Semisimple Lie algebras and root systems

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Abstract

In this paper we introduce Lie algebras and list basic properties and results, including Weyl's theorem. We then introduce the important Lie algebra $\mathfrak{sl}(2)$, and move on to briefly introduce root systems. The main goal of the paper is to prove that semisimple Lie algebras are in one to one correspondance with chrystallographic root systems.

Introduction

I have previously worked with root systems in their own right and in the context of Reflection groups and Coxeter groups, based on Humphreys book of the same title. Therefore some knowledge of root systems will be helpful to fully grasp the finer details of some of the fourth chapter. However we aim to give sufficient details to make the reader who is new to root systems able to follow along.

The structure of the paper is aimed at being the shortest route to proving that semisimple Lie algebras are in one to one correspondance with chrystallographic root systems. In order to reach this goal within a reasonable page number we will state many of the more basic results without proof.

All results are from *Introduction to Lie Algebras and Representation Theory* by James E. Humphreys, which will be refered to simply as Humphreys. The interested reader may look for further reading on root systems in the bibliography.

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Chapter 1

Lie algebras

We start out by giving the general definition of a Lie algebra

Definition 1.0.1. *We call a vector space L over a field F a Lie algebra if there is a map $L \times L \to L$ denoted $(x, y) \mapsto [xy]$ satisfying:*

$$(L1) : \textit{The map is bilinear over } F$$
$$(L2) : \forall x \in L : [xx] = 0$$
$$(L3) : \forall x, y, z \in L : [x[yz]] + [y[zx]] + [z[xy]] = 0$$

A few remarks about notation and language, call the map the **bracket** or the **commutator** of x and y, the motivation for the second name will be obvious when we look at the linear Lie algebras. Furthermore sometimes we will write $[x, y] = [xy]$ in order to avoid ambiguity.
The third criterion $(L3)$ is called the **Jacobi identity**.
Notice that the bilinearity of the bracket ensures that a Lie algebra over F, is an F-algebra.
If the underlying field F is of characteristic greater than 2, then equivivalent criterion to $(L2)$ is $(L2') : [xy] = -[yx]$ due to

$$0 \overset{(L2)}{=} [x + y, x + y] \overset{(L1)}{=} [xx] + [yy] + [xy] + [yx] \overset{(L1)}{=} [xy] + [yx] \iff [xy] = -[yx]$$

Moving on to the standard notions, given two Lie algebras L and L' and an isomorphism between the underlying vector spaces $\phi : L \to L'$, we call ϕ a **Lie algebra isomorphism** if ϕ respects the respective brackets, meaning that for all $x, y \in L$ we have $\phi([xy]) = [\phi(x)\phi(y)]$.
A subalgebra of a Lie algebra is simply a subspace closed within the bracket, i.e. $K \subset L$ a subspace such that for all $x, y \in K$ we have $[xy] \in K$. And naturally K is a Lie algebra in its own right with the restricted bracket.
Untill now we have not assumed the underlying vector spaces to be finite dimensional, but for the remainder of our journey through the theory we will pressume that our vector spaces are finite dimensional, unless otherwise stated.

1.1 Linear Lie algebras

Linear Lie algebras are central in the study of Lie algebras, and built upon $\mathrm{End}(V)$, the set of endomorphisms (linear transformations) on V. We know from linear algebra that $\mathrm{End}(V)$ is a vector space over F with map composition as multiplication, i.e. $xy = x \circ y : V \to V$. Now we define the bracket to be the commutator $[xy] = xy - yx$, which is zero precisely when x and y commute.

Definition 1.1.1. *The general Lie algebra $\mathfrak{gl}(V)$ is the Lie algebra with vectorspace End(V) and the commutator as bracket map.*
*Furthermore any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.*

An advantage of using End(V) is that we can use matrix representations, i.e. $\mathfrak{gl}(V) = \mathfrak{gl}(n, F)$ the set of $n \times n$ matrices over F where $n = \dim V$. When we make this identification we can apply linear algebra and get the standard matrix basis $\varepsilon_{i,j}$ for $1 \leq i, j \leq n$ with a 1 in the (i, j)'th coordinate and zeroes everywhere else. We have $\varepsilon_{i,j}\varepsilon_{k,l} = \delta_{j,k}\varepsilon_{i,l}$ and the bracket

$$[\varepsilon_{i,j}, \varepsilon_{k,l}] = \varepsilon_{i,j}\varepsilon_{k,l} - \varepsilon_{k,l}\varepsilon_{i,j} = \delta_{j,k}\varepsilon_{i,l} - \delta_{l,i}\varepsilon_{k,j} \tag{1.1.1}$$

Note that the zero map $[xy] = 0$ for all $x, y \in L$ satisfies the conditions (L1)-(L3), we call such Lie algebras abelian due to the relation to the commutator in the case of linear Lie algebras.
Another way to construct a finite dimensional Lie algebra L is to take a basis $x_1, x_2, \dots, x_n$, $n = \dim L$ and consider the $a_{i,j,k}$ satisfying $[x_i x_j] = \sum_{k=1}^{n} a_{i,j,k} x_k$ now with some tedious algebra we can show that

$$\begin{aligned}
(L2)\ 0 &= [x_i x_i] = \sum_{k=1}^{n} a_{i,i,k} x_k \iff a_{i,i,k} = 0\ \forall\ i, k \\
(L2')\ 0 &= [x_i x_j] + [x_j x_i] = \sum_{k=1}^{n} (a_{i,j,k} + a_{j,i,k}) x_k \iff a_{i,j,k} = a_{j,i,k}\ \forall\ i, j, k \\
(L3)\ 0 &= [x_i[x_j x_\ell]] + [x_j[x_\ell x_i]] + [x_\ell[x_i x_j]] \\
&= \sum_m \sum_k (a_{j,\ell,k} a_{i,k,m} + a_{\ell,i,k} a_{j,k,m} + a_{i,j,k} a_{\ell,k,m}) x_m \\
&\iff \sum_{k=1}^{n} (a_{j,\ell,k} a_{i,k,m} + a_{\ell,i,k} a_{j,k,m} + a_{i,j,k} a_{\ell,k,m}) = 0
\end{aligned}$$

This interpretation allows us to determine all the Lie algebras of dimension ≤ 2.
Dimension 1 is abelian because let x be the basis then $[xx] = 0$ by (L2) and bilinearity then makes $[yz] = [a_y x, a_z x] = a_y a_z [xx] = 0$.
In dimension two let x, y be a basis of L, then for $w, z \in L$ write $w = x_w x + y_w y$ and $z = x_z x + z_y y$ we get

$$\begin{aligned}
[wz] &= x_w x_z [xx] + x_w y_z [xy] + y_w x_z [xy] + y_w y_z [yy] \\
&= (x_w y_z - y_w y_z)[xy]
\end{aligned}$$

So if $[xy] = 0$ L is abelian, if not let $[xy] = x'$ then x' is a basis of the one dimensional subspace of non abelian elements of L. Extend the basis to x', y' and we get $[x'y'] = ax'$ for some nonzero a in F, now if we scale y' by a^{-1} we still get a basis and $[x', a^{-1}y'] = aa^{-1}x' = x'$. In conclusion we have two types of two dimensional Lie algebras, up to isomorphism, the abelian ones and the ones with basis x_1, x_2 and $a_{1,2,1} = 1, a_{1,2,2} = 0$.

1.2 Ideals

Definition 1.2.1 (Ideal)**.** *For a Lie algebra L, a subspace I of L is called an ideal if for all $x \in L$ and $y \in I$ we have $[xy] \in I$.*

Notice that an ideal is a subalgebra, and that the zero subspace is an ideal. A central example of an ideal is the center $Z(L) := \{z \in L \mid [z\ell] = 0 \ \forall \ell \in L\}$, clearly L is abelian if and only if $L = Z(L)$.
Given ideals I and J we can construct ideals.

$$I + J = \{x + y \mid x \in I, y \in J\}$$
$$[IJ] = \left\{ \sum [x_i y_j] \mid x_i \in I, y_j \in J \right\}$$

An important ideal is the derived algebra denoted $[LL]$. It is immediately clear that L is abelian if and only if $[LL] = 0$.

Definition 1.2.2 (Simple Lie algebra)**.** *We call a Lie algebra L simple if the only proper ideal of L is 0 and if $[LL] = L$.*

It follows immediately from the definition that simple Lie algebras have trivial center.
Having defined ideals it is natural to define the notion of quotients. For a Lie algebra L and an ideal $I \subseteq L$ we define L/I to be the quotient subspace in the usual sense and the bracket for the equivivalence classes is $[x + I, y + I] := [xy] + I$. This is well defined because if $x + I = x' + I$ and $y + I = y' + I$ we can write $x = x' + i_x$ and $y = y' + i_y$ then

$$\begin{aligned}
[x + I, y + I] = [x, y] + I &= [(x' + i_x), (y' + i_y)] + I \\
&= \big([x'y'] + ([i_x, y'] + [x', i_y] + [i_x, i_y])\big) + I \\
&= [x'y'] + I = [x' + I, y' + I]
\end{aligned}$$

Where the small parenthesis is in I by definition of ideals.
For later use we define two important subalgebras.

Definition 1.2.3 (normaliser $N_L(K)$ and centraliser $C_L(X)$)**.** *Given a Lie algebra L, a subalgebra K, and a subset X we define the normaliser $N_L(K)$ and the centraliser $C_L(X)$ to be*

$$N_L(K) = \{x \in L \mid [xK] \subseteq K\}$$
$$C_L(X) = \{\ell \in L \mid [\ell X] = 0\}$$

1.3 Homomorphisms

Defining homomorphisms at this point seems a bit odd, since we already defined isomorphisms. Therefore the definition should come as no surprise.

Definition 1.3.1. *Given two Lie algebra L and L', we call a linear map $\phi : L \to L'$ a homomorphism if it respects the bracket, i.e. $\phi([xy]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. We call a homomorphism a monomorphism if it has trivial kernel and an epimorphism if it is surjective, and an isomorphism if it is both a monomorphism and an epimorphism. We write $L \cong L'$ if there exists an isomorphism $\phi : L \to L'$, or equivivalently and isomorphism $\psi : L' \to L$.*

Note that this is in agreement with the previously given definition of a Lie algebra isomorphism. Now the kernel of a homomorphism $\phi : L \to L'$ is an ideal because if $x \in \ker \phi$ we have $\phi([xy]) = [\phi(x)\phi(y)] = [0\phi(y)] = 0$ for all $y \in L$, so $[xy] \in \ker(\phi)$ as wanted. And as is usual in Algebraic topics, we have a bijection between ideals and homomorphisms for ϕ a homomorphism $\phi \mapsto \ker \phi$ and for I an ideal $I \mapsto (x \mapsto x + I)$. Furthermore we have a standard homomorphism theorem.

Theorem 1.3.2 (Standard homomorphism theorem)**.**

(a) If $\phi : L \to L'$ is a homomorphism of Lie algebras, then $L/\ker(\phi) \cong Im(\phi)$. Furthermore if $I \subseteq \ker(\phi)$ is an ideal of L then there exists a unique homomorphism $\psi : L/I \to L'$ such that $\phi = \psi \circ \pi$, where $\pi : L \mapsto L/I$ is the cannonical map $\pi(x) = x + I$.

(b) If $I \subseteq J$ are ideals of L then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.

(c) If I, J are ideals of L, then $(I + J)/J \cong I/(I \cap J)$.

An important type of homomorphisms are the representations, these are homomorphisms of the form $\phi : L \to \mathfrak{gl}(V)$ where V is a vector space over F. For our purposes the adjoint representation is the most important representation.

1.4 The adjoint representation

Given an F-algebra $\mathfrak{U}$ we call a linear map $\delta : \mathfrak{U} \to \mathfrak{U}$ a derivation if for all $a, b \in \mathfrak{U}$ we have $\delta(ab) = a\delta(b) + \delta(b)a$. We denote the set of all derivations by Der $\mathfrak{U}$, and it is clearly a vector subspace of End $\mathfrak{U}$. A quick calculation shows that the commutator of two derivations is again a derivation, making Der $\mathfrak{U}$ a Lie subalgebra of $\mathfrak{gl}(\mathfrak{U})$.

Definition 1.4.1. *For a Lie algebra L the map ad: $L \to$ Der $L \subset \mathfrak{gl}(L)$, given by ad $x = (y \mapsto [xy])$ is called the adjoint representation.*

The adjoint representation will play a vital role in the theory to come. When we have $K \subseteq L$ of Lie algebras we denote $\mathrm{ad}_K\ x = \mathrm{ad}\ x|_K$ the restriction of the map $(y \mapsto [xy])$ to $y \in K$, but x may still be an element of the bigger Lie algebra L.
To see that the adjoint representation is indeed a homomorphism consider for $x, y, z \in L$

$$\begin{aligned}
[\mathrm{ad}\ x,\ \mathrm{ad}\ y](z) &= \mathrm{ad}\ x\ \mathrm{ad}\ y(z) - \mathrm{ad}\ y\ \mathrm{ad}\ x(z) \\
&= \mathrm{ad}\ x([yz]) - \mathrm{ad}\ y([xz]) \\
&= [x[yz]] - [y[xz]] \\
&= (-[y[zx]] - [z[xy]]) + [y[zx]] \\
&= [[xy]z] = \ \mathrm{ad}\ [xy](z)
\end{aligned}$$

As wanted. Now we record a very usefull proposition, that allows the study of simple Lie algebras to take place within the more convenient linear Lie algebras.

Proposition 1.4.2. *All simple Lie algebras are isomorphic to some linear Lie algebra.*

Proof: Notice that the kernel of the adjoint representation are the x that commute with all other elements, in the sense that $[xy] = 0$ for all $y \in L$, and as such ker ad $= Z(L) = C_L(L)$. Now recall that a simple Lie algebra has trivial center $Z(L) = 0$ so by the standard homomorphism theorem we have an isomorphism $L = L/\ker(\mathrm{ad}) \cong \mathrm{Im}(\mathrm{ad}) \subseteq \mathfrak{gl}(L)$ as wanted. $\square$

Definition 1.4.3. *Automorphism An automorphism on a Lie algebra L is an isomorphism $\phi : L \to L$. Denote by Aut(L) the group of all automorphisms on L.*

Another central definition is for char$F = 0$ and $x \in L$ we call ad x **nilpotent** if there is an $n \in \mathbb{N}$ such that $(\mathrm{ad}\ x)^n = 0$, as usual.

1.5 Solvable and nilpotent Lie algebras

Definition 1.5.1. *For a Lie algebra L we define the derived series $L^{(n)}$ and the descending central series L^n by*

$$L^{(0)} = L, L^{(1)} = [LL], L^{(2)} = [L^{(1)}L^{(1)}], \dots, L^{(n)} = [L^{(n-1)}L^{(n-1)}]$$
$$L^0 = L, L^1 = [LL], L^2 = [LL^1], \dots, L^n = [LL^{n-1}]$$

We call L solvable if there exists an $m \in \mathbb{N}$ such that $L^{(m)} = 0$, and we call L nilpotent if there exists an $m \in \mathbb{N}$ such that $L^m = 0$.

Immediately notice that abelian Lie algebras are both solvable and nilpotent, simply because $L^{(1)} = L^1 = [LL] = 0$.

Proposition 1.5.2. *For L a Lie algebra we have:*

(a) If L is solvable and/or nilpotent then so are all subalgebras and homomorphic images of L.
(b) If I is a solvable ideal of L such that L/I is solvable, then L itself is solvable.
(c) If I and J are solvable ideals of L, then so is I+J.
(d) If $L/Z(L)$ is nilpotent, then so is L.
(e) If L is nilpotent and nonzero, then $Z(L) \neq 0$.

To introduce the next important theorem we call an $x \in L$ ad-nilpotent if ad x is nilpotent, so clearly if L is nilpotent then ad x is nilpotent for all $x \in L$.

Theorem 1.5.3 (Engel)**.** *If all elements of a Lie algebra are ad-nilpotent, then L is nilpotent.*

To prove Engels theorem we need a few more results.

Lemma 1.5.4. *If $x \in \mathfrak{gl}(V)$ is nilpotent as an endomorphism, then so is ad x.*

Engels theorem follows from the following theorem.

Theorem 1.5.5. *Let L be a subalgebra of $\mathfrak{gl}(V)$, where V is finite dimensional. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists nonzero $v \in V$ for which $L.v = 0$.*

Proof of Engels theorem. Given a Lie algebra L with all elements ad-nilpotent, we get that ad $L \subseteq \mathfrak{gl}(L)$ consists purely of nilpotent endomorphisms, the theorem above then grants us an $x \in L$ such that $[Lx] = 0$, which is clearly an action on L. Now we have $x \in Z(L) \neq 0$. We proceed by induction on $\dim L$, and claim that L is nilpotent.
Induction start: For $\dim L = 1$ we previously saw that L is abelian which we noted implies nilpotency.
Induction step: Assume the hypothesis for Lie algebras of dimension less than n, we just saw that $Z(L) \neq 0$ so $\dim(L/Z(L)) < n$ so it is nilpotent. Now 1.5.2(e) implies that L is nilpotent, as wanted. $\square$

Chapter 2

Semisimple Lie algebras

In the following we will need to restrict our choice of underlying field F to be of characteristic 0 and to be algebraically closed, the reason we are demanding algebraic closure is to ensure that all eigenvalues are contained in F.

Definition 2.0.6 (Radical and semisimple)**.**

- *The radical of L denoted Rad(L) is the maximal solvable ideal, with regards to inclusion.*
- *A Lie algebra L is called semisimple if its radical is zero.*

Saying *the* radical is well defined because if I and J are maximal solvable ideals then proposition 1.5.2 implies that also $I + J$ is solvable so either $I \subseteq J$ or $J \subseteq I$ which is a contradiction in both cases, unless they are identical. So the notion of *the maximal solvable ideal* is well defined.
Notice that any simple Lie algebra L is semisimple, because the only proper ideal is zero and $[LL] = L$ implies that simple Lie algebras are not solvable, so $Rad(L) = 0$ as wanted.

Theorem 2.0.7. *Any solvable linear Lie algebra $L \subseteq \mathfrak{gl}(V)$ with $V \neq 0$ and finite dimensional, then V contains a common eigenvector for all endomorphisms in L.*

Corrollary 2.0.8 (Lie's theorem)**.** *If L is a solvable subalgebra of $\mathfrak{gl}(V)$, then L stabilizes a flag in V, i.e. there exists a suitable basis of V such that the matrix representations of the elements of L are upper triangular. Recall that a flag is a chain of subspaces $0 \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$, such that $\dim V_i = i$, and $n = \dim V$.*

Corrollary 2.0.9. *Let L be a solvable Lie algebra, then there exists a chain of ideals of L such that $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$ with $\dim L_i = i$ and $\dim L = n$.*

Corrollary 2.0.10. *Let L be a solvable Lie algebra, then $x \in [LL] \Rightarrow ad_L x$ is nilpotent. In particular, $[LL]$ is nilpotent.*

Notice that $[LL] \subseteq L$ and we defined the notation $\mathrm{ad}_L x$ for subsets the other way around but here it is used to emphasize that we consider $\mathrm{ad}\, x : L \to L$ with $x \in [LL]$.

2.1 Jordan-Chevalley decomposition

In this subsection we do not need the assumption on the characteristic of F. We wish to introduce a very important decomposition, as the title suggests.

Definition 2.1.1 (Semisimple endormorphism)**.** *For finite dimensional vector spaces V, call an endomorphism $x \in$End(V) semisimple if the roots of its minimal polynomial over F are all distinct.*

Recall from linear algebra that $x \in$End(V) is semisimple if and only if x is diagonalizable, and that two commuting diagonalizable endomorphisms can be simultaneously diagonalized, so the sum and difference of semisimple endomorphisms are semisimple. Now for the main result

Proposition 2.1.2. *For a finite dimensional vector space V over F and an endomorphism $x \in$End(V) we have the following*

(a) There exists unique $x_s, x_n \in$End(V) such that $x = x_s + x_n$ where x_s is semisimple and x_n is nilpotent.
(b) There exist polynomials $p, q \in F[y]$ without constant term such that $x_s = p(x)$ and $x_t = q(x)$. In particular x_s and x_n commute with anything that commutes with x.
(c) If $A \subseteq B \subseteq V$ are subspaces, and $x(B) \subseteq A$ then also $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.

Given $x \in$End(V) we call the x_s and x_n from part (a) of the above proposition the Jordan decomposition of x. We have seen that $x \in \mathfrak{gl}(V)$ nilpotent implies that ad x is nilpotent as well, so its is natural to ask the same question for semisimplicity. This turns out to be the case, because for $x \in \mathfrak{gl}(V)$ semisimple choose a basis $(v_1, \ldots, v_n)$ of V such that x has diagonal matrix representation diag$(\alpha_1, \ldots, \alpha_n)$ in this basis, this is possible due to results from linear algebra. Denote by $\varepsilon_{(i,j)}$ the standard basis of $\mathfrak{gl}(V)$ relative to the $(v_1, \ldots, v_n)$ basis such that $\varepsilon_{(i,j)}(v_\ell) = \delta_{j,\ell} v_i$ and ad $x(\varepsilon_{(i,j)}) = (\alpha_i - \alpha_j)\varepsilon_{(i,j)}$ which shows that the matrix representation of ad x, on the n^2 dimensional $\mathfrak{gl}(V)$, is diagonalizable so semisimple as well.

Lemma 2.1.3. *The adjoint representation preserves Jordan decompositions, i.e. ad $(x_s + x_n) =$ ad $x_s +$ ad x_n for $x \in$End(V) and ad x_s, ad $x_n \in$End(End(V)).*

Lemma 2.1.4. *For a finite dimensional F-algebra $\mathfrak{U}$, the vector space Der $\mathfrak{U}$ contains the semisimple and nilpotent parts of all its elements.*

2.2 Cartan's Criterion

Cartan's Criterion proves very usefull when attempting to show that a Lie algebra is solvable. Notice that $[LL]$ nilpotent implies that L is solvable.
The above makes Corollary 2.0.10. a biimplication, L solvable if and only if $[LL]$ is nilpotent, applying Engels theorem we see that L is solvable if and only if $\text{ad}_{[LL]}x$ is nilpotent for all $x \in [LL]$, leading up to the following lemma using the trace map Tr.

Lemma 2.2.1. *For $A \subseteq B \subseteq \mathfrak{gl}(V)$ subspaces with V finite dimensional, define $M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}$. Now if $x \in M$ satisfies Tr$(xy) = 0$ for all $y \in M$, then x is nilpotent.*

Recall the very usefull property of the trace that $\text{Tr}(xy) = \text{Tr}(yx)$, similarly we see that

$$\begin{aligned}\text{Tr}([x,y]z) &= \text{Tr}(xyz - yxz) = \text{Tr}(xyz) - \text{Tr}(y(xz)) \\ &= \text{Tr}(xyz) - \text{Tr}(xzy) = \text{Tr}(xyz - xzy) \\ &= \text{Tr}(x[y,z])\end{aligned}$$

Theorem 2.2.2 (Cartan's criterion)**.** *For a linear lie algebra L over a finite dimensional vector space V, then $Tr(xy) = 0$ for all $x \in [LL]$ and $y \in L$ implies that L is solvable.*

Corrollary 2.2.3. *A lie algebra L with the property that Tr(ad x ad y)=0 for all $x \in [LL]$ and $y \in L$ is solvable.*

Proof: Apply Cartan's Criterion to ad L and apply the standard homomorphism theorem we get ad $L \cong L/\ker(\mathrm{ad})$, now proposition 3(a) implies that $L/\ker(\mathrm{ad})$ is solvable, and recall that $\ker(\mathrm{ad}) = Z(L)$ is a solvable ideal so proposition 3(b) implies that L itself is solvable. □

We will need a nondegenerate symmetric bilinear form later on, the one we will use is after Wilhelm Killing.

Definition 2.2.4 (The Killing form)**.** *The Killing form is defined to be the symmetric bilinear map $\kappa : L \times L \to F$ given by $\kappa : (x, y) \mapsto$ Tr(ad x ad y).*

Applying the property of the trace map we proved in section 4, and remembering that the adjoint representatioon is a homomorphism we see that

$$\begin{aligned}\kappa([xy], z) &= \mathrm{Tr}(\mathrm{ad}\ [xy]\ \mathrm{ad}\ z) = \mathrm{Tr}([\mathrm{ad}\ x, \mathrm{ad}\ y]\mathrm{ad}\ z)\\ &= \mathrm{Tr}(\mathrm{ad}\ x[\mathrm{ad}\ y, \mathrm{ad}\ z])\\ &= \kappa(x, [yz])\end{aligned}$$

Another very fortunate property of the Killing form is the following lemma.

Lemma 2.2.5. *For an ideal I of a Lie algebra L. Denote by κ respectively κ_I the Killing form on L respectively I, then $\kappa_I = \kappa\ |_{I\times I}$ the restriction of κ to $I \times I$.*

Recall that a symmetric bilinear form is nondegenerate if its radical $\{x \in L \mid \kappa(x, y) = 0 \forall y \in L\}$ is zero. It is clear that in our case the radical of κ is an ideal. Recall that we defined semisimple Lie algebras to have $\mathrm{Rad}(L) = 0$ not to be confused with the radical of κ.

Lemma 2.2.6. *A Lie algebra L is semisimple if and only if it has no nonzero abelian ideals.*

Proof: Assume first that L is semisimple, then it has $\mathrm{Rad}(L) = 0$, and by maximality and uniqueness $\mathrm{Rad}(L)$ contains all abelian ideals, so the only abelian ideal is zero.
For the other implication assume that L has no nonzero abelian ideals. Now $\mathrm{Rad}(L)$ is assumed to be the maximal solvable ideal, this means in particular that there is a minimal $n \in \mathbb{N}$ such that $\mathrm{Rad}(L)^{(n)} = [\mathrm{Rad}(L)^{(n-1)}, \mathrm{Rad}(L)^{(n-1)}] = 0$ but then $\mathrm{Rad}(L)^{(n-1)}$ is abelian, and $\mathrm{Rad}(L)^{(n-1)} = 0$ so to avoid contradiction n must be zero, and $\mathrm{Rad}(L)^{(0)} = \mathrm{Rad}(L) = 0$, as wanted. □

Theorem 2.2.7. *A Lie algebra L is semisimple if and only if its Killing form is nondegenerate.*

Proof: For the first implication let Rad $L = 0$, and let S be the radical of the killing form. We have by definition that Tr(ad x ad y) = 0 for all $x \in S$ and $y \in L$ then especially we have it for $x, y \in S$ now we can apply Cartan's Criterion to get that ad S is solvable, and then so is S.
For the other implication let $S = 0$. We employ a smart trick to show that every abelian ideal of L is included in S thus proving that L is semisimple. Let I be an abelian ideal of L and consider for $x \in I$ and $y \in L$ the map ad x ad $y : L \to I$ by the definition of ideal, next notice that (ad x ad $y)^2(z) = [x[y[x[yz]]]] \in [II] = 0$ so the map ad x ad y is nilpotent and has trace zero, so since $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y) = 0$ for all $x \in I$ and $y \in L$ this is precisely the criterion of the radical of the killing form, so $I \subseteq S$ □

Definition 2.2.8 (Direct sum). *Given a Lie algebra L, and a collection of ideals $I_1, I_2, \dots, I_n$ such that $L = I_1 + I_2 + \cdots + I_n$ as ideals and as a direct sum of vector subspaces we say that L is the direct sum of the I_i, and write $L = I_1 \oplus I_2 \oplus \cdots \oplus I_n$.*

Note that this works both ways, if we start out with a collection of Lie algebras sitting inside the same underlying vector space their sum as ideals and diect sum as vector spaces makes the collection a Lie algebra with componentwise bracket map.

Theorem 2.2.9 (Semisimple decomposition into simple ideals). *A semisimple Lie algebra L can be decomposed into simple ideals in the sense that there exists simple Lie algebras $L_1, L_2, \dots, L_n \subseteq L$ such that $L = L_1 \oplus \cdots \oplus L_n$. Furthermore every simple ideal of L is contained in the list $L_1, \dots L_n$.*

Corrollary 2.2.10. *If L is a semisimple Lie algebra then $L = [LL]$ and all ideals and homomorphic images of L are semisimple. Furthermore each ideal of L is a sum of certain simple ideals of L.*

2.3 Abstract Jordan decomposition

We wish to be able to talk about the Jordan decomposition of elements of semisimple L, not just the endomorphisms upon L, to do this we need the following theorem.

Theorem 2.3.1. *For a semisimple Lie algebras L, we have $ad\ L = Der\ L$.*

Recall Lemma 2.1.4. that Der L contains the semisimple and nilpotent parts of all its elements. This means that for semisimple L we can decompose the adjoint representation of $x \in L$ into $\text{ad } x = \text{ad } s + \text{ad } n$ where ad s is semisimple and ad n is nilpotent, the abstract Jordan decomposition is defined to be $x = s + n$. Recall that the kernel of the adjoint representation is an abelian ideal, and as we previously proved, abelian ideals in a semisimple Lie algebra are zero. This makes the abstract Jordan decomposition well defined and uniquely determined by x because the adjoint representation is injective.
Notice that if we are talking about a linear Lie algebra L the elements of L are themselves endomorphisms, so we could run into trouble with the notation $x = x_s + x_n$, however it turns out that there is no cause to be alarmed as the abstract and regular Jordan compositions will coinside in that case.

2.4 Modules

As is common in algebraic subjects the study of modules and of course representations proves to be a fruitful one.

Definition 2.4.1. *For a Lie algebra L, an L-module is a vector space W with a map $L \times W \to W$ denoted $(x, v) \mapsto x.v$ that satisfies the following conditions: for all $x, y \in L, v, w \in W$ and $a, b \in F$.*

(M1) $(ax + by).v = a(x.v) + b(y.v)$
(M2) $x.(av + bw) = a(x.v) + b(y.v)$
(M3) $[xy].v = x.y.v - y.x.v$

An L-submodule is a subspace that is closed within the module multiplication defined above.

Note that representations are in bijection with modules via the equation $x.v = \phi(x)(v)$ with $\phi : L \to \mathfrak{gl}(V)$, because if we start with a module the equation defines a representation, and if we start with a representation the equation defines a module.
We define a homomorphism of $L-$modules to be a map $\phi : W \to W'$ such that $\phi(x.w) = x.\phi(w)$ for all $x \in L$ and $w \in W$.
We call an L-module **irreducible** if the only proper submodule is the zero module, and we call it **completely reducible** if it is a direct sum of irreducible submodules. Notice that L is an L-module with the adjoint representation, and that the L-submodules are the ideals of L. In this context it is clear that the irreducible L-submodules are the simple ideals and the completely reducible L-submodules are the semisimple ideals. Especially a simple Lie algebra L is irreducible as an L-module, and a semisimple Lie algebra L is completely reducible as an L-module.

2.5 Casimir element

We can raise the level of abstraction in what we have previously seen, in order to get a nondegenerate symmetric bilinear form. Let L be a semisimple Lie algebra, and $\phi : L \to \mathfrak{gl}(V)$ a faithful(injective) representation. We define a symmetric bilinear form $\beta : L \times L \to F$ given by $\beta(x, y) = \text{Tr}(\phi(x)\phi(y))$ now recall $\text{Tr}(x[yz]) = \text{Tr}([xy]z)$ for $x, y, z \in \text{End}(V)$, and like before this implies that β is associative, and the radical of β is an ideal, say S. Now $\phi(S) = \phi|_S\ (S) \cong S/\ker(\phi|_S) = S$ by the standard homomorphism theorem, then clearly $S = \{x \in L \mid \beta(\phi(x), \phi(y)) = 0\ \forall y \in L\} = 0$ and β is nondegenerate.
Recall from linear algebra that given a basis $x_1, \ldots, x_n$ of L there exists a unique dual basis $y_1, \ldots, y_n$ such that $\beta(x_i, y_j) = \delta_{i,j}$. These bases are obviously closely related, for $x \in L$ write $[xx_i] = \sum_j a_{i,j}x_j$ and $[xy_i] = \sum_j a_{i,j}y_j$ then

$$\begin{aligned} a_{i,k} &= \sum_j a_{i,j}\beta(x_j, y_k) = \beta([xx_i], y_j) = \beta(-[x_ix], y_k) = \beta(x_i, -[xy_k]) = -\sum b_{k,j}\beta(x_k, y_j) \\ &= -b_{k,i} \end{aligned} \tag{2.5.1}$$

Definition 2.5.1 (Casimir element)**.** *Let L be a Lie algebra. For a faithful representation $\phi : L \to \mathfrak{gl}(V)$ and the nondegenerate bilinear trace form β we define the Casimir element*

$$C_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i) \in End\ (V)$$

Where $(x_i)_{i=1}^n$ is a basis of L and $(y_i)_{i=1}^n$ is the dual basis with respect to β.
Whenever the nondegenerate bilinear form is understood we will write $C_\phi(\beta) = C_\phi$.

The important application of this endomorphism is the following proposition.

Proposition 2.5.2. *The Casimir element commutes wiith the corresponding representation.*

Proof:

$$\begin{aligned}
[\phi(x), C_\phi] &= \sum_i [\phi(x), \phi(x_i)\phi(y_i)] \\
&= \sum_i \big([\phi(x), \phi(x_i)]\phi(y_i) + \phi(x_i)[\phi(x), \phi(y_i)]\big) \\
&= \sum_i \big(\phi([x, x_i]\phi(y_i) + \phi(x_i)\phi[x, y_i]\big) \\
&= \sum_i \Big(\sum_j \phi(a_{i,j}x_j)\phi(y_i) + \sum_j \phi(x_i)\phi(b_{i,j}y_j)\Big) \\
&\overset{(2.5.1)}{=} \sum_{i,j} \big(a_{i,j}\phi(x_j)\phi(y_i) - a_{i,j}\phi(x_j)\phi(y_i)\big) \\
&= 0 \qquad \square
\end{aligned}$$

Not surprisingly the trace of the Casimir element has the nice property that it is the dimension of ϕ viewed as a module:

$$\mathrm{Tr}(C_\phi(\beta)) = \sum_i \mathrm{Tr}(\phi(x_i)\phi(y_i)) = \sum_i \beta(x_i, y_i) = \sum_i 1 = \dim(L)$$

This implies, via Schurs lemma, page 26 of [1](Humphreys), that if ϕ is irreducible we have $C_\phi(\beta) = \dim(L)/\dim(V)$ as a scalar, simply because the matrix representation of $C_\phi(\beta)$ in this case is $a \cdot \mathrm{id}_V$ and $\mathrm{Tr}(a \cdot \mathrm{id}_V) = a{\cdot}\dim(V)$, so $a\dim(V) = \dim(L)$, as wanted. Notice that in this setting the Casimir element is independant of choice of basis.
In case ϕ is unfaithful, not faithful, we can still define the Casimir element. Decompose via theorem 2.2.9 as follows $L = L_1 \oplus \cdots \oplus L_n$, pick out the simple ideals that are in the kernel of ϕ, say without loss of generality $L_1 \oplus \cdots \oplus L_m = \ker(\phi)$ with $m \leq n$, then we can define the Casimir element of ϕ restricted to $L_{m+1} \oplus \cdots \oplus L_n$, and this incarnation of the Casimir element still commutes with ϕ.

2.6 Weyls Theorem

Lemma 2.6.1. *Representations of semisimple Lie algebras have their images contained in* $\mathfrak{sl}(V)$

Proof: Recall that $\mathfrak{sl}(V) \subseteq \mathfrak{gl}(V)$ is the set of traceless endomorphisms. To start out we see that $[\mathfrak{gl}(V), \mathfrak{gl}(V)] \subset \mathfrak{sl}(V)$, because for $a, b \in \mathfrak{gl}(V)$ we have

$$\mathrm{Tr}([ab]) = \mathrm{Tr}(ab - ba) = \mathrm{Tr}(ab) - \mathrm{Tr}(ba) = 0$$

Now corollary 2.2.10 states that $L = [LL]$ so $\phi(L) = \phi([LL]) = [\phi(L)\phi(L)] \subseteq [\mathfrak{gl}(V)\mathfrak{gl}(V)] \subset \mathfrak{sl}(V)$, as wanted. $\square$

Theorem 2.6.2 (Weyl)**.** *Finite dimensional representations of semisimple Lie algebras are completely reducible.*

Theorem 2.6.3 (Weyl,Jantzen)**.** *All finite dimensional L-modules are completely reducible if L is semisimple and the underlying field has characteristic zero and is algebraically closed.*

The two versions of the theorem are obviously equivalent, we aim to prove the second.
Proof: Let V be a finite dimensional L-module and $\phi : L \to \mathfrak{gl}(V)$ the associated representation. Let $W \subset V$ be an L-submodule, we aim to find a complement of W i.e. a $W' \subset V$ such that $V = W \oplus W'$.
Special case 1: $\dim V = \dim W + 1$ and $\phi(x)(V) \subseteq W$ for all $x \in L$
This assumption is clearly equivivalent to $\dim V/W = 1$ since the representation $\psi : L \to V/W$ is given by $\psi(x)\big(v + W\big) = 0 + W \iff \psi(x)(v) \in W$.
Now we proceed by induction on $\dim V$. The goal is to find a v such that $W \cap Fv = 0$ and $V = W \oplus Fv$, because then induction tells us that W is completely reducible, implying that $W \oplus Fv$ is completely reducible aswell.
Special case 1.1: W is trivial in the sense that $\phi(x)(w) = 0$ for all $x \in L$ and $w \in W$.
This implies that $\phi(x)(\phi(y)(w) = 0$ for all $x, y \in L$ and $w \in W$ due to the assumption $\phi(x)(v) \in W$ for all $v \in V$. Then clearly $\phi([x, y]) = 0$ for all x, y and especially $0 = \phi([L, L]) = \phi(L)$ by semisimplicity $L = [LL]$, so any $v \in V \setminus W$ works.
Special case 1.2: W is irreducible but non trivial, i.e. $\phi(L) \neq 0$.
Finally we roll out the Casimir element C_ϕ. Recall that C_ϕ is defined to be a linear combination of terms defined by ϕ, so $C_\phi(V) \subset W$ which implies that $C_\phi \mid_W \in \text{End}\,_L W$. The assumption that W is irreducible then gives that $C_\phi \mid_W (W) = 0$ or $C_\phi \mid_W (W) = W$, in the first case $C_\phi \mid_W (w) = 0$ we get $C_\phi(C_\phi(V)) \subset C_\phi(W) = 0$ so C_ϕ is nilpotent and then has trace 0, which is in contradiction with $\text{Tr}(C_\phi) = \dim\phi(V) \neq 0$ by non triviality assumption. So $C_\phi \mid_W (W) = W$ and then naturally also $C_\phi(V) = W$, now a simple dimension calculation ensures $\dim\ker C_\phi = 1$. Furthermore $C_\phi \mid_W$ is a bijection because it is surjective endomorphism. All in all $\ker C_\phi \cap W = 0$ and $V = W \oplus \ker C_\phi$ with $\ker C_\phi$ one dimensional as wanted.
Special case 1.3: W nontrivial and not irreducible.
Take U a nontrivial proper submodule of W, this is possible because if it is not W is irreducible. In the quotients $0 \neq W/U \subsetneq V/U$ we define $\overline{\phi} : L \to \mathfrak{gl}(V/U)$ by $\overline{\phi}(x) : (v+U) \mapsto (\phi(x)(v) + U)$, clearly $\overline{\phi}(x)(V/U) \subset W/U$ because $\phi(x)(V) \subseteq W$. Looking at dimensions we see that

$$\dim W/U = \dim W - \dim U = \dim V - 1 - \dim U = \dim V/U - 1$$

Which allows us to use induction on $W/U \subsetneq V/U$ and get an element $z \in (V/U) \setminus (W/U)$ such that $\overline{\phi}(x)(z) = 0$ for all $x \in L$. Take a representative v of z, then $v \notin W$ and $\overline{\phi}(x)(z) = \phi(x)(v) + U = 0 + U$ so $\phi(x)(v) \in U$ for all $x \in L$. Define $U' = U + Fv$ then by the previous considerations we have $\phi(x)(U') \subseteq U$ and $\dim U' = \dim U + 1 < \dim W + 1 = \dim V$ so we can apply induction and the special case to get $v' \in U' \setminus U$ such that $\phi(x)(v') = 0$ for all $x \in L$ and $U' = U \oplus Fv'$, now this very v' will do the job in the bigger setting in the sense that $V = W \oplus Fv'$ because $v' \notin W$, since if it is we get $U' = U \oplus Fv' \subseteq W$ but we saw $v \in U \setminus W$, a contradiction.
General case: For arbitrary $0 \neq W \subsetneq V$.
Note first that if we cannot find such a W then V is irreducible which is a special case of completely reducible. Now take the representation $\psi : L \to \mathfrak{gl}(\text{Hom}_F(V, W))$ by $\psi(x)(f) = \phi(x) \circ f - f \circ \phi(x)$ for $x \in L$ and $f \in \text{Hom}_F(V, W)$ and $\phi : L \to \mathfrak{gl}(V)$ the representation of V. Define

$$U = \{f \in \text{Hom}_F(V, W) : \exists \alpha \in F : f \mid_W = \alpha_f id_W\}$$
$$U' = \{f \in U : f \mid_W = 0\}$$

Note $U' = \ker(\zeta)$ where $\zeta : U \to F$ given by $\zeta(f) = \alpha_f$ and $\dim U = \dim U' + 1$. Now for $f \in U, w \in W$,

$$\psi(x)(f)(w) = \phi(x)(f(w)) - f(\phi(x)(w) = \alpha\phi(x)(w) - \alpha\phi(x)(w) = 0$$

This means that $\psi(L)(U) \subseteq U'$ satisfies the conditions of our special case and we get a $f \in U \setminus U'$ such that $\psi(x)(f)$ is the zero map for all $x \in L$ which means that f commutes with $\phi(x)$ for all $x \in L$ and as such $f \in \mathrm{Hom}_L(V, W)$ making $\ker f$ an L module, we wish to use this module as the compliment to W. To see that note that $\alpha_f \neq 0$ due to $f \in U \setminus U'$, then $\ker f = 0$ implying naturally that $W \cap \ker f = 0$ and then $f \mid_W$ is an injective endomorphism so $f(V) = W$ implying finally $V = W \oplus \ker f$. $\square$

An important application of Weyls Theorem is the preservation of the Jordan decomposition

Theorem 2.6.4. *Semisimple finite dimensional linear Lie algebras contain the semisimple and nilpotent parts of their elements, furthermore the abstract and usual Jordan decompositions coincide.*

Corrollary 2.6.5. *For semisimple Lie algebras L and finite dimensional representations $\phi : L \to \mathfrak{gl}(V)$, with V finite dimensional. If $x = s + n$ is the abstract Jordan decomposition of $x \in L$, then $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$.*

Chapter 3

Traceless endomorphisms

A central example of Lie algebras are the traceless endomorphisms $\mathfrak{sl}(n, F) = \mathfrak{sl}(V) \subsetneq \mathfrak{gl}(V)$ with $n = \dim V \geq 2$. Obviously the basis of $\mathfrak{sl}(n, F)$ consists of the matrices $\varepsilon_{i,j}$ for $i \neq j$ and $h_i = \varepsilon_{i,i} - \varepsilon_{i+1,i+1}$ for $1 \leq i < n$ giving $\dim \mathfrak{sl}(n, F) = (n-1) + n^2 - n = n^2 - 1$, notice that this is the vector space dimension, and not necessarily a Lie algebra dimension.
Viewed as a Lie algebra, and later associated to a root system we say these are of type A_{n-1}.

3.1 $\mathfrak{sl}(2)$

For a field F of characteristic > 2 write $\mathfrak{sl}(2) = \mathfrak{sl}(3, F)$ for the root system of type A_2. Now we wish to see that $\mathfrak{sl}(2)$ is simple. We have the basis

$$\begin{aligned} x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{3.1.1}$$

With commutators $[xy] = h,\ [hx] = 2x$ and $[hy] = -2y$, especially x, y and h are eigenvectors of ad h with eigenvalues $2, -2, 0$ respectively. Let $I \subset \mathfrak{sl}(2)$ be an ideal, and $i = ax + by + ch \in I$ by definition I is closed under brackets but $(\text{ad } x)^2(i) = [x[xi]] = [x, (0 + bh - 2cx)] = -2bx$ and likewise $(\text{ad } y)^2(i) = [y[yi]] = -2ay$ so if $b \neq 0$ then $x \in I$ and then also $[xy] = h \in I$ aswell as $[yh] = 2y$ so $I = \mathfrak{sl}(2)$ similarly if $a \neq 0$. Now if $b = a = 0$ we have $ch \in I$ and $[xh], [yh] \in I$ so once again $I = L$. So $\mathfrak{sl}(2)$ is simple, and is generated as Lie algebra by a single element, any one of x, y, h.

3.2 Representations of $\mathfrak{sl}(2)$

We start out by defining weights and weight spaces

Definition 3.2.1. *For a finite dimensional $\mathfrak{sl}(2)$-module V, write $V = \bigoplus_{\lambda \in F} V_\lambda$ where $V_\lambda = \{v \in V \mid \rho(h)(v) = \lambda v\}$ are the eigenspacces of $\rho(h)$. Whenever $V_\lambda \neq 0$ we call λ a **weight** and V_λ a **weight space**.*

Lemma 3.2.2. *If $v \in V_\lambda$ then $\rho(x)(v) \in V_{\lambda+2}$ and $\rho(y)(v) \in V_{\lambda-2}$*

Now since V is finite dimensional there can only be finitely many weights, so the lemma implies that there must be λ such that $V_\lambda \neq 0$ and $V_{\lambda+2} = 0$ such that $\rho(x)(v) = 0$ for all $v \in V_\lambda$, we call these maximal vectors of weight λ.

Lemma 3.2.3. *Define for an irreducible $\mathfrak{sl}(2)$ module V a sequence given by $v_0 \in V_\lambda$ maximal, $v_{-1} = 0$ and $v_i = \frac{1}{i!}\rho(y)^i(v_0)$.*

(a) $\rho(h)(v_i) = (\lambda - 2i)v_i$
(b) $\rho(y)(v_i) = (i+1)v_{i+1}$
(c) $\rho(x)(v_i) = (\lambda - i + 1)v_{i-1}$

This lemma tells us that $\{v_0, v_1, \ldots, v_m\}$ is a basis, where m is minimal such that $v_m \neq 0$ and $v_{m+1} = 0$, this m exists because V is finite dimensional. To see that the v_i are indeed a basis, note that they are all different eigenvectors so linearly independent, they span V because the subset of V spanned by the v_i is an $\mathfrak{sl}(2)-$module so by irreducibility of V it must be the entire thing. On this basis it is clear that x respectively y act as strict upper respectively lower triangular matricies by the above lemma, and h acts diagonally. Another neat application is that if we apply (c) on $i = m+1$ we see that $0 = \rho(x)(v_{m+1}) = (\lambda - (m+1)+1)v_m = (\lambda - m)v_m \Rightarrow \lambda = m$ so the weight of a maximal vector is simply $\dim V - 1 = m$, futhermore (a) implies that for $\lambda = m - 2i$ with $0 \leq i \leq m$ we have $\dim V_\lambda = 1$ with basis v_i, which is summarised in the following theorem.

Theorem 3.2.4. *For irreducible $\mathfrak{sl}(2)$ modules V we have*

(a) $V = \bigoplus_{i=0}^m V_{m-2i}$ *where V_{m-2i} are weight spaces,* $\dim V_{m-2i} = 1$ *and* $m = \dim V - 1$
(b) V has a unique maximal vector of weight m
(c) The action on V is given explicitely by lemma 3.2.3 and as such there is only a single irreducible $\mathfrak{sl}(2)$ module of each dimension ≥ 1.

Chapter 4

Relation to root systems

Now we are finally ready to introduce the relation between semisimple Lie algebras and root systems. In order to avoid self-plagiarism we will presume that the reader, like the author, has knowledge of root systems, and skip many of the proofs. To determine the relation between semisimple Lie algebras and root systems we need maximal toral subalgebras.

Definition 4.0.5 (Toral subalgebra)**.** *A subalgebra $T \subseteq L$ consisting of semisimple elements is called toral. A maximal toral subalgebra H is a toral subalgebra that is maximal with regards to inclusion.*

Lemma 4.0.6. *Toral subalgebras are abelian*

Due to this lemma all the endomorphisms in $\mathrm{ad}_L H$ are pairwise simultaneously diagonalizable, because they commute and F is algebraically closed. Which means that $L = \bigoplus_{\alpha \in H^*} L_\alpha$ where $L_\alpha = \{x \in L \mid [hx] = \alpha(h)x \ \forall h \in H\}$. Now take $\Phi = \{\alpha \in H^* \setminus \{0\} \mid L_\alpha \neq 0\}$, notice that $L_0 = C_L(H)$ the centralizer, but we do not wish to allow 0 to be in our root system so we exclude this case and call the elements of Φ **roots**. We now have a decomposition $L = C_L(H) \oplus \bigoplus_{\alpha \in H^*} L_\alpha$ called the **root space decomposition** of L. The main goal is to show that Φ is a root system and that it alone characterises a unique semisimple Lie algebra.
To start off we want to be able to identify elements of H with roots, elements of H^* the dual vector space, to do that we need the following four results.

Proposition 4.0.7.

- *For all $\alpha, \beta \in H^*$ we have $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$*
- *For all $x \in L_\alpha$ for $\alpha \neq 0$ we have ad x is nilpotent*
- *For all $\alpha, \beta \in H^*, \alpha + \beta \neq 0$ we have $L_\alpha \perp L_\beta$ with regards to the Killing form*

Corrollary 4.0.8. *$\kappa \mid_{L_0}$ is nondegenerate*

Proof: Theorem 2.2.7 states that L is semisimple if and only if its Killing form is nondegenerate, now the proposition implies that $L_0 \perp L_\alpha$ for all $\alpha \in \Phi$, i.e. for $z \in L_0$ and $x \in L_\alpha \neq L_0$ we have $\kappa(z, x) = 0$ so if $\kappa(z, y) = 0$ for $y \in L_0$ then the nondegeneracy of κ on the entirety of L implies that $z = 0$ $\square$

Proposition 4.0.9. *Maximal toral subalgebras are self centralizing.*

Corrollary 4.0.10. *$\kappa \mid_H$ is nondegenerate*

As promised we identify $\phi \in H^*$ with $t_\phi \in H$ defined uniquely by $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$ and more importantly we identify $\Phi \cong \{t_\alpha \mid \alpha \in \Phi\} \subset H$.
Moving on we wish to show that Φ is indeed a root system

4.1 Root systems

Definition 4.1.1. *For an euclidean space E over F with inner product $(\cdot,\cdot)$, a finite set $\Phi \subset E$ is called a root system in E if*

(R1) Φ *is finite, span*$(\Phi) = E$, *and* $0 \notin \Phi$
(R2) If $\alpha \in \Phi$, *then* $F\alpha \cap \Phi = \{\pm\alpha\}$
(R3) If $\alpha \in \Phi$, *the reflection* $\sigma_\alpha(\beta \mapsto \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha)$
(R4) If $\alpha, \beta \in \Phi$, *then* $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

It is also interesting to study root systems without (R4), in which case we call root systems satisfying (R4) Crystallographic. Root systems without (R4) are used in the context of Coxeter groups, in the titles Combinatorics of Coxeter groups by Anders Björner & Francesco Brenti and Reflection Groups and Coxeter Groups by Humphreys. We will go into minimal detail regarding root systems in this paper, but the interested reader can explore the subject further in these works.
We saw in proposition 4.0.7 that $\kappa(L_\alpha, L_\beta) = 0$ for $\alpha, \beta \in H^*$ such that $\alpha + \beta \neq 0$, so also $\kappa(H, L_\alpha) = 0$ for all $\alpha \in \Phi$.
Now to see that our Φ is indeed a root system we mention the following propositions without proof.

Proposition 4.1.2. *For* $\alpha \in \Phi$

(a) Φ *spans* H^*
(b) $-\alpha \in \Phi$
(c) For $x \in L_\alpha$, $y \in L_\beta$ *we have* $[xy] = \kappa(x,y)t_\alpha$
(d) $[L_\alpha, L_{-\alpha}]$ *is one dimensional with basis* t_α
(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$
(f) let $x_\alpha \in L_\alpha \setminus \{0\}$ *then there exists* $y_\alpha \in L_{-\alpha}$ *such that* $x_\alpha, y_\alpha, h_\alpha = [x_\alpha y_\alpha]$ *span a three dimensional subalgebra isomorphic to* $\mathfrak{sl}(2)$ *in the obvious way.*
(g) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ *and* $h_\alpha = -h_{-\alpha}$

Proposition 4.1.3. *For* $\alpha, \beta \in \Phi$

(a) $\dim L_\alpha = 1$, *and* $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ *with* $H_\alpha = [L_\alpha, L_{-\alpha}]$
(b) For all $x_\alpha \in L_\alpha$ *there exists a unique* $y_\alpha \in L_{-\alpha}$ *such that* $[x_\alpha y_\alpha] = h_\alpha$
(c) The only scalar multiples of α *that is in* Φ *are* $\pm\alpha$.
(d) $\beta(h_\alpha) \in \mathbb{Z}$ *and* $\beta - \beta(h_\alpha) \in \Phi$.
(e) If $\alpha + \beta \in \Phi$ *then* $[L_\alpha L_\beta] = L_{\alpha+\beta}$.
(f) If $\beta \neq \pm\alpha$, *let* r, p *be the largest respectively smallest integers such that* $\beta - r\alpha \in \Phi$ *and* $\beta + p\alpha \in \Phi$. *Then all the roots inbetween are roots aswell, i.e.* $\beta + i\alpha \in \Phi$ *for* $-r \leq i \leq p$ *and* $\beta(h_\alpha) = r - q$.
(g) L *is generated as a Lie algebra by the root spaces* L_α.

We wish to define $E_\mathbb{Q} = \operatorname{span}_\mathbb{Q}(\Phi) = H^*$ and $E = \mathbb{R} \otimes_\mathbb{Q} E_\mathbb{Q}$ as an euclidean space. E is an euclidean space because the nondegeneracy of the Killing form allows us to define a nondegenerate bilinear positive definit form on H^* by $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$. Indeed we have seen that Φ spans H so take a basis $\alpha_1, \ldots, \alpha_\ell \in \Phi$ so for all $\Phi \ni \beta = \sum_{i=1}^{\ell} c_{\beta,i}\alpha_i$ with $c_{\beta,i} \in F$, with a bit of linear algebra one can show that $c_{\beta,i} \in \mathbb{Q}$, and that $\dim_\mathbb{Q}(E_\mathbb{Q}) = \ell = \dim_F H^*$. Consider next for $\lambda, \mu \in H^*$

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha\in\Phi} (\alpha, \lambda)(\alpha, \mu) \tag{4.1.1}$$

And especially for $\beta \in \Phi$ we have $(\beta,\beta) = \sum_{\alpha\in\Phi}(\alpha,\beta)^2$ now divide by $(\beta,\beta)^2 \neq 0$ and get $\frac{1}{(\beta,\beta)} = \sum_{\alpha\in\Phi}\left(\frac{(\alpha,\beta)}{(\beta,\beta)}\right)^2 \in \mathbb{Q}$ due to proposition 4.1.3(d) so $(\beta,\beta) \in \mathbb{Q}$ so clearly also $(\alpha,\beta) \in \mathbb{Q}$ so based on (4.1.1) we see that also $(\alpha,\beta) \in \mathbb{Q}$ so by bilinearity we get a nondegenerate positive definit form on $E_{\mathbb{Q}}$ and it extends cannonically to E, as claimed.

Theorem 4.1.4. *For a semisimple Lie algebras L with maximal toral subalgebra H, root system Φ and Euclidean space E then Φ is a root system in the sense of definition 4.1.1.*

4.2 Isomorphism theorem 1

In this section we wish to prove the first part of the main goal of this paper, namely to show that semisimple Lie algebras are completely determined by their root systems, here we make the first step and show that if two semisimple Lie algebras have isomorphic root systems then the algebras them selves are also isomorphic.

Definition 4.2.1. *A root system is called irreducible if it cannot be partitioned into the union of two proper subsets orthogonal to each other.*

Proposition 4.2.2. *Any root system that arises from a simple Lie algebra is irreducible.*

With this proposition and theorem 2.2.9 we can reduce the problem to looking at simple Lie algebras and irreducible root systems

Corrollary 4.2.3. *A semisimple Lie algebra L with maximal toral subalgebra H and root system Φ has decompositions*

- $L = \bigoplus_i L_i$ *with L_i simple, this is just a repeat of theorem 2.2.9*
- $H = \bigoplus_i H_i$ *where $H_i = H \cap L_i$*
- $\Phi = \bigcup_i \Phi_i$ *where Φ_i is the irreducible root system in L_i relative to H_i*

Proof: write $L = \bigoplus_i L_i$ with L_i simple, take a maximal toral subalgebra H in L and decompose it in the following sense $H = H \cap \bigoplus_i L_i =: \bigoplus_i H_i$ where $H_i = L_i \cap H$ is maximal toral in L_i because if it is not it is included in some toral $H_i' \subseteq L_i$ and then we form a bigger toral algebra $\bigoplus_i H_i \oplus H_i' \supsetneq H$ which is a contradiction with the maximality of H. Denote then by Φ_i the root systems in the smaller Lie algebras, but we can extend them in an obvious way. Define for $\alpha \in \Phi_i$ and $h \in H \setminus H_i$ set $\alpha(h) = 0$ so $\alpha \in \Phi$ with $L_\alpha \subseteq L_i$, and if we take $\alpha \in \Phi$ then there is a H_i such that $[H_i L_\alpha] \neq 0$ because if not then $[H L_\alpha] = 0$ and then $L_\alpha \subset H$ which is in contradiction with α being a root. So $L_\alpha \subset L_i$ and $\alpha\mid_{H_i}$ is a root of L_i relative to H_i. $\square$

Definition 4.2.4. *For a semisimple Lie algebra L, and a maximal toral subalgebra H with root system Φ and euclidean space E write (L,H,Φ,E)*

Proposition 4.2.5. *For a quadruple (L,H,Φ,E), fix a basis Δ of Φ, then L is generated, as a Lie algebra, by L_α and $L_{-\alpha}$ for $\alpha \in \Delta$*

Since the L_α are one dimensional the above proposition is equivivalent to saying that L is generated as a Lie algebra by any choice of $x_\alpha \in L_\alpha$ and $y_\alpha \in L_{-\alpha}$.

Theorem 4.2.6. *Given two quadruples (L,H,Φ,E) and (L',H',Φ',E'), with L, L' simple, and an isomorphism $\psi : \Phi \to \Phi'$ there exists an isomorphism $\tilde{\pi} : L \to L'$.*

Proof: The way we do this is to denote $\psi(\alpha) = \alpha'$ and then we have $\psi(\Delta) = \{\alpha' \mid \alpha \in \Delta\}$. If we then choose $x_\alpha \in L_\alpha \setminus \{0\}$ and $x'_\alpha \in L'_{\alpha'} \setminus \{0\}$ we get an isomorphism $\pi_\alpha : L_\alpha \tilde{\to} L'_{\alpha'}$ extend firstly to an isomorphism $\pi : H \tilde{\to} H'$ and finally $\tilde{\pi} : L \tilde{\to} L'$. In this setup the isomorphism is unique, for each set of choices of x_α and x'_α, because we have seen that L is generated by the x_α and y_α for $\alpha \in \Delta$ and that each x_α uniquely determines a y_α such that $[x_\alpha y_\alpha] = h_\alpha$, securing uniqueness.
To prove existence we aim to construct a set $D \subseteq L \oplus L'$ resembling the diagonal $D_L = \{(x, x) \mid x \in L\} \subseteq L \oplus L$ and then get that $L \cong L'$ via the projections. To construct D define $\overline{x_\alpha} = (x_\alpha, x'_\alpha)$, $\overline{y_\alpha} = (y_\alpha, y'_\alpha)$ and $\overline{h_\alpha} = (h_\alpha, h'_\alpha)$, and let D be generated by these. To see that D does in fact resemble the diagonal, recall proposition 4.2.2 that Φ and Φ' are irreducible, it is then a fact of root systems that they have a unique maximal root, say β and β', naturally $\psi(\beta) = \beta'$. Take $x \in L_\beta$ and $x' \in L'_{\beta'}$, and set $\overline{x} = (x, x')$. Define M to be the subspace of $L \oplus L'$ generated by all

$$\text{ad } \overline{y_{\alpha_1}} \cdots \text{ad } \overline{y_{\alpha_m}}(\overline{x}) \tag{4.2.1}$$

For $\alpha_i \in \Delta$. Note that these are elements of $L_{\beta - \sum_{i=1}^m \alpha_i} \oplus L'_{\beta' - \sum_{i=1}^m \alpha'_i}$, and then especially $\dim(M \cap L_\beta \oplus L'_{\beta'}) = 1$ because of the maximality of β in the partial ordering of the roots. This means that $M \subsetneq L \oplus L'$ as $L_\beta \oplus L'_{\beta'}$ is two dimensional.
Claim: D stabilizes M
By definition it is clear that ad $\overline{y_\alpha}$ stabilizes M for $\alpha \in \Delta$ and since $[h_\alpha y_\alpha] = -2y_\alpha$ so does $\overline{h_\alpha}$ for $\alpha \in \Delta$. For $\alpha_j \in \Delta$ just note that

$$\begin{aligned}
&\text{ad } \overline{x_{\alpha_j}} \text{ad } \overline{y_{\alpha_1}} \cdots \text{ad } \overline{y_{\alpha_m}}(\overline{x}) \\
&= \begin{cases} \text{ad } \overline{y_{\alpha_1}} \cdots \text{ad } \overline{y_{\alpha_m}} \text{ad } \overline{x_{\alpha_j}}(\overline{x}) = 0 & \text{if } \alpha_j \neq \alpha_i \text{ for all } 1 \leq i \leq m \\ \text{ad } \overline{y_{\alpha_1}} \cdots (\text{ad } \overline{h_{\alpha_j}} + \text{ad } \overline{y_{\alpha_j}} \text{ad } \overline{x_{\alpha_j}}) \cdots \text{ad } \overline{y_{\alpha_m}}(\overline{x}) & \text{if } \alpha_j = \alpha_i \text{ for some } 1 \leq i \leq m \end{cases}
\end{aligned}$$

In the first case we get a zero because $\beta + \alpha_j$ can never be a root by maximality in the root poset. For the second case we apply the Jacobi identity and we already saw that h_{α_j} stabilizes so the first term in the sum has already been taken care of, and the second term follows by repeated application, proving the claim.
With the claim we have forced D to be a proper subalgebra because if it is not M would be a proper nonzero ideal in contradiction with simplicity of L and L'.
The final step of the proof is to show that the projections of D onto its factors are isomorphisms. They are clearly surjective homomorphisms, to see that they are also injective suppose for contradiction that $D \cap L = D \cap \ker p_2 \neq 0$ where $p_2 : L \oplus L' \to L'$ is the projection. Then we have $(\omega, 0) \in D$ for some $\omega \in L \setminus \{0\}$, denote $\overline{z_{\alpha_i}} = \overline{x_{\alpha_i}}$ or $\overline{y_{\alpha_i}}$ then also $\overline{z_{\alpha_1}} \cdots \overline{z_{\alpha_m}}(\omega, 0) = (z_{\alpha_1} \cdots z_{\alpha_m}(\omega), 0) \in D$ for $\alpha_i \in \Delta$ the $z_{\alpha_1} \cdots z_{\alpha_m}(\omega)$ form a non zero ideal in L so is equal to L by simplicity, then $L \hookrightarrow D$ and in similar fashion $L' \hookrightarrow D$ but that contradicts $D \subsetneq L \oplus L'$.
In conclusion we have $L \cong D \cong L'$ and $L \cong L'$ as wanted. □

The above theorem clearly generalises to semisimple Lie algebras aswell. When we introduce free Lie algebras and how to construct Lie algebras from root systems proving this statement becomes almost trivial, because we can define the Lie algebras solely based upon the root system, so obtaining an isomorphism becomes a simple matter.

To remove the choice of maximal toral subalgebra from the equation we introduce the **Cartan subalgebra** (CSA) which is a Lie subalgebra that is equal to its normalizer and is nilpotent. Certainly maximal toral subalgebras are CSA's, and it turns out that if L is semisimple then CSA's are maximal toral. Humphreys proves in chapters 15 and 16 that all maximal toral subalgebras are conjugate under $\mathrm{Aut}(L)$ which implies that if we choose different maximal toral subalgebras we get isomorphic root systems.

4.3 Free Lie algebras

In this section it is the goal to define the notion of a free Lie algebra defined via generators and relations.

Definition 4.3.1 (Tensor algebra)**.** *For any field F and a vector space V above it define a sequence*

$$T^0(V) = F, T^1(V) = V, T^2(V) = V \otimes V, \ldots, T^m(V) = \bigotimes_{i=1}^{m} V$$

The tensor algebra is $\mathfrak{T}(V) := \bigsqcup_{i=0}^{\infty} T^i(V)$ *with product*
$(v_1 \otimes \cdots \otimes v_k)(v_1' \otimes \cdots \otimes v_\ell') := (v_1 \otimes \cdots \otimes v_k \otimes v_1' \otimes \cdots \otimes v_\ell')$

The tensor algebra has the universal property that given any associative algebra $\mathfrak{B}$ with 1 over F and any F-linear map $\phi : V \mapsto \mathfrak{B}$ there exists a unique homomorphism of F-algebras $\psi : \mathfrak{T}(V) \mapsto \mathfrak{B}$ such the following diagram commutes
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "\phi"] & \mathfrak{T}(V) \arrow[d, "\psi"] \\ & \mathfrak{B} \end{tikzcd}$$

Definition 4.3.2 (Universal enveloping algebra)**.** *For a Lie algebra L we define the universal enveloping algebra of L to be a pair* $(\mathfrak{U},i)$ *where* $\mathfrak{U}$ *is an associative algebra with 1 over F and* $i : L \to \mathfrak{U}$ *is a linear function such that*

$$i([xy]) = i(x)i(y) - i(y)i(x) \tag{4.3.1}$$

and for all other universal enveloping algebras $(\mathfrak{A}, j)$ *there exists a unique* $\phi : \mathfrak{U} \to \mathfrak{A}$ *such that* $\phi(1_{\mathfrak{U}}) = 1_{\mathfrak{A}}$ *and* $\phi \circ i = j$

It is not at all obvious that a the universal enveloping algebra exists

Proposition 4.3.3. *For all Lie algebras L there exists a unique enveloping algebra.*

Proof of existence: Unsurprisingly we use the tensor algebra $\mathfrak{T}(L)$ on L, next let J be the two sided ideal generated by all $x \otimes y - y \otimes x - [xy]$ for $x, y \in L$. Now define $\mathfrak{U}(L) = \mathfrak{T}(L)/J$ with quptient map $\pi : \mathfrak{T}(L) \to \mathfrak{T}(L)/J$. It is clear that there are no scalars in J, it is also clear from the construction that we have a copy (up to isomorphism) of L in $\mathfrak{U}(L)$ which makes the termonology 'enveloping algebra' meaningful. Write $l : L \to \mathfrak{T}(L)$ for the inclusion. Now to see tha $\mathfrak{U}(L)$ has the desired property let $(\mathfrak{B}, j)$ be a pair satisfying definition 4.3.2, we wish to find $\phi : \mathfrak{U}(L) \to \mathfrak{B}$ such that $\phi \circ i = j$. The universal property of $\mathfrak{T}(L)$ gives us a ψ such that the above diagram commutes. Consider next the following diagram

$$\begin{tikzcd} L \arrow[r, "l"] \arrow[dr, "j"'] & \mathfrak{T}(L) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd}
& & \mathfrak{U}(L) \arrow[dd, dashed, "\phi"] \\
L \arrow[r, "l"] \arrow[urr, bend left, "i"] \arrow[drr, bend right, "j"'] & \mathfrak{T}(L) \arrow[ur, "\pi"] \arrow[dr, "\psi"'] & \\
& & \mathfrak{B}
\end{tikzcd}$$

If we can show that $x \otimes y - y \otimes x - [xy] \in \ker \psi$ for all x and y then $\phi(x + J) = \psi(x)$ is well defined and works as intended by commutativity of the topmost and bottommost pockets of the diagram. So to see that we have these elements in the kernel consider $j([xy]) = j(x) \cdot j(y) - j(y) \cdot j(x)$ while $\psi(x \otimes y - y \otimes x) = \psi(x) \cdot \psi(y) - \psi(y) \cdot \psi(x)$ using that ψ is an F-Algebra homomorphism, here $\cdot$ denotes the composition in $\mathfrak{B}$. Commutativity now implies that $\psi(l(x)) = j(x)$ so in turn $\psi(x \otimes y - y \otimes x) = j([xy])$, and then naturally $x \otimes y - y \otimes x - [xy] \in \ker \psi$ as wanted.
Proof of uniqueness: This construction immediately implies uniqueness, consider another pair $(\mathfrak{B}, j)$ satisfying the conditions of definition 4.3.2, then we get homomorphisms $\phi : \mathfrak{U} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{U}$

$$\begin{tikzcd}
& \mathfrak{U} \arrow[dd, dashed, "f"] \\
L \arrow[ur, "i"] \arrow[dr, "i"'] & \\
& \mathfrak{U}
\end{tikzcd}$$

There is a unique homomorphism f, clearly the identity $id_{\mathfrak{U}}$ does the trick, but so does $\psi \circ \phi$ and a similar diagram ensures that $\phi \circ \psi = id_{\mathfrak{B}}$, making ψ and ϕ mutually inverse isomorphisms. $\square$

Definition 4.3.4. *A Lie algebra L over F generated by a set X that has the following property is called **free**: For all Lie algebras M and maps $\phi : X \to M$ there exists a unique homomorphism $\psi : L \to M$ such that $\psi|_X = \phi$.*

Proposition 4.3.5. *Given a set X there exists a free Lie algebra L over F generated by X and it is unique up to isomorphism.*

Proof of uniqueness: This is very simple, given L and L' free Lie algebras generated by X, then the identity map $\phi : X \to L'$ given by $x \mapsto x$ then the free property gives a unique homomorphism that extends ϕ and this map sends generators to generators so is obviously an isomorphism, that is indeed unique.
Proof of existence: Take a vector space V with X as a basis, form the tensor algebra $\mathfrak{T}(V)$ and make it a Lie algebra with the commutator map as Lie operation. Now given another Lie algebra M and a map $\phi : X \to M$, extend this map linearly to $\phi' : V \to M \subseteq \mathfrak{U}(M)$ given by $\phi'(\sum_i f_i x_i) = \sum_i f_i \phi(x_i)$, next we induce $\overline{\phi} : \mathfrak{T}(V) \to \mathfrak{U}(M) = \mathfrak{T}(M)/J$ given by $\overline{\phi}(\otimes_i v_i) = \otimes_i \phi'(v_i) + J$ and finally take $\psi := \overline{\phi}|_L : L \to \overline{\phi}(L) \cong M$ and ψ extends ϕ as wanted. $\square$

Definition 4.3.6. *Given a free Lie algebra L generated by X, and an ideal $R \subseteq L$ generated by r_i we call the quotient $\pi : L \to L/R$ the Lie algebra with generators $x_i \in \pi(X)$ and relations r_i.*

4.4 Isomorphism theorem 2

In the following we wish to construct a Lie algebra with generators and relations inspired by the fact that within every semisimple Lie algebra we can find copies of $\mathfrak{sl}(2)$ built upon the root system. We have already proven that if two semisimple Lie algebras have isomorphic root systems they are in turn isomorphic, so if we from a root system can build a semisimple Lie algebra we immediately get that it is unique up to isomorphism and we establish the 1-1 correspondance we seek.
We start out by finding, what turns out to be, sufficient relations to impose

Proposition 4.4.1 (Relations)**.** *Given a quadruple (L, H, Φ, E) with $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a basis of the root system then L is generated by $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ satisfying for $1 \leq i, j \leq \ell$ and $i \neq j$*

(S1) $[h_i h_j] = 0$
(S2) $[x_i y_j] = \delta_{i,j} h_i$
(S3) $[h_i x_j] = \langle \alpha_j, \alpha_i \rangle x_j$, $[h_i y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$
(S_{ij}^+) $(ad\ x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$
(S_{ij}^-) $(ad\ y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0$

Proof: (S1)-(S3) are obvious from previous propositions. It is a known fact about root system that they are a disjoint union of those that have positive respectively negative coeficients in the base, so no root is of the form $\alpha_i - \alpha_j$, we do however have a string of roots $\alpha_j, \alpha_j + \alpha_i, \alpha_j + 2\alpha_i, \alpha_j + 3\alpha_i, \dots, \alpha_j + q\alpha_i$ where q is such that $\alpha_j + (q+1)\alpha_i$ is not a root, this works because the root systems we are working with are finite, we also know that $q = \langle \alpha_j, \alpha_i \rangle$, now ad x_i maps x_j into $L_{\alpha_i + \alpha_j}$ and $(ad\ x_i)^2(x_j) \in L_{\alpha_j + 2\alpha_i}$ and so on, leading to
$(ad\ x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) \in L_{\alpha_j + (\langle \alpha_j, \alpha_i \rangle + 1)\alpha_i} = 0$, as wanted. □

With proposition 4.4.1 in mind we can proceed. Given a root system Φ with a basis $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, denote by $c_{i,j} = \langle \alpha_i, \alpha_j \rangle$ the Cartan integers. Define a free Lie algebra $\hat{L}$ generated by $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ and define $\hat{R}$ to be the ideal generated by the relations

$$[\hat{h}_i \hat{h}_j], [\hat{x}_i \hat{y}_j] - \delta_{i,j} \hat{h}_i, [\hat{h}_i \hat{x}_j] - c_{j,i} \hat{x}_j, [\hat{h}_i \hat{y}_j] - c_{j,i} \hat{y}_j \tag{4.4.1}$$

Taking these relations clearly amounts to demanding $(S1) - (S3)$ of proposition 4.4.1. Define $L_0 = \hat{L}/\hat{R}$. Next we wish to define a module to better understand the construction, let V' be an ℓ dimensional vector space over F with basis $(v_1, \dots, v_\ell)$ and let V be the free associative algebra on it, which means the algebra of words in the alphabet $v_1, v_2, \dots, v_\ell$ where composition is just putting a word behind another F linearly, we will write $\prod_{k=1}^m v_{i_k} = v_{i_1} v_{i_2} \cdots v_{i_m}$ for brevity. Let $\hat{\phi} : \hat{L} \to \mathfrak{gl}(V)$ be given by

$$\begin{aligned}
&\hat{\phi}(\hat{h}_j) : 1 \mapsto 0, \prod_k v_{i_k} \mapsto -\big(\sum_p c_{i_p,j}\big) \prod_k v_{i_k} \\
&\hat{\phi}(\hat{y}_j) : 1 \mapsto v_j, \prod_k v_{i_k} \mapsto v_j \prod_k v_{i_k} \quad \text{i.e. left multiplication by } v_j \\
&\hat{\phi}(\hat{x}_j) : 1 \mapsto 0, v_i \mapsto 0, \prod_{k=1}^m v_{i_k} \mapsto v_{i_1} \cdot \hat{\phi}(\hat{x}_j)\big(\prod_{k=2}^m v_{i_k}\big) - \delta_{i,j} \big(\sum_{p=2}^m c_{i_p,j}\big) \prod_{k=2}^m v_{i_k}
\end{aligned} \tag{4.4.2}$$

Denote $\hat{K}_0 := \ker \hat{\phi}$

Proposition 4.4.2. $\hat{R} \subseteq \hat{K}_0$

Proof: Notice first that $\hat{\phi}(\hat{h}_i)$ commutes with $\hat{\phi}(\hat{h}_j)$ for $i \neq j$ because they both have diagonal matrix representations so $[\hat{h}_i\hat{h}_j] \in \hat{K}$. Consider

$$\begin{aligned}
&\hat{\phi}(\hat{x}_i)\hat{\phi}(\hat{y}_j)(\prod_{i=1}^m v_{i_k}) - \hat{\phi}(\hat{y}_i)\hat{\phi}(\hat{x}_i)(\prod_{i=1}^m v_{i_1}) \\
&= \hat{\phi}(\hat{x}_i)(v_j \cdot v_{i_1} \cdots v_{i_m}) - v_j \cdot \hat{\phi}(\hat{x}_i)(v_{i_1} \cdots v_{i_m}) \\
&= \begin{cases} v_j \cdot (\hat{\phi}(\hat{x}_i)(v_{i_1} \cdots v_{i_m}) - \hat{\phi}(\hat{x}_i)(v_{i_1} \cdots v_{i_m})) = 0 & \text{if } i \neq j \\ v_j \cdot (\hat{\phi}(\hat{x}_i)(\overline{v_{1\ldots m}}) - (\Sigma_{k=2}^m c_{i_k,i})\overline{v_{1\ldots m}} - v_j \cdot (\hat{\phi}(\hat{x}_i)(\overline{v_{1\ldots m}}) = -(\Sigma_{k=1}^m c_{i_k,i})\overline{v_{1\ldots m}} & \text{if } i = j \end{cases} \\
&= \delta_{j,i}\hat{\phi}(\hat{h}_i)(\prod_{k=1}^m v_{i_k}) \\
&(\hat{\phi}(\hat{x}_i)\hat{\phi}(\hat{y}_j) - \hat{\phi}(\hat{y}_j)\hat{\phi}(\hat{x}_i))(1) = 0 = \delta_{i,j}\hat{\phi}(\hat{h}_i)(1)
\end{aligned}$$

Here $\overline{v_{1\ldots m}}$ is short for $\prod_{k=1}^m v_{i_k}$ to make it fit in a single line in the above calculations, that show $[\hat{\phi}(\hat{x}_i)\hat{\phi}(\hat{y}_j)] - \delta_{i,j}\hat{\phi}(\hat{h}_i) \in \hat{K}_0$. Next up we wish to show $[\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{y}_j)] - c_{j,i}\hat{\phi}(\hat{y}_j) \in \hat{K}_0$.

$$(\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{y}_j) - \hat{\phi}(\hat{y}_j)\hat{\phi}(\hat{h}_i)) = \hat{\phi}(\hat{h}_i)(v_j) = -c_{j,i}v_j = -c_{j,i}\hat{\phi}(\hat{y}_j)(1)$$

Which generalises in the expected way to a generator $\prod_{i=1}^m v_{i_k}$, as wanted. For our final calculation we need the following

$$\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k}) = -(\sum_{k=1}^m c_{i_k,i} - c_{j,i})\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k}) \tag{4.4.3}$$

Which we prove by induciton on m, with the induction start being clear under the convention that $\prod_{k=1}^0 v_{i_k} = 1$. For the induction step

$$\begin{aligned}
\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k}) &= \hat{\phi}(\hat{h}_i)\Big(v_{i_1}\hat{\phi}(\hat{x}_j)(\prod_{k=2}^m v_{i_k}) - \delta_{i_1,j}(\sum_{k=2}^m c_{i_k,j})\prod_{k=2}^m v_{i_k}\Big) \\
&= -(\sum_{k=1}^m c_{i_k,i} - c_{j,i})v_{i_1}\hat{\phi}(\hat{x}_j)(\prod_{k=2}^m v_{i_k}) + \delta_{i_1,j}(\sum_{k=2}^m c_{i_k,i})(\sum_{k=2}^m c_{i_k,j})\prod_{k=2}^m v_{i_k} \\
&= -(\sum_{k=1}^m c_{i_k,i} - c_{j,i})\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k})
\end{aligned}$$

The last equality is true because in case $i_1 = j$ the terms $c_{i_1,i} - c_{j,i} = 0$ in the first sum, cancel out and we get the wanted equality. Now that we have (4.4.3) we can calculate the last relation.

$$\begin{aligned}
[\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{x}_j) - \hat{\phi}(\hat{x}_j)\hat{\phi}(\hat{h}_i)](1) &= 0 \\
[\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{x}_j) - \hat{\phi}(\hat{x}_j)\hat{\phi}(\hat{h}_i)](\prod_{k=1}^m v_{i_k}) &= \Big(-(\sum_{k=1}^m c_{i_k,i} - c_{j,i}) + (\sum_{k=1}^m c_{i_k,i})\Big)\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k}) \\
&= c_{j,i}\hat{\phi}(\hat{x}_j)(\prod_{k=1}^m v_{i_k})
\end{aligned}$$

So $[\hat{\phi}(\hat{h}_i)\hat{\phi}(\hat{x}_j)] - c_{j,i}\hat{\phi}(\hat{x}_j) \in \hat{K}_0$ as wanted. $\square$

With this proposition we can factor the representation $\hat{\phi}$ through L_0 making V an L_0 module, we will need this in the proof of the following theorem.

$$\begin{tikzcd} \hat{L} \arrow[r, "\hat{\phi}"] \arrow[dr, "\psi"] & \mathfrak{gl}(V) \\ & L_0 \arrow[u, "\phi"'] \end{tikzcd}$$

Theorem 4.4.3. *Given a root system Φ with basis $\Delta = \{\alpha_1, \dots \alpha_\ell\}$. Let L_0 be the Lie algebra with generators $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ and relations $(S1)-(S3)$, then the subalgebra H generated by $\{h_i \mid 1 \leq i \leq \ell\}$ is abelian and $L = X \oplus Y \oplus H$ where X and Y are generated by the x_i and y_i.*

Proof: We proceed in steps
Step 1: $\sum_j F\hat{h}_j \cap \ker\hat{\phi} = 0$
Take $\hat{h} = \sum_j a_j\hat{h}_j \in \sum_j F\hat{h}_j \cup \ker\hat{\phi}$, then the eigenvalues of $\hat{h}$ are 0, especially the ones with respect to v_i, $1 \leq i \leq \ell$ but $\hat{h}(v_i) = -\sum_j a_j c_{i,j} = 0$ now it is a general fact from root systems that the Cartan matrix with entries $c_{i,j} = \langle\alpha_i, \alpha_j\rangle$ is invertible so the $c_{i,j}$ are linearly independent in i respectively j, so $a_j = 0$ for all j and $\hat{h} = 0$.
Step 2: $\psi : \hat{L} \to L_0$ maps $\sum_j F\hat{h}_j$ isomorphically onto $\sum_j Fh_j$.
Step 1 implies that the restriction of ψ is injective and the proposition ensures surjectivity.
Step 3: $\sum_j F\hat{x}_j + \sum_j F\hat{y}_j + \sum_j F\hat{h}_j$ maps isomorphically into L_0
In step 1 we saw that the image of $\hat{h}_j$ is nonzero so for a fixed i we get $[x_iy_i] = h_i, [h_ix_i] = 2x_i, [h_iy_i] = -2y_i$, because $c_{i,i} = 2$ for all i. Then $Fx_i + Fy_i + Fh_i \cong \mathfrak{sl}(2)$ and we know that $\mathfrak{sl}(2)$ is simple, so it does not have any Lie algebra ideals, forcing x_i, y_i, h_i to be linearly independent. To see that all $\{x_j, y_j, h_j\}$ are linearly independant we can look at the following system of equations, by applying ad h_1 0 to ℓ times.

$$\begin{aligned}
&\lambda_1x_1 + \cdots + \lambda_\ell x_\ell + \mu_1y_1 + \cdots + \mu_\ell y_\ell + \nu_1h_1 + \cdots + \nu_\ell h_\ell = 0\\
&\lambda_1c_{1,1}x_1 + \cdots + c_{\ell,1}\lambda_\ell x_\ell - c_{1,1}\mu_1y_1 + \cdots - c_{\ell,1}\mu_\ell y_\ell = 0\\
&\lambda_1c_{1,1}^2x_1 + \cdots + c_{\ell,1}^2\lambda_\ell x_\ell + c_{1,1}^2\mu_1y_1 + \cdots + c_{\ell,1}^2\mu_\ell y_\ell = 0\\
&\vdots\\
&\lambda_1c_{1,1}^\ell x_1 + \cdots + c_{\ell,1}^\ell\lambda_\ell x_\ell + (-c_{1,1})^\ell\mu_1y_1 + \cdots + (-c_{\ell,1})^\ell\mu_\ell y_\ell = 0
\end{aligned}$$

This resembles a Vandermonde matrix and with some linear algebra we get that this system has a invertible coefficient matrix if all the $c_{i,1}$ are non zero, but this might not be the case, if it is not, say $c_{j,1} = 0$ we can get some of the coefficients to equal zero with ad h_1 and the j'th ones with ad h_j applied a proper amount of times. So we have $\lambda_i = 0 = \mu_i$, we already say in step 2 that the h_i are linearly independant so we are done.
Step 4: $H = \sum_j Fh_j$ is an abelian subalgebra of L_0.
Step 2 implies that it is a subalgebra and relation (S1) implies that it is indeed abelian.
Step 5: Denote $[x_{i_1} \dots x_{i_k}] = [x_{i_1}[x_{i_2}[\cdots[x_{i_{k-1}}x_{i_k}]\dots]]]$ then
$[h_j[x_{i_1} \dots x_{i_k}]] = (\sum_{t=1}^k c_{i_k,j})[x_{i_1} \dots x_{i_k}]$ and $[h_j[y_{i_1} \dots y_{i_k}]] = (\sum_{t=1}^k c_{i_k,j})[y_{i_1} \dots y_{i_k}]$.
We proceed by induction on k, where the induction start is ensured by the relation (S3). For the

induction step consider

$$\begin{aligned}
[h_j[x_{i_1}\dots x_{i_k}]] &= [h_j[x_{i_1}[x_{i_2}\dots x_{i_k}]]] \\
&= -[x_{i_1}[[x_{i_2}\dots x_{i_k}]h_j]] - [[x_{i_2}\dots x_{i_k}][h_j x_{i_1}]] \\
&= -[x_{i_1}[-(\sum_{t=2}^{k} c_{i_t,j})[x_{i_2}\dots x_{i_k}]]] - [[x_{i_2}\dots x_{i_k}][c_{i_1,j}x_{i_1}]] \\
&= \sum_{t=1}^{k} c_{i_t,j}[x_{i_1}\dots x_{i_k}]
\end{aligned}$$

As wanted. An almost identical argument works in the other case.
Step 6: If $k \geq 2$ then $[y_j[x_{i_1}\dots x_{i_k}]] \in X$ and $[x_j[y_{i_1}\dots y_{i_k}]] \in Y$.
Once more we proceed by induction on k. For the induction start $k = 2$

$$\begin{aligned}
[y_j[x_{i_1}x_{i_2}]] &= -[x_{i_1}[x_{i_2}y_j]] - [x_{i_2}[y_j x_{i_1}]] \\
&= \delta_{i_2,j}[x_{i_1}h_j] - \delta_{i_1,j}[x_{i_2}h_j] \\
&= \delta_{i_2,j}c_{i_1,j}x_{i_1} - \delta_{i_1,j}c_{i_2,j}x_{i_2} \in X
\end{aligned}$$

The induction step follows directly from the Jacobi identity.
Step 7: $X + Y + H = L_0$.
To see that $X + Y + H$ is a subalgebra, we simply refer to the previous three steps to see that it is closed under the bracket operation. And to see that it is the whole thing we simply note that it contains all the generators.
Step 8: The sum from step 7 is direct.
Step 5 shows we can decompose L_0 into eigenspaces of ad H, ensuring that the sum is direct. □

Write $x_{i,j} = (ad\ x_i)^{-c_{j,i}+1}(x_j)$ and $y_{i,j} = (ad\ y_i)^{-c_{j,i}+1}(y_j)$.

Lemma 4.4.4. *In L_0 we have ad $x_k(y_{i,j}) = 0$ for $1 \leq k, i, j \leq \ell$ where $i \neq j$.*

Definition 4.4.5. *In an infinite dimensional vector space V, an endomorphism x is locally nilpotent if for all $v \in V$ there exists a $n \in \mathbb{N}$ such that $x^n(v) = 0$.*

Clearly a locally nilpotent endomorphism x is nilpotent in the usual way if restricted to any finite dimensional subspace W of V stable under x i.e. $xW = W$. Furthermore given two x stable subspaces W, W' we have $\exp x\mid_W = \exp x\mid_{W'}$ on $W \cap W'$, so we can define $\exp x$ as the composite of the local maps.

Theorem 4.4.6 (Serre)**.** *For a root system Φ with basis $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let L be the Lie algebra with generators $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ and relations $(S1) - (S3)$ and $(S_{ij}^{\pm})$. Then L is a finite dimensional semisimple Lie algebra with CSA spanned by $\{h_i \mid 1 \leq i \leq \ell\}$ and corresponding root system Φ*

Proof: We define $L = L_0/R'$, where R' is the ideal generated by $x_{i,j}$ and $y_{i,j}$ for $i \neq j$, note that R' amounts to the relations $(S_{ij}^{\pm})$, and L_0 already had the relations $(S1) - (S3)$ so L has all the relations of proposition 4.4.1. Denote by I respectively J the ideals in X respectively Y generated by $x_{i,j}$ respectively $y_{i,j}$ both included in R'. We proceed in steps.
Step 1: I and J are ideals of L_0
Looking at J we see that $y_{i,j}$ is an eigenvector for ad h_k with eigenvalue $-c_{j,k} + (c_{j,i} - 1)c_{i,k}$ now step 5 of the previous theorem implies that ad $h_k(Y) \subseteq Y$, clearly $J \subseteq Y$ so also ad $h_k(J) \subseteq Y$

in fact in J by the Jacobi identity and a similar argument to step 5. Now step 6 and $(S2)$ implies that ad $x_k(Y) \subseteq Y + H$, but ad $h_k(J) \subseteq J$ and another application of the Jacobi identity we get ad $x_k(J) \subseteq J$.
Step 2: $R' = I + J$.
Clearly $I + J \subseteq R'$, and $I + J$ contains all the generators of $x_{i,j}, y_{i,j}$ so $K \subseteq I + J$.
Step 3: $L = N^- \oplus H \oplus N$ where $N^- = Y/J$ and $N = X/I$.
This is a simple matter, $L := H \oplus X \oplus Y/R'$ and we just saw $R' = I + J$ with $I \subseteq X$ and $J \subseteq Y$, so $L \cong H \oplus X/I \oplus Y/J$.
Step 4: $\sum_i Fx_i + \sum_i Fy_i + \sum_i Fh_i \subseteq L_0$ maps isomorphically into L.
This uses a similar arguemnt to step 3 of the previous theorem, given that H behaves in the same way as it did previously.
Definition: For $\lambda \in H^*$ define $(L)_\lambda = \{x \in L \mid [hx] = \lambda(h)x \ \forall\ h \in H\}$. We saw that $(L_0)_0 = H$, and like previously we get $X = \sum_{\lambda \succ 0}(L_0)_\lambda$ and $Y = \sum_{\lambda \prec 0}(L_0)_\lambda$, where $\prec$ is the partial ordering given by $\lambda = \sum_i k_i\alpha_i \prec 0$ if all $k_i \geq 0$ and likewise for $\succ$.
Step 5: $H = (L)_0$, $N = \sum_{\lambda \succ 0}(L)_\lambda$ and $N^- = \sum_{\lambda \prec 0}(L)_\lambda$ and the $(L)_\lambda$ are finite dimensional.
This is clear from the previous steps.
Step 6: ad x_i and ad y_i are locally nilpotent endomorphisms.
For a fixed i define $M = \{x \in L \mid \exists n_x \in \mathbb{N}(\text{ad } x_i)^{n_x}(x) = 0\}$. Take $x \in M$ and $y \in M$ then lemma 15.1 of Humphreys implies that $(\text{ad } x_i)^{n_x+n_y}([xy]) = 0$, now the relation (S_{ij}^+) implies that M contains all the x_i and

$$(\text{ad } x_i)^{-\langle\alpha_i,\alpha_i\rangle+3}(y_i) = (\text{ad } x_i)^{-\langle\alpha_i,\alpha_i\rangle+2}(h_i) = c_{i,i}(\text{ad } x_i)^{-\langle\alpha_i,\alpha_i\rangle+1}(x_i) = 0$$

So also $y_i \in M$, now the x_i and y_i generate L so $M = L$.
Step 7: $\tau_i = \exp \text{ad } x_i \exp -\text{ad } y_i \exp \text{ad } x_i \in Aut(L)$.
As previously noted locally nilpotent implies well defined exponential, which is an automorphism.
Step 8: For $\lambda, \mu \in H^*$ and $\sigma\lambda = \mu$ for some $\sigma \in \mathcal{W}$ then $\dim L_\lambda = \dim L_\mu$.
Here $\mathcal{W}$ is the Weyl group associated to our root system, the Weyl group is the group of reflections in roots, where $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \langle\beta, \alpha\rangle\alpha$ is a reflection, and if $\alpha_i \in \Delta$ is a simple root then $\mathcal{W}$ is generated by the corresponding reflections. So to prove the claim it is enough to look at σ_{α_i}. Locally on $L_\lambda + L_\mu$ we have τ_i as the usual exponential. It is not too difficult to show that τ_i interchanges these spaces, so also $\dim L_\lambda = \dim L_\mu$.
Step 9: $\dim L_{\alpha_i} = 1$ and $\dim L_{k\alpha_i} = 0$ for $k \notin \{\pm 1, 0\}$.
This is clear in L_0, and step 4 then implies that it also works in L.
Step 10: $\alpha \in \Phi \Rightarrow \dim L_\alpha = 1$ and $\dim_{k\alpha} = 0$ for $k \notin \{\pm 1, 0\}$.
It is a fact of root systems that all roots are conjugates of simple ones under the Weyl group, so this statement follows directly from the previous steps.
Step 11: $L_\lambda \neq 0 \Rightarrow \lambda \in \Phi \cup \{0\}$
If not one can show that there exists a $\sigma \in \mathcal{W}, i \neq j$ such that $(\sigma\lambda)_i > 0$ and $(\sigma\lambda)_j < 0$ meaning it has positive i'th coordinate and negative j'th coordinate, but as previously discussed no root can have this property.
Step 12: $\dim L = \ell + \#\Phi < \infty$
We know that $\dim H = \ell$ and we see that the "root spaces" N and N^- are in one to one correspondance with roots, and since $L = H \oplus N \oplus N^-$ we have the desired dimension formula.
Step 13: L is semisimple.
Let $A \subset L$ be an abelian ideal, we wish to show that $A = 0$. It is clear that ad H stabilises A. Write $L = H + \sum_{\alpha\in\Phi} L_\alpha$ then $A = A \cap H + \sum_{\alpha\in\Phi}(L_\alpha \cap A)$ if there exists a root $\alpha \in \Phi$ such that $L_\alpha \subset A$ then $[L_{-\alpha}L_\alpha] \subset A$ and we can include a copy of $\mathfrak{sl}(2) \hookrightarrow A$, but $\mathfrak{sl}(2)$ is not abelian, a contradiction. So $A \subset H$ and then $[L_\alpha A] = 0$ for all roots α and $A \subset \bigcap_{\alpha\in\Phi} \ker\alpha = 0$.
Step 14: H is a CSA of L, Φ the corresponding root system.

H is an abelian subalgebra so clearly $[HH] = 0$ nilpotent, and self normalising because $L = H \oplus \sum_{\alpha \in \Phi}$ is a direct sum, so $[xH] \subset H$ for all $x \in H$ and by the relations $(S2)$ these are the only elements of L that satisfy this condition. $\square$

Finally we have all the tools to make the bijection between root systems and semisimple Lie algebras, because if we have a root system we can build a semisimple Lie algebra via Serre's theorem, and if we have semisimple Lie algebras with isomorphic root systems they are themselves isomorphic due to theorem 4.2.6. At the end of the proof of theorem 4.2.6 we promised to give another proof using generators and relations, let us delve into it.

Theorem 4.4.7. *For semisimple Lie algebras L and L' with isomorphic root systems there exists an isomorphism between L and L'.*

Proof: Let $\psi : \Phi \to \Phi'$ be the isomorphism between the root systems of L respectively L' induced by H respectively H'. Continuing with the notation from the proof of Serre's theorem and 4.2.6, choose $x_\alpha \in L_\alpha$ and $x'_\alpha \in L'_\alpha$, choose y_α and h_α in L as in Serre's theorem, then define $h'_\alpha = \pi(h_\alpha)$ and choose uniquely $y'_\alpha \in L'$ such that $[x'_\alpha y'_\alpha] = h'_\alpha$ now the choices x'_α, y'_α and h'_α satisfy the relations of 4.4.1 so by construction of the free Lie algebra there exists a unique homomorphism ℓ extending the induced isomorphism $\pi : H \to H'$, sending $x_\alpha, y_\alpha, h_\alpha$ to $x'_\alpha, y'_\alpha, h'_\alpha$. To see that ℓ is indeed an isomorphism recall that $\dim L = \dim H + n = \dim H' + n = \dim L'$ where n is the number of roots in Φ and Φ', now our Lie algebras have the same finite dimension and ℓ maps generators to generators, so it is an isomorphism. $\square$

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