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Persistence of globally attractive periodic solutions under discretization in Hammerstein equations

Bachelor Thesis

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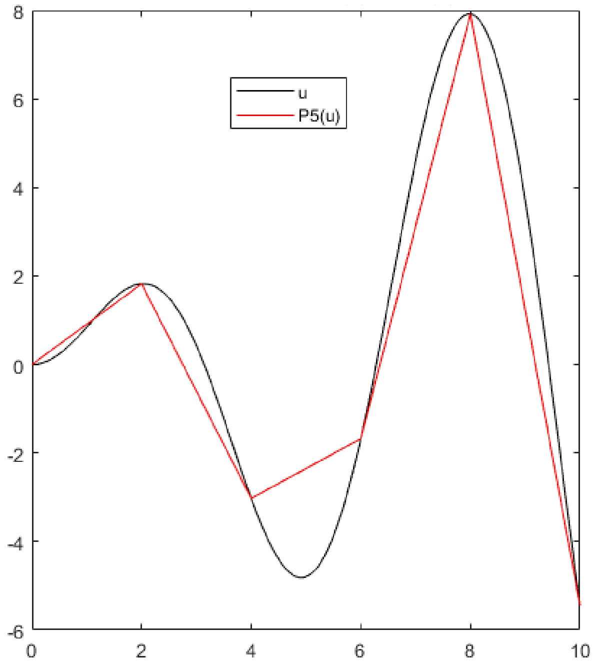
Wiskunde



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Abstract

Hammerstein equations are a type of integrodifference equations (IDEs), which are a type of discrete-time dynamical systems defined on a state space of functions. They have a wide variety of practical applications, such as modeling growth and dispersal of populations. For simulation purposes, appropriate discretization methods need to be applied on IDEs. However, it is still an open question to what extend the dynamics of a discretized IDE resemble the dynamics of the original system. Recently, the first results adressing this question have been published. This thesis elaborates on the discretization methods that can be applied to IDEs, as well as on results of the recent publication, with an emphasis on Hammerstein IDEs.

1 Introduction

Hammerstein integrodifference equations (IDEs) are a class of dynamical systems that are discrete in time and continuous in space. Before giving a precise definition, we consider an application to motivate the theory.

Muskitos are a common cause of the spread of diseases [3]. A common technique that is used to reduce their population size is to release sterilized mosquitoes into the population. Mating between sterilized and wildtype mosquitoes reduces the reproductive potential of the population [8]. To investigate what is the best strategy in mosquitoe release, a mathematical model was proposed that simulates the growth and dispersal of mosquitoes [8].

Example 1.1 For $t \in \mathbb{Z}$ denote with W_t the population size at time step t. We let t be an integer to model with discrete time, and t can be negative to be able to consider values of W_t backward in time. Each time-step represents one generation. Without influence of sterile mosquitos, there is a population growth rate A (which includes offspring per individual and death rate implicitly). Furthermore K is a constant related to the carrying capacity, which is "the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, and other necessities available in the environment" [17]. The model without a spatial effect and without influence of sterile mosquitos is given by

$$W_{t+1} = AW_t e^{-KW_t}.$$

We can imply a spatial effect with a so called dispersal kernal $k : [a,b]^2 \to [0,1]$, where $[a,b]$ is a closed interval and represents a habitat space (the closedness is a biological assumption, which means the habitat of the insects has certain borders they cannot cross. This is not always the case! [8]). The function k is a probability density function and $k(x,y)$ is the probability that an individual moves from location y to location x. The spatio-temporal model without influence of sterile mosquitos becomes:

$$W_{t+1}(x) = \int_a^b k(x,y) AW_t(y) e^{-KW_t(y)} dy \tag{1}$$

Finally, there is a mosquito release ratio $R_t(x)$ that depends on time and place. For more details on this ratio we refer to the article [8]. It can be incorporated into (1) in the following way:

$$W_{t+1}(x) = \int_a^b (1 + \frac{R_t(y)}{W_t(y)})^{-1} k(x,y) AW_t(y) e^{-KW_t(y)} dy \tag{2}$$

In [8] the authors investigated what the most cost-effective strategy is in terms of timing and placing of the muskito release. They did this by building models based on (2). A summary of their results is that an optimal strategy exists and can cause significant suppression (but not extinction) of the mosquito population. $\diamond$

Equation (2) is an example of a Hammerstein IDE. A general Hammerstein IDE is defined as follows. Let p, q be positive integers and $\mathcal{D}$ a non-empty compact subset of $\mathbb{R}^p$ without isolated points. We denote with $C(\mathcal{D}, \mathbb{R}^q)$ the space of continuous functions $u : \mathcal{D} \to \mathbb{R}^q$. For all $t \in \mathbb{Z}$, define the operator $\mathcal{F}_t : C(\mathcal{D}, \mathbb{R}^q) \to C(\mathcal{D}, \mathbb{R}^q)$ by

$$\mathcal{F}_t(u)(x) := \int_{\mathcal{D}} k_t(x,y) f_t(y, u(y)) \mathrm{d}y + h_t(x) \text{ for all } x \in \mathcal{D}, \tag{3}$$

where $k_t : \mathcal{D}^2 \to \mathbb{R}^{q \times p}$ is a continuous function called the *kernal*, $f_t : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^p$ is a continuous function and $h_t : \mathcal{D} \to \mathbb{R}^q$ is a forcing function. The integral is evaluated component-wise. When $h_t \equiv 0$ the equation is said to be homogeneous. Dynamics can be discribed with (3) by defining an initial function $u_0 \in C(\mathcal{D}, \mathbb{R}^q)$ and defining

$$u_{t+1} = \mathcal{F}_t(u_t) \quad \forall t \in \{0, 1, 2, ...\}$$

Remark 1.2. Equation (2) is a homogeneous Hammerstein IDE with $q = 1$, $\mathcal{D} = [a, b]$, $k_t(x, y) = k(x, y)$ and

$$f_t(x,y) = (1 + \frac{R_t(x)}{y})^{-1} A y e^{-Ky}.$$

$\diamond$

Although in applications the homogeneous form of (3) is often called an IDE, mathematically Hammerstein equations are a specific type of IDEs. In their most general form, IDEs involve nonlinearities

$$\mathcal{F}_t(u)(x) := G_t\left(x, \int_{\mathcal{D}} F_t(x, y, u(y)) \mathrm{d}y\right) \qquad \text{for all } t \in \mathbb{Z}, x \in \mathcal{D}. \tag{4}$$

where $G_t : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^q$ and $F_t : \mathcal{D}^2 \times \mathbb{R}^q \to \mathbb{R}^q$ are continuous functions for all $t \in \mathbb{Z}$. Note that (3) is a special case of (4) with

$$G_t(x, y) = y + h_t(x) \quad \text{and} \quad F_t(x, y, z) = k_t(x, y) f_t(y, z).$$

From now on we use IDE to denote a homogeneous Hammerstein IDE. Applications of IDEs such as the mosquito-model from Example 1.1 are common in the field of Theoretical Ecology. IDEs can model growth and dispersal of any population as long as it has non-overlapping generations. This means that growth (by reproduction) and dispersal must occur in separate time-phases. This happens in some insect species (like the mosquitoes), as well as in annual plant species. In such models $u_t(x)$ is the number of individuals on location x and time t. This is a real number (not necessarily an integer), so $q = 1$. The function $f_t : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ denotes a growth function. Furthermore, $k_t : \mathcal{D}^2 \to [0, 1]$ is a probability density function called the *dispersal kernel*, while $k_t(x, y)$ is the probability that an individual disperses from location y to location x.

To give some more examples: in [5] an IDE-model is used to compare different dispersal strategies of populations in an abstract sense. The authors compared two extreme dispersal strategies: to "go everywhere uniformly" or to "always stay in one place". They found that in habitats that vary greatly in time, the first extreme strategy was always optimal, while in time-invariant habitats the latter strategy was. Perhaps this is not a very suprising result, but this is one of the most simplistic examples of applications. More examples can be found in [7], [14], [10] and especially in [9]. A nice application of a non-Hammerstein IDE can be found in [11][p.415]. There an IDE is used to model water waves on liquids of infinite depth.

Before we can formulate interesting mathematical results on IDEs, we need to know more about discrete dynamical systems in general. A general discrete dynamical system is defined as follows. Let (X, d) be a metric space and $f : X \to X$ a map. Given any starting point $x_0 \in X$ we define

$$x_{t+1} = f(x_t) \quad \forall t \in \mathbb{N}, \tag{5}$$

creating a sequence as $t \to \infty$. A sequence $(x_t)_{t\in\mathbb{Z}}$ satisfying (5) for all $t \in \mathbb{Z}$ is called an *entire solution* of (5). Letting $Z := \{(t_1, t_2) \in \mathbb{N}_0^2 \,:\, t_1 \geq t_2\}$, we can also define the map $\varphi : Z \times X \to X$ by

$$\varphi(t, \tau, x_0) = \begin{cases} x_0 & \text{if } t = \tau \\ f^{t-\tau}(x_0) & \text{if } t > \tau \end{cases} \tag{6}$$

We call x_0 the *initial state* of the system and $\varphi(t, \tau, x_0)$ the *state at time t*. The map φ is called the *general solution map* of (5). Note that using a starting time is redundant if f is time-independent (in other words: we can take $\tau = 0$ in (6)), because $\varphi(t, \tau, x_0) = \varphi(t - \tau, 0, x_0)$ independently of τ. However, in an IDE the iteration map is dependent on time, hence we reformulate (5) as

$$x_{t+1} = f_t(x_t) \tag{7}$$

and (6) as

$$\varphi(t, \tau, x_0) = \begin{cases} x_0 & \text{if } t = \tau \\ f_{t-1} \circ \cdots \circ f_\tau(x_0) & \text{if } t > \tau \end{cases} \tag{8}$$

Modelars are interested in the asymptotic behaviour of dynamical system. By this we mean features that give information about the system as t goes to infinity. For example, (7) can have a *fixed point*, i.e. a point $x \in X$ such that $f_t(x) = x$. It is called stable if

$$\lim_{T\to\infty} d(x, f_t^T(x_0)) = 0 \quad \text{for all } x_0 \in X.$$

Stability is an asymptotic behavioural feature. Also (7) can have a *periodic solution*. This is an entire solution for which $x_{t+\theta} = x_t$ holds for all $t \in \mathbb{Z}$ for a fixed $\theta \in \mathbb{N}_0$. A periodic solution $(x_t)_{t\in\mathbb{Z}}$ is called *globally attractive* [13] if

$$\lim_{t\to\infty} d(\varphi\,(t; \tau, x)\,, x_t) = 0 \quad \forall \tau \in \mathbb{Z} \quad \forall x \in X.$$

We will later see that finding periodic solutions can be reduced to a fixed point problem.

How can we investigate the asymptotic behaviour of an IDE-model? When certain conditions are met, there are analytic methods. There are e.g. ways to determine steady states [9] or to set up sufficient conditions for a globally attractive periodic solution [5]. However, as models become more complex, numerical methods are needed. IDE-models are infinite dimensional systems, so in order to simulate them we need to reduce dimension. This is done by discretization of space. In the next section of this thesis, an introduction is given to general discretization methods and how to apply these on IDEs.

Having discretized a (general) IDE, it is an important question to what extend the dynamics of the discretized system reflects the dynamics of the original system. On finite time-intervals usefull error estimates can be given [12], but this tells us nothing about the asymptotic behaviour of the system (as $t \to \infty$). Does this persist under discretization? Recently, the first (to my knowledge) publication adressing this question has been made [13]. The author showed that under certain conditions a globally attractive periodic solution persists. In the third section of this thesis the proof of this result is given in detail. In fact, the goal is to present it in a more accessible way. In the last section a summary and some concluding remarks are given.

Some basic notations: From now on; $k, \kappa, n, m, s, t, \tau$ and θ denote integers. For topological spaces $(V, \mathcal{T}_1)$, $(W, \mathcal{T}_2)$ we denote with $C(V, W)$ the space of functions $f : V \to W$ that are continuous with respect to the topologies $\mathcal{T}_1, \mathcal{T}_2$.

With $\mathcal{D}$ we denote a non-empty compact subset of $\mathbb{R}^k$ ($k \geq 1$) without isolated points. We endow $C(\mathcal{D}, \mathbb{R}^\kappa)$ ($\kappa \geq 1$) with the supremum norm. This makes it a Banach space.

For Banach spaces E, F we denote with $L(E, F)$ the space of bounded linear operators from E to F and with $L(E)$ the space of bounded linear transformations on E. The standard norm on those spaces will be the operator norm. If $v_1, ..., v_n \in E$, we denote with $< v_1, ..., v_n >$ the linear subspace spanned by $v_1, ..., v_n$.

We denote with $\mathbb{R}_+$ the set of non-negative real numbers and define the following set of functions: $\mathfrak{N} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ : \lim_{x \to 0} \gamma(x) = 0\}$.

On a metric space (X, d), with $a \in X$, we define $B_\epsilon(a) := \{x \in X : d(x, a) < \epsilon\}$, the open ball around a with radius ϵ. We denote its closure with $\bar{B}_\epsilon(a)$.

2 Discretization methods

The basic idea of discretization is simple. If we have a dynamical system defined by an operator that acts on an infinite-dimensional space X, and we want to find a fixed point, or at least prove its existence, then we are dealing with an infinite set of equations. In many cases this is impossible to solve analytically. Therefore we construct an operator on a finite-dimensional space X_n that approximates the original operator. We then solve a finite set of equations to find a fixed point in X_n for all $n \in \mathbb{N}$. Hopefully, the resulting sequence of fixed points approximates a fixed point in X. Or to put it more generally: we hope the dynamics of our approximated system resembles the dynamics of the original system.

To be more precise, suppose $\kappa \geq 1$ and $X = C(\mathcal{D}, \mathbb{R}^\kappa)$, and our operator is an integral operator $\mathcal{F} : X \to X$. We want to to find $u \in X$ such that $\mathcal{F}(u) = u$. For every $x \in \mathcal{D}$, $u(x) = \mathcal{F}(u)(x)$ defines an integral equation. We can partitionate $\mathcal{D}$ into a finite amount of pieces and compute the 'average' $u(x)$ on each piece, resulting in finitely many integral equations left to solve. How to do this in detail and what 'averaging' precisely means, depends on the specific discretization method that is used. This chapter elaborates on the different methods that can be used. First, we explain a method in detail, namely *piecewise linear collocation.* Next, we generalise and see which other methods can be used as well. Ideas are taken from [13] for the first paragraph and from [1] for the rest of the section.

Piecewise linear collocation
Let $[a, b] \subset \mathbb{R}$ be a non-empty closed interval and $X = C([a, b], \mathbb{R})$. It is not difficult to extend the method we describe to $C(\mathcal{D}^*, \mathbb{R}^\kappa)$ with $\kappa \geq 1$ and $\mathcal{D}^*$ a k-dimensional rectangle. This is done in Appendix A, but for simplicity we restrict here to $\kappa = 1$ and $\mathcal{D}^* = [a, b]$. Let $n \geq 1$ and define

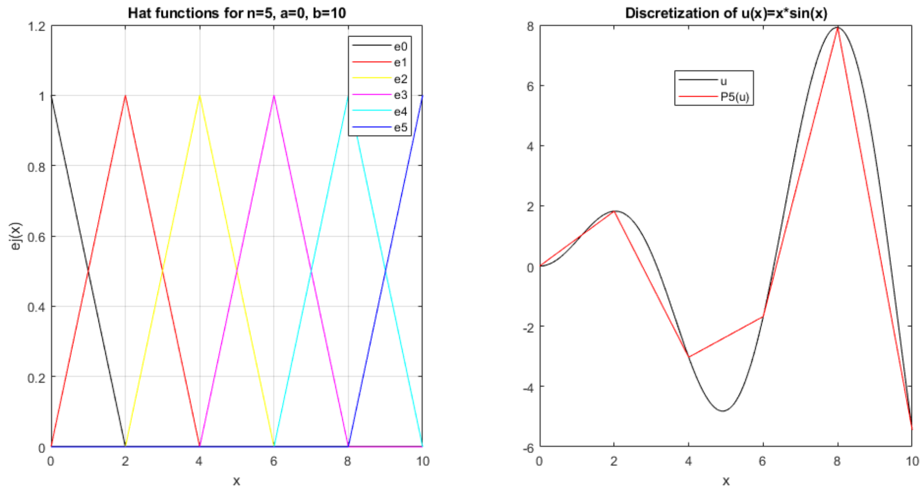

Figure 1: Example of a discretization based on the hat functions for $n = 5$ and $u : [0, 10] \to \mathbb{R}$ defined by $u(x) = x \sin(x)$.

for all $j \in \{0, 1, ..., n\}$:

$$\xi_j := a + j\frac{b-a}{n}.$$

Note that $\xi_0 = a$, $\xi_n = b$ and $\xi_i < \xi_j$ for $i < j$. So $\{\xi_j\}_{j=0}^n$ is a partition of $[a, b]$. The larger n is, the finer the partition is. Next, define the so-called hat functions $e_j : [a, b] \to [0, 1]$ by

$$e_j(x) := \max\{0, 1 - \frac{n}{b-a}|x - \xi_j|\}. \tag{9}$$

See the left part of Figure 1 for an illustration with $n = 5$. Finally, define the projections $P_n : X \to X$ by

$$P_n(u)(x) = \sum_{j=0}^{n} e_j(x)u(\xi_j) \quad \text{for all } u \in X \text{ and } x \in [a, b] \tag{10}$$

The function $P_n(u)$ is the discretization of u (note that this is a piecewise linear function with $P_n(u)(a) = u(a)$ and $P_n(u)(b) = u(b)$, wich explains the name of the method). Clearly, increasing n makes the discretization more accurate as it makes the partition finer. An example is illustrated in Figure 1. In case $\mathcal{F}$ is a Hammerstein operator given by (3) with $h_t \equiv 0$, we can semi-discretize the corresponding IDE by applying (10) to the kernel k. This yields the IDE

$$\begin{aligned} u_{t+1}(x) &= \int_a^b \sum_{j=0}^{n} e_j(x)k_t(\xi_j, y) \cdot f_t(y, u_t(y))\mathrm{d}y \\ &= \sum_{j=0}^{n} \int_a^b e_j(x)k_t(\xi_j, y) \cdot f_t(y, u_t(y))\mathrm{d}y \end{aligned} \tag{11}$$

Finding a fixed point of this system comes down to computing n equations, for which generic methods are available.

Projection methods
The method of piecewise linear collocation belongs to the class of *projection methods* [1]. These methods can be described as follows. Let X be a Banach space and $\{X_n : n \in \mathbb{N}\}$ a collection of finite-dimensional subspaces such that

$$X_n \subset X_{n+1} \text{ and } \bigcup_{n\in\mathbb{N}} X_n = X \tag{12}$$

For all $n \geq 1$ we choose a basis $\{v_0, ...v_n\}$ of X_n and a set of bounded linear functionals $\{\chi_0, ..., \chi_n\}$ that are linearly independent over X_n. Next we define the projections $P_n : X \to X_n$ by

$$P_n(u) = \sum_{j=0}^{n} \chi_j(u)v_j.$$

Note that P_n is bounded for all $n \in \mathbb{N}$ since

$$||P_n|| = || \sup_{||u||=1} \sum_{j=0}^{n} \chi_j(u)v_j|| \leq \sum_{j=0}^{n} ||\chi_j|| \cdot ||v_j||$$

and all χ_j are bounded.

Remark 2.1. Piecewise linear collocation is a projection method with $v_j := e_j$ defined by (9) and $\chi_j(u) := u(\xi_j)$. The e_j form a basis of the space of piecewise linear functions, a subspace of $C([a, b], \mathbb{R})$. If we define X_n as the space spanned by $\{e_0, ..., e_n\}$, then (12) is satisfied. For any $j \in \{0, ..., n\}$ χ_j is clearly linear. Furthermore

$$||\chi_j|| = \sup_{||u||=1} ||u(\xi_j)|| \leq 1$$

and

$$\det[\chi_i(e_j)] = \det[e_j(\xi_i)] = \det I = 1 \neq 0.$$

The latter implies $\{\chi_0, ..., \chi_n\}$ is linearly independent. $\diamond$

Other choices of bases and functionals give other discretization methods. Another example is given by defining P_n to be the truncated Fourier series on the space of 2π-periodic continuous functions. In any case, for an operator $\mathcal{F}_t : X \to X$ we can discretize the system defined by

$$u_{t+1} = \mathcal{F}_t(u_t) \tag{13}$$

to:

$$u_{t+1} = P_n\mathcal{F}_t(u_t) \tag{14}$$

as done with piecewise linear collocation.

As explained earlier we are interested in what features of the dynamics of a dynamical system persist under discretization. Let us make precise what we mean by this.

Definition 2.2. Let Π be a property of (13). We say Π *persists* under the discretization defined by (14) if there exists $N \in \mathbb{N}$ such that (14) forfills Π for every $n \geq N$. $\diamond$

Examples of properties Π are (stability of) periodic solutions or fixed points, or bifurcations. So if (13) has a θ-periodic solution, we say it persists under the discretization defined by (14) if there exists $N \in \mathbb{N}$ such that for every $n \geq N$ the system (14) has a θ-periodic solution as well. In some literature this notion of persistence is called preservation. To quantify the accuracy of a discretization, we introduce the following two notions:

Definition 2.3. For all $n \in \mathbb{N}$ and $t \in \mathbb{Z}$ we define the *local discretization error* [13] by

$$\varepsilon_t^n(u) := \mathcal{F}_t(u) - P_n\mathcal{F}_t(u).$$

We call a discretization method *bounded convergent* [13] if for any bounded set $B \subset X$ we have

$$\lim_{n\to\infty} \sup_{u\in B} ||\varepsilon_t^n(u)|| = 0 \text{ for all } t \in \mathbb{Z}$$

◇

Remark 2.4. Pointwise convergence of a discretization (so $P_n u \to u$ for $n \to \infty$) is sufficient, but not necessary for bounded convergence. As a counterexample, consider the truncated Fourier series mentioned earlier. This defines a discretization that is not pointwise convergent. However, if for bounded $B \subset X$ the set $\mathcal{F}(B)$ consists of functions for which the Fourier series is uniformly convergent (which is quite common), then the resulting discretization is bounded convergent. ◇

In the next section we will need bounded convergence for persistence of a globally attractive periodic solution. Intuitively it should be clear this is a reasonable assumption. On the other hand, discretizations that are not bounded convergent may give better estimates on finite time-intervals. Both have benefits and the best choice depends on the application.

Now that we have a more concrete idea of what discritization can mean, we pass on to the persistence result.

3 Persistence of a globally attractive periodic solution

In this section we will formulate sufficient conditions for persistence of a globally attractive periodic solution in the discretization of a general IDE. We collect the assumptions in a theorem and prove persistence. Next we analyse the assumptions in more detail. Specifically we will look at what conditions are needed for Hammerstein IDEs in order to apply the theorem. First we introduce the setting and some terminology.

For all $t \in \mathbb{Z}$ let U_t be an open convex subset of $C(\mathcal{D}, \mathbb{R}^\kappa)$. In the subsequent theorem we will look at general IDEs defined by

$$u_{t+1} = \mathcal{F}_t(u_t) \tag{15}$$

where $\mathcal{F}_t : U_t \to C(\mathcal{D}, \mathbb{R}^\kappa)$ is defined by

$$\mathcal{F}_t(u)(x) := G_t\left(x, \int_{\mathcal{D}} F_t(x, y, u(y))\mathrm{d}y\right) \qquad \text{for all } t \in \mathbb{Z}, x \in \mathcal{D}, \tag{16}$$

where $G_t : \mathcal{D} \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ and $F_t : \mathcal{D}^2 \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ are continuous functions for all $t \in \mathbb{Z}$. We assume that $\mathcal{F}_t(u) \in U_{t+1}$ for all $u \in U_t$ so that (15) is well-defined regardless of the initial state u_0. Furthermore we assume that there exist $\theta \geq 1$ such that $G_{t+\theta} = G_t$ and $F_{t+\theta} = F_t$ for all $t \in \mathbb{Z}$. This implies $\mathcal{F}_t$ is θ-periodic.

For a short elaboration on differentiation in Banach spaces, including the definition of Fréchet differentiability, see Appendix A. If E, F are Banach spaces, $U \subset E$ is open and an operator $L : U \to F$ is (Frèchet) differentiable on U, we denote with $DL(x)$ the derivative in the point $x \in U$. If the map $DL : U \to L(U, F)$ is continuous, then L is called C^1. The operator L is called *compact* if it maps bounded sets into relatively compact sets.

The following theorem is a reformulated version of Theorem 2.1 from [13].

Theorem 3.1. Consider an IDE of the form (15), with discretization $u_{t+1} = \mathcal{F}_t^n(u_t)$ for all $n \in \mathbb{N}$. Let φ_n denote their general solutions. Suppose the discretization is bounded convergent and θ-periodic, and the following assumptions are satisfied:

(i) There exists a θ-periodic solution u^* such that $\lim_{t\to\infty} \|\varphi_0(t;\tau,u_\tau) - u_t^*\| = 0$ (i.e. u^* is globally attractive).

(ii) The general solution of the IDE satisfies the following properties:

  (I) $\varphi_0(t;\tau,\cdot) : U_\tau \to U_{\tau+t}$ is compact for all $\tau < t$

  (II) $\varphi_0(t;\tau,\cdot)$ is C^1 for all $\tau \le t$

(iii) There exist $a \in (0,1)$ such that $\sigma(D\mathcal{F}_\theta(u_\theta^*) \cdots D\mathcal{F}_1(u_1^*)) \subset B_a(0)$ (the spectrum of the product of derivatives is bounded by a).

and for all s with $1 \le s \le \theta$:

(iv) For all $n \in \mathbb{N}$ the function $\mathcal{F}_s^n : U_s \to C(\mathcal{D},\mathbb{R}^\kappa)$ is compact and C^1, and $D\mathcal{F}_s^n : U_s \to L(C(\mathcal{D},\mathbb{R}^\kappa))$ are bounded uniformly in n (i.e. there exists $M_s > 0$ such that $\|D\mathcal{F}_s^n\| \le M_s$ for all $n \in \mathbb{N}$).

(v) $\lim_{n\to\infty} \|D\varepsilon_s^n(u)\| = 0$ for all $u \in U_s$ (recall that $\varepsilon_s^n(u) := \mathcal{F}_s^n(u) - \mathcal{F}_s(u)$).

(vi) There is $b > 0$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{N}$ such that for all $n \in \mathbb{N}$

  (I) $\|\varepsilon_s^n(u_s^*)\| \le \alpha_1(\frac{1}{n})$

  (II) $\|D\varepsilon_s^n(u_s^*)\| \le \alpha_2(\frac{1}{n})$

  (III) $\|D\mathcal{F}_s^n(u) - D\mathcal{F}_s^n(u_s^*)\| \le \alpha_3(\|u - u_s^*\|)$ for all $u \in B_b(u_s^*)$.

(vii) For all $n \in \mathbb{N}_0$ there is a bounded set $B_n \subset U_s$ such that

  (I) $\bigcup_{n\in\mathbb{N}_0} B_n$ is bounded.

  (II) For all $u \in U_s$ there is $T \in \mathbb{N}$ such that $\varphi_n(s + T\theta, s, u) \in B_n$.

Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have:

(a) The discretized system $u_{t+1} = \mathcal{F}_t^n(u_t)$ possesses a θ-periodic solution u^n

(b) u^n is globally attractive

(c) There is $q \in (a,1)$ and $K > 0$ such that

$$\sup_{t\in\mathbb{Z}} \|u_t^n - u_t^*\| \le \frac{K}{1-q}\alpha_1(\frac{1}{n}) \tag{17}$$

We prepare the proof by formulating two lemmas. The first lemma gives an estimate for the norm of a bounded operator in a Banach space, based on its spectral radius. It is a fact from Functional Analysis that the spectral radius is bounded by the operators norm [4][Th.6.13]. For self-adjoint operators, we even have equality [4][Th.8.8]. But for general bounded operators, this is not always the case. However, we can pass to an equivalent norm in the Banach space to find an estimate for the norm of the operator. More precisely:

Lemma 3.2. Let E be a Banach space with norm $\|\cdot\|$ and $L \in L(E)$ with spectral radius ρ. Then for all $\epsilon > 0$ there exists an equivalent norm $|||\cdot|||$ on E such that

$$|||Lx||| \le (\rho + \epsilon) \cdot |||x||| \tag{18}$$

for all $x \in E$. In particular, based on this norm we have $||L|| \leq \rho + \epsilon$.

Remark. A generalisation of this result can be found in [6][technical lemma, p.6]. The idea of the subsequent proof is based on the proof of that technical lemma.

Proof. Let $\epsilon > 0$ and denote $q := \rho + \epsilon$. Define $|||\cdot|||$ by

$$|||x||| = \sup_{n\geq 0} \frac{||L^n x||}{q^n} \text{ for all } x \in E.$$

Then we have

$$|||Lx||| = \sup_{n\geq 0} \frac{||L^{n+1}x||}{q^n} = q \cdot \sup_{n\geq 0} \frac{||L^{n+1}x||}{q^{n+1}} \leq q \cdot \sup_{n\geq 0} \frac{||L^n x||}{q^n} = q \cdot |||x|||$$

so (18) is satisfied. Furthermore, note that for all $x \in E$ we have

$$||x|| \leq |||x||| \leq \left(\sup_{n\geq 0} \frac{||L^n||}{q^n}\right) ||x||$$

where the second inquality follows from the property that $||L^n x|| \leq ||L^n|| \cdot ||x||$ for all $n \in \mathbb{N}$. Since $\sup_{n\geq 0} ||L^n||^{1/n} = \rho$, we know that $\sup_{n\geq 0} \frac{||L^n||}{q^n}$ exists. Therefore, the norms are equivalent on E. □

The next lemma is a generalisation of the mean value theorem to Banach spaces. It is also known as the mean value inequality and it arises in many forms through literature. In the formulation given below differentiability is assumed, because it simplifies the proof and in Theorem 3.1 we assume it anyway.

Lemma 3.3. Let X and Y be Banach spaces and let $U \subset X$ be open and convex. Suppose the function $f : U \to Y$ is differentiable on U. For all $x_1, x_2 \in U$, write

$$l(x_1, x_2) := \{tx_1 + (1-t)x_2 : t \in [0,1]\}$$

for the line segment joining x_1 and x_2. By convexity this is a subset of U. Then for all $x_1, x_2 \in U$ we have

$$||f(x_1) - f(x_2)|| \leq ||x_1 - x_2|| \cdot \sup_{x \in l(x_1,x_2)} ||Df(x)||.$$

Proof. Let $x_1, x_2 \in U$. Define $\alpha : [0,1] \to Y$ by

$$\alpha(t) = f((1-t)x_1 + tx_2) \text{ for all } t \in [0,1].$$

Note that $\alpha(0) = f(x_1)$ and $\alpha(1) = f(x_2)$. Furthermore the chain rule gives

$$\alpha'(t) = Df((1-t)x_1 + tx_2)(x_2 - x_1) \text{ for all } t \in [0,1].$$

Define on the linear subspace $< \alpha(1) - \alpha(0) >$ the linear functional x^* by

$$x^*(x) = \lambda \cdot ||\alpha(1) - \alpha(0)|| \quad \text{for } x = \lambda(\alpha(1) - \alpha(0)).$$

Note that $||x^*|| = 1$. By the Hahn-Banach theorem [4][th.3.13] there exists $y^* \in Y^*$ such that $||y^*|| = 1$ and $y^*|_{<(\alpha(1)-\alpha(0))>} = x^*$. In particular we have

$$||\alpha(1) - \alpha(0)|| = x^*(\alpha(1) - \alpha(0)) = y^*(\alpha(1) - \alpha(0)) = y^*\alpha(1) - y^*\alpha(0) \tag{19}$$

By the one-dimensional mean-value theorem there exists $c \in [0, 1]$ such that

$$y^*\alpha(1) - y^*\alpha(0) = (y^*\alpha)'(c)(1 - 0) = (y^*\alpha)'(c) \tag{20}$$

and using the chain rule and linearity of y^* gives

$$\begin{aligned}(y^*\alpha)'(c) &= Dy^*(\alpha(c))\alpha'(c) \\ &= y^*(\alpha'(c)) \\ &= y^*[Df((1-c)x_1 + cx_2)(x_2 - x_1)] \\ &\leq ||y^*|| \cdot ||Df(1-c)x_1 + cx_2)|| \cdot ||x_2 - x_1|| \\ &\leq \sup_{x \in l(x_1,x_2)} ||Df(x)|| \cdot ||x_2 - x_1||.\end{aligned}$$

Together with (19) and (20) this proves the result. □

Finally before we start the proof of the theorem, we look at how we can reduce the existence of a periodic solution problem to a fixed point problem (as announced in the introduction).

Remark 3.4 (Relation between periodic solutions and fixed points). Periodic solutions of (15) with period $\theta \in \mathbb{N}$ can be found by finding fixed points of a certain map. Namely, if φ denotes the general solution of (15), then fixed points of

$$\varphi(\tau + \theta, \tau, \cdot) : X \to X$$

define starting points of periodic solutions. Indeed, if $\varphi(\tau + \theta, \tau, x_v) = x_v$, then

$$x_{v+\theta} = f_{\tau+\theta-1}(x_{v+\theta-1}) = ... = f_{\tau+\theta-1} \circ \cdots \circ f_\tau(x_v) = \varphi(\tau + \theta, \tau, x_v) = x_v$$

hence we get a periodic solution with x_v as starting point. ◊

Proof of Theorem 3.1. Consider the parameter set $P = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. For any $p \in P$ we define the maps $c_p : U_\tau \to U_{\tau+\theta} = U_\tau$ by

$$c_p(u) := \begin{cases} \varphi_0(\tau + \theta; \tau, u), & p = 0 \\ \varphi_n(\tau + \theta; \tau, u), & p = \frac{1}{n} \end{cases}$$

for all $u \in U_\tau$ and some fixed $\tau \in \mathbb{Z}$. To prove part (a) (existence of a θ-periodic solution in the discretized system) we will show that any c_p defines a contraction on some neighbourhood of u^*. To do this, we derive some basic properties and several estimates.

Claim 1. The following properties hold:

(a') The maps c_p are C^1.

(b') u^*_τ is a globally attractive fixed point of c_0.

(c') There is an equivalent norm on $C(\mathcal{D}, \mathbb{R}^\kappa)$ and a $q \in (a, 1)$ such that $||Dc_0(u^*_\tau)|| \leq q$.

Proof of claim 1.

(a') Note that (ii)(II) implies that c_0 is C^1. Moreover, for $p \neq 0$ the maps c_p are compositions of the maps $\mathcal{F}^n_\tau$, ..., $\mathcal{F}^n_{\tau+\theta-1}$. By (iv) these are C^1, so the c_p are C^1 as well.

(b') It follows from the definition of c_0 and the θ-periodicity of u^* that u^*_τ is a fixed point of c_0. With (i) we deduce for all $u \in U_\tau$:

$$\lim_{T\to\infty} ||c_0^T(u) - u^*_\tau|| = \lim_{T\to\infty} ||\varphi_0(\tau + T\theta; \tau, u) - u^*_{\tau+T\theta}|| \overset{(i)}{=} 0.$$

(c') By applying the chain rule $(t - \tau - 1)$ times we see that

$$\begin{aligned}\frac{\partial}{\partial u}\varphi_0(t,\tau,u) =& D(\mathcal{F}_{t-1}\circ\cdots\circ\mathcal{F}_\tau)(u)\\ =& D\mathcal{F}_{t-1}((\mathcal{F}_{t-2}\circ\cdots\circ\mathcal{F}_\tau)(u))\cdot D\mathcal{F}_{t-2}((\mathcal{F}_{t-3}\circ\cdots\circ\mathcal{F}_\tau)(u))\\ &\cdots D\mathcal{F}_{\tau+1}(\mathcal{F}_\tau(u))D\mathcal{F}_\tau(u)\\ =& D\mathcal{F}_{t-1}(\varphi_0(t-1,\tau,u))\cdots D\mathcal{F}_\tau(\varphi_0(\tau,\tau,u)).\end{aligned}$$

If we apply this to $c_0 = \varphi_0(\tau + \theta, \tau, \cdot)$ in the point u^*_τ, we get

$$\begin{aligned}Dc_0(u^*_\tau) &= D_3\varphi_0(\tau+\theta,\tau,u^*_\tau)\\ &= D\mathcal{F}_{\tau+\theta-1}(\varphi_0(\tau+\theta-1,\tau,u^*_\tau))\cdots D\mathcal{F}_\tau(\varphi_0(\tau,\tau,u^*_\tau))\\ &= D\mathcal{F}_{\tau+\theta-1}(u^*_{\tau+\theta-1})\cdots D\mathcal{F}_\tau(u^*_\tau)\end{aligned}$$

From (iii) it follows that $\sigma(Dc_0(u^*_\tau)) \subset B_a(0)$. By Lemma 3.2 there exists a $q \in (a, 1)$ such that $||Dc_0(u^*_\tau)|| \le q$. □

Claim 2. There exist $\Gamma \in \mathfrak{N}$ and $\gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ with $\lim_{||x||\to 0}\gamma(x) = 0$ such that for all $u \in B_b(u^*_\tau) \cap U_\tau$ we have:

(d') $||c_p(u^*_\tau) - c_0(u^*_\tau)|| \le \Gamma(p)$

(e') $||Dc_p(u) - Dc_0(u^*_\tau)|| \le \gamma(||u - u^*_\tau||, p)$

Proof of claim 2. For all $t \in \mathbb{Z}$ we have the following. By assumption (iv) and θ-periodicity of $\mathcal{F}^n_t$ we have $||D\mathcal{F}^n_t|| \le M_t$ for some $M_t > 0$ independent of n. This means that for any bounded set $B \subset C(\mathcal{D}, \mathbb{R}^\kappa)$ there is $L(B)_t > 0$ (independent of n) such that $||D\mathcal{F}^n_t(u)|| \le L(B)_t$ for all $u \in B \cap U_t$. If B is also convex, then $B \cap U_t$ is convex, hence lemma 3.3 implies

$$||\mathcal{F}^n_t(u_1) - \mathcal{F}^n_t(u_2)|| \le ||u_1 - u_2||L(B)_t \quad \text{for all } u_1, u_2 \in B \cap U_t. \tag{21}$$

(d') If $p = 0$ the claim is trivial. Assume $p \neq 0$. Note that

$$||c_p(u^*_\tau) - c_0(u^*_\tau)|| = ||\varphi_n(\tau+\theta,\tau,u^*_\tau) - \varphi_0(\tau+\theta,\tau,u^*_\tau)|| \quad (p = \frac{1}{n}). \tag{22}$$

If $\theta = 1$, then (22) equals $||\mathcal{F}^n_\tau(u^*_\tau) - \mathcal{F}_\tau(u^*_\tau)|| = ||\varepsilon^n_\tau(u^*_\tau)||$ and the result folows from (vi)(I) with $\Gamma = \alpha_1$. Assume $\theta > 1$ and define a radius $\rho(\theta) > 0$ such that $\varphi_n(\tau + k, \tau, u^*_\tau) \in B_{\rho(\theta)}(u^*_\tau)$ for all $k \in \{1, ..., \theta\}$. We abbreviate $L(\theta)_t := L(B_{\rho(\theta)}(u^*_\tau))_t$. Define

$$\Gamma_k(x) := \alpha_1(x)\left(1 + \sum_{s=\tau+1}^{\tau+k-1}\prod_{r=s}^{\tau+k-1} L(\theta)_r\right).$$

Since $\alpha_1 \in \mathfrak{N}$ also $\Gamma_t \in \mathfrak{N}$ holds for all $\theta \ge 2$. We will show by mathematical induction on k that

$$||\varphi_n(\tau+k,\tau,u^*_\tau) - \varphi_0(\tau+k,\tau,u^*_\tau)|| \le \Gamma_k(\frac{1}{n}). \tag{23}$$

for all $k \in \{2, ..., \theta\}$. First we look at the case $k = 2$. Using the triangle inequality, (vi)(I) and (21) we deduce

$$\begin{aligned}
||\varphi_n(\tau+2,\tau,u_\tau^*) - &\varphi_0(\tau+2,\tau,u_\tau^*)|| = \\
&= ||\mathcal{F}_{\tau+1}^n(\mathcal{F}_\tau^n(u_\tau^*)) - \mathcal{F}_{\tau+1}(u_{\tau+1}^*)|| \\
&\leq ||\mathcal{F}_{\tau+1}^n(\mathcal{F}_\tau^n(u_\tau^*)) - \mathcal{F}_{\tau+1}^n(u_{\tau+1}^*)|| + ||\mathcal{F}_{\tau+1}^n(u_{\tau+1}^*) - \mathcal{F}_{\tau+1}(u_{\tau+1}^*)|| \\
&\leq L(\theta)_{\tau+1}||\mathcal{F}_\tau^n(u_\tau^*) - u_{\tau+1}^*|| + \alpha_1(\frac{1}{n}) \\
&\leq L(\theta)_{\tau+1}\alpha_1(\frac{1}{n}) + \alpha_1(\frac{1}{n}) \\
&= \alpha_1(\frac{1}{n})(1 + L(\theta)_{\tau+1}) = \Gamma_2(\frac{1}{n}).
\end{aligned}$$

This gives the induction basis. Now assume there exists $k \in \{2, ..., \theta - 1\}$ such that

$$||\varphi_n(\tau+k,\tau,u_\tau^*) - \varphi_0(\tau+k,\tau,u_\tau^*)|| \leq \Gamma_k(\frac{1}{n}). \tag{24}$$

Then we deduce with a similar argumentation:

$$\begin{aligned}
||\varphi_n(\tau+k+1,\tau,u_\tau^*) &- \varphi_0(\tau+k+1,\tau,u_\tau^*)|| = \\
&= ||\mathcal{F}_{\tau+k}^n(\varphi_n(\tau+k,\tau,u_\tau^*)) - \mathcal{F}_{\tau+k}(u_{\tau+k}^*)|| \\
&\leq ||\mathcal{F}_{\tau+k}^n(\varphi_n(\tau+k,\tau,u_\tau^*)) - \mathcal{F}_{\tau+k}^n(u_{\tau+k}^*)|| + ||\mathcal{F}_{\tau+k}^n(u_{\tau+k}^*) - \mathcal{F}_{\tau+k}(u_{\tau+k}^*)|| \\
&\leq L(\theta)_{\tau+k}||\varphi_n(\tau+k,\tau,u_\tau^*) - \varphi_n(\tau+k,\tau,u_\tau^*)|| + \alpha_1(\frac{1}{n}) \\
&\overset{(24)}{\leq} L(\theta)_{\tau+k}\Gamma_k(\frac{1}{n}) + \alpha_1(\frac{1}{n}) \\
&= \Gamma_{k+1}(\frac{1}{n}).
\end{aligned}$$

By mathematical induction (23) holds for all $k \in \{2, ..., \theta\}$, so in particular for $k = \theta$. Choosing $\Gamma = \Gamma_\theta$ completes the proof of (d').

(e') If $p = 0$ the result follows from the fact that c_0 is C^1 (the reader is invited to proof this). Assume $p \neq 0$. We need to show there exists $\gamma : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $\lim_{||x|| \to 0} \gamma(x) = 0$ such that for all $u \in B_b(u_\tau^*) \cap U_\tau$ we have

$$||Dc_p(u) - Dc_0(u_\tau^*)|| = ||D_3\varphi_n(\tau+\theta,\tau,u) - D_3\varphi_0(\tau+\theta,\tau,u_\tau^*)|| \leq \gamma(||u - u_\tau^*||, \frac{1}{n}) \quad (p = \frac{1}{n}).$$

We give a proof by induction on θ. Let $u \in B_b(u_\tau^*) \cap U_\tau$. From (vi)(II) and (vi)(III) we deduce

$$\begin{aligned}
||D_3\varphi_n(\tau+1,\tau,u) - &D_3\varphi_0(\tau+1,\tau,u_\tau^*)|| = \\
&= ||D\mathcal{F}_\tau^n(u) - D\mathcal{F}_\tau(u_\tau^*)|| \\
&\leq ||D\mathcal{F}_\tau^n(u) - D\mathcal{F}_\tau^n(u_\tau^*)|| + ||D\mathcal{F}_\tau^n(u_\tau^*) - D\mathcal{F}_\tau(u_\tau^*)|| \\
&\leq \alpha_3\left(\|u - u_s^*\|\right) + \alpha_2(1/n)
\end{aligned}$$

Put $\gamma(x, y) := \alpha_2(x) + \alpha_3(y)$. Since $\alpha_2, \alpha_3 \in \mathfrak{N}$, this proves the induction basis. The proof of the induction step is comparable to the proof of the induction step in (d'), but rather tedious. Therefore this is given in Appendix B (Proposition B.1). □

Now we are ready to prove that for sufficiently small p (hence for sufficiently large n) c_p defines a contraction map on a neighbourhood of u^*_τ. Choose $\epsilon \in (0, b)$ and $\delta > 0$ such that

$$\Gamma(\delta') \leq \tfrac{1-q}{2}\epsilon \quad \text{and} \quad \gamma(\epsilon', \delta') \leq \tfrac{1-q}{2} \quad \text{for all } \delta' \leq \delta, \epsilon' \leq \epsilon. \tag{25}$$

This is possible because $\lim_{x\to 0} \Gamma(x) = 0$ and $\lim_{||x||\to 0} \gamma(x) = 0$. Let us have a look at the behaviour of c_p on the neighbourhoods $\bar{B}_\epsilon(u^*_\tau)$ for $p \in [0, \delta)$. Suppose $u \in \bar{B}_\epsilon(u^*_\tau)$ and $p \in [0, \delta)$. Then the triangle inequality, (c'), (e') and (25) imply

$$\begin{aligned} ||Dc_p(u)|| &\leq ||Dc_0(u^*_\tau)|| + ||Dc_p(u) - Dc_0(u^*_\tau)|| \\ &\leq q + \gamma(||u - u^*_\tau||, p) \\ &\leq q + \frac{1-q}{2} \\ &= \frac{1+q}{2}. \end{aligned} \tag{26}$$

By Lemma 3.3 and convexity of U_t we have for all $u_1, u_2 \in \bar{B}_\epsilon(u^*_\tau)$ and $p \in [0, \delta)$

$$||c_p(u_1) - c_p(u_2)|| \leq ||u_1 - u_2|| \cdot \sup_{u \in l(u_1,u_2)} ||Dc_p(u)|| \overset{(26)}{\leq} \frac{1+q}{2} \cdot ||u_1 - u_2||. \tag{27}$$

Since $q < 1$ we have $\frac{1+q}{2} < 1$, so c_p satisfies the contraction property. It remains to show that $c_p : \bar{B}_\epsilon(u^*_\tau) \to \bar{B}_\epsilon(u^*_\tau)$ is well-defined. Suppose $u \in \bar{B}_\epsilon(u^*_\tau)$ and $p \in [0, \delta)$. Then (27), (b'), (d') and the triangle inequality imply

$$\begin{aligned} d(c_p(u), u^*_\tau) &= ||c_p(u) - u^*_\tau|| \\ &\leq ||c_p(u) - c_p(u^*_\tau)|| + ||c_p(u^*_\tau) - c_0(u^*_\tau)|| \\ &\leq \frac{1+q}{2}||u - u^*_\tau|| + \Gamma(p) \\ &\overset{(26)}{\leq} \frac{1+q}{2}\epsilon + \frac{1-q}{2}\epsilon = \epsilon. \end{aligned}$$

so $c_p(u) \in \bar{B}_\epsilon(u^*_\tau)$ as we wish. We conclude that $c_p : \bar{B}_\epsilon(u^*_\tau) \to \bar{B}_\epsilon(u^*_\tau)$ is a contraction for all $p \in [0, \delta)$. By the uniform contraction mapping principle [16] there exists a continuous function $c^* : [0, \delta) \to \bar{B}_\epsilon(u^*_\tau)$ such that $c^*(0) = u^*_\tau$ and $c_p(c^*(p)) = c^*(p)$ for all $p \in [0, \delta)$ (so c_p has a unique fixed point for any $p \in [0, \delta)$). Furthermore we have for any $p \in [0, \delta)$ and for any $u \in \bar{B}_\epsilon(u^*_\tau)$

$$\lim_{T\to\infty} ||c^T_p(u) - c^*(p)|| = 0 \quad \text{(local attractivity)} \tag{28}$$

By Remark 3.4 the fixed points correspond to a θ-periodic solution in the system defined by $\mathcal{F}^{1/p}_t$. Since $p \in [0, \delta)$ gives $\frac{1}{p} > \frac{1}{\delta}$, we deduce that for all $n \geq N_1 := \lceil \frac{1}{\delta} \rceil$ the system defined by the discretization $\mathcal{F}^n_t$ has a locally attractive θ-periodic solution u^n. To be precise, it is defined by

$$u^n_t := \varphi_n(t, \tau, c^*(\frac{1}{n})). \tag{29}$$

This proves (a).

So far we have proven that the discretization defined by $\mathcal{F}^n_t$ has a locally attractive periodic solution. To prove global attractivity, i.e. (28) for all $u \in U_\tau$, it suffices to show that any $u \in U_\tau$

eventually gets mapped into the set $\bar{B}_\epsilon(u_\tau^*)$, on which we have local attractivity. That is, for every $u \in U_\tau$ there exists $k \in \mathbb{N}$ such that

$$c_p^k(u) \in \bar{B}_\epsilon\left(u_\tau^*\right). \tag{30}$$

As a consequence of assumption (vii), we only need to show (30) for any $u \in \bigcup_{n\in\mathbb{N}_0} B_n$. For that we need the following claim:

Claim 3. The set $C := \overline{\bigcup_{p\in P} c_p(B_p)}$ is compact, where $B_p = \begin{cases} B_{1/p} & p \neq 0 \\ B_0 & p = 0 \end{cases}$, where B_n with n an integer is defined as in (vii).

Assume this claim for now. The idea of the following proof of global attractivity is taken from [15]. We will prove by contradiction that

$$\exists \delta_0 > 0 : \forall p \in [0, \delta_0) : \forall u \in C : \exists m \in \mathbb{N}_0 : c_p^m(u) \in \bar{B}_\epsilon(u_\tau^*). \tag{31}$$

Assume (31) is not true. Then

$$\forall \delta_0 > 0 : \exists p \in [0, \delta_0), u \in C : \forall m \in \mathbb{N}_0 : ||c_p^m(u) - u_\tau^*|| \geq \epsilon$$

By choosing $\delta_{0,n} = \frac{1}{n}$ for all $n \in \mathbb{N}$ we obtain a sequence $(p_n)_{n\in\mathbb{N}} \subset [0, \delta)$ that converges to 0 and a sequence $(u_n)_{n\in\mathbb{N}} \subset C$ such that for all $n \in \mathbb{N}$ and for all $m \geq 0$ we have

$$||c_{p_n}^m(u_n) - u_\tau^*|| \geq \epsilon \tag{32}$$

Because C is compact, we can take subsequences $(p_{n_j})_{j\in\mathbb{N}}$ of $(p_n)_{n\in\mathbb{N}}$ and $(u_{n_j})_{j\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ that converge to 0 and $u_l \in C$ respectively. However, by global attractivity of u_τ^* w.r.t. c_0, there exists a $k \in \mathbb{N}$ such that

$$||c_0^k(u_l) - u_\tau^*|| < \frac{\epsilon}{3}. \tag{33}$$

By (a') the map $c_{p_{n_j}}^k$ is a composition of C^1-maps, hence C^1 and in particular continuous. Furthermore, bounded convergence of the discretization implies that $c_{p_{n_j}}^k$ converges pointwise to c_0^k. Together with (33) this implies there exists a $K \in \mathbb{N}$ such that for all $j \geq K$ we have

$$||c_{p_{n_j}}^k(u_{n_j}) - u_\tau^*|| \leq ||c_{p_{n_j}}^k(u_{n_j}) - c_{p_{n_j}}^k(u_l)|| + ||c_{p_{n_j}}^k(u_l) - c_0^k(u_l)|| + ||c_0^k(u_l) - u_\tau^*|| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This is in contradiction with (32), so we conclude that (31) holds. Combining this with assumption (vii)(II) we get for all $p \in [0, \delta_0)$ and for all $u \in U_\tau$ there exist $T, m \in \mathbb{N}_0$ such that

$$c_p^{m+T+1}(u) \in \bar{B}_\epsilon\left(u_\tau^*\right).$$

Combining this with the local attractivity (28) yields global attractivity for $p \in [0, \min\{\delta_0, \delta\})$. It remains to prove claim 3.

Proof of claim 3: by compactness of $\mathcal{D}$ and the theorem of Arzelà-Ascoli [4][p.62], it suffices to show that C is closed, bounded and uniformly equicontinuous. Closedness is clear. Note that (ii)(I) and (iv) imply that c_p is compact for all $p \in P$. Define $B := \bigcup_{n\in\mathbb{N}_0} B_n = \bigcup_{p\in P} B_p$, which is bounded due to (vii)(I).

- **Boundedness.** Compactness of c_0 gives that $c_0(B)$ is relatively compact. By Arzelà-Ascoli [4][p.62], we have in particular that $c_0(B)$ is bounded, so there is $r_1 > 0$ such that

$$||c_0(u)|| \leq r_1 \quad \text{for all } u \in B. \tag{34}$$

Bounded convergence of the discretizations imply there is $r_2 > 0$ such that

$$||c_p(u) - c_0(u)|| \leq r_2 \quad \text{for all } u \in B. \tag{35}$$

Using the triangle inequalilty, we get for all $u \in B$:

$$||c_p(u)|| \leq ||c_0(u)|| + ||c_p(u) - c_0(u)|| \overset{(34),(35)}{\leq} r_1 + r_2$$

so $c_p(B) \subset \bar{B}_{r_1+r_2}(0)$. This holds for all $p \in P$, so in particular we have

$$c_p(B_p) \subset \bar{B}_{r_1+r_2}(0)$$

for all $p \in P$. We conclude that $\bigcup_{p \in P} c_p(B_p) \subset \bar{B}_{r_1+r_2}(0)$, hence $C \subset \bar{B}_{r_1+r_2}(0)$, so C is bounded.

- **Uniform equicontinuity.** Let $\epsilon > 0$. By bounded convergence of the discretizations there exists $p_0 \in P$ such that for all $p < p_0$ and for all $u \in U_\tau$ we have

$$||c_0(u) - c_p(u)|| < \frac{\epsilon}{3}. \tag{36}$$

Since $c_0(B)$ is relatively compact, it is uniformly equicontinuous by Arzelà-Ascoli [4][p.62]. So there is $\delta > 0$ such that for all $u \in B$ and for all $x, y \in \mathcal{D}$ we have

$$||x - y|| < \delta \implies ||c_0(u)(x) - c_0(u)(y)|| < \frac{\epsilon}{3}. \tag{37}$$

Using the triangle inequality twice, we get for all $p < p_0$ and for all $u \in B$: if $||x - y|| < \delta$, then

$$\begin{aligned} ||c_p(u)(x) - c_p(u)(y)|| &\leq ||c_p(u)(x) - c_0(u)(x)|| + ||c_0(u)(x) - c_0(u)(y)|| + ||c_0(u)(y) - c_p(u)(y)|| \\ &\overset{(37)}{<} \frac{\epsilon}{3} + 2||c_0(u) - c_p(u)|| \\ &\overset{(36)}{<} \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \end{aligned}$$

We conclude that $\bigcup_{p<p_0} c_p(B)$ is uniformly equicontinuous. Note that

$$C_0 := \bigcup_{p<p_0} c_p(B_p) \subset \bigcup_{p<p_0} c_p(B)$$

so C_0 is uniformly equicontinuous as well. The compactness of c_p gives that $c_p(B_p)$ is relatively compact, hence uniformly equicontinuous. Since

$$C = C_0 \cup \bigcup_{p \geq p_0} c_p(B_p)$$

is a finite union of uniformly equicontinuous sets, we conclude C is uniformly equicontinuous.

This proves claim 3. $\square$

Put $N := \lceil \frac{1}{\min\{\delta_0,\delta\}} \rceil$, then (a) and (b) hold for all $n \geq N$. This means we have proven persistence of the globally attractive solution u^*. Inspired by the proof, we will deduce an estimate of how accurate the discretized periodic solution is, namely (c). From now on assume $n \geq N$. The θ-periodicity of u^n and u^* imply

$$\sup_{t\in\mathbb{Z}} ||u_t^n - u_t^*|| = \max_{0\leq s<\theta} ||u_{\tau+s}^n - u_{\tau+s}^*||.$$

Therefore it suffices to show there exist $K_0, K_1, ..., K_{\theta-1} > 0$ such that

$$||u_{\tau+s}^n - u_{\tau+s}^*|| \leq \frac{K_s}{1-q}\alpha_1(\frac{1}{n}) \quad \text{for all } 0 \leq s < \theta \tag{38}$$

and put $K = \max_{0\leq s<\theta} K_s$. We give a proof by induction on s. For the induction basis we deduce

$$\begin{aligned} ||u_\tau^n - u_\tau^*|| &= ||c^*(\frac{1}{n}) - u_\tau^*|| \\ &= ||c_p(c^*(p)) - c_0(u_\tau^*)|| \\ &\leq ||c_p(c^*(p)) - c_p(u_\tau^*)|| + ||c_p(u_\tau^*) - c_0(u_\tau^*)|| \\ &\overset{(d')}{\leq} \frac{1+q}{2}||c^*(p) - u_\tau^*|| + \Gamma(\frac{1}{n}) \end{aligned}$$

where we have used the contraction property of c_p in the last step. Rearranging this inequality gives

$$||u_\tau^n - u_\tau^*|| = ||c^*(\frac{1}{n}) - u_\tau^*|| \leq \frac{2}{1-q}\Gamma(\frac{1}{n}). \tag{39}$$

Recall from the proof of (d') that

$$\Gamma(\frac{1}{n}) := \begin{cases} \alpha_1(\frac{1}{n}) & \text{if } \theta = 1 \\ \alpha_1(\frac{1}{n})\left(1 + \sum_{s=\tau+1}^{\tau+\theta-1}\prod_{r=s}^{\tau+\theta-1} L(\theta)_r\right) & \text{if } \theta > 1 \end{cases}$$

If $\theta = 1$, then (39) proves (38) directly (we can choose $K = K_0 = 2$), so we may assume $\theta > 1$ from now on. Put

$$K_0 := 2\left(1 + \sum_{s=\tau+1}^{\tau+\theta-1}\prod_{r=s}^{\tau+\theta-1} L(\theta)_r\right).$$

Then (39) can be rewritten as

$$||u_\tau^n - u_\tau^*|| \leq \frac{K_0}{1-q}\alpha_1(\frac{1}{n})$$

and this proves the induction basis. The proof of the induction step is quite similar to the proof of (d'). Therefore it is given in Appendix B (Proposition B.2). We conclude that (c) holds for all $n \geq N$. This completes the proof of Theorem 3.1. $\square$

We can sharpen the result by looking at how condition (ii) can be satisfied. For that, we focus on IDEs defined by Hammerstein operators that work on a space of one-dimensional functions. More precisely, let $a, b \in \mathbb{R}$ with $a < b$ and let $X := C([a,b], \mathbb{R})$. Let $U_t \subset X$ be open and convex for all $t \in \mathbb{Z}$. We look at operators $\mathcal{F}_t : U_t \to X$ defined by

$$\mathcal{F}_t(u) := \int_a^b k_t(x,y) f_t(y, u(y)) \mathrm{d}y \tag{40}$$

for all $t \in \mathbb{Z}$. We will see that appropiate conditions on k_t and f_t imply assumption (ii) of Theorem 3.1.

Proposition 3.5. Let $a, b \in \mathbb{R}$ with $a < b$, $X := C([a,b],\mathbb{R})$, $U \subset X$ open and convex, $k : [a.b]^2 \to \mathbb{R}$ a continuous function and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ a continuous function such that $f(y,\cdot)$ is C^1 for all $y \in [a,b]$. Furthermore we assume that for all $\epsilon > 0$ and all $y \in [a,b]$ there exists $\delta > 0$ such that for all $z_1, z_2 \in \mathbb{R}$:

$$|z_1 - z_2| < \delta \implies |D_2 f(y,z_1) - D_2 f(y,z_2)| < \epsilon.$$

Then the operator $\mathcal{F} : U \to X$ defined by

$$\mathcal{F}(u) := \int_a^b k(\cdot,y) f(y,u(y)) \mathrm{d}y.$$

is compact and C^1 on U.

Proof. Continuity of k and f imply that $\mathcal{F} \in L(X)$. Since $[a,b]$ is compact we can use the theorem of Arzelà-Ascoli [4][p.62] to show compactness of $\mathcal{F}$. Let $B \subset X$ be bounded. Then $\overline{\mathcal{F}(B)}$ is closed and bounded. It remains to show that $\overline{\mathcal{F}(B)}$ is uniformly equicontinuous.

The boundedness of $\overline{\mathcal{F}(B)}$ and $[a,b]$ imply that

$$s := \sup_{y \in [a,b], u \in \overline{\mathcal{F}(B)}} |f(y,u(y))|$$

exists. Furthermore, compactness of $[a,b]$ implies that k is uniformly continuous. Let $\epsilon > 0$. Then there is $\delta > 0$ such that for all $z_1, z_2 \in [a,b]$

$$|z_1 - z_2| < \delta \implies |k(z_1,y) - k(z_2,y)| < \frac{\epsilon}{s(b-a)}. \tag{41}$$

If $u \in \overline{\mathcal{F}(B)}$ and $|z_1 - z_2| < \delta$ we get

$$\begin{aligned} |\mathcal{F}(u)(z_1) - \mathcal{F}(u)(z_2)| &= \left| \int_a^b k(z_1,y) f(y,u(y)) \mathrm{d}y - \int_a^b k(z_2,y) f(y,u(y)) \mathrm{d}y \right| \\ &= \left| \int_a^b (k(z_1,y) - k(z_2,y)) f(y,u(y)) \mathrm{d}y \right| \\ &\overset{(41)}{<} \frac{\epsilon}{s(b-a)} \int_a^b |f(y,u(y))| \, \mathrm{d}y \\ &\leq \frac{\epsilon}{s(b-a)} s(b-a) = \epsilon. \end{aligned}$$

We conclude that $\overline{\mathcal{F}(B)}$ is uniformly equicontinous, hence compact. Therefore $\mathcal{F}$ is compact. For the proof that $\mathcal{F}$ is C^1 we refer to [12][theorem B.8]. □

Corollary 3.6. Consider the IDE

$$u_{t+1} = \mathcal{F}_t(u_t) \tag{42}$$

with right-hand side defined by (40). If for all $t \in \mathbb{Z}$ the functions k_t and f_t are continuous, $f_t(y,\cdot)$ is C^1 for all $y \in [a,b]$, and for all $\epsilon > 0$ and all $y \in [a,b]$ there exists $\delta > 0$ such that for all $z_1, z_2 \in \mathbb{R}$:

$$|z_1 - z_2| < \delta \implies |D_2 f_t(y,z_1) - D_2 f_t(y,z_2)| < \epsilon,$$

then the general solution φ_0 of (42) satisfies the properties stated in (ii), namely:

(I) $\varphi_0(t;\tau,\cdot) : U_\tau \to C(\mathcal{D},\mathbb{R}^q)$ is compact for all $\tau < t$

(II) $\varphi_0(t;\tau,\cdot)$ is C^1 for all $\tau \le t$

Proof. By proposition 3.5 the operator $\mathcal{F}_t$ is compact and C^1 for all $t \in \mathbb{Z}$. Let τ be fixed.

(I) Let $t > \tau$. Then $\varphi_0(t;\tau,\cdot) : U_\tau \to C([a,b],\mathbb{R})$ is a composition of compact operators, hence compact. (The compact operators form a two-sided closed ideal, see [4][p.90]).

(II) If $t = \tau$, then $\varphi_0(t;\tau,\cdot)$ is the identity, which is C^1. If $t > \tau$, then $\varphi_0(t;\tau,\cdot)$ is a composition of C^1 operators, hence C^1.

□

Now we give an example of a model that satisfies the assumptions of Corrollary 3.6 and Theorem 3.1.

Example 3.7. A well known growth function from Theoretical Ecology is the Beverton-Holt growth function [9]. It is defined by

$$f(N) = \frac{RN}{1+kN},$$

where N represents population size and R,k are parameters. A variant of this growth function is defined by

$$f(N) = \frac{R(2-\frac{3}{2}\cos(\frac{s}{2}))N}{1+|N|}, \tag{43}$$

where s is a parameter [13]. Note that the absolute value can be omitted when considering non-negative population sizes only. As a dispersal kernal, define the so-called Laplace kernal [13] by

$$k_{\alpha_t}(x,y) = \frac{\alpha_t}{2}e^{\alpha_t|x-y|}, \tag{44}$$

where $(\alpha_t)_{t\in\mathbb{N}}$ is a 4-periodic sequence. We combine (43) and (44) into an IDE as follows:

$$\mathcal{F}_t(u)(x) = \int_{-2}^{2} k_{\alpha_t}(x,y)f(y)\mathrm{d}y = \int_{-2}^{2} \frac{\alpha_t}{2}e^{\alpha_t|x-y|}\frac{R(2-\frac{3}{2}\cos(\frac{s}{2}))y}{1+|y|}\mathrm{d}y. \tag{45}$$

In [13][Example 3.2] it is shown that (45) satisfies the assumptions (i) and (iii) of Theorem 3.1 (in particular it has a globally attractive periodic solution) and the assumptions of Corrallary 3.6. Hence, by Theorem 3.1 and Corrollary 3.6, the globally attractive periodic solution persists under any bounded convergent discretization satisfying properties (iv)-(vii) of Theorem 3.1. ⋄

4 Conclusions and remarks.

As explained in the introduction, discretizing an IDE is essential in applications. It makes it possible to analyse relatively complicated models. In particular it is possible to study their asymptotic behaviour through simulations. It is therefore an important question to what extend features of the dynamics of IDE-models (such as existence and stability of periodic solutions) persist under discretization. In the third section, some of the first steps towards an answer to this question (formulated first in [13]) were presented. Instead of trying to answer the question directly for general IDEs, the setting was specified to a specific class of IDEs.

In summary, we proved the following result. Consider a θ-periodic IDE defined by (42), with $\mathcal{F}_t$ defined by (40), that possesses a globally attractive θ-periodic solution. Suppose the IDE satisfies:

- The assumptions of Corrolary 3.6.
- Assumption (iii) in Theorem 3.1.

Then, if a bounded convergent discretization satisfies assumptions (iv)-(vii) of Theorem 3.1, the globally attractive solution persists under this discretization. Furthermore, we have an estimate of how close the discretized periodic solution lies to the original periodic solution, namely (17). A few remarks are appropiate.

Remark 4.1 (influence of spectral values). Looking at (iii), one can note that the spectral values of $D\mathcal{F}_\theta(u_\theta^*)\cdots D\mathcal{F}_1(u_1^*)$ essentially influence the accuracy of the estimate (17). If the spectral values are close to 1, then a, and in turn q, have to be close to 1, which causes the expression $\frac{K}{1-q}$ to become large. Furthermore, by looking at how δ was chosen (see equation (25)) we see that if q is close to 1, then δ becomes small, which in turn causes N to become large. In other words, the larger the spectral values are, the larger one has to choose N. $\diamond$

Remark 4.2. The value of K in (17) is dependent on the Lipschitz constants that were defined in the proof of the estimate. The value of those Lipschitz constants depend on the upper bounds M_s of $||D\mathcal{F}_s^n||$ whose existence was assumed in (iv). The third quantity that influences the accuracy of (17) is α_1. Its value is dependent on the local discretization error $\varepsilon_s^n(u_s^*)$. $\diamond$

Remark 4.3 (General IDEs). Theorem 3.1 was formulated for general IDEs and Corrollary 3.6 for Hammerstein IDEs, but in [13] an analagon of Corrollary 3.6 is formulated for general IDEs. The author proves this in [12]. $\diamond$

A natural question is: what discretization methods are appropiate, in order to satisfy assumptions (iv)-(vii) of Theorem 3.1? In [13][Proposition 2.4] it was proven that piecewise linear collocation, which was discribed in section 2 and the appendix, is appropiate. Furthermore, in [13][section 3.2] a similar method is described that works for Hammerstein IDEs (the so-called *bilinear degenerate kernel method*). It could be interesting to look what other projection methods, or other discretization methods, are appropiate in this context.

Recall the mosquito-model of Example 1.1. The operator in this model is only periodic if the mosquito release ratio R_t is a periodic function. This might not be realistic in practice. So in order to make usefull, representative simulations of this model, there is need for a theorem that is similar to Theorem 3.1, but holds for IDEs that are not periodic in time. This could be another subject for future research.

Appendix A

Differentiation in normed vector spaces.
The notions of directional differentiability and differentiability in $\mathbb{R}^p$ can be extended to normed vector spaces. Let V and W be vector spaces with norms $||\cdot||_V$ and $||\cdot||_W$ respectively. Let $U \subset V$ be open and $f : U \to W$ be a function.

Definition A.1 Let $x \in U$. If there exists a bounded linear map $A : V \to W$ such that

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0,$$

then f is called *(Fréchet) differentiable in the point* x and A is called the *(Fréchet) derivative* at x. If f is differentiable in all points $x \in U$, then f is called *differentiable.* $\diamond$

This notion looks quite similar to the notion of differentiability, except for the requirement that A is bounded. But if V and W are finite dimensional, then f is Fréchet differentiable in x if and only if it is differentiable in x, because linear maps on finite dimensional spaces are always bounded.

Definition A.2 Let $x \in U$ and $v \in V$. The *Gateaux differential* of f at the point x in the direction v is defined as

$$df(x,v) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

If $df(x,v)$ exist for every $v \in V$, then f is called *Gateaux differentiable at* x. If f is Gateaux differentiable at any $x \in U$, then f is called Gateaux differentiable. $\diamond$

Lemma A.3. With the notation of definition A.1 and A.2, if f is differentiable, then it is Gateaux differentiable and we have

$$df(x,v) = A(v) \quad \text{for all } x \in U, v \in V.$$

Proof. The proof of this lemma is essentially the same as the proof of its analagon in finite dimension [2][Lemma 1.13, p.6]. Let $x \in U$ and $v \in V$. If $v = 0$ the statement is trivial. Assume $v \neq 0$. Then we know

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0$$

Substitute $h = tv$. By linearity of A we get

$$\begin{aligned} 0 &= \lim_{\|tv\|_V \to 0} \frac{\|f(x+tv) - f(x) - Atv\|_W}{\|tv\|_V} \\ &= \lim_{t \to 0} ||v||_V^{-1} \left\| \frac{f(x+tv) - f(x)}{t} - \frac{tAv}{t} \right\|_W \\ &= ||v||_V^{-1} \lim_{t \to 0} \left\| \frac{f(x+tv) - f(x)}{t} - Av \right\|_W . \end{aligned}$$

So $\lim_{t \to 0} \| \frac{f(x+tv)-f(x)}{t} - Av\|_W = 0$, hence $\lim_{t \to 0} \frac{f(x+tv)-f(x)}{t} = Av$. $\square$

Piecewise linear collocation in higher dimension.
Let $a_i, b_i \in \mathbb{R}$ for all $i \in \{1, ..., k\}$ and $\mathcal{D}^* := \Pi_{i=1}^k [a_i, b_i]$. We will generalise the piecewise linear collocation described in the first chapter to $X := C(\mathcal{D}^*, \mathbb{R}^\kappa)$. Let $n \in \mathbb{N}$ and define for all $j \in \{0, 1, , ..., n\}$

$$\xi_j^i := a_i + j \frac{b_i - a_i}{n}$$

Define the hat functions $e_j^i : [a_i, b_i] \to [0,1]$ by

$$e_j^i(x) := \max\{0, 1 - \frac{n}{b_i - a_i} |x - \xi_j^i|\}.$$

Let I be the multi-index set $\{0,1,...,n\}^k$. For all $\iota \in I$, define $e_\iota : \mathcal{D}^* \to [0,1]$ by

$$e_\iota(x) := \Pi_{i=1}^k e^i_{\iota_i}(x_{\iota_i}).$$

The projections $P_n : X \to X$ then become

$$P_n(u) := \sum_{\iota \in I} e_\iota u(\xi^1_{\iota_1}, ..., \xi^k_{\iota_k}). \tag{46}$$

If we apply (46) to the general IDE-operator (16) we get

$$\mathcal{F}^n_t(u) := P_n \mathcal{F}_t(u) = \sum_{\iota \in I} e_\iota G_t \left(\xi^1_{t_1}, \ldots, \xi^k_{\iota_k}, \int_\Omega f_t \left(\xi^1_{t_1}, \ldots, \xi^k_{\iota_k}, y, u(y) \right) \mathrm{d}y \right) \Bigg) .$$

Appendix B

Proposition B.1. (induction step in the proof of (e')).
Suppose there exist $s \geq 1$ and a function $\gamma_s : \mathbb{R}^2_+ \to \mathbb{R}_+$ with $\lim_{||x|| \to 0} \gamma_s(x) = 0$ such that for all $u \in B_b(u^*_\tau) \cap U_\tau$ we have

$$||D_3\varphi_n(\tau+s,\tau,u) - D_3\varphi_0(\tau+s,\tau,u^*_\tau)|| \leq \gamma_s(||u-u^*_\tau||, \frac{1}{n}) \tag{47}$$

Then the same result holds for $s+1$.

Proof. Abbreviate $\varphi_{j,t}(u) := \varphi_j(\tau+t,\tau,u)$ for all $j \in \mathbb{N}_0$. By the chain rule we have

$$||D_3\varphi_{n,s+1}(u) - D_3\varphi_{0,s+1}(u^*_\tau)|| = ||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u))D_3\varphi_{n,s}(u) - D\mathcal{F}_{\tau+s}(\varphi_{0,s}(u^*_\tau))D_3\varphi_{0,s}(u^*_\tau)||$$

Define the constants $c_1 := D_3\varphi_{0,s}(u^*_\tau)$ and $c_2 := D\mathcal{F}^n_{\tau+s}(u^*_{\tau+s})$. Let $u \in B_b(u^*_\tau) \cap U_\tau$. Then by the triangle inequality, (47), (vi)(II) and (vi)(III) the latter equation is less then or equal to:

$$\begin{aligned}
&||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u))D_3\varphi_{n,s}(u) - D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u))D_3\varphi_{0,s}(u^*_\tau)|| \\
&+||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u))D_3\varphi_{0,s}(u^*_\tau) - D\mathcal{F}_{\tau+s}(\varphi_{0,s}(u^*_\tau))D_3\varphi_{0,s}(u^*_\tau)|| \\
&\leq ||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u))||\gamma_s(||u-u^*_\tau||, 1/n) + c_1||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u)) - D\mathcal{F}_{\tau+s}(\varphi_{0,s}(u^*_\tau))|| \\
&\leq \left[||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u)) - D\mathcal{F}^n_{\tau+s}(u^*_{\tau+s})|| + ||D\mathcal{F}^n_{\tau+s}(u^*_{\tau+s})||\right] \gamma_s(||u-u^*_\tau||, 1/n) \\
&+c_1 \left[||D\mathcal{F}^n_{\tau+s}(\varphi_{n,s}(u)) - D\mathcal{F}^n_{\tau+s}(u^*_{\tau+s})|| + ||D\mathcal{F}^n_{\tau+s}(u^*_{\tau+s}) - D\mathcal{F}_{\tau+s}(\varphi_{0,s}(u^*_{\tau+s}))||\right] \\
&\leq \left[\alpha_3(||\varphi_{n,s}(u) - u^*_{\tau+s}||) + c_2\right] \gamma_s(||u-u^*_\tau||, 1/n) + c_1 \left[\alpha_3(||\varphi_{n,s}(u) - u^*_{\tau+s}||) + \alpha_2(1/n)\right] \\
&=(\gamma_s(||u-u^*_\tau||, 1/n) + c_1)\alpha_3(||\varphi_{n,s}(u) - u^*_{\tau+s}||) + \gamma_s(||u-u^*_\tau||, 1/n)c_2 + c_1\alpha_2(1/n).
\end{aligned} \tag{48}$$

Let $L(s)$ be defined as in the proof of (d'), but with θ replaced by s. Then by (vi)(I) we have

$$\begin{aligned}
||\varphi_{n,s}(u) - u^*_{\tau+s}|| &= ||\mathcal{F}^n_{\tau+s-1}(\varphi_{n,s-1}(u)) - \mathcal{F}_{\tau+s-1}(u^*_{\tau+s-1})|| \\
&\leq ||\mathcal{F}^n_{\tau+s-1}(\varphi_{n,s-1}(u)) - \mathcal{F}^n_{\tau+s-1}(u^*_{\tau+s-1})|| + \alpha(1/n) \\
&\leq L(s)_{\tau+s}||\varphi_{n,s-1}(u) - u^*_{\tau+s-1})|| + \alpha(1/n)
\end{aligned}$$

Repeat this reasoning $s-1$ times. Then we get

$$||\varphi_{n,s}(u) - u^*_{\tau+s}|| \leq \left(\prod_{r=\tau+1}^{\tau+s-1} L_r \right) ||u-u^*_\tau|| + (1 + \sum_{r=\tau+2}^{\tau+s-1} L_r)\alpha_1(1/n)$$

Plugging this into the last line of (48) yields

$$||D_3\varphi_{n,s+1}(u) - D_3\varphi_{0,s+1}(u^*_\tau)|| \le (\gamma_s(||u-u^*_\tau||, 1/n) + c_1)$$
$$\alpha_3(\left(\prod_{r=\tau+1}^{\tau+s-1} L_r\right)||u-u^*_\tau|| + (1+\sum_{r=\tau+2}^{\tau+s-1} L_r)\alpha_1(1/n)) + \gamma_s(||u-u^*_\tau||, 1/n)c_2 + c_1\alpha_2(1/n)$$

Define

$$\gamma(x,y) = (\gamma_s(x,y)+c_1)\alpha_3(\left(\prod_{r=\tau+1}^{\tau+s-1} L_r\right)x + (1+\sum_{r=\tau+2}^{\tau+s-1} L_r)\alpha_1(y)) + \gamma_s(x,y)c_2 + c_1\alpha_2(y)$$

A close look at this formula should convince you that $\lim_{||x||\to 0}\gamma(x) = 0$ is satisfied. This completes the proof. □

Proposition B.2 (induction step in the proof of (c)).
Let $n \ge N$ and let $s \in \mathbb{N}$ be arbitrary. Suppose there exists $K_s > 0$ such that

$$||u^n_{\tau+s} - u^*_{\tau+s}|| \le \frac{K_s}{1-q}\alpha_1(\frac{1}{n}). \tag{49}$$

Then there exists $K_{s+1} > 0$ such that

$$||u^n_{\tau+s+1} - u^*_{\tau+s+1}|| \le \frac{K_{s+1}}{1-q}\alpha_1(\frac{1}{n}). \tag{50}$$

Proof. In the proof of claim 2 (on p.11) we showed that for any bounded, convex set $B \subset C(\mathcal{D}, \mathbb{R}^q)$ there exists a sequence $(L(B)_t)_{t\in\mathbb{Z}} \subset \mathbb{R}_+$ such that

$$||\mathcal{F}^n_t(u_1) - \mathcal{F}^n_t(u_2)|| \le ||u_1-u_2||L(B)_t \quad \text{for all } t\in\mathbb{Z} \text{ and } u_1,u_2 \in B\cap U_t.$$

Define a radius $\rho(s) > 0$ such that $u^n_{\tau+s} \in B_{\rho(s)}(u^*_{\tau+s})$. We abbreviate $L(s) := L(B_{\rho(s)}(u^*_{\tau+s}))$. This establishes

$$||\mathcal{F}^n_{\tau+s}(u^n_{\tau+s}) - \mathcal{F}^n_{\tau+s}(u^*_{\tau+s})|| \le ||u^n_{\tau+s} - u^*_{\tau+s}||L(s)_{\tau+s}. \tag{51}$$

We deduce

$$\begin{aligned}
||u^n_{\tau+s+1} - u^*_{\tau+s+1}|| &= ||\mathcal{F}^n_{\tau+s}(u^n_{\tau+s}) - \mathcal{F}_{\tau+s}(u^*_{\tau+s})|| \\
&\le ||\mathcal{F}^n_{\tau+s}(u^n_{\tau+s}) - \mathcal{F}^n_{\tau+s}(u^*_{\tau+s})|| + ||\mathcal{F}^n_{\tau+s}(u^*_{\tau+s}) - \mathcal{F}_{\tau+s}(u^*_{\tau+s})|| \\
&\overset{(51)}{\le} ||u^n_{\tau+s} - u^*_{\tau+s}||L(s)_{\tau+s} + ||\mathcal{F}^n_{\tau+s}(u^*_{\tau+s}) - \mathcal{F}_{\tau+s}(u^*_{\tau+s})|| \\
&\overset{(vi)}{\le} ||u^n_{\tau+s} - u^*_{\tau+s}||L(s)_{\tau+s} + \alpha_1(\frac{1}{n}) \\
&\overset{(49)}{\le} \left(\frac{K_s}{1-q}L(s)_{\tau+s} + 1\right)\alpha_1(\frac{1}{n}) \\
&= \frac{K_sL(s)_{\tau+s} + 1 - q}{1-q}\alpha_1(\frac{1}{n}).
\end{aligned}$$

Defining $K_{s+1} = K_sL(s)_{\tau+s} + 1 - q > 0$ establishes (50). This proves the induction step in the proof of (c) of Theorem 3.1 in the main text. □

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