian metric on the manifold M. Using these invariant structures, we applied representation theory and the equivariant Darboux Theorem to derive a local model (with a linear torus action) for a Hamiltonian torus action near a fixed point $p \in M$. This local model allowed us to show that the momentum map functions μ_X are Morse–Bott functions with even index and coindex. Finally, we proved by induction on the dimension of the torus that the image $\mu(M)$ for a momentum map of a Hamiltonian torus action is a convex polytope. This polytope Δ is the convex hull of the images of the fixed points.

Chapter 4

Duistermaat–Heckman Theorems

In the previous chapter we have studied the image of the momentum map. In this section we explore two additional ways of studying the momentum map. Firstly, we prove the Duistermaat–Heckman theorem and we use it to relate the image to the reduced symplectic manifolds. Secondly, we prove the Duistermaat–Heckman localization theorem which computes an important integral for Hamiltonian actions in terms of fixed point data.

4.1 Principal Bundles

We follow L. Tu's book *Differential Geometry* [42] for this subsection.

Definition 4.1.1. Let E and M be manifolds. A smooth surjection $\pi : E \to M$ is a **fiber bundle** with fiber F over M if there exists an open covering $\mathcal{U} = (U_i)_{i \in I}$ of M together with diffeomorphisms $\varphi_{U_i} : \pi^{-1}(U_i) \to U_i \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_{U_i}} & U \times F \\ & \underset{\pi}{\searrow} \quad \underset{\mathrm{pr}_1}{\swarrow} & \\ & U. & \end{array}$$

Here $\mathrm{pr}_1 : U \times F \to U$ is the projection onto the first factor. We say that φ_{U_i} are **local trivializations** of the fiber bundle $\pi : E \to M$. The manifold E is called the **total space**, and M is called the **base space**.

Given a fiber bundle $\pi : E \to M$, the existence of local trivializations imply that π is a submersion. Indeed, for any open neighborhood $U_i \in \mathcal{U}$, we have $\pi|_{\pi^{-1}(U)} = \mathrm{pr}_1 \circ \varphi_{U_i}$ which is a composition of submersions. Another consequence of the definition is that the local trivializations restrict to diffeomorphisms of the fibers. More precisely, for all $x \in U_i$, we have that

$$\varphi_{U_i}|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times F$$

is a diffeomorphism. We now restrict our attention to a special type of fiber bundle, namely a fiber bundle where a Lie group acts freely on the total space.

Definition 4.1.2. Let P and M be manifolds, and let G be a Lie group. A **principal G-bundle** is a fiber bundle with fiber G together with a *free* left-action of G on P such that the local trivializations

$$\varphi_{U_i} : \pi^{-1}(U_i) \to U_i \times G$$

are G-equivariant. Explicitly, for $p \in P$ and $g \in G$, we have:

$$\varphi_{U_i}(g \cdot p) = g \cdot \varphi_{U_i}(p).$$

(The left-action of G on $U_i \times G$ is given by $g \cdot (u, h) := (u, gh)$.)

Note that the fibers of π are necessarily G-invariant by equivariance of the local trivializations. Furthermore, we may identify the base space M of a principal G-bundle $\pi : P \to M$ with the quotient manifold P/G. To see this, consider the quotient map $\tilde{\pi} : P \to P/G$, which is a submersion. Since the principal G-bundle $\pi : P \to M$ and the quotient map $\tilde{\pi} : P \to P/G$ are both surjective submersions which respect each other's fibers, it follows that M and P/G are diffeomorphic ([28, Theorem 4.31]).

We will need the following proposition.

Proposition 4.1.3. [9, Theorem 23.4] Let G be a compact Lie group, and let P be a manifold. Suppose G acts freely on P. Then the quotient map $\pi : P \to P/G$ is a principal G-bundle.

Definition 4.1.4. Let $\pi : P \to M$ be a principal G-bundle. We define the **vertical subbundle** of the tangent bundle TP, denoted V, pointwise by:

$$V_p := \ker d\pi_p \subseteq T_pM, \text{ for } p \in P.$$

We call the tangent vectors in $V_p \subseteq T_pM$ **vertical vectors** (at p).

Proposition 4.1.5. Let $\pi : P \to M$ be a principal G-bundle. The vertical vectors are given by the fundamental vectors: $V_p = \{\underline{X}_p : X \in \mathfrak{g}\}$. Moreover, the vertical subbundle is trivial, that is, the vector bundle $V \to P$ is isomorphic to the product vector bundle $P \times \mathfrak{g} \to P$.

Proof. Let $p \in P$ arbitrarily, and consider the orbit map $\Psi_p : G \to P$. Differentiating at the identity yields a map $d(\Psi_p)_e : \mathfrak{g} \to T_pP$, given by $d(\Psi_p)_e(X) = \underline{X}$. Since the fibers of the principal bundle π are G-invariant, we find that $\pi \circ \Psi_p = \pi$. Thus, we obtain that π is constant on the orbits of G. Recall that the fundamental vectors generate the tangent spaces to the orbits, it follows that the fundamental vectors are vertical: $d\pi_p(\underline{X}) = 0$ for all $X \in \mathfrak{g}$.

As G acts freely on P, the orbit map is an immersion. This implies that the dimension of the linear subspace $\{\underline{X}_p : X \in \mathfrak{g}\}$ is equal to the dimension of G and thus equal to the dimension of V_p. We conclude, for all $p \in P$, that

$$\{\underline{X}_p : X \in \mathfrak{g}\} = V_p.$$

The required vector bundle isomorphism is given by

$$P \times \mathfrak{g} \to V, \ (p, X) \mapsto d(\Psi_p)_e(X) = \underline{X}_p,$$

which proves the last assertion. □

The above proposition shows tht the vertical vectors are tangent to the fibers of π, which is the reason we call these vectors vertical. In due course, we describe the notion of a connection on a principal G-bundle, which can be viewed as a choice of subbundle of the tangent bundle TP complementary to the vertical subbundle. Equivalently, a connection can be described by a Lie algebra-valued one-form on P, so we briefly study Lie algebra-valued differential forms.

Definition 4.1.6. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let P be a manifold. We define the space of differential k-forms with values in the Lie algebra by

$$\Omega^k(P, \mathfrak{g}) := \Gamma((\bigwedge^k T^*P) \otimes \mathfrak{g}).$$

A form $\alpha \in \Omega^k(P, \mathfrak{g})$ assigns smoothly to each point p in the manifold P an alternating k-linear map

$$\alpha_p : T_pP \times \cdots \times T_pP \to \mathfrak{g}.$$

Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ of G. Suppose $\alpha \in \Omega^k(P, \mathfrak{g})$ is a Lie algebra-valued k-form on P. For a point $p \in P$ and k tangent vectors $v_1, \ldots, v_k \in T_pP$, we find:

$$\alpha_p(v_1, \ldots, v_k) = \sum_{j=1}^{n} \alpha_p^j(v_1, \ldots, v_k) \cdot X_j \in \mathfrak{g},$$

where $\alpha_p^j(v_1, \ldots, v_k)$ denotes the coefficient of X_j for each j. In this way we obtain n real-valued differential k-forms $\alpha^1, \ldots, \alpha^n \in \Omega^k(P)$, so we may identify α as follows:

$$\alpha = \sum_{j=1}^{n} \alpha^j \otimes X_j \in \Omega^k(P, \mathfrak{g}).$$

The de Rham differential $d : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g})$ extends to Lie algebra-valued forms in the following way:

$$d\alpha := \sum_{j=1}^{n} d\alpha^j \otimes X_j \in \Omega^{k+1}(P, \mathfrak{g}).$$

Definition 4.1.7. Let $\pi : P \to M$ be a principal G-bundle, and let $\alpha \in \Omega^k(P, \mathfrak{g})$ be a Lie algebra-valued k-form on P. Denote the left-action of G on P by ψ.

- The form α is **horizontal** if $i_{\underline{X}}\alpha = 0$ for all $X \in \mathfrak{g}$. Equivalently, the form β pairs to zero on vertical vectors.
- The form α is **equivariant** if, for all $g \in G$, we have:

$$\psi_g^*\alpha = \mathrm{Ad}_g(\alpha).$$

- The form α is **basic** if it is an element of the image of the injective linear map $\pi^* : \Omega^k(M, \mathfrak{g}) \to \Omega^k(P, \mathfrak{g})$. That is, there is a unique $\delta \in \Omega^k(M, \mathfrak{g})$ such that $\pi^*\delta = \alpha$.

The following proposition gives a useful criterion to check whether a form is basic.

Proposition 4.1.8. [42, Theorem 31.12] Let $\pi : P \to M$ be a principal G-bundle, and let $\alpha \in \Omega^k(P, \mathfrak{g})$. The k-form α with values in the Lie algebra is basic if and only if it is horizontal and G-invariant.

Definition 4.1.9. Let $\pi : P \to M$ be a principal G-bundle. A one-form $\alpha \in \Omega^1(P, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ is a **connection one-form** (for π) if it is a G-equivariant form such that $\alpha(\underline{X}) = X$ for all $X \in \mathfrak{g}$.

An **Ehresmann connection** for the principal bundle $\pi : P \to M$ is a subbundle H of the tangent bundle TP complementary to the vertical subbundle:

$$T_pP = V_p \oplus H_p, \text{ for all } p \in P,$$

with the property that H is G-invariant:

$$(d\psi_g)_p(H_p) = H_{gp}, \text{ for all } p \in P, \; g \in G.$$

The subbundle H is called the **horizontal subbundle** of π.

The following proposition asserts that the above definitions of a connection on a principal bundle are equivalent.

Proposition 4.1.10. [42, Theorem 28.1] Let $\pi : P \to M$ be a principal G-bundle.

- Suppose $\alpha \in \Omega^1(P, \mathfrak{g})$ is a connection one-form for π. Then the subbundle H defined pointwise by $H_p := \ker \alpha_p \subseteq T_pP$ is an Ehresmann connection.
- Suppose H is an Ehresmann connection, then $\alpha_p := (d\Psi_p)_e^{-1} \circ \mathrm{pr}_{V_p} : T_pP \to \mathfrak{g}$ defines a connection one-form on P. Here $\Psi_p : G \to P$ is the orbit map, and $\mathrm{pr}_{V_p} : T_pP \to V_p$ the projection onto the vertical subspace at p.

Proposition 4.1.11. Let $\pi : P \to M$ be a principal G-bundle. Then π admits a connection one-form.

Proof. Consider first the trivial principal G-bundle $M \times G \to M$, where the free left-action of G on $M \times G$ is given by $g \cdot (x, h) := (x, gh)$ for $g, h \in G, x \in M$. In view of Proposition 4.1.10, it suffices to exhibit an Ehresmann connection H of the tangent bundle of the total space. For a point $p = (x, g) \in M \times G$, define

$$H_p := T_xM \times \{0\} \subseteq T_p(M \times G),$$

which defines a G-invariant horizontal subbundle by varying p. Let $(\sigma_i)_{i \in I}$ be a smooth partition of unity subordinate to the open cover $\mathcal{U}$ of M associated to the principal bundle $\pi : P \to M$. By the above, there exists a connection one-form α_i on $U_i \times G$ for each $i \in I$. Define $\alpha \in \Omega^1(P, \mathfrak{g})$ pointwise by

$$\alpha_p = \sum_{i \in I} \sigma_i(\pi(p)) \cdot (\varphi_{U_i}^* \alpha_i)_p,$$

which is readily seen to be a connection one-form on P. ☐

We now restrict to the case where $G = \mathbb{T}^n$ is an n-torus, with Lie algebra $\mathfrak{t}$. Suppose $\pi : P \to M$ is a principal $\mathbb{T}^n$-bundle. As before, we use $X_1 := \frac{\partial}{\partial\theta_1}, \dots, X_n := \frac{\partial}{\partial\theta_n} \in \mathfrak{t}$ as a basis for the Lie algebra $\mathfrak{t}$ of the torus. We fix a connection one-form $\alpha \in \Omega^1(P, \mathfrak{t})$, and we now look at some of its properties.

Write $\alpha = \sum_{j=1}^n \alpha_j \otimes X_j$, where the α_j's are ordinary one-forms on P. By differentiating, we find

$$d\alpha = \sum_{j=1}^n d\alpha^j \otimes X_j \in \Omega^2(P, \mathfrak{t}).$$

We claim that $d\alpha$ is basic. In view of Proposition 4.1.8, we need to check that $d\alpha$ is horizontal and $\mathbb{T}^n$-invariant. The connection one-form α is $\mathbb{T}^n$-invariant, as the adjoint action is trivial. Using that the de Rham differential and pullbacks commute, it follows that $d\alpha$ is also $\mathbb{T}^n$-invariant. By application of Cartan's magic formula and $\mathbb{T}^n$-invariance of α, we obtain:

$$i_{\underline{X}} d\alpha = \mathcal{L}_{\underline{X}}\alpha - d i_{\underline{X}}\alpha = -d i_{\underline{X}}\alpha.$$

Since $\alpha(\underline{X}) = X$ for elements of the Lie algebra, it follows that

$$-d i_{\underline{X}}\alpha = -d\Big(\sum_{j=1} \alpha^j(\underline{X}_j) \otimes X_j\Big) = -d\Big(\sum_{j=1}^n 1 \otimes X_j\Big) = 0.$$

Combining the two equations, we conclude that $d\alpha$ is horizontal, so that we have proved that $d\alpha$ is basic. Thus, there exists a unique two-form $\beta \in \Omega^2(M, \mathfrak{t})$ such that

$$\pi^*\beta = d\alpha.$$

Note that $0 = d\pi^*\beta = \pi^* d\beta$, which implies that $d\beta = 0$ by the injectivity of π^*. Writing $\beta = \sum_{j=1}^n \beta^j \otimes X_j$, we see that each ordinary 2-form $\beta_j \in \Omega^2(M)$ represents a cohomology class $[\beta_j] \in H^2_{\mathrm{dR}}(M)$. We can also view $[\beta] \in H^2_{\mathrm{dR}}(M) \otimes \mathfrak{t}$. This discussion leads to the following definition.

Definition 4.1.12. Let $\pi : P \to M$ be a principal $\mathbb{T}^n$-bundle with connection one-form $\alpha \in \Omega^1(P, \mathfrak{t})$. The **curvature** of the connection α is the unique closed two-form $\beta \in \Omega^2(M, \mathfrak{t})$ on the base satisfying $\pi^*\beta = d\alpha$.

We show that the cohomology class of the curvature form does not depend on the connection. For this, suppose we have two connection one-forms $\alpha, \alpha' \in \Omega^1(P, \mathfrak{t})$ with curvature forms $\beta, \beta' \in \Omega^2(M, \mathfrak{t})$, respectively. Then $\alpha - \alpha'$ is horizontal and $\mathbb{T}^n$-invariant, so that there exists a one-form $\tau \in \Omega^1(M, \mathfrak{t})$ such that $\alpha - \alpha' = \pi^*\tau$. Then

$$\pi^*(d\tau) = d\pi^*\tau = d\alpha - d\alpha' = \pi^*(\beta - \beta'),$$

which implies $\beta - \beta' = d\tau$ by injectivity of π^*. We conclude

$$[\beta] = [\beta'] \in H^2_{\mathrm{dR}}(M) \otimes \mathfrak{t}.$$

Thus, we have established an algebraic invariant of the principal $\mathbb{T}^n$-bundle $\pi : P \to M$: the de Rham cohomology class of the curvature form.

Remark 4.1.13. The cohomology class $[\beta] \in H^2_{\mathrm{dR}}(M) \otimes \mathfrak{t}$ of the curvature form has a special name, it is called the **Chern class** of the principal bundle, we denote $c = [\beta]$.

We work out the connection one-form and the curvature form in a familiar example. The following example can be found in *Torus Actions on Symplectic Manifolds* [4, Example V.4.4.]

Example 4.1.14 (A Connection and Curvature Form on the Hopf Fibration). Consider the principal $\mathbb{S}^1$-bundle $\pi_{\mathrm{H}} : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$, where $\mathbb{S}^1$ acts on $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ by

$$s \cdot (z_1, \ldots, z_{n+1}) := (sz_1, \ldots, sz_{n+1}).$$

We compute a connection one-form for π_{H}. If we identify the Lie algebra $\mathfrak{s}$ of the circle with $\mathbb{R}$, the condition that $\alpha \in \Omega^1(\mathbb{S}^{2n+1})$ is a connection-one form reads as follows. Take $X = \frac{\partial}{\partial\theta} \in \mathfrak{s}$, then $\alpha \in \Omega^1(\mathbb{S}^{2n+1})$ must satisfy:

- $i_{\underline{X}}\alpha = 1$;
- $\mathcal{L}_{\underline{X}}\alpha = 0$.

The fundamental vector field generated by $X = \frac{\partial}{\partial\theta} \in \mathfrak{s}$ is given by

$$\underline{X} = 2\pi \sum_{j=1}^{n+1}(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}) \in \mathfrak{X}(\mathbb{S}^{2n+1}),$$

where we write $z_j = x_j + iy_j$ for all j. Define $\alpha \in \Omega^1(\mathbb{S}^{2n+1})$ by

$$\alpha = \frac{1}{2\pi} \sum_{j=1}^{n+1}(-y_j dx_j + x_j dy_j).$$

We compute

$$i_{\underline{X}}\alpha = \frac{1}{2\pi} \sum_{j=1}^{n+1} -y_j dx_j(\underline{X}) + x_j dy_j(\underline{X}_j) = \sum_{j=1}^{n+1} y_j^2 + x_j^2 = 1,$$

so that, by application of Cartan's formula, we have

$$\mathcal{L}_{\underline{X}}\alpha = i_{\underline{X}}d\alpha = \frac{1}{\pi} \cdot i_{\underline{X}}(j^*\omega_0) = -\sum_{j=1}^{n+1}(2y_j dy_j + 2x_j dx_j) = -d(\sum_{j=1}^{n+1}(y_j^2 + x_j^2)) = -d(1) = 0,$$

where $\omega_0 = \sum_{j=1}^{n+1} dx_j \wedge dy_j$ is the symplectic form on $\mathbb{C}^{n+1}$ and $j : \mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ the inclusion map. We conclude that $\alpha \in \Omega^1(\mathbb{S}^{2n+1})$ is a connection one-form for the principal $\mathbb{S}^1$-bundle $\pi_{\mathrm{H}} : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

By the symplectic reduction theorem, we have:

$$d\alpha = \frac{1}{\pi} j^*\omega_0 = \frac{1}{\pi}\pi_{\mathrm{H}}^*\omega_{\mathrm{FS}}.$$

Therefore the curvature form on the base is $\beta := \frac{1}{\pi}\omega_{\mathrm{FS}} \in \Omega^2(\mathbb{C}P^n)$.

4.2 Cohomology Classes of Reduced Symplectic Forms

The goal of this section is to prove the Duistermaat–Heckman Theorem. We follow the proof of the theorem as explained by V. Guillemin in *Moment Maps and Combinatorial Invariants of Hamiltonian Tn Spaces* [17,]. The following theorem, due to A. Weinstein [44], plays an important role in the proof.

Theorem 4.2.1 (**The Equivariant Coisotropic Embedding Theorem** [44]). *Let (M_0, ω_0) and (M, ω) be two symplectic manifolds of dimension $2m$. Let G be a compact Lie group acting in a Hamiltonian fashion on (M_0, ω_0) and (M, ω) with momentum maps $\mu_0 : M_0 \to \mathfrak{g}^*$ and $\mu : M \to \mathfrak{g}^*$, respectively. Let Z be a manifold of dimension $k \leq m$. Suppose $\iota_0 : Z \hookrightarrow M_0$ and $\iota : Z \hookrightarrow M$ are G-equivariant coisotropic embeddings. In addition, suppose that $\iota_0^*\omega_0 = \iota^*\omega$ and $\iota_0^*\mu_0 = \iota^*\mu$. Then there exist G-invariant neighborhoods U_0 of $\iota_0(Z) \subseteq M_0$ and U of $\iota(Z) \subseteq M$, and a G-equivariant symplectomorphism $\phi : U_0 \to U$ satisfying $\phi \circ \iota_0 = \iota$ and $\mu_0 = \mu \circ \phi$.*

Assumptions. Suppose that (M, ω) is a compact symplectic manifold equipped with a Hamiltonian action of a torus $\mathbb{T}^n$, and denote its momentum map by $\mu : M \to \mathfrak{t}^*$. In addition, suppose that $\mathbb{T}^n$ acts freely on the level set $Z := \mu^{-1}(0)$. By the symplectic reduction theorem, we have that

$$(M_{\text{red}},\ \omega_{\text{red}}) = (Z/\mathbb{T}^n,\ \omega_{\text{red}})$$

is a reduced symplectic manifold. Since $\mathbb{T}^n$ acts freely on $\mu^{-1}(0)$ and M is compact, there exists a convex open neighborhood V of 0 in $\mathfrak{t}^*$ such that $\mathbb{T}^n$ acts freely on the level set $\mu^{-1}(\xi)$ for all $\xi \in V$. We denote by

$$(M_\xi,\ \omega_\xi) = (\mu^{-1}(\xi)/\mathbb{T}^n,\ \omega_\xi)$$

the reduced symplectic manifolds for $0 \neq \xi \in V$. Since $\mathbb{T}^n$ acts freely on $\mu^{-1}(\xi)$ $(\xi \in V)$, the level sets $\mu^{-1}(\xi)$ are submanifolds of M of codimension n. Thus, by Proposition 4.1.3, we have that

$$\pi : Z \to M_{\text{red}}$$

is a principal $\mathbb{T}^n$-bundle. We fix a connection one-form $\alpha \in \Omega^1(Z, \mathfrak{t})$ for this principal bundle, and denote by $\beta \in \Omega^2(M_{\text{red}}, \mathfrak{t})$ its curvature form on the base M_{red}.

Theorem 4.2.2 (**The Duistermaat–Heckman Theorem** [12]). *Under the assumptions above, there exists a diffeomorphism $M_\xi \to M_{red}$ for each $\xi \in V$, which allows us to identify the de Rham cohomology groups $H^*_{dR}(M_\xi) \cong H^*_{dR}(M_{red})$. Moreover, using this identification, the cohomology class of the symplectic form ω_ξ varies linearly in $\xi \in V$:*

$$[\omega_\xi] = [\omega_{red}] - \langle \xi, c \rangle \in H^2_{dR}(M_\xi),$$

where $c = [\beta] \in H^2_{dR}(M_{red}) \otimes \mathfrak{t}$ is the Chern class of the principal $\mathbb{T}^n$-bundle $\pi : Z \to M_{red}$.

Proof. The reduced symplectic ω_{red} on M_{red} satisfies $\iota^*\omega = \pi^*\omega_{\text{red}}$, where $\iota : Z \hookrightarrow M$ is the inclusion map and $\pi : Z \to M_{\text{red}}$ is the principal $\mathbb{T}^n$-bundle. By the symplectic reduction theorem, we find that $\iota : Z \hookrightarrow M$ is a $\mathbb{T}^n$-equivariant coisotropic embedding.

Let $X_1 := \frac{\partial}{\partial\theta_1}, \dots, X_n := \frac{\partial}{\partial\theta_n} \in \mathfrak{t}$ be the usual basis for the Lie algebra $\mathfrak{t}$ of the torus. In this basis, we write the connection one-form $\alpha \in \Omega^1(Z, \mathfrak{t}$ as follows:

$$\alpha = \sum_{j=1}^{n} \alpha^j \otimes X_j.$$

By definition of a connection one-form, the ordinary one-forms $\alpha^j \in \Omega^1(Z)$ are $\mathbb{T}^n$-invariant and satisfy $\alpha^j(\underline{X_k}) = \delta_{j,k}$.

We identify $\mathfrak{t}^* = \mathbb{R}^n$, and consider the product manifold $M_0 := Z \times (-\epsilon, \epsilon)^n$ with projections $\mathrm{pr}_Z : M_0 \to Z$ and $\mathrm{pr}_j : M_0 \to (-\epsilon, \epsilon)$ the projection onto the j'th $(-\epsilon, \epsilon)$-factor. For each j, let x_j denote linear coordinates on the j'th $(-\epsilon, \epsilon)$-factor. We define a two-form $\omega_0 \in \Omega^2(M_0)$ on M_0 by:

$$\omega_0 := \mathrm{pr}_Z^*(\pi^*\omega_{\mathrm{red}}) - \sum_{j=1}^{n} d[\mathrm{pr}_j^*(x_j) \cdot \mathrm{pr}_Z^*(\alpha^j)] \in \Omega^2(M_0).$$

The action of $\mathbb{T}^n$ on the manifold $M_0 = Z \times (-\epsilon, \epsilon)^n$ is inherited from the action on Z, it is given by:

$$t \cdot (z, x_1, \dots, x_n) := (tz, x_1, \dots, x_n).$$

Note that ω_0 is $\mathbb{T}^n$-invariant, because the α^j's are $\mathbb{T}^n$-invariant and $\pi^*\omega_{\mathrm{red}} = \iota^*\omega$ is $\mathbb{T}^n$-invariant by symplecticity of the action on Z. Since ω_{red} is closed and $d^2 = 0$, we see that ω_0 is a closed form. We now check that ω_0 is nondegenerate on the submanifold $Z \times \{0\} \subseteq M_0$. We have

$$\omega_0|_{Z\times\{0\}} = \mathrm{pr}_Z^*(\pi^*\omega_{\mathrm{red}}) - \sum_{j=1}^{n} [\mathrm{pr}_j^*(dx_j) \wedge \mathrm{pr}_Z^*(\alpha^j)],$$

so that we have

$$\omega_0|_{Z\times\{0\}}(\frac{\partial}{\partial x_j}, \underline{X_j}) = -1.$$

This implies that ω_0 is nondegenerate along $Z \times \{0\}$. Since nondegeneracy is an open condition, there exists an $\epsilon > 0$ sufficiently small such that ω_0 is nondegenerate on M_0. We conclude that $\omega_0 \in \Omega^2(M_0)$ is a symplectic form on M_0.

We determine a momentum map for the $\mathbb{T}^n$-action on M_0. For $k = 1, \dots, n$, we compute:

$$\begin{aligned} i_{\underline{X_k}}\omega_0 &= i_{\underline{X_k}}\mathrm{pr}_Z^*(\pi^*\omega_{\mathrm{red}}) - \sum_{j=1}^{n} i_{\underline{X_k}} d[\mathrm{pr}_j^*(x_j) \cdot \mathrm{pr}_Z^*(\alpha^j)] \\ &= -i_{\underline{X_k}} d[\mathrm{pr}_k^*(x_k) \cdot \mathrm{pr}_Z^*(\alpha^k)] \\ &= d i_{\underline{X_k}}[\mathrm{pr}_k^*(x_k) \cdot \mathrm{pr}_Z^*(\alpha^k)] \\ &= d[\mathrm{pr}_k^*(x_k)]. \end{aligned}$$

Here we have used the symplectic reduction theorem and the property $\alpha^j(\underline{X_k}) = \delta_{j,k}$ for the second equality, and Cartan's magic formula together with $\mathbb{T}^n$-invariance of α^k for the third equality. We conclude that the action of $\mathbb{T}^n$ on M_0 is Hamiltonian with momentum map μ_0 given by:

$$\mu_0 : M_0 \to \mathbb{R}^n,\ \mu_0(z, x_1, \ldots, x_n) = (x_1, \ldots, x_n).$$

Note that $\iota_0 : Z \to M_0,\ z \mapsto (z, 0)$ is a $\mathbb{T}^n$-equivariant coisotropic embedding.

We check that $\iota_0^* \omega_0 = \iota^* \omega$. Note that $\mathrm{pr}_j \circ \iota_0 = 0$ for all j and $\mathrm{pr}_Z \circ \iota_0 = \mathrm{Id}_Z$. Since the pullback is a contravariant functor and commutes with the de Rham differential, we compute:

$$\begin{aligned}
\iota_0^* \omega_0 &= \iota_0^* \mathrm{pr}_Z^* (\pi^* \omega_{\mathrm{red}}) - \sum_{j=1}^{n} \iota_0^* d[\mathrm{pr}_j^*(x_j) \cdot \mathrm{pr}_Z^*(\alpha^j)] \\
&= (\mathrm{pr}_Z \circ \iota_0)^* (\pi^* \omega_{\mathrm{red}}) - \sum_{j=1}^{n} d[(\mathrm{pr}_j \circ \iota_0)^*(x_j) \cdot (\mathrm{pr}_Z \circ \iota_0)^*(\alpha^j)] \\
&= \pi^* \omega_{\mathrm{red}} = \iota^* \omega.
\end{aligned}$$

By definition of Z, we find that $\iota^* \mu = 0 = \iota_0^* \mu_0$. By virtue of the equivariant coisotropic embedding theorem, we obtain a $\mathbb{T}^n$-equivariant symplectomorphism $\phi : U_0 \to U$ between neighborhoods of $\iota_0(Z)$ and $\iota(Z)$ intertwining the inclusions and momentum maps. This implies that $(M_\xi,\ \omega_\xi)$ is symplectomorphic to the symplectic reduction of the level set $\mu_0^{-1}(\xi) \subseteq (M_0, \omega_0)$. Note that $\mu_0^{-1}(\xi) = Z \times \{\xi\}$ and Z are equivariantly diffeomorphic, which implies that $\mu_0^{-1}(\xi)/\mathbb{T}^n$ and $M_{\mathrm{red}} = Z/\mathbb{T}^n$ are diffeomorphic. In particular, we have established a diffeomorphism between M_{red} and M_ξ, so that we may identify their respective de Rham cohomology groups:

$$H^2_{\mathrm{dR}}(M_{\mathrm{red}}) \cong H^2_{\mathrm{dR}}(M_\xi).$$

By equivariantly identifying $\mu_0^{-1}(\xi) = Z \times \{\xi\}$ with Z, we see that the quotient map $\mu_0^{-1}(\xi) \to \mu_0^{-1}(\xi)/\mathbb{T}^n$ is *identical* to the principal $\mathbb{T}^n$-bundle $\pi : Z \to M_{\mathrm{red}}$. Therefore, we may use π to compute the unique symplectic form on $\mu_0^{-1}(\xi)/\mathbb{T}^n = M_{\mathrm{red}}$. Let $j_\xi : \mu_0^{-1}(\xi) \to M_0$ denote the denote the inclusion map, and write $\xi = (\xi_1, \ldots, \xi_n)$. Recall that the curvature form β of the connection one-form α satisfies $\pi^* \beta = d\alpha$. We find:

$$\begin{aligned}
j_\xi^* \omega_0 &= j_\xi^* \mathrm{pr}_Z^* (\pi^* \omega_{\mathrm{red}}) - \sum_{j=1}^{n} d[j_\xi^* \mathrm{pr}_j^*(x_j) \cdot j_\xi^* \mathrm{pr}_Z^*(\alpha^j)] \\
&= (\mathrm{pr}_Z \circ j_\xi)^* (\pi^* \omega_{\mathrm{red}}) - \sum_{j=1}^{n} d[\xi_j \cdot (\mathrm{pr}_Z \circ j_\xi)^*(\alpha^j)] \\
&= \pi^* \omega_{\mathrm{red}} - \sum_{j=1}^{n} \xi_j \cdot d\alpha^j \\
&= \pi^* \omega_{\mathrm{red}} - \sum_{j=1} \xi_j \cdot \pi^* \beta_j \\
&= \pi^* (\omega_{\mathrm{red}} - \langle \xi, \beta \rangle).
\end{aligned}$$

Thus, we find that (M_ξ, ω_ξ) and $(M_{\mathrm{red}}, \omega_{\mathrm{red}} - \langle \xi, \beta \rangle)$ are symplectomorphic.

Recall that the Chern class $c = [\beta] \in H^2_{\mathrm{dR}}(M_{\mathrm{red}}) \otimes \mathfrak{t}$ is the cohomology class of the curvature form. We conclude:

$$[\omega_\xi] = [\omega_{\mathrm{red}}] - \langle \xi, c \rangle \in H^2_{\mathrm{dR}}(M_\xi).$$

□

Polynomial behavior of the Duistermaat–Heckman measure

In this subsection we study a global invariant of Hamiltonian $\mathbb{T}^n$-spaces, namely the Duistermaat–Heckman measure on the dual $\mathfrak{g}^*$ of the Lie algebra and we show that its density function with respect to the Lebesgue measure on $\mathfrak{g}^*$ is piecewise polynomial. Throughout, let $\mathcal{B}(X)$ denote the Borel measurable subsets on any topological measurable space X.

Definition 4.2.3. Let (M, ω) be a $2m$-dimensional symplectic manifold. We define the **Liouville measure**, denoted by $\mathcal{L}_\omega$, to be the measure on $(M, \mathcal{B}(M))$ that assigns the symplectic volume to each Borel measurable subset $A \subseteq M$. Explicitly, we have

$$\mathcal{L}_\omega(A) := \int_A \frac{\omega^m}{m!}, \ A \in \mathcal{B}(M).$$

Note that the Liouville volumes are invariant under symplectomorphisms.

If the manifold M admits a Hamiltonian action, then we can use the momentum map to define a measure on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. This leads to the following definition.

Definition 4.2.4. Let (M, ω) be a symplectic manifold equipped with a Hamiltonian action of a compact Lie group G, and suppose the associated momentum map $\mu : M \to \mathfrak{g}^*$ is proper. The **Duistermaat–Heckman measure** m_{DH} on $(\mathfrak{g}^*, \mathcal{B}(\mathfrak{g}^*))$ is the pushforward of the Liouville measure under the momentum map:

$$m_{\mathrm{DH}}(B) := \mu_*(\mathcal{L}_\omega)(B) = \mathcal{L}_\omega(\mu^{-1}(B)) = \int_{\mu^{-1}(B)} \frac{\omega^m}{m!}, \ B \in \mathcal{B}(\mathfrak{g}^*).$$

One can integrate compactly-supported smooth functions $h \in C^\infty_c(\mathfrak{g}^*)$ with respect to the Duistermaat–Heckman measure as follows:

$$\int_{\mathfrak{g}^*} h \ dm_{\mathrm{DH}} = \int_M (h \circ \mu) \frac{\omega^m}{m!}. \tag{4.1}$$

To see this, consider first a simple function $h = \sum_{j=1}^N y_j \cdot 1_{B_j}$, where $y_j \in \mathbb{R}$ and $B_j \in \mathcal{B}(\mathfrak{g}^*)$ are disjoint Borel measurable sets. Noting that $1_{\mu^{-1}(B_j)} = (1_{B_j} \circ \mu)$, we obtain:

$$\int_{\mathfrak{g}^*} h \ dm_{\mathrm{DH}} = \sum_{j=1}^N y_j \int_M 1_{\mu^{-1}(B_j)} \frac{\omega^m}{m!} = \int_M \sum_{j=1}^N y_j \cdot (1_{B_j} \circ \mu) \frac{\omega^m}{m!} = \int_M (h \circ \mu) \frac{\omega^m}{m!}.$$

Then apply the sombrero lemma [38, Theorem 8.8] to find the expression for all $h \in C^\infty_c(\mathfrak{g}^*)$.

Now suppose (M, ω, μ) is a compact connected Hamiltonian $\mathbb{T}^n$-space, and in addition suppose that the action is effective. We can now compare two measures on the dual $(\mathfrak{t}^*, \mathcal{B}(\mathfrak{t}^*))$

of the Lie algebra, namely the Duistermaat–Heckman measure m_{DH} and the Lebesgue measure m_λ.

Since the action is effective, the Duistermaat–Heckman measure is absolutely continuous with respect to the (σ-finite) Lebesgue measure m_λ [12, Theorem 3.1]. Then, by the Radon–Nikodym theorem [38, Theorem 20.2], there exists an almost everywhere unique density function $f : \mathfrak{t}^* \to [0, \infty)$ satisfying

$$m_{\mathrm{DH}}(B) = \int_B f(\xi)\ dm_\lambda(\xi),\ \ B \in \mathcal{B}(\mathfrak{t}^*).$$

We denote $f = \frac{dm_{\mathrm{DH}}}{dm_\lambda}$ and one says that f is the Radon–Nikodym derivative for the Duistermaat–Heckman measure. We show that $f = \frac{dm_{\mathrm{DH}}}{dm_\lambda}$ is a polynomial function on each connected component of $\mathfrak{t}^*_{\mathrm{reg}}$.

Theorem 4.2.5. *Let (M, ω, μ) be a Hamiltonian $\mathbb{T}^n$-space with momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that $\mathbb{T}^n$ acts freely on M outside of fixed points. On each connected component of $\mathfrak{t}^*$ the Radon–Nikodym derivative $f = \frac{dm_{DH}}{dm_\lambda} : \mathfrak{t}^* \to [0, \infty)$ of the Duistermaat–Heckman measure is polynomial in ξ.*

Proof. We follow the proof as explained by Cannas da Silva [9, 30.3], and adapt the same notations as in the proof of the Duistermaat–Heckman theorem. Since (M_ξ, ω_ξ) and $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ are symplectomorphic (for $\xi \in (-\epsilon, \epsilon)^n$), we obtain:

$$\mathcal{L}_\omega(M_\xi) = \int_{M_\xi} \exp(\omega_\xi) = \int_{M_{\mathrm{red}}} \exp(\omega_{\mathrm{red}} - \langle \xi, \beta \rangle), \tag{4.2}$$

where $\beta \in \Omega^2(M_{\mathrm{red}}, \mathfrak{t})$ is the curvature form on the base of the principal bundle $\pi : Z \to M_{\mathrm{red}}$ corresponding to the connection one-form $\alpha \in \Omega^1(Z, \mathfrak{t})$.

Let $U \in \mathcal{B}((-\epsilon, \epsilon)^n)$ be a Borel-measurable subset of $(-\epsilon, \epsilon)^n$. Since $(Z \times (-\epsilon, \epsilon)^n, \omega_0)$ and $(\mu^{-1}((-\epsilon, \epsilon)^n), \omega)$ are isomorphic Hamiltonian $\mathbb{T}^n$-spaces, it follows that

$$m_{\mathrm{DH}}(U) = \int_{Z \times U} \exp(\omega_0).$$

By definition $\omega_0 = \mathrm{pr}_Z^*(\pi^* \omega_{\mathrm{red}}) - \sum_{j=1}^n d[\mathrm{pr}_j^*(x_j) \cdot \mathrm{pr}_Z^*(\alpha^j)]$, so we compute

$$\frac{\omega_0^m}{m!} = \frac{1}{(m-n)!} (\mathrm{pr}_Z^*(\pi^*\omega_{\mathrm{red}}) - \sum_{j=1}^n \mathrm{pr}_j^*(x_j) \cdot \mathrm{pr}_Z^*(d\alpha^j))^{m-n} \wedge \mathrm{pr}_Z^*(\alpha^1) \wedge \cdots \wedge \mathrm{pr}_Z^*(\alpha^n) \wedge \mathrm{pr}_1^*(dx_1) \wedge \cdots \wedge \mathrm{pr}_n^*(dx_n).$$

We suppress the pullbacks by projections from the notation. By application of Fubini's theorem ([38, Ch.13]) and using $d\alpha^j = \pi^* \beta^j$, we find

$$\begin{aligned} m_{\mathrm{DH}}(U) &= \int_U [\int_Z \frac{\pi^*(\omega_{\mathrm{red}} - \sum_{j=1}^n x_j \cdot \beta^j)^{m-n}}{(m-n)!} \wedge \alpha^1 \wedge \cdots \wedge \alpha^n] \wedge dx_1 \wedge \cdots \wedge dx_n \\ &= \int_U [\int_Z \frac{\pi^*(\omega_{\mathrm{red}} - \sum_{j=1}^n x_j \cdot \beta^j)^{m-n}}{(m-n)!} \wedge \alpha^1 \wedge \cdots \wedge \alpha^n]\ dm_\lambda(x). \end{aligned}$$

Since the Radon–Nikodym derivative is a.e. unique, we find that

$$f(x) = [\int_Z \frac{\pi^*(\omega_{\text{red}} - \sum_{j=1}^n x_j \cdot \beta^j)^{m-n}}{(m-n)!} \wedge \alpha^1 \wedge \cdots \wedge \alpha^n],$$

so it remains to evaluate this integral. One can show ([17, p.27]) that $\mathcal{L}_{\omega_\xi}(M_\xi)$ is the latter integral, so that we have

$$f(\xi) = \mathcal{L}_{\omega_\xi}(M_\xi), \text{ for } \xi \in (-\epsilon, \epsilon)^n.$$

In view of Equation (4.2) we see that f is polynomial on $(-\epsilon, \epsilon)^n$. Since we may add any constant to the momentum map, we find that f is polynomial on any connected component of the regular values. □

Example 4.2.6 (**Archimedes' observation**)**.** Suppose (M, ω, μ) is a $2n$-dimensional Hamiltonian $\mathbb{T}^n$-space such that the torus acts freely outside fixed points. By the convexity theorem, the regular fibers are connected, so that all the reduced spaces are single points (in view of the dimensions). Hence, the reduced spaces all have Liouville volume equal to one. Therefore, the Radon–Nikodym derivative of the Duistermaat–Heckman measure is equal to the characteristic function of the momentum polytope outside a m_λ-negligible set. By the above theorem, we conclude that the Liouville volume of M is equal to the Lebesgue measure of the momentum polytope:

$$m_{\text{DH}}(\Delta) = \int_M \frac{\omega^n}{n!} = \int_{\mathfrak{t}^*} 1_\Delta(\xi)\ dm_\lambda(\xi) = m_\lambda(\Delta)$$

In the case of the $\mathbb{S}^1$-action on the sphere $\mathbb{S}^2$ by rotations, we recover Archimedes' observation (∼230 B.C.), namely the surface area between two horizontal circles is only dependent on the height.

The purpose of the following example is to show that extending the action to the Delzant case (if possible) is a useful tool in computing the Duistermaat–Heckman measure for non-Delzant actions.

Example 4.2.7. Consider the symplectic $\mathbb{T}^2$-action on the complex projective space $(\mathbb{C}P^2, \omega_{\text{FS}})$ given by

$$(t_1, t_2) \cdot [z_0, z_1, z_2] := [z_0 : -t_1 z_1 : -t_2 z_2].$$

This action is Hamiltonian with momentum map μ given by:

$$\mu : \mathbb{C}P^2 \to \mathbb{R}^2,\ \mu([z_0 : z_1 : z_2]) = \pi(\frac{|z_1|^2}{\sum_{j=0}^3 |z_j|^2}, \frac{|z_2|^2}{\sum_{j=1}^3 |z_j|^2}).$$

Let us now restrict the action to $\{1\} \times \mathbb{S}^1$. This circle action is Hamiltonian with momentum map given by

$$\mu' : \mathbb{C}P^2 \to \mathbb{R},\ \mu' = \text{pr} \circ \mu,$$

where $\text{pr} : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y$ is the projection onto the last coordinate. Denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$ by m_{λ_n}. By the previous example, we know that the Duistermaat–Heckman measure of μ is given by $m_{\text{DH}} = m_{\lambda_2}|_\Delta$, where $\Delta = \mu(M)$ is the momentum polytope. Note that the regular values of μ' are given by the open interval $(0, \pi)$.

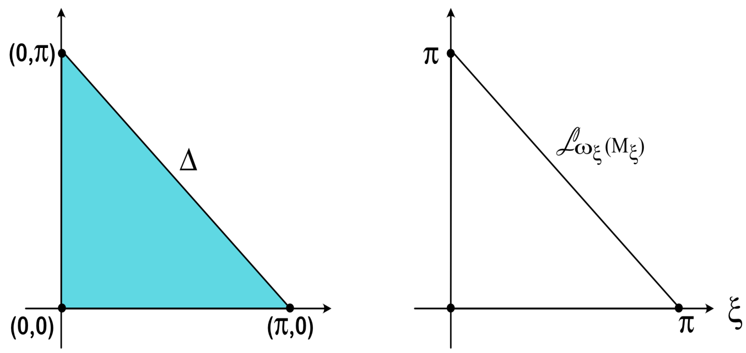


Figure 4.1: Left: The image $\mu(M) = \Delta$ of the momentum map μ. Right: The Radon–Nikodym derivative for the Duistermaat-Heckman measure associated to μ'.

We compute the Radon–Nikodym derivative $f : \mathbb{R} \to [0, \infty)$ of the Duistermaat–Heckman measure m'_{DH} associated to the momentum map μ'. For a Borel set $B \in \mathcal{B}(\mathbb{R})$, we find

$$\begin{aligned} m'_{\mathrm{DH}}(B) &= (\mu')_*(\mathcal{L}_{\omega_{\mathrm{FS}}})(B) = \mathcal{L}_{\omega_{\mathrm{FS}}}(\mu'^{-1}(B)) \\ &= \mathcal{L}_{\omega_{\mathrm{FS}}}(\mu^{-1}(\mathrm{pr}^{-1}(B))) = m_{\mathrm{DH}}(\mathrm{pr}^{-1}(B)) \\ &= \mathrm{pr}_*(m_{\mathrm{DH}})(B) = \mathrm{pr}_*(m_{\lambda_2}|_\Delta)(B). \end{aligned}$$

Thus, the Duistermaat–Heckman measure m'_{DH} is given by the pushforward of the Lebesgue measure on the momentum polytope $\Delta = \mu(M)$ under the projection pr. Let $U \subseteq \mathbb{R}$ be an open subset. Using this characterization of m'_{DH}, we find

$$\begin{aligned} m'_{\mathrm{DH}}(U) &= m_{\lambda_2}|_\Delta(\mathbb{R} \times U) \\ &= \int_\Delta 1_{\mathbb{R}\times U}(x, y)\ dm_{\lambda_2}(x, y) \\ &= \int_0^\pi \int_0^{\pi - y} 1_U(y')\ dx'dy' \\ &= \int_{\Delta'} 1_U(y) \cdot (\pi - y)\ dy \\ &= \int_U 1_{\Delta'}(y) \cdot (\pi - y)\ dm_{\lambda_1}(y), \end{aligned}$$

where we have used Riemann integration for the iterated integral. We conclude that the Radon–Nikodym derivative $f = \frac{dm'_{\mathrm{DH}}}{dm_{\lambda_1}}$ is given by:

$$f : \mathbb{R} \to [0, \infty),\ f(\xi) = 1_{\Delta'}(\xi) \cdot (\pi - \xi).$$

In agreement with Theorem 4.2.5, we find that the Liouville volumes of the symplectic reduced surfaces $(M_\xi, \omega_\xi),\ \xi \in (0, \pi)$ vary linearly in ξ with slope -1, which encodes information about the Chern class of each of the principal $\mathbb{S}^1$-bundles $\mu^{-1}(\xi) \to M_\xi,\ \xi \in (0, \pi)$.

The Duistermaat–Heckman measure has been used as an invariant to obtain classification results for complexity one Hamiltonian $\mathbb{T}$-spaces ($\frac{1}{2} \dim M - \dim \mathbb{T} = 1$). J. Moser [36] has shown that symplectic surfaces are classified by their genus and Liouville area (up to symplectomorphism). If the action of the torus is free on a level set of the momentum map, the Radon–Nikodym derivative of the Duistermaat–Heckman measure returns the Liouville area of the corresponding reduced symplectic surface, so that Moser's classification may be used. For more information, see Y. Karshon and S. Tolman's paper *Centered complexity one Hamiltonian torus actions* [23].

4.3 The Cartan Model: Equivariant Differential Forms

Suppose we have a smooth left action ψ of a compact connected Lie group G on a manifold M:

$$\psi : G \times M \to M.$$

This action induces a left action of G on the algebra of complex-valued differential forms $\Omega^*(M, \mathbb{C})$ on M by the formula

$$G \times \Omega^*(M, \mathbb{C}) \to \Omega^*(M, \mathbb{C}),\ g \cdot \eta := (\psi_{g^{-1}})^* \eta. \tag{4.3}$$

We now introduce the Cartan model of equivariant differential forms, due to H. Cartan.

Definition 4.3.1. Let ψ denote a left action of a compact connected Lie group G on a manifold M. An **equivariant differential form** on M is a polynomial map $\alpha : \mathfrak{g} \to \Omega^*(M, \mathbb{C})$ which is G-equivariant, meaning that

$$\alpha(\mathrm{Ad}_g(X)) = (\psi_{g^{-1}})^*(\alpha(X)),\ \text{for all } g \in G,\ X \in \mathfrak{g}. \tag{4.4}$$

We denote the space of equivariant differential forms on M by $\Omega_G^*(M)$.

Let $\alpha, \beta \in \Omega_G^*(M)$ be two equivariant differential forms, and let $X \in \mathfrak{g}$ be any element of the Lie algebra. We denote by $\alpha(X)_{[j]}$ the (homogeneous) component of $\alpha(X)$ which is of degree j, so that we have

$$\alpha(X) = \sum_{j=1}^{\dim M} \alpha(X)_{[j]} \in \Omega^*(M).$$

The wedge product of two equivariant differential forms is given by

$$(\alpha \wedge \beta)(X) := \alpha(X) \wedge \beta(X),$$

and one readily checks that $\alpha \wedge \beta : \mathfrak{g} \to \Omega^*(M, \mathbb{C})$ is again an equivariant differential form on M, which makes $(\Omega_G^*(M), \cdot \wedge \cdot)$ into an algebra.

Definition 4.3.2. We define the **equivariant differential**, denoted d_G, on $\Omega_G^*(M)$ by the formula

$$(d_G\alpha)(X) = d(\alpha(X)) - i_{\underline{X}}(\alpha(X)),\quad \alpha \in \Omega_G^*(M),\ X \in \mathfrak{g}.$$

An equivariant differential form α is called **equivariantly closed** if $d_G\alpha = 0$, and it is called **equivariantly exact** if there exists an equivariant differential form $\beta \in \Omega_G^*(M)$ such that $d_G\beta = \alpha$.

A first observation is that the equivariant differential defines a map $d_G : \Omega^*_G(M) \to \Omega^*_G(M)$. We check that $d_G^2 = 0$ on $\Omega^*_G(M)$, so that we have justified naming d_G a differential. Let $\alpha \in \Omega^*_G(M)$ arbitarily, and let $X \in \mathfrak{g}$ be any element of the Lie algebra. By equivariance of the form $\alpha : \mathfrak{g} \to \Omega^*(M, \mathbb{C})$ and consideration of the adjoint action, we find

$$\alpha(X) = \alpha(\mathrm{Ad}_{\exp -tX}(X)) = (\psi_{\exp tX})^*(\alpha(X)),$$

which in turn implies that

$$\mathcal{L}_{\underline{X}}(\alpha(X)) = 0.$$

Now, by application of Cartan's formula, we obtain

$$d_G^2(\alpha(X)) = d^2\alpha(X) + i_{\underline{X}} i_{\underline{X}} \alpha(X) - d i_{\underline{X}} \alpha(X) - i_{\underline{X}} d\alpha(X) = -\mathcal{L}_{\underline{X}}(\alpha(X)) = 0.$$

This leads to the following definition.

Definition 4.3.3. We define the **equivariant cohomology** of M (in the Cartan model), denoted $H^*_G(M)$, to be the cohomology of the complex $(\Omega^*_G(M), d_G)$. That is,

$$H^*_G(M) := \frac{\ker d_G}{\mathrm{Im}\ d_G}.$$

Remark 4.3.4. There is a grading on the algebra $\Omega^*_G(M)$ such that the equivariant differential d_G maps $\Omega^k_G(M)$ into $\Omega^{k+1}_G(M)$, but we have omitted this viewpoint here for simplicity. See, for example, *Heat Kernels and Dirac Operators* [8, Chapter 7.1] for more information.

The following example demonstrates the effectiveness of equivariant cohomology in the setting of Hamiltonian actions. Furthermore, this example will play an important role in the proof of the Duistermaat–Heckman Localization Theorem.

Example 4.3.5. Let (M, ω) be a symplectic manifold equipped with a Hamiltonian action of a compact connected Lie group G and denote by $\mu : M \to \mathfrak{g}^*$ its corresponding momentum map. Define a polynomial map $\tilde{\omega} : \mathfrak{g} \to \Omega^*(M)$ by $\tilde{\omega}(X) = \mu_X + \omega$. Note that $\tilde{\omega}$ is of mixed degree:

$$\tilde{\omega}(X)_{[0]} = \mu_X, \text{ and } \tilde{\omega}(X)_{[2]} = \omega.$$

We show that $\tilde{\omega}$ is an equivariantly closed differential form on M. Let $X \in \mathfrak{g}$ and $g \in G$ arbitrarily. Using that the momentum map is equivariant and that the symplectic structure is G-invariant, we compute:

$$\begin{aligned}
\tilde{\omega}(\mathrm{Ad}_g(X)) &= \mu_{\mathrm{Ad}_g(X)} + \omega \\
&= \langle \mu(\cdot), \mathrm{Ad}_g(X) \rangle + \omega \\
&= \langle \mathrm{Ad}^*_{g^{-1}} \mu(\cdot), X \rangle + \omega \\
&= \langle \mu(\psi_{g^{-1}}(\cdot)), X \rangle + \omega \\
&= (\psi_{g^{-1}})^* \mu_X + \omega \\
&= (\psi_{g^{-1}})^*(\tilde{\omega}(X)),
\end{aligned}$$

which implies that $\tilde{\omega} \in \Omega^*_G(M)$. It remains to show that $\tilde{\omega}$ is equivariantly closed:

$$\begin{aligned}
(d_G \tilde{\omega})(X) &= (d - i_{\underline{X}}) \tilde{\omega}(X) \\
&= d\mu_X - i_{\underline{X}} \mu_X + d\omega - i_{\underline{X}} \omega \\
&= d\mu_X - i_{\underline{X}} \omega = 0.
\end{aligned}$$

Here we have used that ω is closed, the interior product vanishes on smooth functions, and the definition of a momentum map for the last equality.

Conversely, suppose that $\mathfrak{g} \to \Omega^*(M),\ X \mapsto J_X + \omega$ is an equivariantly closed differential form on (M, ω), where $J : M \to \mathfrak{g}^*$ is a smooth map. Then one verifies that $J : M \to \mathfrak{g}^*$ defines a momentum map for the Hamiltonian G-action on (M, ω).

Note that we can integrate equivariant differential forms on M by pairing it with an element of the Lie algebra and integrating it over the *top degree* component. This gives rise to a map

$$\int_M : \Omega^*_G(M) \times \mathfrak{g} \to \mathbb{C},\ (\alpha, X) \mapsto \int_M \alpha(X)_{[\dim M]}.$$

For instance, instead of integrating the Liouville volume form $\frac{\omega^m}{m!}$, we can just integrate the mixed degree form $\exp(\omega) = 1 + \omega + \frac{\omega^2}{2} + \cdots + \frac{\omega^m}{m!}$.

4.4 Equivariant Localization

Throughout this section, let M be a manifold with an action of a compact Lie group G. Consider a fundamental vector field $\underline{X}$ on M. We denote by

$$M_0(\underline{X}) := \{p \in M : \underline{X}_p = 0\}$$

the zeros of the fundamental vector field $\underline{X}$ on M. Let $\langle \cdot, \cdot \rangle$ denote a G-invariant Riemannian metric on M, which is obtained by averaging an arbitrary Riemannian metric over G using the Haar measure. We write $||\underline{X}||^2 = \langle \underline{X}, \underline{X} \rangle$.

Proposition 4.4.1. Let M be a $2m$-dimensional manifold equipped with an action of a compact Lie group G. Suppose that $\alpha \in \Omega^*_G(M)$ is an equivariantly closed differential form on M. Then, for all $X \in \mathfrak{g}$, we have that the top degree component $\alpha(X)_{[\dim M]}$ is exact on the open submanifold $M - M_0(\underline{X}) \subseteq M$.

Proof. Let $X \in \mathfrak{g}$ be an element of the Lie algebra, and write $d_X = d - i_{\underline{X}}$ for the resulting operator. We define a one-form $\theta \in \Omega^1(M)$, with respect to the G-invariant metric $\langle \cdot, \cdot \rangle$, pointwise by the formula

$$\theta_p(v) = \langle \underline{X}, v \rangle, \text{ for } v \in T_pM.$$

This one-form θ satisfies $\mathcal{L}_{\underline{X}}\theta = 0$ by invariance. Note that $d_X\theta = d\theta - ||\underline{X}||^2$, which implies that $d_X(d_X\theta) = -\mathcal{L}_{\underline{X}}\theta = 0$. We now show that $d_X\theta$ is invertible on $M - M_0(\underline{X})$ in the space of complex-valued differential forms with respect to the wedge product, that is, we determine $(d_X\theta)^{-1}$ such that

$$(d_X\theta) \wedge (d_X\theta)^{-1} = 1 = (d_X\theta)^{-1} \wedge (d_X\theta).$$

By application of the von Neumann series (the operator-analogue of the geometric series), we obtain on $M - M_0(\underline{X})$ that:

$$\begin{aligned}(d_X\theta)^{-1} &= (d\theta - ||\underline{X}||^2)^{-1} = -||\underline{X}||^{-2}(1 - ||\underline{X}||^{-2}d\theta)^{-1} \\ &= -||\underline{X}||^{-2}(\sum_{k=0}^{m} ||\underline{X}||^{-2k} d\theta^k) = \sum_{k=0}^{m} -||\underline{X}||^{-2(k+1)} d\theta^k.\end{aligned}$$

Indeed, with the dimension of M in consideration we compute:

$$\begin{aligned}(d_X\theta) \wedge (d_X\theta)^{-1} &= (d\theta - ||\underline{X}||^2) \wedge \sum_{k=0}^{m} -||\underline{X}||^{-2(k+1)} d\theta^k \\ &= d\theta \wedge \sum_{k=0}^{m} -||\underline{X}||^{-2(k+1)} d\theta^k + \sum_{j=0}^{m} ||\underline{X}||^{-2j} d\theta^j \\ &= -\sum_{k=0}^{m-1} ||\underline{X}||^{-2(k+1)} d\theta^{k+1} + \sum_{j=0}^{m} ||\underline{X}||^{-2j} d\theta^j = 1,\end{aligned}$$

and similarly $(d_X\theta)^{-1} \wedge (d_X\theta) = 1$. Using this fact, we immediately have that

$$d_X((d_X\theta) \wedge (d_X\theta)^{-1}) = 0. \tag{4.5}$$

Since $d_X(d_X\theta) = 0$, it follows that

$$d_X((d_X\theta) \wedge (d_X\theta)^{-1}) = d_X(d_X\theta) \wedge (d_X\theta)^{-1} + (d_X\theta) \wedge d_X(d_X\theta)^{-1} = (d_X\theta) \wedge d_X(d_X\theta)^{-1}. \tag{4.6}$$

Combining Equation (4.5) and Equation (4.6), we find

$$(d_X\theta) \wedge d_X(d_X\theta)^{-1} = 0,$$

and wedging both sides with $(d_X\theta)^{-1}$ yields $d_X(d_X\theta)^{-1} = 0$. Using $d_X(d_X\theta) = 0$, $d_X(d_X\theta)^{-1} = 0$ and that $\alpha \in \Omega_G^*(M)$ is an equivariantly closed differential form on M, one verifies that

$$d_X(\theta \wedge (d_X\theta)^{-1} \wedge \alpha(X)) = \alpha(X).$$

This implies that

$$d(\theta \wedge (d_X\theta)^{-1} \wedge \alpha(X))_{[\dim M-1]} = \alpha(X)_{[\dim M]},$$

since the interior product operator $i_{\underline{X}}$ lowers the degree of ordinary differential forms. We conclude, for all $X \in \mathfrak{g}$, that $\alpha(X)$ is exact on the open submanifold $M - M_0(\underline{X})$. ☐

Weights convention. Given a $\mathbb{T}$-action on a manifold M and a fixed point $p \in M^{\mathbb{T}}$, we obtain a representation $\mathbb{T} \to \mathrm{GL}(T_pM)$ (cf. section 3.1). This representation is called the **isotropy representation** at a fixed point. We require the induced orientation of this representation to coincide with the orientation on T_pM induced by the symplectic structure.

Localization Theorem 4.4.2. was proven independently by Atiyah–Bott [2] and Berline–Vergne [8].

Theorem 4.4.2 (Atiyah–Bott–Berline–Vergne Localization Theorem [2],[8]**).** *Let M be a compact orientable $2m$-dimensional manifold equipped with an action ψ of a torus $\mathbb{T}^n$. Suppose that the action only has isolated fixed points. In addition, suppose that for each $p \in M^{\mathbb{T}}$ there is a $\mathbb{T}^n$-invariant complex structure J_p on the tangent space T_pM. Let $\alpha \in \Omega_G^*(M)$ be an equivariantly closed differential form on M. Suppose $X \in \mathfrak{t}$ satisfies $M_0(\underline{X}) = M^{\mathbb{T}}$, then we have:*

$$\int_M \alpha(X)_{[\dim M]} = \sum_{p \in M^{\mathbb{T}}} \frac{\alpha(X)_{[0]}(p)}{\Pi_j \langle \lambda_{j,p}, X\rangle}, \tag{4.7}$$

where $0 \neq \lambda_{j,p} \in \mathfrak{t}^*$, $j = 1, \ldots, m$ are the weights associated to the isotropy representation at fixed points $p \in M^{\mathbb{T}}$.

Proof. We follow the proof by N. Berline and M. Vergne [8, Theorem 7.11]. Let $p \in M^{\mathbb{T}}$ be a fixed point of the action. We obtain a complex representation of the torus $\mathbb{T}^n$ on the tangent space (T_pM, J_p). Subsequently, there is a splitting of T_pM into complex lines

$$T_pM = V_1 \oplus \cdots \oplus V_m,$$

such that the torus $\mathbb{T}^n$ acts on each complex line V_j by

$$(\exp Y) \cdot v_j = e^{2\pi i \langle \lambda_{j,p}, Y \rangle} \cdot v_j,$$

where the $\lambda_{j,p} \in \mathfrak{t}^*,\ j = 1, \ldots, m$ are the associated isotropy weights.

By linearizing the action around p using a $\mathbb{T}^n$-invariant Riemannian metric, we obtain a $\mathbb{T}^n$-invariant open neighborhood U_p of p in M and corresponding local complex coordinates $(v_1 = x_1 + iy_1, \ldots, v_m = x_m + iy_m)$ such that the fundamental vector field $\underline{X}$ takes the following form on U_p:

$$\underline{X} = 2\pi \sum_{j=1}^{m} \langle \lambda_{j,p}, X \rangle \cdot (-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}).$$

Since $M_0(\underline{X}) = M^{\mathbb{T}}$, the linearization of the action around fixed points implies that $\langle \lambda_{j,p}, X \rangle \neq 0$ for all $j = 1, \ldots, m$ and $p \in M^{\mathbb{T}}$.

We define a one-form $\theta^{(p)}$ on the neighborhood U_p by

$$\theta^{(p)} := \frac{1}{2\pi} \sum_{j=1}^{m} \frac{1}{\langle \lambda_{j,p}, X \rangle} \cdot (-y_j dx_j + x_j dy_j) \in \Omega^1(U_p).$$

Note that $\theta^{(p)}$ satisfies $\mathcal{L}_{\underline{X}} \theta^{(p)} = 0$ and $\theta^{(p)}(\underline{X}) = \sum_{j=1}^m |v_j|^2$. Using a $\mathbb{T}^n$-invariant partition of unity subordinate to the open cover $\mathcal{U} := (U_p)_{p \in M^{\mathbb{T}}} \cup (M \backslash M_0(\underline{X}))$ we piece the one-forms $\theta^{(p)}$ together to obtain a one-form $\theta \in \Omega^1(M)$ satisfying the following properties [26, 5.2.11]:

- $\mathcal{L}_{\underline{X}} \theta = 0$;
- $\theta = \theta^{(p)}$ on a neighborhood of p contained in U_p;
- $d_X \theta$ invertible on $M - M_0(\underline{X})$.

For a fixed point $p \in M^{\mathbb{T}}$ denote by $B_p^\epsilon := \{v \in U_p : \sum_{j=1}^m |v_j|^2 \leq \epsilon\} \subseteq U_p$ an ϵ-ball in U_p with respect to the local coordinates $v_1, \ldots, v_m$ on U_p. By application of Proposition 4.4.1 and Stoke's theorem, we find

$$\begin{aligned} \int_M \alpha(X)_{[\dim M]} &= \lim_{\epsilon \to 0} \int_{M - \cup_{p \in M^{\mathbb{T}}} [B_p^\epsilon]} \alpha(X)_{[\dim M]} \\ &= \lim_{\epsilon \to 0} \int_{M - \cup_{p \in M^{\mathbb{T}}} [B_p^\epsilon]} d(\theta \wedge (d_X \theta)^{-1} \wedge \alpha(X))_{[\dim M - 1]} \\ &= \lim_{\epsilon \to 0} - \int_{\cup_{p \in M^{\mathbb{T}}} [\partial(B_p^\epsilon)]} (\theta \wedge (d_X \theta)^{-1} \wedge \alpha(X))_{[\dim M - 1]} \\ &= - \sum_{p \in M^{\mathbb{T}}} \lim_{\epsilon \to 0} \int_{\partial(B_p^\epsilon)} (\theta \wedge (d_X \theta)^{-1} \wedge \alpha(X))_{[\dim M - 1]}. \end{aligned} \tag{4.8}$$

The minus sign is a result of switching the orientations of the spheres $\partial(B_p^\epsilon)$ about the fixed points, see Figure 4.2.

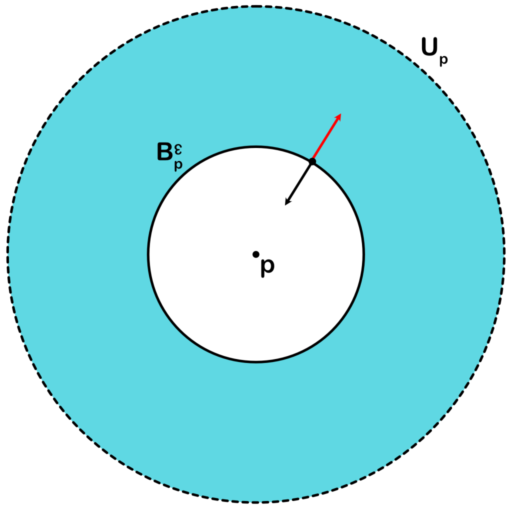


Figure 4.2: The orientation on the manifold M minus the balls around the fixed points is indicated in black, and the usual orientation on the sphere is indicated in red.

It remains to compute the limit of the integral over each sphere. Consider now a neighborhood of a fixed point $p \in M^{\mathbb{T}}$ such that $\theta = \theta^{(p)}$ on this neighborhood. We rescale the local coordinates $v_1, \ldots, v_m$ on this neighborhood by ϵ, and denote the resulting map by m_ϵ. Then we have $m_\epsilon^* \theta = \epsilon^2 \theta$ and $m_\epsilon^*(d_X\theta)^{-1} = \epsilon^{-2}(d_X\theta)^{-1}$, which together imply

$$m_\epsilon^*(\theta \wedge (d_X\theta)^{-1}) = (\theta \wedge (d_X\theta)^{-1}).$$

Note that

$$\lim_{\epsilon \to 0} m_\epsilon^* \alpha(X) = \alpha(X)_{[0]}(p).$$

Viewing the coordinate change m_ϵ as an orientation-preserving diffeomorphism from the unit sphere $\mathbb{S}_1^{2m-1}$ to the ϵ-sphere $\partial(B_p^\epsilon)$ and using $m_\epsilon^*(\theta \wedge (d_X\theta)^{-1} \wedge \alpha(X)) = \theta \wedge (d_X\theta)^{-1} \wedge m_\epsilon^*\alpha(X)$, we compute:

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{\partial(B_p^\epsilon)} -(\theta \wedge (d_X\theta)^{-1} \wedge \alpha(X))_{[2m-1]} \\
&= \alpha(X)_{[0]}(p) \int_{\mathbb{S}_1^{2m-1}} -(\theta \wedge (d_X\theta)^{-1})_{[2m-1]} \\
&= \alpha(X)_{[0]}(p) \int_{\mathbb{S}_1^{2m-1}} (\theta \wedge (1 - d\theta)^{-1})_{[2m-1]} \\
&= \alpha(X)_{[0]}(p) \int_{\mathbb{S}_1^{2m-1}} \theta \wedge (d\theta)^{m-1}.
\end{aligned} \tag{4.9}$$

We find, see for example *Introductory Lectures on Equivariant Cohomology* [43, Chapter 31], that on $\mathbb{S}_1^{2m-1}$ we have:

$$d\theta = \frac{1}{2\pi}\sum_{j=1}^{m}\frac{1}{\langle\lambda_{j,p},X\rangle}(-dy_j\wedge dx_j + dx_j\wedge dy_j) = \frac{1}{\pi}\sum_{j=1}^{m}\frac{1}{\langle\lambda_{j,p},X\rangle}(dx_j\wedge dy_j),$$

$$(d\theta)^{m-1} = \frac{1}{\pi^{m-1}}\sum\frac{(m-1)!}{\langle\lambda_{1,p},X\rangle\cdots\widehat{\langle\lambda_{j,p},X\rangle}\cdots\langle\lambda_{m,p},X\rangle}dx_1\wedge dy_1\wedge\cdots\wedge\widehat{dx_j}\wedge\widehat{dy_j}\wedge\cdots\wedge dx_m\wedge dy_m,$$

$$\theta\wedge(d\theta)^{m-1} = \frac{1}{2\pi^m}\frac{(m-1)!}{\Pi_j\langle\lambda_{j,p},X\rangle}\cdot\eta_{\text{vol}},$$

where $\eta_{\text{vol}}\in\Omega^{2m-1}(\mathbb{S}^{2m-1})$ denotes the standard volume form on the sphere $\mathbb{S}^{2m-1}$. Recall that $\text{vol}(\mathbb{S}^{2m-1}) = \frac{2\pi^m}{(m-1)!}$. Therefore, we have

$$\alpha(X)_{[0]}(p)\int_{\mathbb{S}_1^{2m-1}}\theta\wedge(d\theta)^{m-1} = \frac{\alpha(X)_{[0]}}{\Pi_j\langle\lambda_{j,p},X\rangle}.$$

In view of Equation (4.8) and Equation (4.9), we conclude that

$$\int_M\alpha(X)_{[\dim M]} = \sum_{p\in M^{\mathbb{T}}}\frac{\alpha(X)_{[0]}(p)}{\Pi_j\langle\lambda_{j,p},X\rangle}. \tag{4.10}$$

□

Note that the left-hand side of Equation (4.10) is a *global* quantity (an integral over the entire manifold), whereas the right-hand side is a *localized* quantity (a sum over the fixed point set). In this case, we say that the integral over M *localizes onto* the fixed point set.

Prior to the Atiyah–Bott–Berline–Vergne Localization Theorem, Duistermaat and Heckman proved in the paper *On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space* [12] a localization theorem in the setting of Hamiltonian torus actions.

Theorem 4.4.3 (The Duistermaat–Heckman Localization Theorem [12]**).** *Let (M,ω) be a compact symplectic manifold equipped with a Hamiltonian action of a torus $\mathbb{T}^n$, and denote the associated momentum map by $\mu: M\to\mathfrak{t}^*$. Suppose that the action only has isolated fixed points. If $X\in\mathfrak{t}$ is an element of the Lie algebra such that $M_0(\underline{X}) = M^{\mathbb{T}}$, then we have*

$$\int_M e^{i\langle\mu,X\rangle}\frac{\omega^m}{m!} = (-i)^m\sum_{p\in M^{\mathbb{T}}}\frac{e^{i\langle\mu(p),X\rangle}}{\Pi_j\langle\lambda_{j,p},X\rangle}, \tag{4.11}$$

where $\lambda_{j,p}\in\mathfrak{t}^*,\ j=1,\ldots,m$ are the isotropy weights of the fixed points.

Proof. Let $X\in\mathfrak{t}$ be an element in the Lie algebra of the torus $\mathbb{T}^n$ such that $M_0(\underline{X})$ consists of finitely many points. Define an equivariant differential form $e^{i\tilde{\omega}}\in\Omega^*_G(M)$ by

$$e^{i\tilde{\omega}}:\mathfrak{t}\to\Omega^*(M,\mathbb{C}),\ Y\mapsto\exp(i(\mu_Y+\omega)) = \sum_{j=1}^{m}e^{i\langle\mu,Y\rangle}\frac{(i\omega)^j}{j!}.$$

Since the equivariant differential form $\tilde{\omega}(Y) = \mu_Y + \omega$ is equivariantly closed, it follows that $e^{i\tilde{\omega}}$ is equivariantly closed as well. Applying the Atiyah–Bott–Berline–Vergne Localization Theorem to $e^{i\tilde{\omega}(X)}$ and noting that $(e^{i\tilde{\omega}(X)})_{[\dim M]} = (i)^m e^{i\langle \mu, X\rangle} \frac{\omega^m}{m!}$, we obtain the Duistermaat–Heckman Localization Theorem:

$$\int_M e^{i\langle \mu, X\rangle} \frac{\omega^m}{m!} = (-i)^m \sum_{p \in M^{\mathbb{T}}} \frac{e^{i\langle \mu(p), X\rangle}}{\Pi_j \langle \lambda_{j,p}, X\rangle}.$$

□

In this way we have expressed the integral over $\exp(i(\mu_X + \omega))$ solely in terms of information at the fixed points. Note that we did not make use of the nondegeneracy of the symplectic form ω.

Remark 4.4.4. There exists a similar result for the localization theorems where the fixed point set does not necessarily consist of isolated points [2].

Applications of the Localization Theorems

Isotropy Weights and Hamiltonian Circle Actions

The Duistermaat–Heckman localization theorem implies relationships between the isotropy weights. Let (M, ω, μ) be a compact connected $2m$-dimensional Hamiltonian $\mathbb{S}^1$-space with finitely many fixed points, and let $X \in \mathfrak{s}$ be the element of the Lie algebra such that $\mu = \mu_X$. Inspecting the proof again, we see that the following holds, for all nonzero $t \in \mathbb{R}$:

$$\int_M e^{t\mu} \frac{\omega^m}{m!} = \frac{1}{t^m} \sum_{p \in M^{\mathbb{T}^n}} \frac{e^{t\mu(p)}}{\Pi_j \lambda_{j,p}}.$$

If we view the above expression as an equality of power series in t, we find (in particular) the following two equations:

$$\sum_{p \in M^{\mathbb{T}^n}} \frac{1}{\Pi_j \lambda_{j,p}} = 0 \tag{4.12}$$

and

$$m! \cdot \mathcal{L}_\omega(M) = \sum_{p \in M^{\mathbb{T}^n}} \frac{\mu(p)^m}{\Pi_j \lambda_{j,p}} \tag{4.13}$$

The following familiar example serves as a check for the various conventions we have employed regarding weights and orientations.

Example 4.4.5. Consider the Hamiltonian $\mathbb{S}^1$-space $(\mathbb{S}^2, \omega)$, where the circle acts by rotations. Recall that the momentum map is given by $\mu : \mathbb{S}^2 \to \mathbb{R},\ \mu(\varphi, h) = 2\pi h$. As the orientations induced by the isotropy representations must coincide with the orientation of the sphere, we find that the isotropy weights on the North- and South-pole are given by $\lambda_N = +1$ and $\lambda_S = -1$, respectively. By application of eq. (4.13), we recover the Liouville area of the sphere:

$$\mathcal{L}_\omega(\mathbb{S}^2) = \frac{2\pi}{+1} + (\frac{-2\pi}{-1}) = 4\pi$$

V. Ginzburg has used Equation (4.12) in the paper *Some remarks on symplectic actions of compact groups* [16] to simplify a proof by D. McDuff [31], which states that a symplectic circle action on a compact four-dimensional symplectic manifold is Hamiltonian if and only if it has fixed points.

Relation with the Duistermaat–Heckman measure

We show that the localization theorem is related to the Duistermaat–Heckman measure via a Fourier transform.

Let $\mathbb{T}^n \times M \to M$ be a Hamiltonian action of a torus $\mathbb{T}$ on a $2m$-dim. compact connected symplectic manifold (M, ω) with momentum map $\mu : M \to \mathfrak{t}^*$. By definition of the Fourier transform of a Borel measure (cf. [38, Def. 19.1]), we find that the Fourier transform $\widehat{m_{\mathrm{DH}}}$ of the Duistermaat–Heckman measure is given by

$$\widehat{m_{\mathrm{DH}}}(-X) = \int_{\mathfrak{t}^*} e^{i\langle \xi, X\rangle} dm_{\mathrm{DH}}(\xi), \text{ for } X \in \mathfrak{t}.$$

We substitute $h \in C^\infty(\mathfrak{t}^*)$ defined by $h(\xi) := e^{i\langle \xi, X\rangle}$ in Equation (4.1) to obtain

$$\widehat{m_{\mathrm{DH}}}(-X) = \int_M e^{i\langle \mu, X\rangle} \frac{\omega^m}{m!},$$

which is the same integral that appeared in the localization theorem. This viewpoint has been used by Guillemin, Lerman and Sternberg to express the Radon–Nikodym derivative of the Duistermaat–Heckman measure in terms of fixed point data [18, p.736-741], assuming that the fixed point set is finite.

Localization of the Partition Function of 2d Yang–Mills Theory

The localization principle has been applied to localize quantum partition functions of certain quantum mechanical systems. In order to make sense of localization in this setting, we need a global formulation of quantum mechanics, opposed to the local formulation given by the Schrödinger partial differential equation. We first discuss the path integral formalism for quantum mechanics, and then we briefly discuss how two dimensional Yang–Mills theory, a non-Abelian gauge theory on a surface, fits in with equivariant symplectic geometry.

A global formulation of quantum mechanics is provided by the Feynman path integral formalism, which expresses the probability amplitude of a system prepared in a state $|q\rangle$ at time $t = 0$ to be found in another state $|q'\rangle$ at a later time $t = T$ as an integral over all continuous paths from $|q_1\rangle$ to $|q_2\rangle$. Explicitly, the path integral formalism dictates that the quantum propagator is given by (see [45] for a derivation):

$$W(q_2, q_1; T) := \langle q_2, T \mid q_1, 0\rangle = \int_{q(0)=q_1; q(T)=q_2} \mathcal{D}q(t)\ e^{\frac{i}{\hbar} S[q(t)]}. \tag{4.14}$$

Here $S[q(t)]$ is the classical action functional, that is, the time integral over the Lagrangian:

$$S[q(t)] = \int_0^T dt\ L(q(t), \dot{q}(t)) = \int_0^T dt\ \dot{q}(t)p(t) - H(q(t), p(t)), \tag{4.15}$$

and $\mathcal{D}q(t)$ is the *Feynman measure*.

The classical partition function of a physical system with a $2m$-dim. phase space (M, ω) and a Hamiltonian function $H \in C^\infty(M)$ is given by

$$Z_{cm}(\beta) = \int_M e^{-\beta H} \frac{\omega^m}{m!}, \tag{4.16}$$

where β is a real number proportional to the inverse temperature. Recall that the partition function determines the thermodynamical properties of the system. In the quantum case, the partition function takes the following form [41]:

$$Z_{qm} = \int \mathcal{D}q(t)\ e^{\frac{i}{\hbar} S[q(t)]}\ \delta(q(0) - q(T)), \tag{4.17}$$

where δ represents the Dirac delta distribution so that we only integrate over the paths which are loops (with period T).

Let Σ be a compact orientable surface without boundary, and let $G = \mathrm{SU}(2)$ be the *gauge group*. Since SU(2) is a simply-connected compact Lie group, any principal SU(2)-bundle over a surface admits a *global* trivialization. Thus, we consider the trivial principal G-bundle

$$\pi : G \times \Sigma \to \Sigma, \tag{4.18}$$

where G acts on $G \times \Sigma$ by $g \cdot (h, \sigma) := (gh, \sigma)$.

For such a trivial principal bundle, we can identify the connection one-forms with the space $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ of Lie-algebra valued one-forms on the base Σ. We define the curvature of a connection $A \in \mathcal{A}$ to be the two-form $F_A \in \Omega^2(\Sigma, \mathfrak{g})$ given by:

$$F_A = dA - \frac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g}).$$

The *group of gauge transformations* $\mathcal{G} := C^\infty(\Sigma, G)$ acts on the space $\mathcal{A}$ of connection one-forms, as follows:

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A},\ u \cdot A := \mathrm{Ad}_u(A) + du\ u^{-1}.$$

Analogous to the quantum partition function in Equation (4.17), we define the partition function of two-dimensional quantum Yang–Mills theory on the surface Σ by the following Feynman path integral [46, Equation 1.10]:

$$Z_{\mathrm{YM}}(\epsilon) = \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{A \in \mathcal{A}} \mathcal{D}A \exp\left(\frac{-S_{\mathrm{YM}}[A]}{\epsilon}\right). \tag{4.19}$$

Here ϵ is the Yang–Mills coupling constant, $\mathrm{vol}(\mathcal{G})$ is the volume of the gauge group $\mathcal{G}$, $\mathcal{D}A$ is a path integral measure, and S_{YM} is the $2d$ Yang–Mills action defined by:

$$S_{\mathrm{YM}}(A) := \frac{1}{2}(F_A, F_A),\ A \in \mathcal{A} \tag{4.20}$$

with respect to a suitable L^2-inner product, which is invariant under the action of the group $\mathcal{G}$ of gauge transformations.

We describe the relation between $2d$ Yang–Mills theory and equivariant symplectic geometry. Let $\langle\cdot,\cdot\rangle$ be a Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of $G = \mathrm{SU}(2)$. It turns out that the space $\mathcal{A}$ of connection one-forms is affine (linear modulo a translation), so that we may identify $T_A\mathcal{A} = \mathcal{A}$. In the paper *The Yang–Mills equations over Riemann surfaces* [3] Atiyah and Bott show that $\mathcal{A}$ admits a symplectic structure given by:

$$\Omega : \mathcal{A} \times \mathcal{A} \to \mathbb{R},\ \Omega(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle.$$

The Lie algebra of $\mathcal{G}$ can be identified with the space of smooth maps $\Sigma \to \mathfrak{g}$ and we can interpret $\Omega^2(\Sigma, \mathfrak{g})$ as the dual of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ via the pairing $\langle \xi, F\rangle := \int_\Sigma \langle \xi, F\rangle$ for $\xi \in \Omega^0(\Sigma, \mathfrak{g}) = C^\infty(\Sigma, \mathfrak{g}), F \in \Omega^2(\Sigma, \mathfrak{g})$ [34, p.214]. Furthermore, Atiyah and Bott showed that the action of the group $\mathcal{G}$ of gauge transformations on $\mathcal{A}$ is Hamiltonian with momentum map given by the curvature:

$$\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^* = \Omega^2(\Sigma, \mathfrak{g}),\ \mu(A) = F_A. \tag{4.21}$$

Thus, we see that the Yang–Mills action in Equation (4.20) is proportional to the norm squared of the momentum map $||\mu||^2$. The critical point set of the norm squared of the momentum map has been extensively studied in the setting of Morse–Bott theory by, for example, F. Kirwan [24].

Since the Yang–Mills action is invariant under the group $\mathcal{G}$, it is natural to quotient the physically equivalent solutions, which are related by a gauge transformation. By a formal application of the symplectic reduction procedure we can interpret the *moduli space of flat connections* $\mathcal{M} := \mu^{-1}(0)/\mathcal{G}$ as a symplectic reduced space (with singularities), it turns out that this space is finite-dimensional [3].

By considerations of the Hodge star operator (the complex structure on $\mathcal{A}$), it turns out that the path integral measure $\mathcal{D}A$ and the Liouville/symplectic measure $\exp(\omega)$ coincide [7, p.190]. This allows us to rewrite the partition function of $2d$ Yang–Mills theory in terms of *equivariant symplectic data*, as follows:

$$Z_{YM}(\epsilon) \propto \frac{1}{\mathrm{vol}(\mathcal{G})} \int_\mathcal{A} \exp\left(\Omega - \frac{||\mu||^2}{2\epsilon}\right). \tag{4.22}$$

In *Two dimensional gauge theories revisited* [46] E. Witten derived a non-Abelian localization formula for symplectic integrals of precisely this form. In particular, the partition function $Z_{YM}(\epsilon)$ localizes onto the critical set of the Yang–Mills action $S_{YM} = \frac{1}{2}(\mu, \mu)$, and the connections in the critical set represent either a stable minimum (a flat connection) or connections with nonzero curvature which obey the classical Yang–Mills equations [7, p.217-p.218]. Furthermore, using this approach he recovers as a particular case the result that the partition function $Z_{YM}(0)$ (zero coupling) agrees with the Liouville/symplectic volume $\int_\mathcal{M} \exp(\Omega_{\mathrm{red}})$ of the moduli space $(\mathcal{M}, \Omega_{\mathrm{red}})$ [7, p.226].

Witten's non-Abelian localization principle has been used by L. Jeffrey and F. Kirwan in the paper *Localization and the Quantization Conjecture* [22] to prove the quantization conjecture (quantization commutes with reduction) for certain symplectic manifolds with a Hamiltonian action. For a discussion on the relation between Chern-Simons gauge theory and equivariant symplectic geometry, see the paper *Non-Abelian Localization for Chern-Simons Theory* [7, p.193-213] by C. Beasley and E. Witten.

Summary

In this chapter we studied connection one-forms associated to principal $\mathbb{T}$-bundles. Then we used the equivariant coisotropic embedding theorem to prove the Duistermaat–Heckman theorem, which states that the de Rham cohomology class of the reduced symplectic forms varies linearly. This in turn implied that the Radon–Nikodym derivative of the Duistermaat–Heckman measure (with respect to the Lebesgue measure) is locally polynomial. In order to prove the Duistermaat–Heckman localization theorem, we studied the Cartan model of equivariant cohomology and obtained this theorem as a consequence of the Atiyah–Bott–Berline–Vergne localization theorem. Finally, we discussed some applications and generalizations regarding the localization principle.

Outlook

We have studied Hamiltonian actions on symplectic manifolds, a special class of symplectic actions for which we have a momentum map. We proved the Marsden–Weinstein–Meyer symplectic reduction theorem for free actions, this ensures that the symplectic reduced space is a manifold. The symplectic reduction theorem holds for regular values of the momentum map as well, but then the reduced spaces are generally *orbifolds* [4, p.85].

We have seen that the momentum map gives us smooth functions which reflect the action, so that we could apply Morse–Bott theory to study Hamiltonian actions. This helped us prove the Atiyah–Guillemin–Sternberg convexity theorem for compact Hamiltonian torus spaces: the image of the momentum map is the convex hull of the images of the fixed points. Kirwan [25] generalized this convexity theorem to Hamiltonian actions of arbitrary compact Lie groups on compact symplectic manifolds, see also Sjamaar's paper [39].

Finally, we studied the Cartan model of equivariant cohomology to prove the Atiyah–Bott–Berline–Vergne localization theorem. In the setting of Hamiltonian torus actions, this theorem implied the Duistermaat–Heckman localization theorem. We also compared the Duistermaat–Heckman measure, the pushforward of the Liouville measure under the momentum map, with the Lebesgue measure and we saw that this measure is related to the Duistermaat–Heckman localization theorem by a Fourier transform. It would be interesting to study other models of equivariant cohomology, for example, the Borel model [4, Ch. VI], [21, Ch. 1,2], [43, Ch. 23-26] and the Weil model [21, Ch. 3,4], [43, Ch. 19,20].

The reader should now be sufficiently prepared to take on Delzant's classification theorem [11], [4, IV.4.e], [9, Ch. 28,29]. This theorem asserts, for the case $\dim M = 2\dim \mathbb{T}$, that the momentum polytope completely determines the Hamiltonian $\mathbb{T}$-space.

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