Rewarding Risk in Life-Cycle Investing

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Abstract

Recently, the Dutch government and the social partners came to an agreement about a renewal of the Dutch pension system. The pension accrual of participants will be conditional and depend on the achieved returns. The accrual will also be age-dependent and pension funds are allowed to invest according the life-cycle principle, changing the level of risk for participants of different ages.

We derive the optimal allocation of assets according to the theory of life-cycle investing. We develop a stylized pension fund model along with a financial market model that invests the assets according to the life-cycle principle. We compare 4 different life-cycle strategies with a default strategy. No significant improvement for the pension accrual is found for the life-cycle strategies.

Acknowledgements

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I would also like to thank Bert de Bock for giving me the opportunity to do an internship at InAdmin RiskCo B.V. while writing my thesis. His support and long-term vision really helped formulating my goals and clarifying my thesis. Bert's experience in supervision students paved the way for this result.

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Eppo van der Heijden,
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Contents

Chapter 1

Introduction

This thesis is written during an internship at InAdmin RiskCo B.V. as requirement to obtain the Master's degree in Mathematical Sciences at Utrecht University.

In June 2019, the Dutch government presented the Pension Agreement. This agreement came about by the work of employers' and employees' organizations and the Dutch government in order to change the Dutch pension system. Among other things, the agreement allows pension funds to invest based on the ages of the participants, the so-called *life-cycle investing*. In this thesis, we will model a pension fund that invests according to this principle.

In this chapter, we will give a description of the company InAdmin RiskCo B.V., followed by a problem description, the research question and an outline of the thesis.

1.1 Description of InAdmin RiskCo B.V.

InAdmin RiskCo B.V., founded in 2002 as a consultancy firm, works on the intersection of business, actuary and IT in the financial world. The company started as the distributor of ProductXpress, a workbench for financial product development and calculations, and extended his offerings in, among others, the areas of data quality and reporting for regulators. The company can be characterized by:

- Performing projects for Life Insurance companies and Pension Funds, for the liabilities and assets parts of the Balance sheet.
- Strategic partner of DXC for the implementation of the DXC ProductXpress calculation tool, their most important IT platform for large scale calculations.
- 130 academics trained in mathematics, finance, business studies, economics, econometrics, physics, software development and artificial intelligence.
- Offices in the Netherlands, Portugal and the Philippines.
- Performed projects in 16 jurisdictions in Europe, Asia, Australia, Africa and the Americas.

Examples of projects are:

- Development and implementation of a platform for reporting the liabilities to DNB, for a Dutch system administrator for pension funds;
- Reserve calculations and capital requirements calculations under Solvency II (Redesign of IT landscape for a Dutch insurance company);
- Implementation of product calculation engines for new or existing administration systems, quotation systems and financial planners;
- Investment rule optimizations for banks; using replicating portfolios;
- Multiple data quality audits, using the RiskCo methodology on pension fund administrations;
- Implementation of the rules and calculations for illustration and administration purposes of hundreds of products across many lines of business for a very large international insurance company.

In the Netherlands, RiskCo B.V. worked for, among other companies, ASR, A&O services, Klaverblad, PGGM, Nationale Nederlanden, Vivat and Delta Lloyd. Worldwide, the US based insurance company METLIFE is a customer.

In 2017 and 2018, Aon Hewitt Benefits Administration and InAdmin NV from APG were taken over by RiskCo B.V. to form the new name InAdmin RiskCo B.V. The strategy of InAdmin RiskCo B.V. is to deliver high quality pension administration based on an advanced open new home-built administration platform called RAP.

1.2 Description of the problem

The Dutch pension system has been under pressure for over a decade. The current scheme leads to intergenerational debate about the division of the available pension assets and is insufficiently equipped for the changing labor market and increased differences and preferences of participants. Recently, the government, employers and social partners have come to an agreement about the new pension system. One of the proposed renewals is the transition from the current uniform investment policy to a policy in which the degree of risk is linked to the age of the participant, the so-called life-cycle investing.

The uniform investment policy conflicts with the two goals that pension funds in general have: offer indexation of the pension rights through risky investments and at the same time provide stability of the benefits for participants that are near or in retirement. A uniform investment policy does not assign risk to groups that can and want to take it. Young participants have more time to compensate their rights should a bad return occur. Older participants generally want to secure the wealth they have built up so far, as they can not brush away bad results. Furthermore, the absolute size of negative shocks is bigger for participants near retirement, because they have already built up more.

Life-cycle theory was developed as an answer to finding the optimal allocation of savings in either stocks or bonds over the course of an employee's life. It started off as a retirement plan for an individual who does his own investments, but it has been suggested widely that pension funds should embrace these ideas and include it in their investment policy. However, in most literature, the emphasis is on the individual life-cycle and the effects on the pension collective and risk-sharing have not been thoroughly investigated. In a very social pension system as the Netherlands currently has, these effects should be known.

Furthermore, there have not been many papers that suggest a fair way to compensate participants in a fashion that agrees with the risk different groups take in a life-cycle-based model. In the new pension agreement, the pension funds do not longer have to hold extra funds for the amount of risk they take in their investment strategy, but good and bad return will immediately be divided amongst the members. This thesis aims to formulate a policy that will let the indexation or drawback of participants' rights depend on the risk that is taken in their portfolio.

1.3 Research question

The research question of this thesis will be:

How can risk be shared between different groups in a pension system with life-cycle investing?

A further elaboration and explanation of the research question will be given in Section 2.6.

1.4 Outline

This thesis will consist of three parts.

The first part will contain an introduction to the Dutch pension system. In Chapter 2, the current Dutch pension system is described and the research question will be explained and split up into sub-questions.

The second part concerns the mathematical foundation, the model that is used and the model assumptions. In Chapter 3, we will explain the necessary mathematical background that is used in later chapters. Chapter 4 will focus on the literature and theory of lice-cycle investing. Chapter 4 will be about the methodology in the financial market model and the pension fund model.

The third part will describe the result, conclude the thesis and advice future research. Chapter 6 will be devoted to the results. In Chapter 7, our conclusions will be summarized. Chapter 8 will contain recommendations for future research.

Chapter 2

Dutch pension system, regulations and recent developments

In this chapter, we will describe the structure of the Dutch pension system, the regulations that are in place and recent developments that have been made in the pension system. In Section 2.1, the three pillars of the pension systems are explained and specified for the Dutch case. In Section 2.2, we will explain the different pension schemes and the way that pension entitlement is built up. In Section 2.3, the regulations of the pension system that are cast in the Financial Assessment Framework are explained. Sections 2.4 and 2.5 treat the most recent development in the pension system and in Section 2.6, we will elaborate on the research questions of this thesis.

2.1 The three pillars of the pension system

The Dutch pension system consist of three pillars, similar to many other European countries.

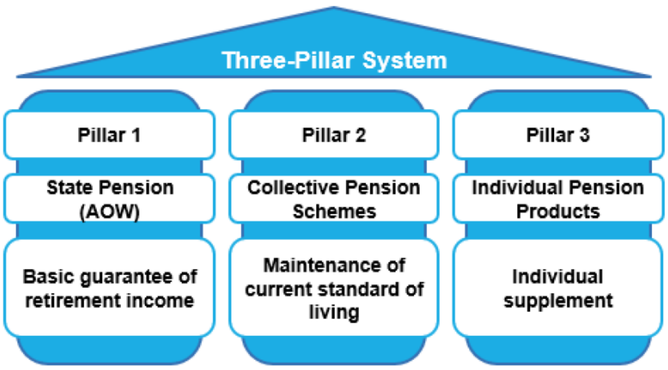


Figure 2.1.1: The three pillars of the Dutch pension system.

The first pillar is the General Old Age Pension Act (AOW). This basic state pension is granted to everyone who lived or worked in the Netherlands at some point of his life. For every year, 2% of a full AOW is accrued, to a maximum of 100%. The value of the AOW depends on whether you live together or not, and it is derived from the minimum wage.

The second pillar consists of the Dutch pension funds, which can usually be linked to a specific company or sector. It is an addition to the state pension and is financed by premiums that employees and employers pay. It is not compulsory by law, but for roughly 90% of all employees, it is an obligation in the employment agreement. The pension fund is obliged to pay pension entitlements till the participants' death. These products can also be offered by insurance companies and by premium pension institutions (PPIs). PPIs are pension executioners that do not bear the financial risks themselves. They are especially interesting for multinationals, because they allow for the centralization of the pension execution.

The third pillar is formed by the individual pension. This can consist of savings, money received from inheritance, excess value on a house or savings with tax discount. In general, the third pillar is a substitute for employees that can not or do not build up pension rights in the second pillar.

2.2 Pension schemes and investment strategies

Pension funds can be categorized by their pension schemes, which can be either a Defined Benefit scheme (DB), a Defined Contribution scheme (DC) or a Collective Defined Contribution scheme (CDC). In a DB scheme, the pension entitlements that you will receive after retirement are fixed. The pension fund works in a collective setting, so indexation and cuts are applied to the all participants. Because the accrual of pension rights highly depends on the financial developments, the pension premiums may vary. In a DC scheme, the premiums are fixed, but the pension benefit is uncertain. Furthermore, the DC schemes are individual plans.

Besides the contrasting DB and DC schemes, there is also a hybrid variant, called the Collective Defined Contribution scheme. In this scheme, The pension premiums are fixed as in a DC scheme, but the pension benefits depend upon the financial state of the entire pension fund.

In the Netherlands, most pension funds follow either a DB or CDC scheme, in contrast to individual DC plans that can for instance be found in the US and the UK.

Pension funds have to invest the wealth according to the 'prudent person' principle. This means that their investment policies have to be in the interest of the active participants and retirees. The risk appetites of different group in the pension funds have to be taken into account when assessing the strategic investment policy. In most cases, pension funds combine these risk profiles into a uniform investment policy.

2.3 Financial Assessment Framework

In the rest of this thesis, we will focus on the second pillar as it is regulated by the Pension Act. In this chapter, we describe the new Financial Assessment Framework (nFTK) which is part of the Pension Act and regulated by the Dutch central bank DNB.

2.3.1 History

In 1952, the Dutch government introduced the first pension law, the Pension and Savings Funds Act (Pensioen- en Spaarfondsenwet). In the following decades, it was regularly adjusted to keep up with social developments, such as increasing labor mobility and the transition from a single-money-maker model to a dual-income model. The law was finally replaced in 2007 by the Pensioen Act (Pensioenwet), to offer employees more transparency, more certainty and more information. An important feature incorporated in the new law was the Financial Assessment Framework (FTK), that specifies recovery and indexation rules according to the financial health of the pension fund. Financial health of a pension fund is measured by the funding ratio, which is based on accurate market information.

Definition 2.3.1. The *future value* (FV) of a cash flow at time t is the nominal value of an asset. The value can be given or estimated by an assumed amount of growth.

Definition 2.3.2. The *present value* (PV) is the current value of a future cash flow at time t given a specific rate of return r. If $FV(t)$ denotes the future cash flow at time t, then

$$PV(t) = \frac{1}{(1+r)^t} FV(t). \tag{2.1}$$

Note that a smaller rate of return means a bigger present value.

Definition 2.3.3. The *funding ratio* (FR) of a pension fund at time t is the ratio of the present value of the pension assets and the present value of the pension liabilities, usually displayed as a percentage. In other words,

$$FR(t) = \frac{PV_A(t)}{PV_L(t)} \cdot 100\%, \tag{2.2}$$

where PV denotes the present value and the subscripts A and L refer to the pension assets and pension liabilities, respectively.

Definition 2.3.4. We call a pension fund *overfunded* if the funding ratio is above 100%. Similarly, a pension fund is *underfunded* if the funding ratio is below 100%.

The new Pension Act was ratified at a time when most pension funds were largely overfunded, with a weighted average funding ratio of 140% in 2007 as seen in Figure 2.3.1. In 2007, the total asset value of the Dutch pension funds was more than 88%[1] of the Dutch GDP. In 2017, this ratio even grew to more than 184%[2].

The 2008 financial crisis and the successive period of extremely low interest rates, as can be seen in Figure 2.3.2, triggered the recovery plans of the FTK that come into force when a pension fund is underfunded for a longer time.

[1] 683 billion euros asset value divided by 774 billion euros GDP.
[2] 1338 billion euros asset value divided by 725.4 billion euros GDP.

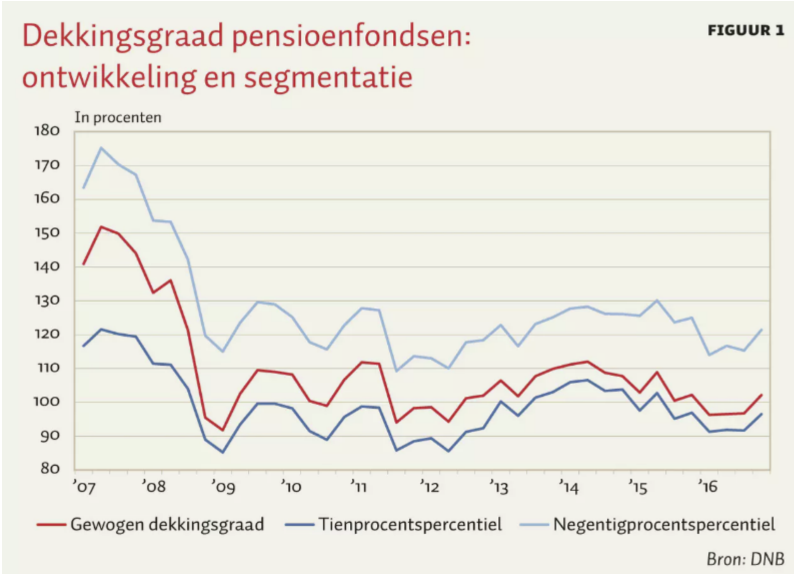


Figure 2.3.1: The weighted average funding ratio (red), 10%-percentile (light blue) and the 90%-percentile (dark blue) of the Dutch pension funds. Image retrieved from [34] on 12th of June, 2019.

2.3.2 nFTK process

The financial position of pension funds can be measured in terms of the funding ratio FR. In the regulations of the nFTK, however, the *policy funding ratio* (PFR) is used for decisions on indexation and pension cuts.

Definition 2.3.5. The *policy funding ratio* (PFR) in month t is the past-12-month-average of the funding ratio:

$$PFR(t) = \frac{1}{12} \sum_{s=t-12}^{t} FR(s). \tag{2.3}$$

At certain levels of the PFR, the nFTK specifies mandatory steps to be taken by pension funds. A schematic overview of these steps and adjustments can be seen in Figure 2.3.3. Let us first remark that besides the PFR, all quantities are function of time t with unit *year*. The Dutch Pension Act dictates that pension funds are obliged to have a *Minimal Required Own Funds* ($MROF$). The calculation of the $MROF$ is specified in the Dutch Pension Law. Its value depends on the investment risk, gross technical provisions[3] (GTP) and management costs (MC) in year t:

- If a pension funds policy generates investment risk, then the $MROF$ is calculated as

$$MROF(t) = 0.04 \cdot GTP(t) \cdot \min\left(\frac{NTP(t)}{GTP(t)}, 0.85\right),$$

[3]Gross technical provisions represent the amount that an insurer requires to fulfil its insurance obligations and settle all expected commitments to policyholders and other beneficiaries arising over the lifetime of the insurer's portfolio of insurance contracts.

Figure 2.3.2: The development of the European Central Bank (ECB) interest rates in % in the period 1999-2019. Recovered from [31] on 12th of June, 2019.

where $NTP(t)$ (net technical provisions) equal the gross technical provisions minus its reinsured part in year t. A pension fund that has more than 15% of its GTP reinsured hence has a $MROF$ of $0.85 \cdot 4\% = 3.4\%$.

- If a pension funds policy does not induce investment risk and the management costs are fixed for more than 5 years, then
$$MROF(t) = 0.01 \cdot GTP(t).$$
- If a pension fund does not induce investment risk and the management costs are fixed for less than five years, then
$$MROF(t) = 0.01 \cdot GTP(t) + 0.25 \cdot MC(t-1).$$

For most pension funds, the $MROF$ will be between 4% and 5%. The *Minimum Required Funding Ratio* ($MRFR$) is defined to be $1 + MROF$. When the PFR is below the $MRFR$ for 5 years, immediate measures have to be taken in order to be above the level again. This means that within six months, pensions are reduced. Hence, in economic hard times, indexation for the working and retired pension participants is at risk. The *Required Funding Ratio* (RFR) will be explained in the next subsection.

2.3.3 Required Own Funds

The *Required Own Funds* (ROF) is a measure for the assets a pension fund should keep in reserve in order to counter economic shocks with a 97.5 % certainty.

The standard model for the ROF has ten risk categories or shocks, denoted by $(S_i)_{i \in \{1,\dots,10\}}$:

- Interest rate risk (S_1),
- Stock and real estate risk (S_2),
- Currency risk (S_3),
- Commodity risk (S_4),
- Credit risk (S_5),
- Technical insurance risk (S_6),

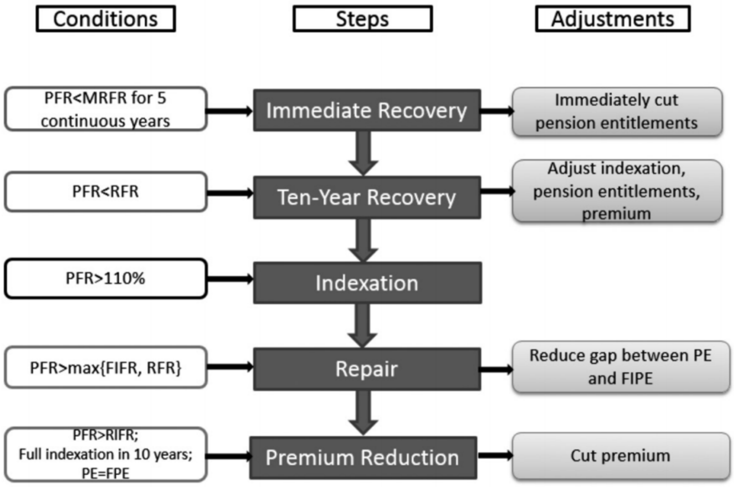


Figure 2.3.3: Schematic overview of the lower bounds on the funding ratios and steps that must be taken according to the nFTK.

- Liquidity risk (S_7),
- Concentration risk (S_8),
- Operational risk (S_9),
- Active management risk (S_{10}).

All of these shocks have to be calculated on a 97.5% level of certainty based on a one-year horizon. This is done by a specific simulation test that is provided by DNB and which is similar for all pension funds. If the risk categories would all have correlation 1 with every other one, the ROF would equal $\sum_{i=1}^{10} S_i$. However, we assume a diversification effect on these shocks, i.e., not all shocks will occur at the same time. Hence the ROF can be calculated as

$$\mathrm{ROF} = \sqrt{\sum_{i=1}^{10} S_i^2 + 2\rho_{1,2}S_1S_2 + 2\rho_{1,5}S_1S_5 + 2\rho_{2,5}S_2S_5}, \tag{2.4}$$

where $\rho_{1,2}$, $\rho_{1,5}$ and $\rho_{2,5}$ are the correlations between subscripted risk categories. These values are fixed by the DNB. Note that the DNB allows for a different calculation of the ROF, if a pension fund can motivate why the standard does not apply to them.

The *Required Funding Ratio* (RFR) is the desired funding ratio such that there is 97.5% certainty that a pension fund can fully pay out all pension rights on a one-year horizon.

Definition 2.3.6. The *Required Funding Ratio* (RFR) is defined as

$$RFR = 1 + ROF. \tag{2.5}$$

The Pension Act specifies what pension funds have to do in case of a funding deficit. A funding deficit occurs when the PFR is below the Required Funding Ratio (RFR). When the Policy Funding Ratio is below the RFR, a Ten-Year recovery plan should be presented and implemented. In this case, indexation, pension entitlements and premium are adjusted. For a deeper explanation of the different shock categories and examples, see [27].

2.4 Pension agreement

Recently, on 5th of June 2019, the government and social partners came to an agreement about the renewal of the Dutch pension system [14]. Although a lot of ideas still need to be worked out, the most important features of the agreement are:

- All pension contract will have to be *Defined Contribution* (DC) or *Collective Defined Contribution* (CDC) instead of *Defined Benefits* (DB). The paid premiums will be converted into a conditional accrual of the pension rights. In contrast to the DB pension contracts, this means that the pension premiums are fixed. Based on a scenario set provided by DNB, the median of all scenario is taken as the average outcome. This is the final pension right that a participant sees. Furthermore, also a bad scenario (5th percentile) and good scenario (95th percentile) are shown to illustrate the conditionality of the accrual. These rights will depend on the returns the pension fund has made and the current and future economic conditions.
- The so-called '*doorsneesystematiek*' will be abolished. In this system, a participant receives the same accrual in each working year. Instead, agents will receive an accrual that depends on their age.
- Pension funds will no longer have to keep Required Own Funds. Instead, the funds can index its participants when the funding ratio is above 100% and will have to cut pension rights if the funding ratio is below 100%.
- Pension funds will no longer be allowed to use premiums that do not cover the cost of the pension rights.
- Participants will be able to withdraw up to 10% of their built-up pension rights around their retirement. This money can be used, for instance, to pay off their mortgage or to go on a big trip.
- Employees with a heavy profession will get the possibility to retire 3 years earlier than the retirement age. Employers will no longer have to pay the fine on the RVU (Early Retirement Scheme) for gross incomes up to 19,000€.
- There will be a possibility to implement life-cycle investing in the new pension system. This means that a pension fund can take different levels of risk for participants of different ages. The concept of life-cycle investing is explained more further on.

2.5 Committee Parameters 2019

Based on the Dutch Pension Act, an independent committee consisting of 5 to 7 members with knowledge in finance and pension planning, is assigned by the government. Members include, for instance, several professors in Actuarial Science and Macro Economics, a former Minister of Finance and a former member of the board of the Pension Federation. This committee is called the *Committee Parameters* and it is assigned at least every five years to report their judgment on the maximal parameters that pension funds can use in their recovery plans and the technical elaboration of the valuation of pension rights. Its formation is laid down in the FTK and the committee members are assigned by the Minister of Social Affairs and Employment. Typically, the members of the committee are pension experts and professors affiliated with pension matters.

The most recent committee was formed on 28 January 2019 and its members were given the task to judge the technical elaboration of the valuation of long-term pension rights and to decide on the values of the following parameters on the basis of the current financial-economic expectations:

- the minimal percentage of the average wage or price index;
- the maximal allowed average return on fixed-income securities;
- the maximal allowed risk premium on, among other things, stocks and real estate; and
- the uniform set with economic scenarios.

2.6 Elaboration research questions

In the current system, the pension funds have to account for the risk in their investment strategy by incorporating the Required Own Funds (Equation 2.4) to the Required Funding Ratio (Definition 2.3.6). The ROF depends on the total capital and the sensitivity to a number of risk categories. If the Policy Funding Ratio (Equation 2.3.5) is above the RFR, the pension fund is allowed to (partially) index all its participants. Clearly, this policy will not be fair if the amount of risk in the strategy of different participants will vary substantially, as will be possible for life-cycle investing. Therefore, we will formulate a new policy that will let the indexation or drawback of participants' rights depend on the risk in their portfolio. This leads to the following question:

How does life-cycle investment affect pension accrual in different age cohorts?

We want to model a pension fund where the participants' savings are invested according to their personal life-cycle. This is an explicit life-cycle scheme, because we explicitly follow each agent's life-cycle. We will test different life-cycle schemes that have been suggested in the literature. Comparison of the different stratgies answers the question:

How do different explicit life-cycle methods perform compared to each other?

We are also interested in the financial health of the entire pension fund, i.e., with the funding ratio. This leads to the following question:

How does life-cycle investment affect the development of the funding ratios of pension funds?

Chapter 3

Mathematical foundation

In this chapter, we will introduce the mathematical foundation of *stochastic calculus* (also known as *Itô calculus*) that will be used in the upcoming chapters. In Section 3.1, we introduce stochastic processes and all kinds of martingales. In Section 3.2, the important concept of Brownian motion is constructed in two different ways. The stochastic integral and stochastic differential equation are defined in Section 3.3. Furthermore, we introduce the statistical methods that are used in this thesis in Sections 3.4 and 3.5. In Appendix A, the measure-theoretic background can be found. The reference for this the stochastic calculus of this chapter is [22].

3.1 Stochastic processes and martingales

First, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this section, we will define *martingales*. We will see below that these martingales are *stochastic processes* with certain defining properties. Let us first define the latter concept.

Definition 3.1.1. Let $T \in \mathbb{R}_+$ and $(E, \mathcal{E})$ some measurable space. A *stochastic process* indexed by $[0, T]$ with values in $(E, \mathcal{E})$ is a collection $X = (X_t)_{t\in[0,T]}$ of random variables from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the state space of the process.

Most of the times, we will take $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let us give definitions for the *variation* of a function and the *quadratic variation* of a stochastic process.

Definition 3.1.2. Let $F : [a, b] \to \mathbb{R}$ be some function, we define the *variation* of F by

$$V_F(t) := \sup\left\{\sum_{i=1}^{n} |F(s_i) - F(s_{i-1})| \;:\; a = s_0 < \cdots < s_n = t\right\},$$

which is the supremum over all partitions of $[a, t]$. We say that F has *bounded variation* on $[a, t]$ if $V_F(t) < \infty$.

Definition 3.1.3. Let X be a stochastic process. The *quadratic variation* $\langle X \rangle = (\langle X \rangle_t)_{t \geq 0}$ is defined by

$$\langle X \rangle_t = \lim_{mesh(\Delta_n) \to 0} \sum_{i=0}^{p_n - 1} (X_{t_{i+1} - X_{t_i}})^2,$$

where $mesh(\Delta_n) = \max\{(t_i^n - t_{i-1}^n) \, : \, i = 1, \ldots, n\}$ is the mesh-size of the partition $\pi = \{0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t\}$ of the interval $[0, t]$, provided it exists.

In the rest of this thesis, we will assume that the quadratic variation of stochastic processes of interest always exists. We can use the quadratic variation of two processes X and Y on the same space to define the *quadratic covariation.*

Definition 3.1.4. Let X, Y be two process on the same probability space. Then the *quadratic covariation* of X and Y is defined as

$$\langle X, Y \rangle = \frac{1}{2}\Big(\langle X + Y \rangle - \langle X \rangle - \langle X \rangle \Big).$$

In order to extend conditional expectation and variance to stochastic processes, we need to have subsequent σ-algebras that "keep up" with the process.

Definition 3.1.5. A *filtration* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of σ-algebras $(\mathcal{F}_t)_{t \geq 0}$ such that for all $0 \leq s \leq t < \infty$, we have that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. We set $\mathcal{F}_\infty = \sigma(\bigcup_{t \in \mathbb{R}_+} \mathcal{F}_t)$.

Definition 3.1.6. A stochastic process X is called *adapted* to filtration $(\mathcal{F}_t)_{t \geq 0}$ is X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$. The smallest filtration to which X_t is adapted is $\mathcal{F}_t^X = \sigma(X_s \, : \, 0 \leq s \leq t)$.

A *stopping time* is a random time that is measurable. Stopping times are often used for stopping rules that decide when stochastic processes are cut off.

Definition 3.1.7. A stopping time $\tau : \Omega \to [0, \infty]$ is a random variable such that for all $t \geq 0$,

$$\{\omega \, : \, \tau(\omega) \leq t\} \in \mathcal{F}_t$$

We are now all set to introduce martingales. Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.1.8. Let $M = (M_t)_{t \geq 0}$ denote a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $t \geq 0$,

- $(M_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$,
- $\mathbb{E}\left[|M_t|\right] < \infty$.

It is called

1. a martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$ if for all $s < t$: $\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s$,
2. a submartingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$ if for all $s < t$: $\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] \geq M_s$,
3. a supermartingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$ if for all $s < t$: $\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] \leq M_s$.

Some stochastic process that are not martingales can be changed into martingales by *localizing* them.

Definition 3.1.9. Let $M = (M_t)_{t\geq 0}$ be a continuous adapted process. M is called a *local martingale* if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ such that $\mathbb{P}(\lim_{n\to\infty} T_n = \infty) = 1$ and $(M_{T_n\wedge t} - M_0)_{t\geq 0}$ is a martingale. We say that $(T_n)_{n\geq 1}$ is localizing the process M.

We can even extend local martingales to *semimartingales* by allowing processes of bounded variation to be added.

Definition 3.1.10. A continuous adapted process X is called a *semimartingale* if it can be decomposed as

$$X_t = X_0 + M_t + V_t,$$

where M is a continuous local martingale and V is an adapted continuous process with bounded variation.

3.2 Brownian motion

Recall that a random variable has *density* f if

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x.$$

In this thesis, these random variables will mostly be *normally (Gaussian) distributed.* Recall that a random variable X is normally distributed with *mean* μ and *variance* σ^2 if its density is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We will now define a Gaussian vector.

Definition 3.2.1. Let $\mathbf{X} = (X_1, \ldots, X_d)'$ be a random vector. $\mathbf{X}$ is called a Gaussian vector if for all $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$, we have that $\lambda_1 X_1 + \cdots + \lambda_d X_d$ is normally distributed.

For a Gaussian vector $\mathbf{X} = (X_1, \ldots, X_d)'$, we can write the expectation as

$$\mu := \mathbb{E}\left[X\right] = (\mu_1, \ldots, \mu_d)',$$

where $\mathbb{E}\left[X_i\right] = \mu_i$ and *covariance matrix* $C = (c_{i,j})_{1\leq i,j\leq d}$ with

$$c_{i,j} = \mathbb{E}\left[(X_i - \mu_i)(X_j - \mu_j)\right].$$

We are now almost set to define Brownian motions. First, we need to define Gaussian processes.

Definition 3.2.2. Let $T \in \mathbb{R}_+ \cup \{\infty\}$. A stochastic process $(X_t)_{t\in[0,T[}$ is called a *Gaussian process* if for all $0 \leq t_1 \leq \cdots \leq t_n < T$, we have that $(X_{t_1}, \ldots, X_{t_n})'$ is a Gaussian vector.

Gaussian processes are uniquely determined by their means and covariance matrices in the following way:

$$\begin{aligned} \mu(t) &= \mathbb{E}\left[X_t\right], \\ c(s,t) &= \mathrm{Cov}(X_s, X_t). \end{aligned}$$

For certain forms of $\mu(t)$ and $c(s,t)$, a Gaussian process is also a Brownian motion.

Definition 3.2.3. A *Brownian motion* $B = (B_t)_{t \in [0,T[}$ is a Gaussian process with $\mu(t) = 0$ and $c(s,t) = s \wedge t$.

From the following proposition, we see that we can characterize Brownian motion also by their behavior on increments.

Proposition 3.2.4. The following are equivalent:

1. $B = (B_t)_{t \in [0,T[}$ is a Brownian motion.
2. (i) $X_0 = 0$ almost surely,
   (ii) for all $n \geq 2$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, we have that $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,
   (iii) for all $s \leq t$, we have that $X_t - X_s \sim \mathcal{N}(0, t-s)$.

Proof. See Proposition 3.1 in [22]. $\square$

3.3 Stochastic calculus

In this section, we define the stochastic integral and stochastic differential equation. We follow the constructive definition for the stochastic integral in means of a limit. Furthermore, we state the Itô formula for continuous semimartingales.

Definition 3.3.1. Let M be a continuous local martingale, $M_0 = 0$ and H an adapted continuous process. The *stochastic integral* of H with respect to M is defined by

$$\int_0^t H_s \mathrm{d}M_s = \lim_{mesh(\pi) \to 0} \sum_{i=0}^{p_n - 1} H_{t_i^n} (M_{t_{i+1}^n} - M_{t_i^n}),$$

where $\pi = \{0 \leq t_0^n < \cdots < t_n^n \leq t\}$ is a partition of $[0,t]$.

A stochastic process that can be written as the sum of a stochastic integral with respect to a Brownian motion and an integral with respect to time is called an *Itô process*.

Definition 3.3.2. Let X be an adapted stochastic process and let B be a Brownian motion. If X can be written as

$$X_t = X_0 + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \mu_s \mathrm{d}s,$$

where σ and μ are adapted process that are integrable, then X is called an *Itô process*.

The quadratic variation of an Itô process can be written in a simple form.

Proposition 3.3.3. Let X be an Itô process. Then

$$\langle X \rangle_t = \int_0^t \sigma_s^2 \mathrm{d}s,$$

or in differential form:

$$\mathrm{d}\langle X \rangle_t = \sigma_t^2 \mathrm{d}t$$

Proof. See [22]. $\square$

Theorem 3.3.4. *Let $X = (X^{(1)}, \cdots X^{(d)})$ be a d-dimensional continuous semimartingale and $f \in C^2(\mathbb{R}^d, \mathbb{R})$. We write $f(X_t)$ for $f(x_1, \ldots, x_d)$, then*

$$f(X_t) = f(X_0) + \sum_{j=1}^{d} \int_0^t \frac{\partial}{\partial x_j} f(X_s) dX_s^{(j)} + \frac{1}{2} \sum_{i,k=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_k} f(X_s) d\langle X^{(i)}, X^{(j)} \rangle_s,$$

or in differential form

$$df(X_t) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} f(X_t) dX_t^{(j)} + \frac{1}{2} \sum_{i,k=1}^{d} \frac{\partial^2}{\partial x_i \partial x_k} f(X_t) d\langle X^{(i)}, X^{(k)} \rangle_t. \tag{3.1}$$

Proof. See [22]. $\square$

An important consequence of Theorem 3.3.4 is the following. Consider $d = 2$ and $Y = (X_t, t)_{t \geq 0}$, then

$$f(X_t, t) = f(X_0, 0) + \int_0^t \frac{\partial}{\partial s} f(X_s, s) \mathrm{d}s + \int_0^t \frac{\partial}{\partial x} f(X_s, s) \mathrm{d}X_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(X_s, s) \mathrm{d}\langle X \rangle_s.$$

Similarly, we can write this in its differential form:

$$\mathrm{d}f(X_t, t) = \frac{\partial}{\partial t} f(X_t, t) \mathrm{d}t + \frac{\partial}{\partial x} f(X_t, t) \mathrm{d}X_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(X_t, t) \mathrm{d}\langle X \rangle_t.$$

Another corollary of Theorem 3.3.4 can be seen as a product rule for Itô calculus. We write it directly in differential form.

Corollary 3.3.5. Let $(X_t, Y_t)_{t \geq 0}$ be a 2-dimensional continuous semimartingale, then

$$\mathrm{d}(X_t Y_t) = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}\langle X, Y \rangle_t.$$

Proof. Take $f(x, y) = xy$, then it follows from 3.3.4. $\square$

In particular, if X and Y are Itô processes with $\sigma = 1$, we can further simplify this because $\mathrm{d}\langle X, Y \rangle_t = \mathrm{d}X_t \mathrm{d}Y_t$. This will be used in later chapters.

3.4 Percentiles of observations

A percentile q divides the probability distribution function into two intervals, one from $]-\infty, q]$ and the other from $]q, \infty[$. For instance, the 75th percentile says that 75% of the observations will lie below this value. In this thesis, these observations will be the outcomes of the simulations that are made in the model.

Definition 3.4.1. The *a-percentile* of a continuous and strictly monotone increasing distribution function F of X is the value ξ_a for which $F_X(\xi_a) = \mathbb{P}(X \leq \xi_a) = a$ for $0 < a < 1$. Hence we can find the a-percentile ξ_a by

$$\xi_a = F_X^{-1}(a).$$

Note that since the function F_X is continuous and strictly monotone increasing, the value ξ_a is unique and the inverse function F_X^{-1} is well-defined.

3.5 Hypothesis testing

In experimentation, we use statistical interference to be able to conclude about the statistical significance of certain outcomes. This is done by so-called *hypothesis testing*. In this set-up, we test several hypotheses against each other: the null hypothesis H_0 and the alternative hypotheses $H_1, H_2, \ldots H_m$, for some $m \in \mathbb{Z}$. In general, the null hypothesis H_0 states that there is no relationship between the measured phenomena. Then, a test statistic T with a certain distribution is determined. This is compared to the realized value t of the experimentation.

To draw conclusions about rejecting or accepting the null hypothesis H_0, we set a significance level α. Often, $\alpha = 0.05$ is chosen. For a two-sided test, the p-value of the realized is determined as follows:

$$p = 2 \min\{\mathbb{P}(T \leq t \mid H_0), \mathbb{P}(T \geq t \mid H_0)\}.$$

Now, if $p < \alpha$, we reject the null hypothesis H_0. Otherwise, the null hypothesis is accepted.

3.5.1 Kruskal-Wallis test

The Kruskal-Wallis test is used to analyze the variance in different groups of data points without assuming a certain parametric distribution of the underlying distribution. First, we rank all N data points and we group all the data in groups $i = 1, \ldots, g$ together. Then, the test statistic is given by

$$H = (N-1)\frac{\sum_{i=1}^{g} n_i(\overline{r_i} - \overline{r})^2}{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(r_{ij} - \overline{r})^2},$$

where

- n_i is the number of observations in group i,
- r_{ij} is the rank among all observations of observation j in group i,
- $\overline{r_i} = \dfrac{\sum_{j=1}^{n_i} r_{ij}}{n_i}$ is the average rank of all observations in group i,
- $\overline{r} = \frac{1}{2}(N+1)$ is the average of all r_{ij}.

3.5.2 Bonferroni correction

For multiple comparisons, we need to correct the significance level α for the number of hypothesis tests that are performed simultaneously. This procedure is called the *Bonferroni correction*. If m hypothesis tests are performed at the same, this is done by replacing α with $\frac{\alpha}{m}$. We can equivalently replace the calculated p-values with pm.

Chapter 4

Life-cycle investing

In this chapter, we motivate and construct the life-cycle methods that we will test in the model. First, we give a short overview of the known literature about life-cycle theory. Then, we explain the role of human capital and utility function theory. Next, we will derive the optimal allocation of stocks according to the life-cycle theory. Finally, we will highlight the different life-cycle methods that we use.

4.1 Life-cycle literature

In this section, we give an overview of the literature on life-cycle theory.

The theory of life-cycle saving and investing has been a matter of academic interest after Keynes built the basics for a macroeconomic theory and associated policy. It arose after portfolio selection theory proved to be useful for investors that want to invest in risky assets. The ideas of portfolio selection originate from the work of Markowitz in 1952, [15], who showed that choosing different assets that are not fully correlated can reduce the investment risk while keeping a high expected return.

Under some mild assumptions, mathematical derivations of optimal portfolio allocation had been constructed in the fifties and sixties. These models, however, were only maximized over one period. The first economists that came up with multi-period models to solve this dynamic portfolio choice problem for the allocation between safe assets (bonds) and risky assets (stocks), were Samuelson [23] and Merton [16]. In their publications, it was shown that a consumer should maintain a constant fraction of savings invested in stocks to receive an optimal pay-off if investment opportunities are constant. This conclusion, however, was stated under the unrealistic assumption that an agent has no labor income.

In 1992, Bodie, Merton and Samuelson, [4], expanded the existing models with flexible labor income. Their main result shows that the fraction of financial wealth invested in stocks should decline with age. This is due to the assumption that human capital, which will be explained in the next section, is seen as a risk-free asset. As human capital declines over time, financial wealth should be shifted towards the safe assets to be in line with the original models Samuelson and Merton proposed. Other papers that assume the safe nature of human capital are Merton [16], Heaton and Lucas [11], Jaganathan and Kocherlacota [13], Campbell and Viceira [7], and Viceira [29]. A survey of recent academic literature on financial planning over the life-cycle can be found in Bovenberg, Koijen, Nijman, and Teulings [5].

Recently, however, empirical evidence has shown that the allocation of wealth to stocks does not completely follow the result given by Bodie, Merton and Samuelson in [4]. Typically, the fraction invested in the risky asset show a humped-shaped pattern, which is low for young agents, increases towards the middle of the working life and then decreases as retirements gets closer. Empirical evidence of this view can, for instance, be seen in Figure 4.1.1.

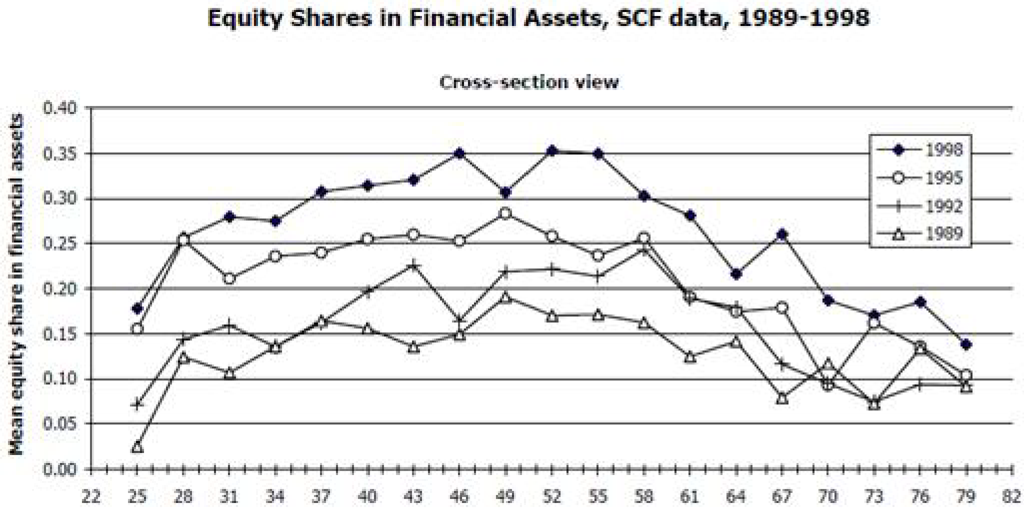


Figure 4.1.1: Equity Shares in Financial Assets, 1989-1998. Source: Ameriks and Zeldes (2004)

We can model a similar pattern by changing our view on human capital: instead of being an implicit holding in a risk-free asset, human capital can be stochastically affected by changes in wage and economic shocks. The risk in human capital can then be decomposed into an economic-wide stochastic part, such as recession, and an individual stochastic part, such as sudden unemployment.

The assumption that human capital is of a risk-free nature follows from assuming that it is not correlated with capital return. Benzoni, Collin-Dufresne and Goldstein [3] claim that in the long run, labor income and capital income are highly cointegrated. This finding implies that the risk in young workers' labor income has a more risky nature and they should therefore hold more financial wealth in risk-free assets. Other papers that study riskiness of labor income are Viceira [29] and Cocco et al. [9].

A recent comparison between the risk-free and risky view of human capital based on Dutch data has been done by Minderhoud, Molenaar and Ponds [18]. In this thesis, the risk-free view of human capital is used.

4.2 Human capital explained

The theory of optimal life-cycle investment has been extensively studied in the academic literature. It has been developed as an answer to the question of what is the amount of money that should be saved for retirement and what is the amount that should be spent now and is therefore relevant for every working person. To answer this question, we must have a good guess of how much wealth the person will earn until he retires. This wealth can be decomposed into two parts: current wealth and future wealth. It is in general not difficult to value the current wealth, because this will mainly consist of savings in a bank, risk-free investments (i.e. bonds) and risky investments (stocks), which are at large valued by the market. However, an important decision to make is how to make an estimate of how much wealth he will *probably* earn in the rest of his life.

It is clear that the future income of an agent cannot be determined deterministically, since there are a lot of uncertainties about the development of the income. An agent can become unemployed, for instance, or switch to a different kind of job with a different wage. We can, however, make an estimate by looking at how much income we expect the agent to earn each year. If we do this for all the future years up to the age of retirement, we can define the human capital $HC(t)$ as the present value of the expected future income:

$$HC(t) = \mathbb{E}\left[\sum_{s=1}^{T-t} \mathrm{PV}(Y_s)\right] = \sum_{s=1}^{T-t} \frac{1}{(1+r_f)^s} \mathbb{E}\left[Y_s\right], \tag{4.1}$$

where Y_s is the (stochastic) annual labor income in year s, r_f is the risk-free return and $T-t$ marks the length of the remaining working life in years. Then the *total wealth* $W(t)$ of an agent at time t is the sum of the human capital $HC(t)$ and the financial wealth $FW(t)$:

$$W(t) = HC(t) + FW(t), \tag{4.2}$$

where the financial wealth is the total of the agents savings and invested assets. The individual's goal is to maximize his consumption $c(t)$ over his entire lifetime, limited by the magnitude of the total wealth $W(t)$. The consumption $c(t)$ is defined by the part of the total wealth $W(t)$ that is not saved. It is also conventional to include an agent's risk appetite in the model. He can influence the risk taken by investing a fraction α_t in a risky asset and $1-\alpha_t$ is a risk-free asset at each time instant t. It will follow that the development of the ratio α_t depends highly on the nature of human capital. In the following section, we will treat two views on human capital.

4.3 Risk-free view on human capital

In the classical portfolio choice problem, the total wealth is invested in two assets, a risk-free and a risky asset. We now assume that the human capital is risk-free. Because it pays out every year "dividends" in the form of labor income, we view it as extra capital in the risk-free asset. In this model, the agent's wealth consist of tradable financial assets and untradable human capital. To see why human capital is untradable, let us assume towards contradiction that it is tradable. The agent sells his future labor income. But then he has no more incentive to work and the sold claims on his future labor income become worthless. Hence human capital must be untradable. A global development of human capital, financial wealth and total wealth over the lifetime is given in Figure 4.3.1.

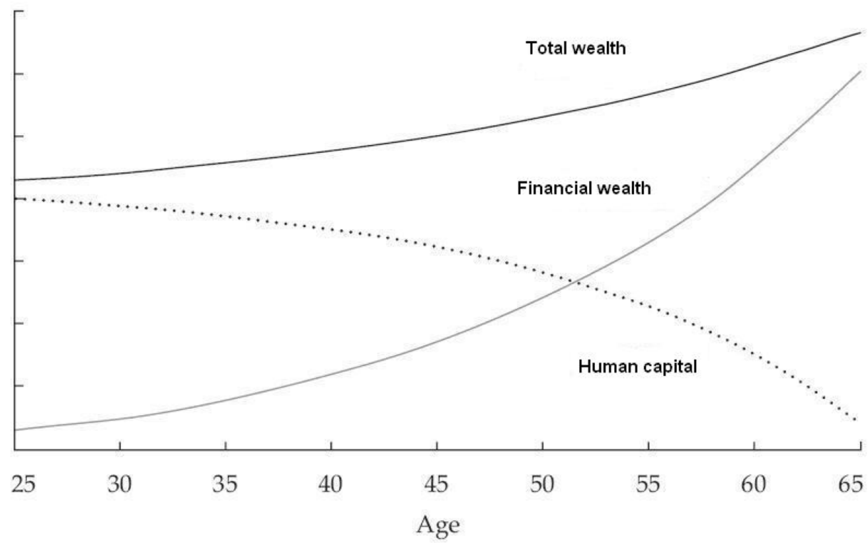


Figure 4.3.1: Development of wealth components over the lifetime.

4.3.1 Utility function

Historically, utility was formulated as a quantitative measurement of the satisfaction one derives from the choice for a certain good. Starting in the field of moral philosophers, it was embraced and adapted by neoclassical economists to a utility function that represent the ordering of a consumer's preference of a set of choices. The functional representation opened the opportunity to use mathematical tools to evaluate and prove properties. If choice y is preferred at least as much as choice x, let us denote that by $x \preceq y$. If y is preferred strictly over x, then we denote $x \prec y$ and if we can not make a choice between x and y, then we write $x \sim y$. Formally, a utility function is a quantification of the preference relation to the real numbers:

Definition 4.3.1. Let X be a set of choices. A *utility function* $U : X \to \mathbb{R}$ represent a preference relation $\preceq$ if and only if for all $x, y \in X$,

$$U(x) \leq U(y) \iff x \preceq y.$$

Note that the existence of a utility function given a preference relation is not clear so far. In order to do this, we need to assume completeness and transitivity of the set of choices.

Axiom 4.3.2. For every pair $x, y \in X$, either $x \preceq y$, $y \preceq x$ or both.

Axiom 4.3.3. For every triple $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

These axioms provide the basic structure on the preferences that allow it to be fully captured by a utility function.

Theorem 4.3.4. *Suppose that X admits a countable order dense subset*[1] *and that Axioms (4.3.2) and (4.3.3) hold, then there exists a utility function $U : X \to \mathbb{R}$ that represents the preference relation $\preceq$.*

Axioms 4.3.2 and 4.3.3 are not enough to let decision-makers make rational choices when confronted with risky outcomes of different choices. Suppose we can assign probabilities to each outcome of a choice made. This allows us to look at linear combinations of outcomes and corresponding probabilities, called *lotteries*.

Definition 4.3.5. Let A_i for $i = 1, 2, \ldots$ denote different outcomes of a random variable X, with corresponding probabilities p_i. A lottery L is any linear combination of the A_i with the p_i as constants, i.e., L can be written as

$$L = \sum_{i=1}^{\infty} p_i A_i.$$

Note that these different outcomes A_i are usually expressed as amounts of money. In the rest of this thesis, we will assume that the different realizations of a lottery are amounts of money or wealth. Since we are in a probabilistic setting now, it makes sense to look at the expectation of the utility of the set of outcomes.

Definition 4.3.6. Let X be a set of choices. If U only has values on a finite or countably infinite set $\{x_1, x_2, \ldots, x_n\}$, then the *expected utility* is

$$\mathbb{E}\left[U(X)\right] = \sum_{i=1}^{n} p_i U(x_i),$$

where $n \to \infty$ for a countably infinite set and p_i is the probability that x_i is realized.

In 1947, John von Neumann and Oskar Morgenstern published sufficient conditions for a decision-maker to act as if he is maximizing his expected utility under a wealth constraint. Let us first give two more axioms.

Axiom 4.3.7. If $x \prec y \prec z$, then there exists a probability $p \in (0, 1)$ such that

$$px + (1 - p)z \sim y.$$

This axiom, known as *continuity*, ensures that there is a tipping point between being better than and worse than a give middle option. With other words, the space of options is in a continuous spectrum.

Axiom 4.3.8. If $x \preceq y$, then for every z and $p \in [0, 1]$,

$$px + (1 - p)z \preceq py + (1 - p)z.$$

[1] A subset is said to be *order dense* if it is dense with respect to the order $\preceq$: i.e., if for all $x, y \in X$ for which $x \preceq y$, there is a $z \in X$ such that $x \preceq z \preceq y$.

This axiom implies that if individuals have a preference between two simple lotteries, this preference remains when compound lotteries are considered.

If Axioms (4.3.2), (4.3.3), (4.3.7)[2] and (4.3.8) are satistied, then any decision-maker is called a *rational agent*. The following theorem holds.

Theorem 4.3.9. *For any rational agent, i.e., decision-maker satisfying Axioms (4.3.2), (4.3.3), (4.3.7) and (4.3.8), there exists a utility function U such that for any two lotteries L, M,*

$$L \prec M \quad \textit{if and only if} \quad \mathbb{E}(U(L)) < \mathbb{E}(U(M)).$$

Proof. See [19]. □

This theorem shows that U can be uniquely determined up to adding a constant and multiplying with a scalar by preferences between simple lotteries. Utility functions as such can then be seen as the ordering individuals give to certain options. The shape of the graph of a utility function will tell us something about the *risk appetite* of agents. Let us look at an example.

Example 4.3.10. An individual has the choice between two options: one with a guaranteed payoff and one without. In the first option, the individual receives €50. In the second option, a fair coin is flipped to determine whether the person receives €100 or nothing. The expected payoff is the same in both scenarios, meaning that a person who is indifferent about risk does not have a preference for one of the two options. However, individuals may have different risk attitudes:

- A *risk averse* person would accept the certain payment over the gamble.
- A *risk neutral* person is indifferent between the two choices.
- A *risk loving* person would go for the gamble instead of the certain payment.

In Figure 4.3.2 we see the shape of the graph for all three categories of risk appetite. In general, the risk averse agent's function is concave, the risk neutral agent's function is linear and the risk loving agent's function is convex.

In this thesis, we will assume that agents are *risk averse*. There are several measure of the risk aversion that a given utility function entails. We will focus on the so-called *Arrow-Pratt measure of relative risk aversion*:

Definition 4.3.11. Let $U : [0, \infty) \to \mathbb{R}$ denote a utility function that is defined for every wealth W. Then the relative risk aversion $R(W)$ is defined as

$$R(W) = \frac{-WU''(W)}{U'(W)}.$$

For one kind of utility, this relative risk aversion is constant, namely the *power utility*.

Definition 4.3.12. The power utility function U is defined as

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma},$$

for $\gamma \geq 0, \gamma \neq 1$.

[2]Instead of this axiom, the *Archimedean property* can be assumed, stating that for all $x \preceq y \preceq z$, there exists a probability $p \in [0, 1]$ such that $(1-p)x + pz \prec y \prec px + (1-p)z$.

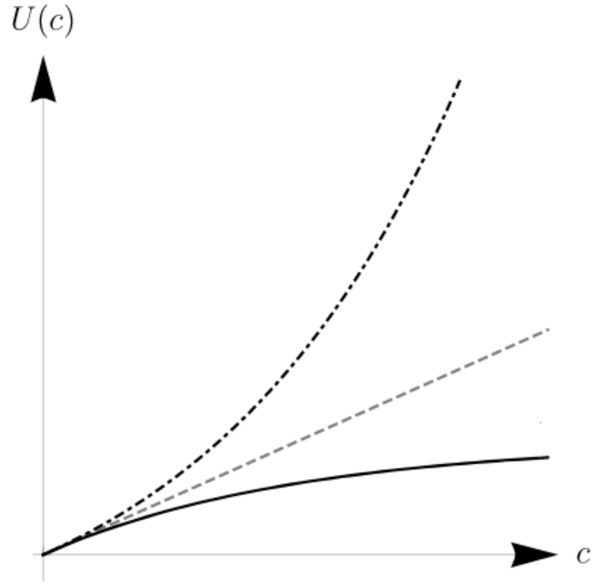


Figure 4.3.2: Utility function of a risk loving (upper graph), a risk neutral (middle graph) and a risk averse (lower graph) person.

4.3.2 Derivation of optimal allocation

Let us now make some assumptions to find an expression for the optimal fraction invested in the risky asset. We start without considering the human capital and will incorporate this later. Assume that there are two assets categories, stocks (risky) and bonds (risk-free). Denote the return of the risk-free asset at time t by $R_{b,t}$ and the return of the risky asset at time t by $R_{s,t}$. We will assume that $R_{b,t}$ is deterministic. The conditional variance of the log return on the risky asset is $\sigma^2_{s,t}$. Suppose the agent invests a fraction α_t in the risky asset and $1-\alpha_t$ in the risk-free asset. The portfolio return $R_{p,t+1}$ at time $t+1$ then equals

$$R_{p,t+1} = \alpha_t R_{s,t+1} + (1-\alpha_t)R_{b,t+1} = R_{b,t+1} + \alpha_t(R_{s,t+1} - R_{b,t+1}). \tag{4.3}$$

The expected portfolio return given the information up to t is $\mathbb{E}_t\left[R_{p,t+1}\right] = R_{b,t+1} + \alpha_t(\mathbb{E}_t\left[R_{s,t+1}\right] - R_{b,t+1})$, and hence by the scaling property of the variance in Proposition A.5.4, the portfolio variance is $\sigma^2_{p,t} = \alpha_t^2\sigma^2_{s,t}$.

We will recall that the total wealth $W_t = W(t)$ is defined as the sum of the human capital $HC(t)$ and the future wealth $FW(t)$. The maximization problem in terms of utility is

$$\max_{\alpha_t} \mathbb{E}_t\left[U(W_{t+1})\right], \tag{4.4}$$

such that it satisfies the wealth constraint

$$W_{t+1} = (1 + R_{p,t+1})W_t. \tag{4.5}$$

We can rewrite this wealth constraint in log form as

$$w_{t+1} = r_{p,t+1} + w_t, \tag{4.6}$$

where $w_{t+1} = \log W_{t+1}$ and $r_{p,t+1} = \log(1 + R_{p,t+1})$. We will use the log form later. This utility is a concave function in wealth W_{t+1}, resembling risk-aversion of investors as we have seen in the previous subsection. There are several functional forms that we could choose to satisfy concavity. We can minimize the options by assuming that the asset returns are log-normally distributed. The results of the maximization problem are then simple if we assume power utility, given by

$$U(W_{t+1}) = \frac{W_{t+1}^{1-\gamma}}{1-\gamma},$$

or equivalently,

$$U(W_{t+1})(1-\gamma) = W_{t+1}^{1-\gamma}, \tag{4.7}$$

where $\gamma := R(W_t)$ is the constant relative risk aversion. We assume that $\gamma > 0, \gamma \neq 1$. This function also satisfies the favorable property that absolute risk aversion declines in wealth. Now note that since log is a strictly increasing function, maximization of the utility is equivalent to maximization the log utility. So rewriting the maximization problem in log form and using Equation 4.7, we want to solve

$$\max_{\alpha_t} \mathbb{E}_t \left[U(W_{t+1})\right](1-\gamma) = \max_{\alpha_t} \log \mathbb{E}_t \left[W_{t+1}^{1-\gamma}\right] \tag{4.8}$$

$$= \max_{\alpha_t} \Big((1-\gamma)\mathbb{E}_t \left[\log W_{t+1}\right] + \frac{1}{2}(1-\gamma)^2 \mathrm{Var}_t \left[\log W_{t+1}\right]\Big), \tag{4.9}$$

where we use Lemma B.1.1 from the Appendix. Now dividing by $1-\gamma$ and using the log form of the wealth constraint from Equation 4.6, we rewrite it as

$$\max_{\alpha_t} \mathbb{E}_t \left[U(W_{t+1})\right] = \max_{\alpha_t} \Big(\mathbb{E}_t \left[r_{p,t+1} + w_t\right] + \frac{1}{2}(1-\gamma)\mathrm{Var}_t \left[r_{p,t+1} + w_t\right]\Big), \tag{4.10}$$

and finally noting that conditional on the information up to time t, w_t is a constant, the maximization can be rewritten as

$$\max_{\alpha_t} \mathbb{E}_t \left[U(W_{t+1})\right] = \max_{\alpha_t} \Big(\mathbb{E}_t \left[r_{p,t+1}\right] + \frac{1}{2}(1-\gamma)\sigma_{p,t}^2\Big), \tag{4.11}$$

where $\sigma_{p,t}^2 = \mathrm{Var}_t \left[r_{p,t+1}\right]$ denotes the conditional portfolio variance. Since we are looking for a relation between the fraction α_t and the return on the two asset categories, we need to find a way to connect the log portfolio return to the log returns of the individual assets. We can rewrite Equation 4.3 such that we have

$$\frac{1 + R_{p,t+1}}{1 + R_{b,t+1}} = 1 + \alpha_t \Big(\frac{1 + R_{s,t+1}}{1 + R_{b,t+1}} - 1\Big).$$

Taking logs, this can be written as

$$r_{p,t+1} - r_{b,t+1} = \log\left(1 + \alpha_t \Big(\frac{1 + R_{s,t+1}}{1 + R_{b,t+1}} - 1\Big)\right),$$

$$= \log\left(1 + \alpha_t \Big(\exp(r_{s,t+1} - r_{b,t+1}) - 1\Big)\right).$$

This equation gives a relation between the log excess return on the risky asset, $r_{s,t+1} - r_{b,t+1}$, and the log excess on the portfolio, $r_{p,t+1} - r_{b,t+1}$. We can approximate this relation by a second-order

Taylor expansion of the function $f_t(x) = \log(1 + \alpha_t(e^x - 1))$ around the point $x = 0$. This gives us the approximation

$$f_t(x) \approx f_t(0) + f_t'(0)x + \frac{1}{2}f_t''(0)x^2.$$

We can easily check that $f_t'(0) = \alpha_t$ and $f_t''(0) = \alpha_t(1 - \alpha_t)$. Furthermore, we can make a simplification by replacing $x^2 = (r_{s,t+1} - r_{b,t+1})^2$ by its conditional variance $\sigma_{s,t}^2$. Combining these observations leads to the relation

$$r_{p,t+1} - r_{b,t+1} = f_t(r_{s,t+1} - r_{b,t+1}) = \alpha_t(r_{s,t+1} - r_{b,t+1}) + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_{s,t}^2. \tag{4.12}$$

In the limit as the time intervals shrink, this approximation is exact. Plugging in this approximation in Equation 4.11 and using the fact that $\sigma_{p,t}^2 = \alpha_t^2\sigma_{s,t}^2$, the maximization function becomes

$$\max_{\alpha_t} \mathbb{E}_t\left[U(W_{t+1})\right] = \max_{\alpha_t} \Big(r_{b,t+1} + \alpha_t(\mathbb{E}_t\left[r_{s,t+1}\right] - r_{b,t+1}) + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_{s,t}^2 + \frac{1}{2}(1 - \gamma)\alpha_t^2\sigma_{s,t}^2\Big). \tag{4.13}$$

Let $\mu_{t+1} := \mathbb{E}_t\left[r_{s,t+1}\right] - r_{b,t+1}$ denote the conditional expected excess return. Define the function $g : [0, 1] \to \mathbb{R}$ by

$$g(\alpha_t) = r_{b,t+1} + \alpha_t\mu_{t+1} + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_{s,t}^2 + \frac{1}{2}(1 - \gamma)\alpha_t^2\sigma_{s,t}^2.$$

Then, taking the derivative gives

$$\begin{aligned} g'(\alpha_t) &= \mu_{t+1} + \frac{1}{2}(1 - \alpha_t)\sigma_{s,t}^2 - \frac{1}{2}\alpha_t\sigma_{s,t}^2 + (1 - \gamma)\alpha_t\sigma_{s,t}^2, \\ &= \mu_{t+1} + \frac{1}{2}\sigma_{s,t}^2 - \alpha_t\gamma\sigma_{s,t}^2. \end{aligned}$$

Now, setting this derivative to zero and solving for α_t yields the solution for the optimal allocation:

$$\alpha_t = \frac{2\mu_{t+1} + \sigma_{s,t}^2}{2\gamma\sigma_{s,t}^2}. \tag{4.14}$$

Note that this is indeed a maximum, since

$$g''(\alpha_t) = -\gamma\sigma_{s,t}^2 < 0,$$

for all α_t because $\gamma, \sigma_{s,t}^2 > 0$. This solution for α_t is independent of the time horizon: suppose we consider an n-period investment. The expected return on the risky asset and the return on the risk-free asset both scale by n, and the variance of the portfolio return equals $\frac{1}{n}\alpha_t^2\sigma_{s,t}^2 n^2 = \alpha_t^2\sigma_{s,t}^2 n$, so Equation 4.14 remains the same.

4.3.3 Including human capital

Let us now include the human capital in the solution. As mentioned before, it can be seen as an implicit holding in a risk-free asset that pays out labor income as dividend. If we now take untradable human capital into account, how does Equation 4.14 change? By investing $\alpha_t W(t)$ in the risky asset, the optimal portfolio allocation is achieved. This means that the adjusted fraction $\hat{\alpha_t}$ of the financial wealth invested in the risky asset should satisfy

$$\hat{\alpha_t} FW(t) = \alpha_t (FW(t) + HC(t)), \tag{4.15}$$

so we can find the optimal fraction by combining this with Equation 4.14:

$$\hat{\alpha_t} = \alpha_t \frac{FW(t) + HC(t)}{FW(t)} = \frac{2\mu_{t+1} + \sigma_{s,t}^2}{2\gamma\sigma_{s,t}^2}\left(1 + \frac{HC(t)}{FW(t)}\right), \tag{4.16}$$

where μ_{t+1} denotes the conditional expected excess return on the risky asset. If the conditional expected excess return and the variance are constant, the development of $\hat{\alpha_t}$ only depends on the human capital and the financial wealth. In the light of Figure 4.3.1, the optimal allocation of financial wealth to the risky asset should decline over time. This is because the human capital declines as 'dividend' is paid out in the form of labor income. This labor income is partially consumed and partially saved to increase financial wealth. The fraction of financial wealth invested in stocks and bonds is shown in Figure 4.3.3.

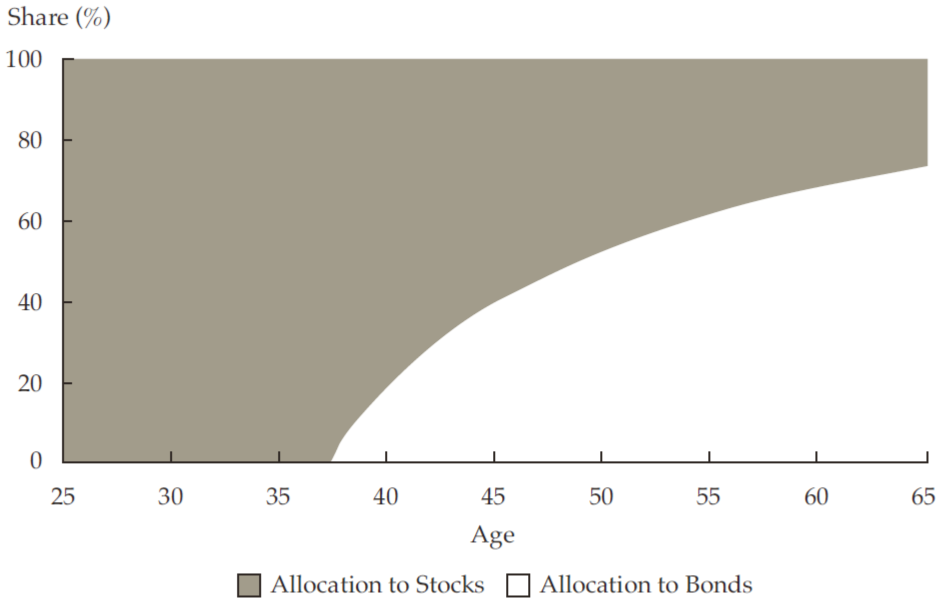


Figure 4.3.3: Fraction of financial wealth invested in stocks and bonds over the lifetime.

4.4 Strategies

In this section, two different ways of determining the allocation of assets are described. These different algorithms will be called *strategies*. As described before, this age-based risk appetite gives rise to a way of compensating agents for the risk they are taking.

We assume that all employees start working at the age of 25 and retire at the age of 67. Furthermore, everyone between 25 and 67 is assumed to work. In all the following, $\overline{g}$ will denote the age at which the investment strategy changes.

4.4.1 Linear decreasing life-cycle

In this strategy, the allocation of assets can be characterized by

$$\alpha_t^{(g)} = \begin{cases} p_{start}, & \text{if } g \leq \overline{g}, \\ -\dfrac{p_{start} - p_{end}}{67 - \overline{g}} g + p_{start} + \dfrac{p_{start} - p_{end}}{67 - \overline{g}} \overline{g}, & \text{if } \overline{g} < g < 67, \\ p_{end}, & \text{if } g \geq 67, \end{cases} \tag{4.17}$$

where $\alpha_t^{(g)}$ is the fraction invested in stocks for the cohort g at time t.

4.4.2 Dynamic life-cycle

The dynamic life-cycle is an individual life-cycle investment strategy, suggested and motivated in [1]. Let R denote the target compounded annual rate of return and r_s the realization of the return for the cohort with age s. Let $\overline{g}$ denote the age at which the portfolio is built down and 67 years the pension age. Allocation of assets is based on the following algorithm:

- For the first $g \leq \overline{g}$ years of an employee's career, a fraction of p_{start} is invested in stocks.
- After $g \in \left[\overline{g}, \frac{1}{2}(\overline{g} + 67)\right]$ years, if $\prod_{s=25}^{g} (1 + r_s) \geq (1 + R)^{g-25}$, set the allocation of assets in stocks to p_{middle}. Otherwise, set to p_{start}.
- After $g \in \left[\frac{1}{2}(\overline{g} + 67)\right], 67]$ years, if $\prod_{s=25}^{g} (1 + r_s) \geq (1 + R)^{g-25}$, set the allocation of assets in stocks to p_{end}. Otherwise, set to p_{start}.
- For $g \geq 67$, the allocation of assets to stocks is fixed at p_{end}.

Chapter 5

Methodology

In this chapter, the underlying financial market model and pension fund model are explained. In Section 5.1, the Uniform Calculation Method Model is laid out and several mathematical derivations are given for the used results. In Section 5.2, we construct a stylized pension fund model that suits the life-cycle setting.

The Dutch population is extracted from the data of the Dutch Central Bureau for Statistics and mortality rates are approximated by the Dutch Royal Actuarial Society. In all following subsections, we will assume that time step $\Delta t = 1$ is used.

5.1 Uniform Calculation Method Model

The financial market in the Netherlands is modelled in this section. We will estimate the stock market, the bond market and the interest rate. In order to estimate these processes, we need to consider the inflation process as well. The assumptions are based on the CPB (Dutch Bureau for Economic Policy Analysis) background document by Nick Draper [8]. The prime symbol denotes the matrix transpose in the following.

We construct a portfolio with a stock index, long-term nominal bonds and nominal money[1]. The interest rate and expected inflation are modelled by two variables that represent the uncertainty and dynamics. These state variables are collected in 2-vector X_t. The instantaneous real interest rate (r_t) and the instantaneous expected inflation (π_t) develop according to

$$r_t = \delta_{0,r} + \delta'_{1,r} X_t,$$

$$\pi_t = \delta_{0,\pi} + \delta'_{1,\pi} X_t,$$

where $\delta_{0,*}$ are scalars that give the starting values and $\delta_{1,*}$ are 2-vectors that give the correlation between interest rate and inflation. The state variables collected in X_t following a mean-reverting process around zero:

$$\mathrm{d}X_t = -KX_t\mathrm{d}t + \Sigma'_X \mathrm{d}Z_t, \tag{5.1}$$

[1]"Nominal" refers to the value of an asset, without taking inflation and other factors into account.

with K a 2×2-matrix of parameters, $\Sigma'_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and Z a 4-dimensional Brownian motion.

The price index (Π_t) is influenced by the inflation and the same Brownian motion Z:

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \sigma'_\Pi \mathrm{d}Z_t, \tag{5.2}$$

with σ_Π a 4-vector and $\Pi_0 = 1$.

We use the following discrete approximation of the Stochastic Differential Equation 5.2 to determine the price inflation in the URM model:

$$\frac{\Pi_{t+1}}{\Pi_t} = \exp\Big((\pi_t - \frac{1}{2}\sigma'_\Pi \sigma_\Pi)\Delta t + \sigma'_\Pi Z(\Delta t)\Big), \tag{5.3}$$

where $Z'(\Delta t)$ is a 4-vector consisting of 4 independent normal random variables $Z_i(\Delta t) \sim \mathcal{N}(0, \Delta t)$, for $i \in \{1, 2, 3, 4\}$.

We further model the risk premium Λ_t as

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t,$$

with Λ_0 a 4-vector and Λ_1 a 4×2-matrix. The present value of a future cash value can be computed by multiplying the future value by a discount factor and taking the expectation. We will use this to calculate present values of future stock and zero coupon bond prices. The nominal stochastic discount factor (ϕ_t) develops as

$$\frac{\mathrm{d}\phi_t}{\phi_t} = -R_t \mathrm{d}t - \Lambda'_t \mathrm{d}Z_t, \tag{5.4}$$

where R_t is the nominal instantaneous interest rate. R_t is also updated by state variables in X_t:

$$R_t = R_0 + R'_1 X_t. \tag{5.5}$$

We will later see that we can express the values of R_0 and R_1 in terms of other parameters in this section.

The stock index (S_t) develops as a geometric Brownian motion:

$$\frac{\mathrm{d}S_t}{S_t} = (R_t + \eta_S)\mathrm{d}t + \sigma'_S \mathrm{d}Z_t, \tag{5.6}$$

where we let $S_0 = 1$, σ_S a 4-vector and η_S the equity risk premium.

In accordance with the approximation of the price inflation in Equation 5.3, we use the following discrete approximation of the Stochastic Differential Equation 5.6 to compute the stock returns in the URM model:

$$\frac{S_{t+1}}{S_t} = \exp\Big((R_t + \eta_S - \frac{1}{2}\sigma'_S \sigma_S)\Delta t + \sigma'_S Z(\Delta t)\Big),$$

where $Z'(\Delta t)$ is a 4-vector consisting of 4 independent normal random variables $Z_i(\Delta t) \sim \mathcal{N}(0, \Delta t)$, for $i \in \{1, 2, 3, 4\}$.

For the nominal zero-coupon bond P, we assume the fundamental pricing equation,

$$\mathbb{E}\left[\mathrm{d}\phi P\right] = 0,$$

which formulates that the expected value of the discount bond price does not change over time. By Corollary 3.3.5, we have

$$\begin{aligned}
\mathbb{E}\left[\mathrm{d}\phi P\right] &= \mathbb{E}\left[P\mathrm{d}\phi + \phi\mathrm{d}P + \mathrm{d}\phi \cdot \mathrm{d}P\right], \\
\mathbb{E}\left[\mathrm{d}\phi P\right] &= \mathbb{E}\left[P \cdot \frac{\mathrm{d}\phi}{\phi} \cdot \phi + \phi \cdot \mathrm{d}P + \frac{\mathrm{d}\phi}{\phi} \cdot \phi \cdot \mathrm{d}P\right], \\
\mathbb{E}\left[\mathrm{d}\phi P\right] &= \mathbb{E}\left[P \cdot \frac{\mathrm{d}\phi}{\phi} \cdot \phi + (\phi + \frac{\mathrm{d}\phi}{\phi} \cdot \phi) \cdot \mathrm{d}P\right] = 0. \qquad (5.7)
\end{aligned}$$

We assume that the nominal zero-coupon bond price is a function of time and state, so $P = P(X,t)$, then by Theorem 3.3.4 and inserting Equation 5.1 we have

$$\begin{aligned}
\mathrm{d}P &= P_X'\mathrm{d}X + P_t\mathrm{d}t + \frac{1}{2}\mathrm{d}X'P_{XX'}\mathrm{d}X, \\
\mathrm{d}P &= P_X'(-KX_t\mathrm{d}t + \Sigma_X'\mathrm{d}Z_t) + P_t\mathrm{d}t\frac{1}{2}Z_t'\Sigma_X P_{XX'}\Sigma_X'\mathrm{d}Z_t, \\
\mathrm{d}P &= \Big(P_X'(-KX_t) + P_t + \frac{1}{2}\Sigma_X P_{XX'}\Sigma_X'\Big)\mathrm{d}t + P_X'\Sigma_X'\mathrm{d}Z_t. \qquad (5.8)
\end{aligned}$$

Substituting Equations 5.4 and 5.8 into Equation 5.7 gives an expectation of several stochastic variables which we can collect in terms of the differentials $\mathrm{d}t$, $\mathrm{d}t^2$, $\mathrm{d}t\mathrm{d}Z_t$ and $\mathrm{d}Z_t^2$. In the limit as $\mathrm{d}t$ tends to 0, the $\mathrm{d}t^2$ and $\mathrm{d}t\mathrm{d}Z_t$ terms disappear and the quadratic terms of the $\mathrm{d}Z_t^2$ term tend to $\mathrm{d}t$. Hence, we see that in order to keep the expectation zero, the $\mathrm{d}Z_t^2$ and $\mathrm{d}t$ terms should also equal 0:

$$P_X'(-KX_t) + P_t + \frac{1}{2}tr(\Sigma_X P_{XX'}\Sigma_X') - PR_t - P_X'\Sigma_X'\Lambda_t = 0. \qquad (5.9)$$

We now include the maturity time T as a variable of P. Let $\tau = T - t$ denote the duration to maturity, then it can be shown that the price of a zero-coupon bond $P = P(X_t, t, T)$ is of the form

$$P(X_t, t, t+\tau) = \exp\Big(A(\tau) + B(\tau)'X_t\Big). \qquad (5.10)$$

Now, we try to find the functions corresponding to A and B. By taking several partial derivatives of Equation 5.10 and by noting that $\mathrm{d}\tau = -\mathrm{d}t$, we find

$$P_X = P \cdot B,$$

$$\begin{aligned}
P_{XX'} &= P \cdot BB', \\
P_t = -P_\tau &= P \cdot (-\dot{A} - \dot{B}'X_t),
\end{aligned}$$

where $\dot{A}$ and $\dot{B}$ denote the derivatives of A and B with respect to τ. Substituting this derivatives and the formulas for R_t and Λ_t in Equation 5.9 and dividing by P gives us

$$B'(-KX_t) + (-\dot{A} - \dot{B}'X_t) + \frac{1}{2}B'\Sigma_X'\Sigma_X B - (R_0 + R_1'X_t) - B'\Sigma_X'(\Lambda_0 + \Lambda_1 X_t) = 0,$$

where we used that $tr(\Sigma_X BB'\Sigma_X') = tr(B'\Sigma_X'\Sigma_X B) = B'\Sigma_X'\Sigma_X B$. Collecting stochastic and non-stochastic terms, we find that

$$(-B'K - R_1' - B'\Sigma_X\Lambda_1 - \dot{B}')X_t + (\frac{1}{2}B'\Sigma_X'\Sigma_X B - R_0 - B'\Sigma_X'\Lambda_0 - \dot{A}) = 0.$$

Since both the stochastic and the non-stochastic terms should equal zero, we can find that

$$\dot{A}(\tau) = \frac{1}{2}B'(\tau)\Sigma_X'\Sigma_X B(\tau) - R_0 - B'(\tau)\Sigma_X'\Lambda_0,$$

$$\dot{B}(\tau) = -R_1 - (K' + \Lambda_1'\Sigma_X)B(\tau).$$

By definition of the nominal zero-coupon bond, we have that the bond price equals 1 if $\tau = 0$. Hence, the solution of to this set of differential equations is

$$B(\tau) = (K' + \Lambda_1'\Sigma_X)^{-1}\left[\exp(-(K' + \Lambda_1'\Sigma_X)\tau)\right]R_1,$$

and

$$A(\tau) = \int_0^\tau \Big(\frac{1}{2}B'(s)\Sigma_X'\Sigma_X B(s) - R_0 - B'(s)\Sigma_X'\Lambda_0\Big)\mathrm{d}s.$$

We assume the portfolio is permanently rebalanced so that the duration to maturity τ of the bonds that are invested is kept constant. For simplicity, let P_t^τ denote the price of a zero-coupon bond with maturity τ. It can be shown that price of a zero-coupon bond follows the funds price dynamics equation,

$$\frac{\mathrm{d}P_t^\tau}{P_t^\tau} = (R_t + B'(t)\Sigma_X'\Lambda_t)\mathrm{d}t + B'(t)\Sigma_X'\mathrm{d}Z_t. \tag{5.11}$$

We approximate the long-term interest rate by the long-term treasury rate. Hence, the annual term structure $T_t^{(\tau)}$ at time t with time to maturity τ can be computed from the zero-coupon bond prices. It is given by:

$$T_t^{(\tau)} = \left(\frac{1}{P_t^\tau}\right)^{1/\tau} - 1. \tag{5.12}$$

We will use the following discrete approximation of Equation 5.11 to compute the zero-coupon bond returns in the URM model:

$$\frac{P_{t+1}^\tau}{P_t^\tau} = \exp\Big((R_t + [B(t)]'\,\Sigma_X'\Lambda_t - \frac{1}{2}\,[B(t)]'\,\Sigma_X'\Sigma_X B(t))\Delta t + [B(t)]'\,\Sigma_X' Z(\Delta t)\Big),$$

where $Z'(\Delta t)$ is a 4-vector consisting of 4 independent normal random variables $Z_i(\Delta t) \sim \mathcal{N}(0, \Delta t)$, for $i \in \{1, 2, 3, 4\}$.

5.2 Pension fund model

We will construct a stylized pension fund that will be used in the model calculations. It is based on the model as presented in [25].

Let us first discuss the notation. Time is denoted by t and a generation g is referred to by the superscript (g). The first year is $t = 2019$. Every year t, a new generation $g = t$ enters the working life. At time t, we denote the number of individuals of generation g by $N_t^{(g)}$. We find the number of individuals in the next year by iteration: $N_{t+1}^{(g)} = {}_1p_t^{(g)} N_t^{(g)}$, where ${}_1p_t^{(g)}$ is the one-year survival probability of generation g at time t. In general, ${}_\tau p_t^{(g)}$ denotes the τ-year survival probability of generation g at time t. The time to retirement of generation g at the start of time $t \geq g$ is denoted by $T_t^{(g)}$. For simplification, we denote $T_g = T_g^{(g)}$. Lastly, we use the fact that $T_t^{(g)} = T_{t-1}^{(g)} - 1$ for $t \geq g + 1$.

We assume that the pension fund follows a pension ambition of 75% of the average wage. This means that in the calculations of the pension fund, parameters such as pension premium have to be set such that this goal will be reached for all pension participants under reasonable assumptions.

The pension fund is modelled as the entire Dutch population. The data is retrieved from the CBS (Dutch Central Bureau for Statistics) [33], and the distribution of people older than 99 years is assumed to decrease linearly. In fact, this is done in the following way. If there are X individuals of age $y - 1$, then there are cX individuals of age y, for $y \in \{99, 100, \ldots, 120\}$ and $c \in [0, 1]$. Since we know that this will in general not give integers, we first try to find a c such that $\sum_{y=99}^{120} f_c(y) \approx 3847$, the number of people older than 98 years, where $f_c : \{99, 100, \ldots, 120\} \to \mathbb{R}_+$ is defined as

$$f_c(y) = \begin{cases} c \cdot N_{98}, & \text{if } y = 99, \\ c \cdot f_c(y-1), & \text{if } y \in \{100, 101, \ldots, 120\}, \end{cases} \tag{5.13}$$

whit $N_{98} = 2826$ the number of people aged 98. We find that a value of $c = 0.57655$ gives $\sum_{y=99}^{120} f_c(y) \approx 3847.73\ldots$ Denoting $[x]$ as the integer nearest to x, we see that $\sum_{y=99}^{120} [f_c(y)] = 3847$. We append the numbers of individuals $[f_c(y)]$ for ages $y \in \{100, 101, \ldots, 120\}$ to the data from the CBS, [33]. The full expanded list can be found in Table 5.2.0.1.

To find future populations, we use the median estimates of the mortality rates of the Dutch Royal Actuarial Society as given by [32]. Since these rates are different for men and women, we use the average value of these rates in our model and we assume that highest possible age reached is 120 years old. By using survival probabilities, we will obtain non-integer population values. We solve this by rounding off to the nearest integer. We assume a constant rate of people born every year of 170,000, which is equal to the number of people born in 2019 rounded off to ten thousands. New generations enter the working life at age 25 and retire at age 67.

The pension base of generation g at the end of period t is denoted by $W_t^{(g)}$. This is the part of the wage over which pension premium is paid. We assume that this pension base increases every year due to a rise in price index Π_t. If we set the initial wage for all generations g at $W_0^g = 1$, then the pension wage is given by

$$W_t^{(g)} = \begin{cases} \Pi_t, & \text{if } g < t \leq g + T_g, \\ 0, & \text{if } t > g + T_g. \end{cases}$$

Furthermore, the pension fund has to decide every year t what fraction β_t of the pension base has to be paid to follow the pension ambitions. Note that in the new pension agreement, this fraction is independent of the generation g. The contribution of generation g at time t denoted by $c_t^{(g)}$ and is given by

$$c_t^{(g)} = N_t^{(g)} \cdot \beta_t \cdot W_t^{(g)}.$$

We set $\beta_t = 0.25$, which is usually assumed to be a good percentage for a pension ambition of 75% of the average wage. We do not assume that the pension fund can alter the fraction.

The indexation I_t is set equal to the price inflation from Equation 5.3. The indexation factor I_t determines $PE_t^{(g)}$, the pension entitlements of generation g at the beginning of period t:

$$PE_t^{(g)} = \begin{cases} PE_{t-}^{(g)} \cdot I_{t-1} + a_{t-1}^{(g)} W_{t-1}^{(g)}, & \text{if } t \leq g + T_g, \\ PE_{t-1}^{(g)} \cdot I_{t-1}, & \text{if } t > g + T_g. \end{cases} \tag{5.14}$$

We start at $PE_g^{(g)} = a_g^{(g)} W_g^{(g)}$. For $a_t^{(g)}$, the pension entitlement as fraction of the pension base built up for the individuals in generation g at time t, we assume an exponential decay. The reason that we assume this specific kind of decay, is based upon the exponential character of return on investments. If interest is compounded continuously, the future value of wealth increases exponentially in time. Thus, the pension base that is built up grows in the opposite direction, which explains exponential decay.

So let $a_t^{(g)} = a_g^{(g)} e^{-\lambda(t-g)}$, where $a_g^{(g)}$ is the fraction built up in the first working year and λ is the rate of decay. We would like $a_t^{(g)}$ to be such that the average fraction over the working years equals 75%, which is in line with the pension ambition. We can do this by assuming that

$$\int_g^{g+42} a_t^{(g)} \mathrm{d}t = \int_g^{g+42} a_g^{(g)} e^{-\lambda(t-g)} \mathrm{d}t = 0.75$$

Let us find a solution to the integral. Let $u = t - g$, then

$$\begin{aligned} \int_g^{g+42} a_t^{(g)} \mathrm{d}t &= \int_g^{g+42} a_g^{(g)} e^{-\lambda(t-g)} \mathrm{d}t \\ &= a_g^{(g)} \int_0^{42} e^{-\lambda u} \mathrm{d}u \\ &= -\frac{a_g^{(g)}}{\lambda} e^{-\lambda u} \Big|_{u=0}^{u=42} \\ &= \frac{a_g^{(g)}(1 - e^{-42\lambda})}{\lambda}. \end{aligned}$$

This solution should equal 0.75. We choose a λ that matches the average return on the investment in our basic life-cycle. For 100,000 simulations, this gives $\lambda \approx 0.066$. We can now find a solution to $a_g^{(g)}$ by fixing this λ. Then $a_g^{(g)} \approx 0.053$ solves the equation. So we use the fraction $a_t^{(g)} = 0.053 \cdot e^{-0.066(t-g)}$ in the model.

Let $\alpha_t^{(g)}$ denote the fraction of capital that generation g had invested in stocks at time t and let $R_{s,t}$ and $R_{b,t}$ be the return on stocks and bonds at time t, respectively. Then, the return on the pension plan's assets of generation g at time t, $R_t^{(g)}$, are given by:

$$R_t^{(g)} = \alpha_t^{(g)} \cdot R_{s,t} + (1 - \alpha_t^{(g)}) \cdot R_{b,t}. \tag{5.15}$$

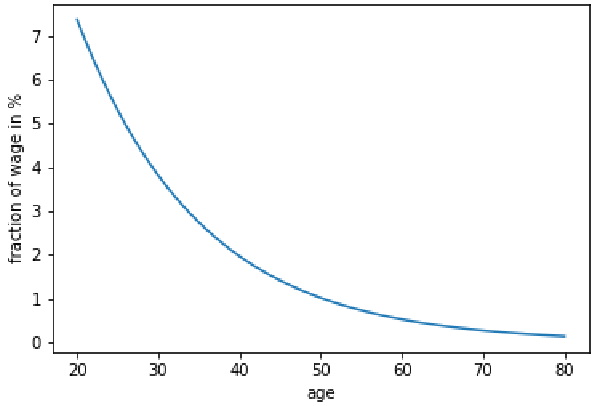


Figure 5.2.1: Pension entitlements as fraction of the wage for different ages. This accrual corresponds to the exponential decay $a_t^{(g)} = 0.053 \cdot e^{-0.066(t-g)}$.

We then find that the assets of generation g at the beginning of time t, $A_t^{(g)}$ are given by:

$$A_t^{(g)} = R_t^{(g)} \cdot A_{t-1}^{(g)} + c_t^{(g)}.$$

We start with an initial wealth for all generations of 1. We will define the total assets of the pension fund as

$$A_t = \sum_{g=t-67}^{t-25} A_t^{(g)}. \tag{5.16}$$

Note that we only include the assets of the working generations. Besides the assets, we need to define the liabilities in our setup to be able to calculate the funding ratio. Since these liabilities include future payouts, we need discount factors. We have discount factors at the start of time t, discounting τ periods back, given by:

$$D_t^{(\tau)} = \frac{1}{\left(1 + d_t^{(\tau)}\right)^{\tau}}, \tag{5.17}$$

where $d_t^{(\tau)}$ is the discount rate. For the liabilities, we follow the DNB and use the simulated term structure $T_t^{(\tau)}$ as given in Equation 5.12. However, the DNB advises to include the use of the Ultimate Forward Rate (UFR) of 2.1% for yields that are more than 30 years away. This advice has recently been confirmed by the Committee Parameters [10]. We assume an interpolated discount factor, that starts converging from the First Smoothing Point at 30 years and converges at 100 years. This is slightly different from the actual setup in the nFTK, where the UFR is an asymptote that is never reached. However, our longest liabilities are 95 years as people start working at age 25 and live no older than 120 years, so we never actually reach the value of the UFR either. We thus compute the discount rate as

$$d_t^{(\tau)} = \begin{cases} T_t^{(\tau)}, & \text{if } t \leq 30, \\ T_t^{(\tau)} + \frac{\tau-30}{100-30}(UFR - T_t^{(\tau)}), & \text{if } 31 \leq t \leq 99, \\ UFR, & \text{if } t \geq 100. \end{cases}$$

We now have to find a good definition for the liabilities L_t. Inspired by Equation 6 of [25], we define the liabilities as

$$L_t = \sum_{g=t-67}^{t} \sum_{\tau=\max\{1,T_t^{(g)}\}}^{120-(t-g)} {}_\tau p_t^{(g)} \cdot N_t^{(g)} \cdot D_t^{(\tau)} \cdot PE_t^{(g)}, \tag{5.18}$$

where ${}_\tau p_t^{(g)}$ denotes the τ-year survival probability of generation g, $N_t^{(g)}$ is the number of people in generation g, $D_t^{(\tau)}$ is the discount factor from Equation 5.17 and $PE_t^{(g)}$ are the pension entitlements of generation g. We modified the definition of the liabilities to only include projected pension payments for the working generations, so that it can be compared to the total assets of Equation 5.16. The Funding Ratio FR_t at time t is now given by

$$FR_t = \frac{A_t}{L_t}.$$

We now want a way to alter the liabilities if $FR_t \neq 1$. That means that the assets are not equal to the liabilities, so we can write $A_t = \delta_t + L_t$, where δ_t denotes the difference between that assets and liabilities at time t. We now note that we can write the Funding Ratio as

$$FR_t = 1 + \frac{\delta_t}{L_t}.$$

We want to change the Funding Ratio in $\widetilde{FR}_t$ so that it is closer to the desired value of 100%. However, we do not want the shocks in pension entitlements to be too big. Hence we spread the indexation over γ years. So the altered Funding Ratio $\widetilde{FR}_t$ is such that

$$\begin{aligned}\widetilde{FR}_t &= 1 + \frac{\delta_t}{L_t} - \frac{1}{\gamma} \cdot \frac{\delta_t}{L_t}, \\ &= \frac{\gamma L_t + (\gamma - 1)\delta_t}{\gamma L_t}.\end{aligned} \tag{5.19}$$

On the other hand, by definition, the altered Funding Ratio $\widetilde{FR}_t$ can be written as

$$\widetilde{FR}_t = \frac{A_t}{\widetilde{L}_t} = \frac{L_t + \delta}{\widetilde{L}_t}, \tag{5.20}$$

for some value of the altered liabilities $\widetilde{L}_t$. Combining Equations 5.19 and 5.20, we can express $\widetilde{L}_t$ as

$$\widetilde{L}_t = \frac{\gamma L_t (L_t + \delta_t)}{\gamma L_t + (\gamma - 1)\delta_t}. \tag{5.21}$$

The only factor in the Equation 5.18 that can be altered are the pension entitlements $PE_t^{(g)}$. Let, as in Equation 5.15, $\alpha_t^{(g)}$ denote the fraction of capital that generation g had invested in stocks at time t. We let the altered pension entitlements be of the form $\widetilde{PE}_t^{(g)} = C_t \cdot \alpha_t^{(g)} \cdot PE_t^{(g)}$, and we want to find the constant C_t that is universal over all working generations. First, we can write $\widetilde{L}_t$

as

$$\widetilde{L}_t = \sum_{g=t-67}^{t} \sum_{\tau=\max\{1,T_t^{(g)}\}}^{120-(t-g)} {}_\tau p_t^{(g)} \cdot N_t^{(g)} \cdot D_t^{(\tau)} \cdot \widetilde{PE}_t^{(g)},$$

$$= \sum_{g=t-67}^{t} \sum_{\tau=\max\{1,T_t^{(g)}\}}^{120-(t-g)} {}_\tau p_t^{(g)} \cdot N_t^{(g)} \cdot D_t^{(\tau)} \cdot (C_t \cdot \alpha_t^{(g)} \cdot PE_t^{(g)}),$$

and since C_t does not depend on g or τ,

$$= C_t \cdot \sum_{g=t-67}^{t} \sum_{\tau=\max\{1,T_t^{(g)}\}}^{120-(t-g)} {}_\tau p_t^{(g)} \cdot N_t^{(g)} \cdot D_t^{(\tau)} \cdot (\alpha_t^{(g)} \cdot PE_t^{(g)}),$$

$$= C_t \cdot L_{t,\alpha},$$

where $L_{t,\alpha} = \sum_{g=t-67}^{t} \sum_{\tau=\max\{1,T_t^{(g)}\}}^{120-(t-g)} {}_\tau p_t^{(g)} \cdot N_t^{(g)} \cdot D_t^{(\tau)} \cdot (\alpha_t^{(g)} \cdot PE_t^{(g)})$. Thus, since $\widetilde{L}_t$ can be written as in Equation 5.21, we can solve for C_t:

$$C_t = \frac{\gamma L_t (L_t + \delta_t)}{L_{t,\alpha}\Big(\gamma L_t + (\gamma - 1)\delta_t\Big)}. \tag{5.22}$$

Hence, we can reset the funding ratio FR_t to $\widetilde{FR}_t$ by replacing $PE_t^{(g)}$ with $\widetilde{PE}_t^{(g)} = C_t \cdot \alpha_t^{(g)} \cdot PE_t^{(g)}$, where C_t is given by Equation 5.22.

When a cohort finally enters the pension age at 67 years, their assets $A_t^{(g)}$ are converted to an insured annuity payment $AP^{(g)}$. We assume that this annuity payment is the same for all years that the retired cohort lives. We can find the value of the annuity payments by noting that the assets $A_t^{(g)}$ should equal the present value of the future annuity payments $AP^{(g)}$. Hence, the present value of a series of annuity payments that is paid each year for n years, is given by

$$A_t^{(g)} = \frac{AP^{(g)}}{1+R} + \frac{AP^{(g)}}{(1+R)^2} + \cdots + \frac{AP^{(g)}}{(1+R)^n}, \tag{5.23}$$

where R is the interested rate. We let $R = R_g$, where R_g is given by Equation 5.5 in the underlying URM model. We see that Equation 5.23 is a geometric series, so by using the geometric series formula, it becomes

$$A_t^{(g)} = \frac{\frac{AP^{(g)}}{1+R} - \frac{AP^{(g)}}{1+R}(\frac{1}{1+R})^n}{1 - \frac{1}{1+R}},$$

which simplifies to

$$A_t^{(g)} = \frac{AP^{(g)} - AP^{(g)}(\frac{1}{1+R})^n}{(1+R) - 1},$$

$$A_t^{(g)} = AP^{(g)} \frac{1 - (\frac{1}{1+R})^n}{R}.$$

Isolating $AP^{(g)}$ gives us the following formula:

$$AP^{(g)} = A_t^{(g)} \frac{R}{1 - (\frac{1}{1+R})^n} \tag{5.24}$$

For simplicity, we will assume a fixed mortality of 85 years to calculate the annuity payments. This implies that $n = 85 - 67 = 18$ years.

Age	Number of people	Age	Number of people	Age	Number of people
0 years	168443	41 years	201027	81 years	90011
1 years	170816	42 years	202851	82 years	84177
2 years	174256	43 years	204259	83 years	76736
3 years	173722	44 years	213259	84 years	70095
4 years	178825	45 years	220643	85 years	62552
5 years	175210	46 years	238587	86 years	57367
6 years	180043	47 years	249933	87 years	50089
7 years	183815	48 years	260536	88 years	44093
8 years	188854	49 years	266020	89 years	35967
9 years	189520	50 years	254909	90 years	30351
10 years	190090	51 years	251946	91 years	24104
11 years	186613	52 years	253097	92 years	19691
12 years	190026	53 years	256392	93 years	15081
13 years	191909	54 years	260512	94 years	11736
14 years	197677	55 years	255864	95 years	8705
15 years	204306	56 years	250557	96 years	6138
16 years	205638	57 years	247542	97 years	4159
17 years	207992	58 years	240203	98 years	2826
18 years	216614	59 years	237366	99 years	1629
19 years	217469	60 years	230261	100 years	939
20 years	218315	61 years	224577	101 years	542
21 years	212979	62 years	219687	102 years	312
22 years	212854	63 years	213299	103 years	180
23 years	214501	64 years	209015	104 years	104
24 years	221276	65 years	204712	105 years	60
25 years	221595	66 years	203761	106 years	35
26 years	224138	67 years	194999	107 years	20
27 years	227080	68 years	194511	108 years	11
28 years	228156	69 years	195091	109 years	7
29 years	220894	70 years	198650	110 years	4
30 years	218629	71 years	206366	111 years	2
31 years	218397	72 years	209871	112 years	1
32 years	217522	73 years	147392	113 years	1
33 years	212882	74 years	151105	114 years	0
34 years	209476	75 years	140466	115 years	0
35 years	203806	76 years	126649	116 years	0
36 years	204401	77 years	117269	117 years	0
37 years	208114	78 years	116317	118 years	0
38 years	210971	79 years	108025	119 years	0
39 years	204001	80 years	101095	120 years	0
40 years	203992				

Table 5.2.0.1: Distribution of ages among the Dutch population, as found in CBS data and extrapolated using the iterative function 5.13.

Chapter 6

Results

In this chapter, we will present and analyze the results that follow from simulations in our model. In presenting the results, we will follow the Pension Communication Law of 2015, that obliges pension funds to communicate the expected pension payment in three scenarios. In Section 6.1, the conditions for the simulations and the method of presentation are explained. In Section 6.2, the results are analyzed and statistically tested for significance.

6.1 Set-up

The Pension Communication Law of 2015 obliges pension funds to communicate the expected pension payment in three scenarios: optimistic, expected and pessimistic. These three scenarios are defined by the a-percentiles for $a = 0.95, 0.5$ and 0.05 respectively. The definition of an a-percentile can be found in Subsection 3.4. So, for instance, the pessimistic scenario corresponds to the simulation such that only 5% of the simulations lie below this value. Since we will simulate values for different ages and different years, we will either take the pension age or the last time point to calculated the percentiles. In Figure 6.1.1, we see an example of these three scenarios in a set of simulations.

We simulate the following 5 strategies, which are explained in Subsection 4.4.1:

1. $\alpha_t^{(g)} = 0.5$ for all t and g.
2. Linear decreasing life-cycle with $\overline{g} = 40$, $p_{start} = 0.9$ and $p_{end} = 0.1$.
3. Linear decreasing life-cycle with $\overline{g} = 40$, $p_{start} = 0.6$ and $p_{end} = 0.4$.
4. Dynamic life-cycle with $\overline{g} = 40$, $R = 1.06$, $p_{start} = 0.9$, $p_{middle} = 0.7$ and $p_{end} = 0.5$.
5. Dynamic life-cycle with $\overline{g} = 40$, $R = 1.06$, $p_{start} = 0.6$, $p_{middle} = 0.4$ and $p_{end} = 0.2$.

First, 42 years are simulated without using the methodology for altering the Funding Ratio FR. This is done to let the pension fund mature and make sure that the retirees will have followed a complete investment cycle. Then, we let the model run for 25 more years. Let t denote the number of years after the burn-in period of 42 years. We simulate the model $N = 500$ times.

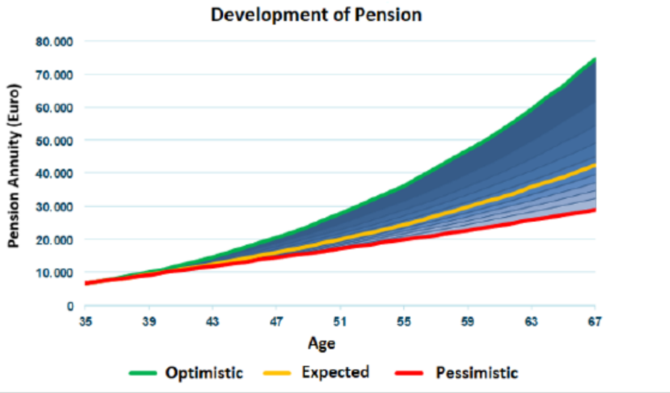


Figure 6.1.1: Different simulations for pension entitlements development, including the optimistic, expected and pessimistic scenario. Image retrieved from [20].

6.2 Statistical tests

The development of the three scenarios of the $FR(t)$ for $t \in \{0, \cdots, 25\}$ is visible in Figure 6.2.1. In Table 6.2.0.1, we see that all strategies in the pessimistic scenario end at $FR = 0.90$. This is due to the altering mechanism that immediately cuts pension entitlements to set the funding ratio back to 90% if the funding ratio gets below 90%. Furthermore, we see that all expected funding ratios are close to each other near the value of 140%. However, in the optimistic scenario, there is a difference between on the one hand the constant strategy 1 and the linear decreasing life-cycle strategies 2 and 3, and on the other hand the dynamic life-cycle strategies 4 and 5. Strategy 4, which keeps the percentage invested in stocks relatively higher than strategy 5, seems to have bigger outliers on the upside than strategy 5. This can be explained by the high amount invested in stocks, as stocks in general generate higher returns. However, we might expect that this would also lead to bigger outliers on the downside of the spectrum. This can not be concluded from this Table, because the funding ratio is set to 90% each time it falls below. We would expect to be able to tell the difference by comparing the pension entitlements $PE_t^{(g)}$.

FR	Pessimistic	Expected	Optimistic
Strategy 1	0.90	1.39	3.58
Strategy 2	0.90	1.41	3.62
Strategy 3	0.90	1.38	3.58
Strategy 4	0.90	1.42	4.24
Strategy 5	0.90	1.37	3.42

Table 6.2.0.1: Values of FR at $t = 25$ for the simulated strategies and the three scenarios.

We perform a Kruskal-Wallis test (explained in Subsection 3.5.2) on the last 10 time points of the *adjusted* mean funding ratios FR for all 5 strategies. First, we filter the funding ratios for all scenarios that end at the 5% highest scenarios. For the remaining scenarios, we calculate the point-wise mean. We set our significance level at 0.05. Since the p-value for the test statistic is $2.1 \cdot 10^{-6}$, we perform a Bonferroni correction to find the corrected p-values for the pairwise hypotheses. The results are shown in Table 6.2.0.2. This gives statistical evidence that there is a difference in variance between the pairs (1,4), (3,4), (2,5) and (4,5). Clearly, strategies 1, 2 and 3 are not mutually correlated, so these strategies exhibit similar behavior. What stands out, is that strategies 4 and 5, the dynamic life-cycles, are mutually correlated, but strategy 4 is also correlated with 1 and 3 and strategy 5 with 2. This shows that the dynamic life-cycle strategies cause the funding ratios to behave capricious. The linear decreasing life-cycle strategies tend to follow a more predictable path, although it can not be said that these paths are more stable than strategy 1.

	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5
Strategy 1	-1.000	1.000	1.000	0.003	0.886
Strategy 2	1.000	-1.000	1.000	0.251	0.405
Strategy 3	1.000	1.000	-1.000	0.002	0.945
Strategy 4	0.003	0.251	0.002	-1.000	0.000
Strategy 5	0.886	0.019	0.945	0.000	-1.000

Table 6.2.0.2: The corrected p-values for the Kruskal-Wallis test for the last 10 time point of the mean funding ratios.

The development of the annuity payments $AP^{(t)}$ for the retired cohort at time $t \in \{0, \cdots, 25\}$ is visible in Figure 6.2.2. We see that for all strategies, the optimistic scenario does not start at a high amount. It rises to that value in the last few years before the simulation ends. The pessimistic scenario however seem to be low over the entire duration. This may imply that once a high annuity payments has been reached, it will not easily decrease anymore. In Table 6.2.0.3, we see that the constant strategy is never at the extremum of the simulated values for the all three displayed scenarios. The linear decreasing life-cycle strategies 2 and 3 seem to have slightly lower expected annuity, but the optimistic scenario generates more annuity payment. The lower expected annuity is a bit unexpected, since the funding ratios did not deviate significantly. For the dynamic life-cycle strategies 4 and 5, it is clear that the pessimistic scenarios generate less annuity than the constant strategy 1. However, we see that strategy 4 has lower expected annuity and higher optimistic annuity, but strategy 5 has higher expected annuity and lower optimistic annuity.

$AP^{(25)}$	Pessimistic	Expected	Optimistic
Strategy 1	206,000	739,000	3,202,000
Strategy 2	208,000	709,000	3,668,000
Strategy 3	204,000	731,000	3,314,000
Strategy 4	121,000	702,000	3,800,000
Strategy 5	197,000	759,000	2,970,000

Table 6.2.0.3: Values of the annuity payments AP at $t = 25$ for the simulated strategies and the three scenarios.

We also need to analyze the development of the pension entitlements $PE_t^{(g)}$. However, since the pension entitlements are influenced by the inflation I_t and the price index Π_t (see Equation 5.14), we need to correct for the level of the price index. Hence, we plot the corrected pension entitlements $PE_t^{(g)}/\Pi_t$ at the last time point $t = 25$ for all simulations. This is done in Figure 6.2.3. It immediately stands out that the dynamic life-cycle strategies 4 and 5 let the pension entitlements drop to near 0 after the age of 40. This can be explained by the set-up for the altering of the pension entitlements. Since the pension entitlements are altered according to the fraction invested in stocks for a cohort g, we see that the participants that are 40 years or older are "punished" in good years because their fraction invested in stocks $\alpha_t^{(g)}$ drops as soon as the become 40. In Table 6.2.0.4, we see that linear decreasing life-cycle strategies 2 and 3 arrive at similar values for the pessimistic and the expected scenario as the constant strategy 1. However, we do see that the accrued pension entitlements in the optimistic scenario are higher than those of strategy 1.

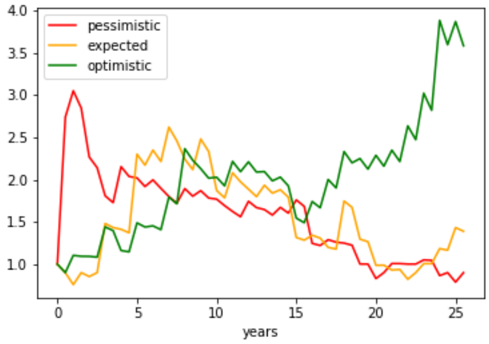
$PE_{25}^{(67)}/\Pi_{25}$	Pessimistic	Expected	Optimistic
Strategy 1	0.23	0.68	2.12
Strategy 2	0.21	0.66	2.34
Strategy 3	0.22	0.66	2.18
Strategy 4	0.00	0.00	0.00
Strategy 5	0.00	0.00	0.00

Table 6.2.0.4: Values of the corrected pension entitlements at $t = 25$ for the retired cohort.

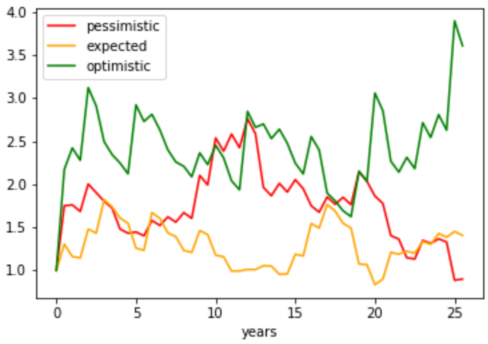
We also adjust the corrected pension entitlements $PE_{25}^{(67)}/\Pi_{25}$ before we calculate the point-wise mean by dropping the 5% highest scenarios. Then, we perform a Kruskal-Wallis test on the adjusted mean to find statistical evidence for a difference in variance of the last 10 time points. We omit comparing strategy 4 and 5 with all the others, since they clearly show a different behavior in the last period of the graphs. The test statistic gives a p-value of 0.011, so we perform a Bonferroni correction to find the pairwise corrected p-values. The results are shown in Table 6.2.0.5. Interestingly, strategy 1 and 3 seem to show no statistical evidence for a difference in variance in the last 10 years. We would expect that strategy 2 and 3 would exhibit similar behavior since they both follow a linear decreasing life-cycle.

	Strategy 1	Strategy 2	Strategy 3
Strategy 1	-1.000	0.010	1.000
Strategy 2	0.010	-1.000	0.105
Strategy 3	1.000	0.105	-1.000

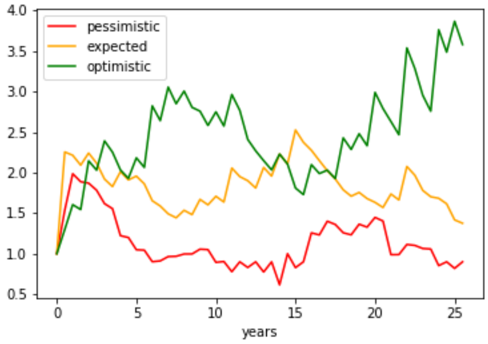
Table 6.2.0.5: The corrected p-values for the Kruskal-Wallis test of the mean corrected pension entitlements $PE_{25}^{(g)}/\Pi_{25}$ for $g = \{58, \ldots, 67\}$.



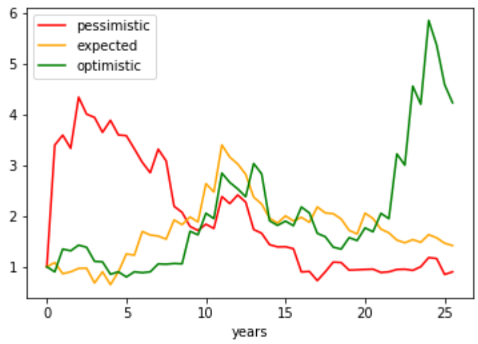
(a) Strategy 1



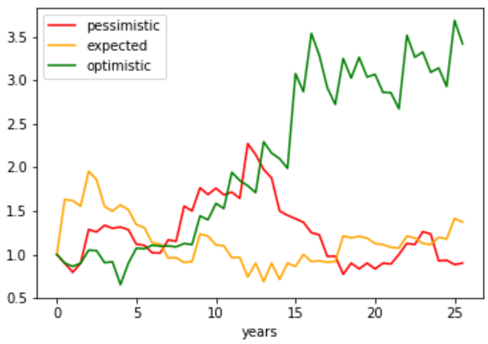
(b) Strategy 2



(c) Strategy 3

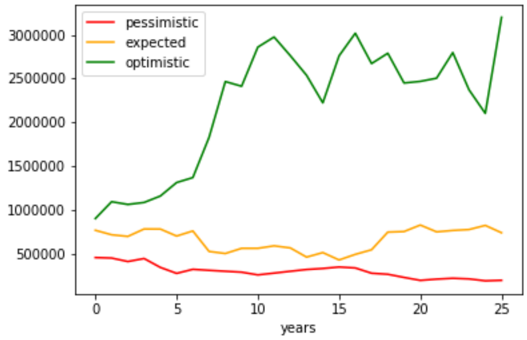


(d) Strategy 4

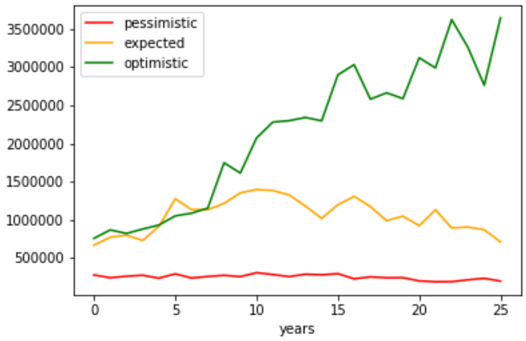


(e) Strategy 5

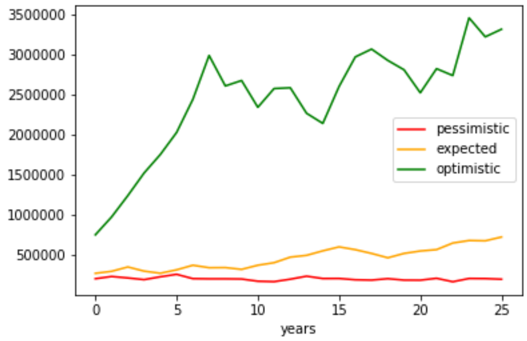
Figure 6.2.1: Development of the funding ratio $FR(t)$, $t \in \{0, \ldots, 25\}$ of the three scenarios for all simulated strategies. By definition, $FR(t) \geq 0$ and by the altering of pension entitlements $PE_t^{(g)}$, the $FR(t)$ has a drift towards 1.



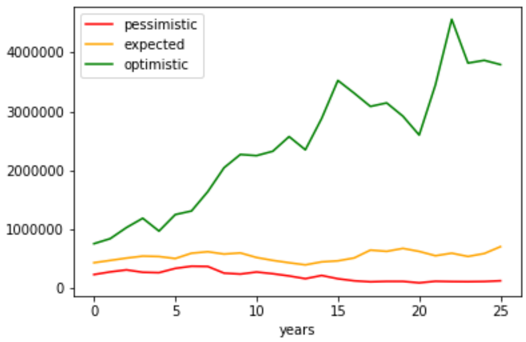
(a) Strategy 1



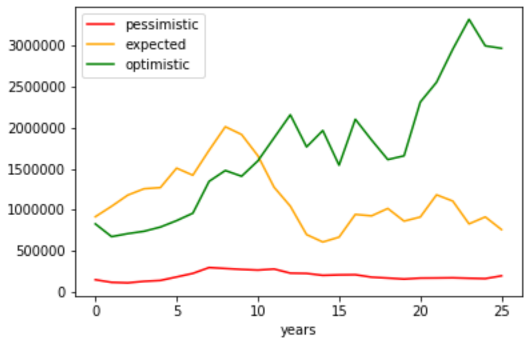
(b) Strategy 2



(c) Strategy 3

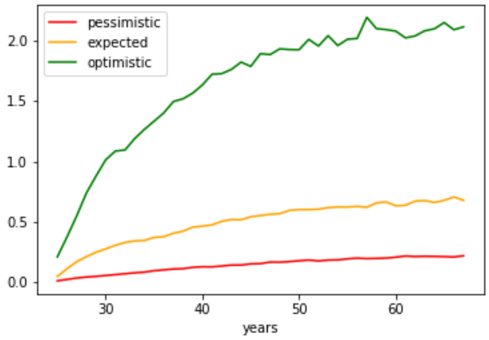


(d) Strategy 4

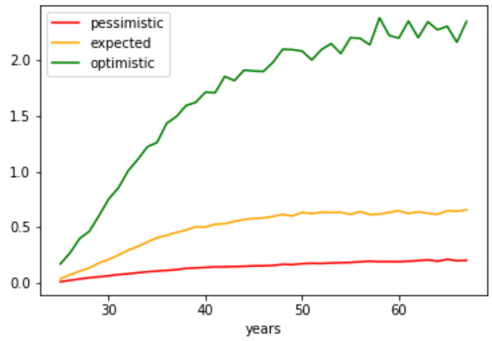


(e) Strategy 5

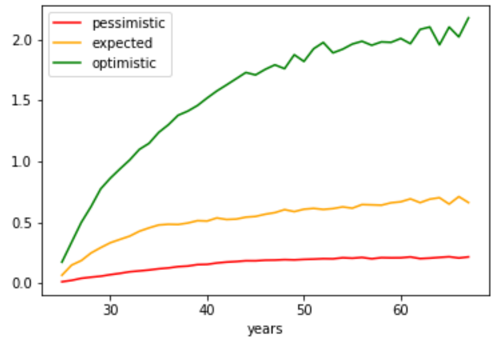
Figure 6.2.2: Development of the annuity payments $AP^{(t)}$, $t \in \{0, \dots, 25\}$ of the three scenarios for all simulated strategies. By definition, we know that $AP^{(t)} \geq 0$.



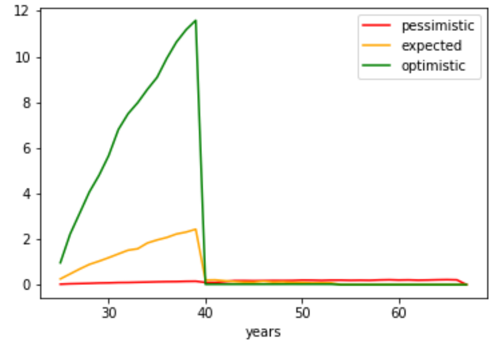
(a) Strategy 1



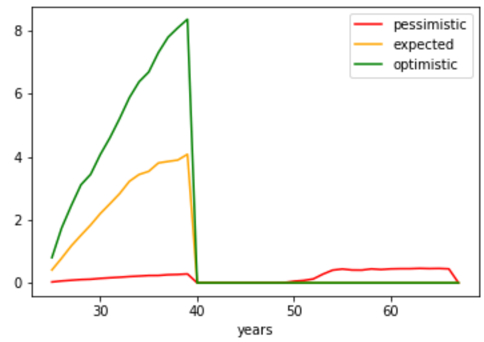
(b) Strategy 2



(c) Strategy 3



(d) Strategy 4



(e) Strategy 5

Figure 6.2.3: Development of the corrected pension entitlements $PE_{25}^{(g)}/\Pi_{25}$, $g \in \{0, \dots, 42\}$ of the three scenarios for all simulated strategies. By definition, we know that $PE_t^{(g)}/\Pi_t \geq 0$ for all $t, g \in \mathbb{N}$.

Chapter 7

Conclusions

In this chapter, we construct conclusions based on the simulations that are treated in Chapter 6. We try to find an answer to the question:

How does life-cycle investment affect pension accrual in different age cohorts?

We have seen in our results that the pension accrual is built up very differently in different life-cycle strategies. The dynamic life-cycle shows a spike downwards as soon the cohort passes the threshold when the investment strategy is changed. This is due to the rewarding system for high investment in stocks that we built into our model. The linear decreasing life-cycles tend to develop similarly to the constant strategy. We expected that the life-cycle strategies would smooth the development of the pension entitlements in the years before a cohort retires. However we did not see this. The second question we answer is:

How do different explicit life-cycle methods perform compared to each other?

The dynamic life-cycles were outperformed by the linear decreasing life-cycles. In choosing the linear decreasing life-cycles, there is no clear distinction between the one that starts at 90% and the one that starts at 60% of the assets invested in stocks. The last subquestion we treat is:

How does life-cycle investment affect the development of the funding ratios of pension funds?

In comparing life-cycle investment with the development of the funding ratios of our model pension fund, we see again that the linear decreasing life-cycle strategies clearly show a different behavior than the dynamic life-cycle strategies. Even though the development of the funding ratios can be rather unpredictable due to the nature of the financial market, the dynamic life-cycle strategies seem to let the funding ratios be more unpredictable than the constant strategy and the linear decreasing life-cycles. The main research question we tackled was the following:

How can risk be shared between different groups in a pension system with life-cycle investing?

In conclusion, we see that risk-sharing is possible between different groups, but the results in this thesis do not show that it will lead to a higher or more stable pension accrual for cohort that near retiring. A reason for this might be that the life-cycle strategies, although differentiating in age of the participants, overall lead to a similar investment fraction in stocks as the constant strategy. Since the indexation rules depend on the financial situation of the entire pension fund, the positive and negatives excesses cancel each other out.

Something we noticed in the results is that the funding ratio simulations are skewed around 1. This can be explained by the altering mechanism that is used to give indexation to participants if the funding ratio is higher than 1, and to cut pension entitlements if the funding ratio is below 1. However, there are some simulation where the funding ratio explodes to values above 10^7. These simulations generate so much wealth in the first years, that the drift towards 1 is not strong enough to get it near a normal value. We decided to share 10% of the surplus each year with the participants, but this is too low in the scenarios where the wealth grows extremely hard. We advice to use higher percentages if the funding ratio gets above a certain value.

Chapter 8

Future research

In this thesis, we limited ourselves to 4 different life-cycles strategies. Including more life-cycles may give a better insight in the subtle differences there are in choosing the right parameters. In this research, we derived the optimal allocation of the fraction in stocks according to the life-cycle principle, but we did not use the exact optimal allocation. Instead, we simplified the development of the allocation to a linear and dynamic life-cycle strategy. These strategies were preferred over a strategy that follows that exact optimal allocation, because we wanted to stay as close to the regulations in the nFTK as possible. Since the exact optimal allocation is based on several immeasurable quantities, its calculation would involve approximating most of these quantities. Therefore, it would have lead to expanding the URM model that is certified by the DNB and have limited the ability to compare different life-cycle strategies. In a future research, historical data could be used to estimate the expected excess return on stocks and the variance of the stock return. Assumptions involving the human capital and the future wealth would then lead to an exact expression for the optimal allocation of stocks. This may improve the development of the pension accrual in pension funds that follow this exact allocation. A method that could be used to find a numerical optimal allocation is *simulated annealing*. For more information on this topic, we refer to [21].

Furthermore, including more assets categories would model the reality better. In many researches including this one, the only two assets categories that are invested are stocks and bonds. No classification is made between different kinds of stocks and bonds, and investments in real estate and commodities are left out. Including these categories may simulate the scenarios more realistically. However, the derivation of the optimal allocation will become more difficult and maybe even impossible to calculate algebraically.

Appendix A

This Appendix is based on [26], more details can be found there. We introduce the definitions of topology and measure theory that are used in Chapter 3. The structure of the section is such that every following section uses what is introduced before.

A.1 Topology

We introduce the notion of *topology* and a *topological space* in this section. Let I be some arbitrary sets of indices. Basic set theory is assumed to be known.

Definition A.1.1. A system $\mathcal{O} = \mathcal{O}(X)$ of subsets of X is called a *topology* if

(i) $\emptyset, X \in \mathcal{O}$,

(ii) $U, V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O}$,

(iii) $U_i \in \mathcal{O}, i \in I \Longrightarrow \bigcup_{i \in I} U_i \in \mathcal{O}$.

The pair $(X, \mathcal{O})$ is called a *topological space*.

A set $U \in \mathcal{O}$ is called an *open set*. A set $F \subseteq X$ is called *closed*, if its complement F^c is open. Let us have a look at the *neighbourhood* of a point $x \in X$.

Definition A.1.2. If $(X, \mathcal{O})$ is a topological space and x is a point in X, a *neighbourhood* of x is a subset V of X that includes an open set U containing p, i.e.,

$$x \in U \subseteq V.$$

Through neighbourhoods, we can define *dense* subsets U of topological space $(X, \mathcal{O})$.

Definition A.1.3. A subset U of a topological space $(X, \mathcal{O})$ is *dense* in X if for any point $x \in X$, any neighbourhood of x contains at least one point of U.

A.2 Measurability

Let us first introduce the notion of a σ-algebra.

Definition A.2.1. Let S be a set. A collection $\Sigma \subseteq 2^S$ is called a *σ-algebra* on S if

(i) $S \in \Sigma$,

(ii) $E \in \Sigma \Rightarrow E^c \in \Sigma$,

(iii) $\forall E_1, E_2, \dots \in \Sigma,\ \bigcup_{n=1}^{\infty} E_n \in \Sigma$.

Note that (iii) implies, specifically, that $E, F \in \Sigma \Rightarrow E \cup F \in \Sigma$. If Σ is a σ-algebra, then (S, Σ) is called a *measurable space* and the elements in Σ are called *measurable sets*.

If $\mathcal{C}$ is some collection of subsets of S, then we will denote by $\sigma(\mathcal{C})$ the smallest σ-algebra containing $\mathcal{C}$. We will say that $\mathcal{C}$ *generates* the σ-algebra $\sigma(\mathcal{C})$. We call a σ-algebra $\mathcal{A}$ a *sub-σ-algebra of* Σ if $\mathcal{A} \subseteq \Sigma$.

We will also need to look at $\mathcal{B} = \mathcal{B}(\mathbb{R})$, the Borel sets of $\mathbb{R}$. Let $\mathcal{O}$ be the collection of all subsets of $\mathbb{R}$ that are open with respect to the usual topology. This means that the open subsets of $\mathbb{R}$ are of the form

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

Then $\mathcal{B} := \sigma(\mathcal{O})$. Now, we can expand the measurable space to a *measure space* by considering a *measure*:

Definition A.2.2. Let (S, Σ) be a measurable space and let $\mu : \Sigma \to [0, \infty]$ be a mapping such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$, for every sequence $(E_n)_{n\in\mathbb{N}}$ of disjoint sets in Σ.

Then the mapping μ is called a *measure* and the triplet (S, Σ, μ) is called a *measure space*.

A.3 Random variables

In this subsection, we will define *random variables* as measurable functions on a underlying *probability* space. Let us first explain what a measurable function is. Let (S, Σ) be a measurable space.

Definition A.3.1. A function $f : S \to \mathbb{R}$ is called *measurable* if $f^{-1}(B) \in \Sigma$ for all $B \in \mathcal{B}$.

Note that in this definition, the measurability of f depends on $\mathcal{B}$ and Σ. We will therefore sometimes speak of Σ-measurable functions, or $\Sigma/\mathcal{B}$-measurable functions to stress this dependency.

Now, we can move to the concept of random variables. These can be defined on a special kind of measurable space, namely a *probability space*.

Definition A.3.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a measure space. If $\mathbb{P}(\Omega) = 1$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then we call Ω the set of outcomes, $\mathcal{F}$ the possible events and $\mathbb{P}$ the probability measure. We will sometimes use this notation to stress that we are working in a probability space. For now, we will stay in this setting to define random variables.

Definition A.3.3. A function $X : \Omega \to \mathbb{R}$ is called a *random variable* if it is $\mathcal{F}$-measurable.

For clarity, we will usually denote random variables by capital letters, such as X, to distinguish them from other measurable functions. Note that random variable X induces a probability measure on the measurable space $(\mathbb{R}, \mathcal{B})$ as it maps elements of Ω to $\mathbb{R}$. This measure $\mu : \mathcal{B} \to [0,1]$ is called the *distribution measure* of the random variable X and it is defined as

$$\mu(B) := \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B)). \tag{A.1}$$

For a collection of sub-σ-algebras on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can define *independence* in the following way:

Definition A.3.4. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be sub-σ-algebras of $\mathcal{F}$. We say that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are *independent* if for all $A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$, we have:

$$\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1) \cdot \dots \cdot \mathbb{P}(A_n).$$

This allows us to define independence of random variables.

Definition A.3.5. Let $X_1, \dots, X_n$ be random variables from the same probability space to $\mathbb{R}$. We call $\{X_1, \dots, X_n\}$ independent if and only if $\{\sigma(X_1), \dots, \sigma(X_n)\}$ are independent.

Measurable functions and random variables in particular can generate σ-algebras:

Definition A.3.6. Let f be an $\mathcal{F}$-measurable function. The *σ-algebra generated by f* is given by

$$\sigma(f) := \{f^{-1}(B) \mid B \in \mathcal{F}\}.$$

The multi-dimensional equivalent of the random variable is called a *random vector.*

Definition A.3.7. A *random vector* is a vector $\mathbf{X} = (X_1, \dots, X_n)'$, whose components X_i, $i = 1, \dots, n$ are random variables on the same probability space.

A.4 Lebesgue integrals

To define the expectation and variance of a random variable X, we need to consider the Lebesgue integral. We will give a short, constructive definition of the Lebesgue integral. More details can, for instance, be found in Chapter 4 of [26]. Consider the measure space (S, Σ, μ) and random variable X. Let us first define the indicator function of a measurable set E:

Definition A.4.1. The *indicator function* $\mathbb{1}_E$ of a measurable set $E \subseteq S$ is defined by

$$\mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

The indicator function of a set E indicates whether an element $x \in S$ is in subset E or not. We can define the Lebesgue integral of an indicator function by the measure of E:

Definition A.4.2. The Lebesgue integral of an indicator function $\displaystyle\int \mathbb{1}_E \, \mathrm{d}\mu \in [0, \infty]$ is defined as

$$\int \mathbb{1}_E \, \mathrm{d}\mu := \mu(E).$$

Defining the integral of the indicator function as the measure allows us to expand the territory to *simple* functions: $f = \sum_k \alpha_k \mathbb{1}_{E_k}$ for $\alpha_k \in \mathbb{R}$ and E_k measurable for $k = 1, 2, \ldots$. The following definition constructs the linearity of the Lebesgue integral.

Definition A.4.3. The Lebesgue integral of a simple function $f = \sum_k \alpha_k \mathbb{1}_{E_k}$ is defined by

$$\int f \,\mathrm{d}\mu = \int \sum_k \alpha_k \mathbb{1}_{E_k} \,\mathrm{d}\mu := \sum_k \alpha_k \mu(E_k).$$

Now, some care needs to be taken to extend the definition of the Lebesgue integral to all measurable functions f. The first step is to define it for all *positive* functions f.

Definition A.4.4. The Lebesgue integral of a positive function f is defined as

$$\int f \,\mathrm{d}\mu = \sup\left(\int g \,\mathrm{d}\mu \;:\; g \leq f,\; g \text{ simple}\right).$$

In this sense, the Lebesgue integral of a positive function f is constructed as some sort of upper bound of the integral of simple functions that approach f. For not strictly positive functions f, let f^+, f^- denote respectively the positive and negative part of f, i.e., such that $f = f^+ - f^-$. Defining the integral for all measurable functions depends on the *integrability* of the function:

Definition A.4.5. A function $f : S \to \mathbb{R} \cup \{\infty\}$ on a measure space (S, Σ, μ) is said to be $\mu-$integrable if it is measurable and if the integrals $\displaystyle\int f^+ \,\mathrm{d}\mu, \int f^- \,\mathrm{d}\mu$ are finite. In this case, we call

$$\int f \,\mathrm{d}\mu := \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu$$

the Lebesgue integral of f.

We are now able to look at the integration of all measurable functions. We are only left with defining the Lebesgue integral over a certain subset A:

Definition A.4.6. The Lebesgue integral of measurable function f over a subset A is defined as

$$\int_A f \,\mathrm{d}\mu := \int \mathbb{1}_A f \,\mathrm{d}\mu.$$

Now, the following properties of the Lebesgue integral can be shown.

Proposition A.4.7. Let $a, b \in \mathbb{R}$ be constants and f, g be integrable functions. Then

(i) Let $N = \{x : f(x) \neq g(x)\}$. If $\mu(N) = 0$, then $\displaystyle\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$.

(ii) $\displaystyle\int (af + bg) \,\mathrm{d}\mu = a \int f \,\mathrm{d}\mu + b \int g \,\mathrm{d}\mu$.

(iii) If $f \leq g$, then $\displaystyle\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$.

Proof. Since f and g are measurable, so is $N \in \Sigma$. Hence (i) follows from

$$\begin{aligned}\int f \, \mathrm{d}\mu &= \int_{\{f=g\}} f \, \mathrm{d}\mu + \int_{\{f \neq g\}} f \, \mathrm{d}\mu \\ &= \int_{\{f=g\}} g \, \mathrm{d}\mu + 0 \\ &= \int_{\{f=g\}} g \, \mathrm{d}\mu + \int_{\{f \neq g\}} g \, \mathrm{d}\mu \\ &= \int g \, \mathrm{d}\mu,\end{aligned}$$

where we used the Markov inequality B.1.2 to conclude that $\int_{\{f \neq g\}} f \, \mathrm{d}\mu = 0$. Property (ii) follows from the linearity of the simple functions that approach the functions f and g. Property (iii) follows from the definition of the Lebesgue integral as supremum. $\square$

A.5 Expectation and variance

We can now use the knowledge of the Lebesgue integral to define expectation and variance. We will switch to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that a random variable $X : \Omega \to \mathbb{R}$ is a measurable function.

Definition A.5.1. The integral or *expectation* of X (with respect to $\mathbb{P}$) is defined as

$$\mathbb{E}(X) = \int_{\Omega} X \mathrm{d}\mathbb{P},$$

provided that the integral is well-defined.

Let us look at some basic properties of the expectation.

Proposition A.5.2. Let $a, b \in \mathbb{R}$ be constants. The expectation of X satisfies the following properties:

(i) If $\mathbb{P}(X \geq 0) = 1$, then $\mathbb{E}(X) \geq 0$.

(ii) For X, Y random variables, $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.

(iii) $\mathbb{E}(X + a) = \mathbb{E}(X) + a$.

Proof. For (i), note that $\Omega = \{X \geq 0\} \cup \{X < 0\}$ and that $\mathbb{P}(X < 0) = 0$. So by Proposition

A.4.7(i),

$$\begin{aligned}
\mathbb{E}(X) &= \int_\Omega X \mathrm{d}\mathbb{P}, \\
&= \int_{\{X \geq 0\}} X \mathrm{d}\mathbb{P} + \int_{\{X<0\}} X \mathrm{d}\mathbb{P}, \\
&= \int_{\{X \geq 0\}} X \mathrm{d}\mathbb{P} + 0, \\
&\geq \int_{\{X \geq 0\}} 0 \, \mathrm{d}\mathbb{P} = 0.
\end{aligned}$$

Statement (ii) follows from Proposition A.4.7(ii). For (iii), we use that $\mathbb{P}(\Omega) = 1$:

$$\begin{aligned}
\mathbb{E}(X+a) &= \int_\Omega (X+a) \mathrm{d}\mathbb{P}, \\
&= \int_\Omega X \mathrm{d}\mathbb{P} + \int_\Omega a \mathrm{d}\mathbb{P}, \\
&= \int_\Omega X \mathrm{d}\mathbb{P} + a, \\
&= \mathbb{E}(X) + a.
\end{aligned}$$

□

From the expectation of X, we can define the *variance* of X, which can informally be thought of as a measure of how far the possible values of X are spread out from the average value.

Definition A.5.3. The *variance* $\mathrm{Var}(X)$ of the random variable X is defined as

$$\mathrm{Var}(X) = \mathbb{E}\Big((X - \mathbb{E}(X))^2\Big),$$

provided that the variance of X is well-defined.

Let us give some helpful properties of the variance.

Proposition A.5.4. Let $a \in \mathbb{R}$ be some constant. The variance of X satisfies the following properties:

(i) $\mathrm{Var}(X) \geq 0$.

(ii) $\mathrm{Var}(X+a) = \mathrm{Var}(X)$.

(iii) $\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$.

Proof. Property (i) follows from Proposition A.4.7(iii):

$$\mathrm{Var}(X) = \mathbb{E}\Big((X - \mathbb{E}(X))^2\Big) \geq \mathbb{E}(0) = 0.$$

For (ii), we see that

$$\begin{aligned}
\mathrm{Var}(X+a) &= \mathbb{E}\Big((X+a-\mathbb{E}(X+a))^2\Big),\\
&= \mathbb{E}\Big((X+a-\mathbb{E}(X)+a)^2\Big),\\
&= \mathbb{E}\Big((X-\mathbb{E}(X))^2\Big) = \mathrm{Var}(X).
\end{aligned}$$

For (iii), we find that

$$\begin{aligned}
\mathrm{Var}(aX) &= \mathbb{E}\Big((aX-\mathbb{E}(aX))^2\Big),\\
&= \mathbb{E}\Big((aX-a\mathbb{E}(X))^2\Big),\\
&= \mathbb{E}\Big(a^2(X-\mathbb{E}(X))^2\Big),\\
&= a^2\mathbb{E}\Big((X-\mathbb{E}(X))^2\Big),\\
&= a^2\mathrm{Var}(X).
\end{aligned}$$

□

The measure the joint variability of two random variables, we can look at their *covariance*.

Definition A.5.5. The *covariance* $\mathrm{Cov}(X, Y)$ of the random variables X and Y is defined as

$$\mathrm{Cov}(X,Y) = \mathbb{E}\Big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\Big),$$

provided that the covariance is well-defined.

A.6 Conditional expectation and variance

In this subsection, we will expand our definitions of expectation and variance to respectively *conditional* expectation and *conditional* variance. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random variable.

Definition A.6.1. The *conditional expectation* of X given event $H \in \mathcal{F}$ with strictly positive probability, is defined as

$$\mathbb{E}(X \mid H) := \frac{\mathbb{E}(\mathbb{1}_H X)}{\mathbb{P}(H)} = \int_{\mathcal{X}} x \,\mathrm{d}\mathbb{P}(x \mid H),$$

where $\mathcal{X}$ is the range of X, and $\mathbb{P}(\cdot \mid H)$ is the probability measure defined for sets A as $\mathbb{P}(A \mid H) = \dfrac{\mathbb{P}(A \cap H)}{\mathbb{P}(H)}$.

Using this definition, we define the conditional variance.

Definition A.6.2. The *conditional variance* of X given event $H \in \mathcal{F}$ with strictly positive probability, is defined as

$$\mathrm{Var}(X \mid H) = \mathbb{E}\Big((X-\mathbb{E}(X \mid H))^2 \mid H\Big).$$

Informally, we can see these conditional quantities as variants given that certain conditions are known to occur.

Appendix B

In this Appendix, two lemmas are given, one of which is proved here.

B.1 Lemmas

The proof of the following lemma is by the hand of the author of this thesis.

Lemma B.1.1. *For a log-normal random variable X, we have*

$$\log \mathbb{E}_t \left[X_{t+1}\right] = \mathbb{E}_t \left[\log X_{t+1}\right] + \frac{1}{2} \, Var_t \left[\log X_{t+1}\right]. \tag{B.1}$$

Proof. Let $Y = \log X$, then Y is normally distributed. Denote the mean and variance of Y conditional on the information up to time t by μ_t and σ_t^2, respectively. Then we have to show that $\mathbb{E}_t \left[X_{t+1}\right] = e^{\mu_t + \frac{1}{2}\sigma_t^2}$. We consider

$$\mathbb{E}_t \left[X_{t+1}\right] = \mathbb{E}_t \left[e^{Y_{t+1}}\right] = \int_{-\infty}^{\infty} e^y \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(y-\mu_t)^2}{2\sigma_t^2}} \, \mathrm{d}y. \tag{B.2}$$

Now note that

$$\begin{aligned}
y - \frac{(y-\mu_t)^2}{2\sigma_t^2} &= -\frac{-2\sigma_t^2 y + y^2 - 2\mu_t y + \mu_t^2}{2\sigma_t^2} \\
&= -\frac{1}{2\sigma_t^2}(y^2 - 2(\mu_t + \sigma_t^2)y + (\mu_t + \sigma_t^2)^2 + \mu_t^2 - (\mu_t + \sigma_t^2)^2) \\
&= -\frac{(y - (\mu_t + \sigma_t^2))^2}{2\sigma_t^2} - \frac{\mu_t^2 - \mu_t^2 - 2\mu_t\sigma_t^2 - \sigma_t^4}{2\sigma_t^2} \\
&= -\frac{(y - \mu_t')^2}{2\sigma_t^2} + \mu_t + \frac{1}{2}\sigma_t^2,
\end{aligned}$$

where $\mu_t' = \mu_t + \sigma_t^2$. So we can rewrite Equation B.2 as

$$\begin{aligned}
\mathbb{E}_t \left[X_{t+1}\right] &= e^{\mu_t + \frac{1}{2}\sigma_t^2} \int_{-\infty}^{\infty} \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(y-\mu_t')^2}{2\sigma_t^2}} \, \mathrm{d}y \\
&= e^{\mu_t + \frac{1}{2}\sigma_t^2}.
\end{aligned}$$

The integral equals 1, because it is the integral of a normal density with parameters μ_t' and σ_t^2. $\square$

The following lemma is known as the *Markov inequality.*

Lemma B.1.2. *Let X be a random variable, $a \in \mathbb{R}$ a constant and $g : \mathbb{R} \to [0, \infty]$ an increasing function. Then* $\mathbb{E}\Big(g(X)\Big) \geq g(a)\mathbb{P}(X \geq a)$.

Proof. See [26]. $\square$

Bibliography

[1] Basu A., Byrne A and Drew M., *Dynamic Lifecycle Strategies for Target Date Retirement Funds*, Journal of Portfolio Management 37:2, 83-96, 2009

[2] Bagliano F., Fugazza C. and Nicodano G., *Pension Funds, Life-Cycle Asset Allocation and Performance Evaluation*, World Bank Publications, 2010

[3] Benzoni L., Collin-Dufresne P. and Goldstein R.S., *Portfolio Choice Over the Life-Cycle when the Stock and Labor Markets are Cointegrated*, FRB of Chicago Working Paper No. 2007-11, 2007

[4] Bodie Z., Merton R.C., and Samuelson W.F., *Labor Supply Flexibility and Portfolio Choice in a Life-Cycle Model*, Journal of Economic Dynamics and Control, 16, pp. 427-449, 1992

[5] Bovenberg L., Koijen R.S.J. and Nijman T., Teulings, C.N., *Saving and investing over the life cycle and the role of collective pension funds*, De Economist, 2007

[6] Campbell J.Y. and Viceira L.M., *Appendix to: Strategic Asset Location: Portfolio Choice for Long-Term Investors*, Oxford University Press, New York, NY, 2001

[7] Campbell J.Y. and Viceira L.M., *Strategic Asset Location: Portfolio Choice for Long-Term Investors*, Oxford University Press, New York, NY, 2001

[8] Draper N., *A financial market model for the Netherlands*, CPB background document, 2014

[9] Cocco, J.F., Gomes, F.J. and Maenhout, P.J., *Consumption and Portfolio Choice over the Life-Cycle*, Review of Financial Studies, 18, 491-533, 2005

[10] Commissie Parameters, *Advies Commissie Parameters*, `https://www.rijksoverheid.nl/documenten/kamerstukken/2019/06/11/advies-commissie-parameters`, last consulted on 12th of May 2020.

[11] Heaton J. and Lucas D., *Market Frictions, Savings Behavior, and Portfolio Choice*, Macroeconomic Dynamics, vol. 1, issue 1, pp. 76-101, 1997

[12] Heston S.L., *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, The Review of Financial Studies, Vol. 6, No. 2, pp. 327-343, 1993

[13] Jagannathan R. and Kocherlakota N.R., *Why should older people invest less in stock than younger people?*, Quarterly Review, Federal Reserve Bank of Minneapolis, vol. 20(Sum), pages 11-23, 1996

[14] Koolmees W., *Kabinetsbrief Principeakkoord vernieuwing pensioenstelsel*, Rijksoverheid, 2019

[15] Markowitz H., *Portfolio Selection*, The Journal of Finance, Vol. 7, No. 1., pp. 77-91, 1952

[16] Merton R.C., *Life Time Portfolio Selection Under Uncertainty: The Continuous Time Case*, Review of Economics and Statistics, 51, pp. 247-257, 1969

[17] Minderhoud I., *Modeling Human Capital in Life-Cycle Portfolio Choice: Riskless or Risky?*, Universiteit van Tilburg, 2009

[18] Minderhoud I., Molenaar R. and Ponds E., *The Impact of Human Capital on Life-Cycle Portfolio Choice: Evidence for the Netherlands*, Netspar Discussion Papers, 2001

[19] Neumann J. von and Morgenstern O., *Theory of Games and Economic Behavior*, Princeton, NJ, Princeton University Press, 1953

[20] Polman F. and Dekker C., *Riskco meeting slides: Uniforme rekenmethodiek (urm)*, 2017

[21] Press W.H., Teukolsky S.A., Vetterling W.T. and Flannery B.P. *Numerical Recipes: The Art of Scientific Computing*, 3rd ed., New York: Cambridge University Press, 2007

[22] Ruszel W.M., *Stochastic Differential Equations*, Lecture Notes, 2019

[23] Samuelson P.A., *Lifetime Portfolio Selection by Dynamic Stochastic Programming*, The Review of Economics and Statistics, 51, pp. 239-246, 1969

[24] Shu L., Melenberg B. and Schumacher J.M., *An Evaluation of the nFTK*, Netspar Industry Paper, 2016

[25] Shu L., Melenberg B. and Schumacher J.M., *An Evaluation of the nFTK. Technical Appendix*, 2016

[26] Spreij P.J.C., *Measure Theoretic Probability*, UvA Lecture Notes, 2016

[27] Telkamp D.J.W., *An Alternative Financial Assessment Framework for Pension Funds: Improving the Current and Future Financial Situation of Pension Participants*, Universiteit van Amsterdam, 2019

[28] Vasicek O., *An equilibrium characterization of the term structure*, Journal of Financial Economics 5, No. 2, pp. 177-188, 188, 1977

[29] Viceira, L.M., *Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income*, Journal of Finance, 56, pp. 433-470, 2001

[30] `https://www.ser.nl/-/media/ser/downloads/adviezen/2019/naar-een-nieuw-pensioenstelsel.pdf`, last consulted on 12th of June, 2019

[31] `https://tradingeconomics.com/euro-area/interest-rate`, last consulted on 12th of June, 2019

[32] Koninklijk Actuarieel Genootschap, Prognosetafel AG2018, `https://www.ag-ai.nl/view.php?action=view&Pagina_Id=885`, 2018, last consulted on 12th of September, 2019

[33] `https://opendata.cbs.nl/statline/#/CBS/nl/dataset/7461bev/table?ts=1552901770671`, last consulted on 12th of September, 2019

[34] `https://esb.nu/esb/20028195/pensioenfondsen-dekkingsgraden-en-deelnemers`, last consulted on 14th of February, 2020