Homotopy groups of spheres using the Pontryagin theorem

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1 Introduction

One of the most important problems of topology is which spaces are homeomorphic to which other spaces. To show that spaces are not homeomorphic is sometimes far from trivial. For example, proving that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ uses constructions of Algebraic Topology.
Algebraic Topology is a branch of mathematics that measures certain properties of spaces by assigning groups or numbers to them. When these numbers or groups for two spaces are not the same then the spaces are not homeomorphic. Of these, the fundamental group is one of the more important ones, and the properties it captures is the kind of holes a space has. So clearly if the fundamental group of two spaces is different then they will not be homeomorphic.

The way the fundamental group $\pi_1(X)$ captures the ideas of holes is very intuitive: map S^1 smooth into your space X and see if you can shrink it smoothly down to a point.
We can generalize this construction to define homotopy groups $\pi_n(X)$, which work in the same way, but then we map S^n into our space X and we see if we can shrink it down to a point.
We still have that X is not homeomorphic to Y if their homotopy groups are not isomorphic.

As we have seen in the course "Topologie en Meetkunde", we were able to calculate the fundamental group of a lot of spaces by starting with the fundamental group of S^1 and using Van Kampen and other theorems. In the same way, even tho there is no extension for Van Kampen to general homotopy groups, we want to first calculate the homotopy groups of spheres S^k. However, this already turns out to be an almost impossible task

When n is smaller than k we see that $\pi_n(S^k)$ is the same as the trivial group because we can move the maps image to not be surjective and then retract $S^k \backslash \{p\}$ onto a point.

When n is the same as k, we see that $\pi_n(S^k) = \mathbb{Z}$ because we can wrap S^k around itself for $l \in \mathbb{Z}$ times, which will not be shrinkable to a point.

When $n > k$ the intuition starts to fall apart. For example $\pi_3(S^2) = \mathbb{Z}$.
So we would like to have some other tools to calculate the homotopy groups of spheres. One of these tools is to look at the framed cobordism group, $\Omega_n^{fr}(X)$, of a space X, which looks at the ways we can "connect" compact submanifolds without boundary. The theory of framed cobordism groups was most notably developed by Lev Pontryagin, a Russian mathematician, who had gone blind when he was 14 years old. He discovered the following beautiful relation:

Theorem 1.1 (Pontryagin Theorem)**.** *As groups,*

$$\Omega_{n-k}^{fr}(S^n) \cong \pi_n(S^k). \tag{1}$$

Proving this theorem will be the main point of this thesis. Afterwards, we will discuss a couple of examples. This thesis will mostly be based on Chapter 7 of *Topology from a differential viewpoint* by Milnor [7], which is based on the original article by Pontryagin [8].

2 Homotopy Groups

To understand what the Pontryagin theorem is about, we first have to introduce smooth homotopy groups. We will define these groups in the same way as we did for the fundamental group in "Topologie en Meetkunde". There is of course a difference between smooth homotopy groups and general continous homotopy groups, however, because of Corollary 17.8.1 of *Differential forms in algebraic topology* by Bott and Tu [1] we see that the two are isomorphic.

Corollary 2.0.1. *Let M be a smooth manifold, then the smooth homotopy groups of M are isomorphic to the continuous homotopy groups of M.*

Because of this we will call the smooth homotopy groups just the homotopy groups and use the notation that is otherwise used for the continuous homotopy groups. Our approach of defining smooth homotopy groups will mostly rely on how continuous homotopy groups are defined in Chapter 4.1 of Hatcher [4].
We first define what we mean with a smooth based homotopy.

Definition 2.1 (Smooth Homotopy with fixed basepoint)**.** *Suppose we have two based manifolds (M, m_0) and (N, n_0) and a two smooth maps*

$$f, g : (M, m_0) \to (N, n_0)$$

then we call these two maps smoothly homotopic with fixed basepoint if there is a smooth map

$$\begin{aligned} &H : (M, m_0) \times I \to (N, n_0) \text{ such that} \\ &H(x, 0) = H_0(x) = f(x) \text{ and} \\ &H(x, 1) = H_1(x) = g(x). \end{aligned}$$

where we note that $H(m_0, t) = n_0$ for all $t \in I$ which is why we call it based. If two based maps f and g are smoothly based homotopic then we write

$$f \simeq g.$$

We will call two maps smoothly homotopic if the homotopy does not have a fixed basepoint. Being smoothly homotopic with fixed basepoint is an equivalence relation, where we write $[M, N]_{(m_0, n_0)}$ as the set of all equivalence classes. We will write $\big((M, m_0) \to (N, n_0)\big)$ as the set of all smooth based maps from M to N.

Theorem 2.2. *Being smoothly homotopic with fixed basepoint is an equivalence relation, so*

1. *reflexivity: $f \simeq f$ for every map $f \in \big((M, m_0) \to (N, n_0)\big)$.*
2. *symmetry: If $f \simeq g$ then $g \simeq f$ for every $f, g \in \big((M, m_0) \to (N, n_0)\big)$.*
3. *transitivity: If $f \simeq g$ and $g \simeq h$ as well then $f \simeq h$ for every $f, g, h \in \big((M, m_0) \to (N, n_0)\big)$.*

To prove transitivity we will need to construct a smooth based homotopy out of two other smooth based homotopies. To do this we define the bump function.

Definition 2.3 (bump function)**.** *The bump function is a smooth map*

$$\begin{aligned}\psi : \mathbb{R} &\to \mathbb{R}\\ x \mapsto \psi(t) &= \begin{cases} 0 & \text{for } t \leqslant 0 \\ \frac{e^{-\frac{1}{t}}}{e^{-\frac{1}{t}}+e^{\frac{1}{1-t}}} & \text{for } 0 < t < 1 \\ 1 & \text{for } t \geqslant 1. \end{cases}\end{aligned}$$

We want to use this function to compose it with smooth maps to get a new smooth map. To do this we first want to define it on I instead of $\mathbb{R}$. So we introduce the diffeomorphism

$$\begin{aligned}L_\epsilon : I &\to [\epsilon, 1-\epsilon]\\ t \mapsto L_\epsilon(t) &= \frac{t-\epsilon}{1-2\epsilon}.\end{aligned}$$

Using this function we create our new smooth bump function

$$\begin{aligned}\tilde{\psi} : I &\to I\\ t \mapsto \tilde{\psi}(t) &= \begin{cases} 0 & \text{for } t \in [0, \epsilon] \\ \psi \circ L_\epsilon(t) & \text{for } t \in (\epsilon, 1-\epsilon) \\ 1 & \text{for } t \in [1-\epsilon, 1] \end{cases}.\end{aligned}$$

Now that we have introduced the bump function we will prove the theorem.

Proof of theorem 2.2. We will prove this theorem in three parts.

1. reflexivity: We see that the smooth homotopy defined by

$$\begin{aligned}H_f : (M, m_0) \times I &\to (N, n_0)\\ (x,t) \mapsto H_f(x,t) &= f(x) \text{ for all } t \in I\end{aligned}$$

is a smooth homotopy between f and itself.

2. symmetry: Suppose f is smoothly homotopic to g by a smooth homotopy H_{fg}. We now construct a new smooth homotopy

$$\begin{aligned}H_{gf} : (M, m_0) \times I &\to (N, n_0)\\ (x,t) \mapsto H_{gf}(x,t) &= H_{fg}(x, 1-t).\end{aligned}$$

We see that this homotopy is the same as H_{fg} but then in the opposite direction. We see that H_{gf} is a homotopy between g and f, so $g \simeq f$.

3. transitivity: Suppose f is homotopic to g by a homotopy H_{fg} and g is homotopic to h by a homotopy H_{gh}. We now construct a new homotopy

$$\begin{aligned} H_{fh} &: (M, m_0) \times I \to (N, n_0) \\ (x,t) &\mapsto H_{fh}(x,t) = \begin{cases} H_{fg}(x, \tilde{\psi}(2t)) & \text{for } t \in [0, \frac{1}{2}] \\ H_{gh}(x, \tilde{\psi}(2t-1)) & \text{for } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

where $\tilde{\psi}$ is the bump function of defintion 2.3. This smooth homotopy takes us first through H_{fg} and afterwards through H_{gh}. We see that H_{fh} is a homotopy between f and h, so $f \simeq h$.

Now that we have proven reflexivity, symmetry and transitivity we see that being based homotopic is indeed an equivalence relation. □

In the same way as the fundamental group, we now want to look at the based maps from S^n to X up to smooth based homotopy.

Definition 2.4 (smooth homotopy groups)**.** *We call* $[S^n, X]_{(s,x_0)}$ *the* n^{th} *smooth homotopy group of* (X, x_0)*, and we write it as* $\pi_n(X, x_0)$.

We first identify S^n with $I^n/\sim$ where we quotient out the boundary of the box, ∂I^n, so more explicitly

$$\begin{aligned} &S^n \cong I^n/\sim \text{ where} \\ &(t_1, \cdots, t_{i-1}, 1, t_{i+1}, \cdots, t_n) \sim (t'_1, \cdots, t'_{i-1}, 1, t'_{i+1}, \cdots, t_n) \text{ and} \\ &(t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n) \sim (t'_1, \cdots, t'_{i-1}, 0, t'_{i+1}, \cdots, t_n) \text{ for all } 1 \leqslant i \leqslant n. \end{aligned}$$

These two spaces are diffeomorphic because the interior of I^n is diffeomorphic to $\mathbb{R}^n$, so $I^n/\sim$ is diffeomorphic to $\mathbb{R}^n \cup \{\infty\} \cong S^n$. Because of this we have an ismomorphism between $[S^n, X]_{(s,x_0)}$ and $[I^n, X]_{(\partial I^n, x_0)}$ which consists of smooth maps from I^n to X, which send the boundary ∂I^n to x_0.

As the name suggests, $\pi_n(X, x_0)$, is a group with addition defined in the following way. We take $[f], [g] \in \pi_n(X, x_0)$ and we take two representatives f and g respectively. We want to define $f + g$ as the new map that first does the map f and the map g afterwards. To do this we look at f and g as if they are maps from I^n to X, both taking ∂I^n to x_0. We then define $f + g$ as

$$\begin{aligned} f + g &: (I^n, \partial I^n) \to (X, x_0) \\ (t_1, ..., t_n) &\mapsto \begin{cases} f(\tilde{\psi}(2t_1), t_2, ..., t_n) & \text{for } t_1 \in [0, \frac{1}{2}] \\ g(\tilde{\psi}(2t_1 - 1), t_2, ..., t_n) & \text{for } t_1 \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

where $\tilde{\psi}$ is the bump function of definition 2.3. We see that this map still maps ∂I^n onto x_0, so it gives us a well-defined map from pointed map from S^n to X. For this operation to define the multiplication on $\pi_n(X, x_0)$ by $[f]+[g] = [f+g]$, we still need to check that it is well defined.

Theorem 2.5. *The multiplication + on $\pi_n(X, x_0)$ is well-defined, so if $f_1 \simeq f_2$ and $g_1 \simeq g_2$ then $(f_1 + g_1) \simeq (f_2 + g_2)$.*

Proof of theorem 2.5. To prove this we first want to prove that if $f_1 \simeq f_2$ then $(f_1 + g) \simeq (f_2 + g)$.
Suppose f_1 and f_2 are homotopic by a homotopy H_f. We now construct a new homotopy

$$\begin{aligned} H : (I^n, \partial I^n) \times I &\to (X, x_0) \\ ((t_1, \cdots, t_p), s) \mapsto H(x, s) &= \begin{cases} (H_f)_s(\tilde{\psi}(2t_1), t_2, \cdots, t_p) & \text{for } t_1 \in [0, \frac{1}{2}] \\ g(\tilde{\psi}(2t_1 - 1), t_2, \cdots, t_p) & \text{for } t_1 \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

where $\tilde{\psi}$ is the smooth bump function from definition 2.3. We note that $H(t, s) = (H_f(s) + g)(t)$, so this gives a homotopy between $H_f(0) + g = f_1 + g$ and $H_f(1) + g = f_2 + g$, so we see that $f_1 + g$ is homotopic to $f_2 + g$.
We can construct a same kind of homotopy such that $f + g_1$ is homotopic to $f + g_2$ when g_1 is homotopic to g_2.
We will now give a proof for the theorem. We see that $f_1 + g_1$ is homotopic to $f_2 + g_1$ by our first part. By our second part we see that $f_2 + g_1$ is homotopic to $f_2 + g_2$. Because being homotopic is an equivalence relation we see that $f_1 + g_1$ is homotopic to $f_2 + g_2$, which was to be proven. $\square$

Theorem 2.6. *The based n^{th}-homotopy group $\pi_n(X, x_0)$ of (X, x_0) is a group with + as multiplication. More explicitly:*

1. *closure: $[f] + [g]$ is an element of $\pi_n(X, x_0)$ for every $[f], [g] \in \pi_n(X, x_0)$.*
2. *associativity: $([f] + [g]) + [h] = [f] + ([f] + [h])$ for every $[f], [g], [h] \in \pi_n(X, x_0)$.*
3. *identity: The identity element is $[const_{x_0}]$ so $[const_{x_0}] + [f] = [f] = [f] + [const_{x_0}]$ for every $[f] \in \pi_n(X, x_0)$*
4. *inverse: For every $[f] \in \pi_n(X, x_0)$ there is an inverse element $[f]^{-1} \in \pi_n(X, x_0)$ such that $[f]^{-1} + [f] = [f] + [f]^{-1} = [const_{x_0}]$.*

To prove associativity, we first state the following lemma.

Lemma 2.7. *Let $\psi : I \to I$ be a smooth map such that $\psi(0) = 0$ and $\psi(1) = 1$. Let $f \in (I^n, \partial I^n) \to (X, x_0)$. Then*

$$\begin{aligned} f \circ (\psi, Id_{I^{n-1}}) : I^n \times I &\to (X, x_0) \\ (t_1, \cdots, t_n) &\mapsto \big(f \circ (\psi, Id_{I^{n-1}})\big)(t) = f(\psi(t_1), t_2, \cdots, t_n) \end{aligned}$$

is homotopic to f.

Proof of lemma 2.7. We see that I is a convex subset of $\mathbb{R}$ so we construct a smooth homotopy

$$\begin{aligned}
H_\psi : (I^n, \partial I^n) \times I &\to (X, x_0) \\
(t,s) &\to H_\psi(t,s) = f(s \cdot t_1 + (1-s)\psi(t_1), t_2, \cdots, t_n)
\end{aligned}$$

which gives a homotopy between $f \circ (\psi, Id_{I^{p-1}})$ and f. $\square$

Proof of theorem 2.6. We will prove this in four parts.

1. closure: The way we defined $f+g$ was by taking f on the first half of I^n and g on the second half, where f and g as maps from $(I^n, \partial I^n)$ to (X, x_0) both map ∂I^n onto $\{x_0\}$. Because of this we see that f maps the boundary of the first half of I^n, being $\partial(I^{n-1} \times [0, \frac{1}{2}])$ onto $\{x_0\}$ and g maps the boundary of the second half of I^n, being $\partial(I^{n-1} \times [\frac{1}{2}, 1])$ onto $\{x_0\}$. Because of this we see that $f+g$ maps ∂I onto $\{x_0\}$, so $f+g$ induces a class $[f+g] \in \pi_n(X, x_0)$.

2. associativity: We note that $(f+g)+h$ is a respeeding of $f+(g+h)$ so by lemma 2.7 we see that they are homotopic, so
$$\begin{aligned}
\big([f]+[g]\big)+[h] = [f+g]+[h] = [\big(f+g\big)+h] = \\
[f+\big(g+h\big)] = [f]+[g+h] = [f]+\big([g]+[h]\big)
\end{aligned}$$
which proves associativity.

3. identity: We see that $[const_{x_0}]+[f] = [const_{x_0}+f]$, so we want to prove that $const_{x_0}+f$ is homotopic to f.
We construct the homotopy
$$H : (I^n, \partial I^n) \times I \to (X, x_0)$$
$$(t,s) \mapsto H(t,s) = \begin{cases} x_0 & \text{for } t_1 \in [0, \frac{1}{2}(1-s)] \\ f(\tilde{\psi}(2t_1-1), t_2, \cdots, t_n) & \text{for } t_1 \in [\frac{1}{2}(1-s), 1] \end{cases}$$
where $\tilde{\psi}$ is the smooth bump function as in definition 2.3. This gives a homotopy between $const_{x_0}+f$ and f, so $[const_{x_0}+f] = [f]$. In the same way we see that $[f] = [f]+[const_{x_0}]$ so we see that $[const_{x_0}]$ is indeed the identity for the group.

4. inverse: We will prove that f^{-1} given by
$$f^{-1}(t_1, t_2, \cdots, t_n) = f((1-t_1), t_2, \cdots, t_n)$$
is an inverse for f, so we want to prove that $const_{x_0}$ is homotopic to $f+f^{-1}$.
We construct the homotopy
$$H_f : (I^n, \partial I^n) \times I \to (X, x_0)$$
$$(t,s) \mapsto H_f(t,s) = \begin{cases} f(2t_1, \cdots, t_n) & \text{for } t_1 \in [0, \frac{1}{2}(1-s)] \\ f^{-1}(1-(2s-2t_1), t_2, \cdots, t_n) & \text{for } t_1 \in [\frac{1}{2}s, s] \\ x_0 & \text{for } t_1 \in [s, 1]. \end{cases}$$

This gives a smooth homotopy between $f + f^{-1}$ and $const_{x_0}$, so we see that $[f^{-1}] + [f] = [const_{x_0}]$. Because of this we see that $[f]^{-1} = [f^{-1}]$.

So we see that $\pi_n(X, x_0)$ is indeed a group. $\square$

Just like we did for the fundamental group, we will now prove that for every choice of basepoint the homotopy groups are isomorphic if the manifold X is smoothly path-connected.

Theorem 2.8. *If X is smoothly path-connected then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for every two basepoints x_0 and x_1 in X.*

To give an outline of the proof we will first define what we mean when we compose a smooth path with a smooth map from $(I^n, \partial I^n)$ to $(X,_0)$. We define this in the same way is it is defined in Hatcher [4] but then to keep it being smooth we respeed both the path and the smooth map. In the same way as is discussed for the continuous case in Hatcher [4] we see that we can construct an isomorphism $\beta_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ which is defined by composing the classes of $\pi_n(X, x_0)$ with a smooth path γ from x_1 to x_0. We note that such a γ exists because X is smoothly path-connected.
We will write this new group as $\pi_n(X)$.

Theorem 2.9. *If X is smoothly simply-connected then $\pi_n(X) \cong [S^n, X]$ where $[S^n, X]$ are the unbased homotopy classes.*

This is proven for the continuous homotopy groups by using Proposition 4A.2 of Hatcher [4] because S^n is a CW-complex and because $\pi_1(X, x_0) = 1$ because X is simply-connected. So by corollary 2.0.1 we see that this is also true for smooth homotopy groups.
For the Pontryagin Theorem 1.1 we look at $\pi_n(S^k)$, which we can write this way because S^k is path-connected for all $k \in \mathbb{N}$.

3 Framed cobordism

We now want to get a better idea of what it means for two compact submanifolds without boundary to be framed cobordant. To do this we first discuss what it means to be cobordant and what a framing is. Afterwards, we will see that framed cobordism is a group. Our construction will be based on the construction given in Milnor [7].
From now on, when we talk about M it will be an m-dimensional compact submanifold without boundary of $\mathbb{R}^{m+k}$.

We will first define what we mean for a submanifold N to be closed

Definition 3.1 (closed submanifold)**.** *Suppose N is a n-dimensional submanifold of M, then we call N closed iff N has no boundary and N is compact.*

We now define what it means to be cobordant.

Definition 3.2 (cobordism)**.** *Suppose we have two n-dimensional closed submanifolds N_1 and N_2 of an ambient manifold M of dimension m. We call N_1 and N_2 cobordant if there exists a $(n+1)$-dimensional compact manifold $W \subset M \times I$ with the property that*

$$\begin{aligned}
\partial W &= \big(N_1 \times \{0\}\big) \cup \big(N_2 \times \{1\}\big), \\
W \cap \big(M \times [0, \epsilon)\big) &= N_1 \times [0, \epsilon) \textit{ and} \\
W \cap \big(M \times [1-\epsilon, 1)\big) &= N_2 \times (1-\epsilon, 1]
\end{aligned}$$

As an example, if N is a closed submanifold of M, then N is cobordant to itself, because $N \times I$ is a $(n+1)$-dimensional manifold in $M \times I$. We see that this is a cobordism between N and itself, so N is indeed cobordant to itself.

Now that we have introduced cobordism, we will define what it means to be framed. First, we need to introduce the normal space.

Definition 3.3 (Normal Space)**.** *Suppose we have an N being an n-dimensional submanifold of M, which is a m-dimensional submanifold of $\mathbb{R}^{m+k}$, then we define its normal space at the point $p \in N$ as the space $T_pN^{\perp}$ being a subspace of T_pM given by*

$$T_pN^{\perp} = \{v \in T_pM \mid \langle v, w\rangle = 0 \textit{ for all } w \in T_pN\}.$$

As an example we look at the S^1 being a submanifold of $\mathbb{R}^2$. We note that S^1 is given by an implicit equation

$$\begin{aligned}
&eq : \mathbb{R} \to \mathbb{R} \\
&(x, y) \mapsto eq(x, y) = x^2 + y^2 - 1
\end{aligned}$$

which is a submersion on all of S^1. We see that for $x \in S^1$ the tangent space at p is given by the kernel of $(d\ eq)_p$, which is given by

$$(d\ eq)_p = \begin{pmatrix} 2x_p & 2y_p \end{pmatrix}.$$

So we see that

$$T_pS^1 = ker\big((d\,eq)_p\big) = \big\{v \in \mathbb{R}^2 \mid x_p \cdot v_1 + y_p \cdot v_2 = 0\big\}.$$

However we see that we can write this as

$$T_pS^1 = \big\{v \in \mathbb{R}^2 \mid \langle v, p\rangle = 0\big\}.$$

So we see that the normal space at a point $p \in S^1$ will consist of all the vectors $\lambda \cdot p$ with $\lambda \in \mathbb{R}$.

Now we will look at what it means for a normal space of a submanifold to be framed.

Definition 3.4 (framing)**.** *A framing of a n-dimensional closed submanifold N of an m-dimensional smooth manifold M is a smooth function which assigns to every point $x \in N$ a basis for the corresponding normal space $(T_xN)^\perp$,*

$$\begin{aligned} \nu : N \ni x &\to base\big((T_xN)^\perp\big) \\ x &\mapsto \nu(x) = \{\nu_1(x), \cdots, \nu_{m-n}(x)\} \end{aligned}$$

We first note that not all manifolds admit a framing. For example, non-orientable manifolds can not be framed.

Now that we have both introduced cobordisms for unframed manifolds and framings, we want to look at what it means to be framed cobordant. For this, we look at framed manifolds, which are tuples of a manifold together with a chosen framing.

Definition 3.5 (framed cobordism)**.** *We call two closed framed submanifolds (N_1, ν_{N_1}) and (N_2, ν_{N_2}) of a manifold M framed cobordant if N_1 and N_2 are cobordant as in definition 3.2 via a cobordism $W \subset M \times I$ and if there is an framing ν_W of W with the condition that*

$$\nu_W\big|_{W \cap \left(M \times [0,\epsilon)\right)}(w) = \nu_W|_{N_1 \times [0,\epsilon)}(x,t) = \big(\nu_{N_1}(x), 0\big) \text{ for } (x,t) \in N_1 \times [0,\epsilon) \text{ and}$$

$$\nu_W\big|_{W \cap \left(M \times (1,1-\epsilon]\right)}(w) = \nu_W|_{N_2 \times (1-\epsilon,1]}(x,t) = \big(\nu_{N_2}(x), 0\big) \text{ for } (x,t) \in N_2 \times (1-\epsilon, 1].$$

If two submanifolds (N_1, ν_{N_1}) and (N_2, ν_{N_2}) are framed cobordant then we will write this as $(N_1, \nu_{N_1}) \overset{\text{fr}}{\cong} (N_2, \nu_{N_2})$.

As an example we will look if (N, ν), being a framed submanifold without boundary of M, is actually framed cobordant to itself. To see that it is, we first construct the submanifold $(N \times I, \nu')$ of $M \times I$ with $\nu'(x,t) = (\nu(x), 0)$ for all $(x,t) \in N \times I$. We see that $(N \times I, \nu')$ is a framed cobordism between (N, ν) and itself, so (N, ν) is framed cobordant to itself.

From this example we already see that framed cobordism is reflexive, so we are already well on our way to prove the following theorem.

Theorem 3.6. *Framed cobordism is an equivalence relation, so more explicitly*

1. *reflexivity:* $(N,\nu) \overset{\text{fr}}{\cong} (N,\nu)$ *for every framed submanifold* (N,ν) *of* M*.*

2. *symmetry: if* $(N_1,\nu_{N_1}) \overset{\text{fr}}{\cong} (N_2,\nu_{N_2})$ *then* $(N_2,\nu_{N_2}) \overset{\text{fr}}{\cong} (N_1,\nu_{N_1})$ *as well, for every framed submanifold* (N_1,ν_{N_1}) *and* (N_2,ν_{N_2}) *of* M*.*

3. *transitivity: if* $(N_1,\nu_{N_1}) \overset{\text{fr}}{\cong} (N_2,\nu_{N_2})$ *and* $(N_2,\nu_{N_2}) \overset{\text{fr}}{\cong} (N_3,\nu_{N_3})$ *then* $(N_1,\nu_{N_1}) \overset{\text{fr}}{\cong} (N_3,\nu_{N_3})$ *as well, for every framed submanifold* (N_1,ν_{N_1})*,* (N_2,ν_{N_2}) *and* (N_3,ν_{N_3}) *of* M*.*

Proof of theorem 3.6. In the previous example we have already proven reflexivity.
It is also symmetric because if we look at the framed cobordism $(W,\mu) \subset M \times I$ between (N_1,ν_{N_1}) and (N_2,ν_{N_2}), then we see that this is also a cobordism in the other direction.

We will now prove transitivity. Suppose we have we have a framed cobordism (W_1,μ_{W_1}) between (N_1,ν_{N_1}) and (N_2,ν_{N_2}) and another framed cobordism (W_2,μ_{W_2}) between (N_2,ν_{N_2}) and (N_3,ν_{N_3}). We now want to prove that there is a framed cobordism $(\tilde{W},\tilde{\mu})$ between (N_1,ν_{N_1}) and (N_3,ν_{N_3}). The idea is to simply take (W_1,μ_{W_1}) and attach (W_2,μ_{W_2}) at its end. So we get more explicitly as definition for $\tilde{W}$ that

$$(\tilde{W},\tilde{\mu}) \cap \big(M \times \{t\}\big) = (W_1,\mu_{W_1}) \cap \big(M \times \{2 \cdot t\}\big) \qquad \text{for } t \in \big[0,\frac{1}{2}\big]$$
$$(\tilde{W},\tilde{\mu}) \cap \big(M \times \{t\}\big) = (W_2,\mu_{W_2}) \cap \big(M \times \{2 \cdot t - 1\}\big) \qquad \text{for } t \in \big[\frac{1}{2},1\big].$$

We see that this gives us a cobordism between (N_1,ν_{N_1}) and (N_3,ν_{N_3}), so framed cobordism is transitive.
It follows that framed cobordism is an equivalence relation. □

We note that if N_1 and N_2 are two submanifolds of M with a different dimension, then there will not exist a submanifold of $M \times I$ connecting the two. We call $\Omega_n^{fr}(M)$ the set of all equivalence classes up to framed cobordism of compact framed n-dimensional submanifolds without the boundary of M. This turns out to be a group with the disjoint union as a product. We refer to Theorem 3.1 of Chapter 4 of Kosinski [6].

Theorem 3.7. $\Omega_n^{fr}(M)$ *is an abelian group when* $2n+1 < m$ *by having disjoint union as product.*

In this case the equivalence class of the empty manifold is the identity class and the inverse of a class $[(N,\nu)]$ is given by the class $[(N,-\nu)]$ where $-\nu$ is the framing of ν but with one basis vector flipped, which gives the same framing but then in the other orientation. We note that when the dimension of our submanifolds gets too big, then $\Omega_n^{fr}(M)$ might not even be a group.

4 The Pontryagin Construction

Now that we know what smooth homotopy groups and what framed cobordism groups are, we will look at the Pontryagin Theorem 1.1. We will prove a broader theorem, however, given by

Theorem 4.1. *Let M be a m-dimensional compact smooth submanifold without boundary of $\mathbb{R}^{m+1}$, then*

$$\Omega_{m-p}^{fr}(M) \cong [M, S^p]$$

We note that this gives the Pontryagin theorem when we take $S^n = M$. To prove this, we first need to look at the regular value theorem, taken from the lecture notes of *Manifolds* [2].

Theorem 4.2 (Regular value theorem)**.** *Suppose M and N are smooth manifolds and we have a smooth map,*

$$f : M \to N$$

Suppose $q \in N$ is a regular value of f. Then the fiber $f^{-1}(q)$ above q is an embedded submanifold of M of dimension

$$dim(f^{-1}(q)) = dim(M) - dim(N)$$

and the tangent space at $p \in f^{-1}(q)$ is given by

$$T_p\big(f^{-1}(q)\big) = ker(df_p).$$

Because $\big(T_p(f^{-1}(q))\big)^{\perp}$ is an $dim(N)$ dimensional subspace of T_pM and because $\big(T_p(f^{-1}(q))\big)^{\perp} \cap T_p(f^{-1}(q) = \{0\}$ we see that $\big(T_p(f^{-1}(q))\big)^{\perp} \oplus T_p(f^{-1}(q) = T_pM$, So $\big(T_p(f^{-1}(q))\big)$ is mapped linearly isomorphic onto T_qN by df_p.

This theorem gives us a relation between maps $f : M \to S^p$ and $(m-p)$-dimensional framed submanifolds (N, ν). Using this relation, we will define two maps. First we will define the map $N* : [M, S^p] \to \Omega_{m-p}^{fr}(M)$ which will send a map to its associated submanifold. Second we will define the map $col* : \Omega_{m-p}^{fr}(M) \to [M, S^p]$ and we will prove that these two maps are eachother inverses.

4.1 Defining N^*

For the Pontryagin theorem we want to look at maps from m-dimensional submanifolds into p-dimensional spheres. So suppose we have a smooth map $f : M \to S^p$ which maps our m-dimensional manifold into a p-dimensional sphere.
This map f induces an equivalence class $[f] \in [M, S^p]$. We will define our map $N^* : [M, S^p] \to \Omega^{fr}_{m-p}(M)$ as $N^*([f]) = N_\bullet(f)$. This new map $N_\bullet$ acts on smooth maps from M to S^p, and gives a equivalence class of framed cobordant $(m-p)$-dimensional submanifold without boundary of M. The way we will construct N^* will be based on Milnor [7] but the fact that we use maps is more in line with Davis and Kirk [3]. This equivalence class is induced by the associated Pontryagin manifold of f. To define this Pontryagin manifold we first need to introduce the pullback framing.

Definition 4.3 (pullback framing)**.** *Suppose that $f : M \to S^p$ has a regular value for a point $y \in S^p$. We choose a framing μ on $T_y S^p$. By the regular value theorem we see there is a unique vector $\nu_i \in T_x f^{-1}(y)$ such that $df_x(\nu_i(p) = \mu_i(y)$. In this way create a basis $\nu(x)$ for $T_x f^{-1}(y)$, which induces a framing ν for $f^{-1}(y)$, which we will call $f^*(\mu)$.*

Now that we have introduced the pullback framing, we need to define what it means for a basis of $T_y S^p$ to be positively oriented. To do this we first define what it means for a basis on $\mathbb{R}^n$.

Definition 4.4 (positively oriented of a basis of $\mathbb{R}^n$)**.** *We first define a linear isomorphism $\mathcal{J}$ which identifies the space of bases of $\mathbb{R}^n$ with $GL_n(\mathbb{R})$ by putting the basis vectors as columns of the matrix in the right order, so more explicitly:*

$$\mathcal{J} : base(\mathbb{R}^n) \to GL_n(\mathbb{R})$$
$$v = v_1, \cdots, v_n \mapsto \left(\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array} \right).$$

We now say that $v \in base(\mathbb{R}^n)$ is positively oriented iff $det(\mathcal{J}) > 0$.

Now that we know what it means for a basis of $\mathbb{R}^n$ to be positively oriented, we will define it for a basis of $T_y S^p$.

Definition 4.5 (positively oriented of a basis of $T_y S^p$)**.** *Say $y \in S^p$. We define the map*

$$\mathcal{I} : base\big(T_y S^p\big) \to base\big(\mathbb{R}^{p+1}\big)$$
$$\nu = \{\nu_1, \cdots \nu_p\} \mapsto \mathcal{I}(\nu) = \{y, \nu_1, \cdots, \nu_2\}.$$

Using $\mathcal{I}$ we define a basis ν of $T_y S^p$ to be positively oriented iff $\mathcal{I}(\nu)$ is positively oriented as in definition 4.4.

Now that we have introduced what it means for a basis of $T_y S^p$ to be positively oriented, we will define the associated Pontrygain manifold of a smooth map f.

Definition 4.6 (Pontryagin manifold)**.** *Suppose we have an m-dimensional manifold M and a smooth map $f : M \to S^p$. Suppose further that f has a regular value $y \in S^p$ and has a positively oriented basis μ of T_yS^p as in definition 4.5. Then we call $(f^{-1}(y), f^*(\mu))$ the associated Pontryagin manifold of f. We note that by the regular value theorem the Pontryagin manifold is an $(m-p)$-dimensional compact boundaryless framed submanifold.*

Perhaps this seems a bit odd. We are talking about the Pontryagin manifold associated to f while we, in fact, chose our regular value $y \in S^p$ and our basis for T_yS^p. We could have chosen any regular value or basis, so why is this well-defined?
This is well-defined because up to framed cobordism, all other Pontryagin manifolds associated to f are the same. We will now define the space

$$\big(M \to S^p\big) = \{\text{smooth maps from } M \text{ to } S^p\}.$$

So we will now define the map $N_\bullet$ as

$$\begin{aligned} N_\bullet : \big(M \to S^p\big) &\to \Omega^{fr}_{m-p}(M) \\ f &\mapsto N_\bullet(f) = [(f^{-1}(y), f^*(\mu)] \end{aligned}$$

where y is an arbitrarily chosen regular value of f and μ is an arbitrarily chosen basis of T_yS^p.

Theorem 4.7 (well-definedness of $N_\bullet$)**.** *Our map $N_\bullet$ is well-defined, or more explicitly: suppose $y' \in S^p$ is another regular value of f and μ' is another positively oriented basis for $T_{y'}S^p$ then $(f^{-1}(y'), f^*(\mu'))$ is framed cobordant to $(f^{-1}(y), f^*(\mu))$,*

$$(f^{-1}(y'), f^*(\mu')) \overset{\text{fr}}{\cong} (f^{-1}(y), f^*(\mu))$$

To prove this we will first prove three lemmas, where the first lemma states that the chosen basis of T_yS^p does not matter up to cobordism, the second states that the regular value does not matter locally and the third states that the map does not matter up to homotopy.

4.1.1 Choosing a basis for T_yS^p

Lemma 4.8. *Suppose $y \in S^p$ is a regular value of f and that both ν_1 and ν_2 are positively oriented bases of T_yS^p, then,*

$$(f^{-1}(y), f^*(\nu_1)) \overset{\text{fr}}{\cong} (f^{-1}(y), f^*(\nu_2))$$

The proof of this theorem is based on the proof given in Milnor [7] but with some clarifications. The idea to prove this is to prove that there is a smooth path through $base(T_yS^p)$ connecting ν and ν', and then we take this path to be

the framing of the cobordism.
To do this, we identify T_yS^p with $\mathbb{R}^p$ in the following way. Suppose ν is a positively oriented basis of T_yS^p. Then we can write an arbitrary $v \in T_yS^p$ as

$$v = \sum_i \big(\lambda_i \cdot \nu_i\big)$$

which gives the linear isomorphism L between T_yS^p and $\mathbb{R}^p$

$$\begin{aligned} L : T_yS^p &\to \mathbb{R}^p \\ v &\mapsto L(v) = \sum_i \big(\lambda_i \cdot e_i\big). \end{aligned}$$

In this way we can also describe another basis ν' of T_yS^p as a basis of $\mathbb{R}^p$

$$\nu'_j = \sum_i \big(\lambda_i^j \cdot \nu_i\big) \text{ which gives}$$

$$L(\nu') = \Big\{ \sum_i (\lambda_i^j \cdot e_i) = L(\nu'_j) \mid 0 < j \leqslant p \Big\}.$$

Because of this we will first prove that there is a smooth path between a postively oriented basis of $\mathbb{R}^p$ and e, being the standard euclidian basis of $\mathbb{R}^p$.

To do this we identify the space of bases of $\mathbb{R}^n$ with $GL_n(\mathbb{R})$ as in definition 4.4. So we now need to prove that $GL_n^+(\mathbb{R})$ is path-connected. We will use the following sublemma:

Sublemma 4.9. *Suppose $A, B \in GL_n(\mathbb{R})$ and suppose A and B are connected by a path γ. Then $det(A)$ and $det(B)$ have the same sign.*

Proof of sublemma 4.9. We will prove this by using contradiction. Suppose that $\det(A)$ and $\det(B)$ have a different sign. We know that a continuous map maps a connected set onto a connected set. Because path-connectedness induces connectedness we know that $Im(\gamma)$ is a connected subset of $GL_n(\mathbb{R})$.
Because $\det : GL_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map, we see that $\det\big(Im(\gamma)\big) = U_\gamma$ is connected as well.
On the other hand, we know that $Im(\det) = \mathbb{R}\backslash\{0\}$ so U_γ cannot be connected because $\det(\gamma(0))$ and $\det(\gamma(1))$ have a different sign. So $\gamma(0) = A$ is not connected to $\gamma(1) = B$ which is in contradiction with the assumption that A and B are path-connected. So we see that $\det(A)$ and $\det(B)$ have the same sign which was to be proven. □

So when constructing a smooth path, we see that the bases will all be positively oriented if we start with a positively oriented basis. We will prove our theorem in two steps: first, we want that every basis is smoothly connected to an orthonormal one and second we want that every orthonormal basis is smoothly connected to the standard euclidian one.

To do this we need to construct a smooth path by composition of other smooth paths. The resulting path does not need to be smooth however, so we give the following definition.

Definition 4.10 (respeeding of paths)**.** *Suppose $\gamma_1, \gamma_2 : I \to M$ are two smooth paths. By respeeding of paths we that we compose γ_1 and γ_2 before composing them to get a new smooth path $\tilde{\gamma}$. So more explicitly*

$$\tilde{\gamma} = \big(\gamma_2 \circ \tilde{\psi}\big) \bullet \big(\gamma_1 \circ \tilde{\psi}\big).$$

where $\tilde{\psi}$ is the smooth bump function from definition 2.3.

Sublemma 4.11. *Every basis is smoothly path connected to an orthonormal basis.*

proof of sublemma 4.11. So suppose that v is a basis for $\mathbb{R}^n$ then we want to have a smooth path $\alpha : I \to base(\mathbb{R}^n)$, where $base(\mathbb{R}^n)$ is the space of all bases of $\mathbb{R}^n$, such that $\alpha(0) = v$ and $\alpha(1)$ is orthonormal.

We will do this in two steps.

1. First we use the Gram-Schmidt procedure to get an orthogonal basis. So suppose we have a basis v of $\mathbb{R}^n$. We first define a_1 as

   $$a_1(t) = v_1.$$

   Then we define a_i inductively as

   $$a_i(t) = \bigg(v_i - \sum_{j=1}^{i-1}\Big(\frac{\langle v_i, a_j(1)\rangle}{\langle a_j(1), a_j(1)\rangle} a_j(1)\Big)\bigg)t + v_i(1-t).$$

   By the Gram-Schmidt procedure we see that $a(1)$ indeed gives an orthogonal basis. All a_i are clearly smooth, so a is a smooth path between v and an orthogonal basis $a(1)$.

2. Now we will normalize our basis. So suppose we have a basis $\nu = (\nu_1, ..., \nu_n)$ of $\mathbb{R}^n$. Then

   $$a_i(t) = \frac{v_i}{||v_i||}t + v_i(1-t)$$

   gives a smooth path from v_i to $\frac{v_i}{||v_i||}$. So as a whole a will be a smooth path from a basis v to a normalized basis v'.

3. The only problem we still have have is that the composition of the two paths is not smooth, but we can make it smooth by respeeding them. We

will call our orthogonalization path α_1 and our normalization path α_2. Using defintion 4.10 gives us a new smooth path α, given by

$$\begin{aligned}\alpha : I &\to base\big(\mathbb{R}^n\big)\\ t &\mapsto \alpha(t) = \big((\alpha_2 \circ \tilde{\psi}) \bullet (\alpha_1 \circ \tilde{\psi})\big)(t)\end{aligned}$$

which gives us a smooth path between a v and an orthonormal basis $\alpha(1) = \nu'$, which was to be proven.

□

By both sublemmas, we already see that every positively oriented basis is smoothly path-connected to a positively oriented orthonormal basis. We define the space $SO_n(\mathbb{R})$ as

$$SO_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) \text{ such that } A \cdot A^t = I \text{ and } \det(A) = 1\}$$

which represents the positively oriented orthonormal bases. So we want to prove the following sublemma:

Sublemma 4.12. *For every positively oriented orthonormal basis, there is a smooth path such that it is connected to the standard basis.*

proof of sublemma 4.12. So suppose that v is an orthonormal basis of $\mathbb{R}^n$ and e is the standard basis.
We will prove this in two parts. First, we construct a smooth path between v and a v rotated in such a way that the new v_1 is the same as e_1. In the second part, we define the total smooth path by rotating the other v_i into place as well. We do this by using an n-dimensional rotationmatrix. To construct this, we first define R_2, which is the rotationmatrix for two dimensions, so rotating through the xy-plane

$$R_2(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

We now define this matrix for n dimensions

$$R_n(\theta) = \left(\begin{array}{c|c} R_2(\theta) & 0 \\ \hline 0 & \mathbb{I}^{n-2} \end{array}\right)$$

which is the rotation through the x_1x_2-plane. We want to rotate through all x_1x_i-planes, with $i \neq 1$, where x_i is the i^{th} axis. Because when we do rotate through all these planes our original vector v_1 will be lined up with e_1.
To define these rotations we first define the linear ismorphisms that define swapping x_i with x_2

$$\Phi_{x_i} = \left(\begin{array}{c|c|c|c} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & 0 \\ \hline 0 & \mathbb{I}^{i-3} & 0 & 0 \\ \hline \begin{pmatrix} 0 & 1 \end{pmatrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbb{I}^{n-i} \end{array}\right)$$

. Using this, the rotationmatrix for the rotation through the x_1x_i is given by

$$R_n^{x_1x_i}(\theta) = \big(\Phi_{x_i}\big)^{-1} \cdot R_n(\theta) \cdot \Phi_{x_i}.$$

The way we use this is by projecting the vector v_1 onto the the plane we want to rotate over and to determine the angle θ it has with respect to e_1. We then create a smooth path γ_{x_i} inductively by

$$\begin{aligned} \gamma_{x_i} &: I \to \mathbb{R}^n \\ & t \mapsto \gamma_{x_i}(t) = R_n^{x_1x_i}(t \cdot \theta) \cdot \big(\gamma_{x_{i+1}}(1)\big) \end{aligned}$$

where we define $\gamma_{x_{n+1}}(1) = v_1$ and $1 < i \leqslant n$. Now we have only rotated v_1, however we want to rotate our basis v. We define our general path inductively by

$$\begin{aligned} \gamma_{x_i}^{v_j} &: I \to \mathbb{R}^n \\ & t \mapsto \gamma_{x_i}^{v_j}(t) = R_n^{x_1x_i}(t \cdot \theta_{\gamma_{x_{i+1}}}) \cdot \big(\gamma_{x_{i+1}}^{v_j}(1)\big) \end{aligned}$$

where we again define $\gamma_{x_{n+1}}^{v_j}(1) = v_j$ and $1 < i \leqslant n$. These paths induce new smooth paths

$$\begin{aligned} \Gamma_{x_i} &: I \to base\big(\mathbb{R}^n\big) \\ & t \mapsto \Gamma_{x_i}(t) = \{\gamma_{x_i}^{v_j}(t) \mid 1 \leqslant j \leqslant n\}. \end{aligned}$$

We see that $\gamma_{x_i}^{v_j}(0) = \gamma_{x_i+1}^{v_j}(1)$ for all $1 \leqslant j \leqslant n$ so $\Gamma_{x_i}(0) = \Gamma_{x_i+1}(1)$. We compose these smooth paths while respeeding as in defintion 4.10 and call the composition Γ_{v_1}, so more explicitly

$$\begin{aligned} \Gamma_{v_1} &: I \to base\big(\mathbb{R}^n\big) \\ & t \mapsto \Gamma_{v_1}(t) = \Big(\big(\Gamma_{x_2} \circ \tilde{\psi}\big) \bullet \cdots \bullet \big(\Gamma_{x_n} \circ \tilde{\psi}\big)\Big)(t). \end{aligned}$$

We now have a new orthonormal basis $\Gamma_{v_1}(1) = v'$ where the last $n-1$ vectors give a basis for a hyperplane spanned by $\{x_2, \cdots x_n\}$ which is linearly isomorphic to $\mathbb{R}^{n-1}$ by forgetting the first coordinate. This also induces a new basis by the map

$$\begin{aligned} \Lambda : \Big\{v \in base\big(\mathbb{R}^n\big) \mid v_1 = e_1\Big\} &\to base\big(\mathbb{R}^{n-1}\big) \\ v &\mapsto \Lambda(v) = \{pr(v_2), \cdots, pr(v_n)\} \end{aligned}$$

We can now want to do the same procedure to get a smooth path $\Gamma_{v_2} : I \to base\big(\mathbb{R}^{n-1}\big)$ and in the same way we inductively make smooth paths Γ_{v_i}.
To do this we take the dimension n as parameter k in our previous formulas and our starting basis v as well, so that we get a smooth path $\Gamma_{v_1}^k : I \times base\big(\mathbb{R}^k\big) \to$

$base\big(\mathbb{R}^k\big)$ which rotates a k-dimensional basis so that v_1 is aligned with e_1. We now want to define smooth paths $\tilde{\Gamma}$ inductively for a fixed v by

$$\begin{aligned} \tilde{\Gamma}_{v_i} &: I \to base\big(\mathbb{R}^{n-i+1}\big) \\ t &\mapsto \tilde{\Gamma}_{v_i}(t) = \Gamma^{n-i+1}_{v_1}(\Lambda^{n-i+2}(\tilde{\Gamma}_{v_{i-1}}(1)), t) \end{aligned}$$

where $1 \leqslant i \leqslant n-1$ and we define $\Lambda^{n+1}(\tilde{\Gamma}_{v_0}) = v$. These are not paths through $base\big(\mathbb{R}^n\big)$, so we define a map that will send them there by

$$\begin{aligned} \Xi^k_n &: base\big(\mathbb{R}^k\big) \to base\big(\mathbb{R}^n\big) \\ v = \{v_1, \cdots, v_k\} &\mapsto \Xi^k(v) = \{e_1, \cdots, e_{n-k-1}, v_1, \cdots, v_k\}. \end{aligned}$$

Now that we have done that, we will define our total path Γ between v and e, where we take the composition while respeeding the parts as in definition 4.10

$$\begin{aligned} \Gamma &: I \to base\big(\mathbb{R}^n)\big) \\ t &\mapsto \Gamma(t) = ((\Xi^2 \circ \tilde{\Gamma}_{n-1}) \bullet \cdots \bullet (\Xi^{n-(i-1)}_n \circ \tilde{\Gamma}_{v_i}) \bullet \cdots \bullet \tilde{\Gamma}_1)(t). \end{aligned}$$

When note this path only aligns the first $n-1$ vectors of v with e, however, we see that there are only two options for v_n which is inside the line spanned by e_n. It is either e_n or $-e_n$. However, because our basis is positively oriented we see that $v_n = e_n$ by sublemma 4.9. So we see that Γ gives us a smooth path between v and e, which was to be proven. $\square$

Now that we have proven this, we are ready to prove lemma 4.8.

Proof of lemma 4.8. Suppose we have an arbitrary positively oriented framing ν_0 of $\mathbb{R}^p$. Using sublemma 4.11 we see that there is a smooth path γ_1 from ν_0 to a positively oriented orthonormal basis ν_1. Using sublemma 4.12 we see that ν_1 is smoothly path connected to the standard basis e by a smooth path γ_2. We create a smooth path $\tilde{\gamma}$ by composing γ_1 and γ_2 as in definition 4.10.

$$\begin{tikzcd}
SO_n(\mathbb{R}) \arrow[dr, phantom, "\ni"] & & \\
& \nu_1 \arrow[r, "\gamma_2"] & e \\
& \nu_0 \arrow[u, "\gamma_1"] \arrow[ur, "\tilde{\gamma}"'] & \\
GL_n^+(\mathbb{R}) \arrow[ur, phantom, "\ni"] & &
\end{tikzcd}$$

Figure 1: diagram for the proof of lemma 4.8

We will now use this path $\tilde{\gamma}$ to construct a cobordism between $(f^{-1}(y), f^*(\nu)$ and $(f^{-1}(y), f^*(\nu')$. We see that there is a smooth path $\tilde{\gamma}$ between $T\big(\nu'\big)$ and $T\big(\nu\big) = e$. So $T^{-1} \circ \tilde{\gamma}$ gives a smooth path between ν' and ν.
We first respeed this new path, so that all the change happens on $[\epsilon, 1-\epsilon]$, so we construct

$$\begin{aligned} \gamma = T^{-1} \circ \tilde{\gamma} \circ \tilde{\psi} : I &\to base\{T_y S^p\} \\ t &\mapsto \gamma(t). \end{aligned}$$

We now see that $W = (f^{-1}(y) \times I, f^*(\gamma))$ is a framed cobordism between $(f^{-1}(y), f^*(\nu))$ and $(f^{-1}(y), f^*(\nu')$ where we define $f^*(\gamma)$ as

$$\begin{aligned} f^*(\gamma) : f^{-1}(y) \times I \ni (x,t) &\to base\Big\{\big(T_{(x,t)}(f^{-1}(y) \times I)\big)^{\perp}\Big\} \\ (x,t) &\mapsto \big(f^*(\gamma)\big)(x,t) = f^*(\gamma_t)(x). \end{aligned}$$

So we see that $\big(f^{-1}(y), f^*(\nu)\big)$ is framed cobordant to $\big(f^{-1}(y), f^*(\nu')\big)$, which is what we wanted to prove. $\square$

$$\begin{tikzcd}
GL_p^+(\mathbb{R}) \arrow[d, phantom, "\rotatebox{-45}{$\ni$}"] & & \\
T(\nu') \arrow[rr, "\tilde{\gamma}"] & & e \\
\nu' \arrow[u, "T"] \arrow[rr, "T^{-1}(\tilde{\gamma})"'] \arrow[d, phantom, "\rotatebox{90}{$\ni$}"] & & \nu \arrow[u, "T"'] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
base(T_y S^p) & & base(T_y S^p)
\end{tikzcd}$$

Figure 2: diagram for the proof of lemma 4.8

Now that we have proven this we will not write down the framing of the Pontryagin manifold anymore because the chosen positively oriented basis of $T_y S^p$ does not matter for the cobordism class of the Pontryagin manifold. We note however that the framing of $f^{-1}(y)$ is still important, but the importance comes from the map f. So from now on we will speak of a Pontryagin manifold $f^{-1}(y)$ where f is a smooth map and y a regular value.

4.1.2 Choosing a regular value locally

Our next step in proving that our map $N^\bullet$ is well-defined is to prove that our choice of regular value locally does not matter for the framed cobordism class.

Lemma 4.13. *Suppose $y \in S^p$ is a regular value of f, then there exists an open $U \ni y$ such that for every $x \in U$*

$$f^{-1}(y) \overset{\text{fr}}{\cong} f^{-1}(x)$$

Our approach for this proof will be based on the proof given in Milnor [7] but with some clarifications. To prove this lemma we first want to have an open around y such that it only consists of regular values. So we want to prove that

$$R_f := \{x \in S^p \mid x \text{ being a regular value of } f\}$$

is open in S^p. Our approach will be to prove that its complement $f(C)$ where C are the critical points of f is closed in S^p. To do this we want to prove the following sublemma.

Sublemma 4.14. *The set of all critical points C for a smooth map $f : M \to N$ with M compact is closed.*

Because M is a compact manifold it is useful to first prove this for maps between euclidean spaces and to use a finite subcover of charts to prove it for the general case.

Sublemma 4.15. *The set of all critical points C for a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.*

We note that the critical points of a smooth map f are the points p such that the matrix representation of df_p does not have maximal rank. So we see that if $m = n$ then $\det(df_p) = 0$ and because det is a continuous map, we see that C would be indeed closed. However when $m \neq n$ we do not have a map like det right away. So we would like to create a square matrix that has the same rank as df_p and create a new continuous map that takes the determinant of this matrix. This map is given by

$$\begin{aligned} h : M &\to \mathbb{R} \\ p &\mapsto \det(df_p \cdot df_p^t) \end{aligned}$$

where df_p^t is the transposed of the matrix representation of df_p. So we now want to prove that $df_p \cdot df_p^t$ has the same rank as df_p. We state the following sublemma.

Sublemma 4.16. *If $L \in M_{m\times n}$ then*

$$rank(L^t \cdot L) = rank(L)$$

To prove this we first note that the null space of a linear map $L \in M_{m\times n}$ is defined as

$$\mathcal{N}(L) = \{v \in \mathbb{R}^n \text{ such that } L(v) = 0 \in \mathbb{R}^m\}.$$

We note that if $L \in M_{m\times n}$ then $L^t \cdot L \in M_{n\times n}$.

Proof of sublemma 4.16. Suppose $x \in \mathcal{N}(L)$, then we see that $(L^t \cdot L)(x) = L^t(L(x)) = 0 \in \mathbb{R}^n$ because $L(x) = 0 \in \mathbb{R}^m$. So we see that $x \in \mathcal{N}(L^t \cdot L)$.

We now suppose that $x \in \mathcal{N}(L^t \cdot L)$. By taking the inner product on $\mathbb{R}^n$ with x with see that

$$\langle x, L^t \cdot L(x) \rangle = \langle L(x), L(x) \rangle = 0 \in \mathbb{R}$$

because $L^t \cdot L(x) = 0$. Because $\langle v, v \rangle = 0$ implies that $v = 0$ we see that $L(x) = 0$, so $x \in \mathcal{N}(L)$.
We have now proven that $\mathcal{N}(L) = \mathcal{N}(L^t \cdot L)$ so we see that $dim(\mathcal{N}(L)) = dim(\mathcal{N}(L^t \cdot L))$ as well. We now use the rank nullity theorem, which sates that $n = dim(\mathcal{N}(L)) + rank(L)$ for every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$.
Using this for L and $L^t \cdot L$ we see that

$$rank(L) = n - dim(\mathcal{N}(L)) = n - dim(\mathcal{N}(L^t \cdot L)) = rank(L^t \cdot L)$$

which was to be proven. □

Proof of sublemma 4.15. Using this sublemma we see that if df_p does not have maximal rank, then $df_p \cdot df_p^t$ has neither, so if and only if p is a critical value of f, then $h(p) = 0$. Because h is a continuous map and $\{0\}$ is closed we see that $h^{-1}(\{0\}) = C$ is closed as well. □

We see that this works in the same way if the domain is an open of $\mathbb{R}^m$, because we could just have taken an open $\Omega \subset \mathbb{R}^m$ for the domain of the map h. We will now prove this statement for arbitrary compact manifolds.

Proof of sublemma 4.14. We know that $f : M \to N$ is a submersion at p iff $f_{\chi,\chi'} = \chi' \circ f \circ \chi^{-1}$ is a submersion around $\chi(p)$ for every chart χ around p and χ' around $f(p)$. We also know that we only have to check this condition for a single chart χ of M around p and a single chart χ' of N around $f(p)$.
Because M is compact we can look at a finite cover of charts χ_i with the domain U_i. We will write the set of critical points of $f|_{U_i}$ as C_{U_i}. We see that C_{U_i} gets mapped by χ onto the critical points of the map $f_{\chi,\chi'}$, which is closed in $\chi(U_i)$.
Because χ^{-1} is a homeomorphism, which takes closed subsets to closed subsets, we see that C_{U_i} is closed as well.
We now note that C_{U_i} is the same as $C \cap U_i$, so we see that

$$C = \bigcup_i (C \cap U_i).$$

We note however that $C \cap U_i$ being closed in U_i for all i does not immediately imply that C is a closed subset of M.
To prove this we look at C^c, which we want to prove to be open in M. We see that

$$C^c \cap \bigcup_i (U_i) = \bigcup_i (U_i \cap C^c)$$

because taking an intersection with a finite amount of unions is distributive. We note now that $U_i \cap C^c$ is the complement of $U_i \cap C$, so it is an open in U_i.

Because of our definition of the subspace topology we see that there is an open $V \subset M$ such that $U_i \cap V = U_i \cap C^c$, and because the intersection of two opens is an open we see that $U_i \cap C^c$ is an open in M. Because a finite union of opens is open, we get that $C^c \cap \bigcup_i \big(U_i\big) = C^c$ is an open in M. Because of this, we see that C is indeed closed in M, which was to be proven. $\square$

Because C is closed in M and M being compact, we see that it is compact in M. Because f is a continuous map from a compact space to a Hausdorff space we see that $f(C)$ is compact in S^p and because S^p is compact, $f(C)$ is closed. So $R_f = \big(f(C)\big)^c$ is indeed open in S^p.

Now that we have proven this we can choose an open ball $B_y \in \mathbb{R}^{p+1}$ around y such that $B_y \bigcap S^p = U_y \subset R_f$.

We want this open U_y because if we would rotate from one point in U_y to another point in U_y, we would not leave it. So this ensures that it will rotate through regular values of f.

proof of lemma 4.13. Suppose $z \in U_y$ different from y, then we can rotate from y to z in a straight line, which we call $r : S^p \times I \to S^p$.

To do this we first note that S^p is diffeomorphic to S^p by a map Φ with $\Phi(y) = e_1$ and where $\Phi(z)$ lays in the x_1x_2-plane, by rotating S^p into place. We now want to construct a smooth map $r : S^p \times I \to S^p$ which rotates the sphere so that e_1 is rotated onto $\Phi(y)$ by

$$\begin{aligned} r : S^p \times I &\to S^p \\ (x,t) &\mapsto r(x,t) = R^p_{x_1,x_2}(t \cdot \theta) \cdot x \end{aligned}$$

where θ is the angle between e_1 and $\Phi(z)$. Here we view S^p as a submanifold of $\mathbb{R}^{p+1}$.
This induces a map $\tilde{r}$ defined as

$$\begin{aligned} \tilde{r} : S^p \times I &\to S^p \\ (x,t) &\mapsto \tilde{r}(x,t) = \big(\Phi^{-1} \circ r \circ (\Phi \times Id_I)\big)(x,t). \end{aligned}$$

We note that $\tilde{r}(x,0) = \big(\Phi^{-1} \circ r \circ (\Phi \times Id_I)\big)(x,0) = \Phi^{-1} \circ r(\Phi(x),0) = \Phi^{-1}(\Phi(x)) = x$ so $\tilde{r}_0 = Id_{S^p}$. We also see that $\tilde{r}(y,1) = \big(\Phi^{-1} \circ r \circ (\Phi \times Id_I)\big)(y,1) = \Phi^{-1} \circ r(e_1,1) = \Phi^{-1}(\Phi(z)) = z$. We also note that, by how we have chosen our open U_y that $\tilde{r}(y,t) \in U_y$ for all $t \in I$, so $\tilde{r}(y,t) \in R_f$ for all t.
We now define the homotopy F as

$$\begin{aligned} F : M \times I &\to S^p \\ (x,t) &\mapsto F(x,t) = \tilde{r}_t \circ f(x) \end{aligned}$$

This is a smooth homotopy between f and $\tilde{r}_1 \circ f$. For each $t \in I$ we see that z is a regular value of $\tilde{r}_t \circ f : M \to S^p$, so for each $t \in I$ we get that $\tilde{r}_t \circ f$ is a submersion for all $x \in (\tilde{r}_t \circ f)^{-1}(z)$. Because of this we see that z is a regular value of F. By lemma 4 of Chapter 2 of Milnor [7] we see that $F^{-1}(z)$ only has boundaries at $t = 0$ and $t = 1$, so $\partial F^{-1}(z) = f^{-1}(z) \times \{0\} \cup (\tilde{r}_1 \circ f)^{-1}(z) \times \{1\}$. So $F^{-1}(z)$ is a framed cobordism between $f^{-1}(z)$ and $\big(\tilde{r}_1 \circ f\big)^{-1}(z) = \big(f^{-1} \circ (\tilde{r}_1)^{-1}\big)(z) = f^{-1}(y)$. So

$$f^{-1}(y) \overset{\text{fr}}{\cong} f^{-1}(z)$$

for all $z \in U_y$, which was to be proven. □

4.1.3 Two homotopic maps

Lemma 4.17. *Suppose $f, g : M \to S^p$ are smoothly homotopic and they both have y as a regular value, then*

$$f^{-1}(y) \overset{\text{fr}}{\cong} g^{-1}(y)$$

Our proof will be based on the proof in Milnor [7] but given with more explanation. Suppose f and g are smoothly homotopic by the smooth homotopy F. We want a smooth homotopy $\tilde{F}$ such that

$$\begin{aligned} \tilde{F}(x,t) &= f(x) \text{ for } t \in [0,\epsilon) \text{ and} \\ \tilde{F}(x,t) &= g(x) \text{ for } t \in (1-\epsilon, 1]. \end{aligned}$$

We create this by respeeding our original F in the following way.

Definition 4.18 (respeeding a homotopy)**.** *Suppose we have a smooth map $F : M \times I \to S^p$, then by respeeding F we mean that we compose F with the bump function $\tilde{\psi}$ of definition 2.3 so that the homotopy only happens between $[\epsilon, 1-\epsilon]$, or more explicitly*

$$\tilde{F} = F \circ (Id_M, \tilde{\psi}).$$

We now want to choose a regular value for $\tilde{F}$ that is close to y. Note that y does not have to be a regular value for F.

Theorem 4.19 (Brown)**.** *Suppose f is a smooth map from M to N, then the set of regular values is everywhere dense in N.*

For a proof we refer to Chapter 2 of Milnor [7].

Proof lemma 4.17. By theorem 4.19 we see that for every $y \in S^p$ and for every open $U_y \ni y$ there is a regular value $z \in R_{\tilde{F}}$ such that $z \in U_y$, because otherwise $S^p \backslash U_y$ would be closed and thus $\overline{R}_{\tilde{F}} \subset S^p \backslash U_y \neq S^p$.

We now take an open U_f and U_g around y like the open U_y we used in lemma 4.13 and we make a new open $U_{fg} = U_f \cap U_g$ around y. By our remark above we see that there is a $z \in U_{fg}$, being a regular value of $\tilde{F}$. We also note that $z \in R_f$ and $z \in R_g$, so it is also a regular value for both f and g.

We get that $\tilde{F}^{-1}(z) \cap \big(M \times [0, \epsilon)\big) = \tilde{F}|^{-1}_{M \times [0,\epsilon)}(z) = f^{-1}(z) \times [0, \epsilon)$ and $\tilde{F}^{-1}(z) \cap \big(M \times (1-\epsilon, 1]\big) = \tilde{F}|^{-1}_{M \times (1-\epsilon,1]}(z) = g^{-1}(z) \times (1-\epsilon, 1]$. By lemma 4 of Chapter 2 of Milnor [7] we see that $\tilde{F}^{-1}(z)$ only has boundaries at $t = 0$ and $t = 1$, so $\partial \tilde{F}^{-1}(z) = f^{-1}(z) \times \{0\} \cup g^{-1}(z) \times \{1\}$. Because of this we see that $F^{-1}(z)$ is a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$. By lemma 4.13 we have that $f^{-1}(y) \overset{\text{fr}}{\cong} f^{-1}(z)$ and $g^{-1}(y) \overset{\text{fr}}{\cong} g^{-1}(z)$. So because being framed cobordant is an equivalence relation we see that $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$. $\square$

$$\begin{tikzcd}
f^{-1}(y) \arrow[r, phantom, "\overset{\text{fr}}{\cong}"] & f^{-1}(z) \arrow[d, "\tilde{F}^{-1}(z)"] \\
g^{-1}(y) \arrow[r, phantom, "\overset{\text{fr}}{\cong}"] & g^{-1}(z)
\end{tikzcd}$$

Figure 3: diagram for the proof of lemma 4.17

Proof of theorem 4.7. Now that we have proven all three lemmas, we are ready to prove $N_\bullet$ is well-defined. Suppose that y and z are two regular values of f. Then we can make a rotation $r_t : S^p \times I \to S^p$ such that

$$r_0 = id_{S^p} \text{ and } r_1(y) = z$$

Using this we see that we see that $r_0 \circ f = f$ is smoothly homotopic to $r_1 \circ f$ by $r_t \circ f : M \times I \to S^p$, so using lemma 4.17 we get that

$$\big(r_0 \circ f\big)^{-1}(z) = f^{-1}(z) \overset{\text{fr}}{\cong} \big(r_1 \circ f\big)^{-1}(z) = f^{-1} \circ \big(r_1\big)^{-1}(z) = f^{-1}(y).$$

So $f^{-1}(z) \overset{\text{fr}}{\cong} f^{-1}(y)$, which was to be proven. $\square$

We have now proven that $N_\bullet$ is well-defined. However, we still need to check that N^* is well-defined.

Theorem 4.20 (well-definedness of N^*)**.** *The map* $N^* : [M, S^p] \to \Omega^{fr}_{m-p}(M)$ *is well-defined, or in other words: suppose* $[f], [g] \in [M, S^p]$ *and* $[f] = [g]$, *then* $N^*\big([f]\big) = N^*\big([g]\big)$.

Proof of theorem 4.20. We first note that $[f] = [g]$ means that f is smoothly homotopic to g. We see that $N^*([f]) = N_\bullet(f) = [N_f]$ and $N^*([g]) = N_\bullet(g) = [N_g]$. We know that $N_\bullet$ is a well-defined map so we can take representatives

$f^{-1}(x)$ of $[N_f]$ and $g^{-1}(x)$ of $[N_g]$, where x is a regular value for both f and g.

To prove that such an x exists we use theorem 4.19. We first take $\tilde{x} \in S^p$ as an arbitrary regular value of g. In the same way as the first part of the proof of lemma 4.17 we see that f has a regular value in any open around $\tilde{x}$. So we take U_g around $\tilde{x}$, which is the same open as U_y used in lemma 4.13. We see that there is a $x \in U_g$ around $\tilde{x}$ and that $x \in R_f$, so x is a regular value for both.

By lemma 4.17 we see that $f^{-1}(x)$ is framed cobordant to $g^{-1}(x)$ because f is homotopic to g, so $N([f]) = N([g])$, which was to be proven. $\square$

$$\begin{tikzcd}
 & {[f]=[g]} & \\
f \arrow[ur, phantom, "\in" sloped] \arrow[d, "N_\bullet"'] & & g \arrow[ul, phantom, "\ni" sloped] \arrow[d, "N_\bullet"] \\
{[N_f]} & & {[N_g]} \\
f^{-1}(x) \arrow[u, phantom, "\in" sloped] & \overset{\text{fr}}{\cong} & g^{-1}(x) \arrow[u, phantom, "\in" sloped]
\end{tikzcd}$$

Figure 4: diagram for the proof of theorem 4.20

4.2 Defining col^*

Now that we have proven that N^* is a well-defined map, we will construct its inverse, col^*. This map will take a class $[(N,\nu)] \in \Omega_{m-p}^{fr}(M)$ of which (N,ν) is a representative and map it onto a class $[f_{(N,\nu)}] \in [M, S^p]$. We will define it as $col^*([N,\nu]) = col(N,\nu)$ where col acts on framed closed submanifolds of M and sends them to the class of their associated map $f_{(N,\nu)}$.
So we want to have a method to construct a map from M to S^p out of a submanifold, which is given by the tubular neighbourhood theorem.

4.2.1 The tubular neighbourhood theorem

Theorem 4.21 (tubular neighbourhood theorem). *For a framed closed submanifold $\big(N,\nu\big)$ of M there is an open V_N of M around N such that V_N is diffeomorphic to $N \times \mathbb{R}^p$ where N corresponds to $N \times \{0\}$ and where each normal frame $\nu(x)$ corresponds to the standard basis of $\mathbb{R}^p$ for every $x \in N$.*

The way we will prove this will be based on the prove given in Milnor [7] but discussed on more detail. To prove this theorem we first want to prove it for M being $\mathbb{R}^m$ because in this situation the ambient tangent space is more intuitive. Suppose we have a n-dimensional framed closed submanifold (N,ν) of $\mathbb{R}^m$ and that $p = m - n$. The map that will be our diffeomorphism between V_n and $N \times \mathbb{R}^p$ will be called ψ and is constructed as

$$\begin{aligned} \psi : N \times \mathbb{R}^p &\to \mathbb{R}^m \\ (x,t) &\mapsto x + t_1 \cdot \nu_1(x) + ... + t_p \cdot \nu_p(x). \end{aligned}$$

We see that this map takes straight lines $(x, \lambda e_i)$ to $x + \lambda \nu_i$. We now give the following sublemma.

Sublemma 4.22. *There is an open $N \times B_\delta^p \subset N \times \mathbb{R}^p$ around $N \times \{0\}$ such that ψ is a local diffeomorphism on it onto its image $\psi(N \times B_\delta^p)$, which is an open around N.*

Proof of sublemma 4.22. To prove this we want to create opens that cover $N \times \{0\}$ such that every open is a diffeomorphically mapped by ψ onto its image. Afterwards, we will create the open $N \times B_\delta^p \subset N \times \mathbb{R}^p$, which will lay inside the union of all the opens.
We will use the inverse function theorem to find these opens. So we want to prove that $d\psi_{(x,0)}$ is an automorphism for every $x \in N$. To see this is the case we first take a chart χ mapping U_x, being an open around $x \in \mathbb{R}^m$, onto $\Omega_{\chi(x)}$ such that $U_x \cap N$ is mapped onto $\Omega_{\chi(x)} \cap (\mathbb{R}^n \times \{0^p\}) \subset \mathbb{R}^m$. We will write χ_N as the chart this induces on N by mapping $U_x \cap N$ onto $\Omega_{\chi_N(x)} \subset \mathbb{R}^n$, where we view $U_x \cap N$ as an open of N. We now take the product chart (χ_N, χ_{Id}) on $N \times \mathbb{R}^p$ which maps the open $\big((U_x \cap N) \times V\big)$ onto $(\Omega_{\chi_N(x)} \times V)$ being an open

of $\mathbb{R}^m$. By looking at ψ under these charts, we see we can write it as

$\psi_{(\chi_N,\chi_{Id}),\chi} = \chi \circ \psi \circ (\chi_N, \chi_{Id})^{-1}$ with
$\psi_{(\chi_N,\chi_{Id}),\chi}(t_1, \cdots, t_n, t_{n+1} \cdots t_{n+p}) =$
$t_1 + \cdots + t_n + t_{n+1}(\nu_1)_{(\chi_N,\chi_{Id}),\chi}(t_1, \cdots, t_n) + \cdots + t_{n+p}(\nu_p)_{(\chi_N,\chi_{Id}),\chi}(t_1, \cdots, t_n)$.

We see that the normal frame induces a linear transformation from $\mathbb{R}^p$ onto itself so the differential of $\psi_{(\chi_N,\chi_I d),\chi}$ at $(t_1, \cdots, t_n, 0, \cdots, 0)$ is given by

$$(d\psi_{(\chi_N,\chi_I d),\chi})_{(t_1,\cdots,t_n,0,\cdots,0)} = \left(\begin{array}{c|c} \mathbb{I}_n & 0 \\ \hline 0 & \Psi_p(t_1, \cdots, t_n) \end{array}\right)$$

where Ψ_p is the linear transformation corresponding to the normal frame.
We now see that

$$\det((d\psi_{(\chi_N,\chi_I d),\chi})_{(t_1,\cdots,t_n,0,\cdots,0)}) = \det(\mathbb{I}_n) \cdot \det(\Psi_p) \neq 0.$$

So $(d\psi_{(\chi_N,\chi_I d),\chi})_{(t_1,\cdots,t_n,0,\cdots,0)}$ is indeed an automorphism, because of which $d\psi_{(x,0)}$ is as well.
So using the inverse function theorem, we see that there is an open $U_{(x,0)}$ around every point $(x, 0) \in N \times \mathbb{R}^p$ such that ψ maps it diffeomorphically to an open $\psi(U_{(x,0)}) \subset \mathbb{R}^m$.

We now want to prove that there is an open $N \times B^p_\delta$ inside the union of all these opens, because then ψ is a local diffeomorphism on it. We will give a prove by contradiction that such an open $N \times B^p_\delta$ exists.
Suppose that there is no $\delta > 0$ such that $N \times B^p_\delta$ is in the union of all the opens, then we see that for every $\delta > 0$ there has to be a $(x_1, y_1) \in N \times B^p_\delta$ such that

$$(x_1, y_1) \notin \bigcup_{x \in N} U_{(x,0)}$$

because otherwise for a certain δ all points of $N \times B^p_\delta$ would be in the union of the opens.
We now construct a sequence out of these points by

$$\alpha_n = (x_1, y_1) \text{ with } (x_1, y_1) \in N \times B^p{}_{\frac{1}{n+1}} \text{ such that}$$
$$(x_1, y_1) \notin \bigcup_{x \in N} U_{(x,0)}$$

Because $\overline{B}^p_1$ and N are both compact we see that $N \times \overline{B}^p_1$ is a compact subset of $\mathbb{R}^m \times \mathbb{R}^p = \mathbb{R}^{m+p}$. Using Bolzano Weierstrass we see that α_n has a converging subsequence, which we will call β_n. We see that $\beta_n = (x_1, y_1) \to (\tilde{x}, 0)$ when $n \to \infty$. Because $(\tilde{x}, 0)$ is a point in $N \times \{0\}$ we see that we have an open $U_{(\tilde{x},0)}$ around it, which is part of the union of opens. Because β_n converges we have an $\tilde{n} \in \mathbb{N}$ such that for every $n > \tilde{n}$ we have that $\beta_n \in U_{(\tilde{x},0)}$, which is in contradiction with the way we constructed β_n. So we see that there is a $\delta > 0$ such that

$$N \times B^p_\delta \subset \bigcup_{x \in N} U_{(x,0)}$$

which proves that there is a $N \times B^p_\delta$ such that ψ is a local diffeomorphism on it onto its image. $\square$

Sublemma 4.23. *There is an open $N \times B^p_\epsilon \subset N \times \mathbb{R}^p$ around $N \times \{0\}$ such that ψ maps it diffeomorphically onto its image $\psi(N \times B^p_\epsilon)$, which is an open around N.*

Proof of sublemma 4.23. Using sublemma 4.22 we see that we already have an open $N \times B^p_\delta \subset N \times \mathbb{R}^p$ such that ψ maps it local-diffeomorphically onto its image. So we still need ψ to be injective on $N \times B^p_\delta \subset N \times \mathbb{R}^p$. We will prove that it is by contradiction.
So suppose that there is no open $N \times B^p_\delta \subset N \times \mathbb{R}^p$ such that ψ is injective on it. Then there are

$$\begin{aligned}
&(x_1, p_1), (x_2, p_2) \in N \times B^p_\delta \text{ such that} \\
&(x_1, p_1) \neq (x_2, p_2) \text{ and} \\
&\psi(x_1, p_1) = \psi(x_2, p_2) \text{ for all } \delta > 0
\end{aligned}$$

because otherwise there would exist an $N \times B^p_\delta$ such that ψ is injective on it, where we could just take $\epsilon = \delta$ as our open $N \times B^p_\epsilon$. So we make a sequence

$$\begin{aligned}
&\alpha_n = ((x_1, p_1), (x_2, p_2)) \text{ with} \\
&(x_1, p_1), (x_2, p_1) \in N \times B^p_{\frac{1}{n+1}} \text{ such that} \\
&(x_1, p_1) \neq (x_2, p_2) \text{ and } \psi(x_1, p_1) = \psi(x_2, p_2).
\end{aligned}$$

Because N is compact, we see that $N \times \overline{B}^p_1$ is a compact subset of $\mathbb{R}^m \times \mathbb{R}^p = \mathbb{R}^{m+p}$. By Bolzano-Weierstrass we see that α_n in $N \times \overline{B}^p_1$ has a converging subsequence β_n. We see that $\beta_n = ((x_1, p_1)(x_2, p_2))_n \to ((\tilde{x}_1, 0), (\tilde{x}_2, 0))$ when $n \to \infty$ and because ψ is injective on $N \times \{0\}$ we see that $\tilde{x}_1 = \tilde{x}_2$. By the definition of convergence we see that for every open $U_{(\tilde{x},0)}$ there is an $\tilde{n} \in \mathbb{N}$ such that for every $n > \tilde{n}$ both points $pr_1 \circ \beta_n = (x_1, p_1)_n$ and $pr_2 \circ \beta_n = (x_2, p_2)$ are in $U_{(\tilde{x},0)}$. However this means that there cannot exist opens $U_{(x,0)}$ as we have proven to exist in the proof of sublemma 4.22, so this gives a contradiction.
We conclude that there has to be an open $N \times B^p_\epsilon \subset N \times \mathbb{R}^p$ around $N \times \{0\}$ such that ψ maps it diffeomorphically onto its image $\psi(N \times B^p_\epsilon)$, which is an open around N. $\square$

Now that we have proven these two sublemmas, we will prove the tubular neighbourhood theorem for our special case.

Proof of theorem 4.21 when $M = \mathbb{R}^m$. Using sublemma 4.23 we see that we already have an open tube around $N \times \{0\}$ such that ψ is a diffeomorphism on it. We will now prove that this open is diffeomorphic to $N \times \mathbb{R}^p$ with the properties as described. We will first show that our open $N \times B^p_\epsilon \subset N \times \mathbb{R}^p$ already has those properties.
We see that $\psi(N \times \{0\}) = N$ so N corresponds to $N \times \{0\}$.
To see that each normal frame $\nu(x)$ corresponds to the standard basis e of $\mathbb{R}^p$, we will look at the speeds of smooth paths. We note that when looking at the path

$$\begin{aligned}\gamma_x : [-1, 1] &\to N \times \mathbb{R}^p \\ t &\mapsto \gamma_x(t) = (x, 0, \cdots, t, \cdots, 0)\end{aligned}$$

where t is taken on t_i, that taking ψ on it we get that

$$\begin{aligned}\psi \circ \gamma_x : [-1, 1] &\to \mathbb{R}^m \\ t &\mapsto \psi \circ \gamma(t) = x + t \cdot \nu_i(x).\end{aligned}$$

By looking at their speeds at $t = 0$ we see that

$$d\psi_{(x,0)}(e_i) = d\psi_{(x,0)}(\frac{d\gamma_x}{dt}(0)) = \frac{d(\psi \circ \gamma_x)}{dt}(0) = \nu_i(x).$$

So each normal frame corresponds to the standard basis e of $\mathbb{R}^p$. Now that we have proven these conditions for our open $N \times B^p_\epsilon \subset N \times \mathbb{R}^p$ we will prove that it is diffeomorphic to $N \times \mathbb{R}^p$. Using the diffeomorphism

$$\begin{aligned}\Phi_1 : B^p_\epsilon &\to B^p_1 \\ v &\mapsto \Phi_1(v) = \epsilon \cdot v\end{aligned}$$

we see that the open ball with radius ϵ is diffeomorphic to the open unit ball. We now give a diffeomorphism between our open unit ball and $\mathbb{R}^p$

$$\begin{aligned}\Phi_2 : B^p_1 &\to \mathbb{R}^p \\ v &\mapsto \Phi_2(v) = \frac{v}{(1 - ||v||^2)}\end{aligned}$$

which rescales B^p_1 into $\mathbb{R}^p$. So by taking the composition of these two diffeomorphisms we get a diffeomorphism between B^p_ϵ and $\mathbb{R}^p$

$$\begin{aligned}\Phi = \Phi_2 \circ \Phi_1 : B^p_\epsilon &\to \mathbb{R}^p \\ v &\mapsto \frac{1}{\epsilon}\left(\frac{v}{(1 - \frac{||v||^2}{\epsilon^2})}\right).\end{aligned}$$

However, this is not yet the diffeomorphism we want, because when we look at a path

$$\begin{aligned}\gamma_i : I &\to B^p_\epsilon \\ t &\mapsto t \cdot e_i\end{aligned}$$

then its speed at $t = 0$ is e_i. However, when looking at its speed after the diffeomorphism we see that its speed at $t = 0$ is $\frac{1}{\epsilon}e_i$, which we do not want. So to fix this we multiply by ϵ after the diffeomorphism Φ, which gives us

$$\begin{aligned}\tilde{\Phi} : B^p_\epsilon &\to \mathbb{R}^p \\ v \mapsto \tilde{\Phi}(v) &= \frac{v}{(1 - \frac{||v||^2}{\epsilon^2})}.\end{aligned}$$

This only changes the speed of our scaling, so it is a diffeomorphism. We see that 0 is mapped to 0 and that e_i corresponds with e_i. This diffeomorphism induces a diffeomorphism between $N \times B^p_\epsilon$ and $N \times \mathbb{R}^p$ with the properties we want it to have. So this proves the tubular neighbourhood theorem in the special case that M is $\mathbb{R}^p$. □

For the general case, we will give an overview of the construction used in Chapter 5 of *Differential Topology* by Hirsch [5]. In this case we have an m-dimensional closed submanifold of $\mathbb{R}^{m+k}$ and a closed framed n-dimensional submanifold (N, ν). One of the main differences is that the manifold M does not have to have a framing. In that case, there is a more general tubular neighbourhood theorem, that we are not going to discuss. So in our case, we take M to be a framed submanifold of $\mathbb{R}^{m+k}$.

Proof of theorem 4.21 when M is framed. Using the tubular neighbourhood theorem for the case when M is $\mathbb{R}^m$ we see that there is an open V_M around M such that it is diffeomorphic to $M \times \mathbb{R}^k$ because M is a submanifold of $\mathbb{R}^{m+k}$. We note that this gives us a map

$$\begin{aligned}\pi_M : V_M = M \times \mathbb{R}^k &\to \mathbb{R}^{m+k} \\ (x, t) &\mapsto \pi_M(x, t) = x.\end{aligned}$$

We now look at the framed submanifold (N, ν). We would like to use the same kind of construction as our case that M is $\mathbb{R}^m$, however, the paths we would get are more difficult to describe because they would be "straight" on M. Our construction will be to first take straight lines, which may not lie in M, but then to project them onto M by using π_M.
So we first use the same construction as we did before by

$$\begin{aligned}\psi_N : N \times \mathbb{R}^{m-n} &\to \mathbb{R}^{m-k} \\ (x, t) &\mapsto x + t_1\nu_1(x) + \cdots + t_{m-n}\nu_{m-n}(x)\end{aligned}$$

Now we look at the part of the image of ψ_N that lies inside V_M

$$U_{V_M} = \{(x, t) \mid \psi_N(x, t) \in V_M\}.$$

We see that U_{V_M} is an open of $N \times \mathbb{R}^{m-k}$ because $U_{V_M} = \psi_N^{-1}(V_M)$ where V_M is open and ψ_N is smooth. We note that $N \times \{0\} \subset U_{V_M}$ because $\psi(x, 0) = x \in$

$N \subset M \subset V_M$. We now construct the map

$$\begin{aligned}\tilde{\psi} : U_{V_M} &\to M \\ (x,t) &\mapsto \tilde{\psi}(x,t) = (\pi_M \circ \psi)(x,t)\end{aligned}$$

which will be an automorphism at $(x,0)$ and in the same way as in the case that $M = \mathbb{R}^m$ we see that there are opens $N \times B_\epsilon^{m-n}$ such that they are mapped diffeomorphically under $\tilde{\psi}$. We see that $\tilde{\psi}(N \times \{0\}) = N$ and because π_M is the identity map on N we see that $d\tilde{\psi}_{(x,0)}(0, e_i) = \psi_{(x,0)}(0, e_i) = \nu_i$. This proves the tubular neighbourhood theorem when M is framed. $\square$

4.2.2 Construction of *col*

Our construction of the map *col* will be based on the construction in Milnor [7], however in our construction we will define *col* as a map, which is more in line with the approach by Davis and Kirk [3]. Now that we have proven the tubular neighbourhood theorem we will use it to construct a smooth map from M to S^p out of a framed closed submanifold (N, ν) of M. To do this we want to use the opens $N \times B_\epsilon^p \subset N \times \mathbb{R}^p$ around $N \times \{0\}$ as used in the proof of theorem 4.21 where we call $\psi(N \times B_\epsilon^p) = V_\epsilon$.

We see that V_ϵ is diffeomorphic to $N \times \mathbb{R}^p$. We now define the projection map

$$\begin{aligned}\pi : V_\epsilon \cong N \times \mathbb{R}^p &\to \mathbb{R}^p \\ (x,y) &\mapsto \pi(x,y) = y\end{aligned}$$

which is clearly smooth and it is also a submersion at 0, because when looking at it under a chart $\tilde{\chi} = \chi \times \chi_{id}$ we see that $(d\pi_{\tilde{\chi}})_{\tilde{\chi}(x)} = \big(0, \mathbb{I}_p\big)$ with $x \in N \times \{0\}$, which is surjective, so π is a submersion.
The only thing we need now is to construct a map between $\mathbb{R}^p$ and S^p such that it has $\big(N, \nu\big)$ as Pontryagin manifold.
We do this by identifying S^p with $\mathbb{R}^p \cup \{\infty\}$. We now construct the map $f^\epsilon_{(N,\nu)}$ as

$$\begin{aligned}f^\epsilon_{(N,\nu)} : M &\to S^p \\ x &\mapsto \begin{cases}\pi(x) & \text{for } x \in V_\epsilon \\ \infty & \text{otherwise}\end{cases}\end{aligned}$$

For now we will drop the (N, ν) out of our notation for $f^\epsilon_{(N,\nu)}$. We note that

$$(f^\epsilon)^{-1}(0) = (\pi \circ \psi^{-1})^{-1}(0) = (\psi \circ \pi^{-1})(0) = \psi(N, 0) = N$$

and because when we have $x \in N$ we see that

$$df^\epsilon_x(\nu_i(x)) = (d\pi_{(x,0)} \circ (d\psi^{-1})_x)(\nu_i(x)) = d\pi_{x,0}(0, e_i) = e_i$$

we get that

$$(df^\epsilon)^*(e) = \nu(x)$$

which proves that f^ϵ has (N,ν) as Pontryagin manifold. We will call $f^\epsilon_{(N,\nu)}$ the associated map of (N,ν) for radius ϵ. To define *col* we will now first give the following notation

$$Sub_n(M) = \big\{\text{framed closed n-dimensional submanifolds of } M\big\}.$$

So we will now define the map *col* as

$$\begin{aligned} col : Sub_{m-p}(M) &\to [M, S^p] \\ (N,\nu) &\mapsto col(N,\nu) = [f^\epsilon_{(N,\nu)}] \end{aligned}$$

where ϵ is the radius of an arbitrarily chosen open $N \times B^p_\epsilon$ around $N \times \{0\}$ such that it is mapped diffeomorphically by ψ, and $f^\epsilon_{(N,\nu)}$ is the corresponding associated map.

Theorem 4.24 (well-definedness of *col*)**.** *Our map col is well-defined, or more explicitly: suppose $N \times B^p_{\epsilon_2}$ is another open around $N \times \{0\}$ such that it is mapped diffeomorphically by ψ then $f^{\epsilon_2}_{(N,\nu)}$ is smoothly homotopic to $f^{\epsilon_1}_{(N,\nu)}$,*

$$f^{\epsilon_1}_{(N,\nu)} \simeq f^{\epsilon_2}_{(N,\nu)}.$$

Proof of theorem 4.24. Our approach to prove this is to make a smooth path between $f^{\epsilon_1}_{(N,\nu)}$ and $f^{\epsilon_2}_{(N,\nu)}$. We will keep our framed submanifold (N,ν) fixed, so we will write $f^\epsilon_{(N,\nu)}$ as f^ϵ.
We first define the space

$$\mathbb{R}^+_\psi = \big\{\epsilon \in \mathbb{R}^+ \mid \psi \text{ is a diffeomorphism on } N \times B^p_\epsilon\big\}.$$

We note that if $0 < \epsilon_2 \leqslant \epsilon_1$ and $\epsilon_1 \in \mathbb{R}^+_\psi$ then we also have that $\epsilon_2 \in \mathbb{R}^+_\psi$, because $N \times B^p_{\epsilon_2} \subset N \times B^p_\epsilon$.
We now construct a path between ϵ_1 and ϵ_2

$$\begin{aligned} \gamma_{\epsilon_1,\epsilon_2} : I &\to \mathbb{R}^+_\psi \\ t &\mapsto \epsilon_1(1-t) + t(\epsilon_2) \end{aligned}$$

where we note that $\gamma_{\epsilon_1,\epsilon_2}(t) \leqslant max\{\epsilon_1,\epsilon_2\}$, so indeed $\gamma_{\epsilon_1,\epsilon_2}(t) \in \mathbb{R}^+_\psi$ for all $t \in I$.
We now construct a map that will send an ϵ to its associated map

$$\begin{aligned} f^\bullet : \mathbb{R}^+_\psi &\to (M \to S^p) \\ \epsilon &\mapsto f^\bullet(\epsilon) = f^\epsilon. \end{aligned}$$

where f^ϵ is the associated map of (N,ν) by choosing $N \times B^p_\epsilon$ as starting open. Using these two maps we create our smooth path

$$\begin{aligned} \gamma : I &\to (M \to S^p) \\ t &\mapsto (f^\bullet \circ \gamma_{\epsilon_1,\epsilon_2})(t) \end{aligned}$$

which defines a smooth homotopy by

$$
\begin{aligned}
F : I \times M &\to S^p \\
t &\mapsto F(t,x) = \big(\gamma(t)\big)(x).
\end{aligned}
$$

We see that $F(0,x) = \big(\gamma(0)\big)(x) = \big(f^\bullet \circ \gamma_{\epsilon_1,\epsilon_2}(0)\big)(t) = f^\bullet(\epsilon_1) = f^{\epsilon_1}$ and that $F(1,x) = \big(\gamma(1)\big)(x) = \big(f^\bullet \circ \gamma_{\epsilon_1,\epsilon_2}(1)\big)(t) = f^\bullet(\epsilon_2) = f^{\epsilon_2}$. So f^{ϵ_1} and f^{ϵ_2} are smoothly homotopic, so we see that *col* is a well-defined map. $\square$

We have now proven that *col* is well-defined. However we still need to check that *col** is well-defined.

Theorem 4.25 (well-definedness of *col**)**.** *The map col** $: \Omega^{fr}_{m-k}(M) \to [M, S^k]$ *is well-defined, or in other words: suppose* $[(N_1,\nu_1)], [(N_2,\nu_2)] \in \Omega^{fr}_{m-k}(M)$ *and* $[(N_1,\nu_1)] = [(N_2,\nu_2)]$*, then col**$\big([(N_1,\nu_1)]\big) =$ *col**$\big([(N_2,\nu_2)]\big)$.

Proof of theorem 4.25. We see that $[(N_1,\nu_1)] = [(N_2,\nu_2)]$ means that there is a framed cobordism $(W,\mu) \subset M \times I$ between (N_1,ν_1) and (N_2,ν_2). Using *col* we get $col(N_1,\nu_1) = [f^{\epsilon_1}_{N_1}]$, $col(N_2,\nu_2) = [f^{\epsilon_2}_{N_2}]$ and $col(W,\mu) = [f^{\epsilon_W}_W]$. Because *col* is a well-defined map we can choose representatives. We take $f^{\epsilon_W}_W$ as a representative of $[f^{\epsilon_W}_W]$, which is a smooth map from $M \times I$ onto S^p.
We see that

$$\psi_W\big((W \times B^p_{\epsilon_W}) \cap (M \times \{0\} \times \mathbb{R}^p)\big) = \psi_N(N \times B^p_{\epsilon_W}) \times \{0\}$$

because W has the induced framing of ν by definition 3.5, so $W \cap \big(M \times \{0\}\big) = N_1 \times \{0\}$ has as framing $\nu_W|_{(N_1 \times \{0\})}(w) = (\nu(x), 0)$. We see that by ψ_W this is mapped precisely onto $\psi_N(N \times B^p_{\epsilon_W}) \times \{0\}$. Because of this and because

$$\psi_W\big((W \times B^p_{\epsilon_W}) \cap (M \times \{0\} \times \mathbb{R}^p)\big) = \psi_W(W \times B^p_{\epsilon_W}) \cap (M \times \{0\})$$

we see that

$$f^{\epsilon_W}_{(W,\nu_W)}(x,0) = f^{\epsilon_W}_{(N_1,\nu_1)}(x)$$

and in the same way we see that

$$f^{\epsilon_W}_{(W,\nu_W)}(x,1) = f^{\epsilon_W}_{(N_1,\nu_1)}(x).$$

So $f^{\epsilon_W}_{(W,\nu_W)}$ gives a smooth homotopy between $f^{\epsilon_W}_{(N_1,\nu_1)}$ and $f^{\epsilon_W}_{(N_2,\nu_2)}$, which proves the well-definedness of *col**. $\square$

4.3 $N^* = (col^*)^{-1}$

Now that we have proven N^* and col^* to be both well-defined maps, we will prove that they are each others inverses.

Lemma 4.26. *$N^* \circ col^* = Id_{\Omega^{fr}_{m-k}(M)}$ or more explicitly $N^* \circ col^*([(N,\nu)]) = [(N,\nu)]$ for all $[(N,\nu)] \in \Omega^{fr}_{m-k}(M)$.*

Proof of lemma 4.26. We see that $N^* \circ col^*([f]) = N^*(col(f)) = N^*([f^{\epsilon}_{(N,\nu)}])$ which gives $N^{\bullet}(f^{\epsilon}_{(N,\nu)}) = [(f^{\epsilon}_{(N,\nu)})^{-1}(0), (f^{\epsilon}_{(N,\nu)})^*(e)] = [(N,\nu)]$ because $\big((f^{\epsilon}_{(N,\nu)})^*(e)\big) = (N,\nu)$ as stated in the construction of the associated map $f^{\epsilon}_{(N,\nu)}$.
Because of this we see that $N^* \circ col^*([(N,\nu)]) = [(N,\nu)]$ for all $[(N,\nu)] \in \Omega^{fr}_{m-k}(M)$, so $N^* \circ col^* = Id_{\Omega^{fr}_{m-k}(M)}$ which was to be proven. □

Lemma 4.27. *$col^* \circ N^* = Id_{[M,S^p]}$ or more explicitly $col^* \circ N^*([f]) = [f]$ for all $[f] \in [M,S^p]$.*

To prove this we first state the following lemma

Lemma 4.28. *Suppose y is a regular value of two smooth maps $f, g : M \to S^p$ and μ is a basis for T_yS^p. If $(f^{-1}(y), f^*(\mu)) = (g^{-1}(y), g^*\mu)$ then $f \simeq g$.*

To prove this we first want to prove the following sublemma.

Sublemma 4.29. *Suppose y is a regular value of two smooth maps $f, g : M \to S^p$ and μ is a basis for T_yS^p. If $(f^{-1}(y), f^*(\mu)) = (g^{-1}(y), g^*\mu)$ and $f = g$ on an open U around $N = f^{-1}(y) = g^{-1}(y)$ then $f \simeq g$.*

Proof of sublemma 4.29. For the proof of this sublemma we will give an overview of the proof given in Milnor [7]. We note $S^p \backslash \{y\}$ is diffeomorphic to $\mathbb{R}^p$ by a map $\mathcal{P}$, which takes the stereographic projection. We will write

$$\tilde{f} = \mathcal{P} \circ f : M \backslash N \to \mathbb{R}^p \text{ and}$$
$$\tilde{g} = \mathcal{P} \circ g : M \backslash N \to \mathbb{R}^p.$$

Because $\mathbb{R}^p$ is convex we see that these two maps are smoothly homotopic by

$$\begin{aligned} \tilde{H} : (M\backslash N) \times I &\to \mathbb{R}^p \\ (x,t) &\mapsto \tilde{H}(x,t) = t \cdot \tilde{f}(x) + (1-t) \cdot \tilde{g}(x). \end{aligned}$$

We now construct our homotopy on all of M

$$\begin{aligned} H : M \times I &\to S^p \\ (x,t) &\mapsto H(x,t) = \begin{cases} f(x) & \text{for } x \in U \\ \mathcal{P}^{-1}\big(\tilde{H}(x,t)\big) & \text{for } x \in M\backslash N \end{cases} \end{aligned}$$

We note that when $x \in U$ then $\tilde{H}(x,t) = \tilde{f}$ for all $t \in I$ so H is a well-defined map. So we see that in this case f is smoothly homotopic to g by H.
We only need to prove now that we can change f locally into g, for which we refer to Milnor [7].

□

So if the Pontryagin manifolds are the same for two maps then they are homotopic.

Proof of lemma 4.27. We see that $col^* \circ N^*([f]) = col^*(N^\bullet(f)) = col^*([f^{-1}(y), f^*(\nu)])$ which gives $col(f^{-1}(y), f^*(\nu)) = [f^\epsilon_{\left(f^{-1}(y), f^*(\nu)\right)}]$ which we will just simply call $[f^\epsilon]$. Taking this f^ϵ as representative, we see that $\left((f^\epsilon)^{-1}(0), f^*(e)\right) = \left(f^{-1}(y), f^*(\nu)\right)$ by our construction of *col*, so using sublemma 4.28 we see that f^ϵ is smoothly homotopic to f. We get that $[f] = [f^\epsilon] = col^* \circ N^*([f])$, so $col^* \circ N^* = Id_{[M,S^p]}$ which was to be proven. $\square$

This proves that $N^* = (col^*)^{-1}$, so N^* is a bijection between $[M, S^p]$ and $\Omega^{fr}_{m-p}(M)$. By taking S^n as M we get the Pontryagin theorem 1.1 for sets, so we still need to prove that it is in fact an isomorphism.
We see that $N^*([f]+[g]) = N^*([f+g])$ and we note that $(f+g)^{-1}(z) = (f + const_{x_0})(z) \cup (const_{x_0} + g)(z)$ when $z \neq x_0$. We also see that $f \simeq f + const_{x_0}$ so $N^*([f]) = N^*([f + const_{x_0}])$ and in the same way $N^*([g]) = N^*([const_{x_0} + g])$. So we get that $N^*([f+g]) = N^\bullet(f+g) = [(f+const_{x_0})(z) \cup (const_{x_0}+g)(z)] = [(f+const_{x_0})(z)] + [(const_{x_0}+g)(z)] = N^*[f] + N^*[g]$ because the pontryagin manifolds are on opposite sides of the sphere and thus they are a disjoint union. So we see that N^* is actually an isomorphism between $[M, S^p]$ and $\Omega^{fr}_{m-p}(M)$, which proves the Pontryagin theorem 1.1.

5 Examples

Theorem 5.1. *$\pi_n(S^k) \cong 1$ for $n < k$.*

proof of theorem 5.1. Using the Pontryagin theorem we see that $\pi_n(S^k) \cong \Omega^{fr}_{n-k}(S^n)$ and because $n-k < 0$ we see that we need to look at negative dimensional framed submanifolds of S^n. We know that the only negative dimensional manifold is $\varnothing$, the empty manifold. Because there is only one submanifold of dimension $n-k$ we see that $\Omega^{fr}_{n-k}(S^n) \cong 1$ and thus $\pi_n(S^k) \cong 1$ as well, which was to be proven. $\square$

Theorem 5.2. *$\pi_k(S^k) \cong \mathbb{Z}$.*

proof of theorem 5.2. Our proof of this theorem will be based on chapter IX.4 of *Differential Manifolds* by Kosinski [6]. Using the Pontryagin theorem we again see that $\pi_k(S^k) \cong \Omega^{fr}_0(S^n)$ so we need to look at the cobordism classes of 0-dimensional framed submanifolds of S^n. These are sets consisting of a finite amount of points where at each point x the framing is just a basis for T_xS^n. We see that cobordisms between 0-dimensional framed submanifolds are a finite number of arcs.
Looking at those arcs we see that there are three cases. The first case is that the arc connects a point in $S^n \times \{0\}$ and another point in $S^n \times \{0\}$. The second case is that the arc connects two points in $S^n \times \{1\}$. The last case is that the arc connects a point $S^n \times \{0\}$ and a point in $S^n \times \{1\}$.

We now construct a bijection from the cobordism group onto $\mathbb{Z}$ by

$$\begin{aligned} deg : \Omega^{fr}_n(S^n) &\to \mathbb{Z} \\ [(N,\nu)] &\mapsto deg(N,\nu) = \sum_{p\in N}(deg(p)) = \sum_{p\in N}(or(\nu(p))) \end{aligned}$$

where we define *or* as

$$\begin{aligned} or : base(S^n) &\to \{+1,-1\} \\ v &\mapsto \begin{cases} +1 & \text{if } v \text{ is positively oriented} \\ -1 & \text{if } v \text{ is negatively oriented} \end{cases} \end{aligned}$$

where orientation is defined as in definition 4.5.
We first need to prove that this map is well-defined, so we need to check that if (N_1,ν_1) is framed cobordant to (N_2,ν_2), then $deg(N_1,\nu_1) = deg(N_2,\nu_2)$.
To do this we look at the three cases of arcs we have and how they change orientation.
We now call our arc A. When orienting A we choose a direction for its tangent space $T_xA \subset T_x(S^p \times I)$.

We now take a positively oriented smooth framing of the tangent space of A, so

$$\begin{aligned} v_A : A \ni x &\to base(T_xA) \\ x &\mapsto v_A(x) \end{aligned}$$

By taking v_A together with the normal framing ν_A of A we get a smooth map

$$\begin{aligned} \mathcal{V} : A \ni x &\to base\big(T_x(S^p \times I)\big) \\ x &\mapsto \mathcal{V}(x) = \{v_A(x), \nu_A(x)\} \end{aligned}$$

We now orient $S^p \times I$ by taking the orientation of S^p together with I, where in the case of I we take e_1 to be positively oriented. We now see that the orientation of $\mathcal{V}(x)$ is the same for all $x \in A$ because otherwise we would have flipped a vector of our basis, which is not possible because then it would not be a basis for a certain point in A. So it is either positively or negatively oriented for all $x \in A$ in the orientation of $S^p \times I$.
We see that if at p_1 the tangent vector $v_A(p_1)$ is inward, then at p_2 the tangent vector $v_A(p_2)$ is outward. Because $\mathcal{V}(p_1)$ and $\mathcal{V}(p_2)$ both have the same orientation, we see that $\nu_A(p_1) = \nu(p_1)$ and $\nu_A(p_2) = \nu(p_2)$ have a different orientation, so $deg(p_1) = -deg(p_2)$. We note that the second case implies the same.
For the third case we see that at p_2 being the point on $S^n \times \{1\}$ the tangent framing $v_A(p_2)$ is in the same direction as on p_1. So by this we see by the same reasoning that $deg(p_1) = deg(p_2)$.
Suppose now that (N_1, ν_1) is cobordant to (N_2, ν_2). When we have an arc of case 1 then this does not change the degree, so we can neglect these arcs when looking at the difference. The same is true for arcs of case 2. For the arcs of case 3 we see that they only connect points with the same orientation, so we can neglect these as well when looking at the difference of the degree. So we see that all arcs do not change the degree, so

$$deg(N_1, \nu_1) = deg(N_2, \nu_2).$$

Because of this we see that $deg : \Omega_0^{fr}(S^p) \to \mathbb{Z}$ is a well-defined map, which was to be proven.

We now want to prove that $deg : \Omega_0^{fr}(S^p) \to \mathbb{Z}$ is a bijection.
We first prove that it is injective.
Suppose (N, ν) has degree 0. Then it consists of as many negatively oriented points as positively oriented points, by which we mean that the basis on them is positively respectively negatively oriented. We can find an open U of I such that there is only one pair of a positively and negatively oriented point in it, which we call p_1 and p_2. Then we can use an arc of case 1 to connect them inside of $U \times S^n$ and use straight arcs for all the other points, so we see that (N, ν) is cobordant to itself with p_1 and p_2 removed.
Doing the same trick multiple times we eventually end up with only two oppositely oriented points, which are cobordant to the empty manifold by an arc of case 3.
In the same way, we see that if (N, ν) has degree $+l$ then it is cobordant to a set of l positively oriented points, and when (N, ν) has degree $-l$ then it is cobordant to a set of l negatively oriented points. Because of this we see that deg

is injective because if $deg(N_1, \nu_1) = deg(N_2, \nu_2)$ implies that $[N_1, \nu_1] = [N_2, \nu_2]$.

We see that deg is surjective as well because $n > 0$, otherwise the only 0-dimensional manifold would be one point but now there can be any finite number of points.
This proves that $deg : \Omega_0^{fr}(S^n) \to \mathbb{Z}$ is an isomorphism of sets.

$\square$

6 References

[1] R. Bott and L. V. Tu. *Differentsialnye formy v algebraicheskoĭ topologii.* "Nauka", Moscow, 1989. ISBN 5-02-013909-2. Translated from the English by I. V. Savelev and G. S. Shmelev, Translation edited and with a preface by A. A. Kirillov.

[2] M. Crainic. Manifolds 2018, 2018. URL `http://www.staff.science.uu.nl/crain101/`. Retrieved: September 2019.

[3] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001. ISBN 0-8218-2160-1. doi: 10.1090/gsm/035. URL `https://doi.org/10.1090/gsm/035`.

[4] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. ISBN 0-521-79160-X; 0-521-79540-0.

[5] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1994. ISBN 0-387-90148-5. Corrected reprint of the 1976 original.

[6] A. A. Kosinski. *Differential manifolds*, volume 138 of *Pure and Applied Mathematics.* Academic Press, Inc., Boston, MA, 1993. ISBN 0-12-421850-4.

[7] J. W. Milnor. *Topology from the differentiable viewpoint.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. ISBN 0-691-04833-9. Based on notes by David W. Weaver, Revised reprint of the 1965 original.

[8] L. S. Pontryagin. Smooth manifolds and their applications in homotopy theory. In *American Mathematical Society Translations, Ser. 2, Vol. 11*, pages 1–114. American Mathematical Society, Providence, R.I., 1959.