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Bachelor Thesis

Quantum dynamics of skyrmions and vortices in antiferromagnets

Author:

J.S. Harms
Natuur- en Sterrenkunde

Supervisor:

Prof. dr. R.A. Duine
Institute for Theoretical Physics

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Abstract

Skyrmions are magnetic textures, that behave like particles. They exist as excitations within two-dimensional magnetic insulators and are promising candidates for future data storage. There has been a strong rise in interest in skyrmions in recent years, which is encouraged by several experimental observations of skyrmions in various magnetic thin-films. In this thesis we study the quantum mechanical propagation of skyrmions in thin film antiferromagnets, using the finite temperature path integral formalism for spins. Using field theoretical tools like the Faddeev-Popov technique, we treat the coordinate of a skyrmion as a dynamical variable and derive an effective action for the skyrmion position. We find that the spin wave fluctuations around a skyrmion give rise to damping of the skyrmion's motion and also gives rise to effective mass contributions.

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Chapter 1

Introduction

From secondary school on we learned from sir Isaac Newton that matter possesses mass. One can distinct between two different types of mass. *Inertial mass* gives the amount of resistance when an object is accelerated. The amount of gravitational force that is exerted on an object is proportional to its *gravitational mass.* Newton also came up with three postulates which *inertially* massive objects should obey [1]. He additionally stated that two *gravitationally* massive bodies feel an attractive force towards one another. The inertial and gravitational mass turn out te be equal in every experiment that is done so this is nowadays taken as an empirical fact. Using his three postulates of motion, Newton was able to explain a lot of open physical problems. The three laws of Kepler could for example be explained by these postulates and the orbit of a many planets could calculated. The observed orbit of Mercury turned out to be different then was calculated using what is now called Classical Mechanics. The explanation came in 1915 when Albert Einstein published his theory of general relativity [2], which gave a geometric view on gravity and generalized Newton's law of gravitation. One of the starting points for this theory was to assume that the correspondence between the inertial and gravitational mass is not a coincidence and must hold. General relativity could explain the measurements done on the perihelion of Mercury. It also states that the mass of matter increases as it's energy rises. Using general relativity, new physical phenomena appeared theoretically, e.g. black holes and bending of light which is referred to as "gravitational lensing".

Still another question remained unanswered; what causes an object to have mass? In 1960 Higgs, Brout and Englert predicted that there is a field (now called Higgs field) which causes matter to have *inertial* mass. The quantum excitations of this field are now known as "Higgs bosons".

In a magnet textures can appear that behave like particles. These magnetic textures are phenomena that move in a collective way through a magnet and may thus be described as a single object. One could now ask themselves the question, can these objects also have a *inertial* mass? Some of them turn out to have an *inertial* mass. Like matter obtains its mass through interaction with the Higgs field, these textures obtain their mass through interaction with the underlying structure of the magnet. A Bloch wall (see Fig. 1.1) is an example of such a collective phenomena that obtains an *inertial* mass through interaction with the underlying structure of the magnet.

The focus of this thesis is on another particle-like magnetic texture, namely skyrmions [4]. There has been a strong rise in interest for skyrmions in recent years, which is stimulated by several experimental observations of skyrmions in various thin-film magnets [5–8]. Magnetic skyrmions are promising candidates for storing information, due to the fact that they are topologically stable in the sense that no continuous local deformation can destroy a skyrmion and they can be moved using relatively low currents.

In previous work the motion and tunnelling effects of Bloch walls [3] and skyrmions [9] within thin-film ferromagnetic insulators are studied using a quantum treatment. In this thesis we will study the inertial mass and the motion of skyrmions within antiferromagnetic insulators, with emphasis on the inertial mass of the skyrmions.

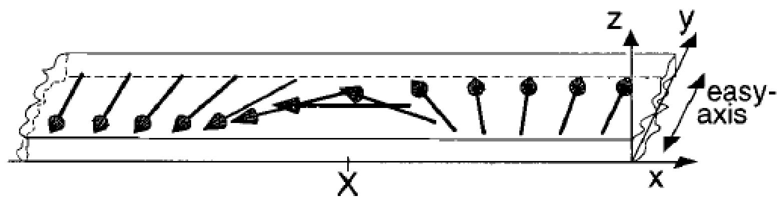

Figure 1.1: Bloch wall configuration in an one-dimensional magnet. A Bloch wall is a particle like texture in an one-dimensional magnet. On one side of the magnet all the spins are point upwards and on the other side all spins are pointing downwards. There is a finite domain where the spins flip from one configuration to another. This domain moves collectively and may thus be described as single object. *Source: Ref. [3].*

The goal of this thesis is to provide a quantum mechanical treatment of skyrmions in thin-film antiferromagnetic insulators. We restrict ourself to spin degrees of freedom only, interactions of spins with e.g. phonons are assumed to be negligibly small (due to low temperatures). By using methods of quantum field theory and in particular the Fadeev-Popov technique for collective coordinates, we are able to treat the skyrmions quantum mechanically.

The main findings of this thesis are (in the zero temperature limit $\beta \to \infty$):

- An antiferromagnetic skyrmion does not experience a Magnus force[1], which is in agreement with Ref. [10]. It will have a mass and for small magnetic fields, the change of mass caused by the magnetic field will be proportional to the square of the magnetic field.
- For magnetic fields above the spin flop field we find vortex configurations (see Fig. 3.7) which will obtain a mass correction that is proportional to the square of the magnetic field and will feel a Magnus force which is proportional to the magnetic field.

The remainder of this thesis is organized as follows. In Chapter 2 we start with a discussion of skyrmions in ferromagnets. In Section 2.1 we discuss a general method to calculate the ferromagnetic-skyrmion profile. Section 2.2 will give a derivation for the Magnus force on a skyrmion. Although this calculation is not really important for later chapters, a basic understanding of the Magnus force is needed and it is recommended to look at Eq. (2.12). Chapter 3 will be organized as follows. In Section 3.1 we describe the dynamics of antiferromagnets and give an action which is only dependent on the Néel vector in Eq. (3.1). We do so by integrating out the magnetization, assuming large wavelengths for both the Néel vector and the magnetization. In Section 3.2 we determine the antiferromagnetic skyrmion and vortex profiles using methods similar to the ones used for ferromagnetic skyrmions in Section 2.1. Using a quantum mechanical treatment, we will calculate the interaction between skyrmions and spin waves in Chapter 4. In Section 4.1 we will expand the action describing skyrmion dynamics in the presence of spin waves up to second order in spin wave fluctuations around the skyrmion configuration. We distinguish three types of actions, the classical one (no spin wave interaction), static interaction and dynamic interaction. In Section 4.2 we use the preceding to calculate the effect of spin wave interactions on the dynamics of the antiferromagnetic skyrmion, we will do so in the absence of a magnetic field and in the presence of a small magnetic field. In Appendix A we will discuss the mathematical reason why skyrmions and Bloch-walls are stable and thus why it makes sense to describe them as objects. We will conclude with a discussion and an outlook.

[1]For intuition one could for now think of the Magnus force as a Lorentz-force, though they are different. While the Lorentz-force will act perpendicular to the velocity of a particle, the Magnus force will give rise to motion perpendicular to the direction of the applied force (see Section 2.2).

Chapter 2

Skyrmions in ferromagnets

A ferromagnet is characterized by an exchange interaction which minimized by alignment of neighboring spins (see Fig. 2.1). In this chapter we derive the profile and the classical[1] *description of the motion of skyrmions in thin-film ferromagnetic insulators. In Section 2.1 we will determine the profile of skyrmions in thin-film ferromagnetic insulators. Section 2.2 will be focused on determining the dynamics of classical ferromagnetic skyrmions, meaning we don't take interactions with e.g. spin waves into account.*

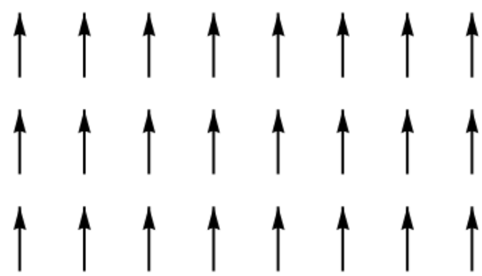

Figure 2.1: Graphical example of ferromagnetic ordering.

2.1 Single skyrmion profile in a ferromagnet

In this section we obtain the stationary single skyrmion profile within a ferromagnet, starting from the Hamiltonian in Eq. (2.1)*. We use classical methods to determine the dependence of the skyrmion profile on given parameters. We assume translational invariance in the $\hat{z}$-direction and use variational principles to obtain a configuration whose winding number equals one and which locally minimizes the energy of the system. This section is largely based on Ref. [11].*

We start by writing out the energy in terms of the functional

$$E[\mathbf{\Omega}] = \int \Bigg\{ -\frac{J}{2}\mathbf{\Omega}\cdot\nabla^2\mathbf{\Omega} + \frac{D}{2}\left(\hat{y}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial x}\right) - \hat{x}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial y}\right)\right) + K(1-\Omega_z^2) + \mu_0 HM(1-\Omega_z) - \mu_0 M\mathbf{\Omega}\cdot\mathbf{H_d}\Bigg\}\mathrm{d}\mathbf{x}. \tag{2.1}$$

[1]Neglecting interactions with e.g. spin waves is meant with classical.

In the above formula $\mathbf{\Omega}(\mathbf{x})$ is the direction of the magnetization at the point $\mathbf{x} \in \mathbb{R}^3$. Furthermore $\mathbf{H}$ is the external magnetic field which is pointed in the $\hat{z}$-direction, J stands for the spin stiffness and D is the Dzyaloshinskii-Moriya interaction constant. Furthermore M is the magnetization constant s.t. $\mathbf{M}(\mathbf{x}) = M\mathbf{\Omega}(\mathbf{x})$ and $\mathbf{H_d}$ is the demagnetization field. Because we assume translational invariance in the z-direction we get $\mathbf{x} \in \mathbb{R}^2$.

We will be neglecting anisotropy and the demagnetization field of the system. This is because these terms have minor contributions to our energy and we first want to understand what is happening in the most simple case. So we end up with the following energy functional

$$E[\mathbf{\Omega}] = \int \Bigg\{ -\frac{J}{2}\mathbf{\Omega}\cdot\nabla^2\mathbf{\Omega} + \frac{D}{2}\left(\hat{y}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial x}\right) - \hat{x}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial y}\right)\right) + \mu_0 HM(1-\Omega_z)\Bigg\}\mathrm{d}\mathbf{x}. \tag{2.2}$$

Assuming we have rotational- and translational symmetry around the $\hat{z}$-axis, it is easier to write Eq. (2.2) into cylindrical coordinates $\mathbf{x} = (\rho, \phi, \theta)$,

$$\mathbf{\Omega}(\mathbf{x}) = sin[\theta]cos[\phi_0]\,\hat{\rho} + sin[\theta]sin[\phi_0]\,\hat{\phi} + cos[\theta]\,\hat{z}. \tag{2.3}$$

Note that ϕ_0 gives the orientation or twist of the skyrmion. For ($\phi_0 = 0$) we get a hedgehog skyrmion and for ($\phi_0 = \frac{\pi}{2}$) we get a vortex skyrmion (see Fig. 2.2).

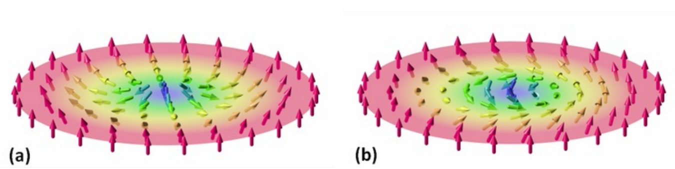

Figure 2.2: Picture of two skyrmions, (a) is a hedgehog skyrmion while (b) is a vortex skyrmion. (*source:wikipedia, authors: Karin Everschor-Sitte and Matthias Sitte*).

The reader should also be aware that in principle ϕ_0 could be dependent on ρ, but we take it to be constant. The reason is that a skyrmion is defined as the lowest energy state which has winding number one, while the winding number of a skyrmion is only depended on the way θ depends ρ. By taking ϕ_0 constant we don't get additional divergences which enlarge the energy. In a nutshell, for a skyrmion profile ϕ_0 is constant.
Now we rewrite Eq. (2.2) in terms of cylindrical coordinates, see Appendix B.1. We assume a thin-film ferromagnet with thickness t. Since the energy in Eq. (2.2) is rotational and translation invariant with respect to the $\hat{z}$-axis we can integrate out ϕ and z. We end up with the following functional

$$\begin{aligned} E[\theta] =& 2\pi t \int \Bigg\{ \frac{J}{2}\left(\left(\frac{\partial\theta(\rho)}{\partial\rho}\right)^2 + \frac{\sin^2\theta(\rho)}{\rho^2}\right) \\ &+ \frac{D}{2}\cos\phi_0\left(\frac{\partial\theta(\rho)}{\partial\rho} + \frac{\sin\theta(\rho)\cos\theta(\rho)}{\rho}\right) + \mu_0 HM(1-\cos\theta(\rho))\Bigg\}\rho\,\mathrm{d}\rho. \end{aligned} \tag{2.4}$$

We want to find functions θ and ϕ_0 which minimizes the energy (2.4). From Eq. (2.4) we are able to see that $\phi_0 \in \{0, \pi\}$. This is because of the fact that $\cos\phi_0$ attains its maximum/minimum at $\phi_0 \in \{0, \pi\}$. The choice of ϕ_0 is depended on the sign of D. If D has positive sign than $\phi_0 = 0$ and if

D has negative sign then $\phi_0 = \pi$. For both cases of ϕ_0 we are looking at a hedgehog skyrmion. The case in which the energy in Eq. (2.4) is minimized must also satisfy the following conditions

$$\begin{aligned} \frac{\delta E[\theta]}{\delta \theta} &= 0, \\ \frac{\delta^2 E[\theta]}{\delta \theta^2} &> 0. \end{aligned} \tag{2.5}$$

By taking the functional derivative, we obtain

$$\begin{aligned} &J\left(\rho \frac{\partial^2 \theta}{\partial \rho^2} + \frac{\partial \theta}{\partial \rho} - \frac{\sin\theta\cos\theta}{\rho}\right) + D\left(\cos\phi_0 \sin^2\theta\right) - \mu_0 H M \rho \sin\theta = 0 \\ \implies &J\left(\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho} - \frac{\sin\theta\cos\theta}{\rho^2}\right) + D\left(\frac{\cos\phi_0 \sin^2\theta}{\rho}\right) - \mu_0 H M \sin\theta = 0. \end{aligned} \tag{2.6}$$

Now we perform the substitution $\tilde{\rho} = \frac{D}{J}\rho$ and multiply Eq. (2.6) by $\frac{J}{D^2}$, thus

$$\frac{\partial}{\partial \rho} \to \frac{D}{J}\frac{\partial}{\partial \tilde{\rho}}, \qquad \frac{\partial^2}{\partial \rho^2} \to \frac{D^2}{J^2}\frac{\partial^2}{\partial \tilde{\rho}^2}, \qquad \frac{1}{\rho} \to \frac{D}{J}\frac{1}{\tilde{\rho}}.$$

We get the following dimensionless Euler Lagrange equation

$$\left(\frac{\partial^2 \theta}{\partial \tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\frac{\partial \theta}{\partial \tilde{\rho}} - \frac{\sin\theta\cos\theta}{\tilde{\rho}^2}\right) + \left(\frac{\cos\phi_0 \sin^2\theta}{\tilde{\rho}}\right) - \frac{h}{2}\sin\theta = 0,$$

with $h = \frac{\mu_0 J H M}{D^2}$ a dimensionless constant. In most physical systems h is of the order one [12]. Now assume $\phi_0 = 0$ and D positive. We are now able to solve this equation numerically, we take the following boundary conditions ($\theta(0) = \pi$ and $\theta(\rho \to \infty) = 0$). For these boundary conditions we obtain Fig. 2.3 for different values of h. Note that a skyrmion has a typical size of $20\frac{J}{D}$.

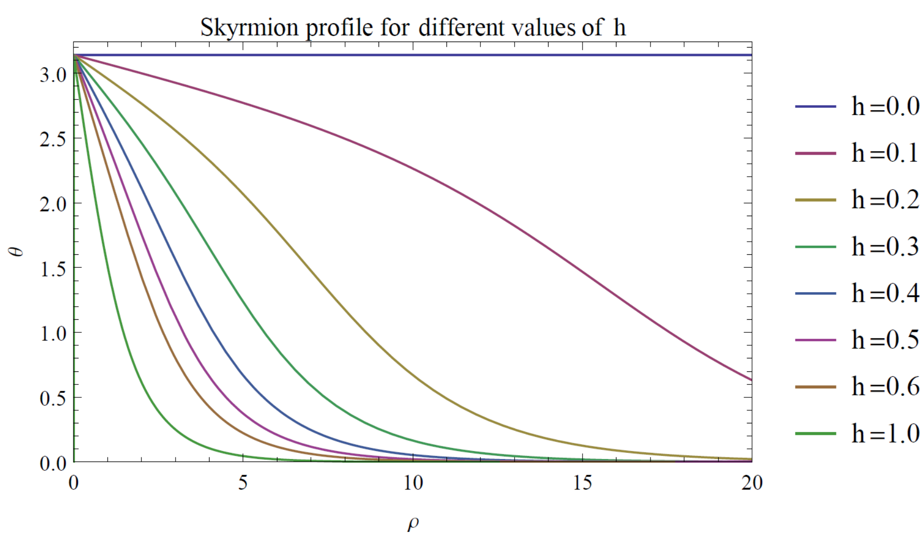

Figure 2.3: Skyrmion profile plotted for $h \in \{0.0, 0.1, 0.2, 0.4, 0.5, 0.6, 1.0\}$. One is able to see that the size of a skyrmion is infinite if a magnetic field is absent. For finite magnetic fields skyrmions have finite size of order $\mathcal{O}(J/D)$.

The above boundary conditions are used since a skyrmion is topologically distinct from the vacuum state of the magnet and has winding number equal to one (see Appendix A). In polar coordinates this winding number (A.1) reduces to

$$n = -(1/2)\cos(\theta)\big|_{\rho=0}^{\rho=\infty}.$$

2.2 Dynamics of a non-interacting ferromagnetic skyrmion

In this section we derive the action of a non-interacting skyrmion w.r.t the skyrmion's position $\mathbf{R}$*. We use that the Hamiltonian is translationally invariant. One will conclude that the skyrmion feels a force which is referred to as Magnus force. In Ref. [13] the same result is obtained but through a different method.*

In Section 2.1 we found skyrmion configurations, e.g. configurations which locally minimizes Hamiltonian (2.2) have a finite size and energy and winding number equal to one (see Appendix A). In this section we denote such a configurations with $\mathbf{\Omega}$. From Refs. [14, 15] we find that the kinetic term or Berry phase of a single spin can be written as

$$\mathcal{K}[\mathbf{\Omega}] = -\hbar S\left(\frac{d\Omega_\alpha}{dt}A_\alpha[\mathbf{\Omega}]\right),$$

with $\mathbf{A}_\alpha[\mathbf{\Omega}]$ a the vector potential of magnetic moment $\mathbf{\Omega}$ s.t. $\nabla_{\mathbf{\Omega}} \times \mathbf{A}[\mathbf{\Omega}] = \mathbf{\Omega}$ and $\langle\hat{\mathbf{S}}\rangle = \hbar S\mathbf{\Omega}$ with $\mathbf{\Omega}\cdot\mathbf{\Omega} = 1$. From Refs. [14, 15] it also follows that

$$\left.\frac{\partial A_\beta}{\partial \Omega_\alpha}\right|_{\mathbf{\Omega}} = \epsilon_{\alpha\beta\gamma}\Omega_\gamma. \tag{2.7}$$

We assume the skyrmion is at position $\mathbf{R}(t)$ at time t. We write $\mathbf{x} = \mathbf{x'} - \mathbf{R}(t)$, with $\mathbf{x'} \in \mathbb{R}^2$ the global coordinates in $\mathbb{R}^2$. Using the preceding we are able to give an expression for the action describing antiferromagnetic dynamics

$$\begin{aligned} S(t) &= \int \mathrm{d}t\,\mathcal{L}(t) = \int \mathrm{d}t\,\{K(t) - H(t)\} \qquad (2.8)\\ &= \iint \mathrm{d}t\,\mathrm{d}\mathbf{x}\left\{\left.\left(-\hbar S\left(\frac{d\Omega_\alpha}{dt}A_\alpha[\mathbf{\Omega}]\right) - \mathcal{H}\right)\right|_{\mathbf{x'}=(\mathbf{x}-\mathbf{R}(t))}\right\}. \end{aligned}$$

Notice that $\int \mathrm{d}\mathbf{x}\,\mathcal{H}$ does not depend on the position of the skyrmion. This is because $\int \mathrm{d}\mathbf{x}\,\mathcal{H}|_t$ gives the same value for every position of the skyrmion. If only a skyrmion is present we find that $\int \mathrm{d}\mathbf{x}\,\mathcal{H}$ is constant. Since we're looking at the dynamics of a non-interacting skyrmion in this section $\mathcal{H}$ does not contribute to the dynamics of the skyrmion. So the action which describes the classical dynamics of a skyrmion is given by

$$\begin{aligned} S_K(t) &= \iint \mathrm{d}t\mathrm{d}\mathbf{x}\left\{-\hbar S\left(\frac{d\Omega_\alpha}{dt}A_\alpha[\mathbf{\Omega}]\right)\right\} \qquad (2.9)\\ &= \iint \mathrm{d}t\mathrm{d}\mathbf{x}\left\{\hbar S\left(\frac{\partial\Omega_\alpha}{\partial x_\beta}A_\alpha[\mathbf{\Omega}]\right)\dot{R}_\beta(t)\right\}\\ &= -\iint \mathrm{d}t\mathrm{d}\mathbf{x}\left\{\hbar S\left(\frac{\partial\dot{\Omega}_\alpha}{\partial x_\beta}A_\alpha[\mathbf{\Omega}] + \frac{\partial\Omega_\alpha}{\partial x_\beta}\dot{A}_\alpha[\mathbf{\Omega}]\right)R_\beta(t)\right\}\\ &= -\iint \mathrm{d}t\mathrm{d}\mathbf{x}\left\{\hbar S\left(\left(\frac{\partial}{\partial x_\beta}\frac{\partial\Omega_\alpha}{\partial x_\gamma}\right)A_\alpha\Big|_{\mathbf{\Omega}} + \frac{\partial\Omega_\alpha}{\partial x_\beta}\frac{\partial A_\alpha}{\partial\Omega_\sigma}\Big|_{\mathbf{\Omega}}\frac{\partial\Omega_\sigma}{\partial x_\gamma}\right)R_\beta(t)\,\dot{R}_\gamma(t)\right\}\\ &= \iint \mathrm{d}t\mathrm{d}\mathbf{x}\left\{\hbar S\left(\frac{\partial\Omega_\alpha}{\partial x_\gamma}\frac{\partial A_\alpha}{\partial\Omega_\sigma}\Big|_{\mathbf{\Omega}}\frac{\partial\Omega_\sigma}{\partial x_\beta} - \frac{\partial\Omega_\alpha}{\partial x_\beta}\frac{\partial A_\alpha}{\partial\Omega_\sigma}\Big|_{\mathbf{\Omega}}\frac{\partial\Omega_\sigma}{\partial x_\gamma}\right)R_\beta(t)\,\dot{R}_\gamma(t)\right\}. \end{aligned}$$

Notice that the third step was acquired by performing partial integration (w.r.t. time t). The fifth step is again acquired by use of partial integration, but this time we assume that all the magnetic moments

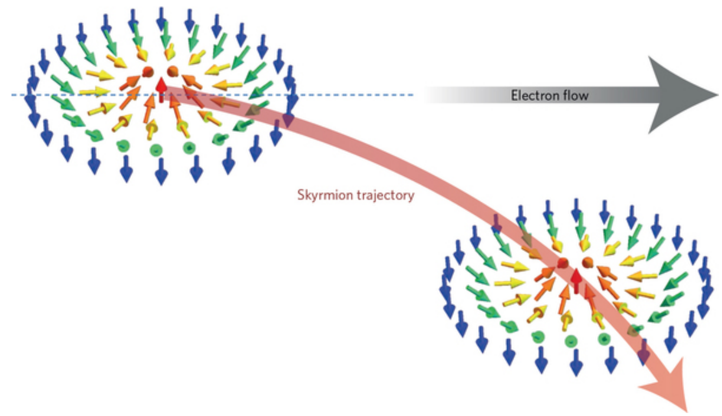

Figure 2.4: Magnus force for ferromagnetic skyrmion with winding number one. *Source:* Ref. [16].

in infinity are pointing in the positive $\hat{z}$-direction. (In other words we are not really considering $\mathbb{R}^2$, but instead the one-point compactification of $\mathbb{R}^2$ which is $\mathbb{S}^2$). By substituting Eq. (2.7) into Eq. (2.9) we obtain

$$\begin{aligned} S_K(t) &= \iint \mathrm{d}t\mathrm{d}\mathbf{x} \left\{ \hbar S \left(\frac{\partial \Omega_\alpha}{\partial x_\gamma} \epsilon_{\sigma\alpha\lambda} \Omega_\lambda \frac{\partial \Omega_\sigma}{\partial x_\beta} - \frac{\partial \Omega_\alpha}{\partial x_\beta} \epsilon_{\sigma\alpha\lambda} \Omega_\lambda \frac{\partial \Omega_\sigma}{\partial x_\gamma} \right) R_\beta(t)\, \dot{R}_\gamma(t) \right\} \\ &= \iint \mathrm{d}t\mathrm{d}\mathbf{x} \left\{ \hbar S \left(\mathbf{\Omega} \cdot \left(\frac{\partial \mathbf{\Omega}}{\partial x_\beta} \times \frac{\partial \mathbf{\Omega}}{\partial x_\gamma} \right) - \mathbf{\Omega} \cdot \left(\frac{\partial \mathbf{\Omega}}{\partial x_\gamma} \times \frac{\partial \mathbf{\Omega}}{\partial x_\beta} \right) \right) R_\beta(t)\, \dot{R}_\gamma(t) \right\} \\ &= \iint \mathrm{d}t\mathrm{d}\mathbf{x} \left\{ 2\hbar S \left(\mathbf{\Omega} \cdot \left(\frac{\partial \mathbf{\Omega}}{\partial x_\beta} \times \frac{\partial \mathbf{\Omega}}{\partial x_\gamma} \right) \right) R_\beta(t)\, \dot{R}_\gamma(t) \right\} \\ &= \int \mathrm{d}t \left\{ 8\pi n \hbar S \left(\epsilon_{\beta\gamma} R_\beta(t)\, \dot{R}_\gamma(t) \right) \right\}. \end{aligned} \tag{2.10}$$

In the above n is the topological index of the skyrmion, which is also referred to as winding number defined in Eq. (A.1). The topological index or winding number is defined as $n = \int \mathbf{\Omega} \cdot \left(\frac{\partial \mathbf{\Omega}}{\partial x_0} \times \frac{\partial \mathbf{\Omega}}{\partial x_1} \right) \mathrm{d}\mathbf{x}$. In reality the space which we consider does have boundaries and the Hamiltonian could dependent on the positions of a skyrmion, for instance due to impurities or electric currents. One can include these effects into an effective potential $V_{eff}(\mathbf{R}(t))$ for the movement of the skyrmion. So we end up with the following action

$$S_{skyrmion}(t) = \int \mathrm{d}t \left\{ 8\pi n \hbar S \left(\epsilon_{\beta\gamma} R_\beta(t)\, \dot{R}_\gamma(t) \right) - V_{eff}(\mathbf{R}(t)) \right\}. \tag{2.11}$$

Writing out the equations of motion for this action we get

$$\left. \begin{aligned} \dot{R}_1 &= \quad \tfrac{1}{16\pi n \hbar S} \tfrac{\partial V_{eff}}{\partial R_2} \\ \dot{R}_2 &= -\tfrac{1}{16\pi n \hbar S} \tfrac{\partial V_{eff}}{\partial R_1} \end{aligned} \right\} \implies 16\pi n \hbar S \epsilon_{ij} \dot{R}_j = -\partial_i V_{eff}. \tag{2.12}$$

These equations of motion yield the result that skyrmions move perpendicular the divergence of the effective potential V_{eff} (applied force), which is referred to as Magnus force, see Fig. 2.4.

Chapter 3

Skyrmions in antiferromagnets

In a ferromagnet the exchange interaction wants to align two neighboring spins. In an antiferromagnet, the exchange interaction forces neighboring spins to point in opposite directions (see Fig. 3.1). In this chapter we will develop the formalism which will be used in Chapter 4 to describe the dynamics of antiferromagnetic skyrmions. In Section 3.1 we describe the dynamics of antiferromagnets and conclude with an action for describing antiferromagnetic motion. In Section 3.2, we use methods similar to the ones used in Section 2.1 to obtain the profile of antiferromagnetic skyrmions.

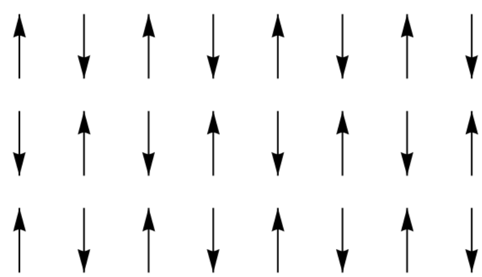

Figure 3.1: A graphical example of antiferromagnetic ordering.

3.1 Dynamics of antiferromagnets

The aim of this section is to describe antiferromagnetic motion in the imaginary time path integral formalism. It turns out that Haldane's mapping is useful for describing antiferromagnet dynamics; a discussion of this mapping will be given in Section 3.1.1. In Section 3.1.3 we work out the kinetic term/Berry phase for our system, using Haldane's mapping. If the applied external magnetic field is small we expect the magnetic moments in the magnet to be small due the antiferromagnetic exchange interaction. This allows us to make an expansion of the Hamiltonian up to second order in the canting field (magnetic moments) $\mathbf{m}$*, this and taking the continuum limit will be done in Section 3.1.4. Section 3.1.5 is dedicated to integrating out these small magnetic moments, such that we are left with an action that is only depended on the Néel vector. This is similar to solving the equations of motion for* $\mathbf{m}$ *and plugging it back into the action. Most methods used in this section are described in Ref. [15].*

3.1.1 Haldane's Mapping

Haldane's mapping is useful to make a distinction between short and long wavelength fluctuations. In this thesis we will be looking at long wavelength fluctuations around a single skyrmion configuration. We start by introducing two continuous vector fields $(\mathbf{n}, \mathbf{m})$, such that:

$$\mathbf{\Omega}_i = \eta_i \hat{\mathbf{n}}(\mathbf{x}_i)\sqrt{1-|\mathbf{m}(\mathbf{x}_i)|^2} + \mathbf{m}(\mathbf{x}_i) \tag{3.1}$$

with $\eta_i = e^{i\mathbf{x}_i\cdot\vec{\pi}}$, $\hat{\mathbf{n}}$ is called the Néel vector and $\mathbf{m}$ the canting field. Furthermore $|\hat{\mathbf{n}}(\mathbf{x}_i)| = 1$ and $\hat{\mathbf{n}}(\mathbf{x}_i)\cdot\mathbf{m}(\mathbf{x}_i) = 0$. One could argue that we have replaced a vector field with two degrees of freedom by a vector-field with six degrees of freedom. Two degrees of freedom are removed by restrictions on the length of $\hat{\mathbf{n}}$ and the inner product between $\mathbf{m}$ and $\mathbf{n}$. Which leaves us with four degrees of freedom, the last two degrees of freedom are fixed by constraining the number of Fourier-components in the measure

$$\mathcal{D}\hat{\mathbf{\Omega}} = \prod_{|\mathbf{q}|\leq\Lambda_{BZ}} \mathrm{d}\hat{\mathbf{n}}_{\mathbf{q}}\mathrm{d}\mathbf{m}_{\mathbf{q}}\delta(\mathbf{m}\cdot\hat{\mathbf{n}})\mathcal{J}[\hat{\mathbf{n}},\mathbf{m}], \tag{3.2}$$

where $\mathcal{J}$ is the Jacobian of transformation (3.1), $\hat{\mathbf{n}}_{\mathbf{q}}$ and $\mathbf{m}_{\mathbf{q}}$ are the Fourier transforms of $\hat{\mathbf{n}}(\mathbf{x})$ and $\mathbf{m}(\mathbf{x})$ respectively. The spherical Brillouin zone radius Λ_{BZ} is chosen such that

$$2\mathcal{N} = 4\sum_{|\mathbf{q}|\leq\Lambda_{BZ}},$$

where $\mathcal{N}$ denotes the number of sites. Assuming a small canting field $\mathbf{m}$ and looking at large correlation lengths ξ/a, we obtain a much smaller cutoff than Λ_{BZ} which we denote with Λ (large wavelength approximation)

$$\xi^{-1} \ll \Lambda \ll \Lambda_{BZ}. \tag{3.3}$$

From Ref. [15, p.131] it follows that the Jacobian in Eq. (3.2) is constant for leading order in $\mathbf{m}$

$$\mathcal{J} \approx S^{-\mathcal{N}}.$$

Note that $\hat{\mathbf{n}}$ and $\mathbf{m}$ are not uniquely defined.

3.1.2 Some intuition behind Haldane's Mapping

This section is included to give some intuition about what the vectors $\hat{\mathbf{n}}$ and $\mathbf{m}$ stand for. Since the choice of neighbors is arbitrary this solution is not exactly the same as the above, but for slowly varying and large wavelength systems they are similar.

During this section we use the phenomenological approach suggested by Ref. [17]. In an antiferromagnet neighbouring spins like to point in opposite direction, which we take care of by assuming there exist two two-dimensional sub-lattices M_1 and M_2. We denote the classical direction of the spins on lattice M_1 with $\mathbf{M}_1$ and the direction of the spins on lattice M_2 with $\mathbf{M}_2$, s.t. $\mathbf{M}_1 \simeq -\mathbf{M}_2$ and $|\mathbf{M}_1| = |\mathbf{M}_2| = M_s$.

First note that in the model we use we assume that the magnetization on each lattice site has equal magnitude. In other words $|\mathbf{M}_1| = |\mathbf{M}_2| = M_s$, we define

$$\mathbf{m}_1 = \mathbf{M}_1/M_s,\ \mathbf{m}_2 = \mathbf{M}_2/M_s. \tag{3.4}$$

For the antiferromagnet it is useful to define the following two vectors:

$$\mathbf{l}(\mathbf{x}_i) = \frac{1}{2}\{\mathbf{m}_1(\mathbf{x}_i) - \mathbf{m}_2(\mathbf{x}_i)\}, \tag{3.5}$$

$$\mathbf{m}(\mathbf{x}_i) = \frac{1}{2}\{\mathbf{m}_1(\mathbf{x}_i) + \mathbf{m}_2(\mathbf{x}_i)\}\}. \tag{3.6}$$

Notice that the above definitions imply that $\mathbf{l} \cdot \mathbf{m} = 0$. It follows that

$$\mathbf{\Omega}_1(\mathbf{x}_i) = \mathbf{l}(\mathbf{x}_i) + \mathbf{m}(\mathbf{x}_i), \tag{3.7}$$

$$\mathbf{\Omega}_2(\mathbf{x}_i) = -\mathbf{l}(\mathbf{x}_i) + \mathbf{m}(\mathbf{x}_i). \tag{3.8}$$

Now we define $\hat{\mathbf{n}}(\mathbf{x}_i) = \mathbf{l}(\mathbf{x}_i)/|\mathbf{l}(\mathbf{x}_i)|$. From $|\mathbf{\Omega}(\mathbf{x}_i)|^2 = |\mathbf{l}(\mathbf{x}_i)|^2 + |\mathbf{m}(\mathbf{x}_i)|^2 = 1$ it follows that $\mathbf{l}(\mathbf{x}_i) = \hat{\mathbf{n}}(\mathbf{x}_i)\sqrt{1 - |\mathbf{m}(\mathbf{x}_i)|^2}$. So the magnetization direction is given by

$$\mathbf{\Omega}_1(\mathbf{x}_i) = \hat{\mathbf{n}}(\mathbf{x}_i)\sqrt{1 - |\mathbf{m}(\mathbf{x}_i)|^2} + \mathbf{m}(\mathbf{x}_i), \tag{3.9}$$

$$\mathbf{\Omega}_2(\mathbf{x}_i) = -\hat{\mathbf{n}}(\mathbf{x}_i)\sqrt{1 - |\mathbf{m}(\mathbf{x}_i)|^2} + \mathbf{m}(\mathbf{x}_i). \tag{3.10}$$

In a antiferromagnet the total magnetization is typically small because of a large exchange interaction, thus $|\mathbf{m}(\mathbf{x}_i)| \ll 1$.

3.1.3 The Kinetic term / Berry phase

From Ref. [3, Appendix A] and Ref. [15, Chapter 10.1] it follows that the Berry phase of the total system is given by the sum over all the independent Berry phases, which is given by

$$\mathcal{A}_B[\{\mathbf{\Omega}\}] = iS\sum_i \omega[\mathbf{\Omega}_i] = iS\sum_i \int_0^\beta \mathbf{A}[\mathbf{\Omega}_i]\partial_\tau \mathbf{\Omega}_i. \tag{3.11}$$

Here the sum over i means a sum over all the lattice points and A is a gauge invariant vector field with the property that $\nabla_\Omega \cdot A_i = \mathbf{\Omega}_i$.
In this section we derive an expression for Eq. (3.11) in terms of Haldane's mapping (3.1), in the large wavelength approximation. Since in this approximation $\mathbf{m}$ is a small variable, we perform an expansion of Eq. (3.11) up to second order in $\mathbf{m}$, which is given by

$$\begin{aligned} \mathcal{A}_B[\{\mathbf{\Omega}\}] &= iS\sum_i \omega[\mathbf{\Omega}_i] = iS\sum_i \omega\left[\eta_i\hat{\mathbf{n}}(\mathbf{x}_i)\sqrt{1 - |\mathbf{m}(\mathbf{x}_i)|^2} + \mathbf{m}(\mathbf{x}_i)\right] \\ &\approx iS\sum_i \eta_i\omega\left[\hat{\mathbf{n}}(\mathbf{x}_i) + \eta_i\mathbf{m}(\mathbf{x}_i)\right] + \mathcal{O}(m^3) \\ &= iS\sum_i\left\{\eta_i\omega\left[\hat{\mathbf{n}}(\mathbf{x}_i)\right] + \frac{\delta\omega}{\delta\hat{\mathbf{n}}_\mathbf{i}}\mathbf{m}(\mathbf{x}_i)\right\} \\ &= i\Upsilon[\hat{\mathbf{n}}] + iS\int \mathrm{d}\tau\sum_i(\hat{\mathbf{n}}_\mathbf{i} \times \partial_\tau\hat{\mathbf{n}}_\mathbf{i} \cdot \mathbf{m}_\mathbf{i}), \end{aligned} \tag{3.12}$$

with

$$\Upsilon[\hat{\mathbf{n}}] = S\sum_i \eta_i\omega[\hat{\mathbf{n}}(\mathbf{x}_\mathbf{i})], \tag{3.13}$$

the Berry phase associated with the Néel vector. In the last line of Eq. (3.12) we used

$$\begin{aligned}
\delta\omega &= \int_0^\beta \mathrm{d}\tau\ \delta\left(\mathbf{A}\cdot\dot{\mathbf{\Omega}}\right) = \int \mathrm{d}\tau\ \delta\mathbf{A}\cdot\dot{\mathbf{\Omega}} + \mathbf{A}\cdot\mathrm{d}_\tau\delta\mathbf{\Omega} \\
&= \int_0^\beta \mathrm{d}\tau\ \left\{\frac{\partial A^\alpha}{\partial\Omega^\beta}\delta\Omega^\beta\dot{\Omega}^\alpha - \frac{\partial A^\alpha}{\partial\Omega^\beta}\dot{\Omega}^\beta\delta\Omega^\alpha + \mathrm{d}_\tau\left(\mathbf{A}\cdot\delta\mathbf{\Omega}\right)\right\} \\
&= \int_0^\beta \mathrm{d}\tau\ \frac{\partial A^\alpha}{\partial\Omega^\beta}\epsilon^{\alpha\beta\gamma}(\dot{\mathbf{\Omega}}\times\delta\mathbf{\Omega})^\gamma = \int \mathrm{d}\tau\ \mathbf{\Omega}\cdot\left(\dot{\mathbf{\Omega}}\times\delta\mathbf{\Omega}\right) \\
&= \int_0^\beta \mathrm{d}\tau\ \delta\mathbf{\Omega}\cdot(\mathbf{\Omega}\times\dot{\mathbf{\Omega}}) \\
\implies \frac{\delta\omega}{\delta\hat{\mathbf{n}}} &= \int_0^\beta \mathrm{d}\tau(\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}}).
\end{aligned} \tag{3.14}$$

In Eq. (3.14) we used that $\nabla_\Omega\mathbf{A} = \mathbf{\Omega}$ and that $\delta\mathbf{\Omega}$ vanishes at $\tau\in\{0,\beta\}$. Since the wavelengths are much bigger than the lattice spacing in the large wavelength approximation, we are able to replace the sum by an integral

$$\mathcal{A}_B[\{\mathbf{\Omega}\}] \approx i\Upsilon[\hat{\mathbf{n}}] + i\frac{S}{a^2}\int \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\ \{\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x})\cdot\mathbf{m}(\mathbf{x})\}. \tag{3.15}$$

3.1.4 The continuum Hamiltonian

In this section we start from the following Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\sum_{\langle ij\rangle}\left\{S^2 J_{ij}\hat{\mathbf{\Omega}}_i\cdot\hat{\mathbf{\Omega}}_j + S^2\mathbf{D}_{ij}\left(\hat{\mathbf{\Omega}}_i\times\hat{\mathbf{\Omega}}_j\right) - S^2\tilde{K}_z^a\hat{\mathbf{\Omega}}_{z,i}^2 - S\mathbf{H}\cdot\mathbf{\Omega}_i\right\}. \tag{3.16}$$

$J_{ij} > 0$ is the Heisenberg exchange interaction (which is positive in the antiferromagnetic case), D_{ij} gives the Dzyaloshinskii-Moriya exchange interaction, K_z gives the anisotropy of the system and $\mathbf{H}$ represents an external magnetic field. We assume that both J_{ij} and D_{ij} have the full lattice symmetry. We use the long wavelength approximation to replace the two-dimensional sum by a two dimensional integral. For the two-dimensional lattice we assume a lattice spacing of a. We rewrite the sum as follows

$$\sum_i F_i \to \frac{1}{a^2}\int \mathrm{d}^2\mathbf{x}\sum_i\delta(\mathbf{x}-\mathbf{x_i})F(\mathbf{x}). \tag{3.17}$$

We will use Haldane's Mapping to obtain a Hamiltonian which is dependent on $\hat{\mathbf{n}}$ as well as $\mathbf{m}$. Furthermore $|\mathbf{m}| \ll 1$ because we assume a large exchange interaction. Since $|\mathbf{m}| \ll 1$ we may expand the Hamiltonian up to second order

$$\mathcal{H}[\hat{\mathbf{\Omega}}] \to \mathcal{H}[\hat{\mathbf{n}},\mathbf{m}] \approx \mathcal{H}[\hat{\mathbf{n}},0] + \int \mathrm{d}\mathbf{x}\frac{\delta\mathcal{H}[\hat{\mathbf{n}},\mathbf{m}]}{\delta\mathbf{m}(\mathbf{x})}\mathbf{m}(\mathbf{x}) + \frac{1}{2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\frac{\delta^2\mathcal{H}[\hat{\mathbf{n}},\mathbf{m}]}{\delta\mathbf{m}(\mathbf{x})\delta\mathbf{m}(\mathbf{x}')}\mathbf{m}(\mathbf{x})\mathbf{m}(\mathbf{x}') + \mathcal{O}(\mathbf{m}^3). \tag{3.18}$$

Performing the expansion up to second order in $\mathbf{m}$ we obtain:

$$\begin{aligned}
\hat{\mathbf{\Omega}}_\mathbf{i}\cdot\hat{\mathbf{\Omega}}_\mathbf{j} &\approx \eta_i\eta_j\hat{\mathbf{n}}_\mathbf{i}\cdot\hat{\mathbf{n}}_\mathbf{j} + (\eta_j\mathbf{m_i}\hat{\mathbf{n}}_\mathbf{j} + \eta_i\mathbf{m_j}\hat{\mathbf{n}}_\mathbf{i}) + [\mathbf{m_i}\mathbf{m_j} - \frac{1}{2}\eta_i\eta_j({\mathbf{m_i}}^2 + {\mathbf{m_j}}^2)] \\
&= \eta_i\eta_j - \frac{1}{2}\eta_i\eta_j(\hat{\mathbf{n}}_\mathbf{i} - \hat{\mathbf{n}}_\mathbf{j})^2 + (\eta_j\mathbf{m_i}\hat{\mathbf{n}}_\mathbf{j} + \eta_i\mathbf{m_j}\hat{\mathbf{n}}_\mathbf{i}) + [\mathbf{m_i}\mathbf{m_j} - \frac{1}{2}\eta_i\eta_j({\mathbf{m_i}}^2 + {\mathbf{m_j}}^2)], \\
\hat{\mathbf{\Omega}}_\mathbf{i}\times\hat{\mathbf{\Omega}}_\mathbf{j} &\approx \eta_i\eta_j\hat{\mathbf{n}}_\mathbf{i}\times\hat{\mathbf{n}}_\mathbf{j} + (\eta_i\hat{\mathbf{n}}_\mathbf{i}\times\mathbf{m_j} - \eta_j\hat{\mathbf{n}}_\mathbf{j}\times\mathbf{m_i}) + \mathbf{m_i}\times\mathbf{m_j} - \frac{1}{2}\eta_i\eta_j({\mathbf{m_i}}^2 + {\mathbf{m_j}}^2)\hat{\mathbf{n}}_\mathbf{i}\times\hat{\mathbf{n}}_\mathbf{j}, \\
\hat{\mathbf{\Omega}}_\mathbf{z,i}^2 &\approx \hat{\mathbf{n}}_\mathbf{z,i}^2 + 2\eta_i\hat{\mathbf{n}}_\mathbf{z,i}\mathbf{m}_\mathbf{z,i} + {\mathbf{m}_\mathbf{z,i}}^2, \\
\hat{\mathbf{\Omega}}_\mathbf{i} &\approx \eta_i\hat{\mathbf{n}}_\mathbf{i} + \mathbf{m_i} - \frac{1}{2}\eta_i\hat{\mathbf{n}}_\mathbf{i}\left|\mathbf{m_i}\right|^2.
\end{aligned} \tag{3.19}$$

Here the difference in the Néel vector may be written as

$$(\hat{\mathbf{n}}_{\mathbf{i}} - \hat{\mathbf{n}}_{\mathbf{j}}) \approx \partial_l \hat{\mathbf{n}}(\mathbf{x_j}) \mathbf{x_{ij}}^l + (\partial_l \partial_k \hat{\mathbf{n}}(\mathbf{x_j})) \mathbf{x_{ij}}^l \mathbf{x_{ij}}^k + h.o.,$$

where $\mathbf{x_{ij}} = (\mathbf{x_i} - \mathbf{x_j})$. Since J_{ij} possesses the full lattice symmetry the first order term in $\mathbf{m}$ of the Heisenberg exchange interaction becomes negligible. For neighboring spins we are able to write

$$\hat{\mathbf{n}}_{\mathbf{i}} \approx \hat{\mathbf{n}}_{\mathbf{j}} + \partial_l \hat{\mathbf{n}}(\mathbf{x_i}) \mathbf{x_{ij}}^l + (\partial_l \partial_k \hat{\mathbf{n}}) \mathbf{x_{ij}}^l \mathbf{x_{ij}}^k + h.o. \tag{3.20}$$

For every lattice point there are surrounding lattice points with which it interacts. So the way to go is to sum over all neighboring lattice sites at each site. We will now see that the zeroth order in Eq. (3.20) plugged into Eq. (3.18) for first order in $\mathbf{m}$ will become zero since $\hat{\mathbf{n}}_{\mathbf{j}} \cdot \mathbf{m_j} = 0$. The first order derivative $\partial_l \hat{\mathbf{n}}$ will drop out in the cross terms, since J_{ij} possesses the full lattice symmetry. For J the first order in $\mathbf{m}$ leaves a term which is proportional to the second derivative $\partial_l \partial_k \hat{\mathbf{n}} \propto \mathcal{O}(\Lambda R_K)^2$, which is negligible by Eq. (3.3). We have defined R_K to be the characteristic range of J and D. We thus obtain

$$\begin{aligned} \sum_{ij} J_{ij} \eta_{ij} \mathbf{m_j} \hat{\mathbf{n}}_{\mathbf{i}} &\approx \sum_{ij} J_{ij} \eta_{ij} \mathbf{m_j} \left\{ \hat{\mathbf{n}}_{\mathbf{j}} + \partial_l \hat{\mathbf{n}}(\mathbf{x_j}) \mathbf{x_{ij}}^l + (\partial_l \partial_k \hat{\mathbf{n}}(\mathbf{x_j})) \mathbf{x_{ij}}^l \mathbf{x_{ij}}^k + h.o. \right\} \\ &\approx \sum_{ij} J_{ij} \eta_{ij} \mathbf{m_j} \left\{ (\partial_l \partial_k \hat{\mathbf{n}}(\mathbf{x_j})) \mathbf{x_{ij}}^l \mathbf{x_{ij}}^k + h.o. \right\} \propto \mathcal{O}(\Lambda R_K)^2, \end{aligned} \tag{3.21}$$

where the first order derivatives in Eq. (3.21) cancel because J possess the full lattice symmetry; what is left is a second order derivative which is negligible. So the exchange interaction does not have a first order term in $\mathbf{m}$. We also note that within the long wavelength approximation the anisotropy first order contribution in $\mathbf{m}$ is negligible. This could be seen by first taken the continuum limit for the first order in $\mathbf{m}$

$$\int_\Lambda \frac{\mathrm{d}\mathbf{x}}{a^2} \left\{ \eta(\mathbf{x}) \hat{\mathbf{n}}_{\mathbf{z}}(\mathbf{x}) \mathbf{m_z}(\mathbf{x}) \right\} = \int_\Lambda \frac{\mathrm{d}\mathbf{q}}{a^2 (2\pi)^2} \mathbf{n_z}(\mathbf{q}) \mathbf{m_z}(\frac{\vec{\pi}}{a} - \mathbf{q}). \tag{3.22}$$

We assume large wavelength in $\hat{\mathbf{n}}$ as well as $\mathbf{m}$, but a/π is a small wave length beyond the introduced cutoff, since π/a gives a momentum which is beyond the momentum cutoff we consider. While either $\hat{\mathbf{n}}$ or $\mathbf{m}$ must be described by a wavelength in the order of a/π we are able to neglect this term within the large wavelength approximation. We will now define

$$\tilde{H} = \left. \frac{\delta \mathcal{H}}{\delta \mathbf{m}} \right|_{\mathbf{m}=0}, \tag{3.23}$$

$$\chi^{-1}_{\mathbf{x},\mathbf{x}'} \approx \frac{1}{S^2} \left. \frac{\delta^2 \mathcal{H}[\hat{\mathbf{n}}, \mathbf{m}]}{\delta \mathbf{m}(\mathbf{x}) \delta \mathbf{m}(\mathbf{x}')} \right|_{\mathbf{m}=0}. \tag{3.24}$$

Since the first order contributions in $\mathbf{m}$ of the Heisenberg exchange interaction and the anisotropy are negligible, we obtain $\tilde{H} \approx \frac{1}{a^2} H$. In principle one should also include the homogeneous Dzyaloshinskii-Moriya interaction here, since this interaction is small in practice we neglect it for now. For χ^{-1} we assume it only depends on the difference in length between x and x', since J and D posses the full lattice symmetry

$$S^2 \int \mathrm{d}^2\mathbf{x} \int \mathrm{d}^2\mathbf{x}' \mathbf{m_x} \chi^{-1}_{\mathbf{x},\mathbf{x}'} \mathbf{m_{x'}} = S^2 \int_{q \leq \Lambda_{BZ}} \frac{\mathrm{d}^2 q}{(2\pi)^2} \mathbf{m_q} \chi^{-1}_{\mathbf{q}} \mathbf{m_{-q}}. \tag{3.25}$$

So in total we now have the following approximate Hamiltonian

$$\begin{aligned} \mathcal{H} \approx E_0^{cl} &+ \int \mathrm{d}\mathbf{x} \left\{ \frac{\rho_s}{2} |\nabla \hat{\mathbf{n}}|^2 + \frac{D}{2} \left(\hat{y} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial x} \right) - \hat{x} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial y} \right) \right) - S^2 \tilde{K}_z \hat{\mathbf{n}}_z^2 - S H_n \cdot \hat{\mathbf{n}}_z \right\} \\ &+ \int \mathrm{d}\mathbf{x}\, S \tilde{\mathbf{H}} \cdot \mathbf{m_x} + \frac{1}{2} \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}' \left\{ \mathbf{m_x} \chi^{-1}_{\mathbf{x},\mathbf{x}'} \mathbf{m_{x'}} \right\}. \end{aligned} \tag{3.26}$$

In Eq. (3.26) we defined $\tilde{K}_z = \frac{K_z^a}{a^2}$ and $\tilde{H} = \frac{H}{a^2}$. In the formula above D is the Dzyaloshinskii-Moriya constant, K_z describes anisotropy, χ^{-1} is the inverse of the uniform susceptibility and ρ_s is the stiffness constant

$$\rho_s = -\frac{S^2}{4Na^2}\sum_{ij} J_{ij}\eta_i\eta_j|\mathbf{x_i}-\mathbf{x_j}|^2. \tag{3.27}$$

Finally, H_n is given by

$$H_n(\mathbf{x}) = \frac{SH}{a^2}\eta(\mathbf{x}), \tag{3.28}$$

where $\eta(\mathbf{x}) = \exp\left(i\mathbf{x}\cdot\frac{\vec{\pi}}{a}\right)$. This implies

$$\int d\mathbf{x}\ H_n(\mathbf{x})\cdot\hat{\mathbf{n}}_z(\mathbf{x}) \to \frac{SH}{2a^2}\int\frac{d\mathbf{q}}{(2\pi)^2}\ \left[\delta\left(\frac{\vec{\pi}}{a}+\mathbf{q}\right)+\delta\left(\frac{\vec{\pi}}{a}-\mathbf{q}\right)\right]\hat{\mathbf{n}}_\mathbf{q}. \tag{3.29}$$

The momentum $\dfrac{\pi}{a}$ is again beyond the momentum cutoff which implies $H_n\cdot\hat{\mathbf{n}}_z$ can be neglected in the large wavelength limit.

3.1.5 The partition function and Green's functions

In this section we derive a expression of the partition or Green's function only in terms of $\hat{\mathbf{n}}$. We do this by integrating over the canting field $\mathbf{m}$. Since we consider the total euclidean action up to quadratic order in $\mathbf{m}$, we are able to perform the integration over $\mathbf{m}^\perp$ by completing the square. We ignore any overall normalization factor. From Eqs. (3.15) and (3.26) we obtain the following Euclidean action

$$\mathcal{S}_E = i\Upsilon[\hat{\mathbf{n}}] + i\frac{S}{a^2}\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\{\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x})\cdot\mathbf{m}(\mathbf{x})\} + \int\mathrm{d}\tau\mathcal{H}[\hat{\mathbf{n}},\mathbf{m}]. \tag{3.30}$$

The partition function for Eq. (3.30) can be simplified by integrating out the $\mathbf{m}$ dependence, which is similar to solving the equations of motion for $\mathbf{m}$ and plugging it back into the action. By completing the square and ignoring any overall normalization factor, we obtain

$$\begin{aligned}
Z &= \int \mathcal{D}\hat{\mathbf{n}}\mathcal{D}\mathbf{m}\ e^{-\mathcal{S}_E[\hat{\mathbf{n}},\mathbf{m}]} \\
&\propto \int\mathcal{D}\hat{\mathbf{n}}\ e^{i\Upsilon[\hat{\mathbf{n}}]}\exp\left[-\int_\Lambda \mathrm{d}\mathbf{x}\left\{\frac{\rho_s}{2}|\nabla\hat{\mathbf{n}}|^2+\frac{D}{2}\left(\hat{y}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial x}\right)-\hat{x}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial y}\right)\right)-\tilde{K}_z\hat{\mathbf{n}}_z^2\right\}\right]\zeta[\hat{\mathbf{n}}], \\
\zeta[\hat{\mathbf{n}}] &= \int_{\Lambda_{BZ}}\mathcal{D}\mathbf{m}^\perp\exp\left[-\int_0^\beta\mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ S\tilde{\mathbf{H}}_{-\mathbf{q}}\mathbf{m}_\mathbf{q}^\mathbf{i} + iS\int_0^\beta\mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ (\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{-\mathbf{q}}\mathbf{m}_\mathbf{q}\right. \\
&\quad\left.-\frac{S^2}{2}\int_0^\beta\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\mathbf{m}_\mathbf{q}\chi_\mathbf{q}^{-1}\mathbf{m}_{-\mathbf{q}}\right] \\
&= \int_{\Lambda_{BZ}}\mathcal{D}\mathbf{m}\delta(\hat{\mathbf{n}}\cdot\mathbf{m})\exp\left(-\int_0^\beta\mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ \left\{\mathcal{L}_{-\mathbf{q}}\cdot\mathbf{m}_\mathbf{q}+\frac{1}{2}\mathbf{m}_\mathbf{q}\chi_\mathbf{q}^{-1}\mathbf{m}_{-\mathbf{q}}\right\}\right)
\end{aligned} \tag{3.31}$$

In the preceding equation we defined $\mathcal{L}_{\mathbf{q}} = S\left\{\tilde{\mathbf{H}}_{\mathbf{q}} - i(\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{\mathbf{q}}\right\}$ and $\delta(\hat{\mathbf{n}}\cdot\mathbf{m}) = \int\mathcal{D}\lambda\exp[i\lambda(\hat{\mathbf{n}}\cdot\mathbf{m})]$. Using these definitions $\zeta[\hat{\mathbf{n}}]$ becomes

$$\begin{aligned}
\zeta[\hat{\mathbf{n}}] &= \int\mathcal{D}\lambda\int_{\Lambda_{BZ}}\mathcal{D}\mathbf{m}\exp\left(-\int_0^\beta \mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ \left\{(\mathcal{L}_{-\mathbf{q}} - i\lambda\hat{\mathbf{n}}_{-\mathbf{q}})\cdot\mathbf{m}_{\mathbf{q}} + \frac{S^2}{2}\mathbf{m}_{\mathbf{q}}\chi_{\mathbf{q}}^{-\mathbf{1}}\mathbf{m}_{-\mathbf{q}}\right\}\right)\\
&\propto \int\mathcal{D}\lambda\exp\left(\int_0^\beta \mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ \frac{1}{2S^2}\chi_{\mathbf{q}}\left\{(\mathcal{L}_{-\mathbf{q}} - i\lambda\hat{\mathbf{n}}_{-\mathbf{q}})\cdot(\mathcal{L}_{\mathbf{q}} - i\lambda\hat{\mathbf{n}}_{\mathbf{q}})\right\}\right)\\
&= \int\mathcal{D}\lambda\exp\left(-\int_0^\beta \mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ \frac{1}{2S^2}\chi_{\mathbf{q}}\left\{(\lambda + i\mathcal{L}_{-\mathbf{q}}\cdot\hat{\mathbf{n}}_{\mathbf{q}})^2 + (\mathcal{L}_{-\mathbf{q}}\cdot\hat{\mathbf{n}}_{\mathbf{q}})^2 - \mathcal{L}_{-\mathbf{q}}\mathcal{L}_{\mathbf{q}}\right\}\right)\\
&\propto \exp\Bigg[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\int_\Lambda\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\ \chi_{\mathbf{q}}\Big\{(\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{-\mathbf{q}}(\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{\mathbf{q}}\\
&+ i\left[(\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{\mathbf{q}}\cdot\tilde{\mathbf{H}}_{-\mathbf{q}} + (\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}})_{-\mathbf{q}}\cdot\tilde{\mathbf{H}}_{\mathbf{q}}\right] + \left(\tilde{\mathbf{H}}_{-\mathbf{q}}\hat{\mathbf{n}}^z_{\mathbf{q}}\right)^2 - \tilde{H}_{-\mathbf{q}}\tilde{H}_{\mathbf{q}}\Big\}\Bigg].
\end{aligned}$$

The variation of $\chi_{\mathbf{q}}$ is small if we consider $\mathbf{q}$ to be in the domain Λ. Thus, we replace it with its zero momentum mode

$$\zeta[\hat{\mathbf{n}}] \propto \exp\left[-\frac{1}{2}\chi_0\int_0^\beta \mathrm{d}\tau\int_\Lambda \mathrm{d}\mathbf{x}\ \left\{|\partial_\tau\hat{\mathbf{n}}(\mathbf{x})|^2 + 2i\tilde{H}(\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x}))^z + \left(\tilde{H}\hat{\mathbf{n}}^z(\mathbf{x})\right)^2 - \tilde{H}^2\right\}\right]. \tag{3.32}$$

The following semi-classical partition function is thus be obtained

$$\begin{aligned}
Z \propto \int\mathcal{D}\hat{\mathbf{n}}\ e^{-i\Upsilon[\hat{\mathbf{n}}]}\exp\Bigg[\int_0^\beta \mathrm{d}\tau\int_\Lambda \mathrm{d}\mathbf{x}\ -\Bigg\{&\frac{1}{2}\chi_0|\partial_\tau\hat{\mathbf{n}}(\mathbf{x})|^2 + i\chi_0\tilde{\mathbf{H}}\cdot(\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x})) + \frac{1}{2}\chi_0\left(\tilde{\mathbf{H}}\cdot\hat{\mathbf{n}}(\mathbf{x})\right)^2\\
&+ \frac{\rho_s}{2}|\nabla\hat{\mathbf{n}}|^2 + \frac{D}{2}\left(\hat{y}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial x}\right) - \hat{x}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial y}\right)\right) - S^2K_z\hat{\mathbf{n}}_z^2\Bigg\}\Bigg].
\end{aligned} \tag{3.33}$$

In the last line we used the identity

$$|\hat{\mathbf{n}}\times\partial_\tau\hat{\mathbf{n}}|^2 = |\partial_\tau\hat{\mathbf{n}}|^2\,.$$

Comparing this result with Ref. [18, Chapter 3] we note that homogeneous Dzyaloshinskii-Moriya interaction is absent. This term is neglected while doing the integration in Eq. (3.31). There should be an extra term in

$$\tilde{H} = \left.\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\right|_{\mathbf{m}=0},$$

which accounts for the homogeneous Dzyaloshinskii-Moriya interaction. From Eq. (3.19) we see that the second order terms in the Dzyaloshinskii-Moriya are negligible by Eq. (3.3), since the outer products turn into second derivatives in the continuum approximation. In practice the homogeneous Dzyaloshinskii-Moriya interaction will be small or one is able to add it manually by changing the magnetic field.

If one would ignore the Dzyaloshinskii-Moriya interaction, the anisotropy and consider small magnetic fields, we would obtain a wave equation with the speed of sound within be given by

$$c = \sqrt{\frac{\rho_s}{\chi_0}}, \tag{3.34}$$

which we will call the asymptotic spin wave velocity.

3.2 Skyrmion configurations in antiferromagnets

This section will be devoted to finding the skyrmion profile in antiferromagnets (see Fig. 3.2), using similar methods as described in Section 2.1. We determine the Euler-Lagrange equations belonging to energy (3.35) *in Section 3.2.1. Section 3.2.2 will be dedicated to finding numerical solutions for the Euler-Lagrange equations obtained in Section 3.2.1, for different values of the magnetic field and for different interaction strengths. For large magnetic fields we are also able to find another class of structures, which are called vortices. Vortices differ from skyrmions, because their boundary conditions are different. We obtain numerical solutions for vortices in Section 3.2.2.*

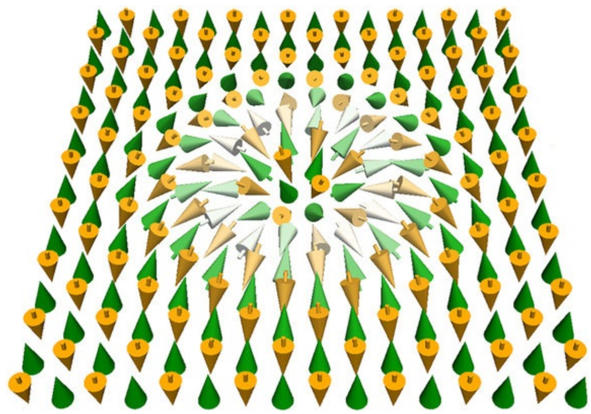

Figure 3.2: Graphical example of an antiferromagnetic skyrmion configuration. The color scale represents the magnetization direction; orange is into the plane, green is out of the plane. *Source:* Ref. [19].

3.2.1 Minimizing the free energy of the antiferromagnet

In this section we determine the Euler-Lagrange equations belonging energy (3.35). The antiferromagnetic energy in the large wavelength approximation is given by Eq. (3.33),

$$\mathcal{F}[\hat{\mathbf{n}}] = N_A \int_\Lambda \mathrm{d}\mathbf{x} \left\{ \frac{\rho_s}{2} \sum_{l=x,y} |\partial_l \hat{\mathbf{n}}|^2 + \frac{D}{2} \left(\hat{y} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial x} \right) - \hat{x} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial y} \right) \right) - K_z \hat{\mathbf{n}}_z^2 + \frac{\chi_0}{2} \left(\tilde{\mathbf{H}} \cdot \hat{\mathbf{n}}(\mathbf{x}) \right)^2 \right\}. \tag{3.35}$$

Now we use the transformation

$$\hat{\mathbf{n}} = \sin[\theta] \cos[\phi_0] \, \hat{\rho} + \sin[\theta] \sin[\phi_0] \, \hat{\phi} + \cos[\theta] \, \hat{z}.$$

The given Hamiltonian is translationally invariant and the skyrmion configuration is a local minimum of the Hamiltonian in Eq. (3.35). Because of this the skyrmion configuration will be rotationally invariant. We assume θ only depends on ρ and ϕ_0 is constant. We also choose the magnetic field to be in the $\hat{z}$-direction. Using the preceding assumptions the Hamiltonian in Eq. (3.35) reduces to [18, Chapter 3]

$$\mathcal{F}[\hat{\mathbf{n}}] = 2\pi N_A \int_\Lambda \left\{ \frac{\rho_s}{2} \left((\partial_\rho \theta)^2 + \frac{\sin^2(\theta)}{\rho^2} \right) + \frac{D}{2} \cos(\phi_0) \left(\partial_\rho \theta + \frac{\cos(\theta)\sin(\theta)}{\rho} \right) - K_z \cos^2(\theta) + \frac{\chi_0}{2} \tilde{H}^2 \cos^2(\theta) \right\} \rho d\rho. \tag{3.36}$$

The skyrmion configuration will be a local minimum in the energy (i.e. $\frac{\delta \mathcal{F}}{\delta \theta} = 0$) with boundary conditions $\theta(0) = \pi$ and $\theta(\infty) = 0$. The Euler-Lagrange equations give

$$\left(\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho} - \frac{\sin(\theta)\cos(\theta)}{\rho^2}\right) + \frac{D}{\rho_s}\cos(\phi_0)\frac{\sin^2(\theta)}{\rho} + \frac{(h^2 - K_z)}{\rho_s}\sin(2\theta) = 0, \tag{3.37}$$

where we defined $h^2 = \frac{\chi_0}{2}\tilde{H}^2$. By performing the substitution $\tilde{\rho} = \sqrt{\frac{K_z}{\rho_s}}\rho$ in Eq. (3.37) and multiplying the total by $\frac{\rho_s}{K_z}$, we obtain the following dimensionless Euler-Lagrange equation

$$\left(\frac{\partial^2 \theta}{\partial \tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\frac{\partial \theta}{\partial \tilde{\rho}} - \frac{\sin(\theta)\cos(\theta)}{\tilde{\rho}^2}\right) + \frac{4D}{\pi D_0}\cos(\phi_0)\frac{\sin^2(\theta)}{\tilde{\rho}} - \left(1 - \frac{h^2}{h_0^2}\right)\sin(2\theta) = 0. \tag{3.38}$$

In the preceding equation we used $D_0 = (4/\pi)\sqrt{\rho_s K_z}$ and $h_0 = \sqrt{K_z}$.

3.2.2 Finding local skyrmion solutions

First note that there are different type of solutions for different values of $\frac{h}{h_0}$. To see this, one must notice that h_0 is the spin flop field. When $h = h_0$ the Néel vector flops into the basal plane and for $h > h_0$ the magnetic field gives rise to a magnetization (see Figs. 3.3 and 3.4). This causes weak ferromagnet behavior. When $h < h_0$ we get solutions with boundary conditions $\theta(0) = \pi$ and $\theta(\infty) = 0$. At the moment $h > h_0$ we also get solutions with boundary conditions $\theta(0) = \pi$ and $\theta(\infty) = \pi/2$. From Figs. 3.5 and 3.7, we are able to give the order of magnitude of antiferromagnetic skyrmions and vortices. Because the width of the profiles is between order $\mathcal{O}(10^0)$ and $\mathcal{O}(10^1)$, it follows via the substitution $\tilde{\rho} = \sqrt{\frac{K_z}{\rho_s}}\rho$ that the width of skyrmions and vortices is of the order $\mathcal{O}(\sqrt{\rho_s/K_z})$ if D/D_0 is of order one.

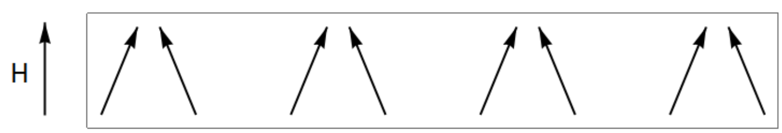

Figure 3.3: Graphical depiction of a antiferromagnetic state when the magnetic field is above the spin flop field ($h > h_0$).

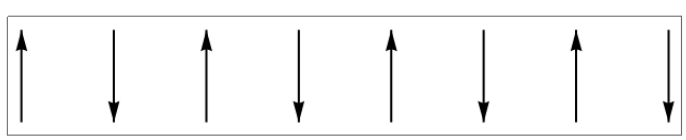

Figure 3.4: Graphical depiction of a antiferromagnetic ordering when the magnetic field is far below the spin flop field ($h < h_0$).

3.2.3 No magnetic field

In first instance we consider the case when $h = 0$. We see that there exist finite skyrmion configurations even in the absence of a magnetic field, which is different from the ferromagnetic case described in Section 2.1 where there is no stable skyrmion configuration in the absence of a magnetic field. In Fig. 3.5 skyrmion profiles for several values of $d = D/D_0$ are plotted. In this figure one is able to see that the skyrmion radius is getting larger as the antisymmetric exchange interaction (DM) is increasing.

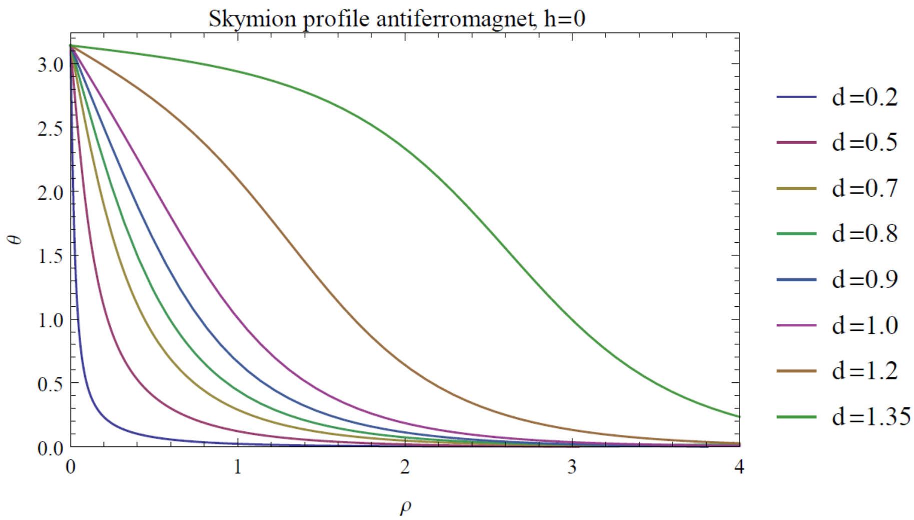

Figure 3.5: Skyrmion configuration for $D/D_0 \in \{0.2, 0.5, 0.7, 0.8, 0.9, 1.0, 1.2, 1.35\}$ and $h = 0$.

3.2.4 Magnetic field $h/h_0 = 0.3$

Now we consider the case in which $h/h_0 = 0.3$. We see that the radii of the skyrmions are slightly larger then in Fig. 3.5. We will continue to see the radius grow as we increase the magnetic field up to $h/h_0 = 1$, where the dimensionless radius will approximately have length 10.

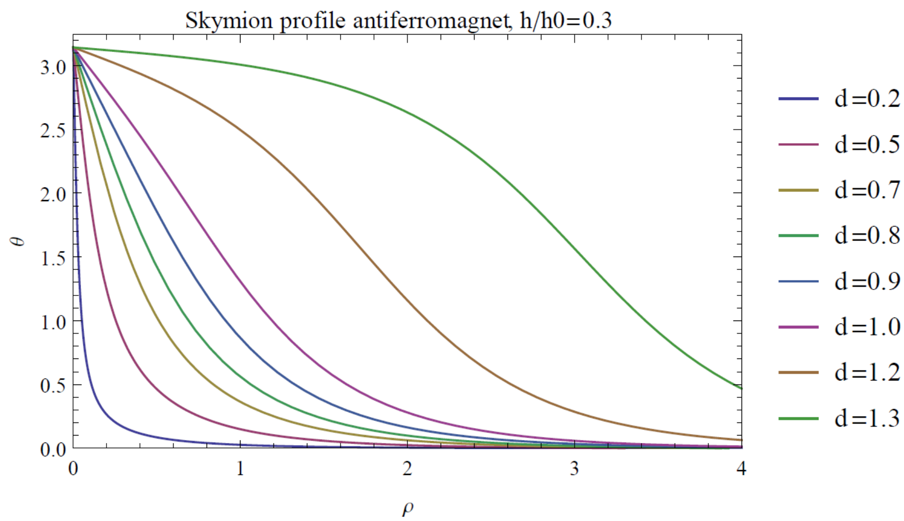

Figure 3.6: Skyrmion configuration for $D/D_0 \in \{0.2, 0.5, 0.7, 0.8, 0.9, 1.0, 1.2, 1.30\}$ with $h/h_0 = 0.3$.

3.2.5 Magnetic field $h/h_0 = 1.2$

If the magnetic field is larger then the spin flop field ($h > h_0$) we obtain vortex solutions. The boundary conditions for vortices are given by $\theta(0) = \pi$ and $\theta(\infty) = \pi/2$, which differ form the ones used for skyrmions. We consider the case in which $h/h_0 = 1.2$. In Fig. 3.7 vortex profiles for several values of $d = D/D_0$ are plotted. Like in the skyrmion case the radius of the solution is getting larger as the antisymmetric exchange interaction (DM) is growing.

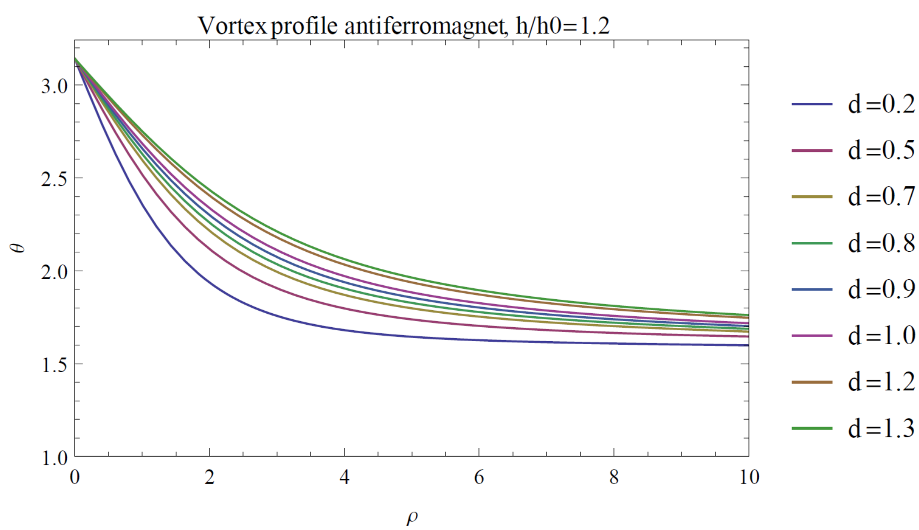

Figure 3.7: Skyrmion configuration for $D/D_0 \in \{0.2, 0.5, 0.7, 0.8, 0.9, 1.0, 1.2, 1.30\}$ with $h/h_0 = 1.2$.

Chapter 4

Interaction between antiferromagnetic skyrmions and spin waves

In this chapter we will discuss the interaction between an antiferromagnetic skyrmion and spin waves using similar methods as used in Refs. [3, 9]. We consider the number of atomic layers in the thin film N_A to be sufficiently large s.t. the spin waves are a small perturbation on the skyrmion, but small enough for the translational invariance in the $\hat{z}$-direction to hold. This chapter aims at deriving an effective action which is only dependent on skyrmion position $\vec{R}$, by integrating out the spin wave fluctuations. Furthermore we assume $\dot{R}/c$ to be small.

4.1 Taking a perturbation of spin-waves around the skyrmion

In this section we describe spin wave fluctuations (see Fig. 4.1) around an antiferromagnetic skyrmion. This will be done by performing the expansion of spin wave fluctuations around the skyrmion up to second order. We categorize the resulting action into three parts, the classical action (no spin wave interactions), the static interaction (describes the static coupling between an antiferromagnetic skyrmion and spin waves) and dynamic interaction (describes the interaction between a moving skyrmion and spin waves) which are denoted by $\mathcal{S}_{cl}, \mathcal{S}_{fl}$ and $\mathcal{S}_I$ respectively.

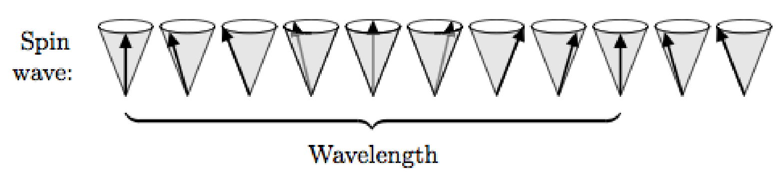

Figure 4.1: Semi-classical picture of a spin wave going through a one-dimensional material, with the spins rotating around their ground state position. Copyright: Addison-Wesley 2000.

In Section 3.1.5 we found that the action describing antiferromagnetic dynamics is given by Eq. (3.33)

$$\mathcal{S}_E[\hat{\mathbf{n}}] = iN_A\Upsilon[\hat{\mathbf{n}}] + N_A\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x}\left\{\frac{1}{2}\chi_0|\partial_\tau\hat{\mathbf{n}}(\mathbf{x})|^2 + i\chi_0\tilde{H}(\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x}))^z\right\} + \mathcal{F}[\hat{\mathbf{n}}], \tag{4.1}$$

$$\mathcal{F}[\hat{\mathbf{n}}] = N_A\int_0^\beta \mathrm{d}\tau\int_\Lambda \mathrm{d}\mathbf{x}\left\{\frac{\rho_s}{2}\sum_{l=x,y}|\partial_l\hat{\mathbf{n}}|^2 + \frac{D}{2}\left(\hat{y}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial x}\right) - \hat{x}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\hat{\mathbf{n}}}{\partial y}\right)\right) - K_z\hat{\mathbf{n}}_z^2 + \frac{1}{2}\chi_0\left(\tilde{\mathbf{H}}\cdot\hat{\mathbf{n}}(\mathbf{x})\right)^2\right\},$$

where N_A is given by the number of antiferromagnetic layers along the $\hat{z}$-direction. For the rest of this chapter we assume the magnetic field to be pointing in the $\hat{z}$-direction. The contribution of Berry phase Υ is negligible in the long wavelength limit, see Appendix B.2.2. From now on we assume that a skyrmion is present in $\hat{\mathbf{n}}$, thus we are looking at a configuration with winding number equal to one. We are still working in the long wavelength approximation and consider low temperatures, such that other effects like phonon interactions can be neglected and spin wave fluctuations will be small. Using the preceding assumptions we are able to write the configuration of the antiferromagnet as the skyrmion configuration plus small deviations from this ground state. Since $\hat{\mathbf{n}}$ is always a unit vector it will be useful to describe it using polar coordinates

$$\hat{\mathbf{n}}(\mathbf{x},\tau) = \{\sin[\theta(\mathbf{x},\tau)]\cos[\phi(\mathbf{x},\tau)], \sin[\theta(\mathbf{x},\tau)]\sin[\phi(\mathbf{x},\tau)], \cos[\theta(\mathbf{x},\tau)]\} \tag{4.2}$$

$$\implies |\partial_\tau\hat{\mathbf{n}}| = [\partial_\tau\theta(\mathbf{x},\tau)]^2 + \sin^2(\theta(\mathbf{x},\tau))[\partial_\tau\phi(\mathbf{x},\tau)]^2, \tag{4.3}$$

$$(\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x}))^z = \sin^2(\theta(\mathbf{x},\tau))\partial_\tau\phi(\mathbf{x},\tau). \tag{4.4}$$

The skyrmion configuration will characterized by the collective dynamical coordinate $\mathbf{R}(\tau)$

$$\theta(\mathbf{x},\tau) = \theta_0(\mathbf{x}-\mathbf{R}(\tau)) + \vartheta(\mathbf{x}-\mathbf{R}(\tau),\tau), \tag{4.5}$$

$$\phi(\mathbf{x},\tau) = \phi_0(\mathbf{x}-\mathbf{R}(\tau)) + \tilde{\varphi}(\mathbf{x}-\mathbf{R}(\tau),\tau). \tag{4.6}$$

Now define

$$\psi(\mathbf{x},\tau) = \begin{pmatrix}\Pi(\mathbf{x},\tau)/\sin(\theta_0)\\ \phi(\mathbf{x},\tau)\sin(\theta_0)\end{pmatrix} \simeq \zeta_0 + \eta, \quad \zeta_0(\mathbf{x},\tau) = \begin{pmatrix}\Pi_0(\mathbf{x},\tau)/\sin(\theta_0)\\ \phi_0(\mathbf{x},\tau)\sin(\theta_0)\end{pmatrix}, \quad \kappa(\mathbf{x},\tau) = \begin{pmatrix}\vartheta(\mathbf{x},\tau)\\ \tilde{\varphi}(\mathbf{x},\tau)\end{pmatrix}. \tag{4.7}$$

We would like to describe our system in canonical coordinates, which are given by $\{\Pi,\phi\}$ (where $\Pi = \cos(\theta)$). Performing the path integral will be much easier while using canonical coordinates. We start be treating a small perturbation in Π in terms of ϑ

$$\Pi \to \Pi_0 + \tilde{\nu}, \;\; \Pi \to \Pi_0 - \sin(\theta_0)\vartheta - \frac{1}{2}\cos(\theta_0)\vartheta^2 \tag{4.8}$$

$$\implies \vartheta \simeq -\frac{1}{\sin(\theta_0)}\tilde{\nu} - \frac{1}{2\sin^2(\theta_0)\tan(\theta_0)}\tilde{\nu}^2. \tag{4.9}$$

Now we introduce a local coordinate transformation which leaves the integration measure invariant

$$\tilde{\eta} = \begin{pmatrix}\tilde{\nu}(\mathbf{x},\tau)\\ \tilde{\varphi}(\mathbf{x},\tau)\end{pmatrix} \to \eta = \begin{pmatrix}\tilde{\nu}(\mathbf{x},\tau)/\sin(\theta_0)\\ \tilde{\varphi}(\mathbf{x},\tau)\sin(\theta_0),\end{pmatrix} \tag{4.10}$$

$$\implies \eta = \begin{pmatrix}\nu(\mathbf{x},\tau)\\ \varphi(\mathbf{x},\tau)\end{pmatrix}. \tag{4.11}$$

This implies $\vartheta \simeq -\nu - \frac{1}{2\tan(\theta_0)}\nu^2$.

The assumption that we have a skyrmion present, plus small deviations from the ground-state makes it useful to do an expansion of $\mathcal{F}$ w.r.t η. Because we know a skyrmion is a local minimum of

$\mathcal{F}$ it follows that $\frac{\delta\mathcal{F}}{\delta\zeta}=0$. Up to second order in $\delta\eta$, $\mathcal{F}$ is written as

$$\mathcal{F}=E_0+\int_0^\beta \mathrm{d}\tau\mathrm{d}\tau' \int_\Lambda \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\ \eta^\intercal\cdot\left\{\left.\frac{\delta^2\mathcal{F}}{\delta\eta^\intercal\delta\eta}\right|_{\eta^\intercal=\eta=0}\right\}\eta+\mathcal{O}(\eta^3), \tag{4.12}$$
$$E_0=\mathcal{F}[\hat{\mathbf{n}}_\mathbf{0}].$$

Note that $\mathcal{F}$ is written as a single integral over a continuous function of $(\mathbf{x},\tau)$. It follows that $\left.\frac{\delta^2\mathcal{F}}{\delta\eta^\intercal\delta\eta}\right|_{\eta^\intercal=\eta=0}$ is diagonal with respect to $(\mathbf{x},\tau)$.

Now we expand $\{\nu,\phi\}$ up to second order for the dynamical terms in Eq. (4.1). Via Appendix C.1 the second order spin wave expansion of $\int \mathrm{d}\tau\mathrm{d}\mathbf{x}|\partial_\tau\hat{\mathbf{n}}|$ becomes,

$$\begin{aligned}\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\,|\partial_\tau\hat{\mathbf{n}}|^2 &= \int \mathrm{d}\tau\mathrm{d}\mathbf{x}\Big\{\,|\partial_\tau\hat{\mathbf{n}}_\mathbf{0}|^2\\ &+\left[\sin(2\theta_0)\dot{\phi}_0^2-2\ddot{\theta}_0\right]\vartheta-2\left[\sin(2\theta_0)\dot{\theta}_0\dot{\phi}_0+\sin^2(\theta_0)\ddot{\phi}_0\right]\tilde{\varphi}\\ &+\vartheta\left(\cos(2\theta_0)\dot{\phi}_0^2-d_\tau^2\right)\vartheta-\tilde{\varphi}\left[\sin(2\theta_0)\dot{\theta}_0 d_\tau+\sin^2(\theta_0)d_\tau^2\right]\tilde{\varphi}+\vartheta\left[2\sin(2\theta_0)\dot{\phi}_0 d_\tau\right]\tilde{\varphi}\Big\}\\ &=\int \mathrm{d}\tau\ \tilde{M}\dot{R}^2+\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\ \Big\{-\eta^\intercal\cdot\partial_\tau^2\eta+\dot{R}^\beta\dot{R}^\gamma\left[\mathbf{J}_{\beta\gamma}\cdot\eta+\eta^\intercal\cdot(\Gamma_{\beta\gamma}+\mathbf{T}_\beta\partial_\gamma-\partial_\beta\partial_\gamma)\,\eta\right]\\ &+\dot{R}^\beta\ \eta^\intercal\cdot(2\partial_\beta\partial_\tau-\mathbf{T}_\beta\partial_\tau)\,\eta\Big\}.\end{aligned} \tag{4.13}$$

The second order expansion of $(\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x}))^z$ is also worked out in Appendix C.1 and is given by

$$\begin{aligned}\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\ (\hat{\mathbf{n}}(\mathbf{x})\times\partial_\tau\hat{\mathbf{n}}(\mathbf{x}))^z &= \int \mathrm{d}\tau\mathrm{d}\mathbf{x}\ \Big\{(\hat{\mathbf{n}}_\mathbf{0}\times\partial_\tau\hat{\mathbf{n}}_\mathbf{0})^z\\ &+\sin^2(\theta_0)\dot{\tilde{\varphi}}+\sin(2\theta_0)\dot{\phi}_0\vartheta+\sin(2\theta_0)\vartheta\dot{\tilde{\varphi}}+\cos(2\theta_0)\dot{\phi}_0\vartheta^2\Big\}\\ &\to\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\ \Big\{(\hat{\mathbf{n}}_\mathbf{0}\times\partial_\tau\hat{\mathbf{n}}_\mathbf{0})^z\\ &+\dot{R}^\beta\left[\sin(2\theta_0)(\partial_\beta\theta_0)\frac{\varphi}{\sin(\theta_0)}-\sin(2\theta_0)(\partial_\beta\phi_0)\vartheta-\vartheta\cos(2\theta_0)(\partial_\beta\phi_0)\vartheta\right]\\ &+\vartheta\sin(2\theta_0)d_\tau\left(\frac{\varphi}{\sin(\theta_0)}\right)\Big\}\\ &=\int \mathrm{d}\tau\ \tilde{\alpha}\epsilon_{ij}R^i\dot{R}^j+\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\ \Big\{-\eta^\intercal\cdot\mathbf{L}\partial_\tau\eta+\dot{R}^\beta\ \left[\eta^\intercal\cdot(\mathbf{M}_\beta+\mathbf{L}\partial_\beta)\,\eta+\mathbf{N}_\beta\cdot\eta\right]\Big\},\end{aligned} \tag{4.14}$$

where[1]

$$\begin{aligned}
\mathbf{J}_{\beta\gamma} &= \Bigg\{ [2(\partial_\beta\partial_\gamma\theta_0) - \sin(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0)] \begin{pmatrix}1\\0\end{pmatrix} \\
&\quad - 2\,[2\cos(\theta_0)(\partial_\beta\theta_0)(\partial_\gamma\phi_0) + \sin(\theta_0)(\partial_\beta\partial_\gamma\phi_0)] \begin{pmatrix}0\\1\end{pmatrix} \Bigg\}, \\
\Gamma_{\beta\gamma} &= \Bigg\{ \left[- \sin^2(\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0) - \frac{1}{2}\cot(\theta_0)(\partial_\beta\partial_\gamma\theta_0) \right] \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \\
&\quad + [(\csc(\theta_0) - 1)(\partial_\beta\theta_0)(\partial_\gamma\theta_0)] \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \\
&\quad + [(2\cos(\theta_0)\cot(\theta_0) + \sin(\theta_0))(\partial_\beta\phi_0)(\partial_\gamma\theta_0) - \cos(\theta_0)(\partial_\beta\partial_\gamma\phi_0)] \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\
&\quad + [\cos(\theta_0)(\partial_\beta\partial_\gamma\phi_0) - \sin(\theta_0)(\partial_\gamma\theta_0)(\partial_\beta\phi_0)] \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \Bigg\}, \\
\mathbf{T}_\beta &= \Bigg\{ 2\cos(\theta_0)(\partial_\beta\phi_0) \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \Bigg\}, \\
\mathbf{L} &= \Bigg\{ \cos(\theta_0) \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \Bigg\}, \\
\mathbf{M}_\beta &= \Bigg\{ \sin^2(\theta_0)(\partial_\beta\phi_0) \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} - \cos(2\theta_0)\csc(\theta_0)(\partial_\beta\theta_0) \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\
&\quad - \frac{1}{2}\sin(\theta_0)(\partial_\beta\theta_0) \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \Bigg\}, \\
\mathbf{N}_\beta &= \Bigg\{ \sin(2\theta_0)(\partial_\beta\phi_0) \begin{pmatrix}1\\0\end{pmatrix} + 2\cos(\theta_0)(\partial_\beta\theta_0) \begin{pmatrix}0\\1\end{pmatrix} \Bigg\}.
\end{aligned} \tag{4.15}$$

In the preceding equations we neglected terms proportional to $\ddot{R}$, since we assume the dynamics of the skyrmion to be slow. We dismissed the constant factors obtained by partial integration w.r.t. τ. The reason is that these factors only depend on the situation at time $\tau = 0$ and $\tau = \beta$, thus they give rise to an overall normalization factor. Because these partial integration factors do not influence the semi-classical equations of motion, we will omit these terms in the first instance. In Eqs. (4.16) and (4.17) we calculate the classical part of the dynamical terms in Eq. (4.1), Eq. (4.16) reduces to a mass and Eq. (4.17) gives rise to a Magnus force, this is worked out in Appendix B

$$\begin{aligned}
\int \mathrm{d}\tau \mathrm{d}\mathbf{x} |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 &= \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left| \partial_\beta \hat{\mathbf{n}}_\mathbf{0} \dot{R}^\beta \right|^2 = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ \partial_\beta \hat{n}_{0,\alpha} \partial_\gamma \hat{n}_0^\alpha \dot{R}^\beta \dot{R}^\gamma \\
&= \int \mathrm{d}\tau \tilde{M}_{\beta\gamma} \dot{R}^\alpha \dot{R}^\gamma, \quad \tilde{M}_{\beta\gamma} = \int \mathrm{d}\mathbf{x}\ \partial_\beta \hat{n}_{0,\alpha} \partial_\gamma \hat{n}_0^\alpha
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z &\to \int \mathrm{d}\tau \dot{R}^i R^j \left\{ \int \mathrm{d}\mathbf{x}\ 2(\partial_i \hat{\mathbf{n}}_\mathbf{0} \times \partial_j \hat{\mathbf{n}}_\mathbf{0})^z \right\} \\
&= \int d\tau \epsilon_{ij} R^i \dot{R}^j \left\{ \int_0^{2\pi} d\phi \int_0^\infty d\rho\ \partial_\rho(\sin^2(\theta_0)) \partial_\phi \phi_0 \right\} = \int d\tau \left\{ \tilde{\alpha} \epsilon_{ij} R^i \dot{R}^j \right\}, \\
\tilde{\alpha} &= \left\{ \int_0^{2\pi} d\phi \int_0^\infty d\rho\ \partial_\rho(\sin^2(\theta_0)) \partial_\phi \phi_0 \right\}.
\end{aligned} \tag{4.17}$$

[1] Note that N_β is orthogonal to the zero-modes. In the $\{\theta, \tilde{\phi}\}$ basis $N_i = \sin(2\theta_0) \begin{pmatrix}-\partial_i\phi_0\\ \partial_i\theta_0\end{pmatrix}$ and the zero-modes are given by $\sigma_i = \begin{pmatrix}\partial_i\theta_0\\ \partial_i\phi_0\end{pmatrix}$. Which implies $N_i^\intercal \cdot \sigma_i = 0$. A more extensive calculation shows $N_i^\intercal \cdot \sigma_j = 0$, for $\{i \neq j\}$, which is due to the factor of two in front of θ_0 in the $\sin(2\theta_0)$.

In Eq. (4.17) we used that the skyrmion configuration is rotationally invariant. Using this rotational invariance it follows that the mass matrix $\tilde{M}$ is diagonal. By writing $\mathbf{x}$ in cylindrical coordinates $(x = \rho\cos(\theta)), y = \rho\sin(\theta))$ we obtain (see Appendix B.1)

$$\begin{aligned} \tilde{M} = \tilde{M}_{xx} = \tilde{M}_{yy} = 2\pi \int_0^\infty d\rho \ \left\{ \rho \left(\partial_\rho \hat{\mathbf{n}}_\mathbf{0}\right)^2 + \frac{1}{\rho}\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}^2 \right\}, \\ \tilde{M}_{xy} = \tilde{M}_{yx} = 0, \end{aligned} \tag{4.18}$$

$$\tilde{\alpha} = 2\pi \int_0^\infty d\rho\, \partial_\rho(\sin^2(\theta_0)) = 2\pi \sin^2(\theta(\rho))\Big|_{\rho=0}^{\rho=\infty}. \tag{4.19}$$

To make the integral in Eq. (4.18) dimensionless and to use the results of Section 3.2 we perform the transformation $\tilde{\rho} = \sqrt{(K_z/\rho_s)}\,\rho$ on Eq. (4.18). By also defining $M = \chi_0 \tilde{M}$ and $\alpha = \chi_0 \tilde{H}\tilde{\alpha}$ we obtain

$$M = 2\pi\chi_0 \int_0^\infty d\tilde{\rho} \ \left\{ \tilde{\rho} \left(\partial_{\tilde{\rho}} \hat{\mathbf{n}}_\mathbf{0}\right)^2 + \frac{1}{\tilde{\rho}}\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}^2 \right\}, \tag{4.20}$$

$$\alpha = 2\pi\tilde{H}\chi_0 \sin^2(\theta(\rho))\Big|_{\rho=0}^{\rho=\infty}. \tag{4.21}$$

Since the dimensionless integral in Eq. (4.20) takes on values between zero and ten for different skyrmion configurations we obtained in Section 3.2, we see that M roughly has order of magnitude $\mathcal{O}(\chi_0)$ if D/D_0 is of order one. For skymions with winding number $n \in \mathbb{Z}$, α in Eq. (4.19) will be zero. So antiferromagnetic skyrmions do not feel a Magnus force, which is in agreement with Ref. [10]. For magnetic fields above the spin-flop field, vortices may arise (see Fig. 3.7). Vortices have different boundary conditions than skyrmions for which α is non-zero, which implies that antiferromagnetic vortices feel a Magnus force. For the boundary conditions of the vortex ($\theta_0(0) = \pi$ and $\theta_0(\infty) = \pi/2$), we see that $\alpha = -2\pi\tilde{H}\chi_0$.

Like in Ref. [9], we split the second order expansion of action in Eq. (4.1) into three parts, $S_E = \mathcal{S}_{cl} + \mathcal{S}_{fl} + \mathcal{S}_I$. The first part describes the skyrmion's dynamics without spin wave interactions

$$\mathcal{S}_{cl} = N_A \left\{ \frac{1}{2} M\dot{R}^2 + i\alpha\epsilon_{ij}R^i\dot{R}^j \right\}, \tag{4.22}$$

where $M = \chi_0\tilde{M}$ and $\alpha = H\tilde{\alpha}$. $\mathcal{S}_{fl}$ describes the interaction between a static skyrmion and spin waves. This static coupling is given by

$$\mathcal{S}_{fl} = N_A\, \eta^\intercal \cdot \mathcal{G}\eta. \tag{4.23}$$

Here we introduced the compact scalar product for operators and functions,

$$\eta^\intercal \cdot \mathcal{G}\eta = \int_0^\beta d\tau d\tau' \int_\Lambda d\mathbf{x}d\mathbf{x}'\ \eta^\intercal(\mathbf{x},\tau)\mathcal{G}(\mathbf{x},\tau,\mathbf{x}',\tau')\eta(\mathbf{x}',\tau'). \tag{4.24}$$

We defined the operator describing the static coupling between the skyrmion and spin waves to be,

$$\mathcal{G} = -\frac{\chi_0}{2}\left\{\partial_\tau^2 + 2i\tilde{H}\mathbf{L}\partial_\tau\right\} + \mathcal{H}, \tag{4.25}$$

$$\mathcal{H} = \frac{\delta^2\mathcal{F}}{\delta\eta^\intercal\delta\eta}\bigg|_{\eta^\intercal=\eta=0}. \tag{4.26}$$

The action associated with the interaction between a moving skyrmion and spin waves is given by

$$\mathcal{S}_I = N_A\left(\eta^\intercal \cdot \mathcal{K}\eta + \mathcal{J}\cdot\eta\right), \tag{4.27}$$

where

$$\mathcal{J} = \frac{\chi_0}{2}\Big\{\dot{R}^\beta \dot{R}^\gamma \mathbf{J}_{\beta\gamma} + 2i\dot{R}^\beta \tilde{H}\mathbf{N}_\beta\Big\}, \tag{4.28}$$

$$\mathcal{K} = \frac{\chi_0}{2}\Bigg\{\dot{R}^\beta \dot{R}^\gamma \left(\Gamma_{\beta\gamma} + \mathbf{T}_\beta \partial_\gamma - \partial_\beta \partial_\gamma\right) + \dot{R}^\beta \Big[(2\partial_\beta \partial_\tau - \mathbf{T}_\beta \partial_\tau) + 2i\tilde{H}\left(\mathbf{M}_\beta + \mathbf{L}\partial_\beta\right)\Big]\Bigg\}. \tag{4.29}$$

Conclusion

We expanded action (4.1) up to second order in $\{\nu, \phi\}$ and distinguished three different types of actions (interactions).

1. *Classical action*, which we denote with $\mathcal{S}_{cl}$ given in Eq. (4.22). The classical action is gives the dynamics of a skyrmion that has no interaction with spin waves. We found that a classical antiferromagnetic skyrmion does not feel a Magnus force and has a non-zero classical mass given by Eqs. (4.18) and (4.19). Vortices also posses a non-zero classical mass, but in addition they feel a Magnus force, both are also described using Eqs. (4.18) and (4.19).

2. *Static interaction*, which is denoted by $\mathcal{S}_{fl}$ Eq. (4.23). The static interaction describes the interaction between spin waves and a skyrmion which is standing still.

3. *Dynamic interaction*, which is denoted as $\mathcal{S}_I$ Eq. (4.27). This describes the interaction between moving skyrmions and spin waves.

We are interested in the effect that spin wave interactions have on the dynamics of a skyrmion. In this section we already found the dynamics for a non-interacting antiferromagnetic skyrmion and calculated the actions and operators to describe the interacting part. In the next sections we will calculate this change of dynamics due to spin wave interactions.

4.2 Skyrmionic dynamics due to spin wave interactions

In this section we integrate out spin wave fluctuations, using the path integral formalism of quantum field theory. This is quite similar to solving the equations of motion for spin waves and plugging it back in. We will use the Faddeev-Popov technique to promote the skyrmion position to a dynamic variable and derive and effective action for the dynamics of the skyrmion. Throughout this chapter we will assume N_A to be large such that spin waves form a small perturbation on the skyrmion. We assume the temperature to be small such that we can neglect other interaction effects like spin phonon interactions and will mainly focus on the zero temperature limit. In Section 4.1 we used that the skyrmion's velocity w.r.t the spin wave velocity is a small parameter $\dot{R}/c$, in this section we continue using this assumption. In Section 4.2.1 we derive the skyrmions dynamics if there is no magnetic field present, while in Section 4.2.2 we consider the effect of a small magnetic field on the dynamics of skyrmions.

4.2.1 Skyrmion dynamics in the absence of a magnetic field

For convenience we start with the action $S_E = S_{cl} + S_{fl} + S_I$ and set the magnetic field equal to zero. So in this section we use

$$\mathcal{G} = -\frac{\chi_0}{2}\partial_\tau^2 + \mathcal{H}, \tag{4.30}$$

$$\mathcal{J} = \frac{\chi_0}{2}\left\{\dot{R}^\beta \dot{R}^\gamma \mathbf{J}_{\beta\gamma}\right\}, \tag{4.31}$$

$$\begin{aligned}\mathcal{K} &= \frac{\chi_0}{2}\left\{\dot{R}^\beta \dot{R}^\gamma \left(\Gamma_{\beta\gamma} + \mathbf{T}_\beta\partial_\gamma - \partial_\beta\partial_\gamma\right) + \dot{R}^\beta \left(2\partial_\beta\partial_\tau - \mathbf{T}_\beta\partial_\tau\right)\right\} \\ &= \frac{\chi_0}{2}\left\{\dot{R}^\beta \dot{R}^\gamma \mathcal{P}_{\beta\gamma} + \dot{R}^\beta \mathcal{N}_\beta\partial_\tau\right\},\end{aligned} \tag{4.32}$$

where $\tilde{S} = \frac{\chi_0}{2}$, $\mathcal{P}_{\beta\gamma} = \Gamma_{\beta\gamma} + \mathbf{T}_\beta\partial_\gamma - \partial_\beta\partial_\gamma$ and $\mathcal{N}_\beta\partial_\tau = \left(2\partial_\beta - \mathbf{T}_\beta\right)\partial_\tau$. Be aware that these operators are Hermitian by construction.

A translation is either described by R or by the "zero modes", $\tilde{\nu} \propto \partial_i\Pi_0$ and $\tilde{\phi} \propto \partial_i\phi_0$. To avoid double counting we impose the constraint that spin wave modes must be orthogonal to the zero modes. This constraint must hold for $i \in \{x, y\}$ and for all time τ, we thus have to impose the following constraint

$$\int d\mathbf{x}\, \partial_i\Pi_0(\mathbf{x}-\mathbf{R},\tau)\tilde{\nu}(\mathbf{x},\tau) = \int d\mathbf{x}\, \partial_i\Pi_0(\mathbf{x}-\mathbf{R},\tau)\sin(\theta_0)\nu(\mathbf{x},\tau) = 0, \tag{4.33}$$

$$\int d\mathbf{x}\, \partial_i\phi_0(\mathbf{x}-\mathbf{R},\tau)\tilde{\varphi}(\mathbf{x},\tau) = \int d\mathbf{x}\, \partial_i\phi_0(\mathbf{x}-\mathbf{R},\tau)\varphi(\mathbf{x},\tau)/\sin(\theta_0) = 0, \tag{4.34}$$

$$\implies \int d\mathbf{x}\, \sigma_i^\intercal \eta(x,\tau) = 0, \tag{4.35}$$

where

$$\sigma_i = \begin{pmatrix}(\partial_i\Pi_0)\sin(\theta_0) \\ (\partial_i\phi_0)/\sin(\theta_0)\end{pmatrix}. \tag{4.36}$$

This is equivalent to the restriction

$$Q_i(\mathbf{R}) = \int d\mathbf{x}\, \sigma_i^\intercal(\mathbf{x}-\mathbf{R},\tau)\psi(\mathbf{x},\tau) = 0. \tag{4.37}$$

We incorporate this constraint into the path integral by means of the Faddeev-Popov technique and promote $\mathbf{R}$ to a dynamic variable. This uses the following identity

$$\int \mathcal{D}R\, \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right) = 1, \tag{4.38}$$

where $\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}$ is the Jacobian. We obtain

$$Z[\mathbf{R}] = \int \mathcal{D}\mathbf{R}\mathcal{D}\eta\, \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right)e^{-S_E[\eta]}. \tag{4.39}$$

Using the second order expansion in η and $\dot{R}$ which is determined in Section 4.1 we obtain

$$Z = \int \mathcal{D}\mathbf{R}\, e^{-\mathcal{S}_{cl}[\mathbf{R}]}F[\mathbf{R}], \tag{4.40}$$

where S_{cl} is the action of a free antiferromagnetic skyrmion and

$$F[\mathbf{R}] = \int \mathcal{D}\eta\, \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right)e^{-N_A\{\eta^\intercal\cdot[\mathcal{G}+\mathcal{K}]\eta+\mathcal{J}\cdot\eta\}}. \tag{4.41}$$

Eq. (4.41) describes the interaction between an antiferromagnetic skyrmion and spin waves. Integrating out these spin waves would give the effect of spin wave interaction on the dynamics of a skyrmion. To integrate out spin waves it's useful to complete the square in Eq. (4.41). Because we are working up to order $\mathcal{O}(\dot{R}^2)$, it is sufficient to shift η by $\rho = (1/2)\mathcal{G}^{-1}\mathcal{J}$ [3]. We denote the introduced variable by $\tilde{\eta} = \eta + \rho$,

$$\begin{aligned}
&\eta^{\mathsf{T}} \cdot [\mathcal{G} + \mathcal{K}]\, \eta + \mathcal{J} \cdot \eta \\
&= \{\tilde{\eta} - (1/2)\mathcal{G}^{-1}\mathcal{J}\}^{\mathsf{T}} \cdot [\mathcal{G} + \mathcal{K}]\, \{\tilde{\eta} - (1/2)\mathcal{G}^{-1}\mathcal{J}\} + \mathcal{J} \cdot \{\tilde{\eta} - (1/2)\mathcal{G}^{-1}\mathcal{J}\} \\
&= \tilde{\eta}^{\mathsf{T}} \cdot [\mathcal{G} + \mathcal{K}]\, \tilde{\eta} - \frac{1}{2}(\mathcal{G}^{-1}\mathcal{J})^{\mathsf{T}} \cdot \mathcal{G}\tilde{\eta} - \frac{1}{2}\tilde{\eta}^{\mathsf{T}} \cdot \mathcal{G}\mathcal{G}^{-1}\mathcal{J} + \mathcal{J} \cdot \tilde{\eta} + \mathcal{O}(\dot{R}^3) \\
&= \tilde{\eta}^{\mathsf{T}} \cdot [\mathcal{G} + \mathcal{K}]\, \tilde{\eta} + \mathcal{O}(\dot{R}^3).
\end{aligned} \tag{4.42}$$

Next we want to consider the contribution of the determinant in Eq. (4.41) to the path integral. Note that σ_i is antisymmetric around $\mathbf{R}$ and $\mathcal{J}$ is symmetric around $\mathbf{R}$. Since $\mathcal{G}$ is invariant under parity transformation, we note that $\int d\mathbf{x}\ \sigma_i^{\mathsf{T}}(\mathbf{x} - \mathbf{R}, \tau)\rho(\mathbf{x}, \tau) \propto \int d\mathbf{x}\ \sigma_i^{\mathsf{T}}(\mathbf{x} - \mathbf{R}, \tau)\mathcal{G}^{-1}\mathbf{J}_{\beta_\gamma} = 0$. This implies $\int d\mathbf{x}\ \sigma_i^{\mathsf{T}}(\mathbf{x} - \mathbf{R}, \tau)\tilde{\eta}(\mathbf{x}, \tau) = 0$, so the newly introduced variable also must be orthogonal to the zero modes. This means that the Gaussian over our newly introduced variable $\tilde{\eta}$ is well defined and the fluctuations have effective size $\mathcal{O}(1/\sqrt{N_A})$.

In the remainder of this section we evaluate Eq. (4.41). We start with determining the effect of $\det\left(\frac{\delta \mathbf{Q}}{\delta \mathbf{R}}\right)$ on the dynamics of the skyrmion, we find

$$\frac{\delta Q_i}{\delta R_j} = -\int d\mathbf{x}\ (\partial_j \sigma_i^{\mathsf{T}}(\mathbf{x} - \mathbf{R}, \tau))\psi(\mathbf{x}, \tau)\delta(t - t'). \tag{4.43}$$

Using the identity $\det = \exp \operatorname{tr} \ln$ and rescaling our variable $\hat{\eta} = \sqrt{N_A}\tilde{\eta}$ we get

$$\det\left(\frac{\delta \mathbf{Q}}{\delta \mathbf{R}}\right) \propto \exp\left\{\operatorname{tr}\ln\left(\mathbf{1} - \frac{\mathcal{Y}_0}{\sqrt{N_A}}\int d\mathbf{x}(\partial_j\sigma_i^{\mathsf{T}})\hat{\eta} + \mathcal{Y}_0\int d\mathbf{x}(\partial_j\sigma_i^{\mathsf{T}})\rho\right)\right\}. \tag{4.44}$$

Where $\int(\partial_j\sigma_j^{\mathsf{T}})\zeta_0 = 1/\mathcal{Y}_0$ and $\int(\partial_x\sigma_y^{\mathsf{T}})\zeta_0 = -\int(\partial_y\sigma_x^{\mathsf{T}})\zeta_0 = 0$. The constant $\exp(\operatorname{tr}\ln 1/\mathcal{Y}_0)$ is taken as a pre-factor. The second term vanishes for the large N_A and the last term gives rise to a pure mass re-normalization, which is of order $\mathcal{O}((N_A)^0)$.

By performing the integration and neglecting irrelevant pre-factors in Eq. (4.41) becomes [20]

$$\begin{aligned}
F[\mathbf{R}] &= e^{-(\Delta M/2)\int d\tau \dot{R}^2}\int \mathcal{D}\hat{\eta}\ \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])e^{-\hat{\eta}^{\mathsf{T}}\cdot[\mathcal{G}+\mathcal{K}]\hat{\eta}} \\
&= e^{-(\Delta M/2)\int d\tau \dot{R}^2}\frac{1}{\sqrt{\det'(\mathcal{G} + \mathcal{K})}}.
\end{aligned} \tag{4.45}$$

The prime in the above equation denotes the omission of the zero-modes. By again using the identity $\det = \exp \operatorname{tr} \ln$ and expanding the logarithm we obtain

$$\begin{aligned}
\frac{1}{\sqrt{\det'(\mathcal{G} + \mathcal{K})}} &= e^{-(1/2)\operatorname{tr}'\ln\left(\mathcal{G}[1+\mathcal{G}^{-1}\mathcal{K}]\right)} \\
&= \frac{1}{\sqrt{\det'(\mathcal{G})}}e^{-(1/2)\operatorname{tr}'[\mathcal{G}^{-1}\mathcal{K}-(1/2)(\mathcal{G}^{-1}\mathcal{K})^2+\mathcal{O}(\dot{R}^3)]}.
\end{aligned} \tag{4.46}$$

The factor $\dfrac{1}{\sqrt{\det'(\mathcal{G})}}$ is independent of $\dot{R}$ and gives the partition function of spin waves interacting with a static skyrmion. Since we are interested in the dynamic properties of a skyrmion, this term can be omitted and just treated as a pre-factor. We evaluate the trace in Eq. (4.46) in the basis of eigenfunctions of $\mathcal{G}$, the eigenfunctions factories into a space- and time depended part,

$$\mathcal{G}\left|n\right\rangle\left|\omega\right\rangle = (\tilde{S}\omega^2 + \epsilon_n)\left|n\right\rangle\left|\omega\right\rangle, \tag{4.47}$$

where $\langle x|n\rangle \langle \tau|\omega\rangle = \psi_n(\mathbf{x})e^{i\omega\tau}/\sqrt{\beta}$ are the normalized eigenvectors of the operator $\mathcal{G}$, with Matsubara frequencies $\omega = 2\pi m/\beta$, for $m \in \mathbb{Z}$. In this basis of eigenvectors of $\mathcal{G}$ the first order term in Eq. (4.46) is given by [2],

$$-\frac{1}{2}\operatorname{tr}'\mathcal{G}^{-1}\mathcal{K} = -\frac{1}{2\beta}\sum_{\omega,n}{}'\int d\tau \frac{\langle\psi_n, \mathcal{K}\psi_n\rangle}{\tilde{S}\omega^2+\epsilon_n} = -\frac{1}{2\beta}\sum_{\omega,n}{}'\frac{\tilde{S}}{\tilde{S}\omega^2+\epsilon_n}\int d\tau \left\{\dot{R}^\beta\dot{R}^\gamma\langle\psi_n, \mathcal{P}_{\beta\gamma}\psi_n\rangle + \dot{R}^\beta\omega\langle\psi_n, \mathcal{N}_\beta\psi_n\rangle\right\}$$
$$= -\frac{1}{2\beta}\sum_{\omega,n}{}'\frac{1}{\omega^2+\hat{\epsilon}_n^2}\int d\tau \dot{R}^\beta\dot{R}^\gamma\langle\psi_n, \mathcal{P}_{\beta\gamma}\psi_n\rangle = -\int_0^\beta d\tau \Delta\tilde{M}\dot{R}^2, \tag{4.48}$$

where we introduced $\langle f, g\rangle = \int d\mathbf{x}\, f^\intercal(\mathbf{x})g(\mathbf{x})$ and

$$\Delta\tilde{M}_{ij} = \frac{\tilde{S}}{2\beta}\sum_{\omega,n}{}'\frac{\langle\psi_n, \mathcal{P}_{ij}\psi_n\rangle}{\tilde{S}\omega^2+\epsilon_n} = \frac{1}{2\beta}\sum_{\omega,n}{}'\frac{\langle\psi_n, \mathcal{P}_{ij}\psi_n\rangle}{\omega^2+\hat{\epsilon}_n^2}, \tag{4.49}$$

with $\hat{\epsilon}_n^2 = \epsilon_n/\tilde{S}$. The summations over ω can be performed using the following exact relation

$$D_\epsilon(\tau) = \frac{2\epsilon}{\beta}\sum_{\omega_n}\frac{e^{i\omega_n\tau}}{\omega_n^2+\epsilon^2} = \frac{\cosh(\epsilon(|\tau|-\beta/2))}{\sinh(\beta\epsilon/2)}, \tag{4.50}$$

where $\omega_n = 2\pi n/\beta$ and the right hand side of Eq. (4.50) will be periodically extended beyond $|\tau| \le \beta/2$. Using identity (4.50) to explicitly evaluate the sum over the Matsubara frequencies in Eq. (4.49), we obtain

$$\Delta\tilde{M}_{ij} = \frac{1}{2\beta}\sum_{\omega,n}{}'\frac{\langle\psi_n, \mathcal{P}_{ij}\psi_n\rangle}{\omega^2+\hat{\epsilon}_n^2} = \frac{1}{4}\sum_{n}{}'\frac{\langle\psi_n, \mathcal{P}_{ij}\psi_n\rangle}{\hat{\epsilon}_n}\frac{\cosh(\beta\hat{\epsilon}_n/2)}{\sinh(\beta\hat{\epsilon}_n/2)}. \tag{4.51}$$

In the low temperature limit ($\beta \to \infty$) Eq. (4.51) becomes

$$\Delta\tilde{M}_{ij} \xrightarrow{\beta\to\infty} \frac{1}{4}\sum_{n}{}'\frac{\langle\psi_n, \mathcal{P}_{ij}\psi_n\rangle}{\hat{\epsilon}_n}. \tag{4.52}$$

We have thus obtained another pure mass renormalization ΔM of order $\mathcal{O}((N_A)^0)$. Please note that $\Delta\tilde{M}_{xx} = \Delta\tilde{M}_{yy}$ and $\Delta\tilde{M}_{xy} = -\Delta\tilde{M}_{yx} = 0$, since we started with a rotationally invariant system. In the absence of a magnetic field, all the damping due to spin waves comes from the remaining terms in Eq. (4.46). Up to second order in $\dot{R}$ we obtain

$$-\frac{1}{4}\operatorname{tr}'(\mathcal{G}^{-1}\mathcal{K})^2 = -\frac{1}{4\beta^2}\sum_{\omega,n,\omega',n'}{}'\frac{\langle\omega|\langle n|\mathcal{K}|n'\rangle|\omega'\rangle\langle\omega'|\langle n'|\mathcal{K}|n\rangle|\omega\rangle}{(\tilde{S}\omega^2+\epsilon_n)(\tilde{S}\omega'^2+\epsilon_{n'})}$$
$$= \frac{\tilde{S}^2}{4\beta^2}\int_0^\beta d\tau\int_0^\beta d\tau'\ \dot{R}^i(\tau)\dot{R}^j(\tau')\sum_{\omega,n,\omega',n'}{}'\frac{\omega\omega'\langle\psi_n, \mathcal{N}_i\psi_{n'}\rangle\langle\psi_{n'}, \mathcal{N}_j\psi_n\rangle}{(\tilde{S}\omega^2+\epsilon_n)(\tilde{S}\omega'^2+\epsilon_{n'})}e^{i(\omega-\omega')(\tau-\tau')}$$
$$= \frac{\tilde{S}^2}{4\beta^2}\int_0^\beta d\tau\int_0^\beta d\tau'\dot{R}^i(\tau)\dot{R}^j(\tau')\sum_{\omega,n,\omega',n'}{}'\frac{\omega\omega'\mathcal{B}_{ij}^{n,n'}e^{i(\omega-\omega')(\tau-\tau')}}{(\tilde{S}\omega^2+\epsilon_n)(\tilde{S}\omega'^2+\epsilon_{n'})}. \tag{4.53}$$

In the above we defined

$$\mathcal{B}_{ij}^{n,n'} = \langle\psi_n, \mathcal{N}_i\psi_{n'}\rangle\langle\psi_{n'}, \mathcal{N}_j\psi_n\rangle. \tag{4.54}$$

[2]Note the first order term in $\dot{R}$ is zero, because $\omega\langle\psi_n, \mathcal{N}_\beta\psi_n\rangle \propto \omega$ and $(\tilde{S}\omega^2+\epsilon_n)$ is symmetric in ω.

Note that we omitted $\mathcal{P}_{\alpha\beta}$ terms, because these will give rise to terms which are of order $\mathcal{O}(\dot{R}^3)$. The damping due to spin waves is thus be given by

$$-\frac{1}{4}\,\mathrm{tr}'(\mathcal{G}^{-1}\mathcal{K})^2 = \int_0^\beta d\tau \int_0^\beta d\tau' \dot{R}_i(\tau)\gamma_{ij}(\tau-\tau')\dot{R}_j(\tau'), \tag{4.55}$$

where we defined

$$\gamma_{ij}(\tau) = \frac{\tilde{S}^2}{4\beta^2} \sum_{\omega,n,\omega',n'}{}' \frac{\omega\omega' \mathcal{B}_{ij}^{n,n'} e^{i(\omega-\omega')\tau}}{(\tilde{S}\omega^2+\epsilon_n)(\tilde{S}\omega'^2+\epsilon_{n'})}. \tag{4.56}$$

Using identity (4.50) we find

$$\begin{aligned}\frac{1}{2\beta}\sum_\omega \frac{\omega e^{i\omega\tau}}{\tilde{S}\omega^2+\epsilon_n} &= (1/\tilde{S})\frac{-i\partial_\tau}{2\beta}\sum_\omega \frac{e^{i\omega\tau}}{\omega^2+\hat{\epsilon}_n^2} \\ &= \frac{-i\partial_\tau}{4\tilde{S}\hat{\epsilon}_n} D_{\hat{\epsilon}_n}(\tau) \simeq \frac{-i}{4\tilde{S}}\frac{\sinh(\hat{\epsilon}_n(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_n/2)},\end{aligned} \tag{4.57}$$

where $\hat{\epsilon}_n^2 = \epsilon_n/\tilde{S}$. A formal expression for the damping kernel can now be obtained by using identity Eq. (4.57) in Eq. (4.53)

$$\gamma_{ij}(\tau) = \frac{1}{16}\sum_{n,n'}{}' \mathcal{B}_{ij}^{n,n'} \frac{\sinh(\hat{\epsilon}_n(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_n/2)} \frac{\sinh(\hat{\epsilon}_{n'}(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_{n'}/2)}. \tag{4.58}$$

In the low temperature limit $\beta \to \infty$ we use the following limits

$$\begin{aligned}\sinh(x) &\to \frac{e^x}{2}, \quad \text{for } x\to\infty \\ \sinh(-x) &\to -\frac{e^{-(-x)}}{2}, \quad \text{for } x\to\infty\end{aligned}$$

Thus in the low temperature limit $\beta\to\infty$ we get

$$\frac{\sinh(\hat{\epsilon}_n(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_n/2)} \to -\frac{e^{-(\hat{\epsilon}_n(|\tau|-\beta/2))}}{e^{\beta\hat{\epsilon}_n/2}} = -e^{-\hat{\epsilon}_n|\tau|}. \tag{4.59}$$

Using Eq. (4.59) to evaluate Eq. (4.58) in the low temperature limit ($\beta\to\infty$) we obtain

$$\begin{aligned}\gamma_{ij}(\tau) =& \frac{1}{16}\sum_{n,n'}{}' \mathcal{B}_{ij}^{n,n'} \frac{\sinh(\hat{\epsilon}_n(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_n/2)} \frac{\sinh(\hat{\epsilon}_{n'}(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_{n'}/2)} \\ \xrightarrow{\beta\to\infty}& \frac{1}{16}\sum_{n,n'}{}' \mathcal{B}_{ij}^{n,n'} e^{-(\hat{\epsilon}_n+\hat{\epsilon}_{n'})|\tau|}.\end{aligned} \tag{4.60}$$

If the dynamics of R are slow compared to the relaxation time of the damping kernel and the temperature is sufficiently small s.t. $\beta \gg \hat{\epsilon}_0^{-1}$, then the damping kernel (4.60) reduces to a pure mass normalization.

$$\begin{aligned}-\frac{1}{4}\,\mathrm{tr}'(\mathcal{G}^{-1}\mathcal{K})^2 &= \int_0^\beta d\tau \int_0^\beta d\tau' \dot{R}_i(\tau)\gamma_{ij}(\tau-\tau')\dot{R}_j(\tau') \\ &\approx \int_0^\beta d\tau\, \dot{R}_i(\tau)\frac{\mathcal{M}_{ij}}{2}\dot{R}_j(\tau),\end{aligned} \tag{4.61}$$

where

$$\begin{aligned}\mathcal{M}_{ij} &\equiv \frac{1}{8}\sum_{n,n'}{}' \mathcal{B}_{ij}^{n,n'} \int_{-\infty}^{\infty} d\mu \; e^{-(\hat{\epsilon}_n + \hat{\epsilon}_{n'})|\mu|} \\ &= \frac{1}{4}\sum_{n,n'}{}' \frac{\mathcal{B}_{ij}^{n,n'}}{\hat{\epsilon}_n + \hat{\epsilon}_{n'}} \simeq \frac{1}{8}\frac{\mathcal{B}_{ij}^{0,0}}{\hat{\epsilon}_0}.\end{aligned} \tag{4.62}$$

Here $\hat{\epsilon}_0$ represents the lowest non-zero eigenenergy of $\mathcal{H}$ Eqs. (4.25) and (4.100) which will be a breathing mode[3]. Please note that this pure mass re-normalization is of order $\mathcal{O}((N_A)^0)$. Since the lowest eigenenergy of $\mathcal{G}$ will be lower than magnon gap, a lower bound on the magnitude of the mass correction $\mathcal{M}$ is given by calculating $\mathcal{M}$ for the magnon gap energy. From Ref. [21] we see that magnon gap has order of magnitude $\mathcal{O}(\sqrt{K_z/\chi_0})$ in the absence of a magnetic field. To determine the order of magnitude of $\mathcal{B}_{ij}^{0,0} = \langle\psi_0, \mathcal{N}_i\psi_0\rangle\langle\psi_0, \mathcal{N}_j\psi_0\rangle$ we assume that the low energy eigenstates of $\mathcal{G}$ approximately have the same size as the skyrmion, thus $\psi_0 \sim e^{-(\sqrt{K_z/\rho_s})\rho}$. By rewriting $\langle\psi_0, \mathcal{N}_i\psi_0\rangle\langle\psi_0, \mathcal{N}_j\psi_0\rangle$ into cylindrical coordinates and using the substitution $\tilde{\rho} = \sqrt{K_z/\rho_s}$ we obtain an order of magnitude given by $\mathcal{O}(\rho_s/K_z)$. The lower bound on the mass in Eq. (4.62) will thus be given by

$$\mathcal{O}(\mathcal{M}) > \mathcal{O}(\rho_s(\chi_0^3/K_z^5)^{1/4}).$$

Using the definition of mass described in Ref. [9, p. 5] as the zero frequency limit of the damping kernel, we obtain that the (AFM) skyrmion mass correction is given by

$$\mathcal{M}_{ij}(\beta) = \frac{1}{4}\sum_{n,n'}{}' \mathcal{B}_{ij}^{n,n'} \frac{\epsilon_m \coth(\beta\epsilon_m/2) - \epsilon_n \coth(\beta\epsilon_n/2)}{\epsilon_m^2 - \epsilon_n^2}. \tag{4.63}$$

Which is valid of one considers slow dynamics of the skyrmion and asymptotically large imaginary times $\tau \gg \epsilon_0^{-1}$.

Conclusion

Considering sufficiently small temperatures and taking into account only the interaction between skyrmions and spin waves up to second order, we obtain several mass renormalizations and a damping kernel of order $\mathcal{O}((N_A)^0)$ described in Eq. (4.60). We thus obtain the following expression for the effective action for a skyrmion interacting with spin waves.

$$\mathcal{S}_{eff}[R] = \int_0^\beta d\tau \left\{ \frac{M_{eff}}{2}\dot{R}^2 \right\} + \int_0^\beta d\tau \int_0^\beta d\tau' \left\{ \dot{R}_i(\tau)\gamma_{ij}(\tau - \tau')\dot{R}_j(\tau') \right\}, \tag{4.64}$$

where

$$M_{eff} \simeq N_A M + \Delta M. \tag{4.65}$$

In the above $N_A M$ will be classical mass of an antiferromagnetic skyrmion described in Eq. (4.18), which is of order $\mathcal{O}(N_A\chi_0)$. Furthermore ΔM will gives a mass renormalization due to spin wave interactions and is given by the total of Eqs. (4.44) and (4.49) which is of order $\mathcal{O}((N_A)^0)$. Using the magnon gap as a lower bound on the magnitude of the mass we obtain

$$\mathcal{O}(\Delta M) > \mathcal{O}((\chi_0^3/K_z)^{1/4}).$$

If we would consider the dynamics of the skyrmion to be much slower than the relaxation time of the damping kernel and the temperature small enough such that $\beta \gg \epsilon_0$. Than the damping kernel will

[3] We are able to distinct between two types of eigenstates of $\mathcal{G}$; breathing modes and scattering states. Breathing modes locally deform the skyrmion and have an eigenenergies which are lower than the magnon gap. Scattering states exist for energies above the magnon gap.

reduce to a pure mass renormalization given by Eqs. (4.62) and (4.63). Thus when considering slow dynamics and low temperatures the action in Eq. (4.64) will thus reduce to

$$\mathcal{S}_{eff}[R] = \frac{M_{eff}}{2}\int_0^\beta d\tau \dot{R}^2, \tag{4.66}$$

where

$$M_{eff} = N_A M + \Delta M + \mathcal{M}. \tag{4.67}$$

With the lower bound on mass $\mathcal{O}(\mathcal{M})$ given by

$$\mathcal{O}(\mathcal{M}) > \mathcal{O}(\rho_s(\chi_0^3/K_z^5)^{1/4}).$$

4.2.2 Skyrmion dynamics for small magnetic fields

In this section we consider $h\sqrt{\chi_0} \ll 1$, thus we start with a small magnetic field being present. The action we begin with is given by $S_E = S_{cl} + S_{fl} + S_I$, with the operators defined in Eqs. (4.25), (4.28) and (4.29)

$$\begin{aligned}
\mathcal{G} &= -\frac{\chi_0}{2}\left\{\partial_\tau^2 + 2i\tilde{H}\mathbf{L}\partial_\tau\right\} + \mathcal{H} \\
\mathcal{J} &= \frac{\chi_0}{4}\left\{\dot{R}^\beta\dot{R}^\gamma\mathbf{J}_{\beta\gamma} + 2i\dot{R}^\beta\tilde{H}\mathbf{N}_\beta\right\}, \\
\mathcal{K} &= \frac{\chi_0}{2}\Bigg\{\dot{R}^\beta\dot{R}^\gamma\left(\Gamma_{\beta\gamma} + \mathbf{T}_\beta\partial_\gamma - \partial_\beta\partial_\gamma\right) \\
&\qquad + \dot{R}^\beta\left[(2\partial_\beta\partial_\tau - \mathbf{T}_\beta\partial_\tau) + 2i\tilde{H}\left(\mathbf{M}_\beta + \mathbf{L}\partial_\beta\right)\right]\Bigg\}.
\end{aligned}$$

We define $h^2 = \frac{1}{2}\chi_0\tilde{H}^2$, $\tilde{S} = \frac{\chi_0}{2}$, $\mathcal{A}_\beta = (\mathbf{M}_\beta + \mathbf{L}\partial_\beta)$, $\mathcal{P}_{\beta\gamma} = \Gamma_{\beta\gamma} + \mathbf{T}_\beta\partial_\gamma - \partial_\beta\partial_\gamma$, $\mathcal{N}_\beta\partial_\tau = (2\partial_\beta - \mathbf{T}_\beta)\,\partial_\tau$ and $\mathcal{W}_\beta = (\mathbf{M}_\beta + \mathbf{L}\partial_\beta)$. Note that all of the above operators are hermitian by construction, for the product we defined in Eq. (4.24). The action will be given by

$$\begin{aligned}
S_{cl} &= N_A S^2\left\{\frac{1}{2}M\dot{R}^2 - i\alpha\epsilon_{ij}R^i\dot{R}^j\right\}, \\
S_{fl} &= N_A\,\eta^\intercal\cdot\mathcal{G}\eta \\
S_I &= N_A\left(\eta^\intercal\cdot\mathcal{K}\eta + \mathcal{J}\cdot\eta\right),
\end{aligned} \tag{4.68}$$

where M and α are defined in Eqs. (4.18) and (4.19). Since we are considering magnetic fields below the spin flop field in this section, we are dealing with skyrmions, which implies that $\alpha = 0$ and thus there is no Magnus force working on the skyrmion. Again we must not take the zero-modes into account because translations are already described by $\mathbf{R}$. We use the Faddeev-Popov technique to promote $\mathbf{R}$ to a dynamic variable

$$\begin{aligned}
Z[\mathbf{R}] &= \int\mathcal{D}\mathbf{R}\; e^{-\mathcal{S}_{cl}[\mathbf{R}]}F[\mathbf{R}], \\
F[\mathbf{R}] &= \int\mathcal{D}\eta\;\delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right)e^{-N_A\{\eta^\intercal\cdot[\mathcal{G}+\mathcal{K}]\eta + \mathcal{J}^\intercal\cdot\eta + \eta^\intercal\cdot\mathcal{J}\}}.
\end{aligned} \tag{4.69}$$

Since we're only looking up to second order in $\{\nu, \phi\}$ and $\dot{R}$, it is sufficient to shift η by $\tilde{\rho} = (\mathcal{G}+\mathcal{K})^{-1}\mathcal{J}$ to complete the square [9],

$$\begin{aligned}&\tilde{\eta} = \eta + \tilde{\rho},\\ &\eta^{\intercal}\cdot[\mathcal{G}+\mathcal{K}]\,\eta + \mathcal{J}^{\intercal}\cdot\eta \;+\eta^{\intercal}\cdot\mathcal{J} = \tilde{\eta}^{\intercal}\cdot[\mathcal{G}+\mathcal{K}]\,\tilde{\eta} - \mathcal{J}^{\intercal}\cdot(\mathcal{G}+\mathcal{K})^{-1}\mathcal{J} + \mathcal{O}(\dot{R}^3).\end{aligned} \tag{4.70}$$

With η defined in Eq. (4.70) the partition function in Eq. (4.69) reduces to

$$\begin{aligned}F[\mathbf{R}] &= \int \mathcal{D}\eta\; \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right)e^{-\mathcal{I}}e^{-N_A\{\tilde{\eta}^{\intercal}\cdot[\mathcal{G}+\mathcal{K}]\tilde{\eta}\}}\\ &= e^{-\mathcal{I}}\frac{1}{\det'[N_A(\mathcal{G}+\mathcal{K})]}.\end{aligned} \tag{4.71}$$

In the above the prime on the determinant denotes the omission of the zero-modes; additionally we defined

$$\mathcal{I} = -N_A\mathcal{J}^{\intercal}\cdot(\mathcal{G}+\mathcal{K})^{-1}\mathcal{J} \approx -N_A\mathcal{J}^{\intercal}\cdot\mathcal{G}^{-1}\mathcal{J}. \tag{4.72}$$

Similar to Eq. (4.44) the determinant in Eq. (4.71) gives rise to a pure mass re-normalization of order $\mathcal{O}((N_A)^0)$ and a topological phase proportional to $\dot{R}$ and h which is also of order $\mathcal{O}((N_A)^0)$

$$\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right) \propto \exp\left\{\operatorname{tr}\ln\left(\mathbf{1} - \frac{\mathcal{Y}_0}{\sqrt{N_A}}\int(\partial_j\sigma_i^{\intercal})\hat{\eta} + \mathcal{Y}_0\int(\partial_j\sigma_i^{\intercal})\tilde{\rho}\right)\right\}, \tag{4.73}$$

where we used the transformation $\hat{\eta} = \sqrt{N_A}\tilde{\eta}$. To evaluate the determinant in Eq. (4.71) we again make use of the identity $\ln\det = \operatorname{tr}\ln$ and expand the logarithm

$$\frac{1}{\sqrt{\det'(\mathcal{G}+\mathcal{K})}} \simeq \frac{1}{\sqrt{\det'(\mathcal{G})}}e^{-(1/2)\operatorname{tr}'[\mathcal{G}^{-1}\mathcal{K}-(1/2)(\mathcal{G}^{-1}\mathcal{K})^2+\mathcal{O}(\dot{R}^3)]}. \tag{4.74}$$

Like in Section 4.2.1 we want to evaluate the trace in Eq. (4.74) in the basis of eigenfunctions of $\mathcal{G}$. To find the eigenfunctions of $\mathcal{G}$ we make use of first order perturbation theory with respect to $\mathbf{L}(-i\partial_\tau)$ for small $\kappa = \sqrt{2\chi_0}h = \chi_0\tilde{H}$. The zeroth order term in the eigenvalue and eigenstates are still given by Eq. (4.47), up to first order in κ the eigenstates and eigenvalues of $\mathcal{G}$ in Eq. (4.25) are given by

$$\begin{aligned}\mathcal{G} &= \mathcal{G}' + \kappa\mathbf{L}(-i\partial_\tau),\\ \tilde{\epsilon}_{n,\omega} &= \tilde{\epsilon}^0_{n,\omega} + \kappa\tilde{\epsilon}^{(1)}_{n,\omega} + \mathcal{O}(\kappa^2),\\ \tilde{\psi}_{n,\omega} &= \tilde{\psi}^0_{n,\omega} + \kappa\tilde{\psi}^{(1)}_{n,\omega} + \mathcal{O}(\kappa^2),\end{aligned}$$

where $\tilde{\epsilon}^0_{n,\omega}$ and $\tilde{\psi}^0_{n,\omega}$ are given by Eq. (4.47). It follows that Ref. [22, p.452,p.464],

$$\tilde{\epsilon}^{(1)}_{n,\omega} = -i\langle n|\,\mathbf{L}\,|n\rangle\,\langle\omega|\,\partial_\tau\,|\omega\rangle = \omega\langle\psi_n, \mathbf{L}\psi_n\rangle = \omega\lambda_{n,n} = 0, \tag{4.75}$$

$$\begin{aligned}\tilde{\psi}^{(1)}_{n,\omega} &= \sum_{\{n',\omega'\}\neq\{n,\omega\}}|n'\rangle|\omega'\rangle\frac{-i\langle n'|\mathbf{L}|n\rangle\langle\omega'|\partial_\tau|\omega\rangle}{\epsilon^0_{n,\omega}-\epsilon^0_{n',\omega'}}\\ &= \sum_{n'\neq n}|n'\rangle|\omega\rangle\frac{\omega\langle n'|\mathbf{L}|n\rangle}{\epsilon^0_n-\epsilon^0_{n'}} = \omega\left\{\sum_{n'\neq n}\frac{\lambda_{n',n}}{\epsilon^0_n-\epsilon^0_{n'}}|n'\rangle\right\}|\omega\rangle,\end{aligned} \tag{4.76}$$

$$\tilde{\epsilon}^{(2)}_{n,\omega} = \omega^2\sum_{n'\neq n}\frac{|\langle n'|\mathbf{L}|n\rangle|^2}{\epsilon^0_n-\epsilon^0_{n'}} = \omega^2\sum_{n'\neq n}\frac{|\lambda_{n',n}|^2}{\epsilon^0_n-\epsilon^0_{n'}}. \tag{4.77}$$

In the previous equations we defined $\lambda_{n,m} = \langle\psi^0_n, \mathbf{L}\psi^0_m\rangle$. In the case that 4.47 has degeneracies, one must take the basis $\{\psi^0_{n,\omega}\}$ such that $\mathbf{L}\partial_\tau$ is diagonal in the degenerate subspace, such that Eq. (4.76) does not blow up in that case.

Using the first order corrections for the eigenvalues and eigenfunctions of $\mathcal{G}$ given by Eqs. (4.75) and (4.76) the first order term in Eq. (4.74) is obtained up to first order in κ,[4]

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}'\,\mathcal{G}^{-1}\mathcal{K} &\approx -\frac{1}{2\beta}\sum_{\omega,n}{}' \int d\tau \frac{\langle\psi_n,\mathcal{K}\psi_n\rangle}{\tilde{S}\omega^2+\kappa^2\tilde{\epsilon}_m^{(2)}+\epsilon_n}\\ &= \frac{1}{2\beta}\int d\tau \sum_{\omega,n}{}' \frac{1}{\omega^2+\hat{\epsilon}_n^2}\Big\{\dot{R}^\beta\dot{R}^\gamma\langle\psi_n^0,\mathcal{P}_{\beta\gamma}\psi_n^0\rangle - i\dot{R}^\beta\omega\langle\psi_n^0,\mathcal{N}_\beta\psi_n^0\rangle - i(\kappa/\tilde{S})\dot{R}^\beta\langle\psi_n^0,\mathcal{A}_\beta\psi_n^0\rangle\\ &+\kappa\dot{R}^\beta\dot{R}^\gamma\left(\omega\langle\psi_n^{(1)},\mathcal{P}_{\beta\gamma}\psi_n^0\rangle + c.c\right) + i\kappa\dot{R}^\beta\left(\omega^2\langle\psi_n^{(1)},\mathcal{N}_\beta\psi_n^0\rangle + c.c.\right)\Big\} + \mathcal{O}(\kappa^2)\\ &\approx \int_0^\beta d\tau\left\{\Delta\tilde{M}\dot{R}^2 + i\kappa\int_0^\beta d\tau\; v_h^\beta\dot{R}_\beta\right\} + \mathcal{O}(\kappa^2).\end{aligned} \tag{4.78}$$

In Eq. (4.78) we obtained the same pure mass renormalization as we obtained in Eq. (4.52). This pure mass renormalization is independent on the magnetic field and the order of magnitude of the mass renormalization is $\mathcal{O}((N_A)^0)$. We also obtain a topological phase of order $\mathcal{O}((N_A)^0)$ that couples linearly to the external magnetic field. Note that this topological phase drops if one considers periodic boundary conditions, additionally it does not contribute to the classical equations of motion. If we would expand Eq. (4.78) up to second order in κ, we would obtain another mass renormalization of order $\mathcal{O}((N_A)^0)$ which couples quadratically to the magnetic field.

Since we consider a non-zero magnetic field $\mathcal{J}$ possesses a term which is linear in $\dot{R}$, this implies that $\mathcal{I}$ defined in Eq. (4.72) is non-vanishing which is entirely due to coupling with the magnetic field. The contribution of $\mathcal{I}$ Eq. (4.72) up to second order in κ is given by

$$\begin{aligned}\mathcal{I} &\approx -N_A\mathcal{J}^\intercal\cdot\mathcal{G}^{-1}\mathcal{J}\\ &= -\kappa^2 N_A\int_0^\beta d\tau\int_0^\beta d\tau'\; R^\beta(\tau)R^\gamma(\tau')\int d\mathbf{x}\int d\mathbf{x}'\;\mathbf{N}_\beta^\intercal(\mathbf{x})\mathcal{G}'^{-1}(\mathbf{x},\mathbf{x}',\tau,\tau')\mathbf{N}_\gamma(\mathbf{x}') + \mathcal{O}(\dot{R}^3,\kappa^3)\\ &= \kappa^2\frac{N_A}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\;\dot{R}^\beta(\tau)\dot{R}^\gamma(\tau')\sum_{\omega,n}\frac{\langle\mathbf{N}_\beta,\psi_n^0\rangle\langle\psi_n^0,\mathbf{N}_\gamma\rangle}{\tilde{S}\omega^2+\epsilon_n}e^{i\omega(\tau-\tau')}\\ &= \int_0^\beta d\tau\int_0^\beta d\sigma\;\dot{R}_i(\tau)\gamma_{ij}^0(\tau-\sigma)\dot{R}_j(\sigma).\end{aligned} \tag{4.79}$$

In the preceding equation we used $\mathcal{G}^{-1} = \mathcal{G}'^{-1} + \mathcal{O}(\kappa)$, where $\mathcal{G}'$, given by Eq. (4.30) describes the static interaction without the presence of a magnetic field. The sum over the Matsubara frequencies in Eq. (4.79) can be evaluated explicitly using Eq. (4.50) to yield

$$\begin{aligned}\gamma_{ij}^0(\tau) &= \frac{1}{\beta}\sum_{\omega,n}\frac{\mathcal{D}_{ij}^n e^{i\omega\tau}}{\omega^2+\hat{\epsilon}_n^2}\\ &= \sum_n\frac{\mathcal{D}_{ij}^n}{2\hat{\epsilon}_n}\frac{\cosh(\hat{\epsilon}_n(|\tau|-\beta/2))}{\sinh(\beta\hat{\epsilon}_n/2)}\\ &\xrightarrow{\beta\to\infty}\sum_n\frac{\mathcal{D}_{ij}^n}{2\hat{\epsilon}_n}e^{-\hat{\epsilon}_n|\tau|}.\end{aligned} \tag{4.80}$$

Where we defined

$$\mathcal{D}_{ij}^n = \frac{\kappa^2 N_A}{\tilde{S}}\;\langle\mathbf{N}_i,\psi_n^0\rangle\langle\psi_n^0,\mathbf{N}_j\rangle. \tag{4.81}$$

[4]We omit terms which are antisymmetric in ω.

Note that in Eqs. (4.79) and (4.80) we did not explicitly take the omission of the zero-modes into account, since $\mathcal{N}_\beta$ is already orthogonal to the zero modes. Thus Eq. (4.79) gives rise to a damping kernel of order $\mathcal{O}((N_A)^1)$ which is quadratically coupled to the magnetic field. If the dynamics of R is slow compared to the relaxation time of the damping kernel and the temperature is sufficiently small s.t. $\beta \gg \hat{\epsilon}_0^{-1}$, then the damping kernel (4.60) reduces to a pure mass normalization

$$\begin{aligned}&\int_0^\beta d\tau \int_0^\beta d\tau' \dot{R}_i(\tau)\gamma^0_{ij}(\tau-\tau')\dot{R}_j(\tau') \\ \overset{\beta\to\infty}{\approx} &\int_0^\beta d\tau\; \dot{R}_i(\tau)\frac{\mathcal{M}^0}{2}\dot{R}_j(\tau).\end{aligned} \tag{4.82}$$

Where we defined the mass tensor as follows

$$\begin{aligned}\mathcal{M}^0_{ij} &\equiv {\sum_n}' \frac{\mathcal{D}^n_{ij}}{\hat{\epsilon}_n}\int_{-\infty}^{\infty} d\mu\; e^{-\hat{\epsilon}_n|\mu|} \\ &= 2{\sum_n}' \frac{\mathcal{D}^n_{ij}}{\hat{\epsilon}_n^2} \simeq \frac{2\mathcal{D}^0_{ij}}{\hat{\epsilon}_0^2}.\end{aligned} \tag{4.83}$$

If we would use the definition of mass described in Ref. [9, p. 5] as the zero frequency limit of the damping kernel, we would exactly obtain the result given in Eq. (4.82).

Since the lowest eigenenergy of $\mathcal{G}$ is smaller than the magnon gap we are able find a lower bound on the order of magnitude of $\mathcal{M}^0$ in Eq. (4.83). To determine the order of magnitude of $\langle \mathbf{N}_i, \psi_n^0\rangle\langle\psi_n^0, \mathbf{N}_j\rangle$ we assume that the low energy eigenstates of $\mathcal{G}$ approximately have the same size as the skyrmion, thus $\psi_0^0 \sim e^{-(\sqrt{K_z/\rho_s})\rho}$. By rewriting $\langle \mathbf{N}_i, \psi_n^0\rangle\langle\psi_n^0, \mathbf{N}_j\rangle$ into cylindrical coordinates and using the substitution $\tilde{\rho} = \sqrt{K_z/\rho_s}$ we obtain an order of magnitude given by $\mathcal{O}(\rho_s/K_z)$. From Ref. [21] it follows that magnon gap has order of magnitude $\mathcal{O}(\sqrt{K_z/\chi_0})$ in the absence of a magnetic field. Using the preceding we are able to see that

$$\mathcal{O}(\mathcal{M}^0) > \mathcal{O}(N_A(\rho_s/K_z)h^2\chi_0/\epsilon_{gap}) = \mathcal{O}(N_A\tilde{H}^2\chi_0^2\sqrt{\chi_0\rho_s^2/K_z^3}) \tag{4.84}$$

gives a lower bound on the order of magnitude of the mass correction due to the magnetic field.

From Section 4.2.1 we know that the second order term in the logarithm expansion Eq. (4.74) gives rise to a damping kernel. We will now again calculate the second order term in Eq. (4.74) up to $\mathcal{O}(\dot{R}^2)$ for small magnetic fields; up to first order in κ we obtain

$$\begin{aligned}\frac{1}{4}\,\mathrm{tr}'(\mathcal{G}^{-1}\mathcal{K})^2 &= {\sum_{\omega,n,\omega',n'}}' \frac{\langle\psi_{n,\omega}|\mathcal{K}|\psi_{n',\omega'}\rangle\langle\psi_{n',\omega'}|\mathcal{K}|\psi_{n,\omega}\rangle}{(\tilde{S}\omega^2+\epsilon_n-\kappa\omega\lambda_{n,n})(\tilde{S}\omega'^2+\epsilon_{n'}-\kappa\omega\lambda_{n',n'})} + \mathcal{O}(\kappa^2) \\ &= \int_0^\beta d\tau\int_0^\beta d\tau'\; \dot{R}_i(\tau)\dot{R}_j(\tau') \\ &{\sum_{\omega,n,\omega',n'}}' \left[\frac{1}{\omega^2+\hat{\epsilon}_n^2}\right]\left[\frac{1}{\omega'^2+\hat{\epsilon}_{n'}^2}\right]\Big\{\langle\psi^0_{n,\omega}+\kappa\psi^{(1)}_{n,\omega}, (\mathcal{N}_i\partial_\tau - i(\kappa/\tilde{S})\mathcal{W}_i)\psi^0_{n',\omega'}+\kappa\psi^{(1)}_{n',\omega'}\rangle \\ &\langle\psi^0_{n',\omega'}+\kappa\psi^{(1)}_{n',\omega'}, (\mathcal{N}_j\partial_\tau - i(\kappa/\tilde{S})\mathcal{W}_j)\psi^0_{n,\omega}+\kappa\psi^{(1)}_{n,\omega}\rangle\Big\} + \mathcal{O}(\kappa^2).\end{aligned} \tag{4.85}$$

The zeroth order in κ will just give Eq. (4.60), the first order in κ will give

$$\begin{aligned}&(\kappa/\tilde{S})\int_0^\beta d\tau\int_0^\beta d\tau'\ \dot{R}_i(\tau)\dot{R}_j(\tau')\\&\frac{1}{\beta^2}\sum_{\omega,n,\omega',n'}{}'\Bigg\{-i\left(\frac{\omega\mathcal{R}_{ij}^{nn'}}{(\omega^2+\hat{\epsilon}_n^2)(\omega'^2+\hat{\epsilon}_n'^2)}e^{i(\omega-\omega')(\tau-\tau')}+c.c.\right)\\&\quad-\tilde{S}\left(\frac{\omega^2\omega'\mathcal{Y}_{ij}^{nn'}}{(\omega^2+\hat{\epsilon}_n^2)(\omega'^2+\hat{\epsilon}_n'^2)}e^{i(\omega-\omega')(\tau-\tau')}+c.c.\right)\Bigg\}.\end{aligned} \tag{4.86}$$

In the preceding equation c.c. stands for complex conjugate. We also defined the following tensors

$$\mathcal{R}_{ij}^{nn'}=\langle\psi_n^0,\mathcal{W}_i\psi_{n'}^0\rangle\langle\psi_n^0,\mathcal{N}_j\psi_{n'}^0\rangle+\langle\psi_n^0,\mathcal{N}_i\psi_{n'}^0\rangle\langle\psi_{n'}^0,\mathcal{W}_j\psi_n^0\rangle, \tag{4.87}$$

$$\begin{aligned}\mathcal{Y}_{ij}^{nn'}&=\langle\psi_n^{(1)},\mathcal{N}_i\psi_{n'}^0\rangle\langle\psi_{n'}^0,\mathcal{N}_j\psi_n^0\rangle+\langle\psi_n^0,\mathcal{N}_i\psi_{n'}^{(1)}\rangle\langle\psi_{n'}^0,\mathcal{N}_j\psi_n^0\rangle\\&+\langle\psi_n^0,\mathcal{N}_i\psi_{n'}^0\rangle\langle\psi_{n'}^{(1)},\mathcal{N}_j\psi_n^0\rangle+\langle\psi_n^0,\mathcal{N}_i\psi_{n'}^0\rangle\langle\psi_{n'}^0,\mathcal{N}_j\psi_n^{(1)}\rangle.\end{aligned} \tag{4.88}$$

Using Eq. (4.50) one is able to perform the sums over ω and ω'. All the terms in Eq. (4.86) give zero as a result. Second order terms in κ will give non-vanishing contributions to the damping kernel which are of order $\mathcal{O}((N_A)^0)$ and thus small compared to the damping kernel found in Eq. (4.80).

Conclusion

We used perturbation theory to calculate the mass correction and the damping kernel for small magnetic fields h. In Eq. (4.80) we found a damping term of order $\mathcal{O}((N_A)^1)$ which is proportional to κ^2. From Eq. (4.85) one is able to see that there are also other non-zero damping terms which couple quadratically to the magnetic field. We did not explicitly calculate this contribution since it will be of order $\mathcal{O}((N_A)^0)$ and thus small compared to the damping kernel described in Eq. (4.80). We also found topological terms in Eq. (4.78) which are of first order for κ. If one would consider periodic boundary conditions in time these topological terms will vanish, they will also not contribute to the semi-classical equations of motion. In principle these topological terms may depend on the winding number of the skyrmions, if this is the case then they are important for interference effects in tunnelling processes where the skyrmion is in a superposition between different winding numbers and these topological terms may not be neglected. If we consider a skyrmion with a definite winding number, no interference will occur and this topological term will just give a complex phase. The action describing the effective dynamics of the skyrmion is given by

$$\begin{aligned}\mathcal{S}_{eff}[R]\approx&\int_0^\beta d\tau\left\{\frac{M_{eff}}{2}\dot{R}^2+i\kappa v_h\dot{R}\right\}\\&+\int_0^\beta d\tau\int_0^\beta d\tau'\left\{\dot{R}_i(\tau)\left[\gamma_{ij}'(\tau-\tau')+\kappa^2\gamma_{ij}^h(\tau-\tau')\right]\dot{R}_j(\tau')\right\},\end{aligned} \tag{4.89}$$

where γ_{ij}' is given by Eqs. (4.60) and (4.85) and describes the damping due to spin waves when no magnetic field is present where the order of magnitude of γ_{ij}' is $\mathcal{O}((N_A)^0)$. The part of the damping kernel that is dependent on the magnetic field is described by γ_{ij}^h which is approximately given by γ_{ij}^0 in Eq. (4.80). The order of magnitude of γ_{ij}^0 is $\mathcal{O}((N_A)^1)$ and thus becomes dominant for sufficiently large magnetic fields.

If the dynamics of the skyrmion is much slower then the relaxation time of the damping kernel and the temperature is sufficiently low $\beta \gg \epsilon_0$, then the damping kernels in Eq. (4.89) will reduce to pure mass renormalizations

$$\mathcal{S}_{eff}[R]\approx\int_0^\beta d\tau\left\{\frac{M_{eff}}{2}\dot{R}^2+i\kappa\alpha\dot{R}\right\}. \tag{4.90}$$

With

$$\begin{aligned} M_{eff} =& N_A M + \Delta M + \mathcal{M} + (\mathcal{M}^0 + \Delta M^0) \\ \overset{\beta \to \infty}{\approx}& (N_A M + \Delta M + \mathcal{M}) + \mathcal{M}^0. \end{aligned} \tag{4.91}$$

The first three terms are independent of the magnetic field and are thus described in Section 4.2.1. The last two terms in Eq. (4.91) give an effective mass renormalization which depends on the square of the magnetic field. We defined $\mathcal{M}^0$ in Eq. (4.83), which is of order $\mathcal{O}((N_A)^1)$ while ΔM^0 has magnitude $\mathcal{O}((N_A)^0)$. In the zero temperature limit ΔM^0 will be negligibly small compared to $\mathcal{M}^0$. We found that the order of magnitude of the mass $\mathcal{M}^0$ is bounded by Eq. (4.84),

$$\mathcal{O}(\mathcal{M}^0) > \mathcal{O}(N_A(\rho_s/K_z)h^2\chi_0/\epsilon_{gap}) = \mathcal{O}(N_A\tilde{H}^2\chi_0^2\sqrt{\chi_0\rho_s^2/K_z^3}).$$

In practice a periodic potential $V_0(\mathbf{R})$ for the skyrmions may arise. This potential may come from different phenomena like the discrete structure of the crystal. Other effects like currents can be taken care of by considering an effective potential $V_{eff}(\mathbf{R})$ for the skyrmion. For low temperatures and slow dynamics of skyrmions the action for a skyrmion may thus be described by

$$\mathcal{S}_{eff}[R] \approx \int_0^\beta d\tau \left\{ \frac{M_{eff}}{2}\dot{R}^2 + i\kappa v\dot{R} + V_{eff}(\mathbf{R}) \right\}. \tag{4.92}$$

4.3 Vortices in antiferromagnets

In Section 3.2 we discussed that an antiferromagnet has a spin flop field h_0. If the magnetic field is below the spin-flop field, skyrmion configurations arise. Above the spin flop field vortices arise (see Fig. 3.7). Like skyrmions these vortices are topologically protected structures, but their boundary conditions are different ($\theta_0(0) = \pi$ and $\theta_0(\infty) = \pi/2$). From Eqs. (4.17) and (4.19) in Section 4.1 we see that a vortex solution within an antiferromagnet feels a Magnus force. Thus in our model the dynamics of an antiferromagnetic vortex is fundamentally different from the dynamics of an antiferromagnetic skyrmion. In Section 3.2 we found that spin flop field for our system is given by $h_0 = \sqrt{K_z}$. If the anisotropy K_z is small and we consider magnetic fields which are comparable to spin flop field, we are able to describe the dynamics of these vortices using the methods we developed in Section 4.2.2. If we consider large magnetic fields and large anisotropies, the perturbative methods used in Section 4.2.2 breaks down. In Appendix D we will discuss the dynamics of skyrmions and vortices for large magnetic fields without explicitly performing the calculations.

4.3.1 Dynamics of vortices for small spin-flop fields

In this section we describe the effective dynamics of a vortex with spin wave interactions. We consider a small spin-flop field (thus weak anisotropy) and consider the magnetic field to be comparable to this spin-flop field. The operators $\mathcal{G}, \mathcal{J}$ and $\mathcal{K}$ are described by Eqs. (4.25), (4.27) and (4.29) where the skyrmion configuration θ_0 is now replaced by the vortex configuration. The entire discussing done in Section 4.2.2, will remain valid since we're still considering small magnetic fields and without further calculation we can give the action for a vortex in an antiferromagnetic insulator with a weak anisotropy. Considering small magnetic fields we obtain

$$\begin{aligned} \mathcal{S}_{eff}[R] \approx & \int_0^\beta d\tau \left\{ \frac{M_{eff}}{2} \dot{R}^2 + i\alpha\epsilon_{ij} R^i \dot{R}^j + i\kappa v_h \dot{R} \right\} \\ & + \int_0^\beta d\tau \int_0^\beta d\tau' \left\{ \dot{R}_i(\tau) \left[\gamma'_{ij}(\tau - \tau') + \kappa^2 \gamma^h_{ij}(\tau - \tau') \right] \dot{R}_j(\tau') \right\}, \end{aligned} \tag{4.93}$$

where $\alpha = -2\pi N_A \tilde{H} \chi_0$ and $M_{eff} = N_A M_{cl} + \Delta M$; here ΔM has order of magnitude $\mathcal{O}((N_A)^0)$. Similar to the conclusion in Section 4.2.2 we see that γ' is of order $\mathcal{O}((N_A)^0)$ and γ^h is approximately given by γ^0 in Eq. (4.80) which is of order $\mathcal{O}((N_A)^1)$ and will thus become the dominant damping for sufficiently large magnetic fields.

If the dynamics of the vortex is slow compared to the damping kernel and the temperature is sufficiently low $\beta \gg \epsilon_0^{-1}$, then the damping kernels in Eq. (4.93) reduce to pure mass renormalizations. In these limits we obtain the following effective action which describes the dynamics of the vortex

$$\mathcal{S}_{eff}[R] \approx \int_0^\beta d\tau \left\{ \frac{M_{eff}}{2} \dot{R}^2 + i\alpha\epsilon_{ij} R^i \dot{R}^j + i\kappa v_h \dot{R} \right\}, \tag{4.94}$$

where

$$\begin{aligned} M_{eff} =& N_A M_{cl} + \Delta M + \mathcal{M} + (\mathcal{M}^0 + \Delta M^0) \\ \overset{\beta \to \infty}{\approx} & (N_A M_{cl} + \Delta M + \mathcal{M}) + \mathcal{M}^0. \end{aligned} \tag{4.95}$$

The first three terms are independent of the magnetic field and give the classical mass of order $\mathcal{O}(N_A \chi_0)$ plus mass corrections of order $\mathcal{O}((N_A)^0)$ due to spin wave interactions which are independent of the magnetic field. Furthermore $\mathcal{M}^0$ gives a mass correction of order $\mathcal{O}((N_A)^1)$ which is quadratically coupled to the magnetic field. We obtained a lower bound on the order of magnitude of $\mathcal{M}^0$ in Eq. (4.84),

$$\mathcal{O}(\mathcal{M}^0) > \mathcal{O}(N_A(\rho_s/K_z)h^2\chi_0/\epsilon_{gap}) = \mathcal{O}(N_A \tilde{H}^2 \chi_0^2 \sqrt{\chi_0 \rho_s^2 / K_z^3}).$$

In reality there might be effects that appear on the length scale of the skyrmion, e.g. spin currents, electrical currents and impurities. We consider include these effects by considering an effective potential for the skyrmion $V_{eff}(\mathbf{R})$, we thus obtain the following action for the dynamics of a vortex in a low magnetic field

$$\mathcal{S}_{eff}[R] \approx \int_0^\beta d\tau \left\{ \frac{M_{eff}}{2}\dot{R}^2 + i\alpha\epsilon_{ij}R^i\dot{R}^j + i\kappa\upsilon_h\dot{R} + V_{eff}(\mathbf{R}) \right\}. \tag{4.96}$$

4.4 Second order expansion of the free energy

In this section we determine the second order expansion of $\mathcal{F}[\hat{\mathbf{n}}]$ in Eq. (4.1) w.r.t. η around skyrmion and vortex configurations. Since skyrmion and vortex configurations are local minima of $\mathcal{F}[\hat{\mathbf{n}}]$, the first order functional derivative in η equals 0. The zeroth order term will just give a energy E_0, which is not interesting for our purposes since we assume the skyrmion is already present and not thermally created. So we are interested in calculating what $\mathcal{H} = \frac{\delta^2 \mathcal{F}}{\delta \eta^\dagger \eta}\Big|_{\eta^\intercal=\eta=0}$ in Eq. (4.26) is.

We start from Eq. (4.1)

$$\mathcal{F}[\hat{\mathbf{n}}] = N_A \int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \left\{ \frac{\rho_s}{2} \sum_{l=x,y} |\partial_l \hat{\mathbf{n}}|^2 + \frac{D}{2}\left(\hat{y}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial \hat{\mathbf{n}}}{\partial x}\right) - \hat{x}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial \hat{\mathbf{n}}}{\partial y}\right)\right) - K_z \hat{\mathbf{n}}_z^2 + \frac{1}{2}\chi_0 \left(\tilde{\mathbf{H}}\cdot\hat{\mathbf{n}}(\mathbf{x})\right)^2 \right\},$$

and take $\tilde{\mathbf{H}}$ to be in the $\hat{z}$-direction. From Appendix C.2 it follows that the exchange interaction becomes,

$$\begin{aligned}\int_0^\beta \mathrm{d}\tau \int_\Lambda d\mathbf{x} \Big\{ &- \nu\nabla^2\nu - \varphi\nabla^2\varphi + \varphi\left(\frac{\cot(\theta_0)}{\rho}(\partial_\rho\theta_0) - (\partial_\rho\theta_0)^2 + \cot(\theta_0)(\partial_\rho^2\theta_0)\right)\varphi \\ &- \nu\frac{4\cos(\theta_0)}{\rho^2}(\partial_\phi\varphi) + \nu\frac{\cos(2\theta_0)}{\rho^2}\nu\Big\}.\end{aligned} \tag{4.97}$$

Where the Dzyloshinskii Moriya interaction reduces to,

$$\begin{aligned}\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \pm \Big\{ &- \varphi\frac{(\partial_\rho\theta_0)}{2\sin^2(\theta_0)}\varphi - \varphi\frac{1}{\sin(\theta_0)\rho}(\partial_\phi\nu) - \nu\frac{\sin(2\theta_0)}{\rho}\nu - \varphi\frac{\cot(\theta_0)}{2\rho}\varphi \\ &- \nu\frac{\cos(2\theta_0)}{\sin(\theta_0)\rho}(\partial_\phi\varphi) - 2\varphi\cot(\theta_0)(\partial_\rho\varphi) + 2\varphi\cot^2(\theta_0)(\partial_\rho\theta_0)\varphi\Big\}.\end{aligned} \tag{4.98}$$

The anisotropy and the magnetic field terms will be given by,

$$-\int_0^\beta \int_\Lambda d\mathbf{x}\ \{\nu\cos(2\theta)\nu\}. \tag{4.99}$$

Now $\mathcal{F} \simeq E_0 + \eta^\dagger \cdot \mathcal{H}\eta + \mathcal{O}(\eta^3)$, where

$$\mathcal{H} = -(\rho_s/2)\nabla^2 + U(\rho) + V(\rho)\partial_\phi. \tag{4.100}$$

The potential terms $U(\rho)$ and $V(\rho)$ are defined by

$$\begin{aligned}U(\rho) = (\rho_s/2)&\left\{\frac{\cos(2\theta_0)}{\rho^2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \left(\frac{\cot(\theta_0)}{\rho}(\partial_\rho\theta_0) - (\partial_\rho\theta_0)^2 + \cot(\theta_0)(\partial_\rho^2\theta_0)\right)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \\ - (D/2)&\left\{\frac{\cot(\theta_0)}{2\rho}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \left(\frac{3(\partial_\rho\theta_0)}{2\sin^2(\theta_0)} - \frac{\cot(\theta_0)}{2\rho} + 2\cot^2(\theta_0)(\partial_\rho\theta_0)\right)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \\ + &\left(\frac{\chi_0}{2}\tilde{H}^2 - K_z\right)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\end{aligned} \tag{4.101}$$

and

$$V(\rho) = \left[-\rho_s\left\{\frac{\cos(\theta_0)}{\rho^2}\right\} + D/2\left\{\frac{\sin(\theta_0)}{\rho}\right\}\right]\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{4.102}$$

Chapter 5

Conclusion, discussion and outlook

5.1 Conclusion

In this thesis we gave a quantum mechanical treatment for skyrmions in antiferromagnetic thin-films. We restricted ourself to spin degrees of freedom and neglected interactions of the magnetization with e.g. phonons. This is valid if one considers low temperatures. In Section 3.1 we assumed long wavelengths for $\hat{\mathbf{n}}$ and $\mathbf{m}$. By integrating out the magnetization in action Eq. (3.30) we were able to derive the action given in Eq. (3.33) which is only dependent on the Néel field $\hat{\mathbf{n}}$. In Chapter 4 we used the Faddeev-Popov technique for collective phenomena, to derive the quantum dynamics of antiferromagnetic skyrmions. In Section 4.1 we expanded spin wave fluctuations around the skyrmions up to second order. We derived that antiferromagnetic skyrmions have a classical mass and feel no Magnus force, which is in agreement with Ref. [10]. This result differs from ferromagnetic skyrmions, which feel a Magnus force and have no classical mass [9]. In Section 4.2.1 we calculated the damping on skyrmions due to interaction with spin waves without a magnetic field. We found that skyrmions experience drag because of interaction with spin waves and also obtain some mass renormalization. The mass renormalization is of order $\mathcal{O}((N_A)^0)$ and mainly depends on the lowest eigenenergy of $\mathcal{G}$ (the static interaction between a skyrmion and spin waves), the classical mass on the other hand is of order $\mathcal{O}((N_A)^1)$. If the dynamics of the skyrmion is slow and temperatures are low, we see that this damping reduces to an additional mass renormalization of order $\mathcal{O}((N_A)^0)$ and the lower bound we gave on this mass is small compared to the classical mass. In Section 4.2.2 we considered the magnetic field as a small perturbation. We found that the mass of antiferromagnetic skyrmions obtains an additional damping of order $\mathcal{O}((N_A)^1)$ which couples quadratically to the external magnetic field. For slow dynamics of the skyrmion and low temperatures this damping kernel reduces to an additional mass of order $\mathcal{O}(\tilde{H}^2 N_A)$. In Section 4.3 we looked at vortices for small spin-flop fields and treated the magnetic field as a small perturbation. In Section 4.1 we found that antiferromagnetic vortices have a classical mass of order $\mathcal{O}(\chi_0 N_A)$ and feel a Magnus force of magnitude $\mathcal{O}(N_A \chi_0 \tilde{H})$. We also see that a vortex in an antiferromagnet insulator with a small anisotropy and external field obtains a mass renormalization independent of the magnetic field of order $\mathcal{O}((N_A)^0)$. Additionally we find that a vortex feels a drag due to spin wave interactions which can be split in two parts, a part which is independent of the magnetic field of magnitude $\mathcal{O}((N_A)^0)$ and a part which couples quadratically to the magnetic field of order $\mathcal{O}((N_A)^1)$. For slow dynamics of the vortex and low temperatures these damping kernels reduce to mass correction of order $\mathcal{O}((N_A)^0)$ and $\mathcal{O}(\tilde{H}^2 N_A)$ respectively. In Appendix D we briefly discuss the dynamics of skyrmions and vortices for large magnetic fields. Using the methods described by Ref. [9] and the fact that N_β in Eq. (4.15) is orthogonal to the zero-modes, we expect slow dynamics of the skyrmions and vortices and sufficiently low temperatures that the mass correction obtained due to interaction with spin waves is of the order $\mathcal{O}(\tilde{H}^2 N_A)$. We expect this mass correction to be larger than the classical mass of the skyrmions and vortices.

5.2 Discussion and Outlook

We started with the Hamiltonian in Eq. (3.16) for thin-film antiferromagnets and rewrote this into an effective two-dimensional model, where we assume translational invariance in the $\hat{z}$-direction. For thin-films this turns out to be a proper approximation.

In this thesis we only took into account the spin degrees of freedom, while neglecting other effects like spin phonon interactions and movement of electrons. These effect's are assumed to be negligibly small or absent for low enough temperatures.

To obtain an action which only depends on the Néel vector we used Haldane's mapping to give the magnetization direction at each point in terms of the Néel vector and the total magnetization. We assumed the total magnetization to be small and expanded the euclidean action up to second order in $\mathbf{m}$ Eq. (3.30), which is a valid assumption if the energy scale of the exchange interaction is much larger than the energy scale of a spin in a magnetic field. We also used the large wavelength approximation, which is valid in the low temperature limit.

In Section 3.1 we left out the homogeneous Dzyaloshinskii Moriya interaction, this term is negligible or can be added by changing the magnetic field. This could possibly also change the direction of the magnetic field and we can thus not consider it to be fully in the $\hat{z}$-direction. Future work could redo integration (3.31) while also taking into account the first order terms of the Dzyaloshinskii-Moriya interaction in $\mathbf{m}$.

Because we work in the low temperature limit we assume the spin wave fluctuations around skyrmions to be small. We thus expanded up to second order in $\{\phi, \nu\}$ and assumed higher order terms are negligible in this limit.

In Section 4.2.2 we neglected some the second order damping terms in κ of order $\mathcal{O}((N_A)^0)$, which would appear in Eqs. (4.78) and (4.85) of we expand them up to second order in κ. The reason is that they are small compared to Eq. (4.80) which is also of second order in κ and has order of magnitude $\mathcal{O}((N_A)^1)$. In the low temperature limit the damping kernel in Eqs. (4.78) and (4.85) will be negligible compared to Eq. (4.80).

A possible way to measure our predictions is through magnetic resonance Ref. [23]. The eigenfrequencies of a skyrmion with respect to a fluctuating magnetic field should depend on the mass of the skyrmion. So one could measure our predictions by measuring the magnetic excitation spectrum of an antiferromagnetic skyrmion.

In future work, it would be useful to determine the eigenvalues and eigenfunctions of $\mathcal{H}$ Eq. (4.100), so one is able to explicitly calculate the mass corrections and damping kernel obtained through interaction with spin waves. Especially the eigenvalues of the breathing modes are important, since those contribute most to the damping and mass renormalizations of the skyrmion. The work of Ref. [24] already determined the eigenspectrum of $\mathcal{H}$ for ferromagnetic skyrmions, similar methods can be used to determine the eigenvalues and eigenvectors for $\mathcal{H}$ in the antiferromagnet case. Instead of looking at fluctuations $\{\tilde{\nu}, \tilde{\phi}\}$, they worked with spinors χ and $\chi^\dagger$ defined by

$$\chi = \frac{1}{2}\begin{pmatrix} \tilde{\phi}\sin(\theta_0) + i\tilde{\nu}/\sin(\theta_0) \\ \tilde{\phi}\sin(\theta_0) - i\tilde{\nu}/\sin(\theta_0) \end{pmatrix} = \begin{pmatrix} \phi + i\nu \\ \phi - i\nu \end{pmatrix}. \tag{5.1}$$

To use Ref. [24] it could be useful to rewrite Eq. (4.15) into spinor form.

One could use the methods described in Ref. [9] to look at the mass of a vortex (see Fig. 3.7). Since we found that N_β in Eq. (4.15) is orthogonal to the zero-modes we expect to find that a vortex obtains an additional mass contribution of the order $\mathcal{O}(\tilde{H}^2 N_A)$ due to interactions with spin waves. In Appendix D we already discussed some of the implications, but did not explicitly calculate the damping kernel.

In this thesis we assumed the magnetic field and the anisotropy to point in the same direction, in future research one could look at the dynamics when both are not pointing in the same direction. For instance anisotropy in the $\hat{z}$-direction and magnetic field in the $\hat{x}$-direction.

One could next start from an effective action in Chapter 4 and then consider a periodic potential for the skyrmion position with period d, e.g.

$$V(R) = V_0 \left[1 - \cos\left(\frac{2\pi R_x}{d}\right)\right] \left[1 - \cos\left(\frac{2\pi R_y}{d}\right)\right].$$

After this one is able to obtain an expression for tunnelling probabilities for skyrmions and look at interference effects due to topological terms. Calculating the transition amplitudes would for instance be interesting, one could calculate how likely tunnelling of a skyrmion for a large potential is. One might continue to find the dispersion relations for skyrmions in weak and strong potentials, using methods described by Ref. [3].

In Section 4.2.2 we neglected the topological phases in the action that did arise from the coupling with the magnetic field. Though these terms are not important for classical movement of the skyrmion they might important for describing interference effects. If these topological phases are dependent on the winding number of the skyrmion, interference effects may occur. One could study if interference between skyrmions or vortices is possible.

One could also look at dynamic skyrmion skyrmion interactions in the presence of spin waves. Two different situations may seem interesting; interactions between skyrmions with the same winding number and interactions between skyrmions with different winding numbers, e.g. $n = 1$ and $n = -1$.

Appendix A

Topological structures within a magnet

In this section we will be talking about topological properties of skyrmions and domain walls. We start with a general discussion of homotopies and homotopy groups Ref.[25]. This mathematical formalism will then be used to give a formal expression of skyrmions and domain walls. This chapter is meant as background knowledge and will not be required for the other chapters. For more details we refer the reader to Ref.[26].

A.1 Defining the topology of the magnet

First we assume that the the magnetic material we look at is very thin and the Hamiltonian is translational invariant in the $\hat{z}$-direction. So the space we describe is effectively 2-dimensional. We start with the map ($\mathbf{\Omega} : \mathbb{R}^2 \to \mathbb{S}^2$) which assigns a magnetic moment to each position in the magnet. The purpose of this chapter is assign "some kind of structure" to the space of all continuous functions $\mathbf{\Omega}$. Now we will also assume that $\mathbf{\Omega}(\{\infty\}) = (0, 0, 1)$, this basically means that all the spins want to point in the same direction at $\{\infty\}$. We will add the point ∞ to $\mathbb{R}^2$, which means that we will now be looking at the the one point compactification of $\mathbb{R}^2$ which is $\mathbb{S}^2$.
So we describe the maps $\mathbf{\Omega} : \mathbb{S}^2 \to \mathbb{S}^2$.

A.2 Homotopy

Let X, Y be two topological spaces and $f, g : X \to Y$ continuous maps. Now these two maps are called homotopic if there exists a continuous function $H : X \times [0, 1] \to Y$ from the product space $X \times [0, 1]$ to Y, such that, $H(X, , 0) = f(x)$ and $H(x, 1) = g(x)$ for $x \in X$. From Ref.[25] it follows that the homotopy relation defines an equivalence relation. From now one could say iff two functions are homotopic then they are in the same homotopy class. We denote the homotopy class of a function f by $[f]$.

A.2.1 Homotopy group

In this section I will shortly explain what a homotopy group is. The nth homotopy group of Y ($\pi_n(Y)$) is the group of homotopy equivalence classes of the functions $f : \mathbb{S}^n \to Y$ (with $n \geq 1$). Let $f, g : \mathbb{S}^n \to Y$ then the group operation will be given by

$$(f + g)(t_1, ..., t_n) = \begin{cases} f(2t_1, .., t_n) & t_i \in [0, 1/2] \\ g(2t_1, .., t_n) & t_i \in [1/2, 1]. \end{cases}$$

Now for the purpose of this thesis we are interested in two homotopy groups namely: $\pi_2(\mathbb{S}^2)$ and $\pi_1(\mathbb{S}^1)$. One is able to find that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. Where $n \in \mathbb{Z}$ boils down to the number of times you wind $\mathbb{S}^n$ around itself. For the 2-dimensional case the winding number can be given by Ref.[, citatie nodig]

$$n = \frac{1}{4\pi} \int \mathbf{\Omega} \cdot \left(\frac{\partial \mathbf{\Omega}}{\partial x} \times \frac{\partial \mathbf{\Omega}}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y. \tag{A.1}$$

A.3 Skyrmions and domain walls

In the case we discussed in the introduction we wanted to give "a structure" to the space of all continuous maps $\mathbf{\Omega} : \mathbb{S}^2 \to \mathbb{S}^2$. We are now able to do this by looking at all the homotopy classes of this space and thus calculating the homotopy group. In A.2.1 we found that $\pi_2(\mathbb{S}^2) = \mathbb{Z}$, this means that structures with winding numbers $n \neq m$ can not be continuously deformed into one another. One is able to see this, because different winding numbers imply different homotopy classes.

A.3.1 Definition of a Skyrmion

Ferromagnetic Skyrmion

A single Skyrmion is defined as the lowest energy state with winding number $n = 1$ or -1. We assume our system is large and lattice spacing is small compared to a skyrmion's typical size. Classically the spins all move continuously in time on a sphere. This means that it is not possible for a single Skyrmion state to decay into a state with a different winding number classically. We thus call the Skyrmion protected by topology.

There also exist higher order Skyrmion which are defined in a similar fashion. The nth order Skyrmion is given by the lowest energy state with winding number n. In this definition the 0th order Skyrmion would be the the vacuum and the $-n$th order Skyrmion will have the same static energy as an nth order Skyrmion. Because the group-action we work with is additive, we can see that it is in principle possible for an 2nd order Skyrmion to decay into two first order Skyrmions. In principle it should also be possible to create a Skyrmion and a anti-Skyrmion ($n = -1$) in a vacuum.

Antiferromagnetic Skyrmion

As we will see in Section 3.1 in the large wavelength limit we get the equivalence relation $\hat{\mathbf{n}} \sim -\hat{\mathbf{n}} \in \mathbb{S}^2$. So a antiferromagnetic configuration in the large wavelength limit can be described by

$$\hat{\mathbf{n}} : \mathbb{S}^2 \to \mathbb{P}^2(\mathbb{R}). \tag{A.2}$$

Where $\mathbb{P}^2(\mathbb{R})$ is called the projective space. From Ref.[25, Proposition 4.1] it follows that the second homotopy group of $\mathbb{P}^2(\mathbb{R})$ is given by $\mathbb{Z}$, i.e. $\pi_2(\mathbb{P}^2(\mathbb{R})) = \mathbb{Z}$. The reason is that $\mathbb{S}^2$ covers $\mathbb{P}^2(\mathbb{R})$ twice and via Ref.[25, Proposition 4.1] it now follows that $\pi_2(\mathbb{P}^2(\mathbb{R})) \simeq \pi_2(\mathbb{S}^2) = \mathbb{Z}$. Where $\simeq$ is denoted as an isomorphism, which can be given by q_*. Where $q_*([f]) = [qf]$ and q is defined to be the quotient map

$$q : \mathbb{S}^2 \to \mathbb{P}^2(\mathbb{R}).$$

Now the generator of the group $\pi_2(\mathbb{S}^2)$ will be mapped to a generator of $\pi_2(\mathbb{P}^2(\mathbb{R}))$ by q_*. Because q is winding number conserving we are able to give a Skyrmion configuration in $\mathbb{S}^2$ and then use q to map it to $\mathbb{P}^2(\mathbb{R})$.

A.3.2 Domain walls

For domain walls we could do a similar talk. In order to have domain walls we must consider a 1-dimensional magnet for which the magnetization can only point within a circle ($\mathbb{S}^1$). All the function $\mathbf{\Omega} : \mathbb{S}^2 \to \mathbb{S}^2$ can be characterized by the homotopy group $\mathbf{\Omega} : \mathbb{S}^1 \to \mathbb{S}^1$. In a similar discussion as the skyrmionic case we would get that it would not be possible for domain walls to decay into states with a different winding number.

A.4 Topological structures in practice

In practice the positions of the spins will not be uniform over $\mathbb{S}^2$, but they will be located on a lattice. A skyrmion will be much larger than the lattice spacing, also the exchange interaction strength between neighboring spins should be large. In this way one is able to approximate the lattice by $\mathbb{S}^2$. In practice the interaction strength between neighboring spins will be finite. Thus there will be a finite energy barrier protecting the skyrmion.

For the assumptions that we make, we use that the interaction energy between two spins quite high, in real life there is always a finite energy associated with the creation and destruction of a skyrmion.

Appendix B

Calculations for effective action anti ferromagnet

B.1 Cylindrical coordinates and the mass of a skyrmion

We start with the following coordinate transformation

$$\{x, y, z\} = \{\rho\cos(\phi), \rho\sin(\phi), z\}\,. \tag{B.1}$$

From this we are able to derive

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \phi} = \cos(\phi)\frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin(\phi)\frac{\partial}{\partial \phi},\\
\frac{\partial}{\partial y} &= \frac{\partial \rho}{\partial y}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \sin(\phi)\frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos(\phi)\frac{\partial}{\partial \phi},\\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial z}
\end{aligned} \tag{B.2}$$

From Section 3.2 we know that a skyrmion configuration can be given by

$$\hat{\mathbf{n}}_{\mathbf{0}} = \sin[\theta_0]\cos[\phi_0]\,\hat{\rho} + \sin[\theta_0]\sin[\phi_0]\,\hat{\phi} + \cos[\theta_0]\,\hat{z}.$$

While the solutions we found were of the hedgehog type, we find that $\phi_0 \in \{0, \pi\}$ and θ_0 is a function of ρ only. Please note we are working in cylindrical coordinates, for Cartesian coordinates we get $\phi_0 = \phi$ or $\phi + \pi$. From (B.2) we now find

$$\begin{aligned}
\frac{\partial \hat{\mathbf{n}}_{\mathbf{0}}}{\partial x} &= \cos(\phi)\frac{\partial \hat{\mathbf{n}}_{\mathbf{0}}}{\partial \rho} + \frac{1}{\rho}\sin(\phi)\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}\\
\frac{\partial \hat{\mathbf{n}}_{\mathbf{0}}}{\partial y} &= \sin(\phi)\frac{\partial \hat{\mathbf{n}}_{\mathbf{0}}}{\partial \rho} - \frac{1}{\rho}\cos(\phi)\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}
\end{aligned} \tag{B.3}$$

Because we know from chapter 3.2 that the skyrmion configuration is rotationally invariant. We will look at the mass tensors

From this it follows that

$$\begin{aligned}
\tilde{M}_{xx} = \tilde{M}_{yy} = \int_0^\infty d\rho \int_0^{2\pi} d\phi\ \rho \left(\partial_i \hat{\mathbf{n}}_\mathbf{0}\right)^2 = \\
\int_0^\infty d\rho \int_0^{2\pi} d\phi\ \left\{\cos^2(\phi)\rho \left(\partial_\rho \hat{\mathbf{n}}_\mathbf{0}\right)^2 + \frac{1}{\rho}\sin^2(\phi)\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}^2\right\} \\
= \pi \int_0^\infty d\rho\ \left\{\rho \left(\partial_\rho \hat{\mathbf{n}}_\mathbf{0}\right)^2 + \frac{1}{\rho}\left\{\sin(\theta_0)\sin(\phi_0)\hat{\rho} - \sin(\theta_0)\cos(\phi_0)\hat{\phi}\right\}^2\right\} \\
\tilde{M}_{xy} = \tilde{M}_{yx} = 0
\end{aligned} \tag{B.4}$$

We used the fact that $\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}$ and $\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}$.

B.2 Magnus force and Berry phase term for antiferromagnetic skyrmion

In this appendix we work out the Berry phase and the Magnus force for a antiferromagnetic skyrmion. We use methods described by Ref. [13].

B.2.1 Magnus force

In Eq. (4.17) we obtained a Magnus which couples linearly to the magnetic field. In this section we perform the calculation. We start from

$$\begin{aligned}
S = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \implies \delta S = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ \delta(\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \\
= \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ \delta\left(\epsilon_{ijk}\hat{\mathbf{n}}_{0,j}\dot{\hat{\mathbf{n}}}_{0,k}\right) \overset{P.I.}{=} \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ \epsilon_{ijk}\left(\delta\hat{\mathbf{n}}_{0,j}\dot{\hat{\mathbf{n}}}_{0,k} - \dot{\hat{n}}_{0,j}\delta\hat{\mathbf{n}}_{0,k}\right).
\end{aligned} \tag{B.5}$$

Now we use

$$\partial_\tau \hat{\mathbf{n}}_0 = -(\partial_\beta \hat{\mathbf{n}}_0)\dot{R}^\beta, \tag{B.6}$$

$$\delta\hat{\mathbf{n}}_0 = -(\partial_\gamma \hat{\mathbf{n}}_0)\delta R^\beta. \tag{B.7}$$

It follows that

$$\begin{aligned}
\delta S = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ \dot{R}^\gamma \delta R^\beta \left\{(\partial_\beta \hat{\mathbf{n}}_0) \times (\partial_\gamma \hat{\mathbf{n}}_0) - (\partial_\gamma \hat{\mathbf{n}}_0) \times (\partial_\beta \hat{\mathbf{n}}_0)\right\} \\
= \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\ 2\dot{R}^\gamma \delta R^\beta \left\{(\partial_\beta \hat{\mathbf{n}}_0) \times (\partial_\gamma \hat{\mathbf{n}}_0)\right\}.
\end{aligned} \tag{B.8}$$

This implies that the action must be given by

$$\begin{aligned}
S \to \int \mathrm{d}\tau \dot{R}^i R^j \left\{\int \mathrm{d}\mathbf{x}\ 2(\partial_i \hat{\mathbf{n}}_\mathbf{0} \times \partial_j \hat{\mathbf{n}}_\mathbf{0})^z\right\} \\
= \int d\tau \epsilon_{ij} R^i \dot{R}^j \left\{\int_0^{2\pi} d\phi \int_0^\infty d\rho\ \partial_\rho(\sin^2(\theta_0))\partial_\phi \phi_0\right\} = \int d\tau \left\{\tilde{\alpha}\epsilon_{ij}R^i \dot{R}^j\right\}, \\
\tilde{\alpha} = \left\{\int_0^{2\pi} d\phi \int_0^\infty d\rho\ \partial_\rho(\sin^2(\theta_0))\partial_\phi \phi_0\right\}.
\end{aligned} \tag{B.9}$$

Thus the above action corresponds gives a Magnus force to a collective phenomena.

B.2.2 Berry phase

In this section we will discus why we are able to neglect the antiferromagnetic Berry phase in the large wavelength limit. The Berry phase we consider is given by Eq. (3.13)

$$\Upsilon[\hat{\mathbf{n}}] = S\sum_i \eta_i \omega[\hat{\mathbf{n}}(\mathbf{x_i})], \tag{B.10}$$

$$\Upsilon[\hat{\mathbf{n}}_0 + \delta\hat{\mathbf{n}}] \approx \Upsilon[\hat{\mathbf{n}}_0] + \frac{\delta\Upsilon[\hat{\mathbf{n}}]}{\delta\hat{\mathbf{n}}_0}\delta\hat{\mathbf{n}}. \tag{B.11}$$

For $\Upsilon[\hat{\mathbf{n}}_0]$ we use a similar method as used in Appendix B.2.1

$$\begin{aligned} \delta\Upsilon[\hat{\mathbf{n}}_0] &= S\sum_i \eta_i \frac{\delta\omega[\hat{\mathbf{n}}(\mathbf{x_i})]}{\delta\hat{\mathbf{n}}_{0,i}}\delta\hat{\mathbf{n}}_{0,i} \\ &\to \int d\tau \int d\mathbf{x}\, \delta R^\beta \dot{R}^\gamma \eta(\mathbf{x}) \left(\partial_\gamma \hat{\mathbf{n}}_0 \times \partial_\beta \hat{\mathbf{n}}_0\right) \cdot \hat{\mathbf{n}}_0 \\ \implies \Upsilon[\hat{\mathbf{n}}_0] &= \int d\tau R^\beta \dot{R}^\gamma \int d\mathbf{x}\, \eta(\mathbf{x}) \left(\partial_\gamma \hat{\mathbf{n}}_0 \times \partial_\beta \hat{\mathbf{n}}_0\right) \cdot \hat{\mathbf{n}}_0. \end{aligned} \tag{B.12}$$

Which gives rise to a negligibly small Magnus force, due to the fact that a skyrmion is much larger than the spacing between lattice sites. The second term in Eq. (B.11) is negligible in the long wavelength approximation, this can be seen by writing it out in momentum space.

Appendix C

Second order expansions of the Berry phase and the Hamiltonian

C.1 Working out the kinetic term expansion

In the calculations below we used the following identity

$$\int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, \xi d_\tau^2 \xi = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, \xi d_\tau \left\{ -\partial_\beta \xi \dot{R}^\beta + \partial_\tau \xi \right\} \approx \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, \xi \left\{ \dot{R}^\beta \dot{R}^\gamma \partial_\beta \partial_\gamma - 2\dot{R}^\beta \partial_\beta \partial_\tau + \partial_\tau^2 \right\} \xi. \quad \text{(C.1)}$$

We also used the following

$$\int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, \dot{\xi}_0^2 = \int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, \partial_\beta \xi_0 \partial_\gamma \xi_0 \dot{R}^\beta \dot{R}^\gamma, \quad \text{(C.2)}$$

where ξ_0 is either θ_0 or ϕ_0.

$$
\begin{aligned}
\int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, |\partial_\tau \hat{\mathbf{n}}|^2 = &\int \mathrm{d}\tau \mathrm{d}\mathbf{x} \Big\{ |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 \\
&+ \left[\sin(2\theta_0)\dot{\phi}_0^2 - 2\ddot{\theta}_0\right] \vartheta - 2\left[\sin(2\theta_0)\dot{\theta}_0\dot{\phi}_0 + \sin^2(\theta_0)\ddot{\phi}_0\right] \tilde{\varphi} \\
&+ \vartheta \left(\cos(2\theta_0)\dot{\phi}_0^2 - d_\tau^2\right) \vartheta - \tilde{\varphi}\left[\sin(2\theta_0)\dot{\theta}_0 d_\tau + \sin^2(\theta_0) d_\tau^2\right] \tilde{\varphi} + \vartheta \left[2\sin(2\theta_0)\dot{\phi}_0 d_\tau\right] \tilde{\varphi} \Big\} \\
= &\int \mathrm{d}\tau \mathrm{d}\mathbf{x} \Big\{ |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 \\
&+ \left[\sin(2\theta_0)\dot{\phi}_0^2 - 2\ddot{\theta}_0\right] \left(-\nu - \frac{1}{2\tan(\theta_0)}\nu^2\right) - 2\left[2\cos(\theta_0)\dot{\theta}_0\dot{\phi}_0 + \sin(\theta_0)\ddot{\phi}_0\right] \varphi \\
&+ \nu \left(\cos(2\theta_0)\dot{\phi}_0^2 - d_\tau^2\right) \nu - \frac{\varphi}{\sin(\theta_0)}\left[\sin(2\theta_0)\dot{\theta}_0 d_\tau + \sin^2(\theta_0) d_\tau^2\right] \frac{\varphi}{\sin(\theta_0)} \\
&- \nu \left[2\sin(2\theta_0)\dot{\phi}_0 d_\tau\right] \frac{\varphi}{\sin(\theta_0)} \Big\} \\
= &\int \mathrm{d}\tau \mathrm{d}\mathbf{x} \Big\{ |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 \\
&+ \dot{R}^\beta \dot{R}^\gamma \Big\{ \left[\sin(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0)_0 - 2(\partial_\beta\partial_\gamma\theta_0)\right] \left(-\nu - \frac{1}{2\tan(\theta_0)}\nu^2\right) \\
&- 2\left[2\cos(\theta_0)(\partial_\beta\theta_0)(\partial_\gamma\phi_0) + \sin(\theta_0)(\partial_\beta\partial_\gamma\phi_0)\right] \varphi \Big\} \\
&+ \nu \left(\dot{R}^\beta \dot{R}^\gamma \cos(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0) - \dot{R}^\beta \dot{R}^\gamma \partial_\beta\partial_\gamma + 2\dot{R}^\beta \partial_\beta \partial_\tau - \partial_\tau^2\right) \nu \\
&- \varphi \Big[2\cot(\theta_0)\dot{\theta}_0 \left(d_\tau - \cot(\theta_0)\dot{\theta}_0\right) \\
&+ \left(-\cot(\theta_0)\dot{\theta}_0 d_\tau + \left(\frac{2}{\sin^2(\theta_0)} - 1\right)\dot{\theta}_0^2 - \cot(\theta_0)\ddot{\theta}_0 + d_\tau^2\right) \Big] \varphi \\
&- \frac{\nu}{\sin(\theta_0)} \left[2\sin(2\theta_0)\dot{\phi}_0 \left(d_\tau - \cot(\theta_0)\dot{\theta}_0\right)\right] \varphi \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \Bigg\{ |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 \\
&+ \dot{R}^\beta \dot{R}^\gamma \Bigg\{ \left[\sin(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0)_0 - 2(\partial_\beta\partial_\gamma\theta_0)\right] \left(-\nu - \frac{1}{2\tan(\theta_0)}\nu^2\right) \\
&- 2\left[2\cos(\theta_0)(\partial_\beta\theta_0)(\partial_\gamma\phi_0) + \sin(\theta_0)(\partial_\beta\partial_\gamma\phi_0)\right]\varphi \Bigg\} \\
&+ \nu \left(\dot{R}^\beta \dot{R}^\gamma \cos(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0) - \dot{R}^\beta \dot{R}^\gamma \partial_\beta\partial_\gamma + 2\dot{R}^\beta \partial_\beta\partial_\tau - \partial_\tau^2\right)\nu \\
&- \varphi\left[\cot(\theta_0)\dot{\theta}_0 d_\tau + \dot{\theta}_0^2 - \cot(\theta_0)\ddot{\theta}_0 + d_\tau^2\right]\varphi \\
&- \nu\left[4\cos(\theta_0)\dot{\phi}_0\left(d_\tau - \cot(\theta_0)\dot{\theta}_0\right)\right]\varphi \Bigg\} \\
&= \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \Bigg\{ |\partial_\tau \hat{\mathbf{n}}_\mathbf{0}|^2 \\
&+ \dot{R}^\beta \dot{R}^\gamma \Bigg\{ \left[\sin(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0)_0 - 2(\partial_\beta\partial_\gamma\theta_0)\right] \left(-\nu - \frac{1}{2\tan(\theta_0)}\nu^2\right) \\
&- 2\left[2\cos(\theta_0)(\partial_\beta\theta_0)(\partial_\gamma\phi_0) + \sin(\theta_0)(\partial_\beta\partial_\gamma\phi_0)\right]\varphi \Bigg\} \\
&+ \nu \left(\dot{R}^\beta \dot{R}^\gamma \cos(2\theta_0)(\partial_\beta\phi_0)(\partial_\gamma\phi_0) - \dot{R}^\beta \dot{R}^\gamma \partial_\beta\partial_\gamma + 2\dot{R}^\beta \partial_\beta\partial_\tau - \partial_\tau^2\right)\nu \\
&- \varphi\Bigg[\dot{R}^\beta \cot(\theta_0)(\partial_\beta\theta_0)\left(\partial_\tau - \dot{R}^\gamma\partial_\gamma\right) + \dot{R}^\beta\dot{R}^\gamma(\partial_\beta\theta_0)(\partial_\gamma\theta_0) - \dot{R}^\beta\dot{R}^\gamma\cot(\theta_0)(\partial_\beta\partial_\gamma\theta_0) \\
&+ \left(\dot{R}^\beta \dot{R}^\gamma\partial_\beta\partial_\gamma - 2\dot{R}^\beta\partial_\beta\partial_\tau + \partial_\tau^2\right)\Bigg]\varphi \\
&- \nu\left[4\dot{R}^\beta\cos(\theta_0)(\partial_\beta\phi_0)\left(-\dot{R}^\gamma\partial_\gamma + \partial_\tau - \dot{R}^\gamma\cot(\theta_0)(\partial_\gamma\theta_0)\right)\right]\varphi \Bigg\} \\
&= \int \mathrm{d}\tau\, \tilde{M}\dot{R}^2 + \int \mathrm{d}\tau\mathrm{d}\mathbf{x}\, \Big\{ -\eta^\intercal\cdot\partial_\tau^2\eta + \dot{R}^\beta\dot{R}^\gamma\left[\mathbf{J}_{\beta\gamma}\cdot\eta + \eta^\intercal\cdot\left(\Gamma_{\beta\gamma} + \mathbf{T}_\beta\partial_\gamma - \partial_\beta\partial_\gamma\right)\eta\right] \\
&+ \dot{R}^\beta\, \eta^\intercal\cdot\left(2\partial_\beta\partial_\tau - \mathbf{T}_\beta\partial_\tau\right)\eta\Big\}.
\end{aligned} \tag{C.3}
$$

$$
\begin{aligned}
\int \mathrm{d}\tau \mathrm{d}\mathbf{x}\, (\hat{\mathbf{n}}(\mathbf{x}) \times \partial_\tau \hat{\mathbf{n}}(\mathbf{x}))^z = &\int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left\{ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \right. \\
&\left. + \sin^2(\theta_0)\dot{\tilde{\varphi}} + \sin(2\theta_0)\dot{\phi}_0\vartheta + \sin(2\theta_0)\vartheta\dot{\tilde{\varphi}} + \cos(2\theta_0)\dot{\phi}_0\vartheta^2 \right\} \\
&\to \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left\{ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \right. \\
&\dot{R}^\beta \left[\sin(2\theta_0)(\partial_\beta\theta_0)\frac{\varphi}{\sin(\theta_0)} - \sin(2\theta_0)(\partial_\beta\phi_0)\vartheta - \vartheta\cos(2\theta_0)(\partial_\beta\phi_0)\vartheta \right] \\
&\left. + \vartheta \sin(2\theta_0) d_\tau \left(\frac{\varphi}{\sin(\theta_0)} \right) \right\} \\
&= \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left\{ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \right. \\
&\dot{R}^\beta \left[2\cos(\theta_0)(\partial_\beta\theta_0)\varphi + \sin(2\theta_0)(\partial_\beta\phi_0) \left(\nu + \frac{1}{2\tan(\theta_0)}\nu^2 \right) - \nu\cos(2\theta_0)(\partial_\beta\phi_0)\nu \right] \\
&\left. - 2\nu\cos(\theta_0) \left(d_\tau - \cot(\theta_0)\dot{\theta}_0 \right) \varphi \right\} \\
&= \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left\{ (\hat{\mathbf{n}}_\mathbf{0} \times \partial_\tau \hat{\mathbf{n}}_\mathbf{0})^z \right. \\
&\dot{R}^\beta \Big[2\cos(\theta_0)(\partial_\beta\theta_0)\varphi - 2\nu\cos(\theta_0)\cot(\theta_0)(\partial_\beta\theta_0)\varphi \\
&+ \sin(2\theta_0)(\partial_\beta\phi_0)\nu + \nu\sin^2(\theta_0)(\partial_\beta\phi_0)\nu \Big] \\
&\left. - 2\nu\cos(\theta_0) \left(\partial_\tau - \dot{R}^\beta\partial_\beta \right) \varphi \right\} \\
&= \int \mathrm{d}\tau\, \tilde{\alpha}\epsilon_{ij} R^i \dot{R}^j + \int \mathrm{d}\tau \mathrm{d}\mathbf{x} \left\{ -\eta^\intercal \cdot \mathbf{L}\partial_\tau\eta + \dot{R}^\beta \; [\eta^\intercal \cdot (\mathbf{M}_\beta + \mathbf{L}\partial_\beta)\,\eta + \mathbf{N}_\beta \cdot \eta] \right\}.
\end{aligned}
\tag{C.4}
$$

C.2 Second order expansion of the free energy

In this appendix we do some of the calculations to determine the second order expansion of $\mathcal{F}[\hat{\mathbf{n}}]$ in Eq. (4.1) w.r.t. η which is used in Section 4.4. Since a skyrmion or vortex configuration is a local minimum of $\mathcal{F}$, the first order term in η equals 0. The zero'th order term will just give a energy E_0, which is not interesting for our purposes because we assume the skyrmion is already there and not thermally created. We will thus restrict ourselves to determining $\mathcal{H} = \frac{\delta^2 \mathcal{F}}{\delta \eta^\dagger \eta}\Big|_{\eta^\intercal=\eta=0}$ in Eq. (4.26)

We start from equation (4.1)

$$\mathcal{F}[\hat{\mathbf{n}}] = N_A \int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \left\{ \frac{\rho_s}{2} \sum_{l=x,y} |\partial_l \hat{\mathbf{n}}|^2 + \frac{D}{2} \left(\hat{y} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial x} \right) - \hat{x} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial y} \right) \right) - K_z \hat{\mathbf{n}}_z^2 + \frac{S^2}{2} \chi_0 \left(\tilde{\mathbf{H}} \cdot \hat{\mathbf{n}}(\mathbf{x}) \right)^2 \right\}.$$

The exchange interaction becomes,

$$\begin{aligned}
&\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \sum_{l=x,y} |\partial_l \hat{\mathbf{n}}|^2 = \int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \left\{ (\nabla\theta)^2 + \sin^2(\theta)(\nabla\phi)^2 \right\}, \\
\text{second order} \rightarrow &\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \big\{ -\vartheta \nabla^2 \vartheta + \sin^2(\theta_0)(\nabla\tilde{\varphi})^2 \\
&\qquad + 2\vartheta \sin(2\theta_0)(\nabla\phi_0)(\nabla\tilde{\varphi}) + \vartheta \cos(2\theta_0)(\nabla\phi_0)^2 \vartheta \big\} \\
\rightarrow &\int_0^\beta \mathrm{d}\tau \int_\Lambda d\phi d\rho\, \rho \big\{ -\nu\nabla^2\nu - \varphi\nabla^2\varphi + \varphi \left(\frac{\cot(\theta_0)}{\rho}(\partial_\rho\theta_0) - (\partial_\rho\theta_0)^2 + \cot(\theta_0)(\partial_\rho^2\theta_0) \right) \varphi \\
&\qquad + 2\vartheta \sin(2\theta_0)(\nabla\phi_0)(\nabla\tilde{\varphi}) + \vartheta \cos(2\theta_0)(\nabla\phi_0)^2 \vartheta \big\} \\
= &\int_0^\beta \mathrm{d}\tau \int_\Lambda d\phi d\rho\, \rho \big\{ -\nu\nabla^2\nu - \varphi\nabla^2\varphi + \varphi \left(\frac{\cot(\theta_0)}{\rho}(\partial_\rho\theta_0) - (\partial_\rho\theta_0)^2 + \cot(\theta_0)(\partial_\rho^2\theta_0) \right) \varphi \\
&\qquad - \nu \frac{4\cos(\theta_0)}{\rho^2}(\partial_\phi\varphi) + \nu \frac{\cos(2\theta_0)}{\rho^2} \nu \big\}.
\end{aligned} \tag{C.5}$$

Where the Dzyloshinskii Moriya interaction reduces to,

$$
\begin{aligned}
&\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \left(\hat{y} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial x} \right) - \hat{x} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \hat{\mathbf{n}}}{\partial y} \right) \right) \\
&= \int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \left\{ \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \cdot (\nabla\theta) + \frac{1}{2}\sin(2\theta) \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix} \cdot (\nabla\phi) \right\}, \\
\text{second order} \rightarrow &\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \Big\{ -(1/2)\tilde{\varphi} \begin{pmatrix} \cos(\phi_0) \\ \sin(\phi_0) \end{pmatrix} \cdot (\nabla\theta_0)\tilde{\varphi} + \tilde{\varphi} \begin{pmatrix} -\sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix} \cdot (\nabla\vartheta) \\
&\qquad - \vartheta \sin(2\theta_0) \begin{pmatrix} -\sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix} \cdot (\nabla\phi_0)\vartheta - \frac{1}{4}\tilde{\varphi}\sin(2\theta_0) \begin{pmatrix} -\sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix} \cdot (\nabla\phi_0)\tilde{\varphi} \\
&\qquad + \vartheta\cos(2\theta_0) \begin{pmatrix} -\sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix} \cdot (\nabla\tilde{\varphi}) - \tilde{\varphi}\sin(2\theta_0) \begin{pmatrix} \cos(\phi_0) \\ \sin(\phi_0) \end{pmatrix} \cdot (\nabla\tilde{\varphi}) \Big\} \\
\rightarrow &\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \pm \Big\{ -(1/2)\tilde{\varphi}(\partial_\rho\theta_0)\tilde{\varphi} + \frac{1}{\rho}\tilde{\varphi}(\partial_\phi\vartheta) - \vartheta\frac{\sin(2\theta_0)}{\rho}\vartheta - \frac{1}{4}\tilde{\varphi}\frac{\sin(2\theta_0)}{\rho}\tilde{\varphi} \\
&\qquad + \vartheta\frac{\cos(2\theta_0)}{\rho}(\partial_\phi\tilde{\varphi}) - \tilde{\varphi}\sin(2\theta_0)(\partial_\rho\tilde{\varphi}) \Big\} \\
\rightarrow &\int_0^\beta \mathrm{d}\tau \int_\Lambda \mathrm{d}\mathbf{x} \pm \Big\{ -\varphi\frac{(\partial_\rho\theta_0)}{2\sin^2(\theta_0)}\varphi - \varphi\frac{1}{\sin(\theta_0)\rho}(\partial_\phi\nu) - \nu\frac{\sin(2\theta_0)}{\rho}\nu - \varphi\frac{\cot(\theta_0)}{2\rho}\varphi \\
&\qquad - \nu\frac{\cos(2\theta_0)}{\sin(\theta_0)\rho}(\partial_\phi\varphi) - 2\varphi\cot(\theta_0)(\partial_\rho\varphi) + 2\varphi\cot^2(\theta_0)(\partial_\rho\theta_0)\varphi \Big\}.
\end{aligned}
\tag{C.6}
$$

Appendix D

Dynamics of skyrmions and vortices for large spin flop fields

Due to lack of time we will not perform all the calculations to obtain closed analytic expressions for the mass renormalizations and damping kernels.

In this section we will consider vortex configurations for large magnetic fields. By neglecting the parts that are independent of the magnetic field, the operators in Eqs. (4.25), (4.28) and (4.29) reduce to

$$\begin{aligned}\mathcal{G} &= -\frac{\chi_0}{2}\left\{2i\tilde{H}\mathbf{L}\partial_\tau\right\} + \mathcal{H}\\ \mathcal{J} &= \frac{\chi_0}{4}\left\{2i\dot{R}^\beta\tilde{H}\mathbf{N}_\beta\right\},\\ \mathcal{K} &= \frac{\chi_0}{2}\left\{2i\tilde{H}\dot{R}^\beta\left(\mathbf{M}_\beta + \mathbf{L}\partial_\beta\right)\right\}.\end{aligned}$$

We see that we cannot split the eigenstates of $\mathcal{G}$ in a time-dependent and a space-dependent part. So it will be a bit harder to perform sums over the Matsubara frequencies. For now we assume

$$\cos(\theta_0(\rho)) \simeq \begin{cases} -1 & \rho < \sqrt{\rho_s/K_z} \\ 0 & \rho > \sqrt{\rho_s/K_z} \end{cases},$$

for vortices. For skyrmions we make the following approximation

$$\cos(\theta_0(\rho)) \simeq \begin{cases} -1 & \rho < \sqrt{\rho_s/K_z} \\ 1 & \rho > \sqrt{\rho_s/K_z} \end{cases}.$$

We can still perform the shift in Eq. (4.70) to complete the square

$$\begin{aligned}Z[\mathbf{R}] &= \int \mathcal{D}\mathbf{R}\, e^{-\mathcal{S}_{cl}[\mathbf{R}]}F[\mathbf{R}],\\ F[\mathbf{R}] &= \int \mathcal{D}\eta\, \delta(Q_1[\mathbf{R}])\delta(Q_2[\mathbf{R}])\det\left(\frac{\delta\mathbf{Q}}{\delta\mathbf{R}}\right)e^{-\mathcal{I}}e^{-N_A\{\tilde{\eta}^\intercal\cdot[\mathcal{G}+\mathcal{K}]\tilde{\eta}\}}\\ &= e^{-\mathcal{I}}\frac{1}{\det'[N_A(\mathcal{G}+\mathcal{K})]}.\end{aligned}$$

Where the prime denotes the omission of the zero-modes and $\mathcal{I}$ is defined by

$$\mathcal{I} = -N_A\mathcal{J}^\intercal\cdot(\mathcal{G}+\mathcal{K})^{-1}\mathcal{J} \approx -N_A\mathcal{J}^\intercal\cdot\mathcal{G}^{-1}\mathcal{J}. \tag{D.1}$$

For large magnetic fields Eq. (D.1) will most probably give a damping kernel of order $\mathcal{O}((N_A)^1)$. For slow dynamics of the vortex and low temperatures $\beta \gg \epsilon_0^{-1}$ we obtain we define the mass correction by the zero frequency limit of the preceding damping kernel Ref.[9]. This mass correction will most probably be quadratic in $\tilde{H}$ and will be of order $\mathcal{O}((N_A)^1)$.

The terms coming from

$$\frac{1}{\det'[N_A(\mathcal{G}+\mathcal{K})]}$$

will be of order $\mathcal{O}((N_A)^0)$ and will be negligible in the zero-temperature limit.

Appendix E

The effective action for domain walls

In this chapter we work through some of the calculations done in Ref.[3].

E.1 The model used in the paper

The model that is used is the Heisenberg spin model with some added spin anisotropies. We will thus write our Hamiltonian:

$$\mathcal{H} = -\tilde{J}\sum_{i,\rho}\mathbf{S}_i \cdot \mathbf{S}_{i+\rho} - \tilde{K}_y \sum_i (S_i^y)^2 + \tilde{K}_z \sum_i (S_i^z)^2. \tag{E.1}$$

For simplicity we will assume that the spins are located on a cubic lattice with lattice constant a. In the above formula the $\mathbf{S}_i$ gives the spin at lattice side i, and $i+\rho$ gives the neighboring lattice sides. The first term in the Hamiltonian gives the exchange interaction an the second and the third term give anisotropy's. In this case assume $\tilde{K}_y, \tilde{K}_z > 0$. We will see that the $\tilde{K}_z$ term will cause the spins to prefer to point in the xy-plane and $\tilde{K}_y$ term will cause the spin to prefer to point parallel to the y-axis.

E.1.1 Taking the continuum limit of the Hamiltonian

Because the system system size of the phenomena we are interested in (Bloch walls/domain walls) is much larger than the lattice spacing a we are allowed to take the continuum limit of Eq. (E.1). This section will be focused on doing this. By using the definition of a Riemann integral and using the argument made above we are able to argue that for sufficiently large phenomena

$$\frac{1}{a}\int \mathrm{d}x \simeq \sum_i . \tag{E.2}$$

From now on we will write down the spin in coherent states basis Ref.[3, Appendix A], and write $\mathbf{\Omega} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. We start finding an expression for the first term in Hamiltonian (E.1). We have $\mathbf{S}_i \cdot \mathbf{S}_{\mathbf{i}+\rho} = S^2\left\{\cos(\theta)\cos(\theta+\delta\theta) + \cos(\delta\phi)\sin(\theta)\sin(\theta+\delta\theta)\right\}$. By taking this expansion op to second order in $\delta\theta$ we get

$$\mathbf{S}_i \cdot \mathbf{S}_{\mathbf{i}+\rho} \simeq S^2\left\{1 - \frac{1}{2}\left(\delta\theta^2 + \sin^2(\theta)\delta\phi^2\right)\right\},$$

Now we make the assumption that the system we consider is almost one dimensional (so needle like) and assume θ, ϕ are only dependent on x- length direction. Notice that $\delta\theta = a\,\frac{\partial\theta}{\partial x}$ and $\delta\phi = a\,\frac{\partial\phi}{\partial x}$, this implies

$$\mathbf{S}_i \cdot \mathbf{S}_{\mathbf{i}+\rho} \simeq a^2 S^2 \left\{ 1 - \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2(\theta)\left(\frac{\partial\phi}{\partial x}\right)^2 \right) \right\}. \tag{E.3}$$

Now combining Eqs. (E.2) and (E.3) we are able to rewrite Eq. (E.1) up to a constant

$$\mathcal{H} = N_A \int \mathrm{d}x \left\{ J\left[(\partial_x\theta)^2 + \sin^2\theta\,(\partial_x\phi)^2 \right] - K_y\left[\sin^2\theta \sin^2\phi - 1 \right] + K_z \cos^2\theta \right\}. \tag{E.4}$$

In the above formula N_A denotes the number of spins in the cross sectional area. The constants are given by $J = \tilde{J}S^2 a$ and $K_y, z = \tilde{K}_{y,z}S^2/a$. It follows that the Euclidean action is given by

$$\mathcal{S}_E = \mathcal{S}_{WZ} + \int_0^\beta \mathrm{d}\tau\,\mathcal{H}, \tag{E.5}$$

with the berry phase term

$$\mathcal{S}_{WZ} = i\frac{SN_A}{a}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\,\dot{\phi}(1 - cos\theta). \tag{E.6}$$

By taking $\delta\mathcal{S}_E = 0$ and rotating to real time we get the following equations of motion

$$\sin\theta\,\partial_t\phi = -\frac{a}{S}\frac{\delta H}{\delta\theta}, \quad \partial_t\theta = \frac{a}{S}\frac{1}{\sin\theta}\frac{\delta H}{\delta\phi} \tag{E.7}$$

The equation in this particular case will be given by

$$\sin\theta\,\partial\phi = -\frac{a}{S}\left\{ \sin(2\theta)\left[J(\partial_x\phi)^2 - K_y\sin^2\phi - K_z \right] - 2J(\partial_x^2\theta) \right\} \tag{E.8}$$

$$\partial_t\theta = -\frac{a}{S}\left\{ J\left[(\partial_x^2\phi)\sin\theta + 4\sin\theta\cos\theta(\partial_x\theta)(\partial_x\phi) \right] + K_y\cos\phi\sin\phi\sin^2\theta \right\} \tag{E.9}$$

E.2 Domain or Bloch wall configurations

In case of a Bloch wall, we want a stationary solution ($\{\partial_t\theta = 0, \partial_t\phi = 0\}$) which connects the anisotropy minima $\phi = \pm\pi/2$, in the $\theta = \pi/2$ plane. We find that Eq. (E.8) will be trivial in this case. On the other hand Eq. (E.9) will give us

$$J\partial_x^2\phi + K_y\sin\phi\,\cos\phi = 0. \tag{E.10}$$

With additional condition $\phi(\pm\infty) = 0$ Eq. (E.10) can be integrated over to obtain

$$\frac{J}{K_y}(\partial_x\phi)^2 - \cos^2\phi = 0 \tag{E.11}$$

It turns out that there are 4 different Bloch wall solutions (see, ref 47 paper):

$$\phi_{QC}(x) = -QC\frac{\pi}{2} + 2\arctan e^{Cx/\delta}, \tag{E.12}$$

in the above formula $\delta = \sqrt{J/K_y}$ and Q, C are respectively denoted as the charge and the chiralty. With $Q = (1/2)\int \mathrm{d}x\,\partial_x(\sin\phi)$ and

$$C = \frac{1}{\pi}\int \mathrm{d}x\,\partial_x\phi \tag{E.13}$$

For Bloch walls $Q, C = \pm 1$, and the energy of this configuration is given by

$$E_0 = 2JN_A\int \mathrm{d}x(\partial_x\phi_{QC})^2 = 4N_A\sqrt{JK_y}\,. \tag{E.14}$$

Note that these solutions are not topologically protected as long the spins are able to move in the z-plane. In the limit of large hard axes ($K_z \gg K_y$), we note that the configuration space becomes a circle and the chiralty C becomes topologically invariant (see Appendix A).

E.3 Relating the action to the Sine-Gordon Model

In the case of domain walls we are interested in the case where $K_z \gg K_y$, thus deviations away from the easy plane are small. So we are now able to expand

$$\theta(x,\tau) \simeq \pi/2 - \vartheta(x,\tau), \tag{E.15}$$

with $|\vartheta| \ll 1$. We will now plug Eq. (E.15) into action (E.5) and expand up to second order

$$\begin{aligned} \cos\theta &\rightarrow\ \vartheta + \mathcal{O}(\vartheta^3) \\ \cos^2\theta &\rightarrow \vartheta^2 + \mathcal{O}(\vartheta^3) \\ (\partial_x\theta)^2 &\rightarrow\ (\partial_x\vartheta)^2 + \mathcal{O}(\vartheta^3) \\ \sin^2\theta(\partial_x\phi)^2 &\rightarrow\ (1-\vartheta^2)(\partial_x\phi)^2 + \mathcal{O}(\vartheta^3) \\ \sin^2\theta\sin^2\phi &\rightarrow\ (1-\vartheta^2)\sin^2\phi + \mathcal{O}(\vartheta^3). \end{aligned}$$

Thus we will end up with the following action

$$\mathcal{S}_{WZ} = i\frac{SN_A}{a}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\,\dot{\phi}(1-\vartheta) \tag{E.16}$$

$$\begin{aligned} \mathcal{H} &= N_A\int \mathrm{d}x\left\{J\left[(\partial_x\vartheta)^2 + (1-\vartheta^2)(\partial_x\phi)^2\right] - K_y\left[(1-\vartheta^2)\sin^2\phi - 1\right] + K_z\vartheta^2\right\} \\ &= N_A\int \mathrm{d}x\left\{J(\partial_x\phi)^2 + K_y\cos^2\phi + \vartheta\,\mathcal{L}\vartheta\right\}, \end{aligned} \tag{E.17}$$

with $\mathcal{L} = -J\partial_x^2 - J(\partial_x\phi)^2 + K_y\sin^2\phi + K_z$ (the second derivative term is obtained by partial integration). Thus the euclidean action will be given by

$$\mathcal{S}_E = N_A\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\left\{i\frac{S}{a}\partial_\tau\phi - i\frac{S}{a}\vartheta\,\partial_\tau\phi + J(\partial_x\phi)^2 + K_y\cos^2\phi + \vartheta\,\mathcal{L}\,\vartheta\right\}. \tag{E.18}$$

We get the following equations of motion

$$\begin{aligned} \partial_t\vartheta &= \frac{a}{S}\left\{4J\vartheta(\partial_x\vartheta)(\partial_x\phi) + 2J(\vartheta^2-1)\partial_x^2\phi) + K_y(\vartheta^2-1)\sin(2\phi)\right\} \\ \partial_t\phi &= \frac{2a}{S}\left\{J(\partial_x\vartheta)^2 + J\vartheta(\partial_x\phi)^2 - K_y\vartheta - 2K_z\vartheta\right\}. \end{aligned} \tag{E.19}$$

Because we are interested in a quantum mechanical description of a Bloch wall, we will use the path integral formalism Ref.[3, Appendix A], we obtain

$$\langle\{\mathbf{\Omega}_b\}|e^{-\beta\mathcal{H}}|\{\mathbf{\Omega}_a\}\rangle = \int \mathcal{D}\phi\mathcal{D}(\cos\theta)e^{-\mathcal{S}_E[\theta,\phi]}. \tag{E.20}$$

Using the assumption that fluctuations in ϑ and ϕ have a wavelength lager than the domain wall with, $\lambda \geq \delta \implies k^2 \leq K_y/J$ (E.12), this means that that hard-axes anisotropy becomes dominant and $\mathcal{L} = K_z(1 + \mathcal{O}(K_y/K_z))$. Using assumption (E.15) we are also able to rewrite $\mathcal{D}(\cos\theta) \simeq \vartheta$, now by also plugging Eq. (E.18) into Eq. (E.20) instead of Eq. (E.5) we are now able to perform Gaussian integrations with respect to ϑ

$$\begin{aligned} \langle\{\mathbf{\Omega}_b\}|e^{-\beta\mathcal{H}}|\{\mathbf{\Omega}_a\}\rangle &= \int \mathcal{D}\phi\mathcal{D}(\cos\theta)e^{-\mathcal{S}_E[\theta,\phi]} \\ \approx \int \mathcal{D}\phi\mathcal{D}\vartheta e^{-\mathcal{S}_E[\vartheta,\phi]} &= \int \mathcal{D}\phi e^{-\mathcal{S}_\phi[\phi]}\int \mathcal{D}\vartheta e^{-\mathcal{S}_\vartheta[\vartheta,\phi]} \simeq \int \mathcal{D}\phi e^{-\mathcal{S}_{SG}[\phi]}, \end{aligned} \tag{E.21}$$

with the restriction that $\phi(x,0) = \phi_a(x)$ and $\phi(x,\beta) = \phi_b(x)$. The action we get which is only dependent on ϕ is given by

$$\mathcal{S}_{SG} = N_A \int \mathrm{d}x\mathrm{d}\tau \left\{ i\frac{S}{a}\partial_\tau\phi + J\left[\frac{1}{c^2}(\partial_\tau\phi)^2 + (\partial_x\phi)^2\right] + K_y\cos^2\phi \right\}, \tag{E.22}$$

with $c = (2a/S)\sqrt{JK_z}$ the asymptotic spin wave velocity. Note that Eq. (E.21) was obtained using

$$\mathcal{S}_\vartheta = N_A \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left\{ \vartheta\,\mathcal{L}\,\vartheta - i\frac{S}{a}\vartheta\,\partial_\tau\phi \right\} \tag{E.23}$$

$$\begin{aligned}\vartheta\mathcal{L}\vartheta - i\frac{S}{a}\vartheta\partial_\tau\phi &= \mathcal{L}^{-1}\left(\mathcal{L}\vartheta - i\frac{S}{2a}\partial_\tau\phi\right)^2 + \mathcal{L}^{-1}\frac{S^2}{(2a)^2}(\partial\tau\phi)^2 \\ &\simeq \frac{1}{K_z}\left(K_z\vartheta - i\frac{S}{2a}\partial_\tau\phi\right)^2 + \frac{S^2}{(2a)^2K_z}(\partial_\tau\phi)^2\end{aligned} \tag{E.24}$$

and also

$$\mathcal{S}_\phi = N_A \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left\{ i\frac{S}{a}\partial_\tau\phi + J(\partial_x\phi)^2 + K_y\cos^2\phi \right\} \tag{E.25}$$

E.4 Interaction between Bloch walls and spin waves

throughout this chapter we assume $\dot{X}/c$ is a small parameter. We also make use of the assumptions of last chapter so it is allowed to use the sin-Gordon action. For now let's consider a single Bloch wall (E.12) $\phi_0(x) = \phi_{Q=1,C=1}(x)$, now recall that $\phi_(x - X)$ is a static solution of $\delta S_{SG} = 0$. Let's assume we have a Bloch wall at position X which is surrounded by spin waves (excitations with winding number 0). We write

$$\phi(x,\tau) = \phi_0(x - X) + \varphi(x - X, \tau) \tag{E.26}$$

We expand action (E.22) up to second order in φ and $\dot{X}$. We get

$$\int \mathrm{d}x\,\phi_0(x - X) = -\pi\dot{X} \implies \int \mathrm{d}\tau\mathrm{d}x\,\partial_\tau\phi \to -\int \mathrm{d}\tau\pi\dot{X} \tag{E.27}$$

$$\begin{aligned}&\int \mathrm{d}x\,(\partial_x\phi)^2 \to \int \mathrm{d}x\left\{(\partial_x\phi_0)^2 + 2(\partial_x\phi_0)(\partial_x\varphi) + (\partial_x\varphi)^2\right\} \\ &= \frac{E_0}{2JN_A} + \int\left\{-2(\partial_x^2\phi_0)\varphi - \varphi\partial_x^2\varphi\right\}\end{aligned} \tag{E.28}$$

$$\begin{aligned}&\int \mathrm{d}x\mathrm{d}\tau(d_\tau\phi)^2 \to \int \mathrm{d}x\mathrm{d}\tau\left\{(d_\tau\phi_0)^2 + 2(d_\tau\phi_0)(d_\tau\varphi) + (d_\tau\varphi)^2\right\} \\ &= \int \mathrm{d}x\mathrm{d}\tau\left\{(\partial_x\phi_0)^2\dot{X}^2 - 2(\partial_x^2\phi_0\dot{X}^2 - \partial_x\phi_0\ddot{X})\varphi - \varphi d_\tau^2\varphi\right\} \\ &\to \int \mathrm{d}\tau\left\{\frac{E_0}{2JN_A}\dot{X}^2 - \int \mathrm{d}x\left[2(\partial_x^2\phi_0)\dot{X}^2\varphi + \varphi\left\{\dot{X}^2\partial_x^2 - \dot{X}\partial_x\partial_\tau + \partial_\tau^2\right\}\varphi\right]\right\}\end{aligned} \tag{E.29}$$

$$\begin{aligned}&\int \mathrm{d}x\cos\phi \to \int \mathrm{d}x\left\{\cos\phi_0 - 2(\cos\phi_0\sin\phi_0)\varphi + \left[\sin^2\phi_0 - \cos^2\phi_0\right]\varphi^2 + \mathcal{O}(\varphi^3)\right\} \\ &= c_1 + \int \mathrm{d}x - 2(\cos\phi_0\sin\phi_0)\varphi + \left\{1 - 2\,\mathrm{sech}^2\left(\frac{x}{\delta}\right)\right\}\varphi^2 + \mathcal{O}(\varphi^3)\end{aligned} \tag{E.30}$$

Note that in Eq. (E.29) we used $\int \mathrm{d}x\varphi d_\tau^2\varphi = \int \mathrm{d}x\varphi d_\tau\left\{-\partial_x\varphi\dot{X} + \partial_\tau\varphi\right\} = \int \mathrm{d}x\left\{\varphi\partial_x^2\varphi\dot{X}^2 - 2\varphi\partial_x\varphi\ddot{X} + \varphi\partial_\tau\left[-\partial_x\varphi\dot{X} + \partial_\tau\right.\right.$
$= \int \mathrm{d}x\varphi\left\{\dot{X}^2\partial_x^2 - \dot{X}\partial_x\partial_\tau + \partial_\tau^2\right\}\varphi$. Also note that the terms which are linear in Eqs. (E.28) and (E.30) cancel each other. The reason is if you plug them in action (E.22) you get $\left\{J\partial_x^2\phi_0 + K_y\sin\phi_0\cos\phi_0\right\}\varphi =$

0, the term between brackets is now 0 because ϕ_0 satisfies the stationary Euler Lagrange in Eq. (E.10). In Eq. (E.30) we used the fact that

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}},\ \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \implies \cos(2\arctan x) = \frac{1-x^2}{1+x^2},\ \sin(2\arctan x) = \frac{2x}{1+x^2}$$
$$\implies \sin^2\phi_0 - \cos^2\phi_0 = \cos^2(2\arctan e^{x/\delta}) - \sin^2(2\arctan e^{x/\delta}) = 1 - \frac{8}{(e^{x/\delta}-e^{-x/\delta})^2} = 1 - 2\,\mathrm{sech}^2\left(\frac{x}{\delta}\right).$$

By plugging Eqs. (E.22) and (E.27) to (E.30) in to Eq. (E.21) we get

$$\langle\{\mathbf{\Omega}_b\}|e^{-\beta\mathcal{H}}|\{\mathbf{\Omega}_a\}\rangle = \int \mathcal{D}X e^{-\mathcal{S}_X[X]}F[X], \tag{E.31}$$

with

$$\mathcal{S}_X[X] = \int \mathrm{d}t \left\{-i\alpha\dot{X} + \frac{M}{2}\dot{X}2\right\} \tag{E.32}$$

the action of the free Bloch wall and

$$F[X] = \int \mathcal{D}\phi\,\delta\left(\int \mathrm{d}x\phi_0'(x-X)\phi(x,\tau)\right)\det\left(\frac{\delta Q}{\delta X}\right)e^{-N_A\int \mathrm{d}x\mathrm{d}\tau\{\varphi\cdot[\mathcal{G}+\mathcal{K}]\cdot\varphi+\mathcal{J}\cdot\varphi\}}. \tag{E.33}$$

describes the interaction between the Bloch wall and spin waves. In Eq. (E.32) $\alpha = \pi S N_A/a$ and $M = \frac{E_0}{c^2} = \frac{N_A S^2}{a^2 K_Z}\sqrt{\frac{K_y}{J}}$. And functional F in Eq. (E.33) describes the interaction between the Bloch wall and spin waves, where the operator

$$\mathcal{G} = -J\partial_x^2 - \kappa\partial_\tau^2 + K_y\left[1 - 2\,\mathrm{sech}^2\left(\frac{x}{\delta}\right)\right]. \tag{E.34}$$

Note we use $\phi_0'' = \partial_x^2\phi_0(x-X)$.

in this case $\kappa = J/c^2$. The operators $\mathcal{G}$ and $\mathcal{P}$ describe the interaction of spin waves with a static Bloch wall. The other two operators are used to describe the dynamic coupling between the Bloch wall and spin waves

$$\mathcal{K} = 2\kappa\dot{X}\partial_x\partial_\tau - \kappa\dot{X}^2\partial_x^2,\ \ \mathcal{J} = -2\kappa\dot{X}^2\phi_0''. \tag{E.35}$$

E.5 Evaluating the damping kernel

We will now evaluate the functional derivative $\frac{\delta Q}{\delta X}$. *Note by ' we mean derivation w.r.t x.*

$$Q[X+\delta X] = \int \mathrm{d}x\mathrm{d}\tau'\,\phi_0'(x-X-\delta X)\phi(x,\tau')\delta(\tau-\tau') \tag{E.36}$$
$$= Q[X] + \int \mathrm{d}x\mathrm{d}\tau'\,\left\{-\phi_0''(x-X)\phi(x,\tau)\delta X + \mathcal{O}(\delta X^2)\right\}\delta(\tau-\tau') \tag{E.37}$$
$$P.I. \implies \frac{\delta Q}{\delta X} = \int \mathrm{d}x\left\{\phi_0'^2(x-X) - \phi_0''(x-X)\varphi(x-X,\tau)\right\}\delta(\tau-\tau'). \tag{E.38}$$

To evaluate $F[X]$ we want to complete the square in exponential. While we are only working up to order $\mathcal{O}(\dot{X}^2/c^2)$ it is sufficient to take $\rho = (1/2)\mathcal{G}^{-1}\mathcal{J}$. Now we have

$$
\begin{aligned}
&(\varphi+\rho).[\mathcal{G}+\mathcal{K}](\varphi+\rho) = &\text{(E.39)}\\
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\varphi.[\mathcal{G}+\mathcal{K}]\rho+\rho.[\mathcal{G}+\mathcal{K}]\varphi+\rho.[\mathcal{G}+\mathcal{K}]\rho = &\text{(E.40)}\\
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\varphi.\mathcal{G}\rho+\rho.\mathcal{G}\varphi+\mathcal{O}(\dot{X}^3) =\\
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\frac{1}{2}\varphi.\mathcal{G}\mathcal{G}^{-1}\mathcal{J}+\frac{1}{2}\mathcal{G}^{-1}\mathcal{J}.\mathcal{G}\varphi+\mathcal{O}(\dot{X}^3) =\\
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\frac{1}{2}\varphi.\mathcal{J}+\frac{1}{2}\mathcal{G}\mathcal{G}^{-1}\mathcal{J}.\varphi+\mathcal{O}(\dot{X}^3) =\\
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\mathcal{J}.\varphi+\mathcal{O}(\dot{X}^3).
\end{aligned}
$$

With $\mathcal{G}$ and $\mathcal{K}$ hermitian. In the second to last step we used the fact that $\mathcal{G}$ is hermitian. *alternative method:*

$$
\begin{aligned}
&\varphi.[\mathcal{G}+\mathcal{K}]\varphi+\varphi.\mathcal{J} =\\
&[(\varphi+\rho)-\rho].[\mathcal{G}+\mathcal{K}][(\varphi+\rho)-\rho]+\mathcal{J}[(\varphi+\rho)-\rho] =\\
&(\varphi+\rho).[\mathcal{G}+\mathcal{K}](\varphi+\rho)-2\rho.[\mathcal{G}+\mathcal{K}](\varphi+\rho)+\mathcal{J}(\varphi+\rho)+\mathcal{O}(\dot{X}^3) =\\
&(\varphi+\rho).[\mathcal{G}+\mathcal{K}](\varphi+\rho)-\mathcal{G}^{-1}\mathcal{J}.[\mathcal{G}+\mathcal{K}](\varphi+\rho)+\mathcal{J}(\varphi+\rho)+\mathcal{O}(\dot{X}^3) =\\
&(\varphi+\rho).[\mathcal{G}+\mathcal{K}](\varphi+\rho)+\mathcal{O}(\dot{X}^3).
\end{aligned}
$$

With ($a.b = \int \mathrm{d}x\mathrm{d}\tau\, a^*b$). Now take $\tilde{\varphi} = \varphi+\rho$. We are now able to rewrite $F[X]$ into the following form

$$
\begin{aligned}
F[X] = \int \mathcal{D}\tilde{\varphi}\ \det\left[\int \mathrm{d}x\{\phi_0''-\phi_0'(\tilde{\varphi}-\rho)\}\delta(\tau-\tau')\right] \qquad \text{(E.41)}\\
\times\,\delta\left(\int \mathrm{d}x\phi_0'(x-X)[\tilde{\varphi}-\rho](x,\tau)\right)e^{-N_A\tilde{\varphi}.[\mathcal{G}+\mathcal{K}]\tilde{\varphi}}.
\end{aligned}
$$

Now by rescaling $\hat{\varphi} = \sqrt{N_A}\tilde{\varphi}$ and using the identity $\det = \exp\operatorname{tr}\ln$, we are able to rewrite $\det\left[\frac{\delta Q}{\delta X}\right]$ as

$$
\begin{aligned}
&\det\left[\int \mathrm{d}x\{\phi_0''-\phi_0'(\tilde{\varphi}-\rho)\}\delta(\tau-\tau')\right] = \exp\left\{\operatorname{tr}\ln\left(\int \mathrm{d}x\{\phi_0''-\phi_0'(\tilde{\varphi}-\rho)\}\delta(\tau-\tau')\right)\right\} \qquad \text{(E.42)}\\
&= \exp\left\{\operatorname{tr}\ln\left(\int \mathrm{d}x\phi_0''-\phi_0'(\tilde{\varphi}-\rho)\right)\right\} \propto \exp\left\{\operatorname{tr}\ln\left(1-\frac{\delta/2}{\sqrt{N_A}}\int \mathrm{d}x\phi_0'\hat{\varphi}-(\delta/2)\int \mathrm{d}x\phi_0''\rho\right)\right\}
\end{aligned}
$$

Note in the above equation we used $\int \mathrm{d}x\phi_0^2 = 2/\delta$. For large N_A the second term becomes negligible and the last term will be proportional to $\dot{X}^2$ and thus gives rise to an effective mass correction. We thus get

$$
\begin{aligned}
F[X] &= e^{-\Delta M\int \mathrm{d}\tau \dot{X}^2}\int \mathcal{D}\hat{\varphi}\ \delta\left(\int \phi_0'\hat{\varphi}\right)e^{-\hat{\varphi}\cdot[\mathcal{G}+\mathcal{K}]\hat{\varphi}} \qquad \text{(E.43)}\\
&= e^{-\Delta M\int \mathrm{d}\tau \dot{X}^2}\frac{1}{\sqrt{\det'(\mathcal{G}+\mathcal{K})}},
\end{aligned}
$$

the prime on the determinant means the omission of the Goldstone zero mode.

Bibliography

[1] Isaac Newton. *Philosophiae naturalis principia mathematica.* Londini : jussu Societatis Regiae ac typis Josephi Streater.

[2] A. Einstein. Erklarung der Perihelionbewegung der Merkur aus der allgemeinen Relativitatstheorie. *Sitzungsber. preuss.Akad. Wiss., vol. 47, No.2, pp. 831-839, 1915*, 47:831–839, 1915.

[3] Hans-Benjamin Braun and Daniel Loss. Berry's phase and quantum dynamics of ferromagnetic solitons. *Phys. Rev. B*, 53:3237–3255, Feb 1996. doi: 10.1103/PhysRevB.53.3237. URL `http://link.aps.org/doi/10.1103/PhysRevB.53.3237`.

[4] T.H.R. Skyrme. A unified field theory of mesons and baryons. *Nuclear Physics*, 31:556 – 569, 1962. ISSN 0029-5582. doi: http://dx.doi.org/10.1016/0029-5582(62)90775-7. URL `http://www.sciencedirect.com/science/article/pii/0029558262907757`.

[5] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni. Skyrmion lattice in a chiral magnet. *Science*, 323(5916):915–919, 2009. ISSN 0036-8075. doi: 10.1126/science.1166767. URL `http://science.sciencemag.org/content/323/5916/915`.

[6] Stefan Heinze, Kirsten Von Bergmann, Matthias Menzel, Jens Brede, André Kubetzka, Roland Wiesendanger, Gustav Bihlmayer, and Stefan Blügel. Spontaneous atomic-scale magnetic skyrmion lattice in two dimensions. *Nature Physics*, 7(9):713–718, 2011.

[7] I. Raičević, Dragana Popović, C. Panagopoulos, L. Benfatto, M. B. Silva Neto, E. S. Choi, and T. Sasagawa. Skyrmions in a doped antiferromagnet. *Phys. Rev. Lett.*, 106:227206, Jun 2011. doi: 10.1103/PhysRevLett.106.227206. URL `https://link.aps.org/doi/10.1103/PhysRevLett.106.227206`.

[8] A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni. Topological hall effect in the a phase of mnsi. *Phys. Rev. Lett.*, 102:186602, May 2009. doi: 10.1103/PhysRevLett.102.186602. URL `https://link.aps.org/doi/10.1103/PhysRevLett.102.186602`.

[9] C. Psaroudaki, S. Hoffman, J. Klinovaja, and D. Loss. Quantum dynamics of skyrmions in chiral magnets. *ArXiv e-prints*, December 2016. URL `https://arxiv.org/abs/1612.01885`.

[10] Joseph Barker and Oleg A. Tretiakov. Static and dynamical properties of antiferromagnetic skyrmions in the presence of applied current and temperature. *Phys. Rev. Lett.*, 116:147203, Apr 2016. doi: 10.1103/PhysRevLett.116.147203. URL `https://link.aps.org/doi/10.1103/PhysRevLett.116.147203`.

[11] A. N. Bogdanov and U. K. Rößler. Chiral symmetry breaking in magnetic thin films and multilayers. *Phys. Rev. Lett.*, 87:037203, Jun 2001. doi: 10.1103/PhysRevLett.87.037203. URL `https://link.aps.org/doi/10.1103/PhysRevLett.87.037203`.

[12] Patrick van Dieten. Quantum fluctuations and degeneracies of two-dimensional magnetic skyrmions, June 2015. Utrecht University.

[13] Michael Stone. Magnus force on skyrmions in ferromagnets and quantum hall systems. *Phys. Rev. B*, 53:16573–16578, Jun 1996. doi: 10.1103/PhysRevB.53.16573. URL `https://link.aps.org/doi/10.1103/PhysRevB.53.16573`.

[14] Rembert A Duine. Lecture notes on spintronics. 2010.

[15] A. Auerbach. *Interacting Electrons and Quantum Magnetism*. Springer-Verlag New York. ISBN 9781461269281.

[16] Gong Chen. Spin-orbitronics: Skyrmion hall effect. *Nature Physics*, 13:112–113, January 2017. doi: 10.1038/nphys4030. URL `http://dx.doi.org/10.1038/nphys4030`.

[17] H Velkov, O Gomonay, M Beens, G Schwiete, A Brataas, J Sinova, and R A Duine. Phenomenology of current-induced skyrmion motion in antiferromagnets. *New Journal of Physics*, 18(7):075016, 2016. URL `http://stacks.iop.org/1367-2630/18/i=7/a=075016`.

[18] Maarten Beens. Current-driven skyrmion motion in antiferromagnets, June 2015. Utrecht University.

[19] Xichao Zhang, Yan Zhou, and Motohiko Ezawa. Antiferromagnetic skyrmion: stability, creation and manipulation. *Scientific reports*, 6, 2016.

[20] Henk TC Stoof, Koos B Gubbels, and Dennis BM Dickerscheid. *Ultracold quantum fields*, volume 1. Springer, 2009.

[21] Scott Bender, Hans Skarsvåg, Arne Brataas, and Rembert A Duine. Enhanced spin conductance of a thin-film insulating antiferromagnet. *arXiv preprint arXiv:1702.00975*, 2017.

[22] Ramamurti Shankar. *Principles of quantum mechanics*. Springer Science & Business Media.

[23] N Ogawa, S Seki, and Y Tokura. Ultrafast optical excitation of magnetic skyrmions. *Scientific reports*, 5:9552, 2015.

[24] Christoph Schütte and Markus Garst. Magnon-skyrmion scattering in chiral magnets. *Phys. Rev. B*, 90:094423, Sep 2014. doi: 10.1103/PhysRevB.90.094423. URL `https://link.aps.org/doi/10.1103/PhysRevB.90.094423`.

[25] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN 9780521795401.

[26] N. D. Mermin. The topological theory of defects in ordered media. *Rev. Mod. Phys.*, 51:591–648, Jul 1979. doi: 10.1103/RevModPhys.51.591. URL `https://link.aps.org/doi/10.1103/RevModPhys.51.591`.

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