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Principal Bundles

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Abstract

A principal bundle is a fiber bundle with a group as typical fiber. Fiber bundles are spaces which locally look like product spaces. In this text, we will classify all numerable principal bundles. That are principal bundles with a suitable partition of unity on the orbit space. In order to do this many tools for principal bundles and partitions of unities are created. We construct universal bundles, using both the Milnor construction and configuration spaces. The latter are used to link principal bundles for symmetric groups with finitely sheeted covering spaces. Lastly, lifting properties on principal bundles are considered.

Contents

1 Introduction

In topology, we are generally happy to proceed locally and thereby study the global structure of locally "nice" spaces. As long as locally our space looks "nice", we usually have a wide variety of tools available. No difference is there in the case of principal bundles. Spaces for principal bundles look locally like a product of some (arbitrary) space B with a topological group. The bundle now locally looks like the projection on B. Precise definitions will be given throughout this text.

Principal bundles are a special case of fiber bundles. As we see in Chapter 7, normal covering spaces are a special case of principal bundles. Another class of examples of principal bundles are frame spaces of vector bundles, see [Die08, Chapter 14.2]. Principal bundles can for example be used to calculate "Čech cohomolgy groups", see [Ful95, Paragraph 15]. As explained in [Mit11, Chapter 10], reductions of structure groups on principal bundles imply several properties on manifolds. In physics, principal bundles can be applied in Gauge theories. This was already implicitly done by Dirac in the 1930th, roughly the time mathematicians started considering principal bundles. Specific applications in physics can be found in [CW06].

The main goal in this text is to classify principal bundles. The classification theorem is present in Chapter 5. The chapters before provide all the required tools to prove this theorem. Afterwards, we first construct so called "universal bundle". In the classification theorem, these bundles are simply assumed to exist. Using specific universal bundles, we are able to prove some properties of universal bundles. Also using universal bundles for symmetric groups, we link finitely sheeted covering spaces to principal bundles. Lastly, I elaborate on lifting properties of principal bundles. The appendix contains some general topological facts used in several proofs.

This text is based on two main sources: The book "Algebraic Topology" by Dieck, [Die08], and the notes "Notes on principal bundles and classifying spaces" by Mitchell, [Mit11]. More sources are evoked on specific topics when required. Chapters 4 and 5 are mostly based on Dieck, while Chapter 3 is more on Mitchell. Chapters 2 and 7 consist more of own work. Other chapters are a mix of the main sources, own work and other sources.

I assume the reader has basic knowledge of topology and abstract algebra. Notions of algebraic topology are briefly recalled before applied. I have tried to be precise throughout the text. Therefore the details might at times be lengthy and technical. I advise the reader who is not interested in all details to skip these technical parts. I hope with this text to interest the reader in the topic of principal bundles.

2 Equivariant Maps

Let G be a topological group. Throughout this text, I assume that G is some topological group. A ***left*** G**-space** is a topological space X with a continuous *left* action $G \times X \to X$. Equivalently, we can define a ***right*** G**-space** as a space X with a continuous *right* action $X \times G \to X$. Notice that we can transform any *left* action into a *right* action and vice versa by setting $gx = xg^{-1}$. So we can restrict ourselves to *left* G-spaces and call them simply G-spaces.

Remark 2.1. The difference between left and right action lies in the order elements of the group act on points in the space. Say $\phi\colon G \times X \to X$ and $\psi\colon X \times G \to X$ are left and right actions respectively. If $g, h \in G$ and $x \in X$, there holds $\phi(gh, x) = \phi(g, \phi(h, x))$ and $\psi(x, gh) = \psi(\psi(x, g), h)$. Hence, the need of taking the inverse above.

For any G-space X, we can define its orbit space X/G (equipped with the quotient topology) and the projection $\pi_X\colon X \to X/G$ which sends every $x \in X$ to its orbit $[x]_X$. By definition of the quotient topology π is a continuous and surjective map. For any open $U \subseteq X$, we have that $\pi_X^{-1}(\pi_X(U)) = \bigcup_{g \in G} gU$, the union of co-sets. By construction of the quotient topology $\pi_X(U)$ is open. Hence, π_X is an open map.

For G-spaces X and Y, considerable maps $X \to Y$ do need to respect both the topology (i.e., be continuous) and the group action. In order to achieve the latter, we demand for a continuous map $f\colon X \to Y$ that it "commutes" with the group actions. More precisely, for all $x \in X$ and $g \in G$, we demand that $f(gx) = gf(x)$. Such a map, we call an G**-equivariant map** or, if the group G is understood, simply an **equivariant map**. This terminology is taken from [Mit11, p. 2]. We notice that a projection to an orbit space is an equivariant map:

Proposition 2.2. *Let X a G-space. Then the projection $\pi\colon X \to X/G$ to the orbit space is an equivariant map. Here G acts trivially on X/G.*

Proof. There holds $\pi(gx) = [gx] = [x] = g[x] = g\pi(x)$ for all $g \in G$ and $x \in X$. By construction, π is continuous. Hence, π is a G-equivariant map. $\square$

General equivariant maps can be factored through the group action and induce a map on the orbit spaces:

Proposition 2.3. *If X and Y are G-spaces and $f\colon X \to Y$ a G-equivariant map, then there exists a unique $\phi\colon X/G \to Y/G$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\pi_X"'] & Y \arrow[d, two heads, "\pi_Y"] \\ X/G \arrow[r, dashed, "\phi"] & Y/G \end{tikzcd}$$

Here π_X and π_Y are the projections from X and Y to their orbit space respectively.

Proof. Define $\phi\colon X/G \to Y/G$ as $\phi([x]_X) = [f(x)]_Y$. This certainly makes the diagram commute. We show that the map ϕ is well-defined. Let $y \in [x]$ for some $x \in X$. There is a $g \in G$ such that $gx = y$ and thus $\phi([y]_X) = [f(y)]_Y = [f(gx)]_Y = [gf(x)]_Y = [f(x)]_Y = \phi([x]_X)$. Hence, ϕ is well-defined.
Suppose $\overline{\phi}\colon X/G \to Y/G$ is another function such that the diagram commutes. For all $\bar{x} \in X/G$, there is an $x \in X$ such that $[x]_X = \bar{x}$. We now have $\phi(\bar{x}) = \phi([x]_X) = [f(x)]_Y = \overline{\phi}([x]_X) = \overline{\phi}(\bar{x})$. Hence, $\phi = \overline{\phi}$ and thus ϕ is unique. $\square$

Proposition 2.3 is in fact nothing else than a restatement of the universal property of the quotient space topology. Hereby, we immediately can conclude that ϕ is continuous. For a given equivariant map $f\colon X \to Y$, we call $\phi\colon X/G \to Y/G$ the **induced** map by f on orbit spaces. Moreover, we call any lift $f\colon X \to Y$ of a given $\phi\colon X/G \to Y/G$ a **G-equivariant map over ϕ**.

Suppose we have a map $\phi\colon X/G \to Y/G$ on the orbit spaces. As we see in the following example, there need not exist an equivariant map over ϕ. We will elaborate more on this case in Chapter 8.

Example 2.4. Let $X = Y = G = \{1, -1\}$, a two point space. Let G act on both X and Y by multiplication. We endow G and Y with the discrete topology, while we endow X with the trivial topology. I leave it to the reader to prove that these actions are continuous and G is a topological group. Note that the orbit space for both X and Y is the one-point space $*$. So for any map $f\colon X \to Y$, we have a well-defined map ϕ on the orbit space making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[d, two heads] \\ * \arrow[r, tail, two heads, "\phi"] & * \end{tikzcd}$$

However, none of the 4 maps $f\colon X \to Y$ is an equivariant map: the two bijections are not continuous, while the two constant mappings don't commute with the group actions. Instead, if we endow also X with the discrete topology, then the two bijections are equivariant maps. $\triangle$

I will now discuss how properties of equivariant maps result in properties of their induced functions and vice versa. First of all note the purely set-theoretical fact that if $f\colon X \to Y$ is surjective, then the induced map $\phi\colon X/G \to Y/G$ is surjective as well. It turns out that the converse of this fact is also true:

Lemma 2.5. *Suppose $f\colon X \to Y$ is a G-equivariant map between G-spaces X and Y. Let $\phi\colon X/G \to Y/G$ the induced map. If the map ϕ is surjective, then f is surjective.*

Proof. Let $y \in Y$. By surjectivity of ϕ and the projection π_X, there exists an $x \in X$ such that $\phi([x]_X) = [y]_Y$. Since $\phi([x]_X) = [f(x)]_Y$, there exists a $g \in G$ such that $y = gf(x) = f(gx)$. $\square$

The following is now trivial:

Corollary 2.6. *If $f\colon X \to Y$ is a G-equivariant map between G-spaces X and Y, then the induced map $\phi\colon X/G \to Y/G$ is surjective if and only if f is surjective.* $\square$

A reader might have already guessed that following surjectivity, there must come injectivity and so it will:

Lemma 2.7. *Let X and Y be G-spaces. For an injective G-equivariant map $f\colon X \to Y$ the induced map $\phi\colon X/G \to Y/G$ is injective.*

Proof. Let $[x]_X, [y]_X \in X/G$ such that $\phi([x]_X) = \phi([y]_X)$. Now there holds $[f(x)]_Y = \phi([x]_X) = \phi([y]_X) = [f(y)]_Y$. So there exists a $g \in G$ such that $f(y) = gf(x) = f(gx)$. Hence, $y = gx$ and thus $[x]_X = [y]_X$. $\square$

Conversely, we have the following:

Lemma 2.8. *Let X and Y be G-spaces such that the action on Y is free (i.e., for all $y \in Y$ and $g, h \in G$ if $gy = hy$, then $g = h$). If for a G-equivariant map $f\colon X \to Y$ the induced map $\phi\colon X/G \to Y/G$ is injective, then f is injective.*

Proof. Let $x, y \in X$ such that $f(x) = f(y)$. There holds $\phi([x]_X) = [f(x)]_Y = [f(y)]_Y = \phi([y]_X)$. By injectivity of ϕ, we see that $[x]_X = [y]_X$. Hence, there exists a $g \in G$ such that $gx = y$ and thus $gf(x) = f(gx) = f(y) = f(x)$. So g is the identity and $x = y$. □

And again the trivial observation:

Corollary 2.9. *Let X and Y be G-spaces such that the action on Y is free. A G-equivariant map $f\colon X \to Y$ is injective if and only if the induced map $\phi\colon X/G \to Y/G$ is injective.* □

The assumption in Lemma 2.8 that the action on the space Y is free, is required in the sense that for any non-free G-space Y, we can find a non-injective equivariant map whose induced map on the orbit spaces is injective: let $X = G$ (as topological spaces) equipped with the action $(g, h) \mapsto gh$. Let $y \in Y$ and $e \neq h \in G$ (e the identity of G) such that $hy = y$. Define $f\colon G = X \to Y$ as $f(g) = gy$. This is a non-injective equivariant map ($f(h) = f(e)$ and $f(gx) = gxy = gf(x)$ for all $g \in G$ and $x \in X = G$). Since the orbit space $G/G = *$, the induced map $\phi\colon X/G \to Y/G$ is injective.
Moreover, if Y is a free G-space, the existence of an equivariant map $f\colon X \to Y$, for some G-space X, forces the action on X to be free. Indeed, if $gx = x$ for a $g \in G$ and $x \in X$, then $gf(x) = f(gx) = f(x)$ and thus g is the identity of G.

For equivariant maps $f, g\colon X \to Y$ between G-spaces X and Y, a ***G*-homotopy** between f and g is an equivariant map $H\colon X \times I \to Y$, (G acting trivially on the I coordinate) such that $H_0 = f$ and $H_1 = g$, i.e., it is a homotopy in the usual sense. When there exists a G-homotopy for equivariant maps f and g, then we say that f and g are ***G*-homotopic**.

Proposition 2.10. *The induced maps $\phi, \phi'\colon X/G \to Y/G$ for G-homotopic equivariant maps $f, g\colon X \to Y$ are homotopic.*

Proof. Let $H\colon X \times I \to Y$ a G-homotopy between f and g and $h\colon X/G \times I \to Y/G$ its induced map on orbit spaces. Notice that we have made the canonical identification of $(X \times I)/G$ with $X/G \times I$, which is possible since G acts trivially on I. The induced functions of $f = H_0$ and $g = H_1$ are h_0 and h_1 respectively. By uniqueness of the induced map, we have that $h_0 = \phi$ and $h_1 = \phi'$. We see that h is a homotopy between ϕ and ϕ'. We conclude that ϕ and ϕ' are homotopic. □

A function $f\colon X \to Y$ between G-spaces X and Y that is both an equivariant map and a homeomorphism, we call a ***G*-equivariant homeomorphism** or shorter a ***G*-homeomorphism**.

Proposition 2.11. *If $f\colon X \to Y$ a G-equivariant homeomorphism between G-spaces X and Y, then f^{-1} is an equivariant map.*

Proof. We have that $gf^{-1}(x) = f^{-1}(f(gf^{-1}(x))) = f^{-1}(gf(f^{-1}(x))) = f^{-1}(gx)$. By assumption f^{-1} is continuous. Hence, f^{-1} is an equivariant map. □

I conclude this section by the following simple (but important) proposition:

Proposition 2.12. *Let X, Y, Z be G-spaces, $f\colon X \to Y$ and $g\colon Y \to Z$ G-equivariant maps. Then their composition $g \circ f$ is also a G-equivariant map.*

Proof. Compositions of continuous functions are continuous thus $g \circ f$ is continuous. If $x \in X$ and $h \in G$, then $g(f(hx)) = g(h(f(x)) = hg(f(x))$, so $g \circ f$ is an equivariant map. □

3 Principal Bundles

For any G-space X, there exists (possibly by the axiom of choice) a map $s\colon X/G \to X$ with the property that the orbit of $s(\bar{x})$ equals $\bar{x}$ for all $\bar{x} \in X/G$ (s is a section of the projection $\pi\colon X \to X/G$). Suppose the action on X is free. Given such a section $s\colon X/G \to X$, we can construct a bijection $\psi\colon G \times X/G \to X$ by setting $\psi(g, \bar{x}) = gs(\bar{x})$. Indeed, for any $x \in X$, the orbits of x and $s([x])$ coincide. Thus, there exists a unique (by freeness of the action) g_x with $x = g_x s([x])$. Now $\psi^{-1}\colon X \to G \times X/G$ defined by $\psi^{-1}(x) = (g_x, [x])$ is an inverse. Notice that for all $g, h \in G$ and $\bar{x} \in X/G$, we have that $\psi(gh, \bar{x}) = g\psi(h, \bar{x})$ and that the following diagram commutes:

$$\begin{tikzcd} G \times X/G \arrow[r, tail, two heads, "\psi"] \arrow[dr, two heads] & X \arrow[d, two heads, "\pi"] \\ & X/G \end{tikzcd}$$

Here the diagonal arrow is the projection on the second coordinate.

This construction (after endowing both G and X with a topology) need not create a homeomorphism (neither ψ nor ψ^{-1} need be continuous). In fact, there even need not exists a homeomorphism between $G \times X/G$ and X. When there does exist such a homeomorphism, we will call X a **trivial G-space** and π a **trivial G-bundle**. A G-space X which locally has this property (a **locally trivial G-space**) paired with the projection $\pi\colon X \to X/G$, we call a **principal G-bundle**. We can generalise this definition by replacing X/G by any space B. It turns out this "generalisation" does not add much, since B will always be homeomorphic to X/G, as we see in this chapter. I will make these definitions precise following [Mit11, p. 2]:

Definition 3.1. Let $p\colon X \to B$ an equivariant map between G spaces X and B where G acts trivially on B. A subset $U \subseteq B$ is called **trivialising** if there exists a G-homeomorphism $\psi\colon G \times U \to p^{-1}(U)$ (where G acts on $G \times U$ by the group multiplication on the first component and trivially on the second) such that the following diagram commutes:

$$\begin{tikzcd} G \times U \arrow[r, tail, two heads, "\psi_U"] \arrow[dr, two heads] & p^{-1}(U) \arrow[d, two heads, "p"] \\ & U \end{tikzcd}$$

Such a G-homeomorphism ψ_U, we call a **trivialisation** for (p, U). If the equivariant map p is understood, we say that ψ_U is a trivialisation for U. An open cover $\mathcal{U}$ of B consisting of trivialising opens is an open **cover trivialising** p. When the equivariant map is understood, the cover $\mathcal{U}$ is an open **trivialising cover**. When an equivariant map $p\colon X \to B$ (G-acting trivially on B) admits an open trivialising cover, then p is a **principal G-bundle over** B. We call a principal bundle $p\colon X \to B$ **trivial** if there exists a trivialisation for B. The group G is called a **structure group**. Every principal bundle we consider will have structure group G, unless specifically mentioned otherwise. ▲

It is obvious that a principal bundle is a fiber bundle with a group as typical fiber. I will illustrate these definitions with some examples. We first look at a trivial case:

Example 3.2. The projection $G \to *$ is a trivial bundle for every topological group. The projection $G \times * \to G$ is a trivialisation. △

Example 3.3. Suppose G is a discrete group and H a subgroup of G. We consider the projection $\pi\colon G \to G/H$. It is obvious that G/H and H have the discrete topology. For every $g \in G$, there holds $\pi^{-1}([g]) = \{hg \mid h \in H\}$. Moreover, the point hg is different for varying $h \in H$. Now the map $\psi\colon \pi^{-1}([g]) \to H \times H/G$ given by $\psi(hg) = (h, [g])$ is a well-defined G-homeomorphism. Hence, π is a principal bundle. △

Example 3.4. We consider spheres. Let $X = S^n$ and equip X with the $\mathbb{Z}_2$ action, where the non-trivial element of $\mathbb{Z}_2$ maps a point to its antipodal point. The group $\mathbb{Z}_2$ has discrete topology. Clearly the orbit space is $\mathbb{R}P^n$. Every open $U \subseteq S^n$ not containing a pair of antipodal points projects down to a trivialising open. This is clear since U is disjoint to the set V of antipodes of U and the group action precisely switches between these sets. In fact, we have a covering space. More on this in Chapter 7. One can do the same analysis when taking $n = \infty$. △

Trivialisations can be restricted to subsets:

Lemma 3.5. *If $U \subseteq B$ is a trivialising set for a G-equivariant map $p\colon X \to B$, then every subset $V \subseteq U$ is also a trivialising set. Moreover, the restriction $\psi_U|_V$ with ψ_U a trivialisation for U is a trivialisation for V is.*

Proof. Notice that $\psi_U(G \times V) = p^{-1}(V)$. Indeed, if $x \in \psi_U(G \times V)$, then $\psi_U^{-1}(x) = (g, p(x))$ for some $g \in G$ and thus $p(x) \in V$. On the other hand if $x \in p^{-1}(V) \subseteq p^{-1}(U)$, then $p(x) \in V$. So $\psi_U^{-1}(x) = (g, p(x)) \in G \times V$ for some $g \in G$. Since $p(\psi_U(g, x)) = x$ for all $(g, x) \in G \times V \subseteq G \times U$, we have the following commuting diagram:

$$\begin{tikzcd} G \times V \arrow[r, tail, two heads, "\psi_U|_V"] \arrow[dr, two heads] & p^{-1}(V) \arrow[d, two heads, "p"] \\ & V \end{tikzcd}$$

Thus $\psi_V \colon = \psi_U|_V$ is a trivialisation for V. □

It is obvious that principal bundles are surjective. Furthermore, principal bundles have the following, in light of Lemma 2.8, useful property, after [Mit11, p. 2]:

Proposition 3.6. *If $p\colon X \to B$ is a principal bundle, then the action of G on X is free.*

Proof. Let $x \in X$ and $g \in G$ such that $gx = x$. Choose a trivialising set $U \subseteq B$ containing $p(x)$ with trivialisation $\psi_U\colon G \times U \to p^{-1}(U)$. Then $\psi_U^{-1}(x) = (h, p(x))$ for some $h \in G$. Now $(h, p(x)) = \psi_U^{-1}(x) = \psi_U^{-1}(gx) = g\psi_U^{-1}(x) = (gh, p(x))$. Thus $gh = h$, so g is the identity of G. □

Following [Die08, p. 329], for a G-space, we X consider the space

$$C(X) := \{(x, gx) \mid x \in X, g \in G\} \subseteq X \times X.$$

We endow $C(X)$ with the subspace topology. In case the action on X is free, there is an obvious map $C(X) \to G$, sending $(x, gx) \mapsto g$. The constructed map $C(X) \to G$ is called the **translation map**.

Proposition 3.7. *If $p\colon X \to B$ is a principal bundle, then the translation map is continuous.*

Proof. Take a trivialising cover $\mathcal{U}$ of B, notice that the family $\mathcal{V} = \{(p^{-1}(U) \times p^{-1}(U)) \cap C(X) \mid U \in \mathcal{U}\}$ form an open cover of $C(X)$. We will prove that the translation map $t\colon C(X) \to G$ is continuous on all elements of $\mathcal{V}$, this directly implies that t is continuous. Let $U \in \mathcal{U}$ and $\psi_U\colon p^{-1}(U) \to G \times U$ a trivialisation. Taking the projection to G of the trivialisation, we obtain an equivariant map $s\colon p^{-1}(U) \to G$. Define the map $t'\colon (p^{-1}(U) \times p^{-1}(U)) \to G$ by $t'(x,y) = s(y)(s(x))^{-1}$ for all $(x,y) \in (p^{-1}(U) \times p^{-1}(U))$. Clearly this map is continuous. Notice that $t'(x,gx) = s(gx)(s(x))^{-1} = g$ for all $x \in X$ and $g \in G$. Hence, $t'|_{C(X)} = t$ and thus t is continuous. $\square$

3.1 Principal Bundles and their Orbit Spaces

In Chapter 2, we considered the projection $X \to X/G$ for a G-space X. It turns out, as we see in this section that a principal bundle is a special case of this. More precisely, for any principal bundle $p\colon X \to B$, the space B is homeomorphic to X/G, see Lemma 3.9. This is noted by [Mit11, p. 2]. Moreover, for a principle bundle $p'\colon X \to B'$ and a space B homeomorphic to B', there exists a principal bundle $p\colon X \to B$, see Lemma 3.8.

Lemma 3.8. *Let $p'\colon X \to B'$ a principal bundle and $\phi\colon B' \to B$ a homeomorphism. Then $p = \phi \circ p'$ is a principal bundle.*

Proof. Note that ϕ is, in fact, a G-homeomorphism since G acts trivially on both B' and B. By Propositions 2.2 and 2.12, the map $p\colon = \phi \circ p'$ is an equivariant map. Take a trivialising cover $\mathcal{U}$ of $p'\colon X \to B'$. Define $\mathcal{V} := \{\phi(U) \mid U \in \mathcal{U}\}$. The family $\mathcal{V}$ is a cover of B. For every $V \in \mathcal{V}$, we have the commuting diagram:

$$\begin{tikzcd}
G \times V \arrow[d, two heads] & G \times U \arrow[l, tail, two heads, "Id_G \times \phi|_U"'] \arrow[r, tail, two heads, "\psi_U"] \arrow[dr, two heads] & p'^{-1}(U) \arrow[d, two heads, "p'"] \\
V & & U \arrow[ll, tail, two heads, "\phi|_U"']
\end{tikzcd}$$

Here $U = \phi^{-1}(V) \in \mathcal{U}$ and $Id_G \times \phi|_U\colon G \times U \to G \times V$ is the homeomorphism defined by $(g,x) \mapsto (g, \phi|_U(x))$ and $\psi_U\colon G \times U \to p'^{-1}(U)$ is a trivialisation. By taking the following sub-diagram, we see that the cover $\mathcal{V}$ with trivialisations $\psi_U \circ (Id_G \times \phi|_U)^{-1}$ form a trivialising cover of p:

$$\begin{tikzcd}
G \times V \arrow[r, tail, two heads, "\psi_U \circ (Id_G \times \phi|_U)^{-1}"] \arrow[dr, two heads] & p^{-1}(V) \arrow[d, two heads, "p = \phi \circ p'|_{p'^{-1}(U)}"] \\
& V
\end{tikzcd}$$

$\square$

Lemma 3.9. *Let $p\colon X \to B$ a principal bundle. Then there exists a unique homeomorphism $\phi\colon X/G \to B$ such that $\phi^{-1} \circ p$ is the projection $\pi\colon X \to X/G$. Moreover, $\pi\colon X \to X/G$ is a principal bundle.*

Proof. Since p is an equivariant map, by Proposition 2.3, we have a unique continuous map $\phi\colon X/G \to B$ such that the following diagram commutes (notice that $B/G \cong B$):

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, two heads] & B \arrow[d, tail, two heads, "Id_B"] \\
X/G \arrow[r, "\phi"] & B
\end{tikzcd}$$

From the diagram it is clear that if ϕ is bijective, then $\phi^{-1} \circ p$ is the projection $X \to X/G$. Moreover, any bijective ϕ with this property makes this diagram commute. Hence, the uniqueness of ϕ.

We will prove that ϕ is a homeomorphism:
For every $b \in B$, there is a trivialising open $U \subseteq B$ containing $p(b)$. Let $\psi_U \colon G \times U \to p^{-1}(U)$ be a trivialisation. We see that $p(\psi_U(e, p(b))) = p(b)$ for all $b \in B$ and e is the identity of G. Hence, p is surjective and thus by Corollary 2.6, ϕ is surjective.

Let $x, y \in X$ such that $\phi([x]) = \phi([y])$. Then also $p(x) = p(y)$. Choose a trivialising open $U \subseteq B$ around $p(x)$ and let $\psi_U \colon G \times U \to p^{-1}(U)$ a trivialisation. Now $x, y \in p^{-1}(U)$ and thus $\psi_U^{-1}(x) = (g_x, p(x))$ and $\psi_U^{-1}(y) = (g_y, p(y))$ for some $g_x, g_y \in G$. We obtain $\psi_U^{-1}(g_y g_x^{-1} x) = g_y g_x^{-1} \psi_U^{-1}(x) = g_y g_x^{-1}(g_x, p(x)) = (g_y, p(y)) = \psi_U^{-1}(y)$. Thus $g_y g_x^{-1} x = y$ and $[x] = [y]$. Hence, ϕ is injective.

Let $W \subseteq X$ open and $w \in p(W)$. There is a trivialising open U containing w. Take a trivialisation $\psi_U \colon G \times U \to p^{-1}(U)$. Write $Pr \colon U \times G \to U$ for the projection. Now the set $W \cap p^{-1}(U)$ is open and thus $Pr(\psi_U^{-1}(W \cap p^{-1}(U)))$ is open. Since $w \in p(W)$, there exists an $x \in W$ with $p(x) = w$. Hereby there holds $x \in p^{-1}(U)$ and $Pr(\psi_U^{-1}(x)) = p(x) = w$ and thus $w \in Pr(\psi_U^{-1}(W \cap p^{-1}(U))) = p(W \cap p^{-1}(U)) \subseteq p(W)$. As $w \in p(W)$ was arbitrary, we know that $p(W)$ is open. We see that p is an open map.
Since the projection $\pi \colon X \to X/G$ is surjective, we have for every open $V \subseteq X/G$ that $\pi(\pi^{-1}(V)) = V$. Hence, $\phi(V) = \phi(\pi(\pi^{-1}(V))) = p(\pi^{-1}(V))$ and thus ϕ is an open mapping. Hence, ϕ is a homeomorphism.
Since ϕ is a homeomorphism and p a principal bundle, by Lemma 3.8, the map $\pi = \phi^{-1} \circ p$ is a principal bundle. $\square$

The fact that for a principal bundle $p \colon X \to B$, the space B is homeomorphic to X/G implies that most assertions in Chapter 2 are also true in the setting of principal bundles. I will highlight the following:

Proposition 3.10 (Compare Proposition 2.3). *Let $p \colon X \to B$ and $p' \colon Y \to B'$ principal bundles and $f \colon X \to Y$ a G-equivariant map. There is a unique function $\phi \colon B \to B'$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p"'] & Y \arrow[d, two heads, "p'"] \\ B \arrow[r, dashed, "\phi"] & B' \end{tikzcd}$$

Proof. Let $\phi_1 \colon X/G \to B$ and $\phi_2 \colon Y/G \to B'$ the unique homeomorphisms from Lemma 3.9 and $\phi' \colon X/G \to X/G$ the unique induced map from Proposition 2.3. We now have the following diagram:

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, two heads, "p"'] \arrow[d, two heads] & Y \arrow[d, two heads] \arrow[dr, two heads, "p'"] & \\ B & X/G \arrow[l, tail, two heads, "\phi_1"] \arrow[r, "\phi'"'] & Y/G \arrow[r, tail, two heads, "\phi_2"'] & B' \end{tikzcd}$$

Here the vertical arrows are the projection to the orbit space. Clearly $\phi = \phi_2 \circ \phi' \circ \phi_1^{-1}$ is a suitable map. Suppose $\psi \colon B \to B'$ has the required property too. We have the following diagram:

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[dl, two heads] \arrow[d, two heads, "p"'] & Y \arrow[d, two heads, "p'"] \arrow[dr, two heads] & \\
X/G \arrow[r, tail, two heads, "\phi_1"'] & B \arrow[r, "\psi"'] & B' & Y/G \arrow[l, tail, two heads, "\phi_2"]
\end{tikzcd}$$

By uniqueness of ϕ', we have that $\phi' = \phi_2^{-1} \circ \psi \circ \phi_1$ and thus $\phi = \psi$. Hence, ϕ is unique. □

The observant reader could have noticed that Lemmas 3.8 and 3.9 describe a bijective correspondence between principal bundles $p\colon X \to B$ for fixed space B and G-space X and homeomorphisms $\phi\colon X/G \to B$. In fact, we can do even better:

Theorem 3.11. *Let B and B' be spaces, X a G-space and $p'\colon X \to B'$ a principal bundle. Then there is a bijective correspondence between principal bundles $p\colon X \to B$ and homeomorphisms $\phi\colon B' \to B$.*

Proof. By Lemma 3.8, we see that the map $\phi \mapsto \phi \circ p'$ sending homeomorphisms $\phi\colon B' \to B$ to principal bundles $p\colon X \to B$ is well-defined. Let $p\colon X \to B$ a principal bundle. Lemma 3.9 gives a unique homeomorphisms $\phi_1\colon X/G \to B$ and $\phi_2\colon X/G \to B'$ such that $\phi_1^{-1} \circ p = \phi_2^{-1} \circ p'$ are the projection $X \to X/G$. Clearly $p = \phi_1 \circ \phi_2^{-1} \circ p'$ and $\phi_1 \circ \phi_2^{-1}\colon B' \to B$ is a homeomorphism. Hence, the mapping is surjective. Let $\phi\colon B' \to B$ another homeomorphism such that $\phi \circ p' = p$. Then $(\phi \circ \phi_2)^{-1} \circ p = \phi_2^{-1} \circ \phi^{-1} \circ \phi \circ p' = \phi_2^{-1} \circ p'$ is the projection $X \to X/G$. Hence, $\phi \circ \phi_2 = \phi_1$ and thus $\phi = \phi_1 \circ \phi_2^{-1}$. So the mapping is injective and thus bijective. □

Notice that this classification theorem classifies for every G-space X, which admits at least one principal bundle, all principal bundles $p\colon X \to B$. In the case that X does not admit a principal bundle or equivalently, if the projection $\pi\colon X \to X/G$ is not a principal bundle, there still are spaces B homeomorphic to X/G (e.g., $B = X/G$). However, there is (obviously) no principal bundle $p\colon X \to B$.

We conclude this section by an explicit example of a principal bundle:

Example 3.12 ([Die08, example 14.1.14])**.** Consider the action $\phi\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ defined by $(k, x) \mapsto x + k$. The group $\mathbb{Z}$ has the discrete topology. Let $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection to the orbit space. Define the (continuous) map $f\colon \mathbb{R} \to S^1$ as $f(x) = \exp(2\pi i x)$ for all $a \in X$. Since $f(x + k) = f(x)$ for all $x \in X$ and $k \in \mathbb{Z}$, the induced map $\overline{f}\colon \mathbb{R}/\mathbb{Z} \to S^1$ is continuous.
For every interval (a, b) with $b - a < 1$, there holds $f^{-1}(f((a, b))) = \sqcup_{i \in \mathbb{Z}}(a + i, b + i)$. Therefore, we have an obvious trivialisation $\psi\colon \sqcup_{i \in \mathbb{Z}} (a + i, b + i) \to \mathbb{Z} \times f((a, b))$ given by $\psi(x) = (\lfloor x - a \rfloor, f(x))$. In particular note that the floor function is continuous on the given domain. This makes f a principal bundle
Let $[x]$ be the orbit of x for all $x \in X$. Now $y \in [x]$ precisely when $f(x) = f(y)$. Indeed, if $y \in [x]$, there exists a $k \in \mathbb{Z}$ such that $y = x + k$ and thus $f(x) = \exp(2\pi i x) = \exp(2\pi i(x + k)) = f(y)$. Conversely, if $f(x) = f(y)$, then $1 = \frac{f(x)}{f(y)} = \exp(2\pi i x)\exp(-2\pi i y) = \exp(2\pi i(x - y)$ and thus $x - y \in \mathbb{Z}$.
Hence, the induced map $\overline{f}$ is injective. Since f is surjective, $\overline{f}$ is surjective and thus bijective. Notice that $\pi([0, 1]) = \mathbb{R}/\mathbb{Z}$ and thus $\mathbb{R}/\mathbb{Z}$ is compact. Since S^1 is Hausdorff, we get that $\overline{f}$ is a homeomorphism. So the orbit space of $\mathbb{Z}$ acting on $\mathbb{R}$ “is” S^1. △

3.2 Bundle Isomorphisms

In the last section, we considered principal bundles for a fixed structure group and fixed G-space X. As showed in the last section, all possible bundles have homeomorphic "orbit" spaces. In this section, we still keep the structure group fixed, but instead of varying the orbit space, we will vary the G-space X. We firstly take the orbit spaces to be homeomorphic:

Theorem 3.13. *Consider two principal bundles $\pi_X \colon X \to X/G$ and $\pi_Y \colon Y \to Y/G$ and an equivariant map $f \colon X \to Y$. If the induced function $\phi \colon X/G \to Y/G$ on the orbit spaces is a homeomorphism, then f is a G-homeomorphism.*

Proof. Since ϕ is a bijection, by Proposition 3.6 and Lemmas 2.5 and 2.8, we see that f is a bijection. It remains to show that f^{-1} is continuous. Let $y \in Y$. Then there are trivialising opens $U \subseteq X/G$ containing $\phi^{-1}(\pi_Y(y))$ and $V \subseteq Y/G$ containing $\pi_Y(y)$ of π_X and π_Y with trivialisation ψ_U and ψ_V respectively. Now $y \in \psi_V(G \times (\phi(U) \cap V)$. Lemma 3.5 shows that the restrictions $\psi_U|_{G\times(U\cap\phi^{-1}(V))}$ and $\psi_V|_{G\times(\phi(U)\cap V)}$ are trivialisations for (π_X, U) and (π_Y, V) respectively. So we have the following commuting diagram:

$$\begin{tikzcd}
G \times (U \cap \phi^{-1}(V)) \arrow[r, tail, two heads, "\psi_U"] \arrow[dr, two heads] & \pi_X^{-1}(U \cap \phi^{-1}(V)) \arrow[r, tail, two heads, "f"] \arrow[d, two heads, "\pi_X"] & \pi_Y^{-1}(\phi(U) \cap V) \arrow[d, two heads, "\pi_Y"] & G \times (\phi(U) \cap V) \arrow[l, tail, two heads, "\psi_V"'] \arrow[dl, two heads] \\
& U \cap \phi^{-1}(V) \arrow[r, tail, two heads, "\phi"] & \phi(U) \cap V &
\end{tikzcd}$$

Define $p := (\psi_V^{-1} \circ f \circ \psi_U)|_{G\times(U\cap\phi^{-1}(V))}$. The diagram gives for all $x \in U \cap \phi^{-1}(V)$ and e, the identity of G, that $p(e, x) = (s(x), \phi(x))$ for some continuous $s \colon U \cap \phi^{-1}(V) \to G$. Proposition 2.12 shows that p is an equivariant map and thus $p(g, x) = g \cdot p(e, x) = g(s(x), \phi(x)) = (g \cdot s(x), \phi(x))$ for all $(g, x) \in G \times (U \cap \phi^{-1}(V))$.

Define $p^{-1} \colon G \times (\phi(U) \cap V) \to G \times (U \cap \phi^{-1}(V))$ as

$$p^{-1}(g, x) = \Big(g \cdot \big(s(\phi^{-1}(x))\big)^{-1}, \phi^{-1}(x)\Big) \text{ for all } (g, x) \in G \times (\phi(U) \cap V).$$

Clearly p^{-1} is continuous. We will show that p^{-1} is an inverse of p. Observe that

$$p\left(p^{-1}(g, x)\right) = p\Big(g \cdot \big(s(\phi^{-1}(x))\big)^{-1}, \phi^{-1}(x)\Big) = \Big(g \cdot \big(s(\phi^{-1}(x))\big)^{-1} \cdot s(\phi^{-1}(x)), x\Big) = (g, x)$$

for all $(g, x) \in G \times (\phi(U) \cap V)$. We also have

$$p^{-1}\Big(p(g, x)\Big) = p^{-1}\Big(g \cdot s(x), \phi(x)\Big) = \Big(g \cdot s(x)\big(s(x)\big)^{-1}, x\Big) = (g, x)$$

for all $(g, x) \in G \times (U \cap \phi^{-1}(V))$. Thus p^{-1} is the inverse of p.

Notice that $f|_{\psi_U(G\times(U\cap\phi^{-1}(V)))} = (\psi_V \circ p \circ \psi_U^{-1})|_{\psi_U(G\times(U\cap\phi^{-1}(V)))}$. Hence, there holds that $f^{-1}|_{\psi_V(G\times(\phi(U)\cap V))} = \psi_U \circ p^{-1} \circ \psi_V^{-1}|_{\psi_V(G\times(\phi(U)\cap V))}$ and thus $f^{-1}|_{\psi_V(G\times(\phi(U)\cap V))}$ is continuous. We have shown that f^{-1} is continuous on an open around y, since $y \in Y$ was chosen arbitrarily, f^{-1} is continuous. □

Taking the orbit spaces equal, we can define the following, after [Mit11, p. 2]:

Definition 3.14. We say two principal bundles $p \colon X \to B$ and $p' \colon Y \to B$ are **isomorphic** when there exists a G-homeomorphism $f \colon X \to Y$ such that $p' \circ f = p$. In other words f "induces" the identity on the "orbit space". ▲

Using the previous lemma, we can characterise isomorphisms:

Corollary 3.15. *Let $p\colon X \to B$ and $p'\colon Y \to B$ principal bundles. Then a G-equivariant map $f\colon X \to Y$ is an isomorphism if and only if $p' \circ f = p$.*

Proof. Left to right is trivial. Suppose the right-hand side. By Lemma 3.9, there are homeomorphisms $\phi\colon X/G \to B$ and $\phi'\colon Y/G \to B$ such that $\phi^{-1} \circ p$ and $\phi'^{-1} \circ p'$ are the projections $X \to X/G$ and $Y \to Y/G$ respectively. We have obtained the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "p"'] & Y \arrow[d, two heads, "p'"] \\
B \arrow[r, tail, two heads, "Id"] \arrow[d, tail, two heads, "\phi"'] & B \arrow[d, tail, two heads, "\phi'"] \\
X/G & Y/G
\end{tikzcd}
$$

Clearly we can restrict to the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "\phi \circ p"'] & Y \arrow[d, two heads, "\phi' \circ p'"] \\
X/G \arrow[r, tail, two heads, "\phi' \circ \phi^{-1}"] & Y/G
\end{tikzcd}
$$

We observe that $\phi' \circ \phi^{-1}$ is the induced map by f on the orbit spaces and $\phi' \circ \phi^{-1}$ is a homeomorphism. Theorem 3.13 shows that the map f is a G-homeomorphism and thus f is an isomorphism. $\square$

We can immediately conclude that any trivialisation $\psi\colon p^{-1}(U) \to G \times U$ for an equivariant map $p\colon X \to B$, is an isomorphism between $p|_{p^{-1}(U)}$ and the projection $G \times U \to U$ and vice versa.

In the introduction of this chapter, we considered sections of the projection $X \to X/G$ for some free G-space X. We managed to construct a bijection $\psi\colon G \times X/G \to X$ with $\psi(gh) = g\psi(h)$ for all $g, h \in G$. However, neither ψ nor ψ^{-1} need be continuous. By the following theorem, we see that for principal bundles the only requirement for ψ to a G-homeomorphism is that the chosen section of the projection $X \to X/G$ is continuous:

Theorem 3.16 ([Mit11, Proposition 2.2])**.** *Let $p\colon X \to B$ a principal bundle. Then there exists a continuous section $s\colon B \to X$ of p if and only if there exists a trivialisation $\psi\colon G \times B \to X$ for p, i.e., p is a trivial G-bundle.*

Proof. Suppose $\psi\colon G \times B \to X$ is a trivialisation. Define $s\colon B \to X$ as $s(x) = \psi(e, x)$ for all $x \in B$, where e is the identity of G. Clearly s is continuous. Now $p(s(x)) = p(\psi(e, x)) = x$ for all $x \in B$, since ψ is a trivialisation. Hence, s is a continuous section of p.
Conversely, suppose that $s\colon B \to X$ is a section of p. Define $\psi\colon G \times B \to X$ as $\psi(g, x) = gs(x)$ for $(g, x) \in G \times B$. We have that $\psi(gh, x) = ghs(x) = g\psi(h, x)$ for all $g, h \in G$ and $x \in B$. Clearly ψ is continuous. Hence, ψ is an equivariant map. The projection $\pi\colon G \times B \to B$ is a principal bundle. Notice that $p(\psi(g, x)) = p(gs(x)) = p(s(x)) = x$ for all $x \in X$ and $g \in G$. We conclude by Corollary 3.15 that ψ is an isomorphism between p and π and thus a trivialisation. $\square$

While this theorem is nice when we already have principal bundles, it does not help us in proving some map is a principal bundle. For this we can instead use the following assertion:

Proposition 3.17. *Let X a G-space. The projection $p\colon X \to X/G$ is a trivial bundle if and only if there exists a continuous section s of p and an equivariant map $f\colon X \to G$.*

Proof. By the preceding theorem and the fact that the projection to G of a trivialisation is an equivariant map, the condition is required. Conversely, suppose we have a continuous section s and an equivariant map f as in the assertion. Define (continuous) maps $\psi\colon G \times X/G \to X$ as $\psi(g,b) = gs(b)$ for $(g,b) \in G \times X/G$, and $\psi^{-1}\colon X \to G \times X/G$ as

$$\psi^{-1}(x) = \Big(f(x)\big(f(s(p(x)))\big)^{-1}, p(x)\Big).$$

As in the last theorem ψ is an equivariant map over the identity on X/G.
We show that ψ and ψ^{-1} are inverses. Therefore, we have a G-homeomorphism as requested. For all $x \in X$, there is an $h \in G$ such that $hx = s(p(x))$. We now see that:

$$\psi(\psi^{-1}(x)) = f(x)\Big(f\big(s(p(x))\big)\Big)^{-1}s(p(x)) = f(x)\Big(f\big(hx\big)\Big)^{-1}hx = f(x)\Big(f\big(x\big)\Big)^{-1}h^{-1}hx = x.$$

For all $(g,b) \in G \times B$, we have:

$$\psi^{-1}(\psi(g,b)) = \bigg(f\big(gs(b)\big)\Big(f\big(s(p(gs(b)))\big)\Big)^{-1}, p\big(gs(b)\big)\bigg) = \bigg(gf\big(s(b)\big)\Big(f\big(s(b)\big)\Big)^{-1}, b\bigg) = (g,b).$$

This finishes the proof. ☐

The condition in the last proposition, can be sharpened: the existence of the equivariant map implies the existence of a section.

Corollary 3.18. *Let X a G-space. If there exists an equivariant map $f\colon X \to G$, then there exists a section of the projection $p\colon X \to X/G$. Hence, the projection p is a principal bundle if and only if there exists an equivariant map $f\colon X \to G$.*

Proof. Define $s\colon X/G \to X$ as $s([x]) = (f(x))^{-1}x$. This clearly is a well-defined continuous section of p. ☐

Applying this assertion, we see that for G-spaces X, which locally admit an equivariant map to G, the projection $X \to X/G$ is a principal bundle.

3.3 Constructions on Principal Bundles

From a given principal bundle, we can construct new principal bundles. In this section, I consider some constructions we will need later on. These include pullbacks, changing the structure group to a subgroup and restrictions. We start with the pullback, following [Mit11, p. 3]:

Let $p\colon X \to B$ a principal bundle and $f\colon C \to B$ a map $f\colon C \to B$. We define the pullback f^*p as the map $p'\colon X \times_B C \to C$, with $X \times_B C = \{(x,c) \mid p(x) = f(c)\}$, by projection on the second coordinate. Endow the subset $X \times_B C \subseteq X \times C$ with the subspace topology and the group action $G \times (X \times_B C) \to (X \times_B C)$ given by $(g,(x,c)) \mapsto (gx,c)$. We observe that $X \times_B C$ is a G-space. Let G act trivially on C. For all $(g,(x,c)) \in G \times (X \times_B C)$, we have that $p'(gx,c) = c = gc = gp'(x,c)$ and thus p' is a G-equivariant map.

Proposition 3.19. *Let $\mathcal{U}$ a trivialising cover for a principal bundle $p\colon X \to B$ and $f\colon C \to B$ a map. Then the cover $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a trivialising cover for the projection $p'\colon X \times_B C \to C$. In particular, p' is a principal bundle.*

Proof. We have the following diagram:

$$\begin{tikzcd} X \times_B C \arrow[r, "Pr_X"] \arrow[d, two heads, "p'"'] & X \arrow[d, two heads, "p"] \\ C \arrow[r, "f"] & B \end{tikzcd}$$

Here $Pr_X : X \times_B C \to X$ is the projection. Note that by surjectivity of p for every $c \in C$, there exists an $x \in X$ such that $p(x) = f(c)$ and thus $(x, c) \in X \times_B C$. This legitimates the double head on p'. By the argument in the text, we know that p' is an equivariant map.

Let $U \in \mathcal{U}$ and ψ a trivialising for U for the bundle p. Define $\psi' : G \times f^{-1}(U) \to p'^{-1}(f^{-1}(U))$ as $\psi'(g, c) = (\psi_U(g, f(c)), c)$ for all $(g, c) \in G \times f^{-1}(U)$. Notice that $f(c) \in U$, $p(\psi(g, f(c))) = f(c)$ and $p'(\psi'(g, c)) = c \in f^{-1}(U)$ for all $(g, c) \in G \times f^{-1}(U)$. Hence, the map ψ' is well-defined. Clearly ψ' is continuous and for all $g, h \in G$ and $c \in f^{-1}(U)$. Furthermore, there holds $h\psi'(g, c) = h(\psi(g, f(c)), c) = (\psi(hg, f(c)), c) = \psi'(hg, c)$ and thus ψ' is a G-equivariant map.

We define the map $\psi'^{-1} : p'^{-1}(f^{-1}(U)) \to G \times f^{-1}(U)$ as $\psi'^{-1}(x, c) = (Pr_G(\psi^{-1}(x)), c)$ for all $(x, c) \in p'^{-1}(f^{-1}(U))$. Notice that $p(x) = f(c) \in U$ for all $(x, c) \in p'^{-1}(f^{-1}(U))$ and thus ψ'^{-1} is well-defined. Clearly ψ'^{-1} is continuous. Furthermore,

$$\psi'^{-1}\Big(\psi'(g, c)\Big) = \psi'^{-1}\Big(\psi(g, f(c)), c\Big) = \Big(Pr_G\Big(\psi^{-1}\big(\psi(g, f(c))\big)\Big), c\Big) = (g, c)$$

for all $(g, c) \in G \times f^{-1}(U)$. For all $(x, c) \in p'^{-1}(f^{-1}(U))$, we have:

$$\psi'\Big(\psi'^{-1}(x, c)\Big) = \psi'\Big(\big(Pr_G(\psi^{-1}(x)), c\big)\Big) = \Big(\psi\big(Pr_G(\psi^{-1}(x)), f(c)\big), c\Big) =$$

$$\Big(\psi\big(Pr_G(\psi^{-1}(x)), p(x)\big), c\Big) = \Big(\psi(\psi^{-1}(x)), c\Big) = (x, c).$$

We conclude that ψ' is a G-homeomorphism and thus a trivialisation for $f^{-1}(U)$ for p'. Hence, the cover $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a trivialising cover for p'. In particular, p' is a principal bundle. $\square$

Pullbacks have the following "uniqueness" property:

Proposition 3.20. *Let $p : X \to B$ and $p' : Y \to B'$ principal bundles, $f : X \to Y$ a G-equivariant map and $\phi : B \to B'$ such that $\phi \circ p = p' \circ f$ (ϕ is the induced function from Proposition 3.10). Then there is an isomorphism $\psi : X \to Y \times_{B'} B$ between p and $\phi^* p'$ such that $f \circ \psi^{-1}$ is the projection $Y \times_{B'} B \to Y$.*

Proof. By assumption, we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p"'] & Y \arrow[d, two heads, "p'"] \\ B \arrow[r, "\phi"] & B' \end{tikzcd}$$

We define $\psi\colon X \to Y \times_{B'} B$ by $\psi(x) = (f(x), p(x))$ for all $x \in X$. Using the diagram, we see that $p'(f(x)) = \phi(p(x))$ and thus the map ψ is well-defined. Furthermore, we see that $\psi(gx) = (f(gx), p(gx)) = g(f(x), p(x)) = g\psi(x)$ for all $g \in G$ and $x \in X$. Hence, ψ is a G-equivariant map. By Proposition 3.19, ϕ^*p' is a principal bundle, and we have by definition $p(x) = \phi^*p'(\psi(x))$. We now have the following diagram:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, two heads, "p"'] & X \times_{B'} B \arrow[d, two heads, "\phi^*p'"] \\ B \arrow[r, "Id"] & B \end{tikzcd}$$

We conclude by Corollary 3.15 that ψ is an isomorphism. Clearly $f \circ \psi^{-1}$ is the projection $Y \times_{B'} B \to Y$. □

Secondly, we consider changing the structure group. We follow [Mit11, p. 5] for this construction. A G-space X can be considered as an H-space for any subgroup H of G. A natural question is: when is the projection $X \to X/H$ a principal bundle? When $X \to X/G$ is a principal G-bundle, not all subgroups H of G give principal bundles. For example, take $X = G = \mathbb{R}$ under addition and $H = \mathbb{Q}$. Suppose the projection $X \to X/H$ is a principal bundle. Since $\mathbb{R}/\mathbb{Q}$ has the trivial topology, we have a trivial bundle. Hence, there is a homeomorphism $\mathbb{R} \to \mathbb{Q} \times (\mathbb{R}/\mathbb{Q})$. However, the right-hand side is a disconnected space.
The necessary and sufficient condition in this case is, as we will see in the next proposition, that the projection $G \to G/H$ is a principal bundle. We call any subgroup of G with this property an **admissible** subgroup. We have seen in Example 3.3 that any subgroup of a discrete group is admissible. Closed subgroups of Lie groups are other examples of admissible subgroups, see [BD03, Theorem 4.3 p. 33].

Proposition 3.21. *Let $p\colon X \to B$ a non-empty principal bundle with structure group G. For every subgroup $H \leqslant G$, the projection $X \to X/H$ is a principal H-bundle if and only if H is admissible.*

Proof. The first half of this proof follows a remark of [Mit11], the second half is different.

Suppose that $X \to X/H$ is a principal bundle. Let $x_0 \in X$. There is an obvious H-equivariant map $G \to X$, taking $g \mapsto gx_0$. Pulling back its induced function $G/H \to X/H$, leaves a principal bundle $X \times_{X/H} G/H \to G/H$. Let $t\colon C(X) \to G$ the translation map (for the bundle p). Notice that if $(x, [g]_H) \in X \times_{X/H} G/H$, then $[gx_0]_H = [x]_H$. Hence, there is an $h \in H$ such that $x = hgx_0$. Now there are H-equivariant maps $G \to X \times_{X/H} G/H$ taking $g \mapsto (gx_0, [g]_H)$ and $X \times_{X/H} G/H \to G$ taking $(x, [g]_H) \to t(x_0, x)$. It is clear both functions are well-defined, and they are inverses of each other. Hereby $G \to G/H$ is a principal bundle (use Lemma 3.8).

Conversely, let H be admissible. Write $\pi\colon G \to G/H$ for the projection. Write $\pi'\colon X \to X/H$ for the projection. For every $x \in X$, there is a trivialising open $U \subseteq B$ containing $p(x)$. There is a G-equivariant map $f\colon p^{-1}(U) \to G$. Now there is a trivialising open $V \subseteq G/H$ for π containing $\pi(f(x))$. Hence, an H-equivariant map $f'\colon \pi^{-1}(V) \to H$. Let $W := f^{-1}(\pi^{-1}(V)) \cap p^{-1}(U)$. Notice that $x \in W$ and that W is closed under the H-action. Hence, $\pi'(W)$ is open and $\pi'^{-1}(\pi'(W)) = W$. The restriction $f' \circ f|_W\colon W \to H$ is an H-equivariant map. Using Corollary 3.18, we see that π' is a principal bundle. □

Lastly, we consider restrictions:

Proposition 3.22. *For a principal bundle $p\colon X \to B$ and a subset $A \subseteq B$, the restriction $p|_{p^{-1}(A)}\colon p^{-1}(A) \to A$ is also a principal bundle.*

Proof. Take a trivialising cover $\mathcal{U}$ of B. Let $\mathcal{V} := \{U \cap A \mid U \in \mathcal{U}\}$. By Lemma 3.5, every $V \in \mathcal{V}$ is trivialising for p and thus for $p|_{p^{-1}(A)}$. Clearly $\mathcal{V}$ is a cover of A.
Since for all $g \in G$ and $x \in p^{-1}(A)$, we have that $p(gx) = gp(x) = p(x) \in A$. We see that $gx \in p^{-1}(A)$. Hence, $p^{-1}(A)$ is closed under the group action. Clearly $p|_{p^{-1}(A)}$ is an equivariant map. We conclude that $p^{-1}(A)$ is a G-space and $p|_{p^{-1}(A)}$ a principal bundle. □

3.4 Trivialising Sets

Trivialising sets for an equivariant map $p\colon X \to B$ can, a priori, be arbitrary subsets of B. Obviously they need be trivialising, but other than that, there are no conditions. In this section, I will provide ways to transform trivialising sets into other, maybe "simpler", trivialising sets. This will be, in particular, useful when considering partitions of unity, see Chapter 4. We have already seen that we can restrict trivialisation to subsets, see Lemma 3.5. In this section, I will consider merging trivialising opens. We firstly consider the case of disjoint unions, secondly gluing on I.

Lemma 3.23. *Let $p\colon X \to B$ a G-equivariant map and $\mathcal{V}$ a set of trivialising opens. If $\mathcal{V}$ is pairwise disjoint (i.e., for all $V_1, V_2 \in \mathcal{V}$ holds $V_1 \cap V_2 = \emptyset$), then $\bigsqcup \mathcal{V} = \bigsqcup_{V\in\mathcal{V}} V$ is a trivialising open.*

Proof. We will construct a G-homeomorphism $\psi\colon G \times \bigsqcup \mathcal{V} \to p^{-1}(\bigsqcup \mathcal{V})$ such that $p(\psi(g, x)) = x$ for all $g \in G$ and $x \in X$.
For every $V \in \mathcal{V}$, there exists a trivialisation ψ_V for V. Define $\psi\colon G \times \bigsqcup \mathcal{V} \to p^{-1}(\bigsqcup \mathcal{V})$ as $\psi|_{G\times V}(g, x) = \psi_V(g, x)$ for all $g \in G$ and $x \in V$ for all $V \in \mathcal{V}$. Notice that this map is well-defined, since $\mathcal{V}$ is pairwise disjoint. For all $x \in \sqcup\mathcal{V}$, there is a $V \in \mathcal{V}$ with $x \in V$ and thus $\psi(gh, x) = \psi_V(gh, x) = g\psi_V(h, x) = g\psi(g, x)$ for all $g, h \in G$. Also $\psi|_{G\times V}$ is clearly continuous and thus ψ is continuous. We conclude that ψ is an equivariant map. Moreover, $p(\psi(g, x)) = p(\psi_V(g, x)) = x$ for all $g \in G$. Clearly $p|_{p^{-1}(\bigsqcup\mathcal{V})}$ and the projection $G \times \bigsqcup \mathcal{V} \to p^{-1}(\bigsqcup \mathcal{V})$ are principal bundles. By Corollary 3.15, ψ is a G-homeomorphism. □

Lemma 3.24 ([Die08, Proposition 3.1.4])**.** *Consider a principal bundle $p\colon X \to B \times I$ (G acting trivially on $B \times I$). Let $U \subseteq B$ and $0 = q_0 < q_1 < \cdots < q_{n-1} < q_n = 1$ such that $U \times [q_{i-1}, q_i]$ is a trivialising set for all $0 < i \leq n$. Then $U \times I$ is trivialising.*

Proof. For $a, b, c \in I$ with $a < b < c$, we will proof that if $U \times [a, b]$ and $U \times [a, c]$ are trivialising, then $U \times [a, c]$ is trivialising. The general case follows by induction on n.
Take trivialisations $\psi_1\colon G \times U \times [a, b] \to p^{-1}(U \times [a, b])$ and $\psi_2\colon G \times U \times [b, c] \to p^{-1}(U \times [b, c])$. Define $\psi'\colon G\times U \to G\times U$ as $\psi'(g, x) = Pr_{G\times U}(\psi_2^{-1}(\psi_1(g, x, b)))$, where $Pr_{G\times U}$ is the projection $G \times U \times I \to G \times U$. The projection $P_{G\times U}$ is an equivariant map and thus by Propositions 2.11 and 2.12, ψ' is an equivariant map.
Define

$$\psi\colon G \times U \times [a, c] \to p^{-1}(U \times [a, c]) \text{ given by } \psi(g, x, s) = \begin{cases} \psi_1(g, x, s) & \text{if } s \in [a, b] \\ \psi_2(\psi'(g, x), s) & \text{if } s \in [b, c] \end{cases}.$$

Notice that $\psi_2(\psi'(g, x), b) = \psi_2(Pr_{G\times U}(\psi_2^{-1}(\psi_1(g, x, b)), b) = \psi_1(g, x, b)$ for all $(g, x) \in G \times U$. Also observe that both $G \times U \times [a, b]$ and $G \times U \times [a, c]$ are closed in $G \times U \times [a, c]$. Hence, by the Pasting Lemma (Lemma A.2), the map ψ is well-defined and continuous. For all $g, h \in G$, $x \in U$, $s \in [a,c]$ holds

$$h\psi(g,x,s) = h \cdot \begin{cases} \psi_1(g,x,s) & \text{if } s \in [a,b] \\ \psi_2(\psi'(g,x),s) & \text{if } s \in [b,c] \end{cases} = \begin{cases} h\psi_1(g,x,s) & \text{if } s \in [a,b] \\ h\psi_2(\psi'(g,x),s) & \text{if } s \in [b,c] \end{cases} =$$

$$\begin{cases} \psi_1(hg,x,s) & \text{if } s \in [a,b] \\ \psi_2(\psi'(hg,x),s) & \text{if } s \in [b,c] \end{cases} = \psi(hg,x,s).$$

Thus ψ is an equivariant map.
Notice that the U coordinate of $\psi'(g,x,s) = Pr_{G\times U}(\psi_2^{-1}(\psi_1(g,x,s)))$ equals x for all points $(g,x,s) \in G \times U \times [a,c]$. Hereby, we get

$$p(\psi(g,x,s)) = \begin{cases} p(\psi_1(g,x,s)) & \text{if } s \in [a,b] \\ p(\psi_2(\psi'(g,x),s)) & \text{if } s \in [b,c] \end{cases} = (x,s).$$

By Proposition 3.22, the restriction $p_{p^{-1}(U\times[a,c])}$ is a principal bundle. We conclude using Corollary 3.15 that ψ is a trivialisation for $U \times [a,c]$ and thus $U \times [a,c]$ is a trivialising set. $\square$

Remark 3.25. In the lemma above we assumed that p is a principal bundle. This is in fact not required: one can construct directly (without using Corollary 3.15) a continuous inverse of ψ. This would show that ψ is a trivialisation. The details are simple but technical and are left for the reader.

With the lemma above, we can classify all principal bundles over I. We will later see that this is part of a more general classification (Theorem 5.5 and Corollary 5.6). The proofs of those assertions will involve the lemma above, but the case for I can be done directly.

Corollary 3.26. *Every principal bundle $p\colon X \to I$ is trivial.*

Proof. Let $\mathcal{U}$ a cover of trivialising opens of I. By the Lebesgue Number Lemma (Lemma A.1), there exists a $1 < n \in \mathbb{N}$ such that for all $a, b \in I$ with $0 < b - a < \frac{1}{n-1}$, $[a,b] \subseteq U$ for some $U \in \mathcal{U}$. Let $q_i = \frac{i}{n}$ for all $0 < i \leq n$. Then the set $[\frac{i-1}{n}, \frac{i}{n}]$ is trivialising for all $0 < i \leq n$ (use Lemma 3.5). Hence, by Lemma 3.24, I is a trivialising set and thus p is trivial. $\square$

4 Partitions of Unity

In our further study of principal bundles, we will need partitions of unity. This section is not meant as a complete review of partitions of unity, merely a “toolbox” for our further studies. We will first consider the general case, afterwards we consider Partitions of Unities related to principal bundles. Most assertions and proofs are inspired from [Die08, Chapter 13.1], the corresponding reference is given in each case.

4.1 The General Case

We define partitions of unity:

Definition 4.1. We say that a collection $\mathcal{U}$ of, not necessarily, opens of a space B is **point finite** if every $x \in B$ is contained in only a finite number of $U \in \mathcal{U}$. More strictly, we say that a collection $\mathcal{U}$ is **locally finite** if for every $x \in B$, there exists an open neighbourhood V of x which intersects only with a finite number of $U \in \mathcal{U}$. It is trivial that a locally finite cover is also point finite.
A point/locally finite **partition of unity** subordinate to an open cover $\mathcal{U}$ of a space B is a family of functions $(t_j : B \to I)_{j \in J}$ with the following properties:

- For all $j \in J$, the support of t_j, denoted by $\mathrm{supp}(t_j) = \overline{t_j^{-1}((0,1])}$, i.e., the closure of $t_j^{-1}((0,1])$ lies in U for some $U \in \mathcal{U}$.
- The collection $\{t_j^{-1}((0,1]) \mid j \in J\}$ is point/locally finite.
- The sum $\sum_{j \in J} t_j(x) = 1$ for all $x \in X$.

Notice that the sum in the third condition makes sense, since for every $x \in B$, there is only a finite number of $j \in J$ such that $t_j(x)$ is non-zero. It should be clear that a locally finite partition of unity is also a point finite one. An open cover $\mathcal{U}$ which admits a point finite partition of unity, also admits a locally finite one, see Corollary 4.4. Motivated by this (yet unproven) fact, we define a **numerable cover** as an open cover that admits either a point or locally finite partition of unity. A partition of unity (whether point or locally finite) subordinated to a cover $\mathcal{U}$, we call a **numeration** of $\mathcal{U}$. ▲

Our first goal is to prove the claim in the text above. We start by characterising locally finite case:

Lemma 4.2 ([Die08, Lemma 13.1.5])**.** *For an open over $\mathcal{U}$ of a space B and some index set J the following assertions are equivalent:*

1. *There exists a locally finite partition of unity indexed by $J \times \mathbb{N}$ subordinate to $\mathcal{U}$*
2. *There exists a set of functions $\{t_{j,n} \mid j \in J, n \in \mathbb{N}\}$ with the following properties:*
    - *For all $j \in J$ and $n \in \mathbb{N}$, there exists a $U \in \mathcal{U}$ such that supp$(t_{j,n}) \subseteq U$.*
    - *For all $x \in B$, there exists a $j \in J$ and $n \in \mathbb{N}$ such that $t_{j,n}(x) > 0$.*
    - *For all fixed $N \in \mathbb{N}$, the collection $\{t_{j,N}^{-1}((0,1]) \mid j \in J\}$ is locally finite.*

Proof. 1 $\implies$ 2 is trivial. Suppose 2. Since $\{t_{j,N}^{-1}((0,1]) \mid j \in J\}$ is locally finite, for all $r \in \mathbb{N}$ the following function is continuous:

$$q_r : B \to I \text{ with } q_r(x) = \sum_{\substack{i \in J \\ 0<n<r}} t_{i,n}(x) \text{ for } x \in B.$$

We immediately see that for all $r \in \mathbb{N}$ and $j \in J$ also the next function is continuous:

$$p_{j,r} \colon B \to I \text{ with } q_r(x) = \max\Big(0, t_{j,r}(x) - r \cdot q_r(x)\Big) \text{ for all } x \in B.$$

By assumption, there exists for every $x \in B$ a $j \in J$ and $n \in \mathbb{N}$ such that $t_{j,n}(x) > 0$. Let $r_0 \in \mathbb{N}$ be minimal with the property that $t_{j,r_0}(x) > 0$ for some fixed $j \in J$. Now $t_{j,n}(x) = 0$ for all $j \in J$ if $n < r_0$ and thus $q_{r_0}(x) = 0$. Hence, $p_{j,r_0}(x) = t_{j,r_0}(x) > 0$. Moreover, we can choose an $N \in \mathbb{N}$ such that $t_{j,r_0}(x) > \frac{1}{N}$ and $r_0 < N$. We have $q_N(x) > t_{j,r_0}(x)$ and thus there exists an open neighbourhood V of x such that $q_N(y) > \frac{1}{N}$ for all $y \in V$. Now for $k > N$ and $y \in V$, we have $k \cdot q_k(x) \geq N t_{j,r_0}(y) > 1$ and thus $p_{j,k}(x) = 0$. We conclude that the collection $\{p_{j,r}^{-1}((0,1]) \mid j \in J, r \in \mathbb{N}\}$ is locally finite.
Define

$$v_{j,r} \colon B \to I \text{ as } v_{j,r}(x) = \frac{p_{j,r}(x)}{\sum\limits_{\substack{i \in J \\ n \in \mathbb{N}}} p_{i,r}(x)} \text{ for all } x \in B.$$

This is well-defined since the $p_{i,n}$'s are locally finite and $\sum_{(i,n) \in J \times \mathbb{N}} p_{i,n}(x) > 0$ for all $x \in X$. Moreover, the map $v_{j,r}$ is continuous (again by the locally finiteness of the $p_{i,n}$'s). Furthermore, $\sum_{(j,r) \in J \times \mathbb{N}} v_{j,r} = 1$ and for all $j \in J$ and $r \in \mathbb{N}$, we have $\operatorname{supp}(v_{j,r}) = \operatorname{supp}(p_{j,r}) \subseteq \operatorname{supp}(t_{j,r})$. We conclude that the $v_{j,r}$'s form a locally finite partition of unity subordinated to $\mathcal{U}$ indexed by $J \times \mathbb{N}$. □

Applying the lemma above to some point finite family of functions, we obtain the following:

Theorem 4.3 ([Die08, Lemma 13.1.7]). *Let B a space and J an index set. If there exists a family of functions $\{t_j \colon B \to I \mid j \in J\}$ such that $\mathcal{U} = \{t_j^{-1}((0,1]) \mid j \in J\}$ is point finite and $\sum\limits_{j \in J} t_j = 1$, then there exists a locally finite partition of unity subordinate to $\mathcal{U}$ indexed by $J \times \mathbb{N}$.*

Proof. Define for each $j \in J$ and $n \in \mathbb{N}$ the (continuous) function $v_{j,n} \colon B \to I$ as

$$v_{j,n}(x) = \max\Big(0, t_j(x) - \frac{1}{n}\Big) \text{ for all } x \in B.$$

For $N \in \mathbb{N}$ and $x \in B$ fixed, there is a finite set $E \subseteq J$ such that for all $j \in (J - E)$ holds $t_j(x) = 0$. We define

$$q_E \colon B \to I \text{ as } q_E(y) = 1 - \sum_{j \in E} t_j(y) \text{ for all } y \in B.$$

The function q_E is continuous since it is a finite sum of continuous functions. Observe that $q_E(x) = 1 - \sum_{j \in E} t_j(x) = 1 - \sum_{j \in J} t_j(x) = 0$. Take the open $V = q_E^{-1}([0, \frac{1}{N}))$. Then $x \in V$ and for all $y \in V$ and $j \in (J - E)$ holds:

$$t_j(y) \leq \sum_{j \in (J-E)} t_j(y) = 1 - \sum_{j \in E} t_j(y) = q_E(y) < \frac{1}{N}.$$

Hence, $v_{j,N}(y) = 0$. Thus there is a finite number of $j \in J$ such that $v_{j,N}^{-1}((0,1]) \cap V \neq \emptyset$. In other words $\{v_{j,N}^{-1}((0,1]) \mid j \in J\}$ is locally finite. Observe that $\operatorname{supp}(v_{j,n}) \subseteq t_j^{-1}([\frac{1}{n}, 1]) \subseteq t_j^{-1}((0,1])$ for all $j \in J$ and $n \in \mathbb{N}$. Notice that for all $x \in B$, there is a $j \in J$ with $t_j(x) > 0$. Thus there exists an $n \in \mathbb{N}$ with $t_j(x) > \frac{1}{n}$. Hence, $v_{j,n}(x) \neq 0$. By applying Lemma 4.2, we obtain a locally finite partition of unity subordinate to $\mathcal{U}$ indexed by $J \times \mathbb{N}$. □

In particular, we can conclude:

Corollary 4.4. *If for an open cover $\mathcal{U}$, there exists a point finite partition of unity subordinate to $\mathcal{U}$, then there also exists a locally finite one. Moreover, if $\mathcal{U}$ admits a countable point finite partition of unity, then it also admits a countable locally finite one.*

Proof. Apply Theorem 4.3 to a (countable) point finite partition of unity. □

We now have proven the claim in the definition of numerable covers. Thus, from here on we can talk about numerable covers without ambiguity. Observe that in Theorem 4.3, we have not assumed that the supports of the t_j's must lay in some $U \in \mathcal{U}$. Instead, we took a cover of preimages $t_j^{-1}((0,1])$. In fact it is not necessary to do this assumption in Lemma 4.2 as the following lemma will make clear:

Lemma 4.5 ([Die08, Lemma 13.1.5])**.** *Suppose that for an open over $\mathcal{U}$ of a space B, there exists a family of functions $\{t_{j,n} \mid j \in J, n \in \mathbb{N}\}$ (for some index set J) with the following properties:*

- *For all $j \in J$ and $n \in \mathbb{N}$, there exists a $U \in \mathcal{U}$ such that $t_{j,n}^{-1}((0,1]) \subseteq U$.*
- *For all $x \in B$, there exists $j \in J$ and $n \in \mathbb{N}$ such that $t_{j,n}(x) > 0$.*
- *For all fixed $N \in \mathbb{N}$, the collection $\{t_{j,N}^{-1}((0,1]) \mid j \in J\}$ is locally finite.*

Then $\mathcal{U}$ is numerable with a numeration indexed by $J \times \mathbb{N}$.

Proof. We argue similar as in Theorem 4.3: Define for each $j \in J$ and $k, n \in \mathbb{N}$ (continuous) functions $v_{j,n,k} \colon B \to I$ as $v_{j,n,k}(x) = \max(0, t_{j,k}(x) - \frac{1}{n})$ for all $x \in B$. Since $\mathbb{N} \times \mathbb{N}$ is countable, up to re-indexing, the $v_{j,N,K}$'s can be indexed by $J \times \mathbb{N}$. For a fixed pair $(N, K) \in \mathbb{N}^2$, there holds $v_{j,N,K}^{-1}((0,1]) \subseteq t_{j,K}^{-1}((0,1])$ for all $j \in J$ and thus $\{v_{j,N,K}^{-1}((0,1]) \mid j \in J\}$ is locally finite. There exists a $U \in \mathcal{U}$ such that $\operatorname{supp}(v_{j,n,k}) \subseteq t_{j,k}^{-1}([\frac{1}{n}, 1]) \subseteq t_j^{-1}((0,1]) \subseteq U$ for all $j \in J$ and $n, k \in \mathbb{N}$. Notice that for all $x \in B$, there is a $j \in J$ and $k \in \mathbb{N}$ with $t_{j,k}(x) > 0$. Thus there exists an $n \in \mathbb{N}$ with $t_{j,k}(x) > \frac{1}{n}$. Hence, $v_{j,k,n}(x) \neq 0$. Apply Lemma 4.2 and we obtain that $\mathcal{U}$ is numerable with a numeration indexed by $J \times \mathbb{N}$. □

The following application of Lemma 4.5 is, in particular, useful when considering homotopies:

Lemma 4.6 ([Die08, Lemma 13.1.6])**.** *Let $\mathcal{U}$ a numerable cover of $B \times I$. Then there exists a numerable cover $\mathcal{V}$ of B such that for all $V \in \mathcal{V}$ there are $0 = q_0 < q_1 < \cdots < q_{M-1} < q_M = 1$ such that for all $0 < i \leq M$, there exists a $U \in \mathcal{U}$ with $V \times [q_{i-1}, q_i] \subseteq U$.*

Proof. Let $\{t_j \mid j \in J\}$ a locally finite partition of unity subordinated to $\mathcal{U} = \{U_j \mid j \in J\}$. Define for every $N \in \mathbb{N}$ and $(k_1, \ldots, k_N) = k \in J^N$ the map $v_k \colon B \to I$ given by

$$v_k(x) = \prod_{i=1}^{N} \min\Big(t_{k_i}(x,s) | s \in \Big[\frac{(i-1)}{N}, \frac{i}{N}\Big]\Big).$$

We will show that $\mathcal{V} := \{v_k^{-1}((0,1]) \mid k \in J^N, N \in \mathbb{N}\}$ fulfills the requirements of the assertion. Notice that every v_k is continuous. Hence, every $V \in \mathcal{V}$ is open.

Let $x \in B$. Define the family $\mathcal{I}$ of all $I' \subseteq I$ for which there exists a $V' \subseteq B$ containing x and a $j \in J$ with $V' \times I' \subseteq t_j^{-1}((0,1])$ and $V' \times I' \cap t_i^{-1}((0,1]) \neq \emptyset$ for a finite number of $i \in J$. For every $s \in I$, there is a $j \in J$ with $(x, s) \in t_j^{-1}((0,1])$. Since $\{t_i^{-1}((0,1]) \mid i \in J\}$ is a locally finite family of opens, there exists opens $V' \subseteq B$ and $I' \subseteq I$ such that $(x,s) \in V' \times I' \subseteq t_j^{-1}((0,1])$ and $V' \times I' \cap t_l^{-1}((0,1]) \neq \emptyset$ for only a finite number of $l \in J$. Hereby $\mathcal{I}$ is a cover of I.

By the Lebesgue Number Lemma (Lemma A.1), there exists a $1 < n \in \mathbb{N}$ such that for all $a, b \in \mathbb{N}$ with $0 < b - a < \frac{1}{n-1}$, there is an $I' \in \mathcal{I}$ with $[a,b] \subseteq I'$. For all $1 \leq i \leq n$, there holds $\frac{i}{n} - \frac{i-1}{n} < \frac{1}{n-1}$. Hence, there is an $\bar{I} \in \mathcal{I}$ with $[\frac{(i-1)}{n}, \frac{i}{n}] \subseteq \bar{I}$ and thus we can choose a $j_i \in J$ with $\{x\} \times \bar{I} \subseteq t_j^{-1}((0,1])$. Let $k = (j_1, \ldots, j_n)$. Then $t_{j_i}(x,s) \neq 0$ for all $s \in [\frac{(i-1)}{n}, \frac{i}{n}]$ and $1 \leq i \leq n$ and thus $\min(t_{j_i}(x,s) | s \in [\frac{(i-1)}{n}, \frac{i}{n}]) \neq 0$, since $[\frac{(i-1)}{n}, \frac{i}{n}]$ is compact. We conclude that $v_k(x) \neq 0$. So $\mathcal{V}$ is an open cover of B.

Take a finite sub-cover $I_1, \ldots, I_m$ of $\mathcal{I}$. Let $N \in \mathbb{N}$ fixed. Let $V_1, \ldots, V_m \subseteq B$ such that for all $1 \leq e \leq m$, there holds $x \in V_e$ and $V_e \times I_e \cap t_l^{-1}((0,1]) \neq \emptyset$ for a finite number of $l \in J$. We have that $x \in V_0 := \bigcap_{1 \leq e \leq m} V_e$ and $V_0 \times I \cap t_l^{-1}((0,1]) \neq \emptyset$ for only a finite number of $l \in J$. Hence, V_0 intersect $v_k^{-1}((0,1])$ for only a finite number of $k \in J^N$. We conclude that for fixed $N \in \mathbb{N}$, the collection $\{v_k^{-1}((0,1]) \mid k \in J^N\}$ is locally finite.

Applying Lemma 4.5 shows that $\mathcal{V}$ is numerable.

For all $V \in \mathcal{V}$, there exists an $M \in \mathbb{N}$ and a $(k_1, \ldots, k_M) = k \in J^M$ such that $V = v_k^{-1}((0,1])$. Define $q_i = \frac{i}{M}$ for $0 \leq i \leq M$. Now for all $1 \leq i \leq M$, there exists a $U \in \mathcal{U}$ such that $v_k^{-1}((0,1]) \times [\frac{(i-1)}{M}, \frac{i}{M}] \subseteq t_{k_i}^{-1}((0,1]) \subseteq U$. □

The partitions of unity constructed in Lemmas 4.2 and 4.5 are in a sense not "bigger" than the original ones (except in the case we had a finite partition). More precisely the cardinality of the partition of unity did not increase. I will now provide a construction which reduces an arbitrary large partitions of unity into countable ones. This countable partition of unity, in general, is not subordinate to the original cover (a cover need not admit a countable refinement), so we will need to transform our cover.

Lemma 4.7 ([Die08, corollary 13.1.9]). *Let $\mathcal{U}$ a numerable cover of a space B. Then there exist a countable numerable cover $\mathcal{V}$ of B with countable numeration, such that every $V \in \mathcal{V}$ can be written as disjoint union of open subsets of elements of $\mathcal{U}$. In other words: we can write every $V \in \mathcal{V}$ as $V = \bigsqcup_{k \in K} W_k$ with K some index set, $W_{k_1} \cap W_{k_2} = \emptyset$ for all $K \ni k_1 \neq k_2 \in K$ and for every $k \in K$, there exists a $U \in \mathcal{U}$ such that $W_k \subseteq U$ and W_k open.*

Proof. Let $\{t_j \mid j \in J\}$ a locally finite partition of unity subordinate to $\mathcal{U}$. In the case that J is finite (in fact countable), the theorem is trivial. Assume J is infinite. Define for every finite nonempty subset $E \subseteq J$ the map $q_E \colon B \to I$ given by

$$q_E(x) = \max\Big(0, \min_{e \in E}\big(t_e(x)\big) - \max_{e \in (J-E)}\big(t_e(x)\big)\Big).$$

Every q_E is continuous since the t_j's form a locally finite partition of unity. Moreover, for every $x \in B$, we have the finite set $E_x := \{j \mid t_j(x) > 0\}$ and there holds $q_{E_x}(x) > 0$.

We now claim the following:

Claim 1. If $E, F \subseteq J$ finite and $x \in B$ such that $q_E(x) \neq 0 \neq q_F(x)$, then $E \subseteq F$ or $F \subseteq E$.

Proof. We argue by contradiction. Suppose there are $e \in E - F$ and $f \in F - E$ such that $0 < q_E(x) \leq t_e(x) - t_f(x)$ and $0 < q_F(x) \leq t_f(x) - t_e(x)$: a contradiction. Hence, $E \subseteq F$ or $F \subseteq E$. ■

By the claim, we immediately get that if $|E| = |F|$ and $q_E^{-1}((0,1]) \cap q_F^{-1}((0,1]) \neq \emptyset$, then $E = F$. Define for every $n \in \mathbb{N}$ the map

$$v_n \colon B \to I \text{ as } v_n(y) = \sum_{\substack{E \subseteq J \\ |E|=n}} q_E(y) \text{ for } y \in B.$$

Notice that this definition makes sense since every $y \in B$: $q_E(y)$ can be non-zero for at most one $E \subseteq J$ with $|E| = n$. Define $\mathcal{V} = \{v_n^{-1}((0,1]) \mid n \in \mathbb{N}\}$. For every $V \in \mathcal{V}$, we have some $n \in \mathbb{N}$ such that

$$V = v_n^{-1}((0,1]) = \bigsqcup_{\substack{E \subseteq J \\ |E|=n}} q_E^{-1}((0,1]).$$

Moreover, there is a $U \in \mathcal{U}$ such that $q_E^{-1}((0,1]) \subseteq U$. Notice that $v_{|E_x|}(x) = q_{E_x}(x) > 0$. Applying Lemma 4.5 to the v_n's, we see that $\mathcal{V}$ is numerable, with a numeration indexed by $\mathbb{N}$. We have shown that $\mathcal{V}$ matches the requirements of the assertion. □

4.2 Partitions of Unity on Principal Bundles

A principal bundle $p \colon X \to B$ has, as defined in Chapter 3, a covering of trivialising opens of B. In this chapter, we are considering covers which admit a partition of unity. When these two coverings coincide, we say that p is a **numerable bundle**. In detail:

Definition 4.8. A principal bundle $p \colon X \to B$ is a **numerable bundle** if there exists cover of trivialising opens $\mathcal{U}$ and a partition of unity subordinate to $\mathcal{U}$. We say that such a cover $\mathcal{U}$ is a **numerable cover of trivialising opens**. A partition of unity subordinate to numerable cover of trivialising opens is a **trivialising numeration**. ▲

We will apply two main results of the previous section in the context of principal bundles: reducing partitions to countable ones (Lemma 4.7) and "gluing" on I (Lemma 4.6). The assertions in this section are, while implicitly used in [Die08], not mentioned or proved. I fill in these omissions.

Theorem 4.9. *Every numerable bundle admits a countable numerable cover of trivialising opens, which admits a countable numeration.*

Proof. Let $p \colon X \to B$ a principal bundle and $\mathcal{U}$ a numerable cover of trivialising opens. Then there exists a countable numerable cover $\mathcal{V}$ with countable numeration as in Lemma 4.7. All we have to show is that every $V \in \mathcal{V}$ is a trivialising open. Let $V \in \mathcal{V}$. Write $V = \sqcup_{k \in K} W_k$ with K some index set, $W_k \subseteq U$ for some $U \in \mathcal{U}$ and W_k open for all $k \in K$ and $W_{k_1} \cap W_{k_2} = \emptyset$ for all $K \ni k_1 \neq k_2 \in K$. By Lemma 3.5, the open W_k is trivialising for every $k \in K$. By Lemma 3.23, the open V is trivialising. □

Theorem 4.10. *Let $p \colon X \to B \times I$ a numerable bundle. Then there exists a countable numerable cover $\mathcal{V}$ of B with countable numeration such that for all $V \in \mathcal{V}$, the subset $V \times I \subseteq B \times I$ is trivialising. For this cover $\mathcal{V}$, the family $\{V \times I \mid V \in \mathcal{V}\}$ is a countable numerable cover of trivialising opens for p, which admits a countable numeration.*

Proof. Take a numerable cover of trivialising opens of $B \times I$. There exists a numerable cover $\mathcal{V}_0$ of B as in Lemma 4.6. By Lemma 4.7, there is a countable numerable cover $\mathcal{V}$ of B which admits a countable numeration $\{t_l \colon B \to I \mid l \in \mathbb{N}\}$. We will show that $\mathcal{V}$ fulfills the requirements.
For every $V_0 \in \mathcal{V}_0$, there exist $0 = q_0 < q_1 < \cdots < q_{n-1} < q_n = 1$ such that for all $0 < i \leq n$, $V_0 \times [q_{i-1}, q_i] \subseteq U$ for some $U \in \mathcal{U}$. Thus by Lemma 3.5, the set $V_0 \times [q_{i-1}, q_i]$ is trivialising

for all $0 < i \leq n$. Hence, by Lemma 3.24, the set $V_0 \times I$ is trivialising for all $V_0 \in \mathcal{V}_0$. Now write $V \in \mathcal{V}$ as $V = \sqcup_{k \in K} W_k$ with K some index set, $W_k \subseteq V_0$ for some $V_0 \in \mathcal{V}_0$ and W_k open for all $k \in K$ and $W_{k_1} \cap W_{k_2} = \emptyset$ for all $K \ni k_1 \neq k_2 \in K$. Again by Lemma 3.5, we see that $W_k \times I \subseteq V_0 \times I$ is trivialising. Hence, by Lemma 3.23, the set $V \times I = \sqcup_{k \in K} W_k \times I$ is trivialising.
Since $\mathcal{V}$ is a countable open cover of B, the family $\mathcal{V}' := \{V \times I \mid V \in \mathcal{V}\}$ is a countable open cover of $B \times I$. Moreover, the family of maps $\{v_l : B \times I \to I \mid l \in \mathbb{N}\}$ with $v_l(x,s) = t_l(x)$ for all $(x,s) \in B \times I$ and $l \in \mathbb{N}$ is a countable numeration of $\mathcal{V}'$. □

By the following lemmas, we see that the constructions on principal bundles in Chapter 3.3 preserve numerability of principal bundles.

Lemma 4.11. *For any map $f : C \to B$, the pullback f^*p of a numerable bundle $p : X \to B$ is again numerable.*

Proof. Take a numerable cover $\mathcal{U}$ of trivialising opens for p. Then by Proposition 3.19, the cover $\mathcal{V} := \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a trivialising cover for f^*p. Let $\{t_j \mid j \in J\}$ a (point finite) numeration for $\mathcal{U}$. Then $\{t_j \circ f \mid j \in J\}$ is a numeration of $\mathcal{V}$. Indeed, for every $c \in C$, there are only a finite number of $j \in J$ such that $t_j(f(c)) \neq 0$ and $\sum_{j \in J} t_j(f(c)) = 1$. For all $j \in J$, there is a $U \in \mathcal{U}$ such that $f^{-1}(t_j^{-1}((0,1])) \subseteq f^{-1}(U)$. □

Lemma 4.12. *Let $p : X \to B$ a numerable G-bundle and H an admissible subgroup of G such that the H-bundle $\pi' : G \to G/H$ is numerable. Then the projection $\pi : X \to X/H$ is a numerable H-bundle.*

Proof. Proposition 3.21 shows that π is a principal H-bundle. Write $p' : X/H \to B$ for the canonical projection. Let $\mathcal{U} = \{U_j \mid j \in J\}$ and $\mathcal{V} = \{V_k \mid k \in K\}$ numerable covers of trivialising opens for p and π' respectively. Let $(u_j)_{j \in J}$ and $(v_k)_{k \in K}$ numerations of $\mathcal{U}$ and $\mathcal{V}$ respectively. Choose for every $U_j \in \mathcal{U}$ an equivariant map $f_j : p^{-1}(U) \to G$. Let $\phi_j : p^{-1}(U)/H \to G/H$ the induced map by f_j. By the proof of Proposition 3.21, the family of opens $\mathcal{W} := \{\pi(f_j^{-1}(\pi'^{-1}(V_k)) \cap p^{-1}(U_j)) \mid V_k \in \mathcal{V} \text{ and } U_j \in \mathcal{U}\}$ is a trivialising cover for π. Define for all $j \in J$ and $k \in K$ the map $w_{j,k} : X/H \to I$ as

$$w_{j,k}(x) = \begin{cases} u_j(p'(x)) \cdot v_k(f_j(x)) & \text{if } p'(x) \in \operatorname{supp}(u_j) \\ 0 & \text{if } u_j(p'(x)) = 0 \end{cases} \quad \text{for all } x \in X/H.$$

By the Pasting Lemma (Lemma A.2), this map is well-defined and continuous. For every $x \in X/H$, there is only a finite number of pairs $(j,k) \in J \times K$ such that $w_{j,k}(x) \neq 0$. Moreover, for every $x \in X/H$, there holds:

$$\sum_{j \in J} \sum_{k \in K} w_{j,k}(x) =$$

$$\sum_{j \in J} \sum_{k \in K} \begin{cases} u_j(p'(x)) \cdot v_k(f_j(x)) & \text{if } p'(x) \in \operatorname{supp}(u_j) \\ 0 & \text{if } u_j(p'(x)) = 0 \end{cases} =$$

$$\sum_{j \in J} \begin{cases} u_j(p'(x)) \cdot \sum\limits_{k \in K} v_k(f_j(x)) & \text{if } p'(x) \in \operatorname{supp}(u_j) \\ 0 & \text{if } u_j(p'(x)) = 0 \end{cases} =$$

$$\sum_{j \in J} \begin{cases} u_j(p'(x)) & \text{if } p'(x) \in \operatorname{supp}(u_j) \\ 0 & \text{if } u_j(p'(x)) = 0 \end{cases} = 1.$$

We conclude by Theorem 4.3 that $\mathcal{W}$ is a numerable cover of X/H. Hence, π is a numerable H-bundle. $\square$

Lemma 4.13. *The restriction $p|_{p^{-1}(A)}$ of a numerable bundle $p\colon X \to B$ for some $A \subseteq B$ is again numerable.*

Proof. By the proof of Proposition 3.22, we know that if $\mathcal{U}$ is a numerable cover of trivialising opens of p, then $\mathcal{V} = \{U \cap A \mid U \in \mathcal{U}\}$ is an open trivialising cover for the restriction. A numeration $(t_j)_{j\in J}$ of $\mathcal{U}$ can obviously be restricted to a numeration of $\mathcal{V}$, by taking the restrictions $(t_j|_A)_{j\in J}$. $\square$

5 Classification of Principal Bundles

This chapter is dedicated to classifying all numerable bundles over a given space B with structure group G. We will see that the isomorphism classes of numerable bundles are in bijection with homotopy classes of maps $B \to BG$. Here BG is a so called classifying space for the group G. A precise definition of a classifying space is given below. It should be clear that for this classification, we need to have a way to turn homotopies on the orbit spaces into isomorphism between bundles. In order to describe this construction, we firstly look at bundles over $B \times I$ for some space B. We observe that these bundles have the following "lifting" property:

Lemma 5.1 ([Die08, Lemma 14.3.1]). *Let $p\colon X \to B \times I$ a numerable bundle and $\phi\colon B \times I \to B \times I$ the map given by $\phi(b, s) = (b, 1)$ for all $(b, s) \in B \times I$. Then there exists an equivariant lift $\Phi\colon X \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ B \times I \arrow[r, "\phi"] & B \times I \end{tikzcd}$$

Proof. By Theorem 4.10, there exists a countable numerable cover $\mathcal{V}$ of B with countable numeration such that $V \times I$ is trivialising for all $V \in \mathcal{V}$. By Corollary 4.4, there exists a countable infinite locally finite partition of unity $\{v_n \colon B \to I \mid n \in \mathbb{N}\}$ subordinate to $\mathcal{V}$.[1] Define for every $n \in \mathbb{N}$ the map

$$q_n \colon B \to I \text{ given by } q_n(b) = \frac{v_n(b)}{\max\limits_{k \in \mathbb{N}}(v_k(b))} \text{ for all } b \in B.$$

Since the v_n's are continuous and form a locally finite partition of unity, every q_n is continuous. Notice for all $n \in \mathbb{N}$ that $\operatorname{supp}(q_n) = \operatorname{supp}(v_n) \subseteq V_n$ for some $V_n \in \mathcal{V}$. Let for all $n \in \mathbb{N}$, $\psi_n \colon G \times V_n \times I \to p^{-1}(V_n \times I)$ a trivialisation. Define $p_B \colon X \to B$ as the B coordinate of $p(x)$ for all $x \in X$. Similarly, define $p_I \colon X \to I$ as the I coordinate of $p(x)$ for all $x \in X$. Write Pr_1, Pr_2, Pr_3 for the projections from $G \times V_n \times I$ to G, V_n and I respectively. Note that $p_B(x) = Pr_2(\psi_n^{-1}(x))$ and $p_I(x) = Pr_3(\psi_n^{-1}(x))$ for all $x \in p^{-1}(V_n \times I)$. Define for all $n \in \mathbb{N}$, the map $R_n \colon X \to X$ as

$$R_n(x) = \begin{cases} \psi_n\Big(Pr_1\big(\psi_n^{-1}(x)\big), p_B(x), \max\big(p_I(x), q_n(p_B(x))\big)\Big) & \text{if } x \in p^{-1}(\operatorname{supp}(q_n) \times I) \\ x & \text{if } x \in X - p^{-1}(q_n^{-1}((0,1]) \times I) \end{cases}.$$

Note that for all $x \in \big(X - p^{-1}(q_n^{-1}((0,1]) \times I)\big) \bigcap \big(p^{-1}(\operatorname{supp}(q_n) \times I)\big)$, there holds $q_n(p_B(x)) = 0$. Hereby, we see:

$$\psi_n\Big(Pr_1\big(\psi_n^{-1}(x)\big), p_B(x), \max\big(p_I(x), q_n(p_B(x))\big)\Big) =$$

$$\psi_n\Big(Pr_1\big(\psi_n^{-1}(x)\big), Pr_2\big(\psi_n^{-1}(x)\big), \max\big(Pr_3(\psi_n^{-1}(x)), q_n(Pr_2(\psi_n^{-1}(x)))\big)\Big) =$$

$$\psi_n\Big(Pr_1\big(\psi_n^{-1}(x)\big), Pr_2\big(\psi_n^{-1}(x)\big), \max\big(Pr_3(\psi_n^{-1}(x)), 0\big)\Big) = x.$$

Since both $X - p^{-1}(q_n^{-1}((0,1]) \times I)$ and $p^{-1}(\operatorname{supp}(q_n) \times I)$ are closed, by the Pasting Lemma (Lemma A.2), R_n is well-defined and continuous. Moreover, for all $n \in \mathbb{N}$ and $x \in X$, there holds

[1] In case we have a finite partition of unity, we add an infinite but countable number of zeroes.

$p_B(R_n(x)) = p_B(x)$. Indeed, if $R_n(x) = x$ it is trivial. Otherwise, since $R_n(x) \in p^{-1}(V_n \times I)$, we have:

$$\begin{gathered} p_B(R_n(x)) = \\ Pr_2\Bigg(\psi_n^{-1}\bigg(\psi_n\Big(Pr_1\big(\psi_n^{-1}(x)\big), p_B(x), \max\big(p_I(x), q_n(p_B(x))\big)\Big)\bigg)\Bigg) = \\ Pr_2\bigg(Pr_1\Big(\psi_n^{-1}(x)\Big), p_B(x), \max\Big(p_I(x), q_n(p_B(x))\Big)\bigg) = \\ p_B(x). \end{gathered}$$

With induction, we conclude that $p_B\big((R_1 \circ \cdots \circ R_k)(x)\big) = p_B(x)$ for all $k \in \mathbb{N}$. Similarly for all $n \in \mathbb{N}$, there holds $p_I\big(R_n(x)\big) = \max\big(p_I(x), q_n(p_B(x))\big)$. Again with induction

$$p_I\Big((R_1 \circ \cdots \circ R_k)(x)\Big) = \max\Big(p_I(x), \max_{n \leq k}\big(q_n(p_B(x))\big)\Big) \text{ for all } k \in \mathbb{N}.$$

We define $\Phi\colon X \to X$ as the composition $\Phi = R_1 \circ R_2 \circ \dots$. For every $x \in X$, there exists an open neighbourhood $W \subseteq B$ of $p(x)$ which intersects $q_n^{-1}\big((0,1]\big) \times I$ for only a finite number of $n \in \mathbb{N}$. Thus there exists an $N \in \mathbb{N}$ such that $q_l(p_B(y)) = 0$ for all $y \in p^{-1}(W)$ and $l > N$. Clearly $R_l(y) = y$ for all $y \in W$ and $l > N$. Hence, $\Phi|_{p^{-1}(W)} = \big(R_1 \circ \cdots \circ R_N\big)|_{p^{-1}(W)}$. Hereby Φ is well-defined and continuous.
We have that $p_I(\Phi(x)) = \max\big(p_I(x), \max_{n\in\mathbb{N}}(q_n(p_B(x)))\big) = \max\big(p_I(x), 1\big) = 1$ and that $p_B(\Phi(x)) = p_B(x)$ and thus $p(\Phi(x)) = \phi(p(x))$ for all $x \in X$. We conclude that Φ is a suitable lift of ϕ. □

For a principal bundle $p\colon X \to B \times I$, we define $p_t\colon X_t := p^{-1}(B \times \{t\}) \to B \times \{t\} \cong B$ for all $t \in I$ as the restriction from p. By Lemma 3.8 and Proposition 3.22, the map p_t is a principal bundle. In case that p is numerable, so is p_t, see Lemma 4.13. These restriction turn out to have the following two "isomorphism" relations:

Corollary 5.2 ([Die08, p. 343])**.** *Let $p\colon X \to B \times I$ a principal bundle. Then the bundle $p_1 \times Id_I\colon X_1 \times I \to B \times I$ is isomorphic to p.*

Proof. Lemma 5.1 gives us the following diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ B \times I \arrow[r, "\phi"] & B \times I \end{tikzcd}$$

Here $\phi\colon B \times I \to B \times I$ is defined as $\phi(b, s) = (b, 1)$. Proposition 3.20 implies that p is isomorphic to $\phi^* p$. Define $\psi\colon X_1 \times I \to X \times_{B\times I} (B \times I)$ as $\psi(x, s) = (x, (p_1(x), s))$. Since $p(x) = (p_1(x), 1) = \phi(p_1(x), s)$ for all $(x, s) \in X_1 \times I$, we see that ψ is well-defined. Moreover, there holds $p_1 \times Id_I = \phi^* p \circ \psi$. We conclude by Corollary 3.15, the map $p_1 \times Id_I$ is isomorphic to $\phi^* f$. □

Corollary 5.3 ([Die08, Lemma 14.3.2])**.** *The restrictions p_0 and p_1 of a numerable bundle $p\colon X \to B \times I$ are isomorphic numerable bundles.*

Proof. By Lemma 5.1, we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ B \times I \arrow[r, "\phi"] & B \times I \end{tikzcd}$$

Here $\phi\colon B\times I \to B\times I$ is defined as $\phi(b,s) = (b,1)$. The restriction $\phi|_{B\times\{0\}}\colon B\times\{0\} \to B\times\{1\}$ is a homeomorphism. We obtain the diagram:

$$\begin{tikzcd} p^{-1}(B\times\{0\}) \arrow[r, "\Phi"] \arrow[d, two heads, "p"'] & p^{-1}(B\times\{1\}) \arrow[d, two heads, "p"] \\ B\times\{0\} \arrow[r, tail, two heads, "\phi"] \arrow[d, tail, two heads] & B\times\{1\} \arrow[d, tail, two heads] \\ B & B \end{tikzcd}$$

The map $B \to B$ completing the diagram is the identity. This is clear by looking at the definition of ϕ. The following sub-diagram together with Corollary 3.15 proves the desired result.

$$\begin{tikzcd} p^{-1}(B\times\{0\}) \arrow[r, "\Phi"] \arrow[d, two heads, "p_0"'] & p^{-1}(B\times\{1\}) \arrow[d, two heads, "p_1"] \\ B \arrow[r, tail, two heads, "Id"] & B \end{tikzcd}$$

□

The assertion above provides a way to transform homotopies into isomorphism: consider a numerable bundle $p\colon Y \to C$ and a homotopy $h\colon B\times I \to C$. Now the pullback h^*p is a bundle over $B\times I$, and we have the isomorphisms as in the assertion.
We now can classify all numerable bundles. We will see that every bundle arises as pullback from a so called **universal bundle**. I give a definition following [Die08, p. 344]:

Definition 5.4. A numerable bundle $pG\colon EG \to BG$ which admits for every numerable bundle $p\colon E \to B$, up to G-homotopy, a unique equivariant map $f\colon E \to EG$ is called a **universal bundle** for G. The space BG is called a **classifying space**. ▲

Define $\mathcal{B}(B,G)$ as the set of isomorphic classes of numerable bundles. By Propositions 2.11 and 2.12, it is clear that isomorphism defines an equivalence relation. In the theorem below we will see that $\mathcal{B}(B,G)$ is a set. Write $[X,Y]$ for the set of homotopy classes of functions $X \to Y$. We arrive that the classification theorem of numerable bundles:

Theorem 5.5 ([Die08, Theorem 14.4.1])**.** *Given a universal bundle $pG\colon EG \to BG$, the map $\alpha\colon [B,BG] \to \mathcal{B}(B,G)$ defined as $[f] \mapsto [f^*pG]$ is a bijection.*

Proof. We first show that for every principal bundle $p\colon E \to B$, there exists a map $\phi\colon B \to BG$ such that ϕ^*pG is isomorphic to p. Notice that this implies surjectivity of α, provided that α is well-defined. This also implies $\mathcal{B}(B,G)$ is a set.
Let $p\colon E \to B$ a principal bundle. By assumption, there exists a G-equivariant map $f\colon E \to EG$. By Proposition 3.10, there is a unique $\phi\colon B \to BG$ with $pG \circ f = \phi \circ p$. Proposition 3.20 shows that ϕ^*pG is isomorphic to p.

Secondly, we show that α is well-defined: let $\phi, \phi' : B \to BG$ two homotopic maps and $h: B \times I \to BG$ a homotopy between ϕ and ϕ'. Lemma 4.11 and Corollary 5.3 show that the restrictions $(h^*pG)_0$ and $(h^*pG)_1$ are isomorphic numerable bundles. For every $s \in I$ we have the following diagram:

$$\begin{tikzcd}
(h^*pG)^{-1}(B \times \{s\}) \arrow[r, hook] \arrow[d, two heads, "(h^*pG)_s"'] & EG \times_{BG} B \arrow[r] \arrow[d, two heads, "h^*pG"'] & EG \arrow[d, two heads, "pG"] \\
B \arrow[r, "i_s"] & B \times I \arrow[r, "h"] & BG
\end{tikzcd}$$

Notice that $h \circ i_s = h_s$ and the top row of the diagram is an equivariant map. By Lemma 4.11 and Proposition 3.20, we conclude that h_s^*pG and $(h^*pG)_s$ are isomorphic numerable bundles for all $s \in I$. Applying this fact for $s = 0, 1$, we get that $\phi^*pG \cong (h^*pG)_0 \cong (h^*pG)_1 \cong \phi'^*pG$.

Lastly, we show that α is injective: let $\phi, \phi' : B \to BG$ such that ϕ^*pG is isomorphic to ϕ'^*pG. Let ψ an isomorphism. We have created the following diagram:[2]

$$\begin{tikzcd}
EG \arrow[d, two heads, "pG"'] & EG \times_{BG} B \arrow[l] \arrow[rr, tail, two heads, "\psi"] \arrow[dr, two heads, "\phi^*pG"'] & & EG \times_{BG} B \arrow[r] \arrow[dl, two heads, "\phi'^*pG"] & EG \arrow[d, two heads, "pG"] \\
BG & & B \arrow[ll, "\phi"] \arrow[rr, "\phi'"'] & & BG
\end{tikzcd}$$

Leaving out the ϕ'^*pG bundle, gives:

$$\begin{tikzcd}
EG \arrow[d, two heads, "pG"'] & EG \times_{BG} B \arrow[l] \arrow[r] \arrow[d, two heads, "\phi^*pG"'] & EG \arrow[d, two heads, "pG"] \\
BG & B \arrow[l, "\phi"] \arrow[r, "\phi'"'] & BG
\end{tikzcd}$$

Where both top arrows are equivariant maps. Since $pG: EG \to BG$ is a universal bundle, the top arrows are G-homotopic. By Proposition 2.10 and Lemma 3.9 and the fact that homotopies factor through homeomorphisms, we conclude that ϕ and ϕ' are homotopic. □

With Corollary 3.26, we concluded that all principal bundles over I are trivial. Using the classification of numerable bundles, we see that the same if true for any contractible space.

Corollary 5.6. *Every numerable bundle over a contractible space is trivial.*

Proof. If B is a contractible space, all maps $B \to BG$ are homotopic. □

[2]One should notice the "abuse of notation": the two spaces $EG \times_{BG} B$ are different, since we have different functions $B \to BG$ and the notation does not carry this information. A similar problem occurs when we consider pullbacks from different bundles $EG \to BG$.

6 Universal Bundles

In the previous chapter, we have classified all principal bundles over a given space B with structure group G. However, we required a universal bundle to be given. In this chapter, we will construct universal bundles. Firstly, a general construction for any topological group, due to Milnor [Mil56b]. Secondly, a construction for finite discrete groups using configuration spaces.

6.1 The Milnor Construction

We first define the Milnor/join space, following [Mil56b, pp. 430–431]:

Definition 6.1. Take a collection of non-empty spaces $(X_j)_{j\in J}$ for some index set J. We consider the set Σ of all sequences $(s_j, x_j)_{j\in J} \in \prod_{j\in J} I \times X_j$ such that only a finite number of s_j's are different from 0 and $\sum_{j\in J} s_j = 1$. We define the **join** $\bigstar_{j\in J} X_j := \Sigma/\!\sim$ where $(s_j, x_j)_{j\in J} \sim (s'_j, x'_j)_{j\in J}$ if $s_j = s'_j$ for all $j \in J$ and $x_j = x'_j$ for all $j \in J$ with $s'_j = s_j \neq 0$. It is clear that $\sim$ defines an equivalence relation.
Instead of $(s_j, x_j)_{j\in J}$, we can use the more suggestive notation $(s_j x_j)_{j\in J}$, with $0x_j = 0x'_j = 0$.

On the space $\bigstar_{j\in J} X_j$, we have for every $i \in J$ "coordinate" functions $t_i : \bigstar_{j\in J} X_j \to I$ taking $(s_j, x_j)_{j\in J} \mapsto s_i$ and $q_i : t_i^{-1}((0, 1]) \to X_i$ taking $(s_j, x_j)_{j\in J} \mapsto x_i$. Notice that the both functions are well-defined. Throughout this chapter, we assume that t_j and q_j are these coordinate functions. We define the **join topology** on $\bigstar_{j\in J} X_j$ as the coarsest topology, such that all the s_j and q_j are continuous. ▲

We can characterise the join topology with the following universal property:

Proposition 6.2. *A function $f : X \to \bigstar_{j\in J} X_j$ is continuous if and only if for all $i \in J$, the maps $t_i \circ f$ and $q_i \circ f$ (where defined) are continuous.*

Proof. The "only if" part is trivial. Suppose that all $t_i \circ f$ and $q_i \circ f$ (where defined) are continuous. Define

$$\mathcal{S} := \{t_j^{-1}(I') \mid I' \subseteq I \text{ open and } j \in J\} \bigcup \{q_j^{-1}(U) \mid U \subseteq X_j \text{ open and } j \in J\}.$$

Clearly $\mathcal{S}$ is a collection of opens in the join topology. Moreover, every topology containing $\mathcal{S}$ makes all t_i and q_i continuous. Hence, $\mathcal{S}$ is a sub-base for the join topology. Hereby every open $V \subseteq \bigstar_{j\in J} X_j$ can be written as a union of finite intersections of elements of $\mathcal{S}$:

$$V = \bigcup_{k\in K} \bigcap_{l=1}^{n_k} S_{k,l} \text{ with } K \text{ some index set and } S_{k,l} \in \mathcal{S}.$$

For every $S \in \mathcal{S}$, there is a $i \in J$ and either an open $I' \subseteq I$ such that $S = t_i^{-1}(I')$ or an open $U \subseteq X_j$ such that $S = q_j^{-1}(U)$. Hereby, $f^{-1}(S) = f^{-1}(t_j^{-1}(I'))$ or $f^{-1}(S) = f^{-1}(q_j^{-1}(U))$. Hence, $f^{-1}(S)$ is open. We observe that $f^{-1}(V) = f^{-1}(\bigcup_{k\in K} \bigcap_{l=1}^{n_k} S_{k,l}) = \bigcup_{k\in K} \bigcap_{l=1}^{n_k} f^{-1}(S_{k,l})$ and thus $f^{-1}(V)$ is open and f is continuous. □

From this point most authors run straightaway to a "correct" join space (one that yields a universal bundle). I will consider slightly more general class of join spaces. However, the used techniques are similar to [Die08, Chapter 14.4]. In Corollary 6.7, we will conclude the classical result.

Assume all X_j's are G-spaces. Then there is a canonical action on the join $\bigstar_{j\in J}X_j$ by acting $(g(s_j,x_j)_{j\in J})\mapsto(s_j,gx_j)_{j\in J}$. Since all q_j's commute with this group action (where q_j is defined) and the action on the I coordinates is trivial, we see, using the previous proposition, that this action is continuous. Hence, the join $\bigstar_{j\in J}X_j$ is a G-space. We have a similar statement for principal bundles:

Theorem 6.3. *Let $(p_j\colon X_j\to B_j)_{j\in J}$ principal bundles with structure group G. Then the projection $\pi\colon\bigstar_{j\in J}X_j\to(\bigstar_{j\in J}X_j)/G$ is a principal bundle. Moreover, if all p_j are numerable bundles, so is π.*

Proof. Choose for all $j\in J$ a trivialising cover $\mathcal{U}_j$ of B_j. Consider the collection of opens $\mathcal{U}:=\{\pi(q_j^{-1}(p_j^{-1}(U)))\bigm| j\in J \text{ and } U\in\mathcal{U}_j\}$. We will see that $\mathcal{U}$ is a trivialising cover.
Suppose $j_0\in J$ and $U\in\mathcal{U}_{j_0}$. Let $V:=q_{j_0}^{-1}(p_{j_0}^{-1}(U))$. Firstly, we show that $\pi^{-1}(\pi(V))=V$. Notice, by construction, that $V\subseteq t_{j_0}^{-1}((0,1])$. It is clear that $V\subseteq\pi^{-1}(\pi(V))$. Let $(t_j,x_j)_{j\in J}=x\in\pi^{-1}(\pi(V))$. Then there is a $(t'_j,y_j)_{j\in J}=y\in V$ such that $\pi(y)=\pi(x)$. Hence, there is a $g\in G$ such that $x=gy$. We deduce that $t_j=t'_j$ and $x_j=gy_j$ for all $j\in J$. In particular $t_{j_0}=t'_{j_0}\neq 0$ and $p_{j_0}(q_{j_0}(x))=p_{j_0}(x_{j_0})=p_{j_0}(gy_{j_0})=p_{j_0}(y_{j_0})=p_{j_0}(q_{j_0}(y))\in U$. We conclude that $x\in V$ and thus $V=\pi^{-1}(\pi(V))$.

Take a trivialisation $\psi\colon G\times U\to p_{j_0}^{-1}(U)$. Let $\psi_G^{-1}\colon p_{j_0}^{-1}(U)\to G$ be the G coordinate of ψ^{-1}. Define $f\colon V\to G$ as $f(x)=\psi_G^{-1}(q_{j_0}(x))$. Clearly f is an equivariant map (both q_{j_0} and ψ_G^{-1} are equivariant maps). Using Corollary 3.18, we see that $\mathcal{U}$ is a trivialising cover for π and thus that π is a principal bundle.

Suppose all p_j's are numerable bundles. Assume without loss of generality that for all $j\in J$, the cover $\mathcal{U}_j$ is numerable. Take for every $j\in J$ a locally finite partition of unity $(v_{j,k})_{k\in K_j}$ subordinate $\mathcal{U}_j$.
Notice that $t_j(gx)=t_j(x)$ for all $x\in\bigstar_{i\in J}X_i$, $g\in G$ and $j\in J$. Using the universal property of quotient space topology, there are (continuous) induced maps $\widetilde{t_j}\colon(\bigstar_{i\in J}X_i)/G\to I$ for all $j\in J$. For all $x\in\bigstar_{i\in J}X_i$, there are only a finite number of $j\in J$ such that $0<t_j(x)=\widetilde{t_j}([x])$ and $\sum_{j\in J}\widetilde{t_j}([x])=\sum_{j\in J}t_j(x)=1$. Theorem 4.3 gives us a locally finite partition of unity $(t'_l)_{l\in L}$ subordinate to

$$\{\widetilde{t_j}^{-1}((0,1])\bigm| j\in J\}=\{\pi(t_j^{-1}((0,1]))\bigm| j\in J\}.$$

We choose for every $l\in L$ a $j_l\in J$ such that $\overline{t_l'^{-1}((0,1])}\subseteq\widetilde{t_j}^{-1}((0,1])$.
Define $v'_{l,k}\colon(\bigstar_{i\in J}X_i)/G\to I$ for all $l\in L$ and $k\in K_{j_l}$ as

$$v'_{l,k}([x])=\begin{cases}t'_l([x])\cdot v_{j_l,k}(p_{j_l}(q_{j_l}(x))) & \text{if } [x]\in\overline{t_l'^{-1}((0,1])}\\ 0 & \text{if } [x]\in t_l'^{-1}(0)\end{cases}\quad\text{for all } [x]\in(\bigstar_{i\in J}X_i)/G.$$

There holds $p_{j_l}(q_{j_l}(gx))=p_{j_l}(gq_{j_l}(x))=p_{j_l}(q_{j_l}(x))$ for all $[x]\in\overline{t_l'^{-1}((0,1])}\subseteq\pi(t_{j_l}^{-1}((0,1]))$ and $g\in G$. Furthermore, there holds $t'_l([x])\cdot v_{j_l,k}(p_{j_l}(q_{j_l}(x)))=0$ if $t'_l([x])=0$. Since both $\overline{t_l'^{-1}((0,1])}$ and $t_l'^{-1}(0)$ are closed, the Pasting Lemma (Lemma A.2) shows that $v'_{l,k}$ is well-defined and continuous.

For all $x\in\bigstar_{i\in J}X_i$, there is only a finite number of $l\in L$ and $k\in K_{j_l}$ such that $t'_l([x])\neq 0$ and $v_{j_l,k}(p_{j_l}(q_{j_l}(x)))$. Hence, $v'_{l,k}([x])\neq 0$ for only a finite number of $l\in L$ and $k\in K_{j_l}$. Notice that for all $x\in\bigstar_{i\in J}X_i$, there holds:

$$\sum_{l\in L}\sum_{k\in K_{j_l}} v'_{l,k}([x]) =$$

$$\sum_{l\in L}\sum_{k\in K_{j_l}} \begin{cases} t'_l([x])\cdot v_{j_l,k}(p_{j_l}(q_{j_l}(x))) & \text{if } [x]\in \overline{t'^{-1}_l((0,1])} \\ 0 & \text{if } [x]\in t'^{-1}_l(0) \end{cases} =$$

$$\sum_{l\in L} \begin{cases} t'_l([x])\cdot \sum_{k\in K_{j_l}} v_{j_l,k}(p_{j_l}(q_{j_l}(x))) & \text{if } [x]\in \overline{t'^{-1}_l((0,1])} \\ 0 & \text{if } [x]\in t'^{-1}_l(0) \end{cases} =$$

$$\sum_{l\in L} \begin{cases} t'_l([x]) & \text{if } [x]\in \overline{t'^{-1}_l((0,1])} \\ 0 & \text{if } [x]\in t'^{-1}_l(0) \end{cases} =$$

$$\sum_{l\in L} t'_l([x]) = 1.$$

For all $l\in L$ and $k\in K_{l_j}$, there exists a $U\in\mathcal{U}_{j_l}$ such that $v_{j_l,k}((0,1])\subseteq U$. Hence, we have that

$$v'^{-1}_{l,k}\big((0,1]\big)\subseteq t'^{-1}_l\big((0,1]\big)\bigcap \pi\Big(q^{-1}_{j_l}(p^{-1}_{j_l}(U))\Big)\subseteq \pi\Big(q^{-1}_{j_l}(p^{-1}_{j_l}(U))\Big)\in\mathcal{U}.$$

Using Theorem 4.3, we see that $\mathcal{U}$ is numerable and thus that π is a numerable bundle. □

Lemma 6.4. *Let $p\colon E\to B$ a numerable bundle for a group G and $(X_j)_{j\in J}$ infinitely many (nonempty) G-spaces. Then there exists an equivariant map $f\colon E\to \bigstar_{j\in J}X_j$.*

Proof. Choose for all $j\in J$ a point $x_{j,0}\in X_j$ and take an injection $r\colon\mathbb{N}\to J$. By Theorem 4.9, there exists a countable cover $\mathcal{U}$ of trivialising opens for p and a partition of unity $(v_n)_{n\in\mathbb{N}}$ subordinated to $\mathcal{U}$. Without loss of generality, we can assume that $\mathcal{U}=\{U_n \mid n\in\mathbb{N}\}$ and $\operatorname{supp}(v_n)\subseteq U_n$ for all $n\in\mathbb{N}$. As all U_n are trivialising, there are equivariant maps $\psi_n\colon U_n\to G$. Define $f\colon E\to\bigstar_{j\in J}X_j$ as $f(x)=(s_j(x),x_j(x))_{j\in J}$ for all $x\in E$ with

$$s_j(x)=\begin{cases} v_n(p(x)) & \text{if } r(n)=j \\ 0 & \text{else}\end{cases} \quad\text{and}\quad x_j(x)=\begin{cases}\psi_n(x)\cdot x_{j,0} & \text{if } r(n)=j \text{ and } p(x)\in U_n \\ x_{j,0} & \text{else}\end{cases}.$$

Notice that f is well-defined, since for all $j\in J$, there is by injectivity of r at most one $n\in\mathbb{N}$ such that $r(n)=j$. Clearly every $t_j\circ f=s_j$ is continuous. Moreover, if $x\in f^{-1}(s_j^{-1}((0,1]))$, then $s_j(x)\neq 0$. Thus there is an $n\in\mathbb{N}$ with $r(n)=j$ and $p(x)\in U_n$. Hence, there holds $q_j(f(x))=x_j(x)=\psi_n(x)\cdot x_{j,0}$ and thus $q_j\circ f$ is continuous (where defined). Proposition 6.2 shows that f is continuous. For all $g\in G$, $j\in J$ and $x\in E$, we have that $s_j(gx)=s_j(x)$. Moreover, for all $j\in J$ and $g\in G$ if $x\in E$ with $s_j(gx)=s_j(x)>0$, then there is an $n\in\mathbb{N}$ with $p(gx)=p(x)\in U_n$ and $r(n)=j$. Hence, $x_j(gx)=\psi_j(gx)\cdot x_{j,0}=g\psi_j(x)\cdot x_{j,0}=g\cdot x_j(x)$. We conclude that $f(gx)=(s_j(gx),x_j(gx))_{j\in J}=(s_j(x),gx_j(x))_{j\in J}=gf(x)$ and thus that f is an equivariant map. □

Lemma 6.5 ([Die08, Proposition 14.4.4])**.** *Consider G-spaces E and X. Any two equivariant maps $f,g\colon E\to\bigstar_{j\in\mathbb{N}}X$ are G-homotopic.*

Proof. Write $f=(s_1x_1,s_2x_2,\dots)$ with $s_j\colon E\to I$ and $x_j\colon s_j^{-1}((0,1])\to X$. For all $k\in\mathbb{N}$ the maps $(s_1x_1,\dots,s_kx_k,0,s_{k+1}x_{k+1},0,\dots)$ and $(s_1x_1,\dots,s_kx_k,s_{k+1}x_{k+1},0,s_{k+2}x_{k+2},0,\dots)$ are G-homotopic with G-homotopy $H^k\colon E\times I\to\bigstar_{j\in\mathbb{N}}X$ given by

$$H^k_t=\big(s_1x_1,\dots,s_kx_k,ts_{k+1}x_{k+1},(1-t)s_{k+1}x_{k+1},ts_{k+2}x_{k+2},(1-t)s_{k+2}x_{k+2},\dots\big).$$

Define $H\colon E\times I \to \bigstar_{j\in\mathbb{N}} X$ as the concatenation[3] $H^1 \cdot H^2 \cdot \ldots$. Notice that this definition makes sense, since for every $x \in E$, there is only a finite number of $k \in \mathbb{N}$ such that H_t^k is different from f for some $t \in I$. For the coordinate maps $t_j \circ H$ and $q_j \circ H$ only the first $j-1$ homotopies are relevant. Hereby, the coordinate maps are continuous. By Proposition 6.2, the map H is continuous. Moreover, H is an equivariant map. Hereby, the maps $(s_1x_1, 0, s_2x_2, 0, \dots)$ and f are G-homotopic. Similarly, when we write $g = (s_1'x_1', s_2'x_2', \dots)$ with $s_j'\colon E \to I$ and $x_j'\colon s_j'^{-1}((0,1]) \to X$, we see that g is G-homotopic to $(s_1'x_1', 0, s_2'x_2', 0, \dots)$. The G-homotopy sending

$$t \mapsto (ts_1'x_1', (1-t)s_1'x_1', ts_2'x_2', (1-t)s_2'x_2', \dots)$$

shows that the maps $(s_1'x_1', 0, s_2'x_2', 0, \dots)$ and $(0, s_1'x_1', 0, s_2'x_2', \dots)$ are G-homotopic. Furthermore, the G-homotopy $t \mapsto (ts_1x_1, (1-t)s_1'x_1', ts_2x_2, (1-t)s_2'x_2', \dots)$ shows that the maps $(s_1x_1, 0, s_2x_2, 0, \dots)$ and $(0, s_1'x_1', 0, s_2'x_2', \dots)$ are G-homotopic. We conclude that f and g are G-homotopic. $\square$

We now conclude the following:

Theorem 6.6 ([Die08, Proposition 14.4.10])**.** *For any non-empty numerable bundle $p\colon X \to B$, the projection $\pi\colon \bigstar_{j\in\mathbb{N}} X \to (\bigstar_{j\in\mathbb{N}} X)/G$ is a universal bundle.*

Proof. By Theorem 6.3, the projection π is a numerable bundle. Lemmas 6.4 and 6.5 show that π is a universal bundle. $\square$

Or even more concretely:

Corollary 6.7 ([Die08, Proposition 14.4.2])**.** *For every topological group G, the projection $\pi\colon \bigstar_{j\in\mathbb{N}} G \to (\bigstar_{j\in\mathbb{N}} G)/G$ is a universal bundle.*

Proof. Take the action $G \times G \to G$ by the group multiplication. Notice that $p\colon G \to *$ is a numerable bundle and apply the previous theorem. $\square$

Remark 6.8. In this construction of universal bundles, I used the axiom of choice several times. While I am perfectly fine with assuming the axiom, one can notice that the axiom is not required when considering $\bigstar_{j\in\mathbb{N}} G$. We can use the exact same proofs, but in this specific case all the choices can be made canonically.

6.2 Properties of Universal Bundles

Having constructed a universal bundle, all other universal bundles inherited properties from this. These include contractibility and universality of quotients of admissible subgroups. This section provides the proofs.

Proposition 6.9 ([Die08, Proposition 14.4.6])**.** *For every non-empty space X, the join $\bigstar_{j\in\mathbb{N}} X$ is contractible.*

Proof. Consider X under the action of the trivial group. This makes X a G-space. Let $x_0 \in X$. Note that the identity on $\bigstar_{j\in\mathbb{N}} X$ and the constant map $\bigstar_{j\in\mathbb{N}} X \to \bigstar_{j\in\mathbb{N}} X$ given by $(t_jx_j)_{j\in\mathbb{N}} \mapsto (1x_0, 0, 0, \dots)$ are equivariant maps. By Lemma 6.5, these two maps are homotopic. $\square$

[3] Homotopies can be concatenated in a similar way as paths can. In this case, we let the first homotopy "work" on $[0, \frac{1}{2}]$, the second on $[\frac{1}{2}, \frac{3}{4}]$ etc. One should notice that this construction is only possible if the "end"-function equals the "start"-function of the next homotopy. This is clear in our case.

Theorem 6.10. *For a universal bundle $p\colon E \to B$, the space E is contractible.*

Proof. Let $pG\colon EG \to BG$ a universal bundle such that EG is contractible. Such a bundle exist by Proposition 6.9 and Corollary 6.7. There exist equivariant maps $f\colon E \to EG$ and $g\colon EG \to E$. Since both p and π are universal bundles, both $g \circ f$ and $f \circ g$ are (G-)homotopic to the identity on E and EG respectively. Hence, E is (G-)homotopy equivalent to EG and thus E is contractible. □

Remark 6.11. The converse of this theorem is also true: Any numerable bundle $p\colon E \to B$ with E contractible is universal. For a proof see [Die08, Theorem 14.4.12]. Using this fact, we see that for a contractible group G (contractible as topological space), the projection $G \to *$ is a universal bundle. Applying Theorem 5.5 gives that every numerable G-bundle is trivial.

Theorem 6.12. *Let $pG\colon EG \to BG$ a universal G-bundle and $H \leqslant G$ an admissible subgroup such that the principal bundle $G \to G/H$ is numerable. Then the projection $pH\colon EG \to EG/H$ is a universal H-bundle.*

Proof. Lemma 4.12 shows that pH is a numerable H-bundle.
Let $p\colon E \to B$ a numerable H-bundle. Since pG is a universal G-bundle, there exists a G-equivariant map $f\colon \bigstar_{i\in\mathbb{N}}G \to EG$. Here we see $\bigstar_{i\in\mathbb{N}}G$ and EG as G-spaces. When seeing $\bigstar_{i\in\mathbb{N}}G$ and EG as H-spaces, the map f is also an H-equivariant map. By Theorem 6.6, the projection $\pi\colon \bigstar_{i\in\mathbb{N}}G \to (\bigstar_{i\in\mathbb{N}}G)/H$ is a universal H-bundle. Hereby, there is an H-equivariant map $g'\colon E \to \bigstar_{i\in\mathbb{N}}G$ and thus $f \circ f'\colon E \to EG$ is an H-equivariant map.
Suppose $g\colon E \to EG$ is another H-equivariant map. Universality of the principal G-bundle $\bigstar_{i\in\mathbb{N}}G \to (\bigstar_{i\in\mathbb{N}}G)/G$ implies that there exists a G-equivariant map $\widetilde{f}\colon EG \to \bigstar_{i\in\mathbb{N}}G$. Again $\widetilde{f}$ is also an H-equivariant map. Both $\widetilde{f} \circ g$ and $\widetilde{f} \circ f \circ f'$ are H-equivariant maps $E \to \bigstar_{i\in\mathbb{N}}G$ and thus they are H-homotopic. Hereby $f \circ \widetilde{f} \circ g$ and $f \circ \widetilde{f} \circ f \circ f'$ are H-homotopic. Since $f \circ \widetilde{f}$ is G-homotopic (thus also H-homotopic) to the identity on EG, the map g is H-homotopic to $f \circ f'$. □

6.3 Configuration Spaces

A second way of producing universal bundles is using configuration spaces. Only, this construction produces universal bundles for just the symmetric groups Σ_n. Together with Theorem 6.12, we obtain universal bundles for all finite groups endowed with the discrete topology. We will see the bundles in question are all numerable. Let us start with a definition of configuration spaces, following [Knu18, p. 3]:

Definition 6.13. The configuration space of n ordered points of a topological space X is:

$$\mathrm{Conf}_n(X) := \{(x_1, \dots, x_n) \in X^n | x_i = x_j \text{ implies } i = j\}.$$

This space admits a natural Σ_n action, by permuting the coordinates. Write $B_n(X)$ for the orbit space. We can immediately note that the Σ_n-action on $\mathrm{Conf}_n(X)$ is free. ▲

In this section, we firstly show that (under mild conditions on the space X) the projection $\mathrm{Conf}_n(X) \to B_n(X)$ is a principal Σ_n-bundle. Secondly, we prove that if $X = \mathbb{R}^\infty$, then we have a universal bundle. The following assertion is inspired by [Hat15, proposition 1.40a]:

Proposition 6.14. *Let G a discrete group and X a G-space. Assume that for every $x \in X$, there exists an open neighbourhood $U \subseteq X$ and such that the co-sets gU for varying $g \in G$ are disjoint. Then the projection $\pi\colon X \to X/G$ is a principal bundle.*

Proof. Take an open set U as in the assertion. Since π is an open map, we know that $\pi(U)$ is open. We see that $\pi^{-1}(\pi(U)) = \sqcup_{g\in G} gU$. Now for every $x \in \sqcup_{g\in G} gU$, there is a unique (!) $g_x \in G$ such that $x \in g_x U$. Define $f\colon \sqcup_{g\in G} gU \to G$ as $f(x) = g_x$. Since for all $g \in G$, we have that $f^{-1}(g) = gU$ and $f(gx) = gf(x)$, we see that f is an equivariant map. With Corollary 3.18, we conclude that π is a principal bundle. □

When X is a Hausdorff, the configuration space has the following two properties. While the properties are seemingly unrelated, the proof depends on the same construction. The argument follows [Knu18, p. 11].

Proposition 6.15. *If X is Hausdorff, then the configuration space $Conf_n(X) \subseteq X^n$ is open and the projection $\pi\colon Conf_n(X) \to B_n(X)$ is a principal bundle.*

Proof. If $n = 1$, there is nothing to check. Suppose $n > 1$. Let $(x_1, \ldots, x_n) = x \in \mathrm{Conf}_n(X)$. By Hausdorffness of X, there exist for all $1 \le i < j \le n$ opens $U_{i,j} \subseteq X$ such that $x_i \in U_{i,j}$ and $U_{i,j} \cap U_{j,i} = \emptyset$. Now define

$$V_i := \bigcap_{\substack{1\le j\le n\\ i\neq j}} U_{i,j}.$$

There holds $x_i \in V_i$ and if $i \neq j$, then $V_i \cap V_j = \emptyset$. Hereby, the set $V_1 \times \cdots \times V_n$ is open in X^n. Moreover, it is subset of $\mathrm{Conf}_n(X)$. Hence, $\mathrm{Conf}_n(X)$ is open in X^n.
For every permutation $\sigma \in \Sigma_n$, we have that $\sigma(V_1 \times \cdots \times V_n) = V_{\sigma(1)} \times \cdots \times V_{\sigma(n)}$. Hence, $(V_1 \times \cdots \times V_n) \cap (V_{\sigma(1)} \times \cdots \times V_{\sigma(n)}) = \emptyset$ if σ is not the identity of Σ_n. Applying Proposition 6.14, we see that π is a principal bundle. □

Now we have principal bundles, we consider numerability. It is well-known that a paracompact Hausdorff space admits a partition of unity for every open cover. See for example [Wil04, p. 152, 20C]. Hence, a principal bundle $p\colon X \to B$ with B paracompact Hausdorff is numerable. Since the map $\pi\colon \mathrm{Conf}_n(X) \to B_n(X)$, for any space X, is a quotient map by a finite group, Hausdorffness is inherited by $B_n(X)$ from $\mathrm{Conf}_n(X)$. The same is true for paracompactness:

Proposition 6.16. *If $Conf_n(X)$ is paracompact, then so is $Conf_n(X)/H$ for any subgroup $H \leqslant \Sigma_n$, in particular $B_n(X)$ is paracompact.*

Proof. Write $\pi\colon \mathrm{Conf}_n(X) \to \mathrm{Conf}_n(X)/H$ for the projection. Take an open cover $\mathcal{U}$ of $\mathrm{Conf}_n(X)/H$. The family $\{\pi^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of $\mathrm{Conf}_n(X)$. Hence, $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$. Consider the open cover $\mathcal{W} = \{\pi(V) \mid V \in \mathcal{V}\}$. This is indeed an open cover of $\mathrm{Conf}_n(X)/H$ as π is a surjective open map. Notice that for all $W \in \mathcal{W}$, there is a $V \in \mathcal{V}$ and $U \in \mathcal{U}$ such that $W = \pi(V) \subseteq \pi(\pi^{-1}(U)) = U$. Thus, $\mathcal{W}$ is a refinement of $\mathcal{U}$.
Let $x \in \mathrm{Conf}_n(X)/H$. Write $\{x_1, \ldots, x_k\} = \pi^{-1}(x)$ for its fibers. There exist disjoint open neighbourhoods $A_1, \ldots, A_k$ of $x_1, \ldots, x_k$ respectively each intersecting only finitely many $V \in \mathcal{V}$. Hereby also $A_1 \cup \cdots \cup A_k$ intersects only a finite number of $V \in \mathcal{V}$.
Consider $\pi(A_1) \cap \cdots \cap \pi(A_k)$. Clearly, there holds $x \in \pi(A_1) \cap \cdots \cap \pi(A_k)$. Moreover, if $\pi(V) \cap (\pi(A_1) \cap \cdots \cap \pi(A_k)) \neq \emptyset$ for some $V \in \mathcal{V}$, then there exists a $y \in V$ such that $\pi(y) \in (\pi(A_1) \cap \cdots \cap \pi(A_k))$. Hence, $y \in A_i$ for some i. Thus, V intersects $A_1 \cup \cdots \cup A_k$. We conclude that $\mathcal{W}$ is locally finite. □

By Theorem 6.10, we know that every universal bundle $p\colon E \to B$ has a contractible space E. Contractible spaces are "highly connected". So in the search for universal bundles, we need to construct spaces with "high connectivity". In the Milnor construction, we created the connectivity by taking products and gluing the zeroes together of some arbitrary (potentially highly unconnected space). In other words, the more spaces you join together, the higher the connectivity. For configuration spaces, we are working the other way round. Since the higher n becomes, the less connected $\mathrm{Conf}_n(X)$ will be. After all we leave out more and more points from X^n. In example, the configuration space $\mathrm{Conf}_1(\mathbb{R}) = \mathbb{R}$ and $\mathrm{Conf}_2(\mathbb{R}) = \mathbb{R}^2 - \{(x, x) \mid x \in \mathbb{R}\}$. For this reason, if we want to construct a universal bundle for configuration spaces, the space X needs to be highly connected. As the example makes clear, even contractibility is not enough. We consider the following space:

Definition 6.17. We define $\mathbb{R}^\infty := \{(x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} \mid x_i = 0 \text{ for all but finitely many } i\}$. We endow $\mathbb{R}^\infty$ with the restriction topology $\mathbb{R}^\infty \subseteq \mathbb{R}^\mathbb{N} = \prod_{i\in\mathbb{N}} \mathbb{R}$. Where $\mathbb{R}^\mathbb{N}$ has the product topology, i.e., the coarsest topology such that all projections are continuous. ▲

Firstly, I will show that $\mathrm{Conf}_n(\mathbb{R}^\infty)$ yields a numerable bundle. We will use the sledgehammer and consider metrisability:

Lemma 6.18. *The space $\mathbb{R}^\mathbb{N}$ and thus also $\mathbb{R}^\infty$ is metrisable.*

Proof. Since $\mathbb{R}$ is homeomorphic to the interval $(0, 1)$, there is a homeomorphism between $\mathbb{R}^\mathbb{N}$ and $(0, 1)^\mathbb{N}$. We will show that $(0, 1)^\mathbb{N}$ is metrisable. Define the metric $d\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}_{\geq 0}$ as

$$d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} \frac{1}{2^i}|x_i - y_i|.$$

It is clear that this is a well-defined metric.
The (product) topology on $\mathbb{R}^\mathbb{N}$ is the coarsest topology such that all projections are continuous. For the topology induced by this metric on, all projections are clearly continuous: any convergent sequence, clearly maps to a convergent sequence in $\mathbb{R}$. Hence, the topology by the metric is finer than the (product) topology on $\mathbb{R}^\mathbb{N}$.
Take a convergent sequence $((x_{i,j})_{i\in\mathbb{N}})_{j\in\mathbb{N}}$ for the (product) topology on $\mathbb{R}^\mathbb{N}$ with some limit $(x_i)_{i\in\mathbb{N}}$. Let $\epsilon > 0$. For every $i \in \mathbb{N}$, there is an N_i such that for all $j > N_i$ holds $|x_{i,j} - x_i| < \frac{\epsilon}{2}$. Moreover, there is an $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\epsilon}{2}$. Now let $M = \max(N, \{N_i \mid i \leq N\})$. Let $m > M$. Using the fact that $x_{i,m}, x_i \in (0, 1)$ for all $i \in \mathbb{N}$, we have that

$$d((x_{i,m})_{i\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} \frac{1}{2^i}|x_{i,m} - x_i| \leq \frac{1}{2^N} + \sum_{1\leq i\leq N} \frac{1}{2^i}|x_{i,m} - x_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, the sequence is also convergent for the metric. Thus, we have found a metric for $\mathbb{R}^\mathbb{N}$. □

Corollary 6.19. *The configuration space $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is metrisable.*

Proof. By the previous lemma and the fact that (finite) products of metrisable spaces are metrisable (see for example [Aze07, Theorem 4]), we know that $(\mathbb{R}^\infty)^n$ is metrisable. Immediately, also $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is metrisable. □

Corollary 6.20. *The projection $\mathrm{Conf}_n(\mathbb{R}^\infty) \to B_n(\mathbb{R}^\infty)$ is a numerable bundle.*

Proof. As $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is metrisable, it is Hausdorff and paracompact. Propositions 6.15 and 6.16 show that we have a numerable bundle. □

The configuration space $\mathrm{Conf}_n(\mathbb{R}^\infty)$ and Milnor/joins spaces, share a similar structure. They both consist of terminating sequences of points in a certain space. Moreover, both are seen as a subspace of some product of spaces. As we will see now, these were exactly the properties required to proof the universality of the Milnor/join spaces. We now use similar proofs to show that $\mathrm{Conf}_n(\mathbb{R}^\infty)$ gives rise to a universal bundle.

Lemma 6.21 (Compare Lemma 6.4). *Let $p\colon E \to B$ a numerable Σ_n-bundle. Then there exists an equivariant map $f\colon E \to Conf_n(\mathbb{R}^\infty)$.*

Proof. By Theorem 4.9, there exists a countable cover $\mathcal{U}$ of trivialising opens for p and a partition of unity $(v_j)_{j\in\mathbb{N}}$ subordinated to $\mathcal{U}$. Without loss of generality, we can assume $\mathcal{U} = \{U_j \mid j \in \mathbb{N}\}$ and $\operatorname{supp}(v_j) \subseteq U_j$ for all $j \in \mathbb{N}$. Since all U_j are trivialising, there are equivariant maps $\psi_j\colon U_j \to \Sigma_n$. We see Σ_n as the permutation on $\{1, \dots, n\}$. Define $f\colon E \to \mathrm{Conf}_n(\mathbb{R}^\infty)$ as

$$f(x) = \begin{pmatrix} (\sigma_1(x))(1)\cdot v_1(p(x)) & (\sigma_2(x))(1)\cdot v_2(p(x)) & \cdots \\ \vdots & \vdots & \dots \\ (\sigma_1(x))(n)\cdot v_1(p(x)) & (\sigma_2(x))(n)\cdot v_2(p(x)) & \cdots \end{pmatrix} \text{ for all } x \in E \text{ with}$$

$$\sigma_j(x) = \begin{cases} (\psi_j(x))^{-1} & \text{if } p(x) \in U_n \\ Id & \text{else} \end{cases}.$$

For all $x \in E$, there exists a $j \in \mathbb{N}$ such that $v_j(p(x)) \neq 0$. Hence, $(\sigma_j(x))(k)\cdot v_j(p(x))$ is different for varying $1 \leq k \leq n$. Moreover: each row terminates. Hereby, f is well-defined. Notice that for all $j, k \in \mathbb{N}$, the map $x \mapsto (\sigma_j(x))(k) \cdot v_j(p(x)$ for $x \in E$ is continuous on U_j and identically zero outside $\operatorname{supp}(v_j)$. By the Pasting Lemma (Lemma A.2), we conclude that f is continuous. For all $\rho \in \Sigma_n$ there holds:

$$\begin{gathered} f(\rho x) = \\ \begin{pmatrix} (\sigma_1(\rho x))(1)\cdot v_1(p(\rho x)) & (\sigma_2(\rho x))(1)\cdot v_2(p(\rho x)) & \cdots \\ \vdots & \vdots & \dots \\ (\sigma_1(\rho x))(n)\cdot v_1(p(\rho x)) & (\sigma_2(\rho x))(n)\cdot v_2(p(\rho x)) & \cdots \end{pmatrix} = \\ \begin{pmatrix} (\sigma_1(x))(\rho^{-1}(1))\cdot v_1(p(x)) & (\sigma_2(x))(\rho^{-1}(1))\cdot v_2(p(x)) & \cdots \\ \vdots & \vdots & \dots \\ (\sigma_1(x))(\rho^{-1}(n))\cdot v_1(p(x)) & (\sigma_2(x))(\rho^{-1}(n))\cdot v_2(p(x)) & \cdots \end{pmatrix} = \\ \rho \begin{pmatrix} (\sigma_1(x))(1)\cdot v_1(p(x)) & (\sigma_2(x))(1)\cdot v_2(p(x)) & \cdots \\ \vdots & \vdots & \dots \\ (\sigma_1(x))(n)\cdot v_1(p(x)) & (\sigma_2(x))(n)\cdot v_2(p(x)) & \cdots \end{pmatrix} = \\ \rho f(x) \end{gathered}$$

The third equality holds because permuting the rows is precisely the inverse of permuting the coefficients. We conclude that f is an equivariant map. $\square$

Lemma 6.22 (Compare Lemma 6.5). *Let E a Σ_n-space. Then any two equivariant maps $f, g\colon E \to Conf_n(\mathbb{R}^\infty)$ are Σ_n-homotopic.*

Proof. We can write $f = (s_1, s_2, \dots)$ with $s_j\colon E \to \mathbb{R}^n$ for all $j \in \mathbb{N}$. For all $k \in \mathbb{N}$, the maps $(s_1, \dots, s_k, 0, s_{k+1}, 0, \dots)$ and $(s_1, \dots, s_k, s_{k+1}, 0, s_{k+2}, 0, \dots)$ are Σ_n-homotopic with the (well-defined) Σ_n-homotopy $H^k\colon E \times I \to \mathrm{Conf}_n(\mathbb{R}^\infty)$ given by

$$H^k_t = \big(s_1, \dots, s_k, ts_{k+1}, (1-t)s_{k+1}, ts_{k+2}, (1-t)s_{k+2}, \dots\big).$$

Define $H\colon E \times I \to \mathrm{Conf}_n(\mathbb{R}^\infty)$ as the concatenation $H^1 \cdot H^2 \cdot \dots$. Notice that this definition makes sense, since for every $x \in E$, there is only a finite number of $k \in \mathbb{N}$ such that H^k_t is different from f for some $t \in I$. For all projections $\mathrm{Conf}(\mathbb{R}^\infty) \to \mathbb{R}$ only finitely many homotopies are relevant. Hence, the universal property of the product topology shows that H

is continuous. Moreover, H is an equivariant map. Hereby, the maps $(s_1, 0, s_2, 0, \dots)$ and f are Σ_n-homotopic. Similarly, when we write $g = (s'_1, s'_2, \dots)$ with $s'_j \colon E \to I$, we see that g is Σ_n-homotopic to $(s'_1, 0, s'_2, 0, \dots)$. By the Σ_n-homotopy $t \mapsto (ts'_1, (1-t)s'_1, ts'_2, (1-t)s'_2, \dots)$, the maps $(s'_1, 0, s'_2, 0, \dots)$ and $(0, s'_1, 0, s'_2, \dots)$ are Σ_n-homotopic. Furthermore, the Σ_n-homotopy given by $t \mapsto (ts_1, (1-t)s'_1, ts_2, (1-t)s'_2, \dots)$ shows that $(s_1, 0, s_2, 0, \dots)$ and $(0, s'_1, 0, s'_2, \dots)$ are Σ_n-homotopic. We conclude that f and g are Σ_n-homotopic. □

Corollary 6.23. *The projection $\pi \colon \mathit{Conf}_n(\mathbb{R}^\infty) \to B_n(\mathbb{R}^\infty)$ is a universal bundle.*

Proof. By Corollary 6.20, the projection π is a numerable bundle. Lemmas 6.21 and 6.22 show that π is universal. □

Corollary 6.24. *For every discrete finite group H, there is an $n \in \mathbb{N}$ such that the projection $\pi \colon \mathit{Conf}_n(\mathbb{R}^\infty) \to \mathit{Conf}_n(\mathbb{R}^\infty)/H$ is a universal bundle for H.*

Proof. Any finite group can be embedded in a Σ_n for some n (Cayley's theorem, see [Arm88, Theorem 8.2]). Since any subgroup of a discrete group is admissible, the previous assertion, Proposition 6.16 and Theorem 6.12 imply the assertion. □

7 Covering Spaces

Till now we considered principal bundles: these were locally trivial spaces by some group. Similar objects are covering spaces: these are locally trivial spaces, by some discrete space. We give a definition:

Definition 7.1. A **covering space** is a map $p\colon X \to B$ such that for every $x \in B$, there exists an open neighbourhood $x \in U \subseteq B$ with $p^{-1}(U) \cong \sqcup_{i\in J} U_j$, where p maps each U_j homeomorphically to U, i.e., the preimage of U is the disjoint union of copies of U. Similar to principal bundles, we call such an open U **trivialising**. ▲

Many authors (e.g., [Hat15, p. 29] and [May07, p. 21]) impose a (path-)connectedness condition on the space X. In this text, I do not impose this. Whenever the condition is required, we impose it "ad hoc". We call such covering spaces **(path-)connected covering spaces**.
As for principal bundle when $p\colon X \to B$ and $p'\colon \widetilde{X} \to B$ are covering spaces, we say that $f\colon X \to \widetilde{X}$ is a **map over** B if $p = p' \circ f$. When such an f is a homeomorphism, we say it is a **covering space isomorphism**, or just **isomorphism** if the context allows so. In case $X = \widetilde{X}$ and f isomorphism, we say that f is a **deck transformation** on X. We denote $\mathrm{Deck}(p)$ for the set deck transformations of a covering space p. The set $\mathrm{Deck}(p)$ has a natural group structure by composition. The deck transformation group also has a natural action on X by sending $(f, x) \mapsto f(x)$ for all $f \in \mathrm{Deck}(p)$ and $x \in X$. A covering space is called **normal** or **regular**, when the group of deck transformations acts transitively on the fibers of p. That is for all $x, y \in X$ with $p(x) = p(y)$, there exists an $f \in \mathrm{Deck}(p)$ such that $f(x) = y$. We shall see that connected normal covering spaces are connected principal bundles.
We first note the following property of covering spaces. The statement and proof are adapted from [Hat15, Proposition 1.34].

Proposition 7.2. *Let $p\colon X \to B$ a (non-empty) connected covering space. Any two deck transformation $f, g\colon X \to X$ are equal if and only if they agree on one point.*

Proof. The condition is obviously necessary. Suppose f and g agree on one point, we shall prove that the set $A = \{x \in X \mid f(x) = g(x)\}$ is both open and closed. By connectedness of X, the assertion follows.
Suppose $x \in X$. There is a trivialising open $U \subseteq B$ containing $p(x)$. Write $p^{-1}(U) = \sqcup_{j\in J} U_j$ such that p maps each U_j homeomorphically to U. Let $j_1, j_2 \in J$ such that $f(x) \in U_{j_1}$ and $g(x) \in U_{j_2}$. Let $N := f^{-1}(U_{j_1}) \cap g^{-1}(U_{j_2})$. Notice that $x \in N$ and N is open. If $f(x) \neq g(x)$, then $U_{j_1} \neq U_{j_2}$. Hence, U_{j_1} and U_{j_2} are disjoint and thus f and g are nowhere equal on N. Hence, A is closed. If $f(x) = g(x)$, then $U_{j_1} = U_{j_2}$. Since $p \circ f = p \circ g$, we have that $p|_{U_{j_1}} \circ f|_N = p|_{U_{j_1}} \circ g|_N$. Since $p|_{U_{j_1}} : U_{j_1} \to U$ is a homeomorphism, there holds $f|_N = g|_N$ and thus A is open. □

Many authors, such as [Die08, p. 349] and [Mit11, p. 9], notice that normal covering spaces are nothing else than principal bundles for discrete groups and vice versa.

Theorem 7.3. *Let G a discrete group and X a connected space. A map $p\colon X \to B$ is a normal covering space with* $\mathrm{Deck}(p) = G$ *(as groups) if and only if it is a principal G-bundle.*

Proof. Suppose p is a normal covering space with $\mathrm{Deck(p)} = G$. By the natural action, X is a G-space and p an equivariant map. Let $U \subseteq B$ a trivialising open for the covering space p. Write $p^{-1}(U) = \sqcup_{j\in J} U_j$ such that p maps each U_j homeomorphically to U. Choose a $j \in J$. Clearly the orbit space $(\sqcup_{j\in J} U_j)/G$ is homeomorphic to U. For all $x \in p^{-1}(U)$, there exists a unique (use the previous proposition) $g \in G$ such that $g^{-1}x \in U_j$. Define $\psi\colon p^{-1}(U) \to G$ as this g. For all $h \in G$ holds $\psi(hx) = h\psi(x)$ and $\psi^{-1}(h) = U_{j'}$ for some $j' \in J$. Thus ψ is an equivariant map and by Corollary 3.18, p is a principal G-bundle.

Conversely, if p is a principal G-bundle, then for any trivialising open U of B (for the principal bundle p), there is a G-homeomorphism over U between $p^{-1}(U)$ and $U \times G = \sqcup_{g \in G} U$. Hence, p maps every sheet in $p^{-1}(U)$ homeomorphically to U. Thus p is a covering space. Notice that acting with a $g \in G$ on X is a deck transformation. For any deck transformation f and $x \in X$, there is a $g \in G$ such that $gx = f(x)$. By the previous proposition, f equals the action by g. We conclude that $\mathrm{Deck}(p) = G$. Let $x, y \in X$ such that $p(x) = p(y)$. Then $x = gy$ for some $g \in G$ and thus p is a normal covering space with deck transformation group G. □

Similar to principal bundles, we say that a covering space $p\colon X \to B$ is **numerable** if there exists a numerable cover $\mathcal{U}$ of trivialising opens. Using the proof of the previous theorem, it is clear that a connected normal numerable covering space is a connected numerable bundle and vice versa. Combining this with the classification of numerable bundles, Theorem 5.5, we obtain the following: for every numerable normal connected covering space over a space B with (discrete) deck transformation group G, there exists a unique homotopy class of maps $[f\colon B \to BG]$, with BG a classifying space such that the pullback over f is isomorphic to the original covering space. In general, this correspondence between homotopy classes of maps and connected covering spaces is not bijective (in fact it is only bijective if $G = *$) since some principal bundles have a disconnected space. In case G is finite, by Corollary 6.24, we can take $BG = \mathrm{Conf}_n(\mathbb{R}^\infty)/G$ for a sufficiently large n.

Another way to view covering spaces in terms of principal bundles is as follows. We consider only finitely evenly sheeted covering spaces. For a principal Σ_n-bundle $p\colon X \to B$, consider the Σ_n space $(X \times \{1, \dots, n\})/\Sigma_n$. Here Σ_n acts on $\{1,,,.,n\}$ in the canonical way and $\{1,,,.,n\}$ has the discrete topology. We claim that the projection $\pi\colon (X \times \{1, \dots, n\})/\Sigma_n \to B$ is an n-sheeted covering space. This construction is similar to [Die08, p. 341] for vector bundles.

Proposition 7.4. *If $p\colon X \to B$ is a principal Σ_n-bundle, then $\pi\colon (X \times \{1, \dots, n\})/\Sigma_n \to B$ given by $\pi([x, i]) = p(x)$ is a (well-defined) n-sheeted covering space.*

Proof. The map π is clearly well-defined. Suppose U is a trivialising open in B for p. We see:

$$\pi^{-1}(U) = (p^{-1}(U) \times \{1, \dots, n\})/\Sigma_n \cong (U \times \Sigma_n \times \{1, \dots, n\})/\Sigma_n \cong U \times (\Sigma_n \times \{1, \dots, n\})/\Sigma_n.$$

Here "$\cong$" means homeomorphic with a homeomorphism over U. The space $(\Sigma_n \times \{1, \dots, n\})/\Sigma_n$ is the discrete space on n points. Hence, π is an n-sheeted covering space. □

It is obvious that isomorphic principal bundles yield under this construction isomorphic covering spaces. Moreover, numerable bundles yield numerable covering spaces. Theorem 5.5 and Corollary 6.23 give a well-defined map from homotopy classes of maps $B \to B_n(\mathbb{R}^\infty)$ to numerable n-sheeted covering spaces $p\colon X \to B$. We now seek to find a map the other way round.

Proposition 7.5. *For any numerable n-sheeted covering space $p\colon X \to B$, there exists a continuous injective map $h\colon X \to B \times \mathbb{R}^\infty$ over B.*

Proof. The proof of Lemma 4.7 works the same for covering spaces. (More precisely Lemmas 3.5 and 3.23 hold for evenly sheeted covering spaces too.) Hence, there is a countable partition of unity $(v_j)_{j \in \mathbb{N}}$ subordinate to a countable open trivialising cover $\mathcal{V} = \{V_j \mid j \in \mathbb{N}\}$. Assume without loss of generality that $\mathrm{supp}(v_j) \subseteq U_j$ for all $j \in \mathbb{N}$. Write $p^{-1}(V_j) = \sqcup_{i=1}^{n} V_{j,i}$ for all $j \in \mathbb{N}$ such that p maps each $V_{j,i}$ homeomorphically to V_j.

There is the continuous (!) map $s_j\colon p^{-1}(V_j) \to \{1,\dots,n\}$ sending $x \in X$ to the unique i such that $x \in V_{j,i}$. Define $h\colon X \to B \times \mathbb{R}^\infty$ as

$$h(x) = \Big(p(x), (x_j(x))_{j\in\mathbb{N}}\Big) \text{ with } x_j(x) = \begin{cases} s_j(x) \cdot v_j(p(x)) & \text{if } p(x) \in \operatorname{supp}(v_j) \\ 0 & \text{if } v_j(p(x)) = 0 \end{cases} \quad \text{for all } x \in X.$$

Using the Pasting Lemma (Lemma A.2), h is well-defined and continuous. We show that h is injective: if $h(x) = h(y)$ for some $x, y \in X$, then $p(x) = p(y)$. Let $j \in \mathbb{N}$ such that $v_j(p(x)) = v_j(p(y)) \neq 0$. Since $x_j(x) = x_j(y)$, there holds $s(x) = s(y)$. Hence, $V_{j,s(x)} = V_{j,s(y)}$ and thus $x = y$. We have found h as requested. □

The map h is as we shall see very similar to the equivariant maps from Lemma 6.4 and Lemma 6.21 in the case of principal bundles. These equivariant maps were unique up to G-homotopy. A similar statement holds for these continuous injective maps, the proof is similar too:

Lemma 7.6 (Compare Lemmas 6.5 and 6.22)**.** *Let $p\colon X \to B$ an n-sheeted covering space. Any two injective continuous maps $h, h'\colon X \to B \times \mathbb{R}^\infty$ over B are homotopic over B as injective maps. That is, there is a homotopy $H\colon X \times I \to B \times \mathbb{R}^\infty$ between h and h' such that H is a map over B and H_t is injective for all $t \in I$.*

Proof. We can write $h = (p, (s_1, s_2, \dots))$ with $s_j\colon X \to \mathbb{R}$. Notice that for all $k \in \mathbb{N}$, the maps $(p, (s_1, \dots, s_k, 0, s_{k+1}, 0, \dots))$ and $(p, (s_1, \dots, s_k, s_{k+1}, 0, s_{k+2}, 0, \dots))$ are homotopic as injective maps over B with the homotopy $H^k\colon X \times I \to \mathbb{R}^\infty$ given by

$$H^k_t = \big(p, (s_1, \dots, s_k, ts_{k+1}, (1-t)s_{k+1}, ts_{k+2}, (1-t)s_{k+2}, \dots)\big).$$

Define $H\colon X \times I \to \operatorname{Conf}_n(\mathbb{R}^\infty)$ as the concatenation $H^1 \cdot H^2 \cdot \dots$. Notice that this definition makes sense, since for every $x \in X$, there is only a finite number of $k \in \mathbb{N}$ such that H^k_t is different from h for some $t \in I$. For all projections $\mathbb{R}^\infty \to \mathbb{R}$ only a finite number of homotopies is relevant. Hence, by using the universal property of the product topology, H is continuous. Moreover, H is a homotopy over B of injective maps. Hereby, the maps $(p, (s_1, 0, s_2, 0, \dots))$ and h are homotopic over B as injective maps. Similarly, when we write $h' = (p, (s'_1, s'_2, \dots))$ with $s'_j\colon X \to I$, we see that h' is homotopic over B to $(p, (s'_1, 0, s'_2, 0, \dots))$ as injective map. By the homotopy $t \mapsto (p, (ts'_1, (1-t)s'_1, ts'_2, (1-t)s'_2, \dots))$, the maps $(p, (s'_1, 0, s'_2, 0, \dots))$ and $(p, (0, s'_1, 0, s'_2, \dots))$ are homotopic over B as injective maps. Furthermore, the homotopy sending $t \mapsto (p, (ts_1, (1-t)s'_1, ts_2, (1-t)s'_2, \dots))$ shows that $(p, (s_1, 0, s_2, 0, \dots))$ and $(p, (0, s'_1, 0, s'_2, \dots))$ are homotopic over B as injective maps. We conclude that h and h' are homotopic over B as injective maps. □

For an n-sheeted covering space $p\colon X \to B$ and a continuous injective map $h\colon X \to B \times \mathbb{R}^\infty$ over B, we can define a map $\alpha\colon B \to B_n(\mathbb{R}^\infty)$: let h' the $\mathbb{R}^\infty$ coordinate of h. Then define $\alpha(b) = h'(p^{-1}(b))$ for all $b \in B$. This construction and the next proposition is inspired by [Han78, p. 241].

Proposition 7.7. *Let p, h and α as above. The map α is well-defined and continuous.*

Proof. For all $b \in B$, the set $p^{-1}(b)$ consists of n different points in X, all mapping down to b. By injectivity of h, the set $h'(p^{-1}(b))$ consists of n different points in $\mathbb{R}^\infty$ (the B coordinate is the same under the map h). Hereby, $\alpha(b)$ defines an element of $B_n(\mathbb{R}^\infty)$.
Take a trivialising open $U \subseteq B$. Write $p^{-1}(U) = U_1 \sqcup \dots \sqcup U_n$ such that p maps each U_i homeomorphically to U. Write $p_i\colon U_i \to U$ for these homeomorphisms. For all $b \in U$, we see that $\alpha(b) = [(p_1^{-1}(b), \dots, p_n^{-1}(b))]$. Here $[.]$ denotes the orbit from a point in $\operatorname{Conf}_n(\mathbb{R}^\infty)$. We conclude that α is continuous. □

Proposition 7.8. *Let p, h and α as above. The homotopy class of α does not depend on h.*

Proof. Let $h, h' \colon X \to B \times \mathbb{R}^\infty$ injective and continuous. By Lemma 7.6, there is a homotopy $H \colon X \times I \to B \times \mathbb{R}^\infty$ over B of injective maps between h and h'. Denote $\alpha, \alpha' \colon B \to B_n(\mathbb{R}^\infty)$ for the maps defined using h and h' respectively. Let H' the $\mathbb{R}^\infty$ coordinate of H. Using the same proof of the previous proposition, we see that the map $\overline{H} \colon B \times I \to B_n(\mathbb{R}^\infty)$ given by $\overline{H}(b,t) = H'(p^{-1}(b), t)$ is well-defined and continuous. Clearly $\overline{H}$ is a homotopy between α and α'. $\square$

For a numerable n-sheeted covering space $p \colon X \to B$, there is a continuous injective map $h \colon X \to B \times \mathbb{R}^\infty$ over B. Using the last two propositions, we can assign to every numerable n-sheeted covering space p, the homotopy class of the map $\alpha_p \colon B \to B_n(\mathbb{R}^\infty)$. Here α_p is defined by the construction above.

Proposition 7.9. *If $p \colon X \to B$ and $p' \colon Y \to B$ are isomorphic, then α_p and $\alpha_{p'}$ are homotopic.*

Proof. Let $\psi \colon X \to Y$ an isomorphism. Let $h \colon X \to B \times \mathbb{R}^\infty$ and $h_0 \colon Y \to B \times \mathbb{R}^\infty$ over B continuous and injective. Write h_0' for the $\mathbb{R}^\infty$ coordinate of h_0. Notice that $h_0 \circ \psi$ is continuous and injective. Also, there holds $h_0'(p'^{-1}(b)) = h_0'(\psi(p^{-1}(b)))$ and thus by Proposition 7.8, α_p and $\alpha_{p'}$ are homotopic. $\square$

The last propositions provide a map between isomorphism classes of numerable n-sheeted covering spaces over B to homotopy classes of maps $B \to B_n(\mathbb{R}^\infty)$. The map is given by the assignment $p \mapsto [\alpha_p]$. By Proposition 7.4, there is a map from isomorphism classes of numerable Σ_n-bundles to classes of numerable n-sheeted covering spaces. The classification of numerable bundles, Theorem 5.5, gives us a bijection between isomorphism classes of numerable Σ_n-bundles and homotopy classes of maps $B \to B_n(\mathbb{R}^\infty)$. Motivated by these assignments, we consider the following diagram:

$$\begin{tikzcd}
\mathcal{B}(B,G) \arrow[rr, "X \mapsto (X \times \{1,\dots,n\})/\Sigma_n"] & & C_n(B) \arrow[dl, "p \mapsto [\alpha_p]"] \\
& {[B, B_n(\mathbb{R}^\infty)]} \arrow[ul, tail, two heads, "\alpha \mapsto \alpha^* \pi"] &
\end{tikzcd}$$

Here $\pi \colon \mathrm{Conf}_n(\mathbb{R}^\infty) \to B_n(\mathbb{R}^\infty)$ is the projection, $\mathcal{B}(B,G)$ is the set of isomorphism classes of numerable principal bundles and $C_n(B)$ the set[4] of isomorphism classes of n-sheeted numerable covering spaces. The diagram is a priori not commutative. To prove commutativity, it is sufficient to show the following two assertions: the map from $C_n(B) \to C_n(B)$ once counterclockwise is the identity and the map $C_n(B) \to [B, B_n(\mathbb{R})^\infty]$ is surjective. It is easy to see that together this shows that the diagram commutes.

In the language of configuration spaces, we can define the following object, after [Han78]: for a space X define:

$$E_n(X) := \{(b,x) \in B_n(X) \times X \mid x \in b\}.$$

We easily see that if X is Hausdorff, then the projection $\pi_n \colon E_n(X) \to B_n(X)$ is an n-sheeted covering space ([Han78, Proposition 2.1]). For any map $f \colon B \to B_n(X)$, we have the pullback $f^*\pi$, which is an n-sheeted covering space (use the proof of Proposition 3.19). In the case where $X = \mathbb{R}^\infty$, we will see that these pullbacks and pullbacks for principal bundles coincide, up to the construction of Proposition 7.4.

[4]At this point, I have not proved that this is a set. We will see this in a moment.

We start by considering pullbacks on covering spaces. The following proposition is inspired by [Han78, Proposition 3.1].

Proposition 7.10. *Let $p\colon X \to B$ an n-sheeted covering space and $h\colon X \to B \times \mathbb{R}^\infty$ a continuous injective map. Then pullback $\alpha_p^* \pi_n$ is isomorphic to p.*

Proof. Write h' for the $\mathbb{R}^\infty$ coordinate of h. We define $\alpha_p\colon B \to B_n(\mathbb{R}^\infty)$ as $\alpha_p(b) = h'(p^{-1}(b))$. Define $\Psi\colon X \to E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B$ by $\psi(x) = ((\alpha_p(p(x)), (h'(x))), p(x))$ for all $x \in X$. This is a well-defined continuous map over B. The map is clearly injective, since h' is injective on orbits of p. For any $((c, z), b) \in E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B$, since $z \in c = h'(p^{-1}(b))$, there exists a $y \in p^{-1}(b)$ such that $h'(y) = z$, now clearly $\Psi(y) = ((c, z), b)$. Since locally there is a continuous map choosing this y (namely locally $\alpha_p^* \pi_n$ is a homeomorphism on its image), Ψ^{-1} is continuous. This makes Ψ a homeomorphism over B. Hence, Ψ is an isomorphism. □

Lemma 7.11. *In the diagram above the map $C_n(B) \to C_n(B)$ once counterclockwise is the identity.*

Proof. Write $\pi\colon \mathrm{Conf}_n(\mathbb{R})^\infty \to B_n(\mathbb{R}^\infty)$ for the projection.
Let $p\colon X \to B$ a numerable covering space. We consider pullbacks over α_p (for any continuous injective map $h\colon X \to B \times \mathbb{R}^\infty$). By Proposition 7.10, there is a covering space isomorphism between p and $\alpha_p^* \pi$. I shall prove that there is a homeomorphsim between $E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B$ and $((\mathrm{Conf}_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B) \times \{1, \dots, n\})/\Sigma_n$ over B. This implies the assertion.
We define:

$$\Psi\colon \Big((\mathrm{Conf}_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B) \times \{1, \dots, n\}\Big)/\Sigma_n \to E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B \text{ given by}$$

$$\Psi\Big(\Big[(((x_1, \dots, x_n), b), i)\Big]\Big) = \big((\pi(x_1, \dots, x_n), x_i), b\big)$$

and

$$\Psi^{-1}\colon E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B \to \Big((\mathrm{Conf}_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B) \times \{1, \dots, n\}\Big)/\Sigma_n \text{ given by}$$

$$\Psi^{-1}((c, x), b) = \Big[((x, x_2, \dots, x_n), b), 1\Big] \text{ where } \{x, x_2, \dots, x_n\} = c.$$

Notice that both maps are well-defined. In particular note that the order of $(x_2, \dots, x_n)$ does not matter as every order defines the same point. Moreover, Ψ and Ψ^{-1} are each others inverses. Clearly Ψ and Ψ^{-1} are continuous maps over B. We conclude that Ψ is a covering space isomorphism. □

This lemma implies directly that the map $C_n(B) \to [B, B_n(\mathbb{R})^\infty]$ in the diagram is injective. Hence, $C_n(B)$ is a set.

Lemma 7.12. *In the diagram above the map $C_n(B) \to [B, B_n(\mathbb{R})^\infty]$ is surjective.*

Proof. Suppose we have a map $\alpha\colon B \to B_n(\mathbb{R}^\infty)$. Then we have the covering space $\alpha^* \pi_n$. There is the obvious continuous injection $h\colon E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B \to B \times \mathbb{R}^\infty$ over B defined by $h((c, x), b) = (b, x)$ for $((c, x), b) \in E_n(\mathbb{R}^\infty) \times_{B_n(\mathbb{R}^\infty)} B$. For all $b \in B$, there holds that $h((\alpha^* \pi_n)^{-1}(b)) = \alpha(b)$. Proposition 7.8 shows that any continuous and injective map $E_n(\mathbb{R}^\infty) \times_{B_n \mathbb{R}^\infty} B \to B \times \mathbb{R}^\infty$ yields the same homotopy class of maps $B \to B_n(\mathbb{R}^\infty)$. We conclude that the map $C_n(B) \to [B, B_n(\mathbb{R})^\infty]$ in the diagram is surjective. □

We have proved the following theorem:

Theorem 7.13. *The diagram above commutes and all arrows are bijections.* □

Obviously this theorem classifies all numerable n-sheeted covering spaces over a certain space B.

8 Lifting Properties

Covering spaces have certain "lifting properties" that is: some maps $f\colon B \to C$ for a covering space $p\colon Y \to C$ can be "lifted" to a map $\widetilde{f}\colon B \to X$ such that $p \circ \widetilde{f} = f$. In this chapter, we study the lifting of maps in the setting of numerable bundles. Firstly, we will consider a slightly different type of lifting, namely in the case we have two principal bundles and a map on the orbit spaces. Example 2.4 showed that, in general, maps on orbit spaces can not be lifted to equivariant maps. On the other hand Lemma 5.1 showed that a specific map can be lifted.

Theorem 8.1 ([Die08, Homotopy Lifting 14.3.4])**.** *Let $p\colon Y \to C$ and $p'\colon X \to B$ numerable bundles, $f\colon X \to Y$ an equivariant map and $h\colon B \times I \to C$ a homotopy such that $p \circ f = h_0 \circ p'$. Then there exists an equivariant map $H\colon X \times I \to Y$ such that $H_0 = f$ and $p \circ H = h \circ (p' \times Id_I)$ i.e., H is a lift of h.*

Proof. We have the following diagram:

$$\begin{tikzcd}
(Y \times_C (B \times I)) \times_{B \times I} B \arrow[r] \arrow[d, two heads, "i_0^*(h^*p)"'] & Y \times_C (B \times I) \arrow[r, "\widetilde{h}"] \arrow[d, two heads, "h^*p"] & Y \arrow[d, two heads, "p"] \\
B \arrow[r, "i_0"] & B \times I \arrow[r, "h"] & C
\end{tikzcd}$$

The diagram consists of two pullbacks. Since the top arrows and f are equivariant maps.

Applying Proposition 3.20 twice gives us an isomorphism $i_0^*(h^*p) \overset{\psi}{\Longleftrightarrow} h_0^*p \overset{\psi'}{\Longleftrightarrow} p'$. This makes the following diagram commute:

$$\begin{tikzcd}
 & Y \times_C B \arrow[r, tail, two heads, "\psi'"] \arrow[dl, tail, two heads, "\psi"'] \arrow[dr] & X \arrow[d, "f"] \\
(Y \times_C (B \times I)) \times_{B \times I} B \arrow[r] \arrow[d, two heads, "i_0^*(h^*p)"'] & Y \times_C (B \times I) \arrow[r, "\widetilde{h}"] \arrow[d, two heads, "h^*p"] & Y \arrow[d, two heads, "p"] \\
B \arrow[r, "i_0"] & B \times I \arrow[r, "h"] & C
\end{tikzcd}$$

Additionally, there holds $i_0^*(h^*p) \circ \psi \circ \psi'^{-1} = p'$. Hence, we have a diagram:

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d, two heads, "p'"'] & Y \times_C (B \times I) \arrow[r, "\widetilde{h}"] \arrow[d, two heads, "h^*p"] & Y \arrow[d, two heads, "p"] \\
B \arrow[r, "i_0"] & B \times I \arrow[r, "h"] & C
\end{tikzcd}$$

Here $\widetilde{h} \circ \iota = f$. By this diagram and Corollary 3.15, p' is isomorphic to the restriction $(h^*p)_0$. Hence, by Corollaries 5.2 and 5.3, the map $p' \times Id_I$ is isomorphic to h^*p. Applying the isomorphism to the diagram above gives:

$$\begin{tikzcd}
X \arrow[r, "\widetilde{\iota}"] \arrow[d, two heads, "p'"'] & X \times I \arrow[r, "\widetilde{H}"] \arrow[d, two heads, "p' \times Id_I"] & Y \arrow[d, two heads, "p"] \\
B \arrow[r, "i_0"] & B \times I \arrow[r, "h"] & C
\end{tikzcd}$$

With $\widetilde{H} \circ \widetilde{\iota} = f$. Define $H\colon X \times I \to Y$ as $H(x,t) = \widetilde{H}(Pr_X(\widetilde{\iota}(x)), t)$. We get that $H_0 = \widetilde{H} \circ \widetilde{\iota} = f$, since the I coordinate of $\widetilde{\iota}(x)$ is 0 for all $x \in X$. Moreover, there holds

$$p(H(x,t)) = p(\widetilde{H}(Pr_X(\widetilde{\iota}(x)), t)) = h(p'(Pr_X(\widetilde{\iota}(x)), t) = h(p'(x), t) \text{ for all } (x,t) \in X \times I.$$

Since $\widetilde{H}$ and $\widetilde{\iota}$ are equivariant maps, so is H. Thus H is a suitable lift of h. □

Using this theorem, we can prove a theorem by [Dol63]:

Corollary 8.2 ([Dol63, Theorem 4.8])**.** *For a numerable bundle $p\colon X \to B$, a map $f\colon B' \to X$ and homotopy $h\colon B' \times I \to B$ with $p \circ f = h_0$, there is a lift $H\colon B' \times I \to X$ with $p \circ H = h$ and $H_0 = f$, i.e., the following diagram commutes:*

$$\begin{tikzcd} B' \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d, two heads, "p"] \\ B' \times I \arrow[r, "h"] \arrow[ur, "H"] & B \end{tikzcd}$$

Proof. Let $\pi\colon G \times B' \to B'$ the trivial (numerable) bundle over B'. Define the G-equivariant map $f'\colon G \times B' \to E$ given by $f'(g,b) = gf(b)$ for all $(g,b) \in G \times B'$. Clearly there holds $p \circ f' = p \circ f \circ \pi = h_0 \circ \pi$. By the previous theorem, we obtain a homotopy $H'\colon G \times B \times I \to X$ with $H'_0 = f'$ and $p \circ H' = h \circ (\pi \times Id_I)$.
Define $H\colon B' \times I \to X$ as $H(b,t) = H'(e,b,t)$ for all $(b,t) \in B' \times I$ (e is the identity of G). Now $p(H(b,t) = p(H'(e,b,t)) = h(\pi(e,b),t) = h(b,t)$ and $H_0(b) = H'(e,b,0) = f'(b)$ for all $(b,t) \in B' \times I$. □

This assertion makes numerable principal bundles into **Hurewicz fibrations**, see [Hur55, p. 2]. Covering spaces have unique lifting, this is not true for principal bundles: the lifts created in Theorem 8.1 and Corollary 8.2 are not necessarily unique. However, for the latter we can precisely say when all lifts are unique:

Theorem 8.3. *Under the assumptions of Corollary 8.2, the lift H is unique if and only if the group G is, as topological space, totally path-disconnected*[5]*.*

Proof. Suppose G is totally path disconnected. We argue by contradiction: let H and H' be different lifts from Corollary 8.2. There is some $(b_0, t_0) \in B' \times I$ such that $H(b_0,t_0) \neq H'(b_0,t_0)$. Recall that for a principal bundle $p\colon X \to B$, there is a continuous map $C(X) \to G$ mapping $(x, gx) \mapsto g$ for all $x \in X$ and $g \in G$, see Proposition 3.7. Since the map $I \to C(X)$ given by $I \ni t \mapsto (H(b_0,t), H'(b_0,t))$ is well-defined and continuous (use Lemma 3.9 and the assumption that $p \circ H = p \circ H'$), this defines a path $\gamma\colon I \to G$. We have that $H(b_0,0) = f(b_0) = H'(b_0,0)$ and thus $\gamma(0)$ is the identity of G. There holds $H(b_0,t_0) \neq H'(b_0,t_0)$ and thus $\gamma(t_0)$ is not the identity of G. Hence, the path γ is not constant, which contradicts the assumption. We conclude that the lift is unique.
Conversely, suppose G is not totally path disconnected. Then there is a non-constant continuous map $\gamma\colon I \to G$. Without loss of generality assume that $\gamma(0) = e$, the identity of G (otherwise multiply with $(\gamma(0))^{-1}$). Let $H\colon B' \times I \to X$ the lift from Corollary 8.2. Now the map $(b,t) \mapsto \gamma(t)H(b,t)$ fulfills the requirements of Corollary 8.2 and is different from H by freeness of the action. □

[5] A **totally path-disconnected** space is a space X such that every path $\gamma\colon I \to X$ is constant. Examples include discrete spaces, $\mathbb{Q}$ as subspace of $\mathbb{R}$ and p-adic groups.

Notice that the first half of the last proof applies also for the lifts created in Theorem 8.1. Hence, for totally path-disconnected groups G, homotopies can be lifted uniquely. However, the second half of the proof can not be applied, since the map $(x,t) \mapsto \gamma(t)H(x,t)$ for $(x,t) \in X \times I$ need not be an equivariant map. Obviously for Abelian groups it will be an equivariant map. Also, when we drop the condition that the lift needs to be an equivariant map, we see that homotopies, in the sense of Theorem 8.1, lift uniquely if and only if G is totally path disconnected. We have obviously proven the uniqueness of lifts on (connected) normal covering spaces, as discrete spaces are totally path disconnected.
The uniqueness of lifts for totally path disconnected groups implies the following theorem, well-known in covering space theory. The proof is similar to standard proofs in covering space theory, for all details I will refer to [FZ07, Proposition 5.1]:

Theorem 8.4. *Let G be totally path disconnected, $p\colon X \to B$ a principal G-bundle, Y a path connected and locally path connected space, $y \in Y$ and $x \in X$. Then a continuous map $f\colon C \to B$ with $f(y) = p(x)$ lifts uniquely to a map $\widetilde{f}\colon C \to X$ with $\widetilde{f}(y) = x$ if and only if the group $f_*(\pi_1(Y,y))$ is a subgroup of $p_*(\pi_1(X,x))$. Here $\pi_1(X,x)$ is the group of homotopy classes of loops starting at x, i.e., the fundamental group of (X,x).*

Having this theorem on hand, one might consider using the techniques of the classical covering space classification on principal bundles with totally path disconnected structure group. However, one runs into some problems:

1. In the theory of principal bundles most of the time, we take the structure group fixed (apart from the admissible subgroups). In the classification of covering spaces, we let instead the amount of sheets vary.

2. In covering space theory and isomorphism is simply a homeomorphism over the space "downstairs". For principal bundles, we also demand that this homeomorphism is an equivariant map. The lift created in the theorem above need not be an equivariant map (in case we have a G-space Y).

3. The construction of a universal covering space does not help us in the case of non-discrete groups. One could consider restricting to the orbit spaces of universal bundles. For this [Mil56a] constructs a universal bundle for a (suitable) given orbit space.

I will leave it for the reader to decide whether these problems can be circumvented.

A Appendix

We used the Lebesgue Number Lemma and Pasting Lemma several times in our proofs. Here I give the proofs:

A.1 Lebesgue Number Lemma

The Lebesgue number lemma, in its most general, form is a tool for compact metric spaces on open covers, see [Wil04, Theorem 22.5]. I prove the special case for I with an induction method:

Lemma A.1 (Lebesgue number lemma). *For every open cover $\mathcal{U}$ of I, there exists an $\epsilon > 0$ such that for all $a, b \in I$ with $0 < b - a < \epsilon$, there is a $U \in \mathcal{U}$ with $[a, b] \subseteq U$.*

We first claim the following:

Claim 1. For every two open intervals $A, B \subseteq I$, there exists an $\epsilon_2 > 0$ such that for all $a, b \in I$ with $0 < b - a < \epsilon_2$ and $[a, b] \subseteq A \cup B$, $[a, b] \subseteq A$ or $[a, b] \subseteq B$.

Proof. Let $A, B \subseteq I$ open intervals and let $a_1, a_2, b_1, b_2 \in I$ such that $A - \{0, 1\} = (a_1, a_2)$ and $B - \{0, 1\} = (b_1, b_2)$. Define

$$\epsilon_2 := \min\left(\max\left(\delta\big(\max(0, a_2 - b_1)\big), a_2 - b_1\right), \max\left(\delta\big(\max(0, b_2 - a_1)\big), b_2 - a_1\right)\right).$$

Here δ is the Kronecker delta. Clearly $\epsilon_2 > 0$. Suppose $a, b \in I$ with $0 < b - a < \epsilon_2$ and $[a, b] \subseteq A \cup B$. We argue by contradiction, say $x, y \in [a, b]$ with $x \in A - B$ and $y \in B - A$. Hence, $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$).
Suppose $x < y$. Now $a_1 \leq x < y \leq b_2$ so $x \leq b_1$. Also $y \geq a_2$ since $x \notin (b_1, b_2) \subseteq B$ and $y \neq (a_1, a_2) \subseteq A$. Now $x \leq a_2 \leq y$ and thus $a_2 \in [x, y] \subseteq [a, b]$. Hence, $a_2 \in A \cup B$ and thus $a_2 \in B$ or $a_2 = 1$. With the observation that $a_2 > 0$ and $b_1 < 1$, we get in both cases $b_1 < a_2$ and thus $\epsilon_2 \leq a_2 - b_1$. We arrive at $y < x + \epsilon_2 \leq b_1 + \epsilon_2 \leq a_2$ which contradicts $y \geq a_2$. Similarly, we get a contradiction if $x > y$. ■

Proof of Lemma A.1. Without loss of generality, we can assume that every $U \in \mathcal{U}$ is path-connected. Otherwise, take the cover of all path-components of all $U \in \mathcal{U}$, which is a refinement of $\mathcal{U}$. Now every $U \in \mathbb{U}$ is an interval. Since I is compact take a finite sub-cover $U_1, \ldots, U_n$ of $\mathcal{U}$. We proceed by induction on n: In the case $n = 1$ any $\epsilon > 0$ would fulfill the requirements.
Suppose that for every $k \in \mathbb{N}$ and all covers $V_1, \ldots, V_k$ of I consisting of intervals, we have an $\epsilon' > 0$ with the required properties. Let $W_1, \ldots, W_{k+1}$ be a cover of intervals of I. If $W_1 = I$ any ϵ would work. Otherwise, we can find a $1 < j \leq k + 1$ such that $W_1 \cup W_j$ is an interval. Indeed, at most one of $\{0, 1\}$ is in W_1. If $0 \in W_1$, then $\sup(W_1) \notin W_1$ and thus choose a $1 < j \leq k + 1$ such that $\sup(W_1) \in W_j$. Else $\inf(W_1) \notin W_1$ and thus choose a $1 < j \leq k + 1$ such that $\inf(W_1) \in W_j$.
Now there is an ϵ' for the cover $W_1 \cup W_j, W_2, \ldots, W_{j-1}, W_{j+1}, \ldots, W_{k+1}$ and an ϵ_2 for (W_1, W_j) as in the claim. Let $\epsilon = \min(\epsilon', \epsilon_2)$. For all $a, b \in I$ with $0 < b - a < \epsilon$, we get that $[a, b] \subset W_i$ for $i \neq 1, j$ or $[a, b] \subseteq W_1 \cup W_j$. In the latter case we get by the claim that $[a, b] \subseteq W_1$ of $[a, b] \subseteq W_j$. □

A.2 Pasting Lemma

The pasting lemma allows us to "glue" continuous functions together into a new continuous function.

Lemma A.2. *Let X and Y topological spaces, $A, B \subseteq X$ both open or both closed, and $f\colon A \to Y$ and $g\colon B \to Y$ continuous such that $f|_{A\cap B} = g|_{A\cap B}$. Then $h\colon A \cup B \to Y$ given by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases} \quad \text{for all } x \in X$$

is well-defined and continuous.

Proof. Since $f|_{A\cap B} = g|_{A\cap B}$, h is well-defined. Say both A and B are closed. If $U \subseteq Y$ is closed, then $h^{-1}(U) = (h^{-1}(U) \cap A) \cup (h^{-1}(U) \cap B) = f^{-1}(U) \cup g^{-1}(U)$ is closed and thus h is continuous. A similar argument holds if both A and B are open. $\square$

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