UNIVERSITY OF UTRECHT

BACHELOR THESIS

Representation Theory in Physics

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Abstract

The main goal of this thesis is to get an insight in the finite-dimensional irreducible complex representations of the special unitary group of order two. This symmetry group plays a fundamental role in physics in various ways. The group together with the unitary group of order one and the special unitary group of order three gives insight in the three fundamental forces and their interactions; namely the electromagnetic force, the weak force and lastly the strong force [4]. The main result of this thesis is that the finite-dimensional irreducible complex representations of the special unitary group of order two can be indexed, up to isomorphism, by a non-negative interger m.

Contents

1 Introduction

In physics observed phenomena are translated to mathematics which in its turn attempts to make useful predictions. Many of those mathematical models start with differential equations that are constructed using the laws of physics, an example of this is Newtons second law that states that the force is equal to the mass times acceleration. In some fields of study these differential equations come about via a Lagrangian that is invariant under some transformations of a symmetry group. These symmetries are called gauge transformations and are of great importance in particle physics [4]. This because elementary particles mathematically appear as the irreducible representations of the internal symmetry group of the standard model [2]. This symmetry group contains the special unitary group of order two on which we will focus in this thesis.

In the coming chapters we will build the basics of the theory of finite-dimensional representation of matrix Lie groups and their Lie algebras. To do so, we will introduce in section one and two of chapter 2 the basic notions of the object of interest, namely matrix Lie groups. After that, we will define the matrix exponential map in section three of this chapter. This lets us define the additional structure of Lie algebras that we will introduce in section four. After that, in chapter 3, the representation theory is build around our objects. Lastly, this all will be applied to an illuminating example that classifies all irreducible finite-dimensional complex representations of the special linear group of order two.

2 Prerequisites

In this chapter, an overview is given of the required mathematics that will be needed to construct the representation theory of matrix Lie groups in chapter 3. In the following sections, we describe the theory of the objects of interest, namely smooth manifold with a smooth group structure and their associated Lie algebras.

2.1 Smooth Manifolds

The general thought of manifolds is that they locally look like flat Euclidean space. Moreover, the theory tries to extend the idea of smoothness of mathematical object and maps between them that globally do not look like $\mathbb{R}^n$. Although the use of the theory of manifolds will not be the focus of this thesis, some formal definitions are still desirable. The following definitions and examples of manifolds in this chapter are gathered from the book 'Introduction to Smooth Manifolds' written by J.M. Lee [5]. Throughout this section an obvious example of a smooth manifold will be made.

Definition 2.1.1. A m-dimensional topological manifold is a topological space M that is Hausdorff, second countable and locally Euclidean of dimension m. More explicitly for the latter, all points in M have a neighbourhood that is homomorphic to an open subset of $\mathbb{R}^n$.

Consider $\mathbb{R}^n$, viewed with the natural topology induced by the Euclidean metric, to be the above topological space M. We see that it trivially abides all the above conditions and thus that it is a canonical example of a topological manifold.

Definition 2.1.2. A map $f : U \to U'$ between two opens $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^m$ is called a diffeomorphism if it is a bijection and both f and f^{-1} are smooth, that is infinitely differentiable.

Definition 2.1.3. A m-dimensional chart of a topological manifold M is a pair (U, φ) consisting of an open subset U of M and a homomorphism $\varphi : U \to U'$ that maps U to an open subset $\varphi(U) = U'$ in $\mathbb{R}^n$.

Definition 2.1.4. Two charts (U, φ) and (V, ψ) are called smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Note that in this definition the domain and image of the transition map are both in Euclidean space. This way it is possible to locally describe the structure of a topological manifold through Euclidean "glasses".

Definition 2.1.5. An atlas on a topological manifold M is a collection $\mathcal{A}$ of m-dimensional charts whose domains cover M. An atlas $\mathcal{A}$ is called a smooth atlas if any two charts of $\mathcal{A}$ are smoothly compatible.

Definition 2.1.6. A smooth atlas $\mathcal{A}$ on a topological manifold M is called maximal if it is not properly contained in any larger atlas. More explicitly, any chart that is smoothly compatible with every other chart in the smooth atlas is contained in the smooth atlas. Given any smooth atlas $\mathcal{A}$ we can make a maximal atlas $\mathcal{A}^{\max}$ by adding all smoothly compatible charts to it.

Extending on the above topological manifold $\mathbb{R}^n$ there exists a natural small atlas for it. Consider the atlas $\mathcal{A}_0$ containing only the n-dimensional chart $(\mathbb{R}^n, \mathbb{1})$ that consist of the whole space together with the identity map. Note that the only possible transition map of this atlas is the identity which is a diffeomorphism, so $\mathcal{A}_0$ is even a smooth atlas. Moreover, the induced maximal atlas $\mathcal{A}_0^{\max}$ are all chart smoothly compatible with the identity chart, these are the diffeomophisms between opens in $\mathbb{R}^n$.

Definition 2.1.7. A smooth m-dimensional manifold is a topological manifold M together with a maximal atlas $\mathcal{A}^{\max}$. We say that the manifold is endowed with the smooth structure of this maximal atlas.

We are now ready to define the canvas on which will paint our objects. With the above defined smooth structure $\mathcal{A}_0^{\max}$ and the Euclidean topological manifold we get the smooth manifold of $\mathbb{R}^n$. In further reading of this thesis when referencing to the manifold $\mathbb{R}^n$, consider it endowed with this standard smooth structure.

Definition 2.1.8. Let M, N be two smooth manifolds and $F : M \to N$ a map. Then F is a smooth map if for every point p in M there exists a chart (U, φ) around p and a chart (U', ψ) around $F(p)$ such that the composition map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(U')$.

Definition 2.1.9. If M and N are two smooth manifolds and $F : M \to N$ a smooth bijection with smooth inverse, then F is a diffeomorphism between M and N. If such a F exists between M and N, call these two manifolds diffeomorphic.

Given an open subset U of a smooth manifold M. If we restrict the charts of the maximal atlas of M to this open subset, then U inherits the properties of a topological manifold and the charts still form a maximal atlas that covers U. Moreover, this induced maximal atlas is a smooth structure on U. We see that open subsets of smooth manifolds naturally have an induced smooth structure which makes them a smooth manifold too. Now it is time to define the objects of interest.

Definition 2.1.10. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with complex entries. Similarly define the set of real matrices $M_n(\mathbb{R})$.

Proposition 2.1.11. *The set $M_n(\mathbb{C})$ is a smooth manifold.*

Proof. Note that we can identify $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. By the above example we have a natural smooth structure which makes this set of complex matrices into a smooth manifold. $\square$

2.2 Matrix Lie Groups

In this section, we endow smooth manifolds with an extra group structure. The following definitions and propositions are drawn from the first two chapters of the lecture notes on Lie groups of E.P. van den Ban [1] unless stated otherwise.

Definition 2.2.1. Let $\mathrm{GL}(n, \mathbb{C})$ denote the complex general linear group containing all $n \times n$ invertible matrices inside $M_n(\mathbb{C})$. Similarly define the real general linear group $\mathrm{GL}(n, \mathbb{R})$.

Proposition 2.2.2. *The complex general linear group is an open subset of $M_n(\mathbb{C})$.*

Proof. To proof this, we show that the complement, the set of non invertible matrices, is closed. A matrix is invertible if and only if its determinant is non zero. Since the determinant is a continuous function, we have that the preimage of $\{0\}$, all non invertible matrices, is also closed in $M_n(\mathbb{C})$. $\square$

As argued earlier in the section on smooth manifolds, this implies that since $\mathrm{GL}(n, \mathbb{C})$ is an open subset of $M_n(\mathbb{C})$ it inherits the smooth structure of $M_n(\mathbb{C})$, this makes the complex general linear group a smooth manifold too. Besides being a manifold, as the name "general linear group" suggests, it is also a group.

Proposition 2.2.3. *The complex general linear group forms a group under matrix multiplication.*

Proof. First note that matrix multiplication is associative and that given two matrices $A, B \in \mathrm{GL}(n, \mathbb{C})$ we have that $\det(AB) = \det(A)\det(B)$, so the multiplication is closed. Lastly, the identity matrix is contained in $\mathrm{GL}(n, \mathbb{C})$ and each element has an inverse by the definition of the set. □

Definition 2.2.4. A Lie group is a smooth manifold M equipped with a group structure and the extra property that both the group multiplication and inversion are smooth maps with respect to the smooth structure of M.

Proposition 2.2.5. *The complex general linear group is a Lie group.*

Proof. As shown above, it is indeed a smooth manifold equipped with a group structure. Moreover, the group product is smooth since it is just the product of polynomials in its components. Secondly, the inversion map is smooth since we can express it as a fraction of polynomial via Cramer's rule. □

Definition 2.2.6. The complex special linear group is the set of complex $n \times n$ invertible matrices with determinant 1, denoted by $\mathrm{SL}(n, \mathbb{C})$. Similarly define the real special linear group $\mathrm{SL}(n, \mathbb{R})$.

Definition 2.2.7. The special unitairy group is the set of unitary $n \times n$ invertible matrices with determinant 1, denoted by $SU(n)$.

The following definition is definition 1.4 drawn from the book of B.C. Hall [3].

Definition 2.2.8. A matrix Lie group is a closed subgroup G of $\mathrm{GL}(n, \mathbb{C})$.

By a similar argument as that of $\mathrm{GL}(n, \mathbb{C})$, we see that $\mathrm{SL}(n, \mathbb{C})$ and $SU(n)$ are subgroups of the complex general linear group. A natural question that arises is whether or not the two groups $\mathrm{SL}(n, \mathbb{C})$ and $SU(n)$ can be made into smooth manifolds which would also make them Lie groups. Using again that the determinant is continuous, we see that both sets are closed subsets of the complex general linear group and thus they are matrix Lie groups. But if they are both closed subsets, we cannot extend the smooth structure on these groups like before in the case of $M_n(\mathbb{C})$. Another approach is needed.

Theorem 2.2.9. *Let G be a Lie group and let H be a subgroup of G. Then the following two assertions are equivalent.*

1. *H is closed in the sense of topology.*
2. *H is a submanifold.*

Proof. For a prove of this theorem see theorem 2.16 in the lecture notes of [1]. □

Corollary 2.2.10. *The special linear group and the special unitary group are Lie groups.*

Now since the general linear group is trivially closed inside itself, it too is a matrix Lie group. In general all matrix Lie groups are Lie groups but the converse is not necessarily true, for a counter example see 1.21 in the book of B.C. Hall [3].

Definition 2.2.11. Given two Lie groups G and H, then a smooth map $\varphi : G \to H$ is called a Lie group homomorphism if it is a homomorphism of groups.

Definition 2.2.12. Given two Lie groups G and H, then a bijective Lie group homomorphism $\varphi : G \to H$ is called a Lie group isomorphism if its inverse is also a Lie group homomorphism.

2.3 The Matrix Exponential Map

In this section, we define and look at the properties of the exponential map for matrices. In a later section, we will use this map to associate a Lie algebra to each matrix Lie group. The following definitions and propositions are drawn from chapter two of the book of B.C. Hall [3].

Definition 2.3.1. Define the matrix exponential map that maps from $M_n(\mathbb{C})$ to itself via the power series, that is

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

Here we use that $X^0 = \mathbb{1}$.

Proposition 2.3.2. *The above series converges for all $n \times n$ matrices with complex entries and the map e^X is continuous.*

Proof. For a prove of this proposition see proposition 2.1 in the book of Hall [3]. □

Proposition 2.3.3. *Given two $n \times n$ complex matrices X and Y we have the following*

1. $e^0 = \mathbb{1}$.
2. $(e^X)^* = e^{X^*}$.
3. e^X *is invertible with inverse* e^{-X}.
4. $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ *for all* $\alpha, \beta \in \mathbb{C}$.
5. *if* $XY = YX$, *then* $e^{X+Y} = e^X e^Y$.
6. *if* $C \in GL(n, \mathbb{C})$, *then* $e^{CXC^{-1}} = Ce^X C^{-1}$.

Proof. For a prove of this proposition see proposition 2.3 in the book of Hall [3]. □

Proposition 2.3.4. *Given a complex $n \times n$ matrix X, then e^{tX} is a smooth curve in $M_n(\mathbb{C})$ for all $t \in \mathbb{R}$ and*

$$\frac{d}{dt} e^{tX} = Xe^{tX}.$$

Proof. For a prove of this proposition see proposition 2.4 in the book of Hall [3]. □

Proposition 2.3.5. *For every invertible $n \times n$ matrix Y there exists a $X \in M_n(\mathbb{C})$ such that $e^X = Y$.*

Proof. For a prove of this proposition see proposition 2.10 in the book of Hall [3]. □

Proposition 2.3.6. *Given a $X \in M_n(\mathbb{C})$, then*

$$det(e^X) = e^{trace(X)}.$$

Proof. For a prove of this proposition see proposition 2.12 in the book of Hall [3]. □

Definition 2.3.7. Call a function $A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ a one-parameter subgroup if the following conditions apply

1. A is continous.
2. $A(0) = \mathbb{1}$.

3. $A(t+s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$.

Theorem 2.3.8. *Given a one-parameter subgroup of $GL(n, \mathbb{C})$, then there exists a unique complex matrix X such that*

$$A(t) = e^{tX}$$

Proof. For a prove of this proposition see proposition 2.14 in the book of Hall [3]. □

2.4 Lie Algebras

Now that we have constructed the objects of interest, we will expand on the extra structures of these matrix Lie groups. In the following section, we will associate to each matrix Lie group a Lie algebra, we will present this as described in the book of B.C Hall [3]. Before we do that, we introduce the abstract notion of a Lie algebras and then define one for each matrix Lie group.

Definition 2.4.1. A finite dimensional real or complex Lie algebra is a finite dimensional real or complex vector space $\mathfrak{g}$ with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that it has the following three properties

1. The map $[\cdot,\cdot]$ is bilinear.
2. The map $[\cdot,\cdot]$ is skew symmetric, that is $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
3. The map $[\cdot,\cdot]$ abides the Jacobi identity given by

$$[X, [Y, Z]] + [Y[Z, X]] + [Z, [X, Y]] = 0.$$

We call the bilinear map the bracket of the Lie algebra and say that two elements of the Lie algebra commute if and only if $[X, Y] = 0$. Note that in the above definition either the vector space is real or complex and that that determines whether or not the Lie algebra is called real or complex.

Definition 2.4.2. A subalgebra of a real or complex Lie algebra is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that it is closed under the Lie bracket, that is $[H_1, H_2] \in \mathfrak{h}$ for all $H_1, H_2 \in \mathfrak{h}$.

Definition 2.4.3. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if for all $X \in \mathfrak{g}$ and any $H \in \mathfrak{h}$ we have that $[X, H] \in \mathfrak{h}$.

Definition 2.4.4. A Lie algebra $\mathfrak{g}$ is called irreducible if it contains no non trivial ideals. That is, the only ideals are $\{0\}$ and $\mathfrak{g}$ itself. A Lie algebra that is irreducible and has a dimension greater than 1 is called simple.

Definition 2.4.5. Given a matrix group G denote the set of all $X \in M_n(\mathbb{C})$ such that e^{tX} lies in G for all real numbers t by $\mathfrak{g}$, the Lie algebra of G.

Proposition 2.4.6. *Given a matrix Lie group and associated Lie algebra $\mathfrak{g}$, then for elements X and Y in $\mathfrak{g}$ we have that*

1. $AXA^{-1} \in \mathfrak{g}$ *for all* $A \in G$.
2. $sX \in \mathfrak{g}$ *for all* $s \in \mathbb{R}$.
3. $X + Y \in \mathfrak{g}$.
4. $XY - YX \in \mathfrak{g}$

Proof. For a proof of this proposition see theorem 3.20 in the book of B.C. Hall [3]. □

Theorem 2.4.7. *Given a matrix Lie group G, then the associated set $\mathfrak{g}$ together with the map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, [X,Y] = XY - YX$$

makes $\mathfrak{g}$ a real Lie algebra.

Proof. Via theorem 2.4.3 and proposition 2.3.3, we know that the set $\mathfrak{g}$ is a real vector space and that the Lie bracket map is well-defined. Moreover, this map is linear in both components if the other components is fixed and thus bilinear. We also have that

$$[X,Y] = XY - YX = -(YX - XY) = -[Y,X]$$

so the bracket is skew symmetric. Lastly we check the Jacobi identity. Given the elemens X, Y, Z in $\mathfrak{g}$ consider

$$\begin{aligned}
[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] =& XYZ - YXZ - ZXY + ZYX + YZX - ZYX - XYZ + XZY \\
&+ ZXY - XZY - YZX + YXZ \\
=&(XYZf - XYZ) + (ZYX - ZYX) + (YZX - YZX) \\
&+ (XZY - XZY) + (ZXY - ZXY) + (YXZ - YXZ) \\
=&0
\end{aligned}$$

□

Definition 2.4.8. Given a finite-dimensional real vector space V, denote by $V_{\mathbb{C}}$ the complexification of V that contains the linear combinations

$$v_1 + iv_2$$

for $v_1, v_2 \in V$.

Proposition 2.4.9. *Given a real finite-dimensional Lie algebra $\mathfrak{g}$, then the Lie algebra bracket of $\mathfrak{g}$ extends uniquely on the complexification $\mathfrak{g}_{\mathbb{C}}$ and the bracket is given by*

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_1] + [X_2, Y_2]).$$

Proof. For a proof of this proposition see proposition 3.37 in the book of B.C. Hall [3]. □

Proposition 2.4.10. *The complex Lie algebra of $SL(n,\mathbb{C})$ or the real Lie algebra of $SL(n,\mathbb{R})$ are denoted by $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{R})$. These are the $n \times n$ complex and respectively real matrices with trace zero.*

Proof. Let X be an element of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ (or the real Lie algebra), then for all real valued t we have that e^{tX} is in $SL(n,\mathbb{C})$. So by the definition of the special linear group it must be that

$$det(e^{tX}) = 1$$

for all real valued t. Now using proposition 2.3.6, we see that

$$det(e^{tX}) = e^{trace(tX)} = 1.$$

Since this must hold for all t we conclude that $trace(X) = 0$. □

Via the above proposition we find the following three basis elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For the complex Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$ we find the six basis elements $\{V_1, V_2, V_3, iV_1, iV_2, iV_3\}$, if viewed via the complexification of the real Lie algebra.

Proposition 2.4.11. *The real Lie algebra of $SU(n)$ denoted by $\mathfrak{su}(n)$ are the $n \times n$ complex matrices X such that $X^* = -X$ and that have trace zero.*

Proof. Similarly as in the proof of proposition 2.4.5, we have that the trace of an element contained in the Lie algebra $\mathfrak{su}(n)$ must have trace 0. Now for the second properties, let X be an element of the Lie algebra. Then we have that e^{tX} is unitary for all real valued t. That means that the inverse of e^{tX} is the adjoint of itself, using part 2 and 3 of proposition 2.3.3 we see that

$$e^{tX^*} = (e^{tX})^* = (e^{tX})^{-1} = e^{-tX}$$

for all real valued t. We conclude that $X^* = -X$. $\square$

Via the above proposition we find that the following basis of the Lie algebra $\mathfrak{su}(2)$.

$$U_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since the Lie algebra of $\mathfrak{su}(2)$ is a real algebra we can complexify it and find, seen as a 6 dimensional real algebra, that the complexification $\mathfrak{su}(2)_\mathbb{C}$ has the basis $U_1, U_2, U_3, iU_1, iU_2, iU_3$ with the commutation relations

$$[U_j, U_k] = 2\epsilon_{jkl}U_l \qquad [iU_j, iU_k] = -2\epsilon_{jkl}U_k \qquad [U_j, iU_k] = 2\epsilon_{jkl}(iU_l).$$

Here ϵ_{jkl} denotes the Levi-Civita symbol which has the possible values $\{-1, 0, 1\}$, depending on the permutation of the indexes. If the permutation is even it has a value 1, if it is odd it has value -1, and 0 if it has any repeating indices.

Definition 2.4.12. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, then a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if it preserves the Lie bracket. That is, for $X, Y \in \mathfrak{g}$ we have that

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)].$$

If this map is also a bijection, then φ is also a Lie algebra isomorphism and the two Lie algebras are isomorphic.

Proposition 2.4.13. *The Lie algebra $\mathfrak{su}(2)_\mathbb{C}$ and $\mathfrak{sl}(2,\mathbb{C})$ are isomorph.*

Proof. Note that we have the following relations

$$\begin{aligned} U_1 &= iV_2 + iV_3, & U_2 &= V_3 - V_2, & U_3 &= iV_1, \\ iU_1 &= -V_2 - V_3, & iU_2 &= iV_3 - iV_2, & iU_3 &= -V_1. \end{aligned}$$

Now since the bracket for Lie algebra of matrix Lie groups is given by the commutator as defined in theorem 2.4.7, we conclude that this change of basis preserves the Lie bracket relations. For example we see that

$$\begin{aligned} [U_1, U_3] =& [iV_2 + iV_3, iV_1] \\ =& -[V_2 + V_3, V_1] \\ =& -[V_2, V_1] - [V_3, V_1] \\ =& 2V_2 - 2V_3 \\ =& -2U_2. \end{aligned}$$

Thus, we have found an isomorphism between the complexification $\mathfrak{su}(2)_{\mathbb{C}}$ and $\mathfrak{sl}(2, \mathbb{C})$. □

Theorem 2.4.14. *Given two matrix Lie groups G and H with associated Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\Phi : G \to H$ is a Lie group homomorphism, then there exists a unique real-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that for all $X \in \mathfrak{g}$*

$$\Phi(e^X) = e^{\phi(X)}.$$

Moreover, we have the following

1. $\phi(AXA^{-1}) = \phi(A)\phi(X)\phi(A)^{-1}$ *for all $X \in \mathfrak{g}$ and $A \in G$.*
2. $\phi([X, Y]) = [\phi(X), \phi(Y)]$ *for all $X, Y \in \mathfrak{g}$.*
3. $\phi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0}$ *for all $X \in \mathfrak{g}$.*

Proof. We only prove the existence and the uniqueness of such a map here. The latter three statements are proven in theorem 3.28 in the book of B.C. Hall [3]. Given such a Lie group homomorphism Φ we know that since it is smooth the following composition is continuous

$$\Phi(e^{tX}) : \mathbb{R} \to H.$$

Moreover, because this composition maps zero to the identity matrix and $\Phi(e^{(t+s)X}) = \Phi(e^{tX})\Phi(e^{sX})$, we conclude that it is a one-parameter subgroup of the general linear group. Via theorem 2.3.8 there must be a unique complex matrix Z such that

$$\Phi(e^{tX}) = e^{tZ}.$$

Now define the unique map $\phi(X) = Z$. It is clear that at t equals one this has the property that for all $X \in \mathfrak{g}$

$$\Phi(e^X) = e^{\phi(X)}.$$

□

Proposition 2.4.15. *Given a matrix Lie group G that is connected with associated Lie algebra $\mathfrak{g}$, then every element A of G can be written in the form*

$$A = e^{X_1}e^{X_2}\ldots e^{X_m}$$

for some finitely many $X_1, X_2, \ldots, X_m$ in $\mathfrak{g}$

Proof. For a proof of this proposition see proposition 3.47 in the book of B.C. Hall [3]. □

3 Representation Theory

In this chapter, we introduce representation theory of Lie Groups and Lie algebras. The general thought of representation theory is that we can represent elements of some set with an additional algebraic structure as linear transformations between vector spaces. In our case, the considered algebraic structures are Lie groups and Lie algebras. Later in this chapter, we will see that a representation on a Lie group naturally extends to a representation on its Lie algebra. The following definitions and examples are drawn from chapter 4 of the book of B.C. Hall [3].

Definition 3.0.1. A finite-dimensional complex representation of a matrix Lie group G is Lie group homomorphism

$$\Pi : G \to GL(V)$$

where V is a finite-dimensional complex vector space. Similarly, if V is a finite real vector space, then call this a finite-dimensional real representation of G.

Definition 3.0.2. A finite-dimensional complex representation of a real or complex Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

where V is a finite-dimension complex vector space. Similarly, if $\mathfrak{g}$ is a real algebra and V a finite-dimensional real vector space, then it would be a finite-dimensional real representation of $\mathfrak{g}$.

Definition 3.0.3. Given a finite-dimensional complex or real representation (Π, V) of a matrix Lie group G. Call a subspace W of V invariant if for all elements X in G we have that

$$\Pi(X)(w) \in W \text{ for all } w \in W.$$

Definition 3.0.4. Given a finite-dimensional complex or real representation of a matrix Lie group G. Call an invariant subspace W nontrivial if $W \neq 0$ or $W \neq V$. A finite-dimensional representation with no nontrivial subspaces is called irreducible.

Definition 3.0.5. Given a finite-dimensional complex or real representation (π, V) of a Lie algebra $\mathfrak{g}$. Call a subspace W of V invariant if for all elements X in $\mathfrak{g}$ we have that

$$\pi(X)(w) \in W \text{ for all } w \in W.$$

Definition 3.0.6. Given a finite-dimensional complex or real representation of a Lie algebra $\mathfrak{g}$. Call an invariant subspace W nontrivial if $W \neq 0$ or $W \neq V$. A finite-dimensional representation with no nontrivial subspaces is called irreducible.

Definition 3.0.7. Given a matrix Lie group G and two representations (Π, V) and (Σ, W) of G. Call a linear map $\varphi : V \to W$ an intertwining map if the following diagram commutes for all elements X in G.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\Pi(X)"'] & W \arrow[d, "\Sigma(X)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

This explicitly means that

$$\Big(\varphi \circ \Pi(X)\Big)(v) = \Big(\Sigma(X) \circ \varphi\Big)(v) \text{ for all } v \in V.$$

Definition 3.0.8. Given an intertwining map between two representations of a matrix Lie group that is also bijective. Then the intertwining map is an isomorphism and call the two representations isomorphic.

Again, these definitions of intertwining maps extend to the finite-dimensional representations of a Lie algebra. We now consider two trivial irreducible complex representations, one for each matrix Lie group and another for a Lie algebra. Define the complex representation over $\mathbb{C}$ with the map $\Pi : G \mapsto \mathrm{GL}(\mathbb{C})$ given by

$$\Pi(X) = \mathbb{1} \text{ for all } X \text{ in } G.$$

Similar for a Lie algebra define the map $\pi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C})$ given by

$$\pi(X) = 0 \text{ for all } X \text{ in } \mathfrak{g}.$$

Clearly, both are representations and irreducible since a one dimensional vector space has no non-trivial subspaces.

Proposition 3.0.9. *Given a matrix Lie group G and (Π, V) a finite-dimension real or complex representation of G. Then there exist a unique representation π for the associated Lie algebra $\mathfrak{g}$ defined on the same vector space such that for all $X \in \mathfrak{g}$*

$$\Pi(e^X) = e^{\pi(X)}.$$

More explicitly we have that

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}$$

and additionally that

$$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A^{-1})$$

for all $X \in \mathfrak{g}$ and $A \in G$.

Proof. Note that this is just an application of theorem 2.4.11 stated in the previous chapter. □

Proposition 3.0.10. *Given a matrix Lie group G that is connected with associated Lie algebra $\mathfrak{g}$. Then for a representation (Π, V) of G and the induced representation (π, V) on $\mathfrak{g}$ we have that Π is irreducible if and only if π is irreducible.*

Proof. Assume that the finite-dimensional representation π is irreducible. We want to show that if W is an invariant subspace of V under the finite-dimensional representation of Π, then W must be $\{0\}$ or the whole space V. Now since G is connected and using proposition 2.4.12, we have for any element A in G that we can write it as the product of m exponentials. So for any w in W we have the following

$$\begin{aligned}
\Pi(A)(w) &= \Pi(e^{X_1}e^{X_2}\dots e^{X_m})(w) \\
&= \Pi(e^{X_1})\Pi(e^{X_2})\dots\Pi(e^{X_m})(w) \\
&= e^{\pi(X_1)}e^{\pi(X_2)}\dots e^{\pi(X_m}(w)
\end{aligned}$$

Because the exponential map is defined by the power series of the upper component, which in this case leaves the vector w invariant, we see that each of the m exponents also leave w invariant. Since Π is irreducible, we conclude that it must be that either $W = \{0\}$ or $W = V$.

Now assume that Π is irreducible and that W is an invariant subspace of V under the finite-dimensional representation of π. Then we have for all elements X in $\mathfrak{g}$ that W is invariant under $\Pi(e^{tX})$ for all real valued t. Now since V is a finite-dimensional vector space W is closed in the topological sense. Using this we see that W must be also invariant under

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}.$$

We conclude either $W = \{0\}$ or it is the whole vector space V. □

Proposition 3.0.11. *Given a real Lie algebra $\mathfrak{g}$ and a finite-dimensional complex representations (π, V). Then there exists a unique extensions of this representation to a complex linear representation on the complexification $\mathfrak{g}_{\mathbb{C}}$ denoted by (π', V). Additionally, the representation on the complexification is irreducible if and only if the representation on $\mathfrak{g}$ is.*

Proof. Note that a possible extension is given by $\pi'(X + iY) = \pi(X) + i\pi(Y)$ for X and Y elements of the Lie algebra $\mathfrak{g}$. The uniqueness of this map follows from the fact that it has to comply with the unique extension of the complexified Lie bracket. Lastly, a complex subspace W of V is invariant under the map $\pi'(X + iY)$ if and only if it is invariant under both $\pi(X)$ and $\pi(Y)$. □

Proposition 3.0.12. *Given a matrix Lie group G and finite-dimensional representations $(\Pi_1, V_1), \cdots, (\Pi_m, V_m)$ of G. Then the direct sum of these m representations is again a finite-dimensional representation of G. This representation is for an element X in G defined by*

$$[\Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_m](X)(v_1, v_2, \ldots, v_m) = (\Pi(X)(v_1), \Pi_2(X)(v_2), \ldots, \Pi_m(X)(v_m)).$$

Proof. Note that the direct sum still preserves the group homomorphism in each component, so this again will be a Lie group homomorphism. □

Definition 3.0.13. Call a finite-dimensional representation of either a matrix Lie group or Lie algebra completely reducible if the representation is isomorphic to the direct sum of finitely many irreducible representations. A matrix Lie group or Lie algebra is said to be completely irreducible if every finite-dimensional representation is completely irreducible.

In the case of finite groups all representations are completely reducible, this is shown in chapter 1 of the book on "Representation Theory" by W. Fulton and J. Harris [6]. For matrix Lie groups this is not always true. We will see that compact matrix Lie groups are completely reducible.

Proposition 3.0.14. *Given (π, V) a completely reducible representation of a matrix Lie group or Lie algebra, then the following holds*

1. *For every invariant subspace U of V there exists an another invariant subspace W such that V is the direct sum of U and W.*

2. *Every invariant subspace of V is completely reducible.*

Proof. For a proof of this proposition see proposition 4.25 in the book of B.C. Hall [3]. □

Definition 3.0.15. Given a matrix Lie group G and a finite-dimensional representation (Π, V) where V is an inner product space. Call the representation unitary if for all elements X in G the operator $\Pi(X) : V \to V$ is unitary.

Proposition 3.0.16. *Given a finite-dimensional unitary representation (Π, V) of a matrix Lie group G, then this representation is completely reducible. Similar for a Lie group $\mathfrak{g}$ we have that, if a finite-dimensional representation (π, V) is unitary, that is $\pi(X)^* = -\pi(X)$ for all X in $\mathfrak{g}$, then it is completely reducible.*

Proof. Given an invariant subspace W of V, we show that the orthogonal complement $W^{\perp}$ of it is again an invariant subspace of V. Notice that since $\Pi(X)$ is unitary for all elements X in G, for a $w \in W$ and $v \in W^{\perp}$ we have that

$$\begin{aligned}\langle \Pi(X)(v), w\rangle &= \langle v, \Pi(X)^*(w)\rangle \\ &= \langle v, \Pi(X^{-1})(w)\rangle \\ &= \langle v, w\rangle = 0.\end{aligned}$$

In the last step, we used that since W is invariant under the representation. We conclude that v is also invariant under the representation and thus that $W^{\perp}$ is also an invariant subspace of V. Now suppose the representation is not irreducible, then it has at least two invariant subspaces W and $W^{\perp}$. If either one of those is again not irreducible this step can be repeated and since V is of finite dimension this process must stop. We see that an unitary finite-dimensional representation must be completely reducible. A similar proof shows this for a finite-dimensional unitary representation of a Lie algebra. □

The following theorem is also known as Weyl's unitary trick and uses, similar to the prove that each representation of a finite group is irreducible, that given any inner product on a vector space of a representation we can make an inner product with the property that the representation is unitary. This prove uses the fact that it is possible to integrate over a compact manifold to construct such a inner product. We will not explain that this is possible but rather just use it. It is explained in the book of B.C.Hall in section 4.4 [3].

Theorem 3.0.17. *Given a compact matrix Lie group G and a finite-dimensional representation (Π, V) of G, then this representation is completely reducible.*

Proof. Given any inner product $\langle \cdot, \cdot \rangle$ on V define the following inner product $\langle \cdot, \cdot \rangle_G : V \times V \to \mathbb{C}$ by

$$\langle v, w\rangle_g = \int_G \langle \Pi(X)(v), \Pi(X)(w)\rangle \alpha(X)$$

Here $\alpha(X)$ is a differential k-form where k is the dimension of the Lie algebra, we will not go into this. We will only prove that this inner product is unitary. Note that for Y in G and v, w in we have that

$$\begin{aligned}\langle \Pi(Y)(v), \Pi(Y)(w)\rangle_G &= \int_G \langle \Pi(X)\Pi(Y)(v), \Pi(X)\Pi(Y)(w)\rangle \alpha(X) \\ &= \int_G \langle \Pi(XY)(v), \Pi(XY)(w)\rangle \alpha(X) \\ &= \int_G \langle \Pi(X)(v), \Pi(X)(w)\rangle \alpha(X) \\ &= \langle v, w\rangle_G\end{aligned}$$

We see that indeed this inner product is unitary and thus via proposition 3.0.16, we conclude that this representation is completely reducible. □

4 Representations of SU(2)

In this chapter, we finalize the work of finding all irreducible representations of the matrix Lie group $SU(2)$. First, we will propose for each non-negative interger m a representation. After that, we will look at how this representation extends to the Lie algebra $\mathfrak{su}(2)$ and its complexification. Lastly, we will prove that these representations are all irreducible representations of $SU(2)$. The prove given below follows the proof given by B.C. Hall in chapter 4 [3].

Definition 4.0.1. Let V_m denote the space of homogeneous polynomials of degree m in two variable. That is

$$V_m = \Big\{ \sum_{i=0}^{m} a_i x^{m-i} y^i \ : \ a_i \in \mathbb{C} \Big\}.$$

where m is a non-negative interger.

Definition 4.0.2. Given an $U \in \mathrm{SU}(2)$ define the maps $\Pi_m(U) : V_m \to V_m$ given by

$$\Pi_m(U)(f(z)) = f(U^{-1}z).$$

Or explicitly, given that $z = (x, y) \in \mathbb{C}^2$

$$\Pi_m(U)(f(z)) = \sum_{i=1}^{m} a_i (U_{11}^{-1}x + U_{12}^{-1}y)^{m-i}(U_{21}^{-1}x + U_{22}^{-1}y)^i.$$

Via the binominal theorem this is the sum of a product of two polynomials, which again gives a polynomial of maximal degree m. So, we see that for all U the map $\Pi_m(U)$ is well-defined.

Proposition 4.0.3. *The map $\Pi_m : SU(2) \to GL(V_m)$ is a Lie group homomorphism and thus a finite-dimensional complex representation.*

Proof. Let A, B be elements in $SU(2)$ and consider for a $f(z)$ in V_M

$$\begin{aligned}
\Big[\Pi_m(A) \circ \Pi_m(B)\Big](f(z)) &= \Pi_m(A)\Big[\sum_{i=1}^{m} a_i (B_{11}^{-1}x + B_{12}^{-1}y)^{m-i}(B_{21}^{-1}x + B_{22}^{-1}y)^i\Big] \\
&= \sum_{i=1}^{m} a_i \Big(A_{11}^{-1}(B_{11}^{-1}x + B_{12}^{-1}y) + A_{12}^{-1}(B_{21}^{-1}x + B_{22}^{-1}y)\Big)^{m-i} \\
&\quad \Big(A_{21}^{-1}(B_{11}^{-1}x + B_{12}^{-1}y) + A_{22}^{-1}(B_{21}^{-1}x + B_{22}^{-1}y)\Big)^{i} \\
&= \sum_{i=1}^{m} a_i \Big(A_{11}^{-1}B_{11}^{-1}x + A_{11}^{-1}B_{12}^{-1}y + A_{12}^{-1}B_{21}^{-1}x + A_{12}^{-1}B_{22}^{-1}y\Big)^{m-i} \\
&\quad \Big(A_{21}^{-1}B_{11}^{-1}x + A_{21}^{-1}B_{12}^{-1}y + A_{22}^{-1}B_{21}^{-1}x + A_{22}^{-1}B_{22}^{-1}y)\Big)^{i} \\
&= f(B^{-1}A^{-1}z) \\
&= \Pi_m(AB)(f(z)).
\end{aligned}$$

This map is also smooth, so we see that Π_m is a Lie group homomorphism. This proves that (Π_m, V_m) is a finite-dimensional complex representation of $SU(2)$. $\square$

Now via proposition 3.0.9, we know that for every representation of the matrix Lie group $SU(2)$ there exists an unique representation (π_m, V_m) of the real Lie algebra $\mathfrak{su}(2)$. Moreover, we can make this map explicit via the same theorem. We have for an element X in $\mathfrak{su}(2)$ and its image U under the exponential map that

$$\begin{aligned}
\pi_m(X)(f(z)) &= \frac{d}{dt}\Pi_m(U)(f(z)))\Big|_{t=0} \\
&= \frac{d}{dt}f(U^{-1}z)\Big|_{t=0} \\
&= \frac{d}{dt}f(U^{*}z)\Big|_{t=0} \\
&= \frac{d}{dt}f(e^{-tX}z)\Big|_{t=0}
\end{aligned}$$

Now, if we identify and substitute $z(t) = (x(t), y(t))$ as the curve in $\mathbb{C}^2$ given by $z(t) = e^{-tX}z$ and use the chain rule we get

$$\pi_m(X)(f) = \frac{df}{dx}\frac{dx}{dt}\Big|_{t=0} + \frac{df}{dy}\frac{dy}{dt}\Big|_{t=0}.$$

Since $dz/dt|_{t=0} = -Xz$ this results in

$$\pi_m(X)(f) = -\frac{df}{dx}(X_{11}x + X_{12}y) - \frac{df}{dy}(X_{21}x + X_{22}y).$$

Now remember that if we complexify the real Lie algebra of $\mathfrak{su}(2)$, which is isomorphic to the Lie algebra $sl(2, \mathbb{C})$, we get the following basis of this complex Lie algebra.

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We now look at how this basis maps under the extended representation (π'_m, V_m) on $\mathfrak{sl}(2, \mathbb{C})$. Notice that

$$\begin{aligned}
\pi'_m(V_1) &= -x\frac{df}{dx} + y\frac{df}{dy}, \\
\pi'_m(V_2) &= -y\frac{df}{dx}, \\
\pi'_m(V_3) &= -x\frac{df}{dy}.
\end{aligned}$$

Furthermore, if we look at how these three operators act on the basis elements $x^{m-i}y^i$ of the complex vector space V_m we see that

$$\begin{aligned}
\pi'_m(V_1)(x^{m-i}y^i) &= (2i - m)x^{m-i}y^i, \\
\pi'_m(V_2)(x^{m-i}y^i) &= (i - m)x^{(m-i)-1}y^{i+1}, \\
\pi'_m(V_3)(x^{m-i}y^i) &= ix^{(m-i)+1}y^{i-1}.
\end{aligned}$$

Notice that the basis vectors of V_m are eigenvectors of the linear operator $\pi'_m(V_1)$ and the other two either raise or lower the exponent of y by 1. Equivalently, they raise or lower the exponent of x by 1.

Proposition 4.0.4. *For each non-negative interger m we have that the finite-dimensional representation (π'_m, V_m) is irreducible.*

Proof. Assume that we have an invariant subspace W of V_m that contains a nonzero vector w. Then we can write w in the above basis as

$$w = \sum_{i=0}^{m} a_i x^{m-i} y^i$$

with a_i some coefficients. Now let k be the smallest integer for which the coefficients a_k in w is nonzero and consider the image of

$$\pi'_m(V_2)^{m-k}(w).$$

This mapping raises the power of y an additional $m-k$ times and lowers the power of x this same amount. So then we see that $a_k x^{m-k} y^k$ gets mapped to a multiple of $x^{(m-k)-(m-k)} y^{k+(m-k)} = y^m$. Since this k was the lowest index for which the coefficient was nonzero, all the terms with higher indexed coefficient are killed. Because W is invariant under this linear operator, we see that y^m is an elements of W. But if this is true by the same argument as above, we see than that for each interger l such that $0 \leq l \leq m$ that

$$\pi'(V_3)^l(y^m)$$

maps onto multiple of the basis vectors $x^l y^{k-l}$ of V_m. We conclude that W is the whole space V_m and thus (π'_m, V_m) is a finite-dimensional complex irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. $\square$

Now we know that the representation on the complexification of $\mathfrak{sl}(2, \mathbb{C})$ is irreducible, we conclude via theorem 3.0.11 that the representation on the real Lie algebra $\mathfrak{su}(2)$ is also irreducible. To make a similar conclusion for the representation of the matrix Lie group $SU(2)$ we need one more thing, compactness.

Proposition 4.0.5. *The special unitary group of order two is compact and connected.*

Proof. Given an element U of $SU(2)$ we have that its inverse of U is the adjoint of itself and that it has determinant equal to 1. More explicitly this gives that U is of the form

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ -\beta & \overline{\alpha} \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and $\alpha^2 + \beta^2 = 1$. If these two are viewed as complex numbers $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$, then we get the condition that $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$. We see that $SU(2)$ is diffeomorphic to S^3 which is compact and connected. We conclude the same for the special linear group of order two. $\square$

Now since $SU(2)$ is compact and connected, we see that it is completely reducible and using the result of proposition 3.0.10 we see that the representation (Π_m, V_m) is irreducible. One question remains, are all irreducible representations on $SU(2)$ of the form (Π_m, V_m)? This is what we will investigate now. To prove this, we show that any representation on $\mathfrak{sl}(n, \mathbb{C})$ has a particular form. The proof that is stated below will follow the proof of theorem 4.32 in the book of B.C. Hall [3].

Theorem 4.0.6. *If (π, V) is an irreducible complex representation of $\mathfrak{sl}(2, \mathbb{C})$ with dimension $m+1$, then this representation is isomorphic to the irreducible complex representation (π_m, V_m) defined earlier.*

Remember that the basis vectors of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ are given by V_1, V_2 and V_3 defined in 2.4.11. These have the following commutation relations

$$[V_1, V_2] = 2V_2 \qquad [V_1, V_3] = -2V_3 \qquad [V_2, V_3] = V_1.$$

Lemma 4.0.7. *Given any representation (π, V) of $\mathfrak{sl}(2,\mathbb{C})$ and u an eigenvector for the linear operator $\pi(V_1)$ with eigenvalue $\alpha \in \mathbb{C}$, then we have the following two equalities*

1. $\pi(V_1)\pi(V_2)(u) = (\alpha+2)\pi(X)(u)$,
2. $\pi(V_1)\pi(V_3)(u) = (\alpha-2)\pi(V_3)(u)$.

Proof. Note that a representation preserves the commutator relations, thus we have that

$$\begin{aligned}\pi(V_1)\pi(V_2)(u) &= [2\pi(V_2) + \pi(V_2)\pi(V_1)](u)\\ &= 2\pi(V_2) + \pi(V_2)(\alpha u)\\ &= (2+\alpha)\pi(V_2)(u)\end{aligned}$$

A similar proof show statement number two. Again we see that $\pi(V_2)$ and $\pi(V_3)$ play the role of a raising and lowering operator. □

Proof. (Theorem 4.0.6) Let (π, V) be any finite-dimensional complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. Since V is a complex vector space the linear operator $\pi(V_1)$ must have at least one eigenvector. Denote this eigenvector by u with eigenvalue α. Now using the previous lemma k times we see that

$$\pi(V_1)\pi(V_2)^k(u) = (\alpha+2k)\pi(V_2)(u).$$

Now since the operator $\pi(V_1)$ working on the finite-dimensional vector space V can only have a finite amount of eigenvalues it must be that there exists an element N in $\mathbb{N}$ such that

$$\pi(V_2)^N(u) \neq 0$$

but

$$\pi(V_2)^{N+1}(u) = 0.$$

Now define the $u_0 = \pi(V_2)^N(u)$ with eigenvalue $\lambda = \alpha + 2N$. This vector has the following properties

$$\pi(V_1)(u_0) = \lambda u_0, \qquad \pi(V_2)(u_0) = 0.$$

Besides this u_0, define the following k vectors via the lowering operator, namely $u_k = \pi(V_3)^k(u_0)$. Using again the above lemma 4.0.6 we see that

$$\pi(V_1)u_k = \pi(v_1)\pi(V_3)^k(u_0) = (\lambda-2k)\pi(V_3)^k(u_0) = (\lambda-2k)u_k.$$

Using the commutation relations we see that

$$\pi(V_2)\pi(V_3)(u_k) = \big[\pi(V_1) + \pi(V_3)\pi(V_2)\big](u_k).$$

We now will prove via induction that for $k > 0$

$$\pi(V_2)u_k = k[\lambda-(k-1)]u_{k-1}.$$

We see for $k = 1$ that $\pi(V_2)\pi(V_3)(u_1) = \lambda u_0$. Assume that the statement holds for $k-1$ and consider

$$\begin{aligned}\pi(V_2)(u_k) &= \pi(V_2)\pi(V_3)^k(u_0)\\ &= \left[\pi(V_2)\pi(V_3)\right]\left(\pi(V_3)^{k-1}(u_0)\right)\\ &= \pi(V_2)\pi(V_3)(u_{k-1})\\ &= \pi(V_1)(u_{k-1}) + \pi(V_3)\pi(V_2)(u_{k-1}).\end{aligned}$$

Using the induction hypothesis we see that

$$\begin{aligned}\pi(V_2)(u_k) &= \big(\lambda - 2(k-1)\big)u_{k-1} + \pi(V_3)\big[(k-1)\big(\lambda-(k-2)\big)\big]u_{k-2}\\ &= \big(\lambda - 2k+2\big)u_{k-1} + \big[\lambda k - \lambda - k^2\big]\pi(V_3)(u_{k-1})\\ &= k[\lambda-(k-1)]u_k.\end{aligned}$$

By the principle of induction we conclude the statement. As before, since V is a finite-dimensional vector space $\pi(V_1)$ has finitely many eigenvectors. Thus there must be a non-negative integer m such that $u_m \neq 0$ but $u_{m+1} = 0$. Lastly, note that if $u_{m+1} = 0$, then we have that $\pi(V_2)u_{m+1} = 0$ so via the last equality it follows that

$$\pi(V_2)u_{m+1} = (m+1)[\lambda - m]u_m = 0.$$

Since $m+1$ and u_k are both nonzero we conclude that $\lambda = m$.

In conclusion, given any finite-dimensional complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ we have an non-negative interger m and nonzero vectors $u_0, \dots, u_m$ such that that

$$\begin{aligned}\pi(V_1)u_k &= (m-2k)u_k,\\ \pi(V_2)u_k &= \begin{cases} k[m-(k-1)]u_{k-1} & \text{if } k>0,\\ 0 & \text{if } k=0.\end{cases}\\ \pi(V_3)u_k &= \begin{cases} u_{k+1} & \text{if } k \leq m,\\ 0 & \text{if } k=m,\end{cases}\end{aligned}$$

Since the eigenvalue of u_k under $\pi(V_1)$ are all distinct they must be linear independent. Furthermore, the span of these vectors is invariant under the image of all three basis vectors of the Lie algebra. So the span of these $m+1$ vectors is invariant under any $\pi(X)$ for X an element in $\mathfrak{sl}(2,\mathbb{C})$. Since by assumption this representation is irreducible we conclude that these $m+1$ vectors span the whole vector space V. Now given any two irreducible representations with the same dimension $m+1$ they are both of the form as described above and thus isomorphic. In particular, this make them also isomorph to the induced π'_m representation. □

To recap this chapter, first we introduced an ad hoc proposition for a finite-dimensional complex representation of $SU(2)$. We proved via an extension on the Lie algebra and its complexification that this representation was indeed irreducible. To prove that all irreducible representation on $SU(2)$ where of this form we looked at any representation on the Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$. Here we saw that any representation, and thus all induced representations coming from the matrix Lie group, where of the form of the induced representation π_m. We thus conclude that the first proposed representations where all irreducible representations on $SU(2)$.

5 Conclusion and Outlook

In conclusion, in this thesis a strong connection is made between the finite-dimensional representations of matrix Lie groups and their associated Lie algebra. For any representation of a matrix Lie group there is one for its Lie algebra. This in such a way that, given some additional conditions, the two representations shared properties like complete irreducibility. Using these facts, we looked at the special unitary matrix Lie group of order two. For this matrix Lie group we identified all finite-dimensional complex irreducible representations and classified them by a non-negative integer m.

An interesting question now arises, could we expand on the ideas in this thesis to classify the irreducible representations of higher dimensions of the special unitary group? The short answer is yes. Although we will not prove any of the claims made here, we will elaborate on them. We already showed that the the special unitary group of order two is compact and connected using a diffeomophisms between the matrix Lie group and the three sphere. More generally, it can be shown that $SU(n)$ is also compact and connected. In the book of B.C. Hall a more geneal proof is given that shows that the complexification of $\mathfrak{su}(n)$ is isomorphic to the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. So like in this thesis we can extend any finite-dimensional representation of the Lie group $SU(n)$ to a finite-dimensional representation of the Lie algebra $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ while still sharing reducibility. The only problem now is that we do not have any suggestions for irreducible representation like the representations (π_m, V_m) on $SU(2)$.

Lets forget about this for a moment. Can we still classify the irreducible representations of $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$? Again the answer is yes, in the chapters 8, 9 and 10 of the book written by B.C. Hall a theory is build that classifies irreducible representations for semisimple Lie algebras using highest weights [3]. These are Lie algebras that can be written as the direct sum of simple Lie algbras. It is also proven there that $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra, so a classification can be made. Although these are not induced from representations of $SU(n)$, these do tell something about the irreducible representations of $\mathfrak{su}(n)$. To link these to the matrix Lie group the converse of theorem 2.4.14 is needed. Each Lie algebra homomorphism induces a unique Lie group homomorphism. This convers is proven for matrix Lie group G that are simply connected in chapter 5 of this same book. Simply connectedness is a topological property on a space that means that it is path-connected and that each closed loop in this on it can be contracted to a point. In the appendix of this same book proposition 13.11 proves that indeed $SU(n)$ is simply connected for all $n \geq 2$. This hints in the direction that in higher dimensions a similar classification can be made for the irreducible representations of special unitary group using highest weights.

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