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Master Thesis

# Calculating the effective potential of a conformal scalar-tensor theory

*Author:*
Barna Fekecs

*Supervisor:*
dr. Tomislav Prokopec

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Faculty of Science

Institute for Theoretical Physics

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**Barna Fekecs**

## Abstract

This thesis investigates a certain Weyl symmetric scalar-tensor model. Previously it's been shown that spontaneous breaking of Weyl symmetry offers an elegant mechanism for inflation. The thesis attempts to extend the classical analysis of its source paper by calculating quantum corrections to the inflaton potential. A flat-space analogue of this potential is successfully determined to one loop order. Future directions in quantization of conformal models are discussed.

# Contents

# 1 Introduction

## 1.1 Quantum gravity as an effective field theory

*Gravity and quantum theory are incompatible.* A well known and often repeated statement about the present state of our best physical theories. In this introduction (and the following thesis) we would like to convince the reader that as many conventional wisdoms, the above statement is only partly true. While the reigning theory of gravitation, general relativity indeed does not fit into the class of renormalizable quantum field theories, like e.g. the components of the Standard Model (SM) do, this does not mean that one cannot extract meaningful predictions from it on the quantum level. The tools of effective field theory are completely appropriate for this purpose. There is of course a caveat - the predictions of effective field theory cannot be extrapolated into arbitrary high energies. While definitely a weakness, this feature is less troubling nowadays than it used to be. On the one hand we know, not just theoretically but experimentally, that effective field theories work reliably, for example in the context of quantum chromodynamics (QCD). On the other hand: virtually no high energy physicist expects the SM to hold at higher energy scales like the GUT scale or the Planck scale. This then implies that the SM itself is an effective field theory, just one that is already renormalizable without its UV completion.

In this thesis we take the effective field theory approach to quantum gravity (QG) and try to extract meaningful physical predictions for a specific model. The application of these techniques to gravity was heralded by John F. Donoghue[1]. A short summary of these ideas is presented here.

Let us first note, that if one wanted to write down a general theory of gravitation based on Riemannian curvature and general covariance, the Ricci scalar would not be the only allowed term in the action, as it is in the case of the Einstein-Hilbert action. The general action would take a form like

$$S = \int d^4x\sqrt{-g}\Big\{\Lambda + \frac{2}{\kappa^2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + ... + \mathcal{L}_{matter}\Big\} \tag{1}$$

The Riemann squared term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be eliminated by combining it with $R^2$ and $R_{\mu\nu}R^{\mu\nu}$ to give the Gauss-Bonnet term, which vanishes in four dimensions. The reason standard general relativity doesn't use these higher order terms is that, generally speaking, curvature is small. By that we mean: in every classical physical situation where general relativity might be applicable. This then implies that $R^2$'s contribution would be much less, and the contribution of higher order terms even smaller accordingly. A classic paper by Stelle[2] puts the experimental bounds on $c_1$ and $c_2$ as $< 10^{74}$. One can thus see that, if these couplings are not extremely large, they can easily remain unnoticed at ordinary scales. This is the conclusion effective field theory gives us in the classical case.

Higher order terms acquire a more important role in the context of quantization and especially, the renormalization procedure. The divergent term of pure gravity was first calculated to one loop order by 't Hooft and Veltman[3], using the background field method and dimensional regularization (we use the same method in this thesis). They determined that the divergent part of the Lagrangian is

$$\mathcal{L}^{(div)}_{1loop} = \frac{1}{8\pi^2(D-4)}\Big\{\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\Big\} \tag{2}$$

which is clearly divergent when $D \to 4$. What is peculiar about this result, which arises from one loop fluctuations, is that it contains entirely new types of terms compared to the classical Einstein-Hilbert action, which is linear in the Ricci scalar and contains no

Ricci tensor. This means that the removal of these divergences requires new types of counter-terms not found in the classical action. Which in turn means that the divergence cannot be absorbed by the redefinition of bare coupling constants, as it is normally the case for renormalizable field theories. The advantage of effective field theory is that we can come to terms with this. We just introduce the new counter-terms to the action, absorb the divergence and use this action while keeping two caveats in mind. One, this model is unreliable in the IR limit therefore in every calculation one should make explicit the energy scale. And two, the coupling constants in effective field theory are to be determined experimentally, therefore those cannot be regarded as predictions of this model.

Following this prescription, a number of important and interesting calculations become possible: an amplitude for graviton-graviton scattering can be obtained[4] or quantum corrections to the gravitational potential can be calculated[5].

## 1.2 A short note on asymptotic safety

At this point it's important to say a few words about asymptotic safety. Asymptotic safety is a concept, first proposed in 1979 by Steven Weinberg[6], that allows a non-renormalizable quantum field theory to behave 'reasonably' in the UV-limit, that is, not produce nonphysical infinities. This is achieved by postulating the existence of a 'non-trivial fixed point' in the renormalization group (RG) flow. This describes a scenario where in the UV-limit Lagrangian parameters collectively flow toward a point in parameter space, as opposed to infinitely diverging in arbitrary directions. As long as

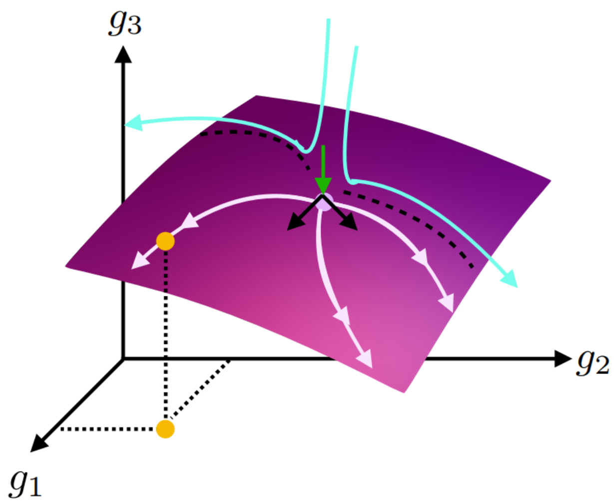


Figure 1: A fixed point (light purple dot) in the space of coupling constants (theory space). The directed curves correspond to the RG group flow, and point from the UV to the IR by convention. The purple surface is known as the critical (hyper-)surface, which contains all physically realizable trajectories.[8]

such a point exists, asymptotic safety provides non-perturbative methods to calculate physical observables. While asymtotic safety was always meant to be a tool to tackle QG, it has been shown to work in other contexts, which is a strong indirect evidence in support of the idea. A direct proof for the existence of a non-trivial fixed point in Einstein gravity has not yet been found, asympotic safety in QG is an active area of research[7][9].

A special feature of fixed points is that they always correspond to quantum field theories that are scale-invariant and sometimes even conformal. This is the reason asymptotic safety is mentioned here; the following thesis takes the opposite route in a sense. We start with a classically scale-invariant (in fact Weyl-symmetric) model, and exploit the fact that in this case pertubative methods are allowed. Classically, it has been shown that there is a scale dynamically generated by condensates. Our hope is that we'll find a UV conformal fixed point perturbatively, such that Weyl symmetry is restored in the UV limit. In this sense, the thesis pursues a goal similar to that of the asymptotically safe Einstein gravity program.

## 1.3 Weyl and conformal symmetries

In 1918, only three years after Einstein published his general theory of relativity, German mathematician Hermann Weyl proposed an extension to it [10]. In it he attempted to unify the two forces of nature known at the time, gravity and electromagnetism by means of introducing a novel symmetry (later named *Weyl symmetry*). If one required this symmetry to hold locally, a new vector field would naturally emerge which Weyl identified as the electromagnetic vector potential. This procedure should ring familiar to any student of field theory: in his attempt Weyl introduced the concept of *gauge*. While the attempt ultimately failed (Einstein pointed out that Weyl symmtery is not obeyed in reality), it kick-started the research of *gauge theories*, which constitute the bedrock of modern physics. In this section I'll introduce Weyl symmetry, discuss the difference between it and conformal symmetry and finally attempt to show why it is such an attractive feature in the context of field theory.

### 1.3.1 Weyl transformation

Weyl transformation is local rescaling of the metric tensor, such that

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu}, \; x^\mu \to x^\mu \tag{3}$$

where $\theta(x)$ is an arbitrary function. Changing length scales has of course implications to the field content of the model. Scalar, spinor and vector fields transform as

$$\begin{aligned} \phi &\to e^{\Delta_\phi \theta(x)} \phi \\ \psi &\to e^{\Delta_\psi \theta(x)} \psi \\ V_\mu &\to e^{\Delta_V \theta(x)} V_\mu \end{aligned} \tag{4}$$

where in $D$-dimensions the scaling dimensions are $\Delta_\phi = -\frac{(D-2)}{2}$, $\Delta_\psi = -\frac{(D-1)}{2}$ and $\Delta_V = -\frac{(D-4)}{2}$. If we want a field theory that is invariant under this transformation, our next step is to construct a covariant derivative.

$$D_\mu \phi(x) \to (D_\mu \phi(x))' = e^{-\theta(x)} D_\mu \phi(x) \tag{5}$$

From this it follows that the covariant derivative has to take the form

$$D_\mu \phi = \partial_\mu \phi + \frac{D-2}{2} \mathcal{T}_\mu \phi \tag{6}$$

which in the case of a generalised field $\mathbf{\Psi}$ of an arbitrary scaling dimension $\Delta_{\mathbf{\Psi}}$

$$D_\mu \mathbf{\Psi} = \partial_\mu \mathbf{\Psi} + (\Delta_g - \Delta_{\mathbf{\Psi}}) \mathcal{T}_\mu \mathbf{\Psi} \tag{7}$$

where $\Delta_g$ is the geometrical dimension, which for a tensor field of type $\binom{p}{q}$ is defined as $\Delta_g = q - p$[11]. The new vector object $\mathcal{T}_\mu$ is a gauge field that, consistently with the above definitions, transforms as

$$\mathcal{T}_\mu \to \mathcal{T}_\mu + \partial_\mu \theta(x) \tag{8}$$

Now, anyone acquainted with electrodynamics can see that this procedure is completely analogous with the one involving the Abelian gauge group $U(1)$. This clarifies why Weyl himself tried to identify the group of his transformations with it. There are subtle differences, for example if one derives the equations of motion for the two gauge fields. Then it can be seen that while $\mathcal{T}_\mu$ is sourced by a longitudinal current, $A_\mu$ is sourced by a purely transverse current up to leading order.

While reading professional literature about scale invariant theories, one encounters the word "conformal" a lot. Indeed, the difference between these three expressions "scale invariance", "Weyl symmetry" and "conformal symmetry" might not be immediately clear. Let's address this in the following subsection.

### 1.3.2 Conformal symmetry

Weyl symmetry refers to theories invariant Weyl transformations defined above, while scale invariance only means that a theory does not contain any dimensionful parameters. Conformal symmetry refers to something more specific. Conformal symmetry is defined as the group of coordinate transformations which leave the metric invariant up to a conformal factor[12].

$$x' = F(x), \; g_{\mu\nu} = \Omega^2(x') g'_{\rho\sigma} \partial_\mu F^\rho \partial_\nu F^\sigma \tag{9}$$

The infinitesimal version of this transformation is $x^\mu \to x^\mu + \xi^\mu$, which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha = \Omega^2(x) g_{\mu\nu} \tag{10}$$

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi^\mu$. This is a proper Lie group, with 15 generators in flat space-time.

Field theories respecting this symmetry, CFTs, are a fashionable area of research nowadays, mainly because of string theory, where they characterize dynamics on the string world-sheet, and AdS/CFT correspondence, a conjectured duality between $D$-dimensional gravitating bulk systems, and $(D-1)$-dimensional hyper-surfaces that accommodate CFTs.

In the following chapters we shall only discuss Weyl symmetry, on one hand because the work this thesis is based on uses Weyl symmetry, but on the other it is the strongest condition: Weyl symmetry automatically implies conformal symmetry[12].

## 1.4 Coleman-Weinberg mechanism

The Coleman-Weinberg mechanism is a realization of spontaneous symmetry breaking in QFT. Spontaneous symmetry breaking is the physical process where a system obeys equations of motion or Lagrangian that respect a symmetry, but the system's lowest energy state does not. The cause of this effect lies in the shape of the potential. Students of physics will be familiar with the most often used example: the Mexican hat potential. The zero field configuration does not correspond to a minimum (it is in fact a local maximum), so the field gains a non-zero vacuum expectation value. What Coleman and Weinberg[13] showed is that there are cases where this effect can be realized even if the classical potential has a minimum at zero. In other words, field theories that appear to have symmetric vacua at the classical (tree) level can still go through spontaneous

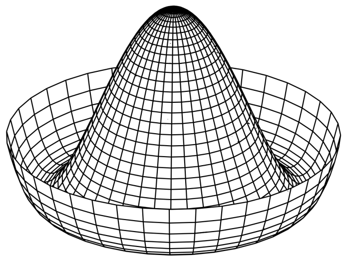

Figure 2: Mexican hat potential[14]

symmetry breaking due to quantum corrections to the potential. If one wants to know about the true vacuum states of a theory, she needs to consult the effective (Coleman-Weinberg) potential, which contains quantum corrections up to an arbitrary order of perturbation expansion.

While the case of the Mexican hat potential and its $U(1)$ symmetry is geometrically intuitive, in principle any continuous symmetry of the classical action can be broken by quantum corrections. In this thesis we shall focus on Weyl symmetry - one of our main goals is to see how quantum corrections introduce a scale spontaneously into a classically scale-free theory.

## 1.5 Inflation

Cosmic inflation is an extension of the big bang theory (BB), which is the main paradigm of contemporary cosmology. It hypothesises an era of exponential expansion in the very early universe. The first formulations of this theory have been put forth by Alexei Starobinsky and Alan Guth, who were searching for solutions to a set of problems related to the classical BB.

### 1.5.1 Flatness problem

According to most recent measurements, the geometry of our observable universe is nearly flat, that is, the parameter $k$ in the FLRW metric (defined below 11) is very close to zero. $k$ has dimensions of length$^{-2}$, and is readily understood as the Gaussian curvature of the space when $a(t) = 1$. This is is somewhat unexpected as previous epochs of cosmology all had dynamics that would have made any curvature grow. Therefore bounds on the present curvature parameter, $\Omega_k \equiv -\frac{k}{a_0^2 H_0^2} < 10^{-3}$, imply bounds for past states that are several orders of magnitude more strict. An initial condition this special demands explanation, and it gets one in inflation. During the exponential expansion, curvature decreases exponentially. Therefore most bunched up initial states would produce a nearly flat geometry after a suitably long inflationary phase.

### 1.5.2 Horizon problem

The cosmic microwave background (CMB) is very isotropic. While it's average temperature is roughly $2.7K$, the largest variation in it is of micro-Kelvin order. A logical inference would then be that this glow is the imprint of regions that were in thermal equilibrium across the whole visible sky. When we consult classical BB models however, we realize that this could not have been so - two points in the sky could only ever have been in causal contact if they're separated by less than $1.6°$. How does one explain then the striking uniformity in temperature between regions that never exchanged any information? An answer can be found in inflation: the space that corresponds to the observable universe today was tiny in the early inflationary period and thus had plenty of time to equilibrate.

#### 1.5.3 Dynamics of inflation

In order to square exponential expansion with general relativity we need to introduce a driving field - the inflaton. Most working models take this field to be either one or several scalar fields. In the following we will assume familiarity with the Friedmann-Lemaitre-Robertson-Walker(FLRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right] \tag{11}$$

Then the pressure and energy density of a single spatially homogeneous scalar field are given as

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \quad \rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{12}$$

The energy conservation equation of FLRW-cosmologies $\dot{\rho} = -3H(\rho + p)$ then takes the form

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \tag{13}$$

A general inflationary phase can be defined by an accelerating scale factor: $\ddot{a} > 0$. This criterion can be expressed in terms of the Hubble parameter as $|\dot{H}| << H^2$. If we then enforce this condition on 13, we end up with two conditions for $V(\phi)$ and its derivatives:

$$\left| \frac{V'(\phi)}{V(\phi)} \right| << \sqrt{16\pi G}, \quad \left| \frac{V''(\phi)}{V(\phi)} \right| << 24\pi G \tag{14}$$

These are known as the 'flatness conditions'. They are the key features of modern 'slow roll' inflationary models.

The moral of this section is the following: inflation is a pretty vague mechanism that can realized by a plethora of potentials and inflatons of various types. In spite of this fact, the cosmology community is (with some notable exceptions) in agreement that inflation happened, because it solves the two puzzles discussed above. Another, even stronger argument comes from the CMB and the large scale distribution of matter in the universe - in inflationary models, quantum fluctuations magnified by the expansion act as seeds for growing structures. This theory has been strongly vindicated by recent astronomical surveys, like the Planck Collaboration[15].

A lot of current research focuses on motivating inflation with good, credible new physics. Identifying the inflaton and explaining the shape of its potential are the main task for these projects. As it happens, the basis material for this thesis is also provided by a paper of this kind.

# 2 Preliminary concepts

## 2.1 Derivation of the graviton propagator

### 2.1.1 Derivation of the graviton Lagrangian

There are two main approaches to general relativity, one that may be called "top down" and another that is more "bottom up". When first introducing general relativity, Einstein followed the first path and thus made it well known. For our purposes however, the second path is just as practical (if not more), and so here we choose to follow it.

Here we take as our starting point the concept of a spin-2 particle. Fierz and Pauli have managed to write down an action for both the massive and massless case of this (linear) model in 1939. Then a project more than a decade long has been carried out by Gupta, Thirring, Feynman and Weinberg. It was finally concluded by Deser in 1970 when he managed to reproduce the non-linear Einstein equations from a particle point of view. We shall not discuss the details of these works as we only need the linear Fierz-Pauli version for our one-loop calculation.

The action is always a scalar quantity $\Rightarrow$ we need an expression with all indices contracted. We also expect in each term a product of two derivatives of $h$, much like in the case of the scalar particle.

1. $\partial_\sigma h_{\mu\nu} \partial^\sigma h^{\mu\nu}$
2. $\partial_\sigma h_{\mu\nu} \partial^\nu h^{\mu\sigma}$
3. $\eta^{\rho\sigma} \partial_\nu h^{\mu\nu} \partial_\sigma h_{\rho\mu}$
4. $\partial_\nu h^{\mu\nu} \partial_\mu h$
5. $\partial_\mu h \partial^\mu h$

where, if we neglect surface terms, we can show that 2 and 3 are actually the same term. Integrate 2 by parts

$$\begin{aligned} &\int d^D x \; \partial_\sigma h_{\mu\nu} \partial^\nu h^{\mu\sigma} = \int_S d^{D-1}x \; \hat{n}_\sigma h_{\mu\nu} \partial^\nu h^{\mu\sigma} - \int d^D x \; h_{\mu\nu} \partial_\sigma \partial^\nu h^{\mu\sigma} \\ &= - \int d^D x \; h_{\mu\nu} \partial_\sigma \partial^\nu h^{\mu\sigma} = - \int_S d^{D-1}x \; \hat{n}^\nu h_{\mu\nu} \partial_\sigma h^{\mu\sigma} \\ &+ \int d^D x \; \partial^\nu h_{\mu\nu} \partial_\sigma h^{\mu\sigma} \; \Rightarrow 3 \end{aligned} \tag{15}$$

The form of the action that we assume is the following:

$$S = \int d^D x \; [a \partial^\sigma h^{\mu\nu} \partial_\sigma h_{\mu\nu} + b \partial_\nu h^{\mu\nu} \partial_\sigma h^\sigma_\mu + c \partial_\nu h^{\mu\nu} \partial_\mu h + d \partial_\mu h \partial^\mu h] \tag{16}$$

Let's vary this with respect to $h_{\alpha\beta}$. Notice that there are no 'undifferentiated' $h$-terms in this action

$$\Rightarrow \delta S = - \int d^D x \; \left[ \partial_\gamma \left( \frac{\partial \mathcal{L}}{\partial(\partial_\gamma h_{\alpha\beta})} \right) \right] \delta h_{\alpha\beta} \tag{17}$$

$$\begin{aligned} \partial_\gamma \left( \frac{\partial \mathcal{L}}{\partial(\partial_\gamma h_{\alpha\beta})} \right) &= 2a \partial^2 h^{\alpha\beta} + b(\partial^\beta \partial_\mu h^{\mu\alpha} + \partial^\alpha \partial_\mu h^{\beta\mu}) + c(\partial^\beta \partial^\alpha h \\ &+ \eta^{\alpha\beta} \partial_\nu \partial_\mu h^{\mu\nu}) + 2d \eta^{\alpha\beta} \partial^2 h \end{aligned} \tag{18}$$

Now this expression should be contracted with $\delta h_{\alpha\beta}$ and then functionally differentiated with respect to it. The important step here is the symmetrization $\Rightarrow$ $\delta h_{\alpha\beta}$ should be symmetric, so only the symmetric part of this equation gives the equation of motion correctly. Thankfully this expression is already symmetric $\alpha$ and $\beta$, so we can set it to 0 to get the vacuum equation of motion or we can put in a source term on the RHS: $-\lambda T^{\alpha\beta}$. Now use the fact that if $T^{\alpha\beta}$ is a conserved current as $\partial_\beta T^{\alpha\beta} = 0$, then the LHS also has to be identically zero.

$$\begin{aligned} &2a \partial_\beta \partial^2 h^{\alpha\beta} + b(\partial^2 \partial_\sigma h^{\sigma\alpha} + \partial_\beta \partial^\alpha \partial_\nu h^{\beta\nu}) + c(\partial^2 \partial^\alpha h + \partial^\alpha \partial_\nu \partial_\gamma h^{\gamma\nu}) + 2d \partial^\alpha \partial^2 h \\ &\Rightarrow (2a+b) \partial_\sigma \partial^2 h^{\sigma\alpha} = 0 \\ &\quad (c+b) \partial^\alpha \partial_\mu \partial_\nu h^{\mu\nu} = 0 \\ &\quad (c+2d) \partial^\alpha \partial^2 h = 0 \end{aligned} \tag{19}$$

We have 4 coefficients and 3 equations, so we need to choose a value for one of them. Traditionally we set $a = \frac{1}{2}$,

$$\Rightarrow a = \frac{1}{2}, \quad b = -1, \quad c = 1, \quad d = -\frac{1}{2} \tag{20}$$

Thus we have our quadratic action, also known as the Fierz-Pauli action:

$$S_{FP} = \int d^D x \left[ -\frac{1}{4}\partial^\sigma h^{\mu\nu}\partial_\sigma h_{\mu\nu} + \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\sigma h^\sigma_\mu - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h + \frac{1}{4}\partial_\mu h \partial^\mu h \right] \tag{21}$$

This can be shown to be the correct quadratic expansion of $\sqrt{-g}R$. Because in later parts we will use this times one half, let us include now that factor so that we have

$$S = \int d^D x \; \frac{1}{2} \left[ -\frac{1}{4}\partial^\sigma h^{\mu\nu}\partial_\sigma h_{\mu\nu} + \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\sigma h^\sigma_\mu - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h + \frac{1}{4}\partial_\mu h \partial^\mu h \right] \tag{22}$$

Now let's add a gauge-fixing term. The most general Gaussian gauge-breaking term which is local and and bilinear in the fields and derivatives if of the form:

$$\Delta S_{gauge\ fixing} = \int d^D x \; -\frac{1}{2a}\left(\partial^\mu h_{\mu\nu} - \frac{b}{2}\partial_\nu h\right)\eta^{\nu\rho}\left(\partial^\sigma h_{\sigma\rho} - \frac{b}{2}\partial_\rho h\right) \tag{23}$$

$$\begin{aligned} S_{FP+gf} = \int d^D x \left[ -\frac{1}{8}\partial^\sigma h^{\mu\nu}\partial_\sigma h_{\mu\nu} + \left(\frac{1}{4} - \frac{1}{2a}\right)\partial_\nu h^{\mu\nu}\partial_\sigma h^\sigma_\mu - \left(\frac{1}{4} - \frac{b}{2a}\right)\partial_\nu h^{\mu\nu}\partial_\mu h \right. \\ \left. + \left(\frac{1}{8} - \frac{b^2}{8a}\right)\partial_\mu h \partial^\mu h \right] = \int d^D x \; \frac{1}{2} h_{\mu\nu}\mathcal{L}^{\mu\nu\rho\sigma}_{gauge\ fixed} h_{\rho\sigma} \end{aligned} \tag{24}$$

where in the last line we defined $\mathcal{L}^{\mu\nu\rho\sigma}_{gauge\ fixed}$, the gauge fixed Lichnerowicz operator.

### 2.1.2 The propagator

We would like to find the object (the propagator) ${}_{\rho\sigma}\Delta_{\alpha\beta}(x, x')$ for which the following equation is satisfied:

$$\mathcal{L}^{\mu\nu\rho\sigma}_{g\ fixed}(x) i[{}_{\rho\sigma}\Delta_{\alpha\beta}](x, x') = i\delta^{(\mu}_\alpha \delta^{\nu)}_\beta \delta^D(x - x') \tag{25}$$

To find it it is beneficial to go to momentum space, where we can use the methods of linear algebra straightforwardly. So let us make the standard substitution: $\partial_\mu \to ik_\mu$, and factor out the $h$-terms in the original gauge-free action first:

$$\begin{aligned} \int d^D x \; \frac{1}{2} h_{\mu\nu} [\frac{1}{2}\eta^{\mu(\rho}\eta^{\sigma)\nu}k^2 - \frac{1}{4}\eta^{\mu\rho}k^\nu k^\sigma - \frac{1}{4}\eta^{\mu\sigma}k^\nu k^\rho - \frac{1}{4}\eta^{\nu\rho}k^\mu k^\sigma - \frac{1}{4}\eta^{\nu\sigma}k^\mu k^\rho \\ - \frac{1}{2}\eta^{\rho\sigma}k^\mu k^\nu - \frac{1}{2}\eta^{\mu\nu}k^\rho k^\sigma + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}k^2] h_{\rho\sigma} = \int d^D x \; \frac{1}{2} h_{\mu\nu}\mathcal{L}^{\mu\nu\rho\sigma}_{free}(k) h_{\rho\sigma} \end{aligned} \tag{26}$$

At this point one needs to split $h$-terms into irreducible representations of the Lorentz group. These correspond to degrees of freedom of different spin $J$ and parity $P$. This is important because otherwise it would be immensely difficult to invert the tensor structure of the Lichnerovicz operator. This way the parts of different spin and parity do not mix and therefore can be addressed individually. For a symmetric rank $(0, 2)$ tensor field we expect the following elements:

- a transverse traceless tensor $h_{ij}^{TT}$, $J = 2$, $P = +$
- a transverse vector $h_{iN}$, $J = 1$, $P = -$
- a scalar $w = h_{NN}$, $J = 0$, $P = +$ (longitudinal)
- and another scalar $s = h_{ii}$, $J = 0$, $P = +$ (transverse)

where $x_N$ is the direction of the momentum and $x_i$, $i = 1, ..., D$ are transverse to it. The basic building blocks of our projectors will be

$$\ell_{\mu\nu} \equiv \frac{\partial_\mu \partial_\nu}{\partial^2} \equiv \frac{k_\mu k_\nu}{k^2}, \qquad t_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \equiv \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \tag{27}$$

They satisfy the properties below

$$\begin{gathered} t^{\mu\nu} t_{\nu\lambda} = t^{\mu}_{\lambda}, \qquad \ell^{\mu\nu} \ell_{\nu\lambda} = \ell^{\mu}_{\lambda}, \qquad t^{\mu\nu} \ell_{\nu\lambda} = 0, \\ t^{\mu\nu} t_{\mu\nu} = D - 1, \qquad \ell^{\mu\nu} \ell_{\mu\nu} = 1 \end{gathered} \tag{28}$$

It is important to note that any Lorentz-covariant type-$(2, 0)$ tensor can be decomposed into elements proportional to these projectors:

$$S_{\mu\nu} = S^T t_{\mu\nu} + S^L \ell_{\mu\nu} \tag{29}$$

This result will be useful later. For now, since the Lichnerowicz operator has 4 covariant indices, we'll have to use higher order projectors. In terms of the above defined operators, we have

$$\begin{aligned} P^{(2)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(t_{\mu\rho} t_{\nu\sigma} + t_{\mu\sigma} t_{\nu\rho}) - \frac{1}{D-1} t_{\mu\nu} t_{\rho\sigma} \\ P^{(1)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(t_{\mu\rho} \ell_{\nu\sigma} + t_{\mu\sigma} \ell_{\nu\rho} + t_{\nu\rho} \ell_{\mu\sigma} + t_{\nu\sigma} \ell_{\mu\rho}) \\ P^{(0,ss)}_{\mu\nu\rho\sigma} &= \frac{1}{D-1} t_{\mu\nu} t_{\rho\sigma}, \qquad P^{(0,ww)}_{\mu\nu\rho\sigma} = \ell_{\mu\nu} \ell_{\rho\sigma} \\ P^{(0,sw)}_{\mu\nu\rho\sigma} &= \frac{1}{\sqrt{D-1}} t_{\mu\nu} \ell_{\rho\sigma}, \quad P^{(0,ws)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{D-1}} \ell_{\mu\nu} t_{\rho\sigma} \end{aligned} \tag{30}$$

When contracted by the last two and first two indices respectively, the spin-0 operators satisfy the algebra

$$\begin{aligned} &P^{\alpha\beta\mu\nu}_{(ss)} P^{(ss)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ss)\rho\sigma}, \ P^{\alpha\beta\mu\nu}_{(ss)} P^{(sw)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(sw)\rho\sigma}, \ P^{\alpha\beta\mu\nu}_{(sw)} P^{(ws)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ss)\rho\sigma}, \\ &P^{\alpha\beta\mu\nu}_{(sw)} P^{(ww)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(sw)\rho\sigma}, \ P^{\alpha\beta\mu\nu}_{(ww)} P^{(ww)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ww)\rho\sigma}, \ P^{\alpha\beta\mu\nu}_{(ws)} P^{(ss)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ws)\rho\sigma}, \\ &P^{\alpha\beta\mu\nu}_{(ws)} P^{(sw)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ww)\rho\sigma}, \ P^{\alpha\beta\mu\nu}_{(ww)} P^{(ws)}_{\mu\nu\rho\sigma} = P^{\alpha\beta}_{(ws)\rho\sigma}, \text{ and every other pair} = 0 \end{aligned} \tag{31}$$

$$\Rightarrow \frac{1}{2} \mathcal{L}^{\mu\nu\rho\sigma}_{free} = \frac{-k^2}{4} (\mathbb{1}^{\mu\nu\rho\sigma} - P^{(1)\mu\nu\rho\sigma} - P^{(ww)\mu\nu\rho\sigma} - (D-1) P^{(ss)\mu\nu\rho\sigma}) \tag{32}$$

But there is a completeness relation

$$P^{(2)} + P^{(1)} + P^{(ss)} + P^{(ww)} = \mathbb{1} \tag{33}$$

thus we have

$$\frac{1}{2} \mathcal{L}^{\mu\nu\rho\sigma}_{free} = \frac{-k^2}{4} \left( P^{(2)} - (D-2) P^{(ss)} \right)^{\mu\nu\rho\sigma} \tag{34}$$

This by itself is not invertible, that is why we need the gauge-fixing term

$$\Delta\mathcal{L}^{\mu\nu\rho\sigma}_{gauge\ fixing} = \frac{-k^2}{2a}\left[\frac{k^{(\mu}\eta^{\nu)(\rho}k^{\sigma)}}{k^2} - \frac{b}{2}\left(\eta^{\mu\nu}\frac{k^\rho k^\sigma}{k^2} + \eta^{\rho\sigma}\frac{k^\mu k^\nu}{k^2}\right) + \frac{b^2}{4}\eta^{\mu\nu}\eta^{\rho\sigma}\right]$$
$$= \frac{-k^2}{2a}\left[\frac{1}{2}P^{(1)} + \left(\frac{b}{2}-1\right)^2 P^{(ww)} + \frac{b}{2}\left(\frac{b}{2}-1\right)\sqrt{D-1}(P^{(sw)} + P^{(ws)}) + \frac{b^2}{4}(D-1)P^{(ss)}\right] \tag{35}$$

Let's add this up with the expression for the unfixed Lichnerovicz:

$$\frac{1}{2}\mathcal{L}^{\mu\nu\rho\sigma}_{g\ fixed} = \frac{-k^2}{2}\Bigg[\frac{1}{2}P^{(2)} + \frac{1}{2a}P^{(1)} - \left(\frac{(D-2)}{2} - \frac{b^2}{4a}(D-1)\right)P^{(ss)}$$
$$+\frac{1}{a}\left(\frac{b}{2}-1\right)^2 P^{(ww)} + \frac{b}{2a}\left(\frac{b}{2}-1\right)\sqrt{D-1}(P^{(sw)} + P^{(ws)})\Bigg] \tag{36}$$

We have to note here that projectors belonging to the same spin come with a matrix structure, so in our case for spin-0 we have

$$P^{(0)} = \begin{bmatrix} P^{(ss)} & P^{(sw)} \\ P^{(ws)} & P^{(ww)} \end{bmatrix} \tag{37}$$

This we need to keep in mind as in our inverted operator the spin-0 coefficients will be the elements of the inverse spin-0 coefficient matrix. Having noted we can now write down the inverse Lichnerowicz operator, which is proportional to the graviton propagator.

$${}_{\mu\nu}\Delta_{\rho\sigma} = \frac{-1}{k^2}\Bigg[2P^{(2)} + 2aP^{(1)} - \frac{2}{D-2}P^{(ss)} + \frac{a(D-2) - \frac{b^2}{4}(D-1)}{(D-2)(\frac{b}{2}-1)^2}P^{(ww)}$$
$$+\frac{b\sqrt{D-1}}{2(D-2)(\frac{b}{2}-1)}(P^{(sw)} + P^{(ws)})\Bigg] \tag{38}$$

so we finally get the graviton propagator in position space:

$$i[{}_{\mu\nu}\Delta_{\rho\sigma}](x,x') = \{2t_{\mu(\rho}t_{\sigma)\nu} - \frac{2}{D-1}t_{\mu\nu}t_{\rho\sigma}$$
$$-\frac{2}{(D-1)(D-2)}\left[\eta_{\mu\nu} - \frac{Db-2}{b-2}\frac{\partial_\mu\partial_\nu}{\partial^2}\right]\left[\eta_{\rho\sigma} - \frac{Db-2}{b-2}\frac{\partial_\rho\partial_\sigma}{\partial^2}\right]$$
$$+4a\frac{\partial_{(\mu}t_{\nu)(\rho}\partial_{\sigma)}}{\partial^2} + \frac{4a}{(b-2)^2}\frac{\partial_\mu\partial_\nu\partial_\rho\partial_\sigma}{\partial^4}\}i\Delta(x,x') \tag{39}$$

## 2.2 The quantum effective action

It's important to say a few words about the method we are using to extract the one loop momentum and gauge dependence of the observables. The constant and linear terms in the perturbation expansion which will also be dropped, which elicits an explanation. We are interested in the quantum effective action up to one loop order [23][24][25]. In a simple scalar model the partition function is defined as

$$Z[J] = \int \mathcal{D}\varphi \exp\left[iS[\varphi] + i\int d^Dx J\varphi\right] = \exp[iW[J]] \tag{40}$$

where $W[J]$ is the generating functional. Differentiating this with respect to $J$ we can obtain arbitrary Green's functions

$$\langle 0|\varphi(x)|0\rangle_J \equiv \frac{\delta}{\delta J(x)} W[J] = \frac{1}{Z}\int \mathcal{D}\varphi e^{iS[\varphi]+i\int d^D x J\varphi}\varphi(x) \tag{41}$$

Then the quantum effective action is defined through a Legendre transformation as

$$\Gamma(\langle\varphi\rangle_J) = W[J] - \int d^d x J(x)\langle\varphi(x)\rangle_J \tag{42}$$

As its name suggest, we can use the quantum action to derive the quantum equation of motion

$$\frac{\delta}{\delta\langle\varphi(y)\rangle_J}\Gamma(\langle\varphi\rangle_J) = \int d^D x \frac{\delta W[J]}{\delta J(x)}\frac{\delta J(x)}{\delta\langle\varphi(y)\rangle_J} - \int d^D x \frac{\delta J(x)}{\delta\langle\varphi(y)\rangle_J}\langle\varphi(x)\rangle_J - J(y) = -J(y) \tag{43}$$

which is in a sense a dual equation to (41). Of course we can't evaluate these expressions to arbitrary degree. For this reason we introduce the background field expansion

$$\varphi = \bar{\varphi} + \eta \tag{44}$$

such that the classical background $\bar{\varphi}$ should satisfy the equation

$$\left.\frac{\delta(S[\varphi] + \int d^D x J(x)\varphi(x))}{\delta\varphi(x)}\right|_{\varphi=\bar{\varphi}} = 0 \tag{45}$$

Now one needs to expand the action in $\eta$ up to quadratic order to obtain the following approximation

$$\begin{aligned} Z = e^{iW[J]} &= \int \mathcal{D}\varphi e^{iS[\varphi]+i\int d^D x J\varphi} \\ &\approx e^{i(S[\bar{\varphi}]+J\bar{\varphi})}\int \mathcal{D}[\eta]\, e^{\frac{i}{2}\int d^D x\, d^D y\, \eta(x)\mathcal{L}^{(2)}\eta(y)} \end{aligned} \tag{46}$$

Thus we get that the generating functional in this approximation is

$$W[J] = \left[S[\bar{\varphi}] + \int J\bar{\varphi}\right] + \frac{i}{2}\mathrm{Tr}\log\left[\mathcal{L}^{(2)}[\bar{\varphi}]\right] \tag{47}$$

where

$$\mathcal{L}^{(2)}[\bar{\varphi}] := \left.\frac{\delta^2\mathcal{L}[\varphi]}{\delta\varphi(x)\delta\varphi(y)}\right|_{\varphi=\bar{\varphi}} \tag{48}$$

Through this expression we can define the propagator $G_F$

$$\int d^D y\, \mathcal{L}^{(2)}(x,y) i G_F(y,z) = i\delta^D(x-z) \tag{49}$$

where one needs to follow the Feynman $i\epsilon$ prescription. We can see that linear terms don't contribute because they imply an odd integral which then by 45 evaluates to zero. And now that we have an approximation for $W[J]$, we can do the Legendre transform to get an expression for the quantum effective action to one loop order

$$\Gamma(\varphi) = S[\varphi] + \frac{i}{2}\mathrm{Tr}\log\left[\mathcal{L}^{(2)}[\varphi]\right] \tag{50}$$

This derivation outlines the method that we'll employ and generalize for the case of a scalar-tensor theory.

Although the linear order terms do not contribute in this calculation, the classical equations of motion (which they satisfy) have been derived and it is found that the first order terms indeed vanish. One note to make here is that for $\delta h$ to vanish, the artificial cosmological constant is already needed.

# 3 The classical model

## 3.1 A rather general conformal action

Our starting point is the Jordan frame action from our primary source paper [16]:

$$
\begin{aligned}
S_J = \int d^D x \sqrt{-g} \bigg[ & \alpha R^2 + \zeta R_{(\mu\nu)} R^{\mu\nu} + \frac{\xi}{2} \phi^I \phi^J \delta_{IJ} R \\
& - \frac{\lambda}{4} \left( \phi^I \phi^J \delta_{IJ} \right)^2 - \frac{1}{2} g^{\mu\nu} \delta_{IJ} \nabla_\mu \phi^I \nabla_\nu \phi^J - \frac{\sigma}{4} \mathcal{T}_{\mu\nu} \mathcal{T}^{\mu\nu} \bigg]
\end{aligned}
\tag{51}
$$

where $\alpha, \zeta, \xi, \lambda$ and $\sigma$ are dimensionless coupling constants and we allow for $\mathcal{N}$ scalar fields $\phi^I$ with $I = 1, ..., \mathcal{N}$.

## 3.2 Weyl geometry

We work in a special Weyl-symmetric geometry (see details in [17]), therefore some objects in the action need to be unambiguously defined, starting with the torsion

$$
\Gamma^\lambda_{[\mu\nu]} = T^\lambda_{\mu\nu} \tag{52}
$$

that under a local Weyl transformation changes like

$$
g_{\mu\nu} \longrightarrow e^{2\theta} g_{\mu\nu}, \qquad T^\lambda_{\mu\nu} \longrightarrow T^\lambda_{\mu\nu} + \delta^\lambda_{[\mu} \partial_{\nu]} \theta \tag{53}
$$

One can now define a torsion trace that will act as a gauge field associated with local Weyl invariance

$$
\mathcal{T}_\mu \equiv -\frac{2}{D-1} T^\lambda_{\mu\lambda} \tag{54}
$$

This also implies the need for a covariant derivative which we denote in the action by unaccented $\nabla$, and which acts on scalars like

$$
\nabla_\mu \phi \equiv (\partial_\mu - \Delta_\phi \mathcal{T}_\mu) \phi \tag{55}
$$

where $\Delta_\phi = -(D-2)/2$ is the scaling dimension of a canonically normalized scalar field. A shorthand notation for the field strength tensor of the gauge field has also been used

$$
\mathcal{T}_{\mu\nu} \equiv \partial_\mu \mathcal{T}_\nu - \partial_\nu \mathcal{T}_\mu \tag{56}
$$

Finally, the special Ricci scalar in terms of the "standard" objects (denoted by circle over) is

$$
R = g^{\alpha\beta} R^\lambda_{\alpha\lambda\beta} = \overset{\circ}{R} + 2(D-1) \overset{\circ}{\nabla}{}^\mu \mathcal{T}_\mu - (D-1)(D-2) g^{\mu\nu} \mathcal{T}_\mu \mathcal{T}_\nu \tag{57}
$$

## 3.3 An on-shell equivalent action

Having introduced the elements of the action, we now bring it to a form easiest to work with. For starters, we restrict the number of scalar fields $\mathcal{N}$ to one and set $\zeta$ to zero as a first step towards the Einstein frame. This also makes the model classically stable. The next step is to use Lagrange multipliers to get rid of the quadratic Ricci term. Let's make the substitution

$$
\chi = R, \text{ and add to the action } + \frac{\omega^2}{2} (R - \chi) \tag{58}
$$

If we then derive the equation of motion for $\chi$, we can use that to express $\chi$

$$
\chi = \frac{1}{4\alpha} (\omega^2 - \xi \phi^2) \tag{59}
$$

Thus, the new form of the action is

$$
\begin{aligned}
S_E = \int d^D x\sqrt{-g}\Big[ &- \left(\frac{\xi^2}{16\alpha} + \lambda\right)\phi^4 + \frac{\xi}{8\alpha}\omega^2\phi^2 - \frac{\omega^4}{16\alpha} \\
&+ \frac{\omega^2}{2}\left(\mathring{R} + 2(D-1)\mathring{\nabla}^\mu \mathcal{T}_\mu - (D-1)(D-2)\mathcal{T}_\mu\mathcal{T}^\mu\right) \\
&- \frac{1}{2}g^{\mu\nu}(\partial_\mu - \Delta_\phi\mathcal{T}_\mu)\phi(\partial_\nu - \Delta_\phi\mathcal{T}_\nu)\phi - \frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}\Big]
\end{aligned}
\tag{60}
$$

# 4 Expansion and gauge fixing

## 4.1 Expansion to quadratic order

Now we can start to expand this action up to to second order around the background. The following are some standard Taylor expansions of expressions found in the action

$$
\begin{aligned}
(\bar{\phi} + \delta\phi)^4 &\approx \bar{\phi}^4 + 4\bar{\phi}^3\delta\phi + 6\bar{\phi}^2\delta\phi^2 \\
(\bar{\phi} + \delta\phi)^2 &\approx \bar{\phi}^2 + 2\bar{\phi}\delta\phi + \delta\phi^2
\end{aligned}
\tag{61}
$$

$$
\frac{\xi}{8\alpha}\omega^2\phi^2 \approx \frac{\xi}{8\alpha}(\bar{\omega}^2\bar{\phi}^2 + 2\bar{\omega}^2\bar{\phi}\delta\phi + 2\bar{\omega}\bar{\phi}^2\delta\omega + 4\bar{\omega}\bar{\phi}\delta\omega\delta\phi + \bar{\omega}^2\delta\phi^2 + \bar{\phi}^2\delta\omega^2)
\tag{62}
$$

Let's look at the expansion of the $\phi$-kinetic term. Our working assumption is that $\mathcal{T}_\mu$ is already a perturbation field.

$$
\begin{aligned}
&- \frac{1}{2}g^{\mu\nu}(\partial_\mu - \Delta_\phi\mathcal{T}_\mu)(\bar{\phi} + \delta\phi)(\partial_\nu - \Delta_\phi\mathcal{T}_\nu)(\bar{\phi} + \delta\phi) \\
&= -\frac{1}{2}g^{\mu\nu}[\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \partial_\mu\delta\phi\partial_\nu\delta\phi + \Delta_\phi^2\bar{\phi}^2\mathcal{T}_\mu\mathcal{T}_\nu + (\partial_\mu\bar{\phi}\partial_\nu\delta\phi + \partial_\nu\bar{\phi}\partial_\mu\delta\phi) \\
&- \Delta_\phi\bar{\phi}(\mathcal{T}_\mu\partial_\nu\bar{\phi} + \mathcal{T}_\nu\partial_\mu\bar{\phi}) - \Delta_\phi\delta\phi(\mathcal{T}_\mu\partial_\nu\bar{\phi} + \mathcal{T}_\nu\partial_\mu\bar{\phi}) - \Delta_\phi\bar{\phi}(\mathcal{T}_\mu\partial_\nu\delta\phi + \mathcal{T}_\nu\partial_\mu\delta\phi)]
\end{aligned}
\tag{63}
$$

Four out of these seven terms contain a derivative of the background field $\bar{\phi}$. If this does not depend on any spacetime coordinates, these terms should become zero. Note however that in any model of cosmic expansion, we would need $\bar{\phi}(t)$, so we would keep these terms. They will be dropped for now. What is left happens to be all second order in perturbations, so their indices are only contracted by the background metric.

$$
\Rightarrow -\frac{1}{2}\bar{g}^{\mu\nu}(\partial_\mu\delta\phi\partial_\nu\delta\phi - \Delta_\phi\bar{\phi}(\mathcal{T}_\mu\partial_\nu\delta\phi + \mathcal{T}_\nu\partial_\mu\delta\phi) + \Delta_\phi^2\bar{\phi}^2\mathcal{T}_\mu\mathcal{T}_\nu)
\tag{64}
$$

Now we need to address the volume term $\sqrt{-g}$. We're employing the "shortwave approximation" [20]. This means that we can split the metric into a background plus perturbations

$$
g_{\mu\nu} \approx \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}
\tag{65}
$$

$\sqrt{-g}$'s expansion up to second order is given below, where $\kappa$ just tracks the order of expansion

$$
\sqrt{-g} \approx \sqrt{-\bar{g}}\left(1 + \frac{\kappa}{2}h^\lambda_\lambda + \frac{\kappa^2}{8}(h^\lambda_\lambda)^2 - \frac{\kappa^2}{4}h^\lambda_\nu h^\nu_\lambda\right)
\tag{66}
$$

In the following I will drop the indices of traces $h^\lambda_\lambda \to h$. Multiplying out further and keeping terms up to second order

$$
\begin{aligned}
&- \left(\frac{\xi^2}{16\alpha} + \lambda\right)\sqrt{-g}\phi^4 \approx -\left(\frac{\xi^2}{16\alpha} + \lambda\right)(\bar{\phi}^4 + 4\bar{\phi}^3\delta\phi + 6\bar{\phi}^2\delta\phi^2 + \frac{\kappa\bar{\phi}^4}{2}h + 2\kappa\bar{\phi}^3\delta\phi h \\
&+ \frac{\kappa^2\bar{\phi}^4}{8}h^2 - \frac{\kappa^2\bar{\phi}^4}{4}h^\lambda_\nu h^\nu_\lambda)
\end{aligned}
\tag{67}
$$

$$\frac{\xi}{8\alpha}\sqrt{-g}\omega^2\phi^2 \approx \frac{\xi}{8\alpha}(\bar{\omega}^2\bar{\phi}^2 + 2\bar{\omega}^2\bar{\phi}\delta\phi + 2\bar{\omega}\bar{\phi}^2\delta\phi + 4\bar{\omega}\bar{\phi}\delta\omega\delta\phi + \bar{\omega}^2\delta\phi^2 \\ + \bar{\phi}^2\delta\omega^2 + \frac{\kappa\bar{\omega}^2\bar{\phi}^2}{2}h + \kappa\bar{\omega}^2\bar{\phi}\delta\phi h + \kappa\bar{\phi}^2\bar{\omega}\delta\omega h + \frac{\kappa^2\bar{\omega}^2\bar{\phi}^2}{8}h^2 - \frac{\kappa^2\bar{\omega}^2\bar{\phi}^2}{4}h^\lambda_\nu h^\nu_\lambda) \tag{68}$$

$$-\frac{1}{16\alpha}\sqrt{-g}\omega^4 \approx -\frac{1}{16\alpha}(\bar{\omega}^4 + 4\bar{\omega}^3\delta\omega + 6\bar{\omega}^2\delta\omega^2 + \frac{\kappa\bar{\omega}^4}{2}h + 2\kappa\bar{\omega}^3\delta\omega h \\ + \frac{\kappa^2\bar{\omega}^4}{8}h^2 - \frac{\kappa^2\bar{\omega}^4}{4}h^\lambda_\nu h^\nu_\lambda) \tag{69}$$

Now we have all the terms except for the Ricci scalar. It will be convenient to first calculate the $\omega h$ coupling term and then introduce the "pure" Ricci scalar separately. In this first attempt our gravitational background is Minkowski ($\sqrt{-g} = 1$), therefore to zeroth order the Ricci scalar is zero. To first order it is (following [20])

$$R^{(1)}(h) = \kappa\left(-\partial^2 h + \partial^\mu\partial^\nu h_{\mu\nu}\right) \tag{70}$$

Multiplying it with the Minkowski metric determinant expanded to first order and $\omega^2$

$$\frac{1}{2}\omega^2\sqrt{-g}R \approx \frac{\bar{\omega}^2}{2}(-\kappa\partial^2 h + \kappa\partial^\mu\partial^\nu h_{\mu\nu} - \frac{\kappa^2}{2}h\partial^2 h + \frac{\kappa^2}{2}h\partial^\mu\partial^\nu h_{\mu\nu}) \\ + \frac{\kappa\bar{\omega}}{2}\delta\omega(-\partial^2 h + \partial^\mu\partial^\nu h_{\mu\nu}) \tag{71}$$

We were looking for the second term on the RHS. Now we can consider the expansion of $\frac{\sqrt{-g}}{2}R$ to quadratic order. For this we'll use an expression that we already discussed, the massless Fierz-Pauli Langrangian. (This can be found in [21] and several other sources [22], but it can also be derived from the second order expanded Ricci tensor given in [20] by a couple of integrations by parts.)

$$S_{FP} = \int d^D x\, \frac{\kappa^2}{2}\left[-\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda} + \frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\mu h\partial^\mu h\right] \tag{72}$$

$$S^{(2)} = \int d^D x\Big[-\frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2}\left((D-1)(D-2)\bar{\omega}^2 + \Delta_\phi^2\bar{\phi}^2\right)\mathcal{T}_\mu\mathcal{T}^\mu \\ + \left(\frac{\xi\bar{\omega}^2}{8\alpha} - 6\bar{\phi}^2\left(\frac{\xi^2}{16\alpha} + \lambda\right)\right)\delta\phi^2 + \frac{1}{8\alpha}(\xi\bar{\phi}^2 - 3\bar{\omega}^2)\delta\omega^2 + \frac{\xi\bar{\phi}\bar{\omega}}{2\alpha}\delta\omega\delta\phi \\ - \frac{1}{2}\partial_\mu\delta\phi\partial^\mu\delta\phi + \kappa\bar{\phi}\left(\frac{\xi\bar{\omega}^2}{8\alpha} + 2\bar{\phi}\left(\frac{\xi^2}{16\alpha} + \lambda\right)\right)\delta\phi h + \frac{\kappa\bar{\omega}}{8\alpha}\left(\xi\bar{\phi}^2 - \bar{\omega}^2\right)\delta\omega h \\ + \frac{\kappa^2}{4}\left(\left(\frac{\xi^2}{16\alpha} + \lambda\right)\bar{\phi}^4 + \frac{\bar{\omega}^2}{16\alpha}\left(\bar{\omega}^2 - 2\xi\bar{\phi}^2\right)\right)(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2) \\ + \frac{\kappa^2\bar{\omega}^2}{2}\left[-\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda} + \frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\mu h\partial^\mu h\right] \\ + \frac{\kappa\bar{\omega}}{2}\delta\omega(-\partial^2 h + \partial^\mu\partial^\nu h_{\mu\nu}) + \left[2(D-1)\bar{\omega}\delta\omega\partial^\mu\mathcal{T}_\mu + \Delta_\phi\bar{\phi}\mathcal{T}^\mu\partial_\mu\delta\phi\right]\Big] \tag{73}$$

## 4.2 Treatment of graviton masses

Examining the action we can see that it contains two types of "graviton masses":

- terms proportional to $h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2$
- mass-like vertices $\delta\phi h$ and $\delta\omega h$

The first one we can address by introducing a suitable cosmological constant $\Lambda$. This is necessary if we want to maintain a flat Minkowski background, which otherwise would not follow from the matter-energy content of our acion. The second type is somewhat more problematic because it can't be balanced adding a standard term. If, however, we derive the equations of motion of the original action (60) we get:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_E}{\delta\phi} = -\left(\frac{\xi^2}{4\alpha}+4\lambda\right)\phi^3 + \frac{\xi}{4\alpha}\omega^2\phi - \Delta_\phi^2\phi\mathcal{T}_\mu\mathcal{T}^\mu + \partial^2\phi - \Delta_\phi\phi\partial_\mu\mathcal{T}^\mu = 0 \tag{74}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta S_E}{\delta\omega} &= \frac{\omega}{4\alpha}(\xi\phi^2-\omega^2) + \omega\left(\overset{\circ}{R} - (D-1)(D-2)\mathcal{T}_\mu\mathcal{T}^\mu\right)\\ &\quad + 2(D-1)(2\omega\partial_\mu\mathcal{T}^\mu + \mathcal{T}^\mu\partial_\mu\omega) - 2\kappa\partial_\mu\omega\partial^\mu h - 2\kappa\omega\partial^2 h\\ &\qquad + \kappa\partial^{(\mu}\omega\partial^{\nu)}h_{\mu\nu} + \kappa\omega\partial^\mu\partial^\nu h_{\mu\nu} = 0\end{aligned} \tag{75}$$

(For simplicity, we used a first order expansion of the Ricci scalar $R$ in the cases where integration by parts implied that derivatives would act on $R$'s coefficients.) Now, set the derivative terms (since the background field is constant) and the torsion trace field (since it is already a perturbation field) to zero, and we get from both equations expressions which are proportional to the coefficients of the vertices:

$$\frac{\kappa\bar{\omega}}{8\alpha}\left(\xi\bar{\phi}^2-\bar{\omega}^2\right)\delta\omega h \qquad \text{and} \qquad \kappa\bar{\phi}\left(\frac{\xi\bar{\omega}^2}{8\alpha} - \bar{\phi}^2\left(\frac{\xi^2}{8\alpha}+2\lambda\right)\right)\delta\phi h \tag{76}$$

Since the background fields have to obey the classical equations of motion, these coupling constants are both equal to zero and thus can be disregarded.

**Note**: the above argument is incorrect. The classical, 'zeroth order' equations cannot regulate quadratic fluctuations this way. This error probably comes from an oversight in section 2.2, where we didn't take into account the source dependent nature of the field expectation value $\langle\varphi\rangle$. Now this 'on-shell approximation' is only correct when the classical vacuum and the quantum vacuum coincide. Unfortunately we realized this too late, so from here on we continue with this assumption.

As for the first type of "mass", the cosmological constant needed to flatten our spacetime is the following:

$$\Lambda = -\left(\frac{\xi^2}{16\alpha}+\lambda\right)\bar{\phi}^4 + \frac{\bar{\omega}^2}{4\alpha}\left(2\xi\bar{\phi}^2-\bar{\omega}^2\right) \tag{77}$$

## 4.3 Gauge fixing

Gauge fixing of the gravitational sector has already been discussed in section 2.1.1. It's time to address the fixing term for the torsion trace gauge field. Previous calculations indicated that modelling the torsion trace by only a longitudinal component and applying average gauge fixing does not lead to an invertible kinetic matrix. The approach followed here prescribes an $R_\xi$ gauge fixing term (inspired by similar one introduced by 't Hooft) of the form $-\frac{1}{2\zeta}[\partial^\mu\mathcal{T}_\mu + \zeta(A\delta\omega + B\delta\phi)]^2$. The constants $A$ and $B$ can be chosen freely, which allows us to eliminate derivative couplings of $\mathcal{T}_\mu$ to $\delta\phi$ and $\delta\omega$.

$$
\begin{aligned}
S^{(2)} = \int d^D x \Big[ & -\frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2}M_{\mathcal{T}}^2\mathcal{T}_\mu\mathcal{T}^\mu - \frac{1}{2}M_\phi^2\delta\phi^2 - \frac{1}{2}M_\omega^2\delta\omega^2 + V_{\phi\omega}\delta\omega\delta\phi \\
& - \frac{1}{2}\partial_\mu\delta\phi\partial^\mu\delta\phi + (\Lambda - \frac{1}{2}M_h^2)(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2) + \frac{\kappa\bar{\omega}}{2}\delta\omega(\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h) \\
& + \frac{\kappa^2\bar{\omega}^2}{2}\left[-\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda} + \frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\mu h\partial^\mu h\right] \\
& + \left[2(D-1)\bar{\omega}\delta\omega\partial^\mu\mathcal{T}_\mu + \Delta_\phi\bar{\phi}\mathcal{T}^\mu\partial_\mu\delta\phi\right] - \frac{1}{2\zeta}[\partial^\mu\mathcal{T}_\mu + \zeta(A\delta\omega + B\delta\phi)]^2 \\
& - \frac{1}{2a}\left(\partial^\mu h_{\mu\nu} - \frac{b}{2}\partial_\nu h\right)\eta^{\nu\rho}\left(\partial^\sigma h_{\sigma\rho} - \frac{b}{2}\partial_\rho h\right)\Big]
\end{aligned} \tag{78}
$$

where

$$
\begin{aligned}
& M_\phi^2(\bar{\phi},\bar{\omega}) = 3\left(\frac{\xi^2}{16\alpha} + \lambda\right)\bar{\phi}^2 - \frac{\xi\bar{\omega}^2}{4\alpha} \\
& M_\omega^2(\bar{\phi},\bar{\omega}) = \frac{1}{4\alpha}(3\bar{\omega}^2 - \xi\bar{\phi}^2),\; V_{\phi\omega}(\bar{\phi},\bar{\omega}) = \frac{\xi\bar{\phi}\bar{\omega}}{2\alpha} \\
& M_h^2(\bar{\phi},\bar{\omega}) = -2\Lambda = \left(\frac{\xi^2}{8\alpha} + 2\lambda\right)\bar{\phi}^4 - \frac{\bar{\omega}^2}{\alpha}\left(\xi\bar{\phi}^2 - \frac{\bar{\omega}^2}{2}\right) \\
& M_{\mathcal{T}}^2(\bar{\phi},\bar{\omega}) = (D-2)\left((D-1)\bar{\omega}^2 + \frac{(D-2)}{4}\bar{\phi}^2\right) \\
& A = 2(D-1)\bar{\omega},\; B = \frac{(D-2)}{2}\bar{\phi}
\end{aligned} \tag{79}
$$

Let us write down the same action, but with the graviton-mass term and the kinetic coupling to the torsion trace dropped, as explained above

$$
\begin{aligned}
S^{(2)} = \int d^D x \Big[ & -\frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2}M_{\mathcal{T}}^2\mathcal{T}_\mu\mathcal{T}^\mu - \frac{1}{2}M_\phi^2\delta\phi^2 - \frac{1}{2}M_\omega^2\delta\omega^2 \\
& + V_{\phi\omega}\delta\omega\delta\phi - \frac{1}{2}\partial_\mu\delta\phi\partial^\mu\delta\phi + \frac{\kappa\bar{\omega}}{2}\delta\omega(\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h) \\
& + \frac{\kappa^2\bar{\omega}^2}{2}\left(-\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda} + \frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\mu h\partial^\mu h\right) \\
& - \frac{\zeta}{2}\left(4(D-1)^2\bar{\omega}^2\delta\omega^2 + 2(D-1)(D-2)\bar{\phi}\bar{\omega}\delta\phi\delta\omega + \frac{(D-2)^2}{4}\bar{\phi}^2\delta\phi^2\right) \\
& - \frac{1}{2\zeta}(\partial^\mu\mathcal{T}_\mu)^2 - \frac{1}{2a}\left(\partial^\mu h_{\mu\nu} - \frac{b}{2}\partial_\nu h\right)\eta^{\nu\rho}\left(\partial^\sigma h_{\sigma\rho} - \frac{b}{2}\partial_\rho h\right)\Big]
\end{aligned} \tag{80}
$$

### 4.4 The first attempt

The procedure taken in the previous chapter was the second one that was tried. Originally the torsion trace was modelled by a scalar field, representing only it's longitudinal component

$$
\mathcal{T}_\mu = \partial_\mu\phi_0 \tag{81}
$$

This follows the source paper [16] more closely, where they justify this approximation in more detail. A short explanation is that transverse modes are only sourced by higher order interactions.

Ultimately, this approach didn't lead to a solution, because even after gauge fixing the kinetic matrix appeared singular and thus not invertible. The gauge fixing procedure was also inspected by an external academic consultant, but no errors were uncovered. Whether the reason for this malfunction is something physical or simply just a mistake in our calculations, is not yet known. For the sake of completeness, we present the partial results obtained.

Substituting 81 into 51, and then defining an on-shell equivalent by the same Lagrange multiplier method as in the previous chapter, we get the action

$$\begin{aligned} S_E = \int d^D x\sqrt{-g}[-\left(\frac{\xi^2}{16\alpha}+\lambda\right)(\delta_{IJ}\phi^I\phi^J)^2+\frac{\xi}{8\alpha}\omega^2\delta_{IJ}\phi^I\phi^J \\ +\frac{\omega^2}{2}\overset{\circ}{R}-\frac{\omega^4}{16\alpha}+3\omega^2\overset{\circ}{\nabla}{}^2\phi^0-\frac{1}{2}g_{AB}g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B] \end{aligned} \tag{82}$$

where

$$g_{AB}d\phi^A d\phi^B = (6\omega^2+\Delta_\phi^2\phi^I\phi^J\delta_{IJ})(d\phi^0)^2 - 2\Delta_\phi\phi_I d\phi^I d\phi^0 + \delta_{IJ}d\phi^I d\phi^J \tag{83}$$

After executing integration by parts, introducing spherical coordinates in field space and making some substitutions, we get that the $\mathcal{N}=1$ action can be brought to the form

$$\begin{aligned} S = \int d^D x\sqrt{-g}[-\left(\frac{\xi^2}{16\alpha}+\lambda\right)\rho^4+\frac{\xi}{8\alpha}\omega^2\rho^2+\frac{\omega^2}{2}\overset{\circ}{R}-\Lambda-\frac{\omega^4}{16\alpha} \\ -\frac{1}{2}g^{\mu\nu}[(6\omega^2+\Delta_\phi^2\rho^2)\partial_\mu\varphi\partial_\nu\varphi-\Delta_\phi(1+\Delta_\phi)\rho(\partial_\mu\rho\partial_\nu\varphi+\partial_\mu\varphi\partial_\nu\rho) \\ +\frac{(2+\Delta_\phi)\Delta_\phi^3\rho^2}{6\omega^2+\Delta_\phi^2\rho^2}\partial_\mu\rho\partial_\nu\rho+\frac{6\Delta_\phi\rho\omega}{6\omega^2+\Delta_\phi^2\rho^2}(\partial_\mu\rho\partial_\nu\omega+\partial_\mu\omega\partial_\nu\rho)-\frac{36\omega^2}{6\omega^2+\Delta_\phi^2\rho^2}\partial_\mu\omega\partial_\nu\omega] \\ +\text{gauge fixing terms}] \end{aligned} \tag{84}$$

Expanding to quadratic order results in

$$\begin{aligned} S^{(2)} = \int d^D x[-\frac{1}{2}M_\rho^2(\bar\rho,\bar\omega)\delta\rho^2-\frac{1}{2}(M_\omega^2(\bar\rho,\bar\omega)+m^2)\delta\omega^2+V_{\rho\omega}(\bar\rho,\bar\omega)\delta\omega\delta\rho \\ +\frac{\kappa\bar\omega}{2}\delta\omega(\partial^\mu\partial^\nu h_{\mu\nu}-\partial^2 h)+\frac{\kappa^2\bar\omega^2}{2}(-\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda}+\frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda} \\ -\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h+\frac{1}{4}\partial_\mu h\partial^\mu h)-\frac{1}{2}\eta^{\mu\nu}[K_{\varphi\varphi}(\bar\rho,\bar\omega)\partial_\mu\delta\varphi\partial_\nu\delta\varphi \\ +K_{\rho\varphi}(\bar\rho)(\partial_\mu\delta\rho\partial_\nu\delta\varphi+\partial_\mu\delta\varphi\partial_\nu\delta\rho)+K_{\rho\rho}(\bar\rho,\bar\omega)\partial_\mu\delta\rho\partial_\nu\delta\rho \\ +K_{\rho\omega}(\bar\rho,\bar\omega)(\partial_\mu\delta\rho\partial_\nu\delta\omega+\partial_\mu\delta\omega\partial_\nu\delta\rho)+K_{\omega\omega}(\bar\rho,\bar\omega)\partial_\mu\delta\omega\partial_\nu\delta\omega] \\ -\frac{1}{2a}(\partial^\mu h_{\mu\nu}-\frac{b}{2}\partial_\nu h)\eta^{\nu\rho}(\partial^\sigma h_{\sigma\rho}-\frac{b}{2}\partial_\rho h)] \end{aligned} \tag{85}$$

where

$$\begin{aligned} &M_\rho^2(\bar\rho,\bar\omega) = 12\left(\frac{\xi^2}{16\alpha}+\lambda\right)-\frac{\xi\bar\omega^2}{4\alpha},\ M_\omega^2(\bar\rho,\bar\omega)=\frac{3\bar\omega^2-\xi\bar\rho^2}{4\alpha} \\ &V_{\rho\omega}(\bar\rho,\bar\omega)=\frac{\xi\bar\rho\bar\omega}{2\alpha},\ K_{\varphi\varphi}(\bar\rho,\bar\omega)=6\bar\omega^2+\Delta_\phi^2\bar\rho^2,\ K_{\rho\rho}(\bar\rho,\bar\omega)=\frac{(2+\Delta_\phi)\Delta_\phi^3\bar\rho^2}{6\bar\omega^2+\Delta_\phi^2\bar\rho^2} \\ &K_{\rho\varphi}(\bar\rho)=-\Delta_\phi(1+\Delta_\phi)\bar\rho,\ K_{\rho\omega}(\bar\rho,\bar\omega)=\frac{6\Delta_\phi\bar\rho\bar\omega}{6\bar\omega^2+\Delta_\phi^2\bar\rho^2}, \\ &K_{\omega\omega}(\bar\rho,\bar\omega)=-\frac{36\bar\omega^2}{6\bar\omega^2+\Delta_\phi^2\bar\rho^2},\ \Lambda=-\left(\frac{\xi^2}{16\alpha}+\lambda\right)\bar\rho^4+\frac{1}{8\alpha}\left(\xi\bar\omega^2\bar\rho^2-\frac{\bar\omega^4}{2}\right) \end{aligned} \tag{86}$$

and where the gauge fixing terms are the last term in the action (for the graviton), and $\frac{m^2}{2}$ which is meant to fix $\omega$. Note that the quadratic graviton terms have been 'eliminated' here by the same method as in the main part of the thesis. See the last term $\Lambda$ in the list above.

# 5 Calculating the effective action

The strategy we shall follow here to determine the effective action starts by calculating the propagators first. Note that this is not the only (or the usual) way. We could also get there by a functional determinant of the quadratic kinetic matrix (defined below). However, the method involving the propagator has the advantage that it can be carried out by basically only relying on linear algebra.

## 5.1 Equations of motion

We vary the quadratic action with respect to the fluctuation fields to derive their equations of motion:

$$\frac{\delta S^{(2)}}{\delta(\delta\phi)} = \partial^2\delta\phi - \left(M_\phi^2 + \zeta\Gamma\right)\delta\phi + \left(V_{\phi\omega} + \zeta\Delta\right)\delta\omega = 0 \tag{87}$$

$$\frac{\delta S^{(2)}}{\delta(\delta\omega)} = -\left(M_\omega^2 + \zeta\Theta\right)\delta\omega + \left(V_{\phi\omega} + \zeta\Delta\right)\delta\phi + \frac{\kappa\bar{\omega}}{2}(\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h) = 0 \tag{88}$$

$$\frac{\delta S^{(2)}}{\delta \mathcal{T}_\nu} = \sigma\partial_\mu(\partial^\mu\mathcal{T}^\nu - \partial^\nu\mathcal{T}^\mu) + \frac{1}{\zeta}\partial^\nu\partial_\alpha\mathcal{T}^\alpha - M_{\mathcal{T}}^2\mathcal{T}^\nu = 0 \tag{89}$$

$$\frac{\delta S^{(2)}}{\delta h_{\alpha\beta}} = \frac{\kappa\bar{\omega}}{2}(\eta^{\alpha\beta}\partial^2 - \partial^\alpha\partial^\beta)\delta\omega - \mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed}h_{\mu\nu} = 0 \tag{90}$$

where

$$\Gamma = \frac{(D-2)^2}{4}\bar{\phi}^2, \quad \Delta = -(D-1)(D-2)\bar{\phi}\bar{\omega}, \quad \Theta = 4(D-1)^2\bar{\omega}^2 \tag{91}$$

and

$$\begin{aligned}\mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed} &= \frac{\kappa^2\bar{\omega}^2}{4}\eta^{\mu(\alpha}\eta^{\beta)\nu}\partial^2 + (\frac{\kappa^2\bar{\omega}^2}{2} - \frac{b}{a})\left[\eta^{\mu\nu}\partial^\beta\partial^\alpha + \eta^{\alpha\beta}\partial^\mu\partial^\nu\right] \\ &- \frac{1}{2}(\frac{\kappa^2\bar{\omega}^2}{2} - \frac{b^2}{2a})\eta^{\alpha\beta}\eta^{\mu\nu}\partial^2 - (\frac{\kappa^2\bar{\omega}^2}{2} - \frac{1}{a})\partial^{(\mu}\eta^{\nu)(\alpha}\partial^{\beta)}\end{aligned} \tag{92}$$

which we can express in $k$-space or in terms of transverse and longitudinal projectors

$$\begin{aligned}\mathcal{L}^{\alpha\beta\mu\nu}_{gauge\ fixed} &= (-k^2)\left[\frac{\kappa^2\bar{\omega}^2}{4}\eta^{\mu(\alpha}\eta^{\beta)\nu} + \frac{1}{4}\left(\frac{1}{a} - \frac{\kappa^2\bar{\omega}^2}{2}\right)\left(\eta^{\mu\rho}\frac{k^\nu k^\sigma}{k^2}\right.\right. \\ &+\eta^{\nu\sigma}\frac{k^\mu k^\rho}{k^2} + \eta^{\mu\sigma}\frac{k^\nu k^\rho}{k^2} + \left.\eta^{\nu\rho}\frac{k^\mu k^\sigma}{k^2}\right) + \frac{1}{2}\left(\frac{b^2}{2a} - \frac{\kappa^2\bar{\omega}^2}{2}\right)\eta^{\mu\nu}\eta^{\rho\sigma} \\ &+\left.\frac{1}{2}\left(\frac{\kappa^2\bar{\omega}^2}{2} - \frac{b}{a}\right)\left(\eta^{\mu\nu}\frac{k^\rho k^\sigma}{k^2} + \eta^{\rho\sigma}\frac{k^\mu k^\nu}{k^2}\right)\right]\end{aligned} \tag{93}$$

Let us make the substitution $ik_\mu = \partial_\mu$

$$-(k^2 + M_\phi^2 + \zeta\Gamma)\delta\phi + (V_{\phi\omega} + \zeta\Delta)\delta\omega = 0 \tag{94}$$

$$-(M_\omega^2 + \zeta\Theta)\delta\omega + (V_{\phi\omega} + \zeta\Delta)\delta\phi + \frac{\kappa\bar{\omega}}{2}(\eta^{\mu\nu}k^2 - k^\mu k^\nu)h_{\mu\nu} = 0 \tag{95}$$

$$\frac{\kappa\bar{\omega}}{2}(k^\alpha k^\beta - \eta^{\alpha\beta}k^2)\delta\omega - \mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed}(k)h_{\mu\nu} = 0 \tag{96}$$

and then the independent equation

$$-\sigma k^2\mathcal{T}^\nu + (\sigma - \frac{1}{\zeta})k^\nu k_\alpha \mathcal{T}^\alpha - M^2_{\mathcal{T}}\mathcal{T}^\nu = 0 \tag{97}$$

## 5.2 The kinetic matrix

Let's organize the system of three equations into a matrix equation

$$\begin{pmatrix} (k^2 + M^2_\phi + \zeta\Gamma) & -(V_{\phi\omega} + \zeta\Delta) & 0^{\mu\nu} \\ -(V_{\phi\omega} + \zeta\Delta) & (M^2_\omega + \zeta\Theta) & \frac{\kappa\bar{\omega}}{2}(k^\mu k^\nu - \eta^{\mu\nu}k^2) \\ 0^{\alpha\beta} & \frac{\kappa\bar{\omega}}{2}(k^\alpha k^\beta - \eta^{\alpha\beta}k^2) & -\mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed}(k) \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\omega \\ h_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{98}$$

To get the propagator, we need to introduce the Green's function matrix and eventually invert the matrix above:

$$\begin{pmatrix} (k^2 + M^2_\phi + \zeta\Gamma) & -(V_{\phi\omega} + \zeta\Delta) & 0^{\mu\nu} \\ -(V_{\phi\omega} + \zeta\Delta) & (M^2_\omega + \zeta\Theta) & \frac{\kappa\bar{\omega}}{2}(k^\mu k^\nu - \eta^{\mu\nu}k^2) \\ 0^{\alpha\beta} & \frac{\kappa\bar{\omega}}{2}(k^\alpha k^\beta - \eta^{\alpha\beta}k^2) & -\mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed}(k) \end{pmatrix} iG \stackrel{?}{=} iI \tag{99}$$

We make the ansatz that our inverse takes the form

$$G = \begin{pmatrix} A & B & \mathcal{D}_{\rho\sigma} \\ B & C & E_{\rho\sigma} \\ \mathcal{D}_{\mu\nu} & E_{\mu\nu} & G_{\mu\nu\rho\sigma} \end{pmatrix} \tag{100}$$

such that

$$\begin{pmatrix} (k^2 + M^2_\phi + \zeta\Gamma) & -(V_{\phi\omega} + \zeta\Delta) & 0^{\mu\nu} \\ -(V_{\phi\omega} + \zeta\Delta) & (M^2_\omega + \zeta\Theta) & \frac{\kappa\bar{\omega}}{2}(k^\mu k^\nu - \eta^{\mu\nu}k^2) \\ 0^{\alpha\beta} & \frac{\kappa\bar{\omega}}{2}(k^\alpha k^\beta - \eta^{\alpha\beta}k^2) & -\mathcal{L}^{\alpha\beta\mu\nu}_{gaugefixed}(k) \end{pmatrix} \begin{pmatrix} A & B & \mathcal{D}_{\rho\sigma} \\ B & C & E_{\rho\sigma} \\ \mathcal{D}_{\mu\nu} & E_{\mu\nu} & G_{\mu\nu\rho\sigma} \end{pmatrix}$$
$$= \mathbf{I} \equiv \begin{pmatrix} 1 & 0 & 0_{\rho\sigma} \\ 0 & 1 & 0_{\rho\sigma} \\ 0^{\alpha\beta} & 0^{\alpha\beta} & \delta^\alpha_{(\rho}\delta^\beta_{\sigma)} \end{pmatrix} \tag{101}$$

This results in 18 scalar equations.

$$
\begin{aligned}
&(1,1) \quad (k^2 + M_\phi^2 + \zeta\Gamma)A - (V_{\phi\omega} + \zeta\Delta)B = 1 \\
&(1,2) \quad (k^2 + M_\phi^2 + \zeta\Gamma)B - (V_{\phi\omega} + \zeta\Delta)C = 0 \\
&(1,3a) \; (k^2 + M_\phi^2 + \zeta\Gamma)\mathcal{D}^T - (V_{\phi\omega} + \zeta\Delta)E^T = 0 \\
&(1,3b) \; (k^2 + M_\phi^2 + \zeta\Gamma)\mathcal{D}^L - (V_{\phi\omega} + \zeta\Delta)E^L = 0 \\
&(2,1) \quad -(V_{\phi\omega} + \zeta\Delta)A + (M_\omega^2 + \zeta\Theta)B - \frac{\kappa\bar{\omega}}{2}(D-1)\mathcal{D}^T k^2 = 0 \\
&(2,2) \quad -(V_{\phi\omega} + \zeta\Delta)B + (M_\omega^2 + \zeta\Theta)C - \frac{\kappa\bar{\omega}}{2}(D-1)E^T k^2 = 1 \\
&(2,3a) \; -(V_{\phi\omega} + \zeta\Delta)\mathcal{D}^T + (M_\omega^2 + \zeta\Theta)E^T - \frac{\kappa\bar{\omega}}{2}k^2 G_{ss} = 0 \\
&(2,3b) \; -(V_{\phi\omega} + \zeta\Delta)\mathcal{D}^L + (M_\omega^2 + \zeta\Theta)E^L - \sqrt{D-1}\,\frac{\kappa\bar{\omega}}{2}k^2 G_{sw} = 0 \\
&(3,1a) \; -\frac{\kappa\bar{\omega}}{2}k^2 B - \left((D-2)\frac{\kappa^2\bar{\omega}^2}{4} + (D-1)\frac{b^2}{4a}\right)k^2\mathcal{D}^T \\
&\qquad\qquad + \frac{b}{2a}\left(\frac{b}{2}-1\right)^2 k^2\mathcal{D}^L = 0 \\
&(3,1b) \; k^2\left[(D-1)\frac{b}{2a}\left(\frac{b}{2}-1\right)\mathcal{D}^T + \frac{1}{a}\left(\frac{b}{2}-1\right)^2\mathcal{D}^L\right] = 0 \\
&(3,2a) \; -\frac{\kappa\bar{\omega}}{2}k^2 C + k^2\left[\left((D-1)\frac{b^2}{4a} - (D-2)\frac{\kappa^2\bar{\omega}^2}{4}\right)E^T\right. \\
&\qquad\qquad \left. + \frac{b}{2a}\left(\frac{b}{2}-1\right)E^L\right] = 0 \\
&(3,2b) \; k^2\left[(D-1)\frac{b}{2a}\left(\frac{b}{2}-1\right)E^T + \frac{1}{a}\left(\frac{b}{2}-1\right)^2 E^L\right] = 0 \\
&(3,3a) \; \frac{\kappa^2\bar{\omega}^2}{4}G_2 k^2 = 1, \qquad (3,3b) \; \frac{1}{2a}G_1 k^2 = 1 \\
&(3,3c) \; -\frac{\kappa\bar{\omega}}{2}(D-1)k^2 E^T - \left((D-2)\frac{\kappa^2\bar{\omega}^2}{4} + (D-1)\frac{b^2}{4a}\right)G_{ss}k^2 \\
&\qquad\qquad + \sqrt{D-1}\frac{b}{2a}\left(\frac{b}{2}-1\right)G_{ws}k^2 = 1 \\
&(3,3d) \; -\frac{\kappa\bar{\omega}}{2}\sqrt{D-1}\,k^2 E^L + \sqrt{D-1}\frac{b}{2a}\left(\frac{b}{2}-1\right)G_{ww}k^2 \\
&\qquad\qquad - (D-1)\left(\frac{\kappa^2\bar{\omega}^2}{4} + \frac{b^2}{4a}\right)G_{sw}k^2 = 0 \\
&(3,3e) \; k^2\left[\sqrt{D-1}\frac{b}{2a}\left(\frac{b}{2}-1\right)G_{ss} + \frac{1}{a}\left(\frac{b}{2}-1\right)^2 G_{ws}\right] = 0 \\
&(3,3f) \; k^2\left[\sqrt{D-1}\frac{b}{2a}\left(\frac{b}{2}-1\right)G_{sw} + \frac{1}{a}\left(\frac{b}{2}-1\right)^2 G_{ww}\right] = 1
\end{aligned}
\tag{102}
$$

The solution for this system is

$$
\begin{aligned}
A &= \frac{(D-1)k^2 + (D-2)(M_\omega^2 + \zeta\Theta)}{Q} \\
B &= \frac{(D-2)(V_{\phi\omega} + \zeta\Delta)}{Q} \\
C &= \frac{(D-2)(k^2 + M_\phi^2 + \zeta\Gamma)}{Q} \\
\mathcal{D}^T &= \frac{-2(V_{\phi\omega} + \zeta\Delta)}{\kappa\bar{\omega}Q} \\
\mathcal{D}^L &= \frac{2b(D-1)(V_{\phi\omega} + \zeta\Delta)}{(b-2)\kappa\bar{\omega}Q} \\
E^T &= \frac{-2(k^2 + M_\phi^2 + \zeta\Gamma)}{\kappa\bar{\omega}Q} \\
E^L &= \frac{2b(D-1)(k^2 + M_\phi^2 + \zeta\Gamma)}{(b-2)\kappa\bar{\omega}Q} \\
G^{(2)} &= \frac{4}{\kappa^2\bar{\omega}^2k^2} \\
G^{(1)} &= \frac{2a}{k^2} \\
G^{(ss)} &= \frac{4\left((V_{\phi\omega} + \zeta\Delta)^2 - (M_\omega^2 + \zeta\Theta)(k^2 + M_\phi^2 + \zeta\Gamma)\right)}{\kappa^2\bar{\omega}^2k^2Q} \\
G^{(sw)} = G^{(ws)} &= \frac{-8b\sqrt{D-1}\left((V_{\phi\omega} + \zeta\Delta)^2 - (M_\omega^2 + \zeta\Theta)(k^2 + M_\phi^2 + \zeta\Gamma)\right)}{(b-2)\kappa^2\bar{\omega}^2k^2Q} \\
G^{(ww)} &= \frac{4a}{(b-2)^2k^2} \\
&\quad - \frac{4b^2(D-1)\left((M_\omega^2 + \zeta\Theta)(k^2 + M_\phi^2 + \zeta\Gamma) - (V_{\phi\omega} + \zeta\Delta)^2\right)}{(b-2)^2\kappa^2\bar{\omega}^2k^2Q}
\end{aligned}
\tag{103}
$$

where

$$
\begin{aligned}
Q = (D-1)k^4 &+ k^2\left[(D-1)(M_\phi^2 + \zeta\Gamma) + (D-2)(M_\omega^2 + \zeta\Theta)\right] \\
&+ (D-2)\left[(M_\phi^2 + \zeta\Gamma)(M_\omega^2 + \zeta\Theta) - (V_{\phi\omega} + \zeta\Delta)^2\right]
\end{aligned}
\tag{104}
$$

## 5.3 Determinant and trace

Now we have all the coefficients to build $G$, the propagator matrix. Our goal is to construct the second term of the effective action, which is $-\frac{i}{2}\mathrm{Tr}\log[G]$. A well-known matrix identity is $\mathrm{Tr}[A] = \log[\det[\exp A]]$, which then immediately implies $\mathrm{Tr}[\log[B]] =$

log[det[$B$]]. We will try to evaluate the RHS of the last equation for $G$.

$$
\begin{aligned}
&\det[G]_{\mu\nu\rho\sigma} \\
&= A(CG_{\mu\nu\rho\sigma} - E_{\mu\nu}E_{\rho\sigma}) - B(BG_{\mu\nu\rho\sigma} - \mathcal{D}_{\mu\nu}E_{\rho\sigma}) + \mathcal{D}_{\rho\sigma}(BE_{\mu\nu} - C\mathcal{D}_{\mu\nu}) \\
&= (AC - B^2)G^{(2)}P^{(2)}_{\mu\nu\rho\sigma} + (AC - B^2)G^{(1)}P^{(1)}_{\mu\nu\rho\sigma} \\
&+ \Big((AC - B^2)G^{(ss)} + (D-1)(2B\mathcal{D}^T E^T - A(E^T)^2 - C(\mathcal{D}^T)^2)\Big) P^{(ss)}_{\mu\nu\rho\sigma} \\
&+ \Big((AC - B^2)G^{(sw)} + \sqrt{D-1}\,\big(B(\mathcal{D}^T E^L + \mathcal{D}^L E^T) \\
&- AE^T E^L - C\mathcal{D}^T\mathcal{D}^L\big)\Big)(P^{(sw)}_{\mu\nu\rho\sigma} + P^{(ws)}_{\mu\nu\rho\sigma}) \\
&+ \Big((AC - B^2)G^{(ww)} + 2B\mathcal{D}^L E^L - A(E^L)^2 - C(\mathcal{D}^L)^2\Big) P^{(ww)}_{\mu\nu\rho\sigma}
\end{aligned} \tag{105}
$$

After substituting in and considerable simplification, the determinant is given as

$$
\begin{aligned}
\det[G] &= \frac{4(D-2)}{\kappa^2\bar{\omega}^2 k^2 Q}P^{(2)} + \frac{2(D-2)a}{k^2 Q}P^{(1)} - \frac{4}{\kappa^2\bar{\omega}^2 k^2 Q}P^{(ss)} \\
&+ \frac{4\sqrt{D-1}\,b}{(b-2)\kappa^2\bar{\omega}^2 k^2 Q}(P^{(sw)} + P^{(ws)}) + \frac{4a(D-2)\kappa^2\bar{\omega}^2 - 4b^2(D-1)}{(b-2)^2\kappa^2\bar{\omega}^2 k^2 Q}P^{(ww)}
\end{aligned} \tag{106}
$$

Next we should evaluate the logarithm of this expression. With the projection tensors still hanging on, we should consider this as a matrix logarithm. Then the case of the $P^{(2)}$ and $P^{(1)}$ is rather easy: since they satisfy the condition $P^{(n)}P^{(m)} = \delta_{nm}P^{(m)}$, they act like diagonal matrix elements and the matrix logarithm commutes with them. The case of the spin-0 projectors is a little more complicated. As it has been already mentioned (2.1.2), these come with a $2 \times 2$ matrix structure

$$
P^{(0)} = \begin{bmatrix} P^{(ss)} & P^{(sw)} \\ P^{(ws)} & P^{(ww)} \end{bmatrix} = \begin{bmatrix} \frac{t_{\mu\nu}}{\sqrt{D-1}} \\ \ell_{\mu\nu} \end{bmatrix} \bigotimes \begin{bmatrix} \frac{t_{\rho\sigma}}{\sqrt{D-1}} \\ \ell_{\rho\sigma} \end{bmatrix} \tag{107}
$$

Before taking the logarithm, this matrix has to be diagonalized. After we can utilize the pleasant property of matrix logarithms:

$$
\log(M) = \log(S^{-1}JS) = S^{-1}(\log J)S \tag{108}
$$

The matrix in our case takes the form

$$
\frac{4}{\kappa^2\bar{\omega}^2 k^2 Q}\begin{bmatrix} -1 & \frac{b\sqrt{D-1}}{b-2} \\ \frac{b\sqrt{D-1}}{b-2} & \frac{a(D-2)\kappa^2\bar{\omega}^2 - b^2(D-1)}{(b-2)^2} \end{bmatrix} \tag{109}
$$

We need to diagonalize this matrix. For that we need to find the eigenvectors so that we can construct the rotation matrices. One needs to proceed with caution, as the matrix has a special basis as shown in 107, and any vector thus would come as

$$
\mathbf{v} = \begin{bmatrix} \frac{t^{\rho\sigma}}{\sqrt{D-1}} \\ \ell^{\rho\sigma} \end{bmatrix} \tag{110}
$$

while the identity matrix is defined as

$$
I = \begin{bmatrix} \eta_{\mu(\rho}\eta_{\sigma)\nu} & 0 \\ 0 & \eta_{\mu(\rho}\eta_{\sigma)\nu} \end{bmatrix} \tag{111}
$$

Taking all of this into account, we write down the equation for the eigenvectors $(M - \lambda I)\mathbf{v} = 0$, and we get two equations

$$
(M_{11} - \lambda)v_1 + M_{12}v_2 = 0, \quad M_{12}v_1 + (M_{22} - \lambda)v_2 = 0 \tag{112}
$$

Eigenvectors are only determined up to a scaling factor, therefore we are free to choose $v_2 = 1$. Then we get two solutions for $v_1$

$$v_1 = \frac{-M_{12}}{(M_{11} - \lambda)}, \quad v_1 = \frac{-(M_{22} - \lambda)}{M_{12}} \tag{113}$$

If we then require these to be equal, we end up with an equation for the eigenvalues

$$\begin{aligned} &(M_{11} - \lambda)(M_{22} - \lambda) = M_{12}^2 \\ &\Rightarrow \lambda_{\pm} = \frac{(M_{11} + M_{22})}{2} \pm \frac{\sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2}}{2} \end{aligned} \tag{114}$$

We can then construct the rotation matrices that diagonalize $M$.

$$\begin{aligned} S &= \begin{bmatrix} \mathbf{v}_+ & \mathbf{v}_- \end{bmatrix} = \begin{bmatrix} \frac{-(M_{22}-\lambda_+)}{M_{12}} & \frac{-(M_{22}-\lambda_-)}{M_{12}} \\ 1 & 1 \end{bmatrix} \\ S^{-1} &= \frac{1}{\lambda_- - \lambda_+} \begin{bmatrix} -M_{12} & \lambda_- - M_{22} \\ M_{12} & M_{22} - \lambda_+ \end{bmatrix} \end{aligned} \tag{115}$$

We take the logarithm of the two diagonal elements

$$\begin{aligned} \log A = \log &\left[ \frac{2}{\kappa^2 \bar{\omega}^2 k^2 Q} \left( \left( \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2} - 1 \right) \right.\right. \\ &\left.\left. - \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)^2} \right) \right] \\ \log B = \log &\left[ \frac{2}{\kappa^2 \bar{\omega}^2 k^2 Q} \left( \left( \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2} - 1 \right) \right.\right. \\ &\left.\left. + \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)^2} \right) \right] \end{aligned} \tag{116}$$

Having then the two transformation matrices $S^{-1}$ and $S$ on the 'logarithmed' diagonal matrix, we get, again, on the diagonal, the following

$$\begin{aligned} d_1 &= \frac{1}{2} \frac{(\log A - \log B)\left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)}{\sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)^2}} + \frac{1}{2}(\log A + \log B) \\ d_2 &= \frac{1}{2} \frac{(\log B - \log A)\left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)}{\sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a - (D-1)b^2}{(b-2)^2}\right)^2}} + \frac{1}{2}(\log A + \log B) \end{aligned} \tag{117}$$

The reason why we don't care about the off-diagonal elements this time is the next step: we shall take the trace of the projection tensors, and both $P^{(sw)}$ and $P^{(ws)}$ is traceless. In hopes of significant simplification we employ the logarithm identity $\log A + \log B = \log AB$, and indeed find that

$$\frac{1}{2}(\log A + \log B) = \frac{1}{2} \log AB = \frac{1}{2} \log \left[ -\frac{16(D-2)a}{(b-2)^2 \kappa^2 \bar{\omega}^2 k^2 Q^2} \right] \tag{118}$$

$\log A/B$ and $\log B/A$ cannot be brought to such a nice form

$$\begin{aligned}
\log A - \log B &= \log \frac{A}{B} \\
&= \log \left[ \frac{\left(\frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2} - 1\right) - \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}}{\left(\frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2} - 1\right) + \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}} \right] \\
\log B - \log A &= \log \frac{B}{A} \\
&= \log \left[ \frac{\left(\frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2} - 1\right) + \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}}{\left(\frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2} - 1\right) - \sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}} \right]
\end{aligned} \tag{119}$$

Let us zoom out now and try to see where we are in our efforts of trying to evaluate the effective action

$$\begin{aligned}
&\log[\det[G_{\mu\nu\rho\sigma}]] = \log\left[\frac{4(D-2)}{\kappa^2\bar{\omega}^2 k^2 Q}\right] P^{(2)} + \log\left[\frac{2(D-2)a}{k^2 Q}\right] P^{(1)} \\
&+ \frac{1}{2}\left( \frac{\log[\frac{A}{B}]\left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)}{\sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}} + \log\left[\frac{-16(D-2)a}{(b-2)^2\kappa^2\bar{\omega}^2 k^2 Q^2}\right]\right) P^{(ss)} \\
&+ \frac{1}{2}\left( \frac{\log[\frac{B}{A}]\left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)}{\sqrt{\frac{4(D-1)b^2}{(b-2)^2} + \left(1 + \frac{(D-2)\kappa^2\bar{\omega}^2 a-(D-1)b^2}{(b-2)^2}\right)^2}} + \log\left[\frac{-16(D-2)a}{(b-2)^2\kappa^2\bar{\omega}^2 k^2 Q^2}\right]\right) P^{(ww)} \\
&+ \text{something} \times P^{(sw)} + \text{something} \times P^{(ws)}
\end{aligned} \tag{120}$$

Now we shall take the trace of every projection tensor and finally get rid of the Lorentz-indices. The traces of the projectors are give as

$$\begin{aligned}
\eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(2)}_{\mu\nu\rho\sigma} &= \frac{(D+1)(D-2)}{2}, & \eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(1)}_{\mu\nu\rho\sigma} &= D-1 \\
\eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(ss)}_{\mu\nu\rho\sigma} &= 1, & \eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(sw)}_{\mu\nu\rho\sigma} &= 0 \\
\eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(ws)}_{\mu\nu\rho\sigma} &= 0, & \eta^{\mu(\rho}\eta^{\sigma)\nu} P^{(ww)}_{\mu\nu\rho\sigma} &= 1
\end{aligned} \tag{121}$$

When we sum up the terms all over again, we notice that the two most cumbersome terms only differ by a factor of $\log(A/B)$ and $\log(B/A)$ respectively, which is equivalent to a sign difference. Therefore they cancel each other out.

$$\begin{aligned}
\eta^{\mu(\rho}\eta^{\sigma)\nu} \log[\det[G]_{\mu\nu\rho\sigma}] &= \frac{(D+1)(D-2)}{2} \log\left[\frac{4(D-2)}{\kappa^2\bar{\omega}^2 k^2 Q}\right] \\
+ (D-1)\log\left[\frac{2(D-2)a}{k^2 Q}\right] &+ \log\left[-\frac{16(D-2)a}{(b-2)^2\kappa^2\bar{\omega}^2 k^2 Q^2}\right]
\end{aligned} \tag{122}$$

where

$$\begin{aligned}
Q = (D-1)k^4 &+ k^2\left[(D-1)(M_\phi^2 + \zeta\Gamma) + (D-2)(M_\omega^2 + \zeta\Theta)\right] \\
&+ (D-2)\left[(M_\phi^2 + \zeta\Gamma)(M_\omega^2 + \zeta\Theta) - (V_{\phi\omega} + \zeta\Delta)^2\right]
\end{aligned} \tag{123}$$

## 5.4 Euclidean intergral

The next step towards the effective action is taking a D-dimensional integral of this expression in momentum space. Now, one could have been worried about the negative sign under the log in the last term, but using the logarithmic product identity $\log AB = \log A + \log B$, every constant coefficient term 'splinters off'. These are then integrated according to the rules of dimensional regularization. Let us recall an important results here

$$\lim_{\omega\to 2} \int d^{2\omega}k (k^2)^{\lambda-1} = 0, \text{ for } \lambda = 0, 1, 2, ... \tag{124}$$

This equation is meant strictly in the context of dimensional regularization. It was conjectured by 't Hooft and Veltman[26] that this substitution would lead to consistent results in dim-reg, and proven by Capper and Leibbrandt[27]. This allows us to disregard not only the integrals of constants $C\int d^D k$ but the massless 'tadpole integrals' as well $\int d^D k (k^2)^{-1}$. In the end we are left with

$$\begin{aligned} &\int \frac{d^D k}{(2\pi)^D}\, \eta^{\mu(\rho}\eta^{\sigma)\nu} \log[\det[G]_{\mu\nu\rho\sigma}] \\ &= -\left(\frac{(D+1)(D-2)}{2} + (D-1) + 2\right) \int \frac{d^D k}{(2\pi)^D} \log Q \end{aligned} \tag{125}$$

This is a surprising result. It means that at least in the one-loop effective action there is no graviton gauge dependece. The only gauge parameter still in play is $\zeta$, which we introduced with the $R_\xi$ gauge fixing in 78. In the end only the massive poles count, which are dependent on background field values. To find the massive poles, we turn to Q

$$\begin{aligned} \tilde{Q} = k^4 + k^2 &\left[(M_\phi^2 + \zeta\Gamma) + \frac{D-2}{D-1}(M_\omega^2 + \zeta\Theta)\right] \\ &+ \frac{D-2}{D-1}\left[(M_\phi^2 + \zeta\Gamma)(M_\omega^2 + \zeta\Theta) - (V_{\phi\omega} + \zeta\Delta)^2\right] \end{aligned} \tag{126}$$

where we have divided by $D-1$ for convenience. It's a quartic polynomial with two degenerate roots, which means it can be written like

$$\tilde{Q} = (k^2 + m_1^2)(k^2 + m_2^2) \tag{127}$$

where the positions of the two massive poles are

$$\begin{aligned} m_\pm^2 = &\frac{1}{2}\left((M_\phi^2 + \zeta\Gamma) + \left(\frac{D-2}{D-1}\right)(M_\omega^2 + \zeta\Theta)\right) \\ \pm &\frac{1}{2}\sqrt{\left((M_\phi^2 + \zeta\Gamma) - \left(\frac{D-2}{D-1}\right)(M_\omega^2 + \zeta\Theta)\right)^2 + 4\left(\frac{D-2}{D-1}\right)(V_{\phi\omega} + \zeta\Delta)^2} \end{aligned} \tag{128}$$

Now we must solve the following integral

$$\int \frac{d^D k}{(2\pi)^D} \log(k^2 + m^2) \tag{129}$$

Following the prescription of dimensional regularization, we execute a Wick rotation. We replace the Minkowskian magnitude of the $k$ four-vector $\mathbf{k}^2 - k^{0\,2}$ by a Euclidean alternative $\mathbf{k}^2 + k_E^2$, where $k^0 = ik_E$. In summary,

$$\int d^D k \quad \Rightarrow \quad i\int d^{D-1}\mathbf{k} \int dk_E \equiv i\int d^D q \tag{130}$$

To solve our integral, we use the standard results[28]

$$i\int d^Dq\frac{1}{(q^2+m^2)^\alpha}=i\pi^{D/2}\frac{\Gamma(\alpha-\frac{1}{2}D)}{\Gamma(\alpha)}(m^2)^{\frac{D}{2}-\alpha} \tag{131}$$

Then

$$i\int\frac{d^Dq}{(2\pi)^D}\log\left(q^2+m^2\right)=I(m^2)\quad\Bigg|\frac{\partial}{\partial m^2} \tag{132}$$

$$\begin{aligned}I'(m^2)&=i\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2}\\ \Rightarrow\int dm^2 I'(m^2)+\cancel{C}&=\frac{i}{D(D-2)}\frac{\Gamma(2-\frac{D}{2})}{4\pi^{\frac{D}{2}}}(m^2)^{\frac{D}{2}}\end{aligned} \tag{133}$$

## 5.5 Regularization

To extract the divergent parts, we set the number of dimensions $D$ to 4, and perturb it by an infinitesimal amount. Then $D-4$ approaches zero, and we can isolate poles that diverge in this limit. Using again two well-known formulae[28]

$$\begin{aligned}&(1)\quad\Gamma(z)=z^{-1}-\gamma_E+\left(\frac{1}{12}\pi^2+\frac{1}{2}\gamma_E^2\right)z+\mathcal{O}(z^2)\\ &(2)\quad x^\varepsilon=1+\varepsilon\log x+\mathcal{O}(\varepsilon^2),\qquad 0\leq\varepsilon\ll 1\end{aligned} \tag{134}$$

and a standard Taylor expansion for the $D$s in the fraction in the front, we get

$$\begin{aligned}&\int\frac{d^4k}{(2\pi)^4}\log\left(k^2+m^2\right)=i\int\frac{d^4q}{(2\pi)^4}\log\left(q^2+m^2\right)\\ &\approx-\frac{im^4}{16\pi^2}\mu^{D-4}\left(\frac{1}{D-4}+\frac{\gamma_E}{2}-\frac{3}{4}+\frac{1}{2}\log\frac{4\pi m^2}{\mu^2}\right)\end{aligned} \tag{135}$$

Now we can write down the regularized (but not yet renormalized) effective action

$$\begin{aligned}&\Gamma=S_{cl}-\frac{i}{2}\mathrm{Tr}\log\left[G\right]=S_{cl}\\ &+\frac{1}{2}\left(\frac{(D+1)(D-2)}{2}+(D-1)+2\right)\left(\frac{m_+^4}{32\pi^2}\mu_+^{D-4}\left(\frac{2}{D-4}+\gamma_E-\frac{3}{2}+\log 4\pi+\log\frac{m_+^2}{\mu_+^2}\right)\right.\\ &\left.+\frac{m_-^4}{32\pi^2}\mu_-^{D-4}\left(\frac{2}{D-4}+\gamma_E-\frac{3}{2}+\log 4\pi+\log\frac{m_-^2}{\mu_-^2}\right)\right)\end{aligned} \tag{136}$$

To renormalize this action, we need to introduce suitable counterterms. To see which ones exactly, let's write down the two masses in terms of fields and coupling constants

$$\begin{aligned}m_\pm^2=&\left(\frac{1}{2}(\zeta+3\lambda)+\frac{\xi}{96\alpha}(9\xi-8)\right)\bar{\phi}^2+\left(\frac{1}{4\alpha}+12\zeta-\frac{\xi}{8\alpha}\right)\bar{\omega}^2\\ \pm&\frac{1}{96\alpha}\Big[\left(48\alpha(\zeta+3\lambda)+\xi(9\xi+8)\right)^2\bar{\phi}^4+24\left(4608\alpha^2\zeta^2-13824\alpha^2\zeta\lambda\right.\\ -&\left.864\alpha\zeta\xi^2-2352\alpha\zeta\xi-144\alpha\lambda\xi-9\zeta^3-288\alpha\lambda+38\xi^2-96\alpha\zeta-16\xi\right)\bar{\phi}^2\bar{\omega}^2\\ +&144\left(96\alpha\zeta+\xi+2\right)^2\bar{\omega}^4\Big]^{1/2}\end{aligned} \tag{137}$$

## 5.6 Conditions on the coupling constants

Let's stop for a little here and analyze this expression. More precisely, let's determine the conditions on the parameters that would guarantee that we don't get tachyonic modes - that is that our mass-squared terms are always positive (and also real). This expression takes the form

$$m_{\pm}^2 = A\bar{\phi}^2 + B\bar{\varpi}^2 \pm C\sqrt{D^2\bar{\phi}^4 + E\bar{\phi}^2\bar{\varpi}^2 + F^2\bar{\varpi}^4} \tag{138}$$

Here $A$, $B$, $C$, $D$, $E$ and $F$ could be any real constants, the only thing we know is that the squares are positive. The first condition will come from the fact that the expression under the square-root must be positive. That is

$$\begin{aligned} & D^2\bar{\phi}^4 + E\bar{\phi}^2\bar{\varpi}^2 + F^2\bar{\varpi}^4 > 0 \\ & \Rightarrow D^2\frac{\bar{\phi}^2}{\bar{\varpi}^2} + F^2\frac{\bar{\varpi}^2}{\bar{\phi}^2} > -E \\ & D^2x^2 + F^2\frac{1}{x^2} > -E \\ & x^4 + \frac{E}{D^2}x^2 + \frac{F^2}{D^2} > 0 \\ & x^2 = z \Rightarrow z^2 + \frac{E}{D^2}z + \frac{F^2}{D^2} > 0 \end{aligned} \tag{139}$$

This quadratic inequality is satisfied when it's determinant determinant is negative

$$\frac{E^2}{D^4} - 4\frac{F^2}{D^2} < 0 \Rightarrow \frac{E^2}{D^2} - 4F^2 < 0 \Rightarrow \left(\frac{E}{D} + 2F\right)\left(\frac{E}{D} - 2F\right) < 0 \tag{140}$$

which implies

$$\begin{aligned} & \left(\frac{E}{D} + 2F\right) < 0 \wedge \left(\frac{E}{D} - 2F\right) > 0 \\ \text{or } & \left(\frac{E}{D} + 2F\right) > 0 \wedge \left(\frac{E}{D} - 2F\right) < 0 \end{aligned} \tag{141}$$

$$\begin{aligned} & \frac{E}{D} + 2F = \frac{24(9216\alpha^2\zeta^2 - 9\zeta^3 - 1536\alpha\zeta\xi + \xi^2(64 + 9\xi))}{48\alpha(\zeta + 3\lambda) + \xi(8 + 9\xi)} \\ & \frac{E}{D} - 2F = \\ & \frac{-24(9\zeta^3 + 27648\alpha^2\zeta\lambda + \xi(32 + 3\xi(3\xi - 4)) + 96\alpha(3\lambda(\xi + 2) + \zeta(2 + 3\xi(6\xi + 11))))}{48\alpha(\zeta + 3\lambda) + \xi(8 + 9\xi)} \end{aligned} \tag{142}$$

## 5.7 The massive tosion trace

### 5.7.1 Propagator

$$\frac{\delta S^{(2)}}{\delta \mathcal{T}_\nu} = \sigma\partial_\mu(\partial^\mu\mathcal{T}^\nu - \partial^\nu\mathcal{T}^\mu) + \frac{1}{\zeta}\partial^\nu\partial_\alpha\mathcal{T}^\alpha - M_{\mathcal{T}}^2\mathcal{T}^\nu = 0 \tag{143}$$

Go to momentum space

$$\begin{aligned} & -\sigma k^2\mathcal{T}^\nu + (\sigma - \frac{1}{\zeta})k^\nu k_\alpha\mathcal{T}^\alpha - M_{\mathcal{T}}^2\mathcal{T}^\nu = 0 \\ & \left[(k^2 + \tilde{M}^2)\eta^{\mu\nu} - \left(1 - \frac{1}{\zeta\sigma}\right)k^\mu k^\nu\right]\mathcal{T}_\mu = 0, \text{ where } \tilde{M}^2 = \frac{M_{\mathcal{T}}^2}{\sigma} \end{aligned} \tag{144}$$

We want to find the propagator that satisfies

$$\left[(k^2+\tilde{M}^2)\eta^{\mu\nu}-\left(1-\frac{1}{\zeta\sigma}\right)k^\mu k^\nu\right]iD_{\mu\alpha}=i\delta^\nu_\alpha \tag{145}$$

and from Lorentz-covariance we know that $D_{\mu\nu}$ takes the form

$$D_{\mu\nu}=a\eta_{\mu\nu}+bk_\mu k_\nu \tag{146}$$

$$\begin{aligned}&\left[(k^2+\tilde{M}^2)\eta^{\mu\nu}-\left(1-\frac{1}{\zeta\sigma}\right)k^\mu k^\nu\right](a\eta_{\mu\alpha}+bk_\mu k_\alpha)=\delta^\nu_\alpha\\ &\Rightarrow a=\frac{1}{k^2+\tilde{M}^2},\ b=\left(\frac{\zeta\sigma-1}{k^2+\zeta M_{\mathcal{T}}^2}\right)\frac{1}{k^2+\tilde{M}^2}\end{aligned} \tag{147}$$

where we note that in $b$ the first mass squared has no tilde, so it's the original one. So know we can write down the k-space propagator

$$D_{\mu\nu}=\frac{1}{k^2+\tilde{M}^2}\eta_{\mu\nu}+\left(\frac{\zeta\sigma-1}{k^2+\zeta M_{\mathcal{T}}^2}\right)\frac{1}{k^2+\tilde{M}^2}k_\mu k_\nu \tag{148}$$

This can be written in terms of the standard transverse and longitudinal projectors $t_{\mu\nu}$, $\ell_{\mu\nu}$

$$D_{\mu\nu}=\frac{1}{k^2+\tilde{M}^2}t_{\mu\nu}+\frac{1}{k^2+\tilde{M}^2}\left\{1+\left(\frac{\zeta\sigma-1}{k^2+\zeta M_{\mathcal{T}}^2}\right)k^2\right\}\ell_{\mu\nu} \tag{149}$$

Here we can clearly see that there are two modes with two different masses: a transverse mode whose mass is gauge-independent, and a longitudinal mode where the position of the mass-pole is explicitly dependent on the $R_\xi$ gauge parameter $\zeta$.

### 5.7.2 One loop effective action

Following the formula $-\frac{i}{2}\mathrm{Tr}\log[G]$ we now take the logarithm of this expression. Because $t_{\mu\nu}$ and $\ell_{\mu\nu}$ are orthogonal projectors, they commute with the logarithm

$$\log(D_{\mu\nu})=\log\left(\frac{1}{k^2+\tilde{M}^2}\right)t_{\mu\nu}+\log\left(\frac{1}{k^2+\tilde{M}^2}\left\{1+\left(\frac{\zeta\sigma-1}{k^2+\zeta M_{\mathcal{T}}^2}\right)k^2\right\}\right)\ell_{\mu\nu} \tag{150}$$

Taking the trace in this context means two steps: first, contraction of free indices which in this case are the two Lorentz-indices of the projectors. Second, a D-dimensional integral over the momentum $k$. The trace of the projectors $t_{\mu\nu}$ and $\ell_{\mu\nu}$ are $D-1$ and 1 respectively.

$$\begin{aligned}\mathrm{Tr}\log(D_{\mu\nu})=(D-1)\int\frac{d^Dk}{(2\pi)^D}\log\left(\frac{1}{k^2+\tilde{M}^2}\right)+\int\frac{d^Dk}{(2\pi)^D}\log\left(\frac{1}{k^2+\tilde{M}^2}\right)\\ +\int\frac{d^Dk}{(2\pi)^D}\log\left(1+\frac{(\zeta\sigma-1)k^2}{k^2+\zeta M_{\mathcal{T}}^2}\right)\end{aligned} \tag{151}$$

After a little manipulation this can be brought to the form

$$\mathrm{Tr}\log(D_{\mu\nu})=-\left\{(D-1)\int\frac{d^Dk}{(2\pi)^D}\log(k^2+\tilde{M}^2)+\int\frac{d^Dk}{(2\pi)^D}\log(k^2+\zeta M_{\mathcal{T}}^2)\right\} \tag{152}$$

where we repeat that $\tilde{M}^2=M_{\mathcal{T}}^2/\sigma$. Again, we must solve the integral

$$\int\frac{d^Dk}{(2\pi)^D}\log\left(k^2+m^2\right) \tag{153}$$

We have shown previously that

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\log\left(k^2+m^2\right) &= i\int \frac{d^4q}{(2\pi)^4}\log\left(q^2+m^2\right)\\ &\approx -\frac{im^4}{32\pi^2}\mu^{D-4}\left(\frac{2}{D-4}+\gamma_E-\frac{3}{2}+\log 4\pi+\log\frac{m^2}{\mu^2}\right)\end{aligned}\tag{154}$$

We can thus determine the massive vector contribution to the effective action

$$\begin{aligned}-\frac{i}{2}\mathrm{Tr}\log[D_{\mu\nu}] &= \frac{3\tilde{M}^4}{64\pi^2}\tilde{\mu}^{D-4}\left(\frac{2}{D-4}+\gamma_E-\frac{3}{2}+\log\frac{\tilde{M}^2}{\tilde{\mu}^2}\right)\\ &+\frac{\zeta^2M_{\mathcal{T}}^4}{64\pi^2}\mu_{\mathcal{T}}^{D-4}\left(\frac{2}{D-4}+\gamma_E-\frac{3}{2}+\log\frac{\zeta M_{\mathcal{T}}^2}{\mu_{\mathcal{T}}^2}\right)\end{aligned}\tag{155}$$

Here we remind ourselves that

$$M_{\mathcal{T}}^2=(D-2)\left((D-1)\bar{\omega}^2+\frac{(D-2)}{4}\bar{\phi}^2\right)\overset{D=4}{\longrightarrow}6\bar{\omega}^2+\bar{\phi}^2\tag{156}$$

## 5.8 Renormalization

Let's write down only the divergent part in the effective action

$$\Gamma_{div}=\left\{\frac{5}{16\pi^2}\left(m_+^4\mu_+^{D-4}+m_-^4\mu_-^{D-4}\right)+\frac{1}{32\pi^2}\left(3\tilde{M}^4\tilde{\mu}^{D-4}+\zeta^2M_{\mathcal{T}}^4\mu_{\mathcal{T}}^{D-4}\right)\right\}\left(\frac{1}{D-4}\right)\tag{157}$$

Since both $m_\pm^2$ contain square-roots, it is fortunate that they appear here as a sum so that the square root terms drop out or get squared like $(A+B)^2+(A-B)^2=2A^2+2B^2$, where B would be the square-root term. Thus we get that

$$\begin{aligned}m_+^4+m_-^4 &= \left((\zeta+3\lambda)^2+\frac{3\xi^2}{8\alpha}(\zeta+3\lambda)+\frac{\xi^2}{2304\alpha^2}\left(64+81\xi^2\right)\right)\bar{\phi}^4\\ &+\left(48\zeta^2-\frac{3\xi}{2\alpha}(11\zeta+\lambda)-\frac{\xi}{6\alpha^2}\left(1-2\xi+\frac{9\xi^2}{16}\right)\right)\bar{\phi}^2\bar{\omega}^2\\ &+\left(\left(\frac{1}{2\alpha}+24\zeta\right)^2+\frac{\xi^2}{16\alpha^2}\right)\bar{\omega}^4\end{aligned}\tag{158}$$

$$3\tilde{M}^4\tilde{\mu}^{D-4}+\zeta^2M_{\mathcal{T}}^4\mu_{\mathcal{T}}^{D-4}=\left(\frac{3}{\sigma^2}\tilde{\mu}^{D-4}+\zeta^2\mu_{\mathcal{T}}^{D-4}\right)(6\bar{\omega}^2+\bar{\phi}^2)^2\tag{159}$$

For the sake of simplicity, we identify the two pairs of arbitrary reference masses: $\mu_+$ with $\mu_-$, and $\tilde{\mu}$ with $\mu_{\mathcal{T}}$, such that we have

$$\Gamma_{div}=\frac{1}{16\pi^2}\left\{5\mu^{D-4}\left(m_+^4+m_-^4\right)+\frac{\tilde{\mu}^{D-4}}{2}\left(\frac{3}{\sigma^2}+\zeta^2\right)M_{\mathcal{T}}^4\right\}\left(\frac{1}{D-4}\right)\tag{160}$$

From this we can clearly see that our counter-term Lagrangian will contain three terms: each proportional to $\phi^4$, $\omega^4$ and $\phi^2\omega^2$.

Let's define the relevant bare Lagrangian

$$\begin{aligned}\mathcal{L}_{bare}=\sqrt{-g}\Big[&-\frac{\beta_0}{4!}\phi_0^4+\frac{\gamma_0}{4}\omega_0^2\phi_0^2-\frac{\chi_0}{4!}\omega_0^4\\ &-\frac{\zeta}{2}\left(36\omega_0^2+12\phi_0\omega_0+\phi_0^2\right)+\left(\frac{\omega_0^2}{2}\overset{\circ}{R}+\Lambda\right)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0\Big]\end{aligned}\tag{161}$$

Here $\phi_0$ and $\omega_0$ are the bare fields, $\beta_0$, $\gamma_0$ and $\chi_0$ are bare coupling constants which are easily related back to the original ones in 60, $\zeta$ is the $R_\xi$ gauge fixing parameter and $\mathcal{L}_{\mathcal{T}}$ is the decoupled massive vector sector. The renormalized fields are defined as

$$\phi \equiv Z_\phi^{-1/2}\phi_0, \qquad \omega \equiv Z_\omega^{-1/2}\omega_0 \tag{162}$$

Then we can express the above bare action as

$$\begin{aligned}\mathcal{L}_{bare} = \sqrt{-g}\Big[ & -\frac{\beta_0}{4!}Z_\phi^2\phi^4 + \frac{\gamma_0}{4}Z_\omega Z_\phi\omega^2\phi^2 - \frac{\chi_0}{4!}Z_\omega^2\omega^4 - \frac{1}{2}Z_\phi g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ & - \frac{\zeta}{2}\left(36Z_\omega\omega^2 + 12\sqrt{Z_\phi Z_\omega}\phi\omega + Z_\phi\phi^2\right) + \left(\frac{1}{2}Z_\omega\omega^2\overset{\circ}{R} + \Lambda\right)\Big]\end{aligned} \tag{163}$$

We can then implement renormalization if we define the bare coupling constants in terms of the physical ones plus some counter parameters that neutralize the infinities

$$Z_\phi \equiv 1+\delta Z_\phi,\ Z_\omega \equiv 1+\delta Z_\omega,\ Z_\phi^2\beta_0 \equiv \beta+\delta\beta,\ Z_\omega^2\chi_0 \equiv \chi+\delta\chi,\ Z_\phi Z_\omega\gamma_0 = \gamma+\delta\gamma \tag{164}$$

Fortunately, the divergent terms here only involve polynomials of the scalar fields. Since there is no local contribution proportional to the box operator, or $k^2$ in momentum space, we can safely assume the $Z$-coefficients to be one. Then we can just subtract exactly the divergent part with the help of a counter-term, consistently with MS, the minimal subtraction scheme

$$\begin{aligned}\mathcal{L}_{ct} = & \frac{-1}{16\pi^2}\left(\frac{1}{D-4}\right)\Bigg\{ \\ & \left[5\mu^{D-4}\left((\zeta+3\lambda)^2 + \frac{3\xi^2}{8\alpha}(\zeta+3\lambda) + \frac{\xi^2}{2304\alpha^2}\left(64+81\xi^2\right)\right) + \frac{\tilde{\mu}^{D-4}}{2}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\phi^4 \\ & + \left[5\mu^{D-4}\left(48\zeta^2 - \frac{3\xi}{2\alpha}(11\zeta+\lambda) - \frac{\xi}{6\alpha^2}\left(1-2\xi+\frac{9\xi^2}{16}\right)\right) + 6\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\phi^2\omega^2 \\ & + \left[5\mu^{D-4}\left(\left(\frac{1}{2\alpha}+24\zeta\right)^2 + \frac{\xi^2}{16\alpha^2}\right) + 18\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\omega^4\Bigg\}\end{aligned} \tag{165}$$

$$\begin{aligned}\delta\beta = & \frac{3}{2\pi^2}\left[5\mu^{D-4}\left((\zeta+3\lambda)^2 + \frac{3\xi^2}{8\alpha}(\zeta+3\lambda) + \frac{\xi^2}{2304\alpha^2}\left(64+81\xi^2\right)\right)\right. \\ & \left. + \frac{\tilde{\mu}^{D-4}}{2}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left(\frac{1}{D-4}\right) \\ \delta\gamma = & -\frac{1}{4\pi^2}\left[5\mu^{D-4}\left(48\zeta^2 - \frac{3\xi}{2\alpha}(11\zeta+\lambda) - \frac{\xi}{6\alpha^2}\left(1-2\xi+\frac{9\xi^2}{16}\right)\right)\right. \\ & \left. +6\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left(\frac{1}{D-4}\right) \\ \delta\chi = & \frac{3}{2\pi^2}\left[5\mu^{D-4}\left(\left(\frac{1}{2\alpha}+24\zeta\right)^2 + \frac{\xi^2}{16\alpha^2}\right) + 18\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left(\frac{1}{D-4}\right)\end{aligned} \tag{166}$$

Thus we get the renormalized effective action in four dimensions, in terms of the original parameters $\alpha, \xi, \lambda$, $\sigma$ and one gauge parameter $\zeta$

$$
\begin{aligned}
\Gamma = \int d^4x\sqrt{-g}\Bigg[ & -\left(\frac{\xi^2}{16\alpha} + \lambda + \frac{1}{16\pi^2}\left[5\mu^{D-4}\left((\zeta+3\lambda)^2 + \frac{3\xi^2}{8\alpha}(\zeta+3\lambda)\right.\right.\right. \\
& + \left.\left.\frac{\xi^2}{2304\alpha^2}\left(64+81\xi^2\right)\right) + \frac{\tilde{\mu}^{D-4}}{2}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left.\left(\frac{1}{D-4}\right)\right)\phi^4 + \left(\frac{\xi}{8\alpha}\right. \\
& - \frac{1}{16\pi^2}\left[5\mu^{D-4}\left(48\zeta^2 - \frac{3\xi}{2\alpha}(11\zeta+\lambda) - \frac{\xi}{6\alpha^2}\left(1-2\xi+\frac{9\xi^2}{16}\right)\right)\right. \\
& \left.\left. +6\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left(\frac{1}{D-4}\right)\right)\omega^2\phi^2 \\
& - \left(\frac{1}{16\alpha} + \frac{1}{16\pi^2}\left[5\mu^{D-4}\left(\left(\frac{1}{2\alpha}+24\zeta\right)^2 + \frac{\xi^2}{16\alpha^2}\right)\right.\right. \\
& \left.\left. +18\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}+\zeta^2\right)\right]\left(\frac{1}{D-4}\right)\right)\omega^4 + \frac{\omega^2}{2}\left(\mathring{R}+\Lambda\right) \\
& - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\zeta}{2}\left(36\omega^2+12\phi\omega+\phi^2\right) - \frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2\zeta}(\partial^\mu\mathcal{T}_\mu)^2 \\
& - \left(3\omega^2+\frac{1}{2}\phi^2\right)\mathcal{T}_\mu\mathcal{T}^\mu + \frac{5\mu^{D-4}}{32\pi^2}\left(m_+^4(\phi,\omega)\left(\gamma_E - \frac{3}{2} + \log 4\pi + \log\frac{m_+^2(\phi,\omega)}{\mu^2}\right)\right. \\
& \left. +m_-^4(\phi,\omega)\left(\gamma_E - \frac{3}{2} + \log 4\pi + \log\frac{m_-^2(\phi,\omega)}{\mu^2}\right)\right) \\
& + \frac{(6\omega^2+\phi^2)^2}{64\pi^2}\tilde{\mu}^{D-4}\left(\frac{3}{\sigma^2}\left(\gamma_E - \frac{3}{2} + \log\frac{6\omega^2+\phi^2}{\tilde{\mu}^2\sigma}\right)\right. \\
& \left.\left. +\zeta^2\left(\gamma_E - \frac{3}{2} + \log\frac{\zeta(6\omega^2+\phi^2)}{\tilde{\mu}^2}\right)\right)\right]
\end{aligned}
\tag{167}
$$

where $D-4$ has only been left wherever it should be understood as a limit tending to zero, and where we left $m_\pm^2$ unexpressed, because they are very long expressions, see

$$
\begin{aligned}
& m_\pm^2(\bar{\phi},\bar{\omega}) = \left(\frac{1}{2}(\zeta+3\lambda) + \frac{\xi}{96\alpha}(9\xi-8)\right)\bar{\phi}^2 + \left(\frac{1}{4\alpha}+12\zeta-\frac{\xi}{8\alpha}\right)\bar{\omega}^2 \\
& \pm \frac{1}{96\alpha}\Big[\left(48\alpha(\zeta+3\lambda)+\xi(9\xi+8)\right)^2\bar{\phi}^4 + 24\left(4608\alpha^2\zeta^2 - 13824\alpha^2\zeta\lambda\right. \\
& \left. -864\alpha\zeta\xi^2 - 2352\alpha\zeta\xi - 144\alpha\lambda\xi - 9\zeta^3 - 288\alpha\lambda + 38\xi^2 - 96\alpha\zeta - 16\xi\right)\bar{\phi}^2\bar{\omega}^2 \\
& + 144\left(96\alpha\zeta+\xi+2\right)^2\bar{\omega}^4\Big]^{1/2}
\end{aligned}
\tag{168}
$$

# 6 Discussion and conclusion

With this result we conclude this thesis. One can, of course, consider a number of follow-up steps and improvements. We shall discuss these here.

The first course of action would be the analysis of the running of these renormalized coupling constants. We could determine the RG-flow of the theory, and find out if there really is an UV fixed point as we suspect. Taking another step in this direction, this action could be RG-improved - that is, the logarithmic depencies on the arbitrary reference masses $\mu$ and $\tilde{\mu}$ could be resummed in a self-consistent way. This would eliminate a major redundancy in our description because these reference masses (also called renormalization scale) are not physical, they are only there as computational tools.

While these steps would eluminate the UV behaviour, there are also interesting possibilities in the IR. Determining how the quantum vacuum breaks the apparent symmetries of a model is what the Coleman-Weinberg mechanism (ambitiously described in the Introduction) is all about. Before anything else however, we should correct the mistake pointed out in 76. Unfortunately, because this current approximation is only valid when the quantum vacuum follows the structure of the classical one, there is little to none we can validly say about it this scheme. The $\delta\phi h$ vertex would probably complicate the diagonalization of the kinetic matrix a great deal, but including it is ultimately the correct approach. Then the correct effective masses produced by the field condensates could be calculated.

Finally it is important to note that this whole model came up in the context of inflation - a phenomenon associated with (approximately) de Sitter spacetimes. Thus, if we want to talk about the related cosmology, we would have to put this model on curved spacetime. In that case, background fields might still be assumed to be spatially constant, but they would have a relevant time-dependece. This fact by itself implies an appreciably more complicated action. Then the curved space formulation of quantum field theory would need to be used, a theory significantly more advanced than the standard Minkowski treatment outlined in this thesis. One of the difficulties arising in curved space is the determination of the propagators. As it turns out, they can only be expressed in terms of special functions. In the case of the graviton, gauge fixing is also limited to one gauge parameter.

In conclusion, while the exact physical relevance of the results in this thesis might be questioned, we can confidently say that all methods presented are sound and together constitute an effective toolbox for calculations in the area of quantum gravity and Weyl-symmetric quantum field theories.

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# Appendix

## A Massless scalar propagator

Consider the action[18]:

$$S[\phi] = \int d^D x \left( -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) \right) \tag{169}$$

then the Euler-Lagrange equations are the following

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \Rightarrow \partial^2 \phi = 0 \tag{170}$$

We need to find the propagator that correspond to this operator such that it is defined as

$$\partial^2 i\Delta(x, x') = i\delta^D(x - x') \tag{171}$$

By translation symmetry we can easily see that $i\Delta(x, x') = i\Delta(x - x')$ must hold. Let us now got to Fourier space:

$$i\tilde{\Delta}(k^\mu) = \int d^D e^{-ik.x} i\Delta(x - x') \tag{172}$$

where $k.(x-x') = \eta_{\mu\nu}k^\mu(x^\nu - x'^\nu)$. Let us now see our equation in Fourier representation

$$-k^2 i\tilde{\Delta}(k^\mu) \equiv -k^\mu k_\mu i\tilde{\Delta}(k^\mu) = i \tag{173}$$

This can easily be solved algebraically

$$\Rightarrow i\tilde{\Delta}(k^\mu) = \frac{-i}{k^\mu k_\mu} \tag{174}$$

Now let's transform this result back to position space

$$\begin{aligned} \Delta(x - x') &= \int \frac{d^D k}{(2\pi)^D} e^{ik.(x-x')} \left( \frac{-1}{k_\mu k^\mu} \right) \\ &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}.(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0 \Delta t} \frac{1}{k_0^2 - k^2} \end{aligned} \tag{175}$$

where $k = |\vec{k}|$ and $t - t' = \Delta t$. The $k^0$ integral is singular at $k^0 = \pm k$, in order to evaluate it we need to analytically continue it to the complex plane.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0 \Delta t} \frac{1}{2k} \left( \frac{1}{k^0 - k + i\epsilon} \right) - \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0 \Delta t} \frac{1}{2k} \left( \frac{1}{k^0 + k - i\epsilon} \right) \tag{176}$$

where we followed Feynman's prescription with the infinitesimal additions such that our calculation leads to the correct Feynman propagator. From this point on we'll consider the two terms (**I** and **II**) separately and add them up at the end.

#### A.0.1 I when $\Delta t < 0$

First, we choose the upper half-plane to evaluate this contour integral.

$$\oint \frac{dk^0}{2\pi} e^{-ik^0 \Delta t} \frac{1}{2k} \left( \frac{1}{k^0 - k + i\epsilon} \right) \tag{177}$$

No residue, this part contributes 0.

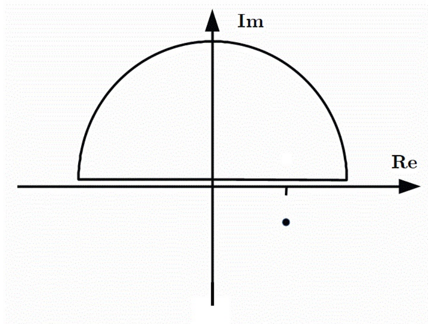


Figure 3: Contour for **I** when $\Delta t < 0$

### A.0.2 I when $\Delta t > 0$

Now we choose the lower half-plane, and there is a residue.

$$\oint \frac{dk^0}{2\pi} e^{-ik^0 \Delta t} \frac{1}{2k} \left( \frac{1}{k^0 - k + i\epsilon} \right) = -\frac{2\pi}{2\pi} i \left( \frac{1}{2k} e^{-i(k - i\epsilon)\Delta t} \right) = -\frac{i}{2k} e^{-ik\Delta t} \tag{178}$$

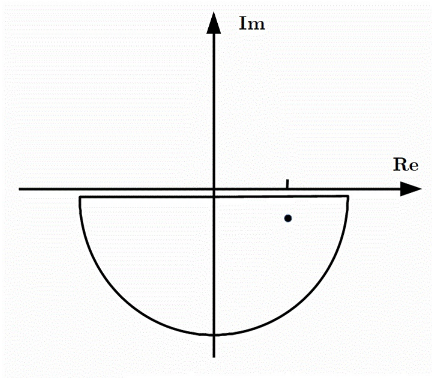


Figure 4: Contour for **I** when $\Delta t > 0$

### A.0.3 II when $\Delta t < 0$

Now we pick the upper half-plane. Here there is also a residue.

$$\oint \frac{dk^0}{2\pi} e^{-ik^0 (t-t')} \frac{1}{2k} \left( \frac{1}{k^0 + k - i\epsilon} \right) = \frac{2\pi}{2\pi} i \left( \frac{1}{2k} e^{-i(-k + i\epsilon)\Delta t} \right) = \frac{i}{2k} e^{+ik\Delta t} \tag{179}$$

### A.0.4 II when $\Delta t > 0$

This should also yield zero.

### A.0.5 Spherical coordinates

$$J = -\Theta(\Delta t) \frac{i}{2k} e^{-ik\Delta t} - \Theta(-\Delta t) \frac{i}{2k} e^{+ik\Delta t} \tag{180}$$

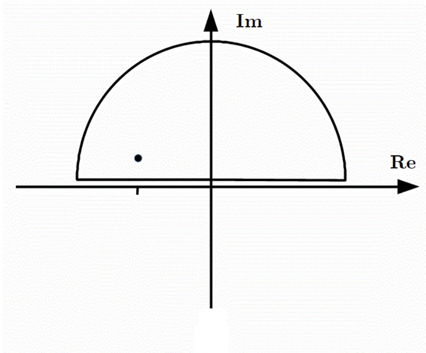


Figure 5: Contour for **II** when $\Delta t < 0$

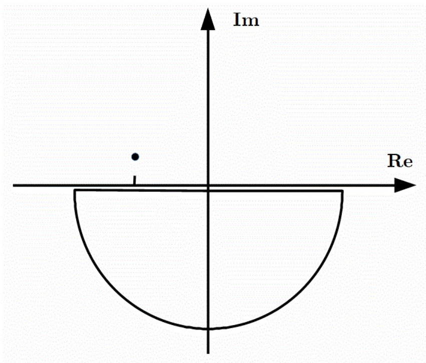


Figure 6: Contour for **II** when $\Delta t > 0$

$$\Rightarrow \Delta(x-x') = -\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}.(\vec{x}-\vec{x}')} \frac{i}{2k} \left( \Theta(\Delta t) \frac{i}{2k} e^{-ik\Delta t} + \Theta(-\Delta t) \frac{i}{2k} e^{+ik\Delta t} \right) \tag{181}$$

Having integrated out the frequency component and having expressions with $k = |\vec{k}|$ left in them, we should go to an $n-1$ dimensional spherical coordinate system where $k$is the radius. Let us also choose $\vec{x} - \vec{x}' = ||\vec{x} - \vec{x}'||$ to point to the 'last direction' $(0,0,...,1)$, so that we can write

$$\vec{k}.(\vec{x} - \vec{x}') = kr\cos(\varphi_{n-1}) \quad \text{where} \quad r = ||\vec{x} - \vec{x}'|| \tag{182}$$

The volume of the element is given by

$$\int d^n x = \int dr r^{n-1} \int_0^{2\pi} d\varphi_1 \int_0^{\pi} d\varphi_2 \sin\varphi_2 ... \int_0^{\pi} d\varphi_{n-1} \sin^{n-2}\varphi_{n-1} \tag{183}$$

$$\begin{aligned} \Delta(x-x') = & \frac{-1}{(2\pi)^{D-1}} \int dk k^{D-2} \int_0^{2\pi} d\varphi_1 \int_0^{\pi} d\varphi_2 \sin\varphi_2 ... \\ \times & \int_0^{\pi} d\varphi_{D-2} \sin^{D-3}\varphi_{D-2} e^{ikr\cos(\varphi_{D-2})} \frac{i}{2k} \left( \Theta(\Delta t) e^{-ik\Delta t} + \Theta(-\Delta t) e^{+ik\Delta t} \right) \end{aligned} \tag{184}$$

Let us use the identity

$$\int_0^\pi d\theta \sin^k\theta = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}k)}{\Gamma(1+\frac{1}{2}k)} \tag{185}$$

to evaluate $\int_0^\pi d\varphi_2 \sin\varphi_2 ... \int_0^\pi d\varphi_{D-1} \sin^{D-4}\varphi_{D-1}$.

$$\begin{aligned} &\Rightarrow \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2})}{\cancel{\Gamma(1+\frac{1}{2})}} \frac{\Gamma(\frac{1}{2})\cancel{\Gamma(\frac{1}{2}+1)}}{\cancel{\Gamma(1+1)}} \frac{\Gamma(\frac{1}{2})\cancel{\Gamma(\frac{1}{2}+\frac{3}{2})}}{\cancel{\Gamma(1+\frac{3}{2})}} ... \frac{\Gamma(\frac{1}{2})\cancel{\Gamma(\frac{1}{2}+\frac{1}{2}(D-4))}}{\Gamma(1+\frac{1}{2}(D-4))} \\ &= \frac{\Gamma(\frac{1}{2})^{D-4}\Gamma(1)}{\Gamma(1+\frac{1}{2}(D-4))} = \frac{\pi^{(\frac{D-4}{2})}}{\Gamma(\frac{D}{2}-1)} \end{aligned} \tag{186}$$

$$\begin{aligned} &\Rightarrow \Delta(x-x') = \frac{-1}{(2\pi)^{D-1}} \frac{\pi^{(\frac{D-4}{2})}}{\Gamma(\frac{D}{2}-1)} \int dk k^{D-2} \int_0^{2\pi} d\varphi_1 \\ &\times \int_0^\pi d\varphi_{D-2} \sin^{D-3}\varphi_{D-2} e^{ikr\cos(\varphi_{D-2})} \frac{i}{2k} \left(\Theta(\Delta t)e^{-ik\Delta t} + \Theta(-\Delta t)e^{+ik\Delta t}\right) \end{aligned} \tag{187}$$

Now to evaluate the last integral, the one of $\varphi_{D-2}$, we need to use the formula

$$\begin{aligned} J_\nu(z) &= \frac{1}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \int_0^\pi e^{iz\cos\theta}(\sin\theta)^{2\nu} d\theta \\ \Rightarrow \int_0^\pi e^{iz\cos\theta}(\sin\theta)^{2\nu}d\theta &= \Gamma(\nu+\frac{1}{2})\sqrt{\pi}\left(\frac{2}{z}\right)^\nu J_\nu(z) \end{aligned} \tag{188}$$

So we substitute this identity into our expression for the propagator

$$\begin{aligned} \Delta(x-x') &= \frac{-1}{(2\pi)^{D-2}} \frac{\pi^{\frac{D-3}{2}}}{\cancel{\Gamma(\frac{D}{2}-1)}} \cancel{\Gamma\left(\frac{D-3}{2}+\frac{1}{2}\right)} \\ &\times \int_0^\infty dk k^{D-2} \left(\frac{2}{kr}\right)^{\frac{D-3}{2}} J_{(\frac{D-3}{2})}(kr) \frac{i}{2k} \left(\Theta(\Delta t)e^{-ik\Delta t} + \Theta(-\Delta t)e^{+ik\Delta t}\right) \end{aligned} \tag{189}$$

$$\begin{aligned} \mathbf{I} &= \int_0^\infty dk k^{D-2} \left(\frac{2}{kr}\right)^{\frac{D-3}{2}} J_{(\frac{D-3}{2})}(kr) \frac{i}{2k} \Theta(\Delta t)e^{-ik\Delta t} \\ &= i2^{\frac{D-5}{2}} \int_0^\infty dk \sqrt{kr} r^{\frac{1-D}{2}} k^{\frac{D-5}{2}} J_{(\frac{D-3}{2})}(kr)\Theta(\Delta t)e^{-ik\Delta t} \end{aligned} \tag{190}$$

This is a Hankel transform and the relevant entry from tables[19] is

| $f(x)$ | $\int_0^\infty f(x)J_\nu(xy)\sqrt{xy}dx \quad y>0$ |
|---|---|
| $x^{\nu-\frac{1}{2}}e^{-ax} \quad Re[a]>0, Re[\nu]>-1$ | $2^\nu \frac{1}{\sqrt{\pi}}\Gamma(\nu+\frac{1}{2})y^{\nu+\frac{1}{2}}(a^2+y^2)^{-\nu-\frac{1}{2}}$ |

This is where we need the analytic continuation. We need to add $-i\epsilon$, so that $Re[a]$ is indeed positive.

$$\mathbf{I} = i2^{\frac{D-5}{2}}\Theta(\Delta t)\frac{2^{\frac{D-3}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{D}{2}-1\right)(r^2-(\Delta t - i\epsilon)^2)^{1-\frac{D}{2}} \tag{191}$$

**II** is evaluated the same way, except there we add $+i\epsilon$

$$\mathbf{II} = i2^{\frac{D-5}{2}}\Theta(-\Delta t)\frac{2^{\frac{D-3}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{D}{2}-1\right)(r^2-(\Delta t + i\epsilon)^2)^{1-\frac{D}{2}} \tag{192}$$

Thus we have the propagator in position space

$$
\begin{aligned}
i\Delta(x-x') = \frac{1}{4\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right)[\Theta(\Delta t)(r^2-(\Delta t-i\epsilon)^2)^{1-\frac{D}{2}} \\
+\Theta(-\Delta t)(r^2-(\Delta t+i\epsilon)^2)^{1-\frac{D}{2}}]
\end{aligned}
\tag{193}
$$