2-dimensional topological quantum field theories

and their categorical equivalence with commutative Frobenius algebras

by

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Introduction

The goal of this bachelor thesis is proving the canonical equivalence between the category of commutative Frobenius algebras $\mathbf{cFA}_{\Bbbk}$ and the category of 2-dimensional topological quantum field theories **2d-TQFTs**. This work is highly inspired by the work of Joachim Kock [6]. The structure of the thesis is based upon it, as is the proof. References to his work should be studied for more techinical details.

This thesis consists of 7 chapters. It will start by looking at the history of topological quantum field theories (TQFTs) and their meaning within physics. Then, in chapters 2 & 3 we will describe cobordisms and Frobenius algebras, the base components of our categories that we are trying to link. In these chapters we will look at their definition and derive the properties that we need. In chapter 4 we will take a look at category theory and define monoidal categories. This theory will be used to define TQFTs in chapter 5, and gives a good way to represent the categories that we are interested in. Then, in chapter 6 we will give the proof of the canonical equivalence. This proof will be quite straight forward after all the preparation that we did. After the proof, we will work out a few examples in which we can use this equivalence. Finally, in chapter 7 we will look at how this theory can be made more abstract to include more categories. The last chapter will primarily show how our result can be expanded to a more abstract result. However, no hard proofs are given.

1 Historical and physics background

TQFTs are part of a far larger subject in physics called quantum field theories (QFTs). QFT is an important part of modern physics. It is used to describe and predict the beheaviour of fundamental particles. It combines theories about quantum mechenics and field theories such as electromagnetism. TQFTs follow from the study of string theory, a theory studied since the late 1960's, wich sees one dimensional strings as the real fundemental particles. The great thing about string theory is that it gives new insights to important physical principles. These are principles such as quantum mechanics, general relativity, gauge theories and supersymmetry [3]. These fundamental string particles can be described with the use of TQFTs. You can think of quantum field theory as an infinite dimensional generalization of ordinary quantum mechanics [8].

Within QFTs, there are a lot of different directions in which research is done. We will focus on the topological variant. The connection between quantum physics and topology is not too strange, as in both not only purely local aspects are important but also their global counterpart [2]. This vague viewpoint is not enough to justify the relation between the two, but gives a simple explanation why one would be interested to look into their connection. If we look closer at the connection between topology and QFT we find a lot of usefull facts. For example, invarients are very important within physics and being a topological invariant ensures that and object is invariant under many other infinitesimal transformations. Thus, it can be enough to prove that an object is topological invariant to see that certain relations hold. This gives us a convenient way to work with invariants in TQFTs [7]. Furthermore, theories such as Donaldson theory and Floers theory rely on invarients. With the use of TQFTs and topological invarients we can couple these theories [7].

There are two ways we try to define quantum field theories. First through quantization of path integrals and second through an axiomatic description. The first method is less elegant than the second, but for high dimensional systems it is sometimes the only way to describe such theory. Also, through this route the theory follows more logically and directly out of physical observations. The approach through axioms gives a strong mathematical foundation to build on, yet it is not directly clear why we chose these axioms [3]. In this thesis we will use the axiomatic approach, as we are most interested in the mathematical properties of TQFTs. Furthermore, we work in two dimensions which lends itself very well for an axiomatic definition as we have the topological classification of surfaces in two dimensions [5].

The axioms for TQFTs were first written down by Atiyah in 1988. He used the same approach on producing these axioms as Graeme Segal did for Conformal field theories (CFTs) [2]. A year later Dijkgraaf noted the categorial equivalence that stands as the center of this thesis [4]. According to Dijkgraaf, 2d-TQFTs do not have a direct physical application, but much of the geometry used in these theories comes back in a different way in conformal field theory. This way 2d-TQFTs give a simple way to get introduced to conformal field theories. Which is an important field of study, because it provides examples of highly nontrivial, nonperturbative interacting field theories that are exactly solvable.

To conclude, TQFTs are a part of QFTs which are used to describe strings within string theory. It combines theories from Floers and Donaldson. 2d-TQFTs themselves do not have direct physical applications, but it gives a good start in understanding CFTs. The axioms for the theory were first written by Atiyah in 1988 and the categorial equivalence with Frobenius algebras was noted a year later by Dijkgraaf.

2 Cobordisms

The first step in our journey of understanding TQFTs is looking at cobordisms. We do this because the definition of a TQFT relies on them. 2-dimensional TQFTs can even be seen as a linear representation of the symmetric monoidal category $(\mathbf{2Cob}, \amalg, \emptyset, T)$ where **2Cob** is the category of 2-dimensional cobordisms. This result will be discussed later in the thesis. We will focus on 2-dimensional cobordism as these are the only ones we need. The exploration of the concept of cobordisms follows the line of the book from Joachim Kock [6].

To define oriented cobordisms, from now on refered to as cobordisms, we first need the notion of in-boundaries and out-boundaries.

Definition: Given Σ a closed submanifold and the boundary of M with codimension 1 and assume they are both oriented. Furthermore, assume that Σ is connected. Then for a point $x \in \Sigma$ let $[v_1, \ldots, v_n]$ be a positive basis for $T_x\Sigma$. Now choose a $w \in T_xM$ such that $[v_1, \ldots, v_n, w]$ is a positive basis for T_xM. Σ is called an in-boundary if the vector w points inwards. Inwards in the sense that when we look through charts at the half-space $H^n = \{x \in \mathbb{R}^n | x_1 \geq 0\}$. Σ is called an out-boundary if the vector w points outwards.

2.1 Definition of a cobordism

Given two $(n-1)$-dimensional oriented closed manifolds Σ_0, Σ_1. These are said to be cobordant if there exists a n-dimensional oriented manifold M, such that $\Sigma_0 \amalg \Sigma_1$ is the boundary of M together with smooth maps

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically onto the in-boundary of M, and Σ_1 maps diffeomorphically onto the out-boundary of M. M is called the cobordism between Σ_0 and Σ_1.

In the 2-dimensional case, we see that $\Sigma_{0,1}$ are both 1-dimensional, and the only 1-dimensional closed manifolds are circles. So $\Sigma_{0,1}$ are both the union of a finite number of circles. A few examples of 2-dimensional cobordism are shown in figure 1. We see that cobordisms can have holes in them or have multiple connected components. It is good to note that the convention is to put the in-boundaries on the left and the out-boundaries on the right. Also, we should realize that we can distinguish the different boundaries as they are ordered.

2.2 The category of cobordisms

Now that we have defined cobordisms our next step should be defining the category of cobordisms **2Cob**. This category will eventually help giving a simpler definition of **2TQFTs**. The category **2Cob** is the category where the objects are closed oriented 1-manifolds, so the finite union of oriented circles, and the arrows are oriented cobordism classes. Note that we need cobordism classes instead of cobordisms themselves. We need them because our composition relation will be defined up to diffeomorphisms and we are only interested in the topological properties of a cobordism.

To be precise two cobordisms are said to be equivalent if there exist an orientation preserving diffeomorphism $\psi : M \to M'$ making the diagram in figure 2.2 commute.

To see that **2Cob** is actually a category, we have to check a couple of things. First, we need to check that there exists an identity arrow for each object and we have to show how to compose two arrows and check associativity.

We define the composition of two cobordisms by gluing. When we have two cobordisms $M : \Sigma_0 \Rightarrow \Sigma_1$ and $M' : \Sigma_1 \Rightarrow \Sigma_2$, we can get a new cobordism $MM' : \Sigma_0 \Rightarrow \Sigma_2$ by gluing M and M' along Σ_1. We read MM' from left to right. This might seems like a strange notation. However, this way it translates easier to figures. It is clear that this operation is associative, because gluing right first and then left is the same as first gluing left and then right. You could even glue them at the same time. Of course, there are more precise definitions needed to make sure that gluing does not result in multiple smooth structures on the manifold. This result is to be expected for people familiar with topology. Therefore, I will refer to the book from Joachim Kock [6] for this proof.

A logical choice for the identity arrow is the cylinder cobordism denoted as $\Sigma \times I$. Topologically nothing happens when you glue a cylinder to a cobordism. You can find a diffeomorphism that 'retracts' the cylinder into the original cobordism. This shows that the new cobordism is equivalent with the old one and thus, that compo-

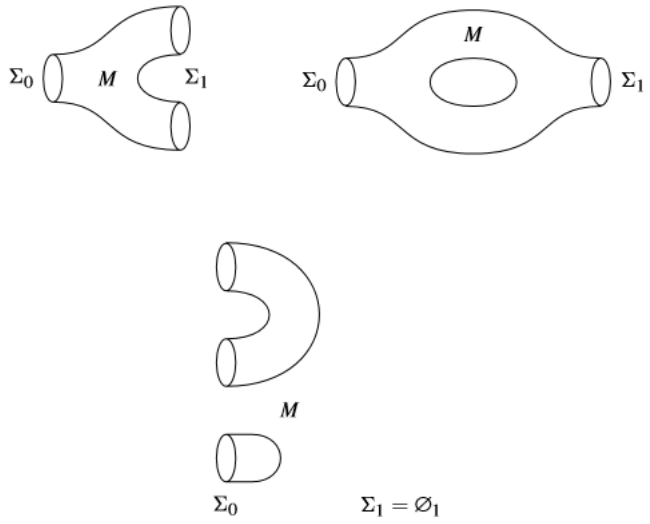

Figure 1: Three examples of 2-dimensional cobordisms.

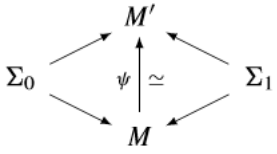

Figure 2: The diagram that should commute if M, M' equivalent.

sition with the cylinder does not change the cobordism. Giving the properties that we expect from the identity arrow.

We like to define one more operation for cobordisms, namely the disjoint union. We do not need it in the categorical sense, but the disjoint union makes it easier to split up cobordisms in simpler pieces. The definition is as following. If we have cobordism $M : \Sigma_0 \Rightarrow \Sigma_1$ and $M' : \Sigma_0' \Rightarrow \Sigma_1'$ then $M \amalg M' : \Sigma_0 \amalg \Sigma_0' \Rightarrow \Sigma_1 \amalg \Sigma_1'$. Where we have the cobordisms parallel to each other. If we look back at our examples in figure 1 we note that the third example can also be seen as the disjoint union of two cobordisms. One that is U-tube and one that is simply a cap.

2.3 Generators for 2Cob

We want to represent TQFTs with **2Cob**,because we can describe **2Cob** with only 6 generators. This follows from the topological classification of surfaces in 2 dimensions. This result states that two connected, compact oriented surfaces without boundary are diffeomorphic if and only if they have the same genus [5]. In the case of our cobordisms we do have boundaries in the form of circles. These circles can move around freely without changing the topology, which gives the more general result that two connected compact oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus, the same number of in-boundaries, and the same number of out-boundaries [6].

We claim that **2Cob** is generated by the following six cobordisms under composition and disjoint union. The cobordisms as shown in figure 3 are named left cap, left pair of pants, cylinder, right pair of pants, right cap and twist respectively. Note that the cylinder is technically not a generator while it is the identity. However, we inculde it to make the decompositions of certain cobordisms clearer.

To prove this claim, we will first start with connected cobordisms. Connected cobordisms are completely described by their number of in-boundary circles m, genus g and out-boundary circles n. If we are given m, n, g, we can construct a cobordism with our generators as follows.

First, attach a left pair of pants to two highest in-boundary circles and attach a cylinder to the others. Then $m - 1$ circles remain. Repeat this process m times. Now you are left with one circle. Attach a right pair of pants to it and then a left pair of pants. Then we still have one circle, but also genus 1. Repeat this process g times. Now we construct from the n out-boundaries with right pair of pants a structure to one out-boundary in the same way we did for the in-boundaries. For the case $m = 5, g = 4$ and $n = 4$ we get the result as shwon in figure 4.

Figure 3: The six generators of **2Cob**

Now one might think that the disconnected part is simple. We could just generate the connected parts as below and take the disjoint union. Well it is a little more complicated than that. If we have for example the cobordism in figure 5 we see that the two connected components are tangled up. This prevents us from using our method. We would like to untangle the parts to get a simple disjoint union of connected cobordisms as in figure 6. We can do this by permuting the circles with the twist generator. We know from group theory that all permutations can be obtained by simple transpositions [1]. So, we first would make a permuting cobordism on the left which would order the circles such that the connected components are together. We then make another permuting cobordism which does the same on the right. In between these cobordisms we are left with a simple disjoint union of untangled connected cobordisms on which we can use our method of constructing cobordisms. For the example below we would find the cobordism as figure 7.

Thus, we see that with the help of permutation cobordisms we can get simple disjoint unions of connected cobordisms. We conclude that **2Cob** is indeed generated by our six cobordisms.

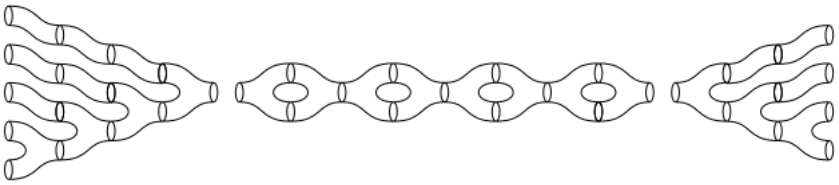

Figure 4: Constructing a cobordism with 5 in-boundary circles, 4 out-boundary circles and genus 4 with the generator cobordisms.

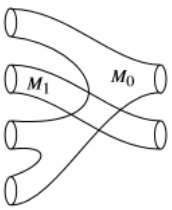

Figure 5: Example where the cobordism is not simply the disjoint union of connected cobordism.

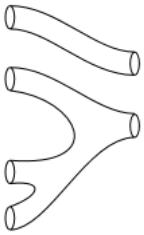

Figure 6: The cobordism of the previous example where the connected parts are untangled.

2.4 Relations on 2Cob

As a result of the classification of surfaces we get relations on **2Cob**. These relations are useful to check whether two cobordisms are indeed equivalent, and it gives us a great way to compare **2Cob** with Frobenius algebras, because they have similar relations. This we will discuss later in the thesis.

The most important relations are shown in figure 8. That these relations hold follows direct from the classification of surfaces. All the relations below have genus zero left and right. Furthermore, they have the same number of in-boundaries and out-boundaries on each side of the equality. The most upper two relations are called *sewing a disc*. Then the next two are called *associativity* and *coassociativity*, followed by the *Frobenius relation*, and the last two are *commutativity* and *cocommutativity*.

We need a few more relations in order to relate every cobordism. We need relations involving the twist to untangle cobordisms. First, we get the relation that moves the twist past the caps as shown in figure 9. The next relation moves a twist past a pair of pants. Here we have four variations namely left or right and over

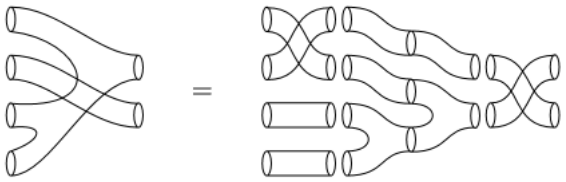

Figure 7: The composition of figure 5 in terms of the generators.

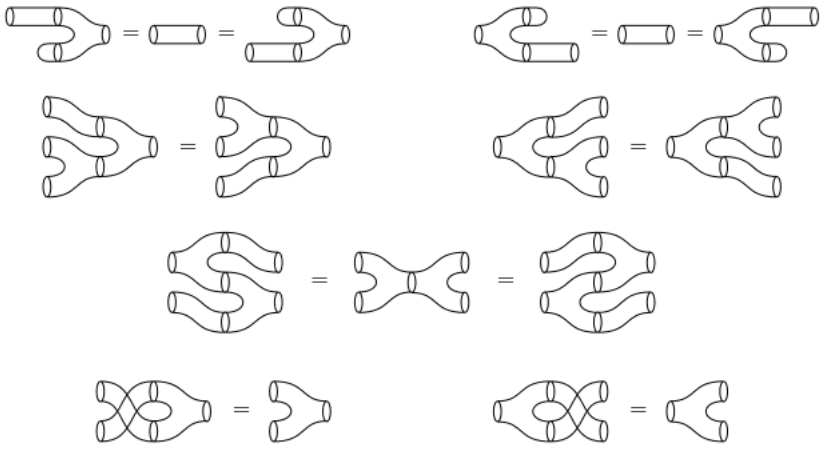

Figure 8: A bunch of standard relations for **2Cob**.

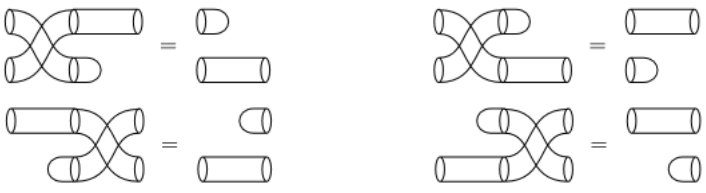

Figure 9: The relation that describes how to move a twist past a cap.

or under as shown in figure 10. We need just one final relation. The relation that moves the twist past another twist. This relation is shown in figure 11.

The proof of the relations below is again simply counting the in-boundaries, the out-boundaries for all of the connected components to see that the relations hold. It is however not straight forward to see that these relations are sufficient to relate any two equivalent cobordisms. This proof relies on being able to relate any cobordism to the standard form as in figure 4 . The proof however is quite messy and consist of moving pieces systematically. It does not contribute to a better understanding of cobordisms. Therefore, we refer to the book from Joachim Kock [6] for this technical proof.

We will work out an example to give a sense of the use of the relations. This example is given as exercise in the book of Joachim Kock. The to be proven is the relation in figure 12. To prove this relation, we first add an extra twist map by using the lower right relation in figure 8. Then, we move the twist past the pair of pants with the relation in figure 10. These two steps are depicted in figure 13. Now, we use the Frobenius relation followed by removing the twist map in the same way we created it earlier. Finally, we apply the Frobenius relation two more times to find the final result. These steps are depicted in figure 14.

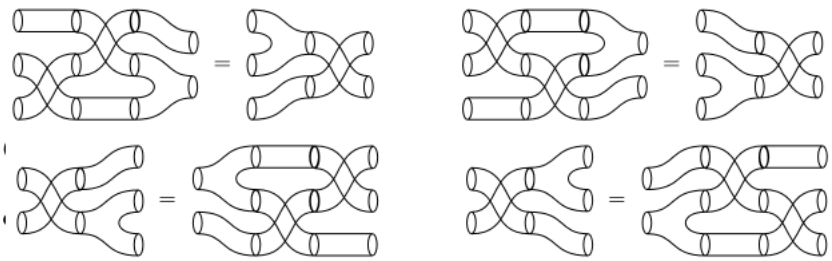

Figure 10: The relation that describes how to move a twist past a pair of pants.

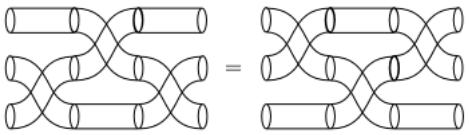

Figure 11: The relation that describes how to move a twist past another twist.

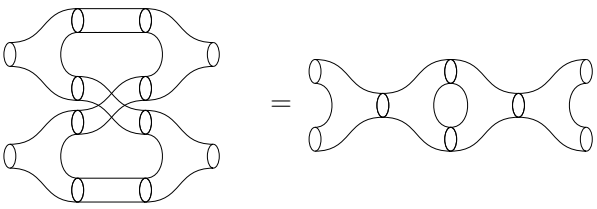

Figure 12: A new relation to be proven as example.

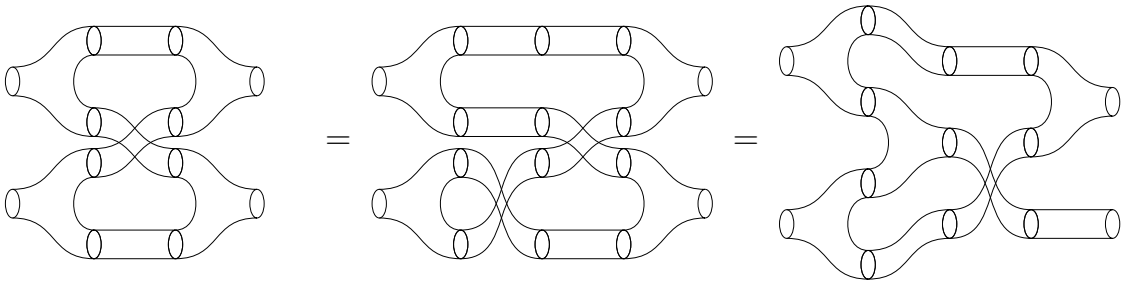

Figure 13: First steps in de proof of the relation in figure 12.

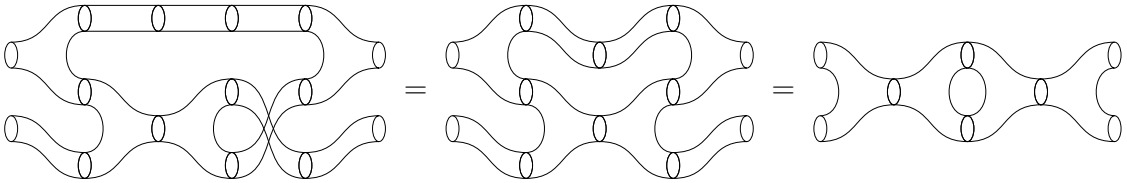

Figure 14: Conclusion of the proof of the relation in figure 12.

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3 Frobenius Algebra

A Frobenius algebra is a vector space that is both an algebra, and a coalgebra in a compatible way. Furthermore, Frobenius algebras can be defined in any monoidal category. This gives us a way to connect them with TQFTs. In this chapter we will discuss the definition of a Frobenius algebra and at a couple of examples. Then we will discuss the relations within the Frobenius algebra, and end by discussing the category of Frobenius algebras.

The reader is expected to know the basic notions of vector spaces, algebras, and modules as they will not be explained in this thesis. We will however, discuss the pairing of vector spaces and in particular nondegenerate pairings. Here and further on we use $\Bbbk$ as the ground field.

Definition: A bilinear pairing of two vector spaces V, W is a linear map $\beta : V \otimes W \to \Bbbk, \quad \beta(v, w) = \langle v|w\rangle$. This pairing is called nondegenerate if V and W have the same dimension and one of the following conditions holds

$$\langle v|w\rangle = 0, \forall v \in V \Rightarrow w = 0$$
$$\langle v|w\rangle = 0, \forall w \in W \Rightarrow v = 0.$$

3.1 Definition of a Frobenius algebra

There are multiple equivalent definitions of a Frobenius algebra. We will discuss two of them. The first through the Frobenius form, and the second through the Frobenius pairing.

Definition 1: A Frobenius Algebra is a pair (A, ϵ), with A an $\Bbbk$-Algebra of finite dimension and ϵ a linear functional $\epsilon : A \to \Bbbk$ with the requirement that $\mathrm{Null}(\epsilon)$ has no nontrivial left ideals. This functional ϵ is called the Frobenius form.

Definition 2: A Frobenius algebra is an $\Bbbk$-algebra A of finite dimension, together with an associative and nondegenerate pairing $\beta : A \otimes A \to \Bbbk$ called the Frobenius pairing.

We note that an algebra A can have multiple Frobenius forms or Frobenius pairings, and thus multiple Frobenius structures on it. The connection between these two definitions is that $\beta = \epsilon \circ \mu$, with μ the multiplication on A acquired from the algebraic structure. Deciding whether the kernel of ϵ contains no nontrivial left ideals is not always an easy task. Therefore, it is nice to have the second definition if a basis is given. Then, we only have to write down the matrix defined by β and see if it is invertible. We also have commutative Frobenius algebras. A Frobenius algebra is commutative if $\beta(a, b) = \beta(b, a)$ or in other notation $\epsilon(a.b) = \epsilon(b.a)$

Two simple examples of Frobenius algebras are the trivial Frobenius algebra and a Frobenius algebra over $\mathbb{R}$ using $\mathbb{C}$ as the field. For the trivial Frobenius algebra let $A = \Bbbk$ and let $\epsilon : \Bbbk \to \Bbbk$ be the identity. The kernel of this Frobenius form only consist of 0, and clearly does not consist any nontrivial left ideals. For the second example let $A = \mathbb{C}, \Bbbk = \mathbb{R}$. Then a Frobenius form is $\epsilon : a + bi \to a$. Here the kernel

consists of all the purely imaginary numbers which does not contain any nontrivial left ideals.

Further on in this chapter, we will work out more interesting examples with the use of coordinates. For more general examples the reader is directed towards the book of Joachim Kock [6].

3.2 Comultiplication on Frobenius algebras

As stated above a Frobenius algebra is vector space that is both an algebra, and a coalgebra. This means that there should be a way to construct a comultiplication, and a counit out of the multiplication, and our Frobenius structure. We will do this with the help of graphical calculus as inspired by the book of Joachim Kock [6].

We start with only 5 maps

$$\begin{aligned} \mu &: A \otimes A \to A \\ \eta &: \Bbbk \to A \\ \epsilon &: A \to \Bbbk \\ \beta &: A \otimes A \to \Bbbk \\ id_A &: A \to A. \end{aligned}$$

These are the multiplication μ, and unit η from the algebraic structure, the Frobenius form ϵ, and Frobenius pairing β from the Frobenius structure, and the identity map id_A. We directly see that ϵ could be a candidate for the counit, because it goes from $\Bbbk$ to A. This fact will be discussed later.

All these maps are of powers of A so, we can represent them with pictures. For example the function given $\mu \circ (\eta \otimes \mathrm{id}_A)$ can be shown as in figure 15. Note that all pictures should be read from left to right.

We represent our maps as in figure 16. Gluing these pictures means that we evaluate the function after each other. These maps have certain properties. Some of these properties can be displayed in terms of our pictures. We start with the associativity, and the unit axiom. These are depicted in figure 17. Not all the properties of our functions can be described with pictures. For example, that the kernel of ϵ does not contain any left ideals can not be described using pictures. However, these properties can be described using the Frobenius pairing, as there properties are equivalent.

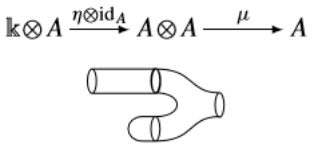

Figure 15: The function $\mu \circ (\eta \otimes \mathrm{id}_A)$ presented graphically.

There are two more properties of the Frobenius pairing that we can put into pictures. First, we have the associativity as pictured in figure 18. We can also find this relation by putting a cap on the associativity relation in figure 17, because we know that $\beta = \epsilon \circ \mu$. Another property of the Frobenius paring that we can put in pictures is the nondegeneracy. It ensures that there exist a copairing γ such that composition gives the identity. This is expressed in figure 19 through pictures.

For convenience, we will also define the three point function. This will make our construction of the comultiplication easier. We define the three point function as in figure 20. Now that we have all the pieces we can start using them to prove new relations and define a comultiplication.

We first will look at the relation in figure 21. This proof is quite simple. The steps will be described to the reader. However one should draw the pictures themselves, if they are not convinced. The proof is as follows. We first use the definition of the three point function. This gives us a snake form as is figure 19. By replacing

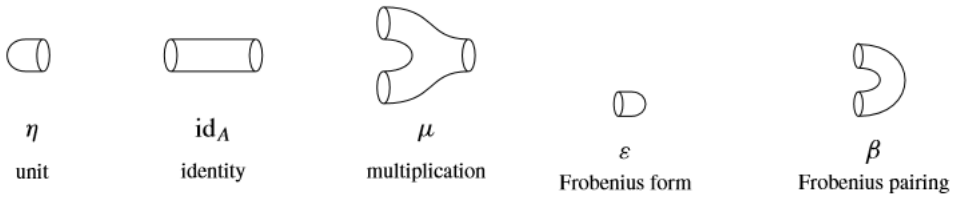

Figure 16: A dictionary of our functions as pictures.

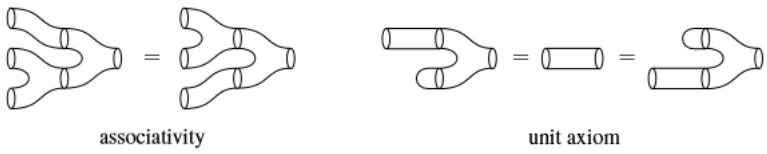

Figure 17: The associativity, and unit axiom properties in picture form.

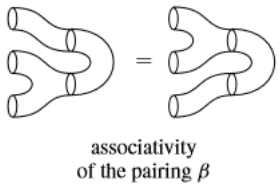

Figure 18: The associativity property of the Frobenius pairing in picture form.

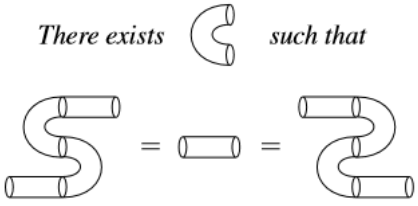

Figure 19: The nondegeneracy property of the Frobenius pairing in picture form.

this with the identity, we directly get the multiplication.

Now we are ready to define the comultiplication as in figure 22. We see that this is indeed a comultiplication as it goes from A to $A \otimes A$. However we need to check that it is coassociative, and the counit axiom holds if we combine it with the counit. We will see that if we choose the Frobenius form ϵ as the counit these relations hold. First, we look at the counit axiom in figure 23 and directly conclude that it holds. For the proof of the coassociativity, we first use the definition of the comultiplication as in figure 22. Then, we use the associativity of the multiplication as in figure 17, and use the definition again to find our result. This can be seen in figure 24.

The multiplication, and the comultiplication together satisfy the *Frobenius condition* as in figure 25. The relation is proven by using the definition of the comultiplication as in figure 22 then, associativity as in figure 17. Then, again the definition.

This Frobenius condition gives us a new defenition for a Frobenius algebra. If we have a vector space A with a multiplication μ, comultiplication δ, unit η, counit ϵ,

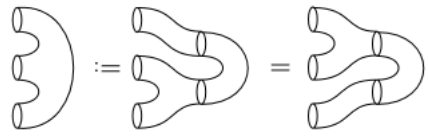

Figure 20: The definition of the three point function.

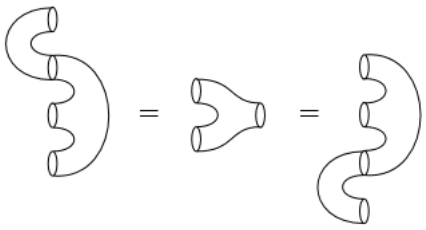

Figure 21: A relation containing the multiplication and the three point function.

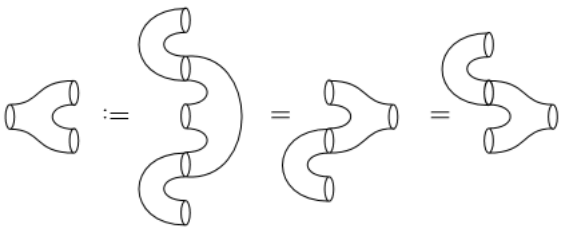

Figure 22: The definition of the comultiplication.

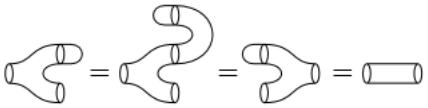

Figure 23: Proof that the counit axiom holds for the comultiplication.

and suppose the Frobenius relation holds. Then, we have a Frobenius algebra with ϵ the Frobenius form.

To support this claim, we have to check 2 things. We have to check that $\beta = \epsilon \circ \mu$ gives a nondegenerate pairing, and that the Frobenius relation implies that μ is associative. This is proven by establishing the snake relation. We do this by putting caps on two parts of our Frobenius relation, and using the (co)unit axiom as in figure 26. For the associativity, we use the fact that we can rewrite the multiplication and comultiplication in terms of each other. Then, by using the Frobenius relation as in figure 27, we see that μ is associative. Now we see that (A, ϵ) is indeed a Frobenius algebra.

3.3 The tensor product of Frobenius algebras

In the next chapter we will discuss Frobenius algebras in the context of monoidal categories. For monoidal categories we will need a way to combine two Frobenius

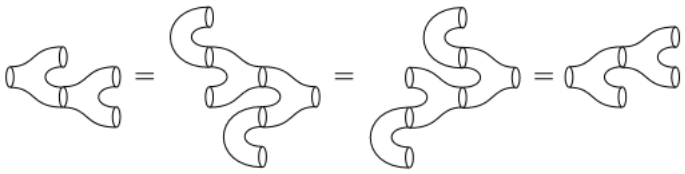

Figure 24: Proof that the coassociativity holds for the comultiplication.

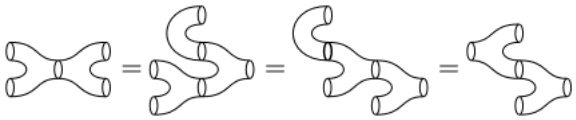

Figure 25: The Frobenius condition and its proof.

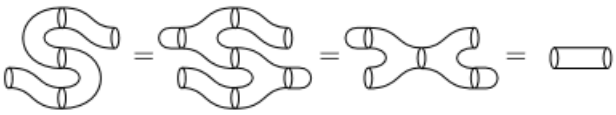

Figure 26: Using the Frobenius relation to prove the nondegeneracy of β.

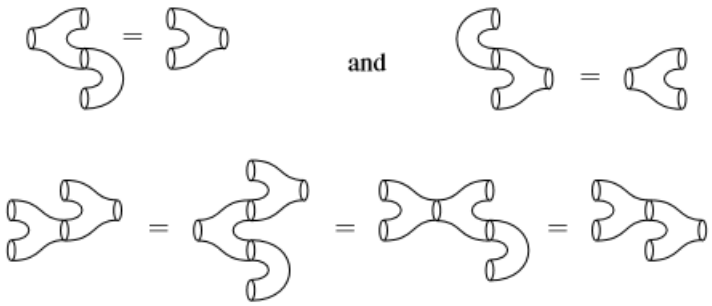

Figure 27: Using the Frobenius relation to prove the associativity of μ.

algebras. This is done by taking the tensor product. Given two Frobenius algebras $(A, \epsilon), (A', \epsilon')$ consider the tensor product $A \otimes A'$. Then, component wise multiplication makes this into an algebra. So,

$$(A \otimes A') \otimes (A \otimes A') \to (A \otimes A')$$
$$(x \otimes x') \otimes (y \otimes y') \to (xy \otimes x'y').$$

In our graphical representation, we can see this as in figure 28. Now we have to check that this map is associative. This amounts to writing the claim out in its graphical form using a couple of twist maps, and filling the associativity in for the parts we know. This is left for the reader to check. The comultiplication is defined in a simular fashion, but by reversing the picture in figure 28. By checking the Frobenius relation we can conclude that the tensor product is a Frobenius algebra. For this case this means the relation as depicted in figure 29. The proof of this relation is also just filling in the Frobenius relation for (A, ϵ), and (A', ϵ') then using twist maps. This proof will be left to the reader as well. The reader could look up the proof in the book of Joachim Kock [6].

3.4 The handle operator

We have the components needed to prove our desired end result. However, there is one more interesting function we would like to discuss. This function is left as an exercise to explore in the book of Joachim Kock [6].

We discuss the handle operator $\omega = \mu \circ \delta$ which is depicted in graphical sense in figure 30. If, we would compare this with cobordisms this is the part where the genus comes from. Furthermore $\omega : A \to A$ so, it can also be seen as an element of the algebra. Through graphical calculus, we can even proof that this element is central. We do this by proofing the relation in figure 31.

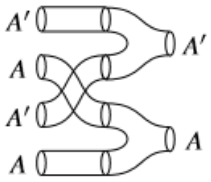

Figure 28: The tensor product multiplication of two algebras in graphical form.

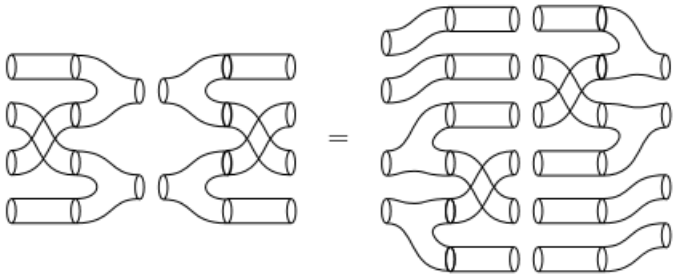

Figure 29: The Frobenius relation for the tensor product.

The proof of this relation follows by using the associativity of the multiplication, followed by using the Frobenius relation, and then again the associativity. This is depicted in figure 32.

3.5 Tensor calculus

Another way to look at the functions we have is through tensors. This gives us a good way to work with coordinates. For example, if we have for A the basis $\{T_0, T_1, \ldots, T_n\}$, and we have the multiplication $\mu : A \otimes A \to A$, then μ is completely determined by its behaviour on the canonical basis of $A \otimes A$ in the form of $T_i \otimes T_j$. So, we can write μ in the following way.

$$T_i \otimes T_j \to T_k \mu_{ij}^k$$

Here we represented μ by the tensor μ_{ij}^k. Note that we make use of the Einstein summation convention. If, we want to couple this with our earlier graphical approach you could picture it as in figure 33 for μ, β.

To describe a Frobenius algebra we will have a bunch of operators. The multiplication μ, the Frobenius pairing β, the coFrobenius pairing γ, and the comultiplication

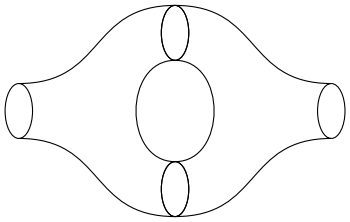

Figure 30: The handle operator in graphical sense.

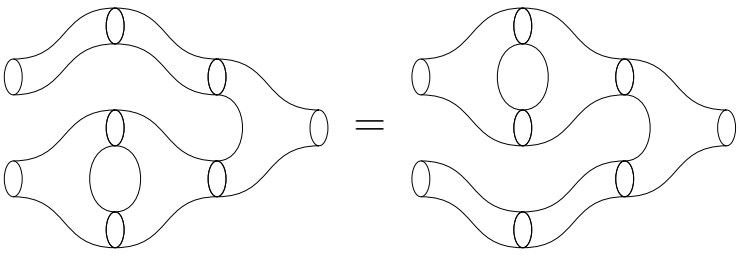

Figure 31: Relation needed to be proven to show that ω as an element is central.

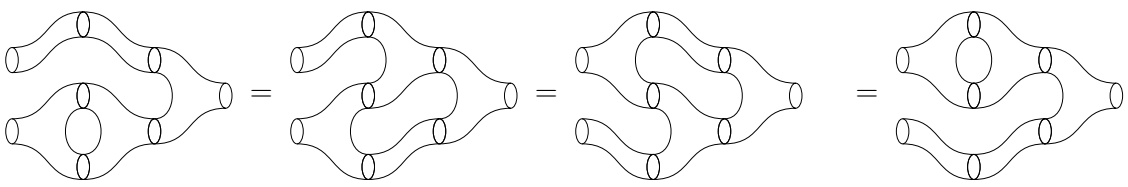

Figure 32: Proof of the relation in figure 31.

δ. Between them, we have the following relations from our graphical proofs

$$\begin{aligned}\beta_{ij}\gamma^{jk} &= \phi_i^k \\ \delta_k^{ij} &= \gamma^{ie}\mu_{ek}^j \\ \mu_{ij}^k &= \beta_{ie}\delta_j^{ek}.\end{aligned}$$

The first relation is the snake relation, and thus ϕ_j^i is the Kronecker delta function (we already used δ). The second, and third relation are the definition of the multiplication and comultiplication with the use of the Frobenius pairing as in figure 22. We can also use these coordinates to calculate the element ω that follows from the handle operator. It is calculated with $\omega = T_i\gamma^{ij}T_j$.

Now that we have a nice way to work with coordinates, we will look at a couple of examples of Frobenius algebras that are given as exercises in the book of Joachim Kock [6].

3.5.1 Example 1

Given the symmetric group algebra $\Bbbk\langle x, y\rangle(x^2-1, y^2-1, xyx-yxy)$, with the basis $\{1, x, y, xy, yx, xyx\}$. We take as Frobenius form the function that sends $1 \to 1$ and the other basis elements to zero.

A priori, it is not clear that the given function is actually a Frobenius form. We will check this by calculating the β tensor, but first we will calculate the multiplication tensor. Because, we have 6 basis elements we have to calculate the 36 multiplications.

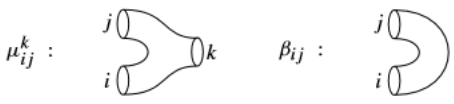

Figure 33: Picture of how the tensors relate to our graphical approach

$$
\begin{aligned}
1 \otimes 1 &= 1 & xy \otimes 1 &= xy \\
1 \otimes x &= x & xy \otimes x &= xyx \\
1 \otimes y &= y & xy \otimes y &= xy^2 = x \\
1 \otimes xy &= xy & xy \otimes xy &= xyxy = yx \\
1 \otimes yx &= yx & xy \otimes yx &= xy^2x = 1 \\
1 \otimes xyx &= xyx & xy \otimes xyx &= xyxyx = y \\
x \otimes 1 &= x & yx \otimes 1 &= yx \\
x \otimes x &= x^2 = 1 & yx \otimes x &= yx^2 = y \\
x \otimes y &= xy & yx \otimes y &= yxy = xyx \\
x \otimes xy &= x^2y = y & yx \otimes xy &= yx^2y = 1 \\
x \otimes yx &= xyx & yx \otimes yx &= yxyx = xy \\
x \otimes xyx &= x^2yx = yx & yx \otimes xyx &= yx^2yx = x \\
y \otimes 1 &= y & xyx \otimes 1 &= xyx \\
y \otimes x &= yx & xyx \otimes x &= xyx^2 = xy \\
y \otimes y &= y^2 = 1 & xyx \otimes y &= xyxy = yx \\
y \otimes xy &= yxy = xyx & xyx \otimes xy &= xyx^2y = x \\
y \otimes yx &= y^2x = x & xyx \otimes yx &= xyxyx = y \\
y \otimes xyx &= yxyx = xy & xyx \otimes xyx &= xyx^2yx = 1
\end{aligned}
$$

We see that every multiplication gives another basis element. Thus, 36 of 216 values of the operator μ_{ij}^k are 1, and the rest are 0. The ones that will become zero are left out, but for the rest we get

$$
\begin{aligned}
&\mu_{11}^1 = \mu_{12}^2 = \mu_{13}^3 = \mu_{14}^4 = \mu_{15}^5 = \mu_{16}^6 = \mu_{21}^2 = \mu_{22}^1 = \mu_{23}^4 = \mu_{24}^3 = \mu_{25}^6 = \mu_{26}^5 = \\
&\mu_{31}^3 = \mu_{32}^5 = \mu_{33}^1 = \mu_{34}^6 = \mu_{35}^2 = \mu_{36}^4 = \mu_{41}^4 = \mu_{42}^6 = \mu_{43}^2 = \mu_{44}^5 = \mu_{45}^1 = \mu_{46}^3 = \\
&\mu_{51}^5 = \mu_{52}^3 = \mu_{53}^6 = \mu_{54}^1 = \mu_{55}^4 = \mu_{56}^2 = \mu_{61}^6 = \mu_{62}^4 = \mu_{63}^5 = \mu_{64}^2 = \mu_{65}^3 = \mu_{66}^1 = 1.
\end{aligned}
$$

The operator β_{ij} can be written as a matrix, and with the multiplication given we directly see that

$$
\beta = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Now that we have β it is easily checked that the given ϵ is indeed a Frobenius form, because this matrix is invertible. The inverse being our operator γ, which here gives the same matrix as β. Now if we calculate δ we find

$$
\begin{aligned}
&\delta_1^{11} = \delta_2^{12} = \delta_3^{13} = \delta_4^{14} = \delta_5^{15} = \delta_6^{16} = \delta_1^{22} = \delta_2^{21} = \delta_3^{24} = \delta_4^{23} = \delta_5^{26} = \delta_6^{25} = \\
&\delta_1^{33} = \delta_2^{35} = \delta_3^{31} = \delta_4^{36} = \delta_5^{32} = \delta_6^{34} = \delta_1^{45} = \delta_2^{43} = \delta_3^{46} = \delta_4^{41} = \delta_5^{44} = \delta_6^{42} = \\
&\delta_1^{54} = \delta_2^{56} = \delta_3^{51} = \delta_4^{55} = \delta_5^{51} = \delta_6^{53} = \delta_1^{66} = \delta_2^{64} = \delta_3^{65} = \delta_4^{62} = \delta_5^{63} = \delta_6^{61} = 1.
\end{aligned}
$$

Let us also calculate the handle element for this example.

$$\begin{aligned}\omega &= T_i\gamma^{ij}T_j \\ &= T_1 \otimes T_1 + T_2 \otimes T_2 + T_3 \otimes T_3 + T_5 \otimes T_4 + T_4 \otimes T_5 + T_6 \otimes T_6 \\ &= 1 + x^2 + y^2 + xyyx + xyyx + xyxxyx \\ &= 1 + 1 + 1 + 1 + 1 + 1 \\ &= 6.\end{aligned}$$

This is clearly a central element, and if we check for the element x by applying the comultiplication and multiplication we see

$$\begin{aligned}\mu \circ \delta(T_2) &= \mu^k_{ij}\delta^{ij}_2 T_k \\ &= (\mu^k_{12} + \mu^k_{21} + \mu^k_{35} + \mu^k_{43} + \mu^k_{56} + \mu^k_{64})T_k \\ &= (T_2 + T_2 + T_2 + T_2 + T_2 + T_2) \\ &= 6x.\end{aligned}$$

This is indeed the same as multiplication with the element ω.

3.5.2 Example 2

For this example, we will again discuss the symmetric group algebra $\Bbbk\langle x, y\rangle(x^2 - 1, y^2 - 1, xyx - yxy)$, with the basis $\{1, x, y, xy, yx, xyx\}$. However, now we take as Frobenius form the map that sends $xyx \to 1$, and the rest to zero. This example shows that one algebra can have multiple Frobenius structures as noted in section 3.1.

The multiplication operator stays the same as it only depends on the algebraic structure. The Frobenius pairing changes. We can directly find the corresponding matrix by looking at our multiplication. We see

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is invertible and its inverse γ is as follows

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, this function ϵ is also a Frobenius form for our algebra. While μ stays the same δ changes if you have another Frobenius pairing. Now we find for δ

$$\delta_1^{16} = \delta_2^{14} = \delta_3^{15} = \delta_4^{12} = \delta_5^{13} = \delta_6^{11} = \delta_1^{24} = \delta_2^{26} = \delta_3^{22} = \delta_4^{25} = \delta_5^{21} = \delta_6^{23} = \\ \delta_1^{35} = \delta_2^{33} = \delta_3^{36} = \delta_4^{31} = \delta_5^{34} = \delta_6^{32} = \delta_1^{43} = \delta_2^{45} = \delta_3^{41} = \delta_4^{46} = \delta_5^{42} = \delta_6^{44} = \\ \delta_1^{52} = \delta_2^{51} = \delta_3^{54} = \delta_4^{53} = \delta_5^{56} = \delta_6^{55} = \delta_1^{61} = \delta_2^{62} = \delta_3^{63} = \delta_4^{64} = \delta_5^{65} = \delta_6^{66} = 1.$$

As last we calculate the element ω

$$\begin{aligned}\omega &= T_i \gamma^{ij} T_j \\ &= T_1 \otimes T_6 + T_2 \otimes T_4 + T_3 \otimes T_5 + T_4 \otimes T_3 + T_5 \otimes T_2 + T_6 \otimes T_1 \\ &= xyx + y + x + x + y + xyx \\ &= 2(xyx + x + y)\end{aligned}$$

It is not directly clear that this is a central element. So, we will check what happens if we multiply from the left and the right with the basis elements (we exclude 1 as this is a central element on its own).

$$\begin{aligned} x \otimes 2(xyx + x + y) &= 2(yx + 1 + xy) \\ 2(xyx + x + y) \otimes x &= 2(xy + 1 + yx) \\ y \otimes 2(xyx + x + y) &= 2(xy + yx + 1) \\ 2(xyx + x + y) \otimes y &= 2(yx + xy + 1) \\ xy \otimes 2(xyx + x + y) &= 2(y + xyx + x) \\ 2(xyx + x + y) \otimes xy &= 2(x + y + xyx) \\ yx \otimes 2(xyx + x + y) &= 2(x + y + xyx) \\ 2(xyx + x + y) \otimes yx &= 2(y + xyx + x) \\ xyx \otimes 2(xyx + x + y) &= 2(1 + xy + yx) \\ 2(xyx + x + y) \otimes xyx &= 2(1 + yx + xy) \end{aligned}$$

This shows that ω is a central element. Let us also check that we get the same result if we apply comultiplication, and multiplication on x

$$\begin{aligned}\mu \circ \delta(T_2) &= \mu_{ij}^k \delta_2^{ij} T_k \\ &= (\mu_{14}^k + \mu_{26}^k + \mu_{33}^k + \mu_{45}^k + \mu_{51}^k + \mu_{62}^k) T_k \\ &= (T_4 + T_5 + T_1 + T_1 + T_5 + T_4) \\ &= 2(xy + yx + 1).\end{aligned}$$

We see that this also coincides with what we calculated above.

3.6 The category of Frobenius algebras

The next step is to use Frobenius algebras to form the category of Frobenius algebras. Here the objects are Frobenius algebras, but we still have to find out what the arrows are. Therefore, we define the Frobenius algebra homomorphism.

Definition: A Frobenius algebra homomorphism $\theta : (A, \epsilon) \to (A', \epsilon')$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. In particular it preserves the Frobenius form such that $\epsilon = \epsilon' \circ \theta$.

Define $\mathbf{FA}_{\Bbbk}$ as the category of Frobenius algebras with as object Frobenius algebras, and as arrows Frobenius algebra homomorphisms. Let $\mathbf{cFA}_{\Bbbk}$ be the full subcategory of all commutative Frobenius algebras. Now we have all the information we need to couple $\mathbf{cFA}_{\Bbbk}$ to TQFTs.

4 Monoidal categories

We need one more tool to complete our result. TQFTs can be defined as functors from 2Cob to $\mathbf{Vect}_{\Bbbk}$. If, we want to put these in a category we will obtain a monoidal functor category. Therefore, we will derive the theory needed to understand these objects. The approach of this chapter was inspired by the book of Joachim Kock [6].

4.1 Monoids

Monoidal categories flow from monoids so it is logical to first consider them. A monoid is a set M with a binary operation (denoted by .) that is associative, and has a neutral element. So, it is a group without the condition that an inverse element has to exist. It is also possible to express this in term of commutative diagrams. This is useful when we make the step to monoidal categories.

The definition of a monoid with diagrams is a set M together with two functions

$$\mu : M \times M \to M, \quad \eta : 1 \to M.$$

such that the three diagrams in figure 34 commute. Between monoids there exist monoid homomorphisms. These are maps $\theta : M \to M'$ that commute with the binary operation. So,

$$(\theta(a)).(\theta(b)) = \theta(a.b) \quad \text{and} \quad \theta(e) = e'.$$

This gives rise to the category **Mon** where the objects are monoids, and the arrows are monoid homomorphisms.

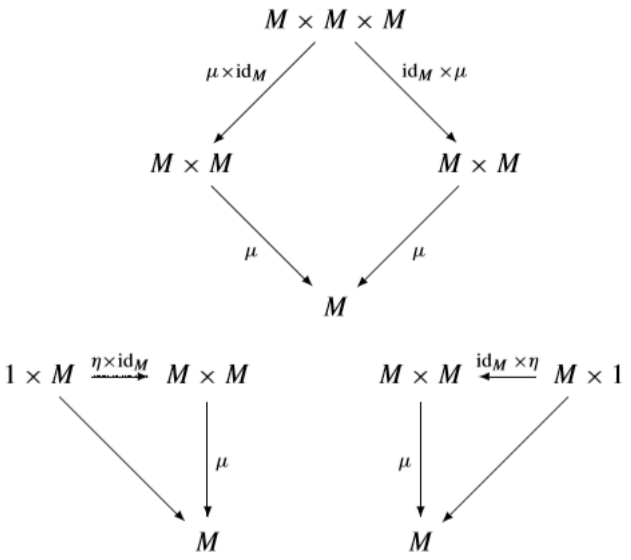

Figure 34: These diagrams should commute if M is a monoid.

4.2 Definition of monoidal categories

There is a simular definition for a monoidal category. Now we have a category V instead of a set M. So, we have a category V together with two functors

$$\mu : V \times V \to V, \quad \eta : 1 \to V,$$

such that the three diagrams in figure 35 commute. It is important that μ is now a functor. It does not only work on the objects of the category, but also on the arrows. We use the following notation

$$\begin{aligned} \mu : V \times V &\to V \\ (X, Y) &\to X \square Y \\ (f, g) &\to f \square g. \end{aligned}$$

In applications $\square$ will be $\times, \otimes, \amalg$ or something similar. Because, μ is a functor it respects compositions, and identity arrows so,

$$\begin{aligned} ff' \square gg' &= (f \square g)(f' \square g') \\ \mathrm{id}_X \square \mathrm{id}_Y &= \mathrm{id}_{X \square Y} \end{aligned}$$

hold as relations. If, we denote the object image of η as I then, we can reformulate the triangular diagrams in the following way.

$$I \square X = X = X \square I, \quad \mathrm{id}_I \square f = f = f \square \mathrm{id}_I$$

for all arrows, and objects in V. Same as with the Frobenius algebra denoting only V gives insufficient information. We need information about μ, η as well. So, we will denote a monoidal category as the triple $(\mathbf{V}, \square, I)$.

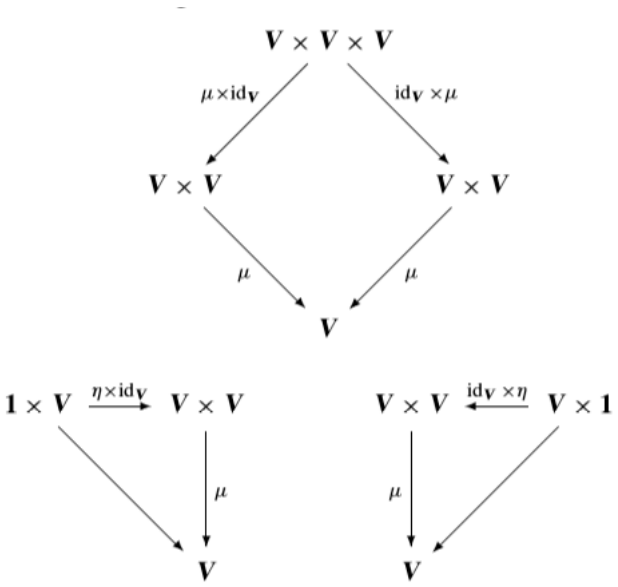

Figure 35: These diagrams should commute is V is a monoidal category.

We had monoid homomorphisms for monoids, and now we will define monoidal functors for monoidal categories. Given $(\mathbf{V}, \Box, I)$ and $(\mathbf{V}', \Box', I')$ a monoidal functor is a functor $F : \mathbf{V} \to \mathbf{V}'$ that commutes with the structure. In terms of commuting diagrams we have figure 36. In terms of objects and arrows we have

$$\begin{aligned} F(X)\Box' F(Y) &= F(X \Box Y) \\ F(I) &= I' \\ F(f)\Box' F(g) &= F(f \Box g). \end{aligned}$$

This gives rise to yet another category. The category of monoidal categories **MonCat**. Here the objects are monoidal categories, and the arrows are monoidal functors.

4.3 Frobenius algebras

We can make the category $\mathbf{FA}_{\Bbbk}$ into a monoidal category by using the tensor product. In the last chapter we have seen that the tensor product between two Frobenius algebras is also a Frobenius algebra, and choose the trivial Frobenius algebra as the neutral object. Then, $(\mathbf{FA}_{\Bbbk}, \otimes, \Bbbk)$ is a monoidal category.

We can do the same for the category $\mathbf{cFA}_{\Bbbk}$. The tensor product of commutative Frobenius algebras is also commutative, and so is the trivial Frobenius algebra. So, $(\mathbf{cFA}_{\Bbbk}, \otimes, \Bbbk)$ is a monoidal category.

4.4 2-dimensional cobordisms

The category **2Cob** can also be made into a monoidal category. For this, we use the disjoint union $\amalg$. We have already seen that the disjoint union of cobordisms are still cobordisms. For the unit element, we can choose the empty set $\emptyset$. This is strictly a cobordism, and for a cobordism M the following relation clearly holds $M \amalg \emptyset = M$. Thus, we see that $(\mathbf{2Cob}, \amalg, \emptyset)$ gives a monoidal category.

4.5 Symmetric monoidal categories

Now that we have monoidal categories, we can look at symmetric monoidal categories. Here we need to add one more piece of structure, namely the twist maps. The definition is as follows.

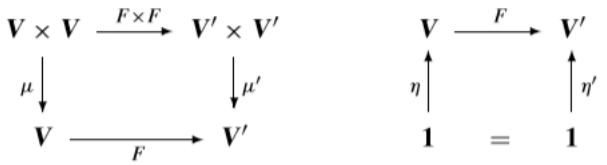

Figure 36: The diagrams that should commute for a monoidal functor.

Definition: A monoidal category $(\mathbf{V}, \square, I)$ is called symmetric, if for each pair of object X, Y there is given a twist map

$$\tau_{X,Y} : X \square Y \to Y \square X$$

such that the following three axioms hold.

1. The maps are natural, this means that for every pair of arrows $f : X \to X'$ and $g : Y \to Y'$ the diagram in figure 37 should commute.
2. For every three object X, Y, Z, the diagrams in figure 38 should commute.
3. We have $\tau_{X,Y}\tau_{Y,X} = id_{X \square Y}$.

This extra structure should also be preserved by our monoidal functors. So, a symmetric monoidal functor from $(\mathbf{V}, \square, I, \tau)$ to $(\mathbf{V}', \square', I', \tau')$ is a monoidal functor $F : \mathbf{V} \to \mathbf{V}'$ with the condition

$$F(\tau_{X,Y}) = \tau'_{F(X),F(Y)}.$$

This gives rise to the category of symmetric monoidal categories **SymMonCat**.

An example for a symmetric monoidal category is $(\mathbf{2Cob}, \amalg, \emptyset, T)$ where the twist map $T_{\Sigma,\Sigma'}$ is the cobordism denoted in figure 39. Another example is the monoidal category of vector spaces $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$. This carries a canonical symmetry σ defined by

$$V \otimes W \to W \otimes V$$
$$v \otimes w \to w \otimes v.$$

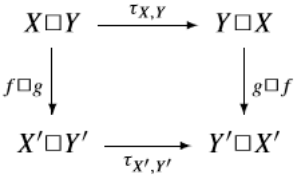

Figure 37: This diagram should commute for the twist map τ to be natural.

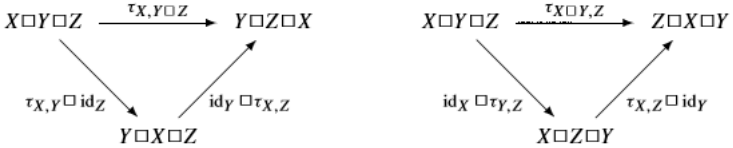

Figure 38: These diagrams should commute for the twist map τ.

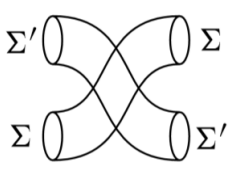

Figure 39: The twist cobordism T.

4.6 Monoidal functor categories

We now want to take a look at a category where we have the monoidal functors as objects, but then we need a new arrow. This will be the monoidal natural transformation. Given two monoidal categories $(\mathbf{V}, \Box, I)$, $(\mathbf{V}', \Box', I')$ and we have two functors F, G, then a natural transformation $u : F \Rightarrow G$ is called monoidal, if for every two object X, Y in $\mathbf{V}$ we have

$$u_X \Box' u_Y = u_{X \Box Y} \quad \text{and} \quad u_I = \mathrm{id}'_I.$$

With the natural transformations as arrows we can define monoidal categories. For two monoidal categories $(\mathbf{V}, \Box, I)$, and $(\mathbf{V}', \Box', I')$, there is a category **MonCat**($\mathbf{V}$,$\mathbf{V}'$) where the objects are monoidal functors between $\mathbf{V}$ and $\mathbf{V}$. Together with monoidal natural transformations as arrows. Similarly, we can do the same for symmetric monoidal categories. We then get the categories **SymMonCat**($\mathbf{V}$,$\mathbf{V}'$).

For this thesis an important class of monoidal functors are the linear representations. By definition, a linear representation of a symmetric monoidal category $(\mathbf{V}, \Box, I, \tau)$ is a symmetric monoidal functor $(\mathbf{V}, \Box, I, \tau) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$ with σ the canonical symmetry on vector spaces. So the set of all linear representations of $\mathbf{V}$ are the objects of a category which we denote

$$\mathbf{Repr}_{\Bbbk}(\mathbf{V}) = \mathbf{SymMonCat}(\mathbf{V}, \mathbf{Vect}_{\Bbbk}).$$

5 Topological quantum field theories

Now that we have done a lot of preparation, we can finally define TQFTs. We will give two definitions. The first is based on the axioms of Atiyah [2]. These axioms can be summarized in a very concise manner using symmetric monoidal categories. This will be used for our second definition.

5.1 Definition of a TQFT

A n-dimensional TQFT is a rule $\mathscr{A}$ which associates to each closed oriented $(n-1)$-manifold Σ an vector space $\mathscr{A}(\Sigma)$, and it associates to every oriented cobordism $M : \Sigma_0 \to \Sigma_1$ a linear map $\mathscr{A}(M)$ from $\mathscr{A}(\Sigma_0)$ to $\mathscr{A}(\Sigma_1)$. The rule $\mathscr{A}$ must satisfy the following 6 axioms [6].

1. Two equivalent cobordism must have the same image:

$$\text{If} \quad M \cong M' \quad \text{then} \quad \mathscr{A}(M) = \mathscr{A}(M').$$

2. The cylinder $\Sigma \times I$, seen as the cobordism from Σ to itself, must be sent to the identity map of $\mathscr{A}(\Sigma)$.

3. Given a decomposition of a cobordism $M = M'\tilde{M}$, then the linear map decomposes as well such that

$$\mathscr{A}(M) = (\mathscr{A}(M')) \circ (\mathscr{A}(\tilde{M})).$$

4. The disjoint union of manifolds or cobordisms is associated with the tensor product of the vector spaces.

$$\Sigma = \Sigma' \amalg \tilde{\Sigma} \Rightarrow \mathscr{A}(\Sigma) = \mathscr{A}(\Sigma') \otimes \mathscr{A}(\tilde{\Sigma})$$
$$M = M' \amalg \tilde{M} \Rightarrow \mathscr{A}(M) = \mathscr{A}(M') \otimes \mathscr{A}(\tilde{M})$$

5. The empty manifold $\Sigma = \emptyset$ is send to the groundfield $\Bbbk$.

6. The TQFT $\mathscr{A}$ should send symmetries of cobordisms to the canonical symmetry of vector spaces.

$$\mathscr{A}(\tau_{M \amalg M'}) = \sigma_{\mathscr{A}(M) \otimes \mathscr{A}(M')}$$

The first two axioms describe the topological nature of the field theory. Axioms 3 and 4 describe the quantum mechanical principle that the state space of two independent systems together is given by the tensor product of their individual state spaces.

5.2 Functorial definition

We have gone through a lot of different categories, and that work now pays of. TQFTs can be defined through symmetric monoidal functors. A n-dimensional topological quantum field theory is a symmetric monoidal functor from $(\mathbf{nCob}, \amalg, \emptyset, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$.

How does this conform with the definition build on axioms one might ask. Well the first axiom implies that we have a well defined map $\mathscr{A}$ from **nCob** to $\mathbf{Vect}_{\Bbbk}$. Then axioms 2 and 3 make sure that $\mathscr{A}$ is a functor. They say it should respect compositions, and identities. Axiom 4 gives the associative function μ and axiom 5 ensures the neutral elements gets send to the neutral element. As last axiom 6 gives that we have a symmetric monoidal functor. Thus, all the information of the axioms is included within this new definition.

Now by using the information from the last chapter, we can conclude

$$\mathbf{nTQFT}_{\Bbbk} = \mathbf{Repr}_{\Bbbk}(\mathbf{nCob}) = \mathbf{SymMonCat}(\mathbf{nCob}, \mathbf{Vect}_{\Bbbk}).$$

In this thesis we are mostly interested in the two dimensional case, so

$$\mathbf{2TQFT}_{\Bbbk} = \mathbf{Repr}_{\Bbbk}(\mathbf{2Cob}) = \mathbf{SymMonCat}(\mathbf{2Cob}, \mathbf{Vect}_{\Bbbk}).$$

6 Frobenius algebras and 2d TQFTs

We have looked at both Frobenius algebras, and 2d-TQFTs. It is time to couple them. We have seen in the last chapter that a TQFT $\mathscr{A}$ is a monoidal functor from **2Cob** to $\mathbf{Vect}_{\Bbbk}$. This functor is completely determined by what it does on the generators of **2Cob**. So, such a functor is completely determined, when we specify what vector space A is the image of **1** the single circle, and what linear map each of the generators is send to. The rest follows from the properties of a symmetric monoidal functor. For example, it implies that **2** the disjoint union of two circles is send to $A \otimes A$. Furthermore, we know that the twist cobordism is send to the regular twist map σ for vector spaces. So, for a TQFT $\mathscr{A}$ we have the information as in figure 40. Here we use the notation that $A^n = A \otimes \ldots \otimes A$ with n A's.

Thus, the TQFT gives us a certain vector space A with a couple of linear maps. These linear maps also have the relations they had as cobordisms. If, we compare the relations for 2 dimensional cobordisms with the graphical calculus relations of the commutative Frobenius algebra, we see that they coincide. In other words, we have a multiplication, comultiplication, unit, and counit together with the knowledge that the Frobenius relation holds. So, we see that the image under $\mathscr{A}$ of **1** is a commutative Frobenius algebra.

Should we have a Frobenius algebra A then this also gives us a TQFT. We can define the TQFT by defining that **1** goes to A and the generators are send to the corresponding functions of the Frobenius algebra. Using the fact that the relations of the cobordisms coincide with the axioms for the Frobenius algebra, we see that this definition makes sense. If, we get a Frobenius algebra from a TQFT, and use this to get a TQFT, we clearly get the TQFT we started with. Thus, we have found a bijective correspondence between 2d-TQFTs and commutative Frobenius algebras.

This correspondence also works for the arrows. The arrows in **2TQFT** are natural transformations. A natural transformation between TQFTs $\mathscr{A}, \mathscr{B}$ consists of linear maps $A^n \to B^n$. These are determined by a single linear map from $A \to B$. The maps $A^n \to B^n$ follow from the first power as a result of the monoidal structure of the functor. The fact that this map is natural gives us that it is compatible with

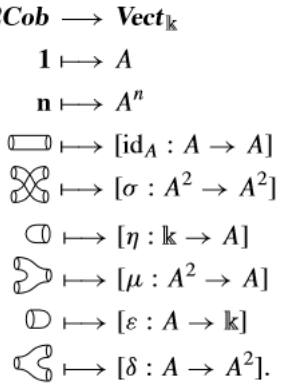

Figure 40: The information that a TQFT gives.

the arrows of **2Cob**. These arrows are built up from the generators. So, we get four commutative diagrams. For the left pair of pants and the left cap we have the diagrams in figure 41. That these diagrams commute is the same as saying that $A \to B$ is a $\Bbbk$-algebra homomorphism. For the right pair of pants and right cap we get the condition that $A \to B$ is a $\Bbbk$-coalgebra homomorphism . Together this gives that $A \to B$ is a Frobenius algebra homomorphism. The diagram of the twist map is by definition commuting because, we have symmetric monoidal functors. Thus, by our correspondence the natural transformations are send to Frobenius algebra homomorphisms the arrows of $\mathbf{cFA}_{\Bbbk}$.

The same arguments can be used in reverse direction to see that a Frobenius algebra homomorphism gives a natural transformation from $A \to B$. This together gives us the result that there is a canonical equivalence of categories

$$\mathbf{2TQFT}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

If, we see $\mathbf{2TQFT}_{\Bbbk}$ as the category of representations of the skeletal version of **2Cob**, then we even have an isomorphism of categories.

Now that we have this result, we will discuss a couple of examples where we can use this equivalence to calculate topological invariants for certain TQFTs. We note that each closed surface is a cobordism $\emptyset \to \emptyset$ so its image is a linear map $\Bbbk \to \Bbbk$, in other words a constant. This constant is the topological invariant of the closed surface. These closed surfaces have only two defining characteristics. The number of connected components and the genus of each of these components. So, there are not to many invariants to be calculated. In higher dimensions this gives far more interesting problems.

6.1 Example 1

We first discuss the TQFT that follows out of the Frobenius algebra defined by the algebra $A = \Bbbk[t]/t^2$ with as basis $\{1, t\}$, and the Frobenius form such that $t \to 1, 1 \to 0$. Now our sidetrack with the handle operator pays off. To get a surface with genus g we only have to apply the handle element g times to the image of η, and finish by using the Frobenius form. This is pictured in figure 42 for genus 3.

So, let us first calculate the handle element. For this we will need β and γ. It is easily seen that

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and thus} \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Figure 41: The commutative diagrams that have to be checked for the left pair of pants and the right cap.

Then, with $\omega = T_i \gamma^{ij} T_j$ we find $\omega = t \otimes 1 + 1 \otimes t = 2t$. We also note that the image of η must be 1, because of the unit-axiom. Then, we see that the sphere gives invariant 0, the torus gives invariant 2 and all the surfaces with higher genus give invariant 0. This is because applying the handle operator more than once gives something with the factor t^2 in it which is equal to zero in our algebra. If, we have a disjoint union of surfaces the invariant is simply adding the invariants of the components. We can summarize the invariants of this TQFT as $2n$ with n the number of tori in our disjoint union.

6.2 Example 2

In this example we discuss the TQFT defined by the Frobenius algebra with as algebra $A = \Bbbk[t]/(t^2 - 1)$, with basis $\{1, t\}$, and Frobenius form the map such that $1 \to 1$ and $t \to 0$. Now we will directly calculate the operators β, γ as well as the handle operator.

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and thus} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

With this we find $\omega = 1 \otimes 1 + t \otimes t = 2$. So this TQFT gives for a connected component the invariant 2^g with g the genus. We see that this TQFT can count the genus if we have one connected component. If, we have more than one connected component this will not always be determinable.

6.3 Example 3

Lastly, we discuss the TQFT defined by the Frobenius algebra with as algebra $A = \Bbbk[t]/(t^2 - 1)$, with basis $\{1, t\}$, but now with the Frobenius form the map such that $1 \to 0$ and $t \to 1$. Then, we find for β, γ and the handle operator.

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and thus} \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

With this we find $\omega = t \otimes 1 + t \otimes 1 = 2t$. This looks a lot like example 1, but there is one key difference. Now only the surfaces with even genus give 0 as their invariant. The surfaces with odd genus give as invariant 2^g. So, this TQFT detects whether a surface has an even or odd genus and if this genus is odd then, we also know what the genus is.

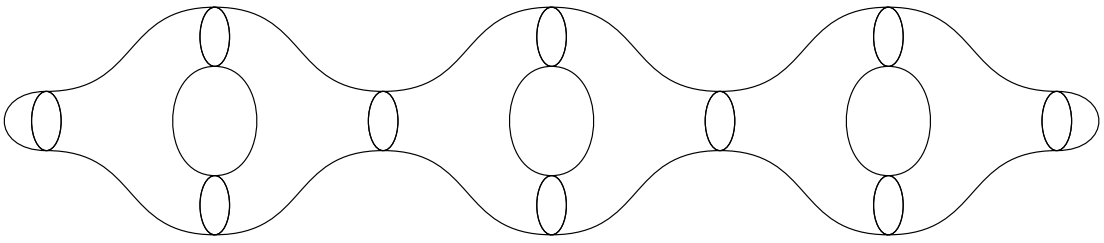

Figure 42: Using the handle operator 3 times to create a closed surface with genus 3.

7 Epilogue

All throughout this thesis, we worked to show that $\mathbf{2TQFT}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}$. However, this is quite a focused result. The question remains if there is a more general result, and the awser is yes. We can use some of the same techniques to prove that

$$\mathbf{SymMonCat(\Phi,V)} \simeq \mathbf{cMon(V)}.$$

Here is Φ the free symmetric monoidal category on a commutative monoid and $\mathbf{V}$ is a symmetric monoidal category. In this epilogue, we will take a glance at the proof of this statement. The necessary objects and ingredients for the proofs are given. However, the details are left out. For these details the reader is refered to the work of Joachim Kock [6] as this chapter was inspired by his work.

7.1 Finite ordinals

We start by looking at totally ordered finite sets. A totally ordered set S is a set with a relation $\leq$ between its elements which, is transitive,reflexive, anti-symmetric and for all two element $x, y \in S$ we have $x \leq y$ or $y \leq x$. Together with order preserving maps these sets form a category **FinOrd**. Since two finite ordered sets are isomorphic if, they have the same number of elements we can describe **FinOrd** by its skeleton. This is the category that consist of one element from each isomorphic class. Say that $\mathbf{n}$ is the ordered set with n elements. Then we can give the skeleton of **FinOrd** as

$$\Delta = \{\mathbf{0,1,2,}\ldots\}.$$

This is called the category of finite ordinals. The arrows in this category are still order preserving maps. We can add a monoidal structure to this category by defining $+$ as the disjoint union. The order in this new set we get by simply saying that for $S + S'$ and any $s \in S, s' \in S'$ that $s \leq s'$.

As we have seen for cobordisms, we can sometimes represent a monoidal category with the help of generators and relations. Therefore, we note that $\mathbf{1}$ is a terminal object in Δ as there is exactly one function $\mathbf{n} \to \mathbf{1}$. We denote

$$\mu^{(n)} : \mathbf{n} \to \mathbf{1}.$$

The monoidal category $(\Delta, +, \mathbf{0})$ is generated by $\mu^{(0)} : \mathbf{0} \to \mathbf{1}$, and $\mu^{(2)} : \mathbf{2} \to \mathbf{1}$. If, we look at these generators graphically together with the identity we get figure 43. Now it is clear that a map $\mathbf{m} \to \mathbf{n}$ is just a sum of n arrows to $\mathbf{1}$ as lines are not permitted to cross over each other otherwise, order will not be preserved. Furthermore, $\mu^{(n)}$ can be constructed by using $\mu^{(2)}$ multiple times, and if an element is not reached it is the image of $\mu^{(0)}$. This gives all the ingredients needed for this proof.

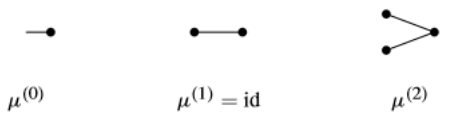

Figure 43: Graphical representation of the generators of Δ.

Thus, we see that Δ is generated by these 2 generators. Now we can have a look at the relations on Δ. There are only two actual relation. We have the relation between $\mu^{(0)}$ and $\mu^{(2)}$, as well as the associativity relation, both shown in figure 44. That these relations are sufficient should be checked, but this does not come as a surprise and will be omitted here.

7.2 Finite cardinals

Next we discuss the symmetric case of ordinals, named cardinals. The category of cardinals Φ has the same objects as for ordinals, but we lose the ordering. So, there is forgetful functor from **FinOrd** $\to$ **Finset** which makes sure that $\Delta \hookrightarrow \Phi$ is an embedding. The monoidal structure also stays the same so we even have an embedding $(\Delta, +, \mathbf{0}) \hookrightarrow (\Phi, +, \mathbf{0})$. We note that $+$ is the coproduct on Φ, and with that we get a unique symmetric structure on $(\Phi, +, \mathbf{0})$.

The generators for Φ are almost the same as for Δ. We only have to add the twist. This can be seen by showing that every arrow in Φ can be factored as a permutation followed by an order preserving map. A quick way to realize this is to look at the graphical point of view by making a vertical cut so far to the right that the lines do not cross each other anymore as in figure 45. Every permutation can be found by having a single twist, and the order preserving maps, we get from the generators from Δ. Thus, the generators of Φ are the ones shown in figure 46.

The relations that we then have are the same we have for Δ, and a few extra relations involving the twist. First of all, we have the relations from the twist on its own as shown in figure 48. We also have that Φ is a symmetric monoidal category therefore the twist map has a naturality. This gives the relations of using the twist

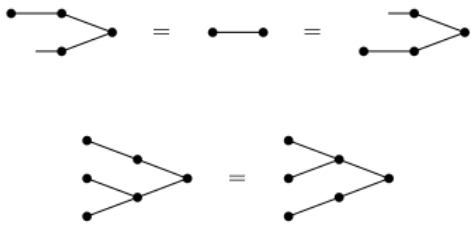

Figure 44: The relations on Δ.

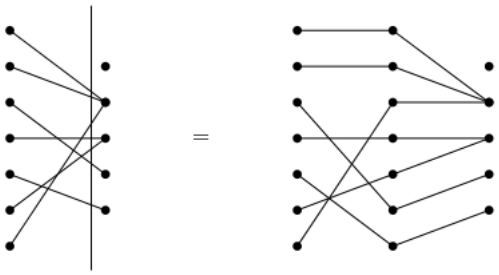

Figure 45: Example of factorizing an arrow of Φ into a permutation and an order preserving map.

parallel with the identity and the commutativity relation. These are shown in figure 49. These are the same relations we had for the twist cobordisms. If we look at the graphical presentation of Φ one might expect that this corresponds with **1Cob**. However, in **1Cob** it is possible to have a situation as in figure 47. This is not the case for Φ.

7.3 Monoids

We have already seen the definition of a monoid in chapter 5, but this definition was based on sets. There is a more general definition for elements from any arbitrary monoidal category. The definition is very similar. Let $(\mathbf{V}, \Box, I)$ be a monoidal category. Then a monoid in $\mathbf{V}$ is an object M together with two arrows

$$\mu : M \Box M \to M, \quad \eta : I \to M$$

such, that the diagrams in figure 50 commute. We then also have monoid homomorphism in $\mathbf{V}$ between two monoids (M, μ, η) and (M', μ', η'). This is a function $\phi : M \to M'$ such that the diagrams in figure 51 commute. This gives rise to a new category. The category of monoids **Mon(V)**. Here the objects are monoids in $\mathbf{V}$ and the arrows are monoid homomorphisms.

This **Mon(V)** is canonically a monoidal category if $(\mathbf{V}, \otimes, I, \tau)$ is a symmetric

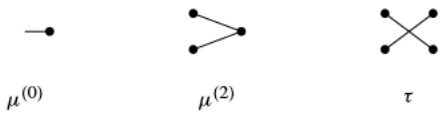

Figure 46: The generators of Φ.

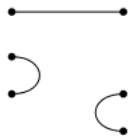

Figure 47: Example why Φ does not coincide with **1Cob**.

Figure 48: The relations on Φ involving only the twist.

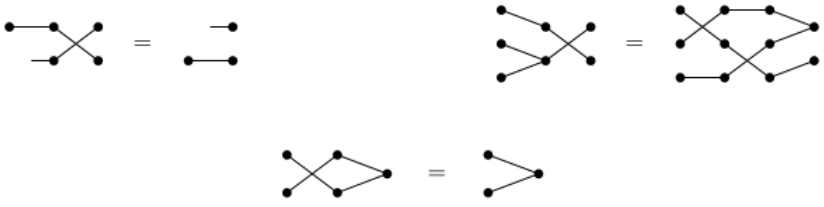

Figure 49: The relations on Φ that follow out of the naturality of the twist.

monoidal category. We can see this by the following construction of the multiplication. If, we have (M, μ, η), and (M', μ', η') two monoids in $\mathbf{V}$. Then we define $(M \square M', \mu \square \mu', \eta \square \eta')$ by setting $M \square M' = M \otimes M'$, $\eta \square \eta' = \eta \otimes \eta'$, and define the new multiplication $\mu \square \mu'$ as in figure 52. In this definition we use the twist map. That is why we need a symmetric monoidal category. If, we put our new multiplication in a drawing we get figure 53. This drawing is exactly what we had when we proved that the tensor product of two algebras was still an algebra. The proof that we used their also applies here.

If, we have such a monoidal category $\mathbf{Mon(V)}$, we could ask ourselves what it means for a monoid M to be commutative. Normally we would say that we can

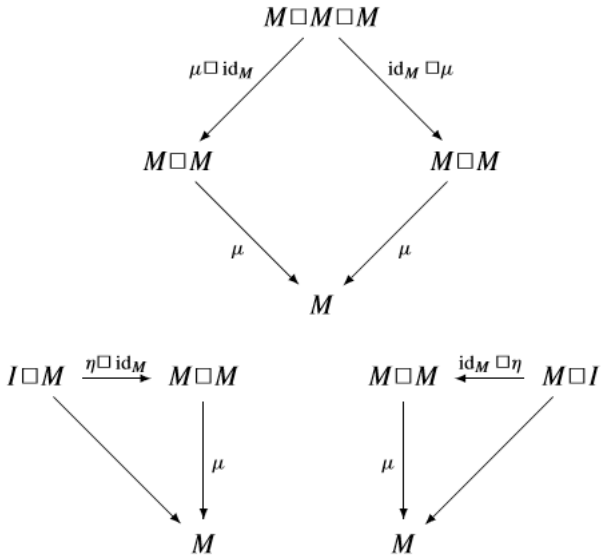

Figure 50: The diagrams that need to commute such that M is a monoid.

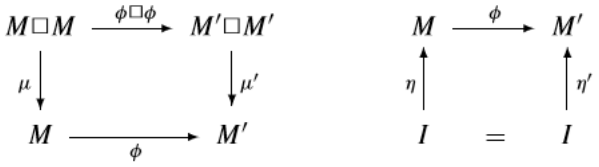

Figure 51: The diagrams that need to commute such that ϕ is a monoid homomorphism.

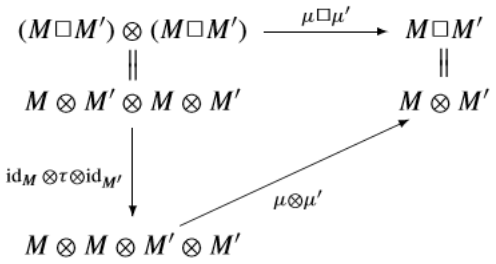

Figure 52: The defenition of $\mu \square \mu'$.

switch elements, but we do not know whether M contains any elements at all as it could be anything. So, we say that a monoid is commutative if the multiplication is compatible with the twist map. This means that the diagram in figure 54 should commute.

7.4 $\Bbbk$-algebras

With this theory we can show that

$$\mathbf{Mon(Vect}_{\Bbbk}\mathbf{)=Alg}_{\Bbbk}.$$

This will be crucial in getting our final result. To get this result we first claim that a monoid in the category of abelian groups $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is a ring. Note that a monoid in $\mathbf{Ab}$ already has the group structure of a ring. The multiplication follows from the map μ that we get from the monoidal structure. One can check that this satifies all the conditions for a ring. The reader could read this proof back in the book of Joachim Kock [6]. The monoid homomorphisms are with this identification even ring homomorphisms. Thus, we find

$$\mathbf{Mon(Ab)=Ring}.$$

Now let R be a commutative ring, and thus a commutative monoid in $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z}, \tau)$. Then, a R-module is an abelian group M with an R-action. If, we connect this internally to a monoidal category, we can find that these R-modules form a monoidal category $(\mathbf{Mod}_R, \otimes_R, R)$. If, we use the fact that abelian groups can be regarded as $\mathbb{Z}$-modules, and rings as $\mathbb{Z}$-algebras, we can check that an R-algebra is a monoid in $(\mathbf{Mod}_R, \otimes_R, R)$. Now if we choose R as our fixed field $\Bbbk$. Then, we see that the modules are $\Bbbk$-vector spaces. With that we see that a monoid in $(\mathbf{Vect}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ is a $\Bbbk$-algebra an thus,

$$\mathbf{Mon(Vect}_{\Bbbk}\mathbf{)=Alg}_{\Bbbk}.$$

If, we use the canonical symmetry τ on $\mathbf{Vect}_{\Bbbk}$ then we have

$$\mathbf{cMon}(\mathbf{Vect}_{\Bbbk}) = \mathbf{cAlg}_{\Bbbk}.$$

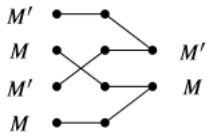

Figure 53: The definition of $\mu\square\mu'$ in graphical form.

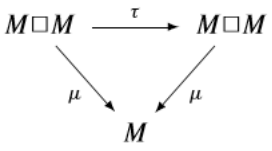

Figure 54: The diagram that should commute when M is a commutative monoid in $\mathbf{V}$.

This notion could also be broadened by looking at graded vector spaces and algebras. Then, we walk the same path to see that the monoids in the category of graded vector spaces give graded algebras. The results regarding commutativity are not direct however. If, we use the canonical symmetry then, we see that certain graded algebras are not commutative. We should therefore also take a graded symmetry such as the Koszul sign change κ. If, we do that we see that we get the category of graded-commutative algebras.

$$\mathbf{cMon}(\mathbf{grVect}_{\Bbbk}, \kappa) = \mathbf{gr\text{-}cAlg}_{\Bbbk}$$

This proof is anything but complete, yet the intention of this chapter is not to give detailed proof. The intention is to give some intuition in how our main result can be broadened. For a more precise proof the reader is directed to the book of Joachim Kock [6].

7.5 Monoidal categories generated from monoids

If, we have a monoid (M, μ, η) in a monoidal category $(\mathbf{V}, \square, I)$, then this monoid generates a subcategory of $\mathbf{V}$ denoted by $\langle M \rangle$. The elements of this subcategory are $I, M, M\square M, M\square M\square M, \ldots$. The arrows in this category follow from the monoidal operators μ, η on M itself.

7.6 Monoids in Δ

In the beginning of this chapter we introduced Δ and Φ. Now we will see how they fit in the picture. We will discuss what the monoids are in $(\Delta, +, \mathbf{0})$. Remember that the elements in Δ are ordered sets $\mathbf{n}$ and the arrows are ordered maps. We see that $\mathbf{n} = 0$ gives the trivial monoid. We see that $\mathbf{n} = 1$ also gives a monoid, by using the unique maps $\mu = \mu^{(2)}$, and $\eta = \mu^{(0)}$ as in 8.1 to get $(\mathbf{1}, \mu, \eta)$. However, there are no more other monoids. If, we would have $\mathbf{n} \geq 2$ then, we would need something as in figure 55 for the unit axioms. We directly see that these maps do not exist because, the lines cross and are therefore not order preserving maps. So, we see that $(\Delta, +, \mathbf{0})$ contains one nontrivial monoid, namely $(\mathbf{1}, \mu, \eta)$.

We that the subcategory generated by $(\mathbf{1}, \mu, \eta)$ is Δ itself, and because this is the case Δ is also called a free monoidal category. This means that any monoid in any monoidal category $\mathbf{V}$ is the image of $\mathbf{1}$ under a unique monoidal functor $\Delta \to \mathbf{V}$. Thus, we have for every monoid of $\mathbf{V}$ a functor from Δ to $\mathbf{V}$ and between these functors we have natural transformations. This leads to a new canonical equivalence of categories

$$\mathbf{MonCat}(\Delta, \mathbf{V}) \simeq \mathbf{Mon}(\mathbf{V}).$$

Figure 55: The multiplication map μ for $\mathbf{2}$.

Here is **MonCat**(Δ,**V**) the category of functors from Δ to **V** with as arrows natural transformations and **Mon(V)** the category of monoids in **V** with monoid homomorphisms as arrows. With the equivalence we determined earlier we see

$$\mathbf{MonCat}(\Delta, \mathbf{Vect}_{\Bbbk}) \simeq \mathbf{Mon}(\mathbf{Vect}_{\Bbbk}) \simeq \mathbf{Alg}_{\Bbbk}.$$

7.7 Monoids in Φ

If, we look at all the monoids in Φ then, we see that there are a lot more then on Δ. The maps that were not permitted for Δ are fine for Φ. The reason for this is that + is the coproduct in Φ. This guarantees that every object has a unique monoid structure [6]. However, we can still speak of a free monoidal category. Every object generates a monoidal subcategory, but these are all isomorphic. Thus, Φ is generated by $(\mathbf{1}, +, 0)$, and the other monoids do not give any new information. With this the symmetric monoidal category is still free. This gives us the following canonical equivalence of categories

$$\mathbf{SymMonCat}(\Phi,\mathbf{V}) \simeq \mathbf{cMon}(\mathbf{V}).$$

The equivalence stated in the beginning of this chapter. Note that **V** is here a symmetric monoidal category. This equivalence follows the same as above, but now the symmetric version. Now if we chose $\mathbf{V} = \mathbf{Vect}_{\Bbbk}$ then we get the following equivalence

$$\mathbf{Repr}_{\Bbbk}(\Phi) = \mathbf{SymMonCat}(\Phi, \mathbf{Vect}_{\Bbbk}) \simeq \mathbf{cMon}(\mathbf{Vect}_{\Bbbk}) = \mathbf{cAlg}_{\Bbbk}.$$

7.8 Connection with TQFTs

In our endeavor with TQFTs, we have the symmetric monoidal category $(\mathbf{2Cob}, \amalg, \emptyset, T)$. Here the circle is a commutative monoid. We can see it as the image of the symmetric monoidal functor $(\Phi, +, 0, \tau) \to (\mathbf{2Cob}, \amalg, \emptyset, T)$. Where we define that **1** goes to the circle. The rest of the functor follows from the condition that it is symmetric and monoidal.

We proved that a Frobenius algebra corresponds to a functor $\mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$. If, we combine it with the functor $\Phi \to \mathbf{2Cob}$ from above we get a functor $\Phi \to \mathbf{Vect}_{\Bbbk}$ which corresponds to an commutative algebra. This is the same algebra we would get if we would forget the Frobenius structure on the algebra we had. So, we find our result of TQFTs by giving more structure to Φ with a symmetric monoidal functor. This process can be reversed by looking at the forgetful functor $\mathbf{cFA}_{\Bbbk} \to \mathbf{cAlg}_{\Bbbk}$.

This concludes our journey into TQFTs, and their categorical description.

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