Utrecht University

Mathematics

BSc Thesis

Boundary values of holomorphic functions on the upper half-plane

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17th January 2020

Introduction

To any continuous function $f : \mathbb{R}\times]0,\infty[\to \mathbb{C}$ we can associate a linear form on the space of compactly supported smooth functions. Namely, the map

$$\phi \mapsto \int_{\mathbb{R}} f(x,y)\phi(x)\,\mathrm{d}x. \tag{1}$$

We want to know what happens when we let y go to 0, or more precisely whether

$$\lim_{y\downarrow 0}\int_{\mathbb{R}} f(x,y)\phi(x)\,\mathrm{d}x \tag{2}$$

exists for all $\phi \in C_c^\infty(\mathbb{R})$. If this is the case, then the map

$$\phi \mapsto \lim_{y\downarrow 0}\int_{\mathbb{R}} f(x,y)\phi(x)\,\mathrm{d}x \tag{3}$$

defines a linear form on $C_c^\infty(\mathbb{R})$ as well.
Now the map (1) defines a so called distribution. In the case that (2) exists for all $\phi \in C_c^\infty(\mathbb{R})$, the map (3) is also a distribution, which we will denote by $f(\cdot,0)$. In this case we say that f has a distributional boundary value.

It is a classical result that holomorphic functions on the upper half plane having a certain growing behavior towards the real line, have a distributional boundary value. Let $\mathcal{O}_*(H_+)$ be the space of holomorphic functions on the upper half plane having such growing behavior and let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$. The assignment $f \mapsto f(\cdot + i0)$ gives us a linear operator

$$\beta : \mathcal{O}_*(H_+) \to \mathcal{D}'(\mathbb{R}). \tag{4}$$

Erik van den Ban expected that the map β could be applied to embed the holomorphic discrete series representations in the principal series representations of $\mathrm{SL}(2,\mathbb{R})$.

The aim of this thesis is to study the boundary value map (4) and to investigate its relation with the natural action of $\mathrm{SL}(2,\mathbb{R})$ on the upper half plane by Möbius transformations. The second aim was to realize the mentioned embedding of the holomorphic discrete series representations into the principal series representations.

The first aim has been reached. With this result the mentioned embedding can indeed be realized, as was shown to me by Erik van den Ban. Due to restrictions on the time available to me, I have not been able to finish my study of the proof and to give a written account of it.

Contents

Chapter 1

Complex analysis

In this chapter we shall deal with some basic concepts of complex analysis. In the first section we shall deal with one dimensional complex analysis. We shall go over the notion of complex differentiability and analytic functions, derive the Cauchy-Riemann equations and we shall state the Cauchy integral formula. This first section will be mainly based on the book of Lang [11]. In the second section we will introduce the notion of a multidimensional analytic function and a complex manifold. In the third section we shall look at the fractional linear transformations.
The notion of a holomorphic function will be needed in Chapter 7. Section 2 is mainly provided to justify the proceedings in Chapter 11.

1.1 Complex analysis in one variable

We shall begin by saying what it means for a function to be complex differentiable.

Definition 1.1. Let U be an open subset of $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a function and $z \in U$. We say that f is complex differentiable in z if

$$\lim_{h \to z} \frac{f(z+h) - f(z)}{h}$$

exists. In this case this limit is denoted by $f'(z)$ or $\frac{df}{dz}(z)$ as in the case of real differentiablity. We say that f is differentiable or holomorphic if f is complex differentiable in all $z \in U$. The space of holomorphic functions on U is denoted by $\mathcal{O}(U)$. If D is a closed subset of $\mathbb{C}$ we say that $f : D \to \mathbb{C}$ is holomorphic, if f is holomorphic on some open neighbourhood of D.

Since $\mathbb{C} \simeq \mathbb{R}^2$, we can ask ourselves about the relation of being complex differentiable and being differentiable as function from $\mathbb{R}^2 \to \mathbb{R}^2$. We shall now inquire into this. In our approach we shall follow Lang [11]. Therefore first assume that $f : U \to \mathbb{C}$ is a holomorphic function. We write $f = f_1 + if_2$, where $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$. Now let $z = x + iy \in U$

and let $w = u + iv$. We write $f'(z) = a + ib$. Just as in the real case the condition of being complex differentiable in a point z implies that there is a function $\rho : \mathbb{C} \to \mathbb{C}$ such that

$$\rho(w) = f(z + w) - f(z) - wf'(z),$$

and

$$\lim_{w \to 0} \frac{|\rho(w)|}{|w|} = 0.$$

So we have that

$$f(z + w) - f(z) = wf'(z) + \rho(w) = (au - bv) + i(av + bu) + \rho(w).$$

Now consider $F : U \to \mathbb{R}^2$ the vector field associated with f, i.e. the function that is defined as $F(x, y) := (f_1(x + iy), f_2(x + iy))$. We notice that

$$F(x + u, y + v) - F(x, y) = (au - bv, av + bu) + R(x, y),$$

where $R(u, v) := (\mathrm{Re}(\rho(u + iv)), \mathrm{Im}(\rho(u + iv)))$, with $\lim_{(u,v)\to 0} \|(u, v)\|^{-1}\|R(u, v)\| = 0$. We thus conclude that F is differentiable in (x, y), with Jacobi matrix given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

From this we conclude that the functions f_1 and f_2 have to satisfy the following differential equations, known as the Cauchy-Riemann equations:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \tag{1.1}$$

Conversely a function $f = f_1 + if_2 : U \to \mathbb{C}$, with functions $f_1, f_2 : U \to \mathbb{R}$, is holomorphic if f_1 and f_2 are continuously differentiable and satisfy the Cauchy-Riemann equations, see [11] Chapter I.

For reasons that will become clear later, we introduce the differential operators

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.2}$$

It follows from the Cauchy-Riemann equations that $f : U \to \mathbb{C}$ is holomorphic if and only if $\partial_{\bar{z}} f = 0$. If f is holomorphic it also follows from the Cauchy-Riemann equations that $f' = \partial_z f$.

A certain class of holomorphic functions are functions that are defined by a converging power series. The derivative of such a function is given by the formal derivative of this power series, where the formal derivative of a power serie $\sum_{n=0}^{\infty} a_n z^n$ is given by

$$\sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Definition 1.2. We say that a function $f : U \to \mathbb{C}$ is analytic in $z_0 \in U$ if there is a $r > 0$, such that $D(z_0; r) \subset U$, and a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

with $a_n \in \mathbb{C}$, that is convergent on the disc $D(z_0; r)$, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for all $z \in D(z_0; r)$. The function f is said to be analytic if f is analytic in every $z \in U$.

Notice that an alalytic function is holomorphic, as follows from the above discussion. As a direct consequence of Definition 1.2 we have

Corollary 1.3. *Let U be a connected subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be analytic. Then if there is a $z \in U$ such that $\frac{d^k}{dz^k} f(z) = 0$, for all $k \geq 0$, then $f = 0$.*

This means in particular that if $f, g : U \to \mathbb{C}$ are analytic functions on some open and connected subset U of $\mathbb{C}$ and $f\big|_V = g\big|_V$, for some open $V \subset \mathbb{C}$, then $f = g$.
It turns out that every holomorphic function is also analytic, as can be proved with the Cauchy integral formula.

Theorem 1.4 (Cauchy integral formula)**.** *Let $\overline{D}$ be a closed disc and let $f : \overline{D} \to \mathbb{C}$ be a holomorphic function. Let $\partial\overline{D}$ denote the orientated boundary of $\overline{D}$, orientated counter clock wise. Then for every $z \in D$*

$$f(z) = \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{f(z)}{z - \zeta} \, \mathrm{d}\zeta.$$

Another consequence of the Cauchy integral formula, which we shall need in Chapter 11, is the following theorem.

Theorem 1.5 (Louville's theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded holomorphic function. Then is f constant.*

1.2 Multidimensional complex analysis

The material discussed in this section is draw from [15] Chapter 1 and Chapter 5.

Definition 1.6. Let $U \subset \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}$ be a function. Then f is said to be analytic if each point $z_0 = ({z_0}^1, ..., {z_0}^n)$ has an open neighbourhood D on which f is given by a convergent power series of the form

$$\sum_{k_1,...,k_n \geq 1} a_{k_1,...,k_n}(z^1 - {z_0}^1)^{k_1} \cdots (z^n - {z_0}^n)^{k_n}$$

Lemma 1.7 ([15], Lemma 1.3). *Let $U \subset \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}$ be a continuous function. Then is f on U analytic if and only if f is holomorphic in each of its variables.*

Definition 1.8. Let U, V be open subsets of $\mathbb{C}^n$ and let $f : U \to V$ be a bijective holomorphic map. Then f is called a bi-holomorphic map or an analytic isomorphism if f^{-1} is holomorphic.

Definition 1.9. Let X be a $2n$ dimensional topological manifold. Let $\mathcal{A}$ be an atlas of X. We call $\mathcal{A}$ a holomorphic atlas if for any pair of charts (U, κ) and (V, λ), such that $U \cap V \neq \emptyset$, the map $\lambda \circ \kappa^{-1} : \kappa(U \cap V) \to \lambda(U \cap V)$ is bi-holomorphic. A complex manifold is a $2n$ dimensional topological manifold equipped with a holomorphic atlas. Notice that in particular X is a smooth manifold.

Example 1.10. A classical example is the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$. Recall that $\mathbb{P}^n(\mathbb{C})$ is the space of one dimensional linear subspaces of $\mathbb{C}^{n+1}$. Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^1(\mathbb{C})$ be the map that sends each point z to the subspace spanned by z. We consider $\mathbb{P}^1(\mathbb{C})$ with the quotient topology. Now for any $z = (z_0, ..., z_n) \in \mathbb{C}^{n+1}\backslash\{0\}$ we let

$$[z_0 : ... : z_n] = \pi(z).$$

We consider the the subsets $U_i = \{[z_0 : ... : z_n] \in \mathbb{P}^1(\mathbb{C}) \mid z_i \neq 0\}$, for $0 \leq i \leq n$. Then is the map $\kappa_i : U_i \to \mathbb{C}^n$, given by

$$\kappa_i([z_0 : ... : z_n]) = \left(\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i}\right)$$

a well defined homeomorphism. By a straightforward computation we find that $\kappa_j \circ \kappa_i^{-1} : \kappa_i(U_i \cap U_j) \to \kappa_j(U_i \cap U_j)$, is given by

$$\kappa_j \circ \kappa_i^{-1}(w_1, ..., w_n) = \left(\frac{u_1}{u_j}, ..., \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, ..., \frac{u_{i-1}}{u_j}, \frac{1}{u_j}, \frac{u_i}{u_j}, ..., \frac{u_n}{u_j}\right),$$

where we have assumed that $j > i$ for convenience. One immediately sees that this is a bi-holomorphic function.
In the above example we have omitted some details, but for those I refer the reader to Chapter 1 of [12].

Finally we want to mention that the inverse and implicit function theorem hold also in the holomorphic setting (see [6]). The submersion and immersion theorems hold in the holomorphic setting, as well.

1.2.1 Differential forms and the Dolbeault operator

We shall now briefly discuss de Rham operator on complex manifolds. The material discussed in this subsection can be found in Chapter 9 of [15]. Let M be a n dimensional complex manifold. We look at the complexification of the cotangnet space T_p^*M, of M at a point $p \in M$. We notice that $T_p^*M \otimes \mathbb{C} \simeq (T_pM \otimes \mathbb{C})^*$. We also have that

$$\bigwedge^k (T_p^*M \otimes \mathbb{C}) \simeq \left(\bigwedge^k T_p^*M\right) \otimes \mathbb{C}.$$

From this it then also follows that $\Omega^k(M) \otimes \mathbb{C} \simeq \Gamma^\infty((\bigwedge^k T_p^*M) \otimes \mathbb{C})$. We shall denote this latter space by $\Omega^k_{\mathbb{C}}(M)$.

It follows that de Rham operator d extends to a map $\Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$, which has the same properties as de Rham operator. Now given a coordinate chart (U, κ) of M we have the

the local frame $dx^1, ..., dx^n, dy^1, ..., dy^n$ of T^*M, where $z^j = x^j + iy^j$, for all $1 \leq j \leq n$. This is then also a frame of $T^*M \otimes \mathbb{C}$. Instead of looking at the local frame $dx^1, ..., dx^n, dy^1, ..., dy^n$ we can also look at the frame $dz^1, ..., dz^n, d\bar{z}^1, ..., d\bar{z}^n$, where

$$dz^j := dx^j + idy^j \qquad \text{and} \qquad d\bar{z}^j := dx^j - idy^j.$$

Now for any $f \in C^\infty(M, \mathbb{C})$

$$df = df_1 + idf_2,$$

where $f_1 = \text{Re}(f)$ and $f_2 = \text{Im}(f)$. On U we then have that

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j.$$

It follows that for a smooth k-form $\omega \in \Omega^k_{\mathbb{C}}(M)$ we have, on U,

$$\omega(z) = \sum_{|I|+|J|=k} \omega_{I,J}(z) dz^I \wedge d\bar{z}^J$$

And thus

$$\begin{aligned} d\omega(z) &= \sum_{|I|+|J|=k} d\omega_{I,J}(z) \wedge dz^I \wedge d\bar{z}^J \\ &= \sum_{|I|+|J|=k} \sum_{l=1}^{n} \frac{\partial \omega_{I,J}}{\partial z^l}(z) dz^l \wedge dz^I \wedge d\bar{z}^J + \sum_{|I|+|J|=k} \sum_{l=1}^{n} \frac{\partial \omega_{I,J}}{\partial \bar{z}^l}(z) d\bar{z}^l \wedge dz^I \wedge d\bar{z}^J. \end{aligned}$$

on U.

Definition 1.11. We say that $\omega \in \Omega^k_{\mathbb{C}}(M)$ is of type (p, q) if, in a given coordinate chart, it can be written as

$$\omega(z) = \sum_{|I|=p} \sum_{|J|=q} \omega_{I,J}(z) dz^I \wedge d\bar{z}^J$$

The space of all forms of type (p, q) is denoted by $\Omega^{(p,q)}_{\mathbb{C}}(M)$.

It follows from the following lemma that that the above definition is independent of the choice of coordinates.

Lemma 1.12. *Let $T : U \to V$ be a bi-holomorphic map between open sets $U, V \subset \mathbb{C}^n$. Then $T^* dz^j = \sum_{k=1}^n \frac{\partial F^j}{\partial w^k} dw^k$ and $T^* \,\mathrm{d}\bar{z}^j = \sum_{k=1}^n \frac{\partial F^j}{\partial \bar{w}^k} \,\mathrm{d}\bar{w}^k$.*

The above lemma implies that $T^*(dz^I) = \sum_{|J|=|I|} g_J dw^J$, for some functions $g_J : U \to \mathbb{C}$. So we indeed see that Definition 1.11 is independent on the chosen coordinates.

It follows from Definition 1.11 that $\Omega^k_{\mathbb{C}}(M)$ decomposes as

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{l=0}^{k} \Omega^{(k-l,l)}_{\mathbb{C}}(M).$$

In particular for each k and each p, q such that $p + q = k$, there is a canonical projection

$$\pi^{p,q}_k : \Omega^k_{\mathbb{C}}(M) \to \Omega^{(p,q)}_{\mathbb{C}}(M).$$

We notice that

$$d : \Omega^{(p,q)}_{\mathbb{C}}(M) \to \Omega^{(p+1,q)}_{\mathbb{C}}(M) \oplus \Omega^{(p,q+1)}_{\mathbb{C}}(M).$$

Using de Rham operator and the projections, defined as above, we define the operators

$$\begin{aligned} \partial &:= \pi^{p+1,q} \circ d : \Omega^{(p,q)}_{\mathbb{C}}(M) \to \Omega^{(p+1,q)}_{\mathbb{C}}(M), \\ \overline{\partial} &:= \pi^{p,q+1} \circ d : \Omega^{(p,q)}_{\mathbb{C}}(M) \to \Omega^{(p,q+1)}_{\mathbb{C}}(M). \end{aligned}$$

Notice that $d = \partial + \overline{\partial}$. The operator $\overline{\partial}$ is known as the Dolbeault operator and will be important for us later on. Given a chart (U, κ), we have that

$$\partial\omega(z) = \sum_{|I|+|J|=k} \sum_{l=1}^{n} \frac{\partial \omega_{I,J}}{\partial z^l}(z) dz^l \wedge dz^I \wedge d\bar{z}^J \tag{1.3}$$

$$\bar{\partial}\omega(z) = \sum_{|I|+|J|=k} \sum_{l=1}^{n} \frac{\partial \omega_{I,J}}{\partial \bar{z}^l}(z) d\bar{z}^l \wedge dz^I \wedge d\bar{z}^J \tag{1.4}$$

for any $\omega \in \Omega^{(p,q)}_{\mathbb{C}}(M)$.

1.3 Fractional linear transformations

In this section we shall discuss the fractional linear transformations. This section will be based on [11] and a lecture given by Erik van den Ban in the 2018 summer school of Utrecht University. In this section we will stick to the convention that for $g \in \mathrm{GL}(n, \mathbb{C})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We shall now first introduce the Riemann sphere.

Consider $\mathbb{P}^1(\mathbb{C})$. Let $\kappa_1 : U_1 \to \mathbb{C}$ be as in the above example and let $\varphi := {\kappa_1}^{-1}$. One readily verifies that the complement of the image of $\mathbb{C}$ under φ consists of the single point $[1:0]$. We now let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and we extend φ to a function $\widehat{\varphi} : \widehat{\mathbb{C}} \to \mathbb{P}^1(C)$, by defining $\widehat{\varphi}(\infty) = [1:0]$. We can put a complex structure on $\widehat{\mathbb{C}}$ by requiring $\widehat{\varphi}$ to be a bi-holomorphic map. The complex manifold $\widehat{\mathbb{C}}$ is called the Riemann sphere.

The group $\mathrm{GL}(2, \mathbb{C})$ acts in a natural way on $\mathbb{C}^2$ via the usual matrix multiplication. It is clear that $\mathbb{C}^2 \backslash \{0\}$ is an invariant under this action. We notice that this action maps lines through the origin to lines through the origin, and hence the action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^2 \backslash \{0\}$ induces an action on $\mathbb{P}^1(\mathbb{C})$, given by

$$g \cdot [z_1 : z_2] = [az_1 + bz_2 : cz_1 + dz_2] \qquad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We see that the action of cg is the same as that of g, for any $c \in \mathbb{C}^*$, so we can restrict our attention to the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$. Now via the map $\widehat{\varphi}$ the action of $\mathrm{SL}(2, \mathbb{C})$ transfers to an action on $\widehat{\mathbb{C}}$, by transformations F_g, such that $g \cdot \widehat{\varphi} = \widehat{\varphi} \circ F_g$.

Lemma 1.13. *Let $g \in \mathrm{SL}(2, \mathbb{C})$. The bi-holomorphic transformation $F_g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by the following rules*

(1) For $z \in \mathbb{C}$

$$F_g(z) = \frac{az + b}{cz + d}$$

if $cz + d \neq 0$ and $F_g(z) = \infty$ if $cz + d = 0$

(2) For $z = \infty$ we have that $F_g(z) = \frac{a}{c}$ if $c \neq 0$ and $F_g(z) = \infty$ if $c = 0$.

Proof. We first assume that $z \in \mathbb{C}$. We notice that

$$\widehat{\varphi}(F_g(z)) = g \cdot \widehat{\varphi}(z) = g \cdot [z : 1] = [az + b : cz + d]. \tag{1.5}$$

If $cz+d = 0$ then $\hat{\varphi}(F_g(z)) = [az+b : 0]$ and we thus conclude that $F_g(z) = \infty$. If $cz+d \neq 0$ then

$$\widehat{\varphi}(F_g(z)) = [\frac{az+b}{cz+d} : 1]$$

and hence $F_g(z) = \frac{az+b}{cz+d}$.
Now let $z = \infty$. Then

$$\widehat{\varphi}(F_g(z)) = g \cdot \widehat{\varphi}(z) = g \cdot [1 : 0] = [a : c],$$

from which the last part of the lemma follows. $\square$

The transformations F_g as in the above lemma are known as the fractional linear transformations.

For $a, b \in \mathbb{C}$ we define the maps $M_a, T_b, J : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as

$$\begin{aligned} T_b(z) &:= z + b \\ J(z) &:= -\frac{1}{z} \\ M_a(z) &:= az \end{aligned}$$

Note that these are fractional linear transformations. The following theorem is an slightly adapted version of theorem 5.1 in chapter VII in [11], and says that every fractional linear transformation can be realized as a composition of the above maps.

Theorem 1.14. *Given $g \in \mathrm{GL}(2, \mathbb{C})$ let F_g be the corresponding fractional linear transformation. Then there exist complex numbers α, β, γ such that, either $F_g = T_\beta \circ M_\alpha$ or*

$$F_g = T_\gamma \circ M_\alpha \circ J \circ T_\beta.$$

Furthermore, if $g \in GL(2, \mathbb{R})$, then α, β, γ can be chosen to be real. If in addition $\det(g) > 0$, then α can be chosen to be larger 0.

Proof. First suppose $c = 0$. Then $d \neq 0$ and $F_g(z) = (az + b)/d$. We thus we see that $F_g = T_\beta \circ M_\alpha$, with $\alpha = a/d$ and $\beta = b/d$. It is obvious that α and β are real, if $g \in \mathrm{GL}(2, \mathbb{R})$. We also see that $\alpha > 0$ if $\det(g) > 0$.
We now consider the case that $c \neq 0$. We see that the map $F_{\frac{1}{c}g} = F_g$. So without loss of generality we can assume that $c = 1$. We set $\beta = d$. We then have to solve

$$\frac{az + b}{z + d} = \frac{-\alpha}{z + d} + \gamma$$

or stated differently, we have to solve the equation $az + b = -\alpha + \gamma z + \gamma d$. We see that this equation is solved for $\gamma = a$ and $\alpha = ad - b$. And again it follows that α, β and γ can be chosen to be real, if $g \in \mathrm{GL}(2, R)$ and that $\alpha > 0$ if $\det(g) > 0$. $\square$

We can restrict the action of $\mathrm{SL}(2, \mathbb{C})$ on $\widehat{\mathbb{C}}$ to an action of $\mathrm{SL}(2, \mathbb{R})$.

Lemma 1.15. *Let $g \in \mathrm{SL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ then*

$$\mathrm{Im} F_g(z) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

Proof. The proof is by a direct computation and is left for the reader. $\square$

From the above lemma it follows that the action of $\mathrm{SL}(2, \mathbb{R})$ on $\widehat{\mathbb{C}}$ is not transitive. We let $H_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, the upper half plane, $H_- = \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$, the lower half plane, and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, the extended real line.

Lemma 1.16. *The orbits of the action $\mathrm{SL}(2, \mathbb{R})$ in $\widehat{\mathbb{C}}$ are H_+, H_- and $\hat{\mathbb{R}}$.*

Proof. We first prove that given a $z = x + iy \in H_+$ there is a $g \in \mathrm{SL}(2, \mathbb{R})$ such that $F_g(i) = z$. Let $a = \sqrt{y}$, $b = x\sqrt{y^{-1}}$, $c = 0$ and $d = \sqrt{y^{-1}}$. Then we see that $F_g(i) = \frac{\sqrt{y}i + x\sqrt{y^{-1}}}{\sqrt{y^{-1}}} = x + iy$ and $ad - cb = 1$. The previous argument and Lemma 1.15 show that H_+ is an orbit. By applying complex conjugation we see that H_- is also a $\mathrm{SL}(2, \mathbb{R})$ orbit.
We now notice that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot 0$$

is equal to $\tan\theta$ if $\theta \notin 1/2\pi + \pi\mathbb{Z}$ and to ∞ when $\theta \in 1/2\pi + \pi\mathbb{Z}$. This and Lemma 1.15 show that $\widehat{\mathbb{R}}$ is an orbit. $\square$

Lemma 1.17. *The stabalizer of i in $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2)$.*

Proof. Let $g \in \mathrm{SL}(2,\mathbb{R})$ such that $f_g(i) = i$. Then $\frac{ai+b}{ci+d} = i$ or equivalently $ai + b = id - c$ or $(a-d)i + b + c = 0$. Thus, since $a, b, c, d \in \mathbb{R}$, we have that $a = d$ and $b = -c$. Now notice that $1 = \det(M) = ad - bc = a^2 + b^2$. This can only be the case if $a = \cos(\theta)$ and $b = \sin(\theta)$, for some $\theta \in \mathbb{R}$. $\square$

Chapter 2

Lie groups and Lie algebras

In chapters 11 and 12 we shall be interested in certain representations of $\mathrm{SL}(2, \mathbb{R})$. Therefore a basic knowledge of Lie groups and their representations will be useful. In this chapter we will focus our attention to Lie groups. We shall develop the theory along the lines of Chapters 2 and 3 of [1]. We shall first give a definition of a Lie group and show that $\mathrm{SL}(n, \mathbb{R})$ is a Lie group. After that we shall work towards introducing the exponential map and the Lie algebra. In section 2.2 we shall inquire about the conditions on a smooth group action of a Lie group on a manifold that are sufficient to guaranty the existence of a smooth manifold structure on the quotient space. This will be useful in the discussion of chapter 10 which is essential for chapter 11 and 12.

2.1 Lie groups and Lie algebras

Definition 2.1. A smooth manifold G equipped with the structure of a group such that the multiplication and inversion maps are smooth is called a Lie group.

Example 2.2. We notice that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathrm{Mat}(n, \mathbb{R})$, so $\mathrm{GL}(n, \mathbb{R})$ is a submanifold of $\mathrm{Mat}(n, \mathbb{R})$. We now notice that the multiplication map is the restriction of a bilinear map, and hence smooth. Now the inverse map is also smooth, since the coefficients of the inverse of an invertible matrix are rational functions of the coefficients of the matrix. Hence $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

Example 2.3. The map $\det : \mathrm{Mat}(n, \mathbb{R}) \to \mathbb{R}$, given by $A \mapsto \det A$, is a polynomial function in the coefficients of $A \in \mathrm{Mat}(n, \mathbb{R})$ and hence a smooth function. We notice that, for all $H \in \mathrm{Mat}(n, \mathbb{R})$, we have

$$\det(I + tH) = 1 + t\,\mathrm{tr}(H) + t^2 R(t, H),$$

where $R(t, H)$ is a polynomial in t and the coefficients of H. So we conclude that $T_I \det(H) = \mathrm{tr}(H)$. From this it follows that

$$T_A \det(H) = \det(A)\mathrm{tr}(A^{-1}H),$$

for all $A \in \mathrm{GL}(n, \mathbb{R})$. It thus follows that det is a submersion on $\mathrm{GL}(n, \mathbb{R})$, and hence is $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ a submanifold of $\mathrm{GL}(n, \mathbb{R})$. It thus follows that $\mathrm{SL}(n, \mathbb{R})$ is a Lie group.

In this thesis we will focus our attention on the Lie group $\mathrm{SL}(2, \mathbb{R})$.

Definition 2.4. Let G and H be Lie groups. A map $\varphi : G \to H$ is called a Lie group homomorphism if it is a smooth map that is also a homomorphism in the algebraic sense. The map φ is called a Lie group isomorphism if it is a diffeomorphism and a Lie group homomorphism.

Lemma 2.5. *Let G and H be Lie groups and $\varphi : G \to H$ a continuous group homomorphism. Then is φ a smooth map and hence a Lie group homomorphism.*

Let M be a smooth manifold and $X : M \to TM$ a smooth vector field. Recall that an integral curve is a differentiable curve $\gamma : I \to G$, with I an open interval, such that

$$\frac{\mathrm{d}\gamma}{dt}(t) = X(\gamma(t)), \qquad \text{for all } t \in I.$$

Now let G be a Lie group. Recall that a vector field V is left invariant if $T_{g'}l_g V(g') = V(gg')$, for all $g, g' \in G$, where $l_g : G \to G$, $x \mapsto gx$ is the left translation. For an $X \in T_eG$ we can define a left invariant vector field $V_X : G \to T_eG$, by $V_X(g) = T_e l_g(X)$. The map $X \mapsto V_X$ is actually a linear isomorphism from T_eG to the space of left invariant vector fields on G. For a proof I refer the reader to [12] Theorem 8.37 or [1] Lemma 3.1.

Lemma 2.6 ([1], lemma 3.2)**.** *Let $X \in T_eG$. Then the domain of the integral curve α_X equals $\mathbb{R}$. Moreover, $\alpha_X(s+t) = \alpha_X(s)\alpha_X(t)$, for all $s, t \in \mathbb{R}$. Finally, the map $(t, X) \mapsto \alpha_X(t)$ is smooth.*

With the previous lemma in mind, the following definition makes sense.

Definition 2.7. Let G be a Lie group. The exponential map $\exp : T_eG \to G$ is defined by

$$\exp(X) := \alpha_X(1),$$

where $\alpha_X : \mathbb{R} \to G$ is again the integral curve corresponding to the left invariant vector field $g \mapsto T_e l_g X$.

Lemma 2.8 ([1], lemma 3.6)**.** *Let G be a Lie group. Then for all $t, s \in \mathbb{R}$ and all $X \in \mathfrak{g}$ we have*

$$\begin{aligned}\exp(tX) &= \alpha_X(t)\\ \exp((t+s)X) &= \exp(tX)\exp(sX).\end{aligned}$$

Consider the map $\mathcal{C}_x : G \to G$ given by $\mathcal{C}_x(g) = xgx^{-1}$. Notice that this is the composition of smooth maps, namely the maps $l_x : G \to G$, $g \mapsto xg$ and $r_{x^{-1}} : G \to G$, $g \mapsto gx^{-1}$, and hence smooth. So we can consider its differential at the identity element $T_e\mathcal{C}_x : T_eG \to T_eG$. Since $\mathcal{C}_x$ is a diffeomorphism, $T_e\mathcal{C}_x \in \mathrm{GL}(T_eG)$. The assignment $x \mapsto T_e\mathcal{C}_x$ is a map $G \to \mathrm{GL}(T_eG)$. We denote this map by Ad. Lemma 4.4 of [1] says that Ad is a Lie group homomorphism, and hence in particular that Ad is a smooth map.
Differentiating Ad at the identity element gives us a linear map $T_e\mathrm{Ad} : T_eG \to \mathrm{End}(T_eG)$, which we denote by ad. We are now ready to give the following definition.

Definition 2.9. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the bracket, that is bi-linear, anti-symmetric and satisfies the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \qquad (\text{for all } X, Y, Z \in \mathfrak{g}).$$

Lemma 2.10 ([1], lemma 4.9 an corollary 4.11)**.** *The vectorspace T_eG equipped with the map $[X,Y] := \mathrm{ad}(X)Y$ is a Lie algebra.*

In the remaining of the text we shall refer to $(T_eG, [\cdot,\cdot])$ as the Lie algebra of G.

Example 2.11. Consider again $\mathrm{SL}(n,\mathbb{R})$. Recall that $\det : \mathrm{SL}(n,\mathbb{R}) \to \mathbb{R}$ is a smooth submersion. Since $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ we know that the tangent space of $\mathrm{SL}(n,\mathbb{R})$ in I is $\ker(T_I \det) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$.

Definition 2.12. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebra's and let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a linear map. Then ψ is called a Lie algebra homomorphism if $\psi([X,Y]) = [\psi(X), \psi(Y)]$, for all $X, Y \in \mathfrak{g}$.

Lemma 2.13 ([1], lemma 4.16)**.** *Let G and H be Lie groups and let $\varphi : G \to H$ be a Lie group homomorphism. Then is the map $\phi_* := T_e\phi$ a Lie algebra homomorphism. We further have that the following diagram is commutative.*

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\varphi_*"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

In particular the above lemma implies that

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \end{array}$$

is a commutative diagram. Stated differently we have, for $X \in \mathfrak{g}$, that $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$.

2.2 Quotients of manifolds by group actions

In this section we will briefly discuss a few results concerning manifold structures on orbit spaces of a smooth group actions. We will state the necessary definitions and main results. For a more detailed discussion see [12] Chapter 21 or [1] Chapter 11 up to 15.

Definition 2.14. A topological group is a topological space with a group structure, such that the multiplication map $G \times G \to G$, $(x, y) \mapsto xy$ and the inversion map $G \to G$, $x \mapsto x^{-1}$ are continuous.

Definition 2.15. An action of a topological group G on a manifold M is called a proper action if the map $G \times M \to M \times M$ given by $(g, p) \mapsto (g \cdot p, p)$ is proper, where we recall that a continuous map is proper if the inverse image of a compact set is compact.

Definition 2.16. An action of a group G on a space X is called free if the only element of G that fixes any element of X is the identity element. So stated differently if $g \cdot x = x$, for some $x \in X$, then $g = e$.

Theorem 2.17 ([12], Quotient manifold theorem)**.** *Let G be a Lie group that acts freely and properly on a manifold M. Then the orbit space M/G, equipped with the quotient topology, is a topological manifold and M/G has a unique smooth structure making the projection map $\pi : M \to M/G$ into a submersion.*

Lemma 2.18 ([1], Lemma 14.1)**.** *Let H be a closed subgroup of G. Then the right action of H on G is proper and free.*

Corollary 2.19 ([1], Corollary 14.2). *Let G be a Lie group and H a closed subgroup. Then G/H has a unique structure of a smooth manifold such that the canonical projection $\pi : G \to G/H$ is a smooth submersion.*

Let M be a manifold equipped with a smooth left action of the Lie group G. For $x \in M$ the stabilizer G_x, of x, is defined as

$$G_x := \{g \in G \mid g \cdot x = x\}.$$

Since G_x is the pre-image of x under the map $G \to M$, $g \mapsto gx$, the stabilizer is a closed subgroup. So G/G_x has the structure of a smooth manifold. The map $\alpha_x : g \mapsto gx$ factors through a bijection $\bar{\alpha}_x$ of G/G_x onto the orbit G_x.

Theorem 2.20 ([1], Orbit stabilizer theorem). *Let the Lie group G act transitively on the manifold M, and let $x \in M$. Then $\alpha_x : G \to M$, $g \mapsto gx$ induces a diffeomorphism between G/G_x and M.*

Chapter 3

Vector bundles

In this chapter we shall introduce the notion of smooth vector bundles. Smooth vector bundles will be needed to introduce the induction process in Chapter 10. They will also give us a nice framework for introducing the generalized sections. We shall closely follow [12] in formulation in the first part. The second part is based on some explanation given by Erik van den Ban in some private meetings, but the basic definitions can also be found in [8] Chapter 4.

Definition 3.1. Let M be a smooth manifold. A vector bundle of rank k over M, over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$, is a smooth manifold E together with a smooth surjective map $p : E \to M$ such that

(i) For each $x \in M$, the fiber $E_x = p^{-1}(x)$ has the structure of a k dimensional vector space over the field $\mathbb{K}$.

(ii) For each $x \in M$ there exists a neighbourhood $U \subset M$ of x and a diffeomorphism $\tau : p^{-1}(U) \to U \times \mathbb{K}^k$, satisfying the conditions

- $pr_U \circ \tau = p$, where $pr_U : U \times \mathbb{C}^k \to U$ is is the projection;
- for each $y \in U$ the restriction of τ to E_y is a vector space isomorphism from E_y to $\{y\} \times \mathbb{K}^k$.

The map τ is called a local trivialization over U.

If M and E are complex manifolds, $p : E \to M$ a holomorphic submersion and we can chose the trivializations to be bi-holomorphic, we say that $p : E \to M$ is a holomorphic vector bundle.
The most trivial example of a vector bundle is $p : M \times \mathbb{C}^k \to M$, where M is a smooth manifold and p the projection $(m, v) \mapsto m$. Other examples are the tangent and the cotangent bundle (see [12]).

Lemma 3.2 ([12], Vector bundle chart lemma)**.** *Let M be a manifold and assume that for every $x \in M$ we are given a vector space E_x of dimension k. Let $E = \coprod_{x\in M} E_x$, and let $\pi : E \to M$ be the projection map that maps each element of E_x to x. Assume further that we are given the following data:*

(i) an open cover $\{U_\alpha\}_{\alpha\in A}$ of M

(ii) for each $\alpha \in A$ a bijective map $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^k$, whose restriction to each E_x is a vector space isomorphism from E_x to $\{x\} \times \mathbb{K}^k$.

(iii) for all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{K}, k)$, such that the map $\tau_\alpha \circ \tau_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^k \to (U_\alpha \cap U_\beta) \times \mathbb{K}^k$ is of the form

$$\tau_\alpha \circ \tau_\beta^{-1}(x, v) = (x, \tau_{\alpha\beta(x)} v).$$

Then E has an unique topology and smooth structure making it into a manifold and a smooth vector bundle of rank k over M, with π as projection and $\{(U_\alpha, \tau_\alpha)\}$ as local trivialization.

This lemma is very useful for constructing new vector bundles from existing ones. Let $p : E \to M$ and $q : F \to M$ be vector bundles. From the lemma it follows that we can define vector bundles

$$E \oplus F, \qquad E \otimes F, \qquad E^*, \qquad \bigwedge^k E$$

in a canonical way. The fibers of these bundles are given by,

$$E_x \oplus F_x, \qquad E_x \otimes F_x, \qquad E_x^*, \qquad \bigwedge^k E_x,$$

respectively.

Definition 3.3. Let $p : E \to M$ and $q : L \to N$ be smooth vector bundles. A smooth map $F : E \to L$ is called a bundle homomorphism, if there is a $f : M \to N$ such that $f \circ p = q \circ F$,

$$\begin{array}{ccc} E & \xrightarrow{F} & L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

with the property that $F|_{E_x} : E_x \to L_{f(x)}$ is a linear map. In this case we say that F is a bundle homomorpism over f. We say that F is a bundle isomorphism when F is a bijection and the inverse F^{-1} of F is also a bundle homomorphism.

Proposition 3.4. *Let $p : E \to M$ and $p' : E' \to M$ be smooth vector bundles over M and let $F : E \to E'$ be a bijective smooth vector bundle homomorphism. Then F is a smooth bundle isomorphism.*

Proof. To establish smoothness of F^{-1} it is sufficient to proceed locally. We can thus assume that $E = U \times \mathbb{K}^k$ and $E' = U \times \mathbb{K}^k$. Since $p = p' \circ F$, we conclude that $F(x, v) = (x, A(x)v)$ for some $A : U \to \mathrm{GL}(\mathbb{K}^k)$. It is readily confirmed that A is smooth and therefore also $x \mapsto A(x)^{-1}$. We immediately conclude that $F^{-1}(y, w) = (y, A(y)^{-1}w)$, which is a smooth function. $\square$

3.1 Pull-back vector bundle

Let $p : E \to M$ be a vector bundle and let $f : N \to M$ be a smooth map. Now there is a vector bundle $q : E' \to N$, such that there is a bundle morphism F over f such that the restriction of F on a fiber is a linear isomorphism. We let $f^*(E)$ be the submanifold of $N \times E$ given by

$$f^*(E) := \{(x, v) \in N \times E \mid p(v) = f(x)\}.$$

We now claim that $q : f^*(E) \to N$, given by the restriction of projection of $N \times E \to N$, is a vector bundle. We will therefore show the existence of local trivializations. For this take again a local trivialization (U, τ) of E around $f(x)$. We define the function $\tilde{\tau} : f^{-1}(U) \times p^{-1}(U) \to f^{-1}(U) \times U \times \mathbb{K}^k$, given by $\tilde{\tau}(x', w) = (x, \tau(w))$. Restricting $\tilde{\tau}$ to $f^*(E) \cap (f^{-1}(U) \times p^{-1}(U))$ gives us our trivialization. The fibers of q are given by $\{x\} \times E_{f(x)}$, where E_y denotes the fiber over $y \in M$. These fibers inherit a linear structure making the map $\tilde{f}_x : f^*(E)_x \to E_{f(x)}$ into a linear isomorphism, where $\tilde{f} : f^*(E) \to E$ is given by the restriction of the projection $N \times E \to E$ to $f^*(E)$. Accordingly $\tilde{f}$ is a vector bundle morphism over f and thus we have the following commutative diagram

$$\begin{tikzcd} f^*(E) \arrow[r, "\tilde{f}"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Definition 3.5. The vector bundle $q : f^*(E) \to N$ is called the pull-back bundle.

Now the pull-back bundle is uniquely determined, up to a vector bundle isomorphism, by the following universal property.

Lemma 3.6. *Let $q' : L \to N$ be a vector bundle and $F : L \to E$ a vector bundle homomorphism over f, then there exists an unique vector bundle homomorphism $G : L \to f^*(E)$ such that the following diagram commutes*

$$\begin{tikzcd} L \arrow[dr, "G"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "q'"'] & & \\ & f^*(E) \arrow[r, "\tilde{f}"] \arrow[d, "q"] & E \arrow[d, "p"] \\ & N \arrow[r, "f"] & M \end{tikzcd}$$

In particular we have that if F_x is a linear isomorphism, for every $x \in N$, then G is a vector bundle isomorphism.

Proof. First assume that $G, G' : L \to f^*(E)$ are both vector bundle morphisms such that the above diagram commutes. Then $q \circ G' = q' = q \circ G$ and $\tilde{f} \circ G = F = \tilde{f} \circ G$. Since $\tilde{f} : (x, v) \mapsto v$ and $q : (x, v) \mapsto x$, we conclude that $G(x, v) = G'(x, v) = (q'(w), F(w))$.
Consider the function $G : L \to N \times E$, given by $w \mapsto (q'(w), F(w))$. This is clearly a smooth map. We further notice that, for $w \in L_x$, we have $G(w) \in f^*(E)_x$, so G is a function from L to $f^*(E)$. It is now readily verified that, with G as just defined, the above diagram commutes. We now are left to show that $G|_{L_y}$ is a linear map from L_y to $f^*(E)_y$, for $y \in N$. Therefore we notice, that since $F = \tilde{f} \circ G$ and $G|_{L_x} : L_x \to f^*(E)_x$, we have that $F|_{L_x} = \tilde{f}_x \circ G|_{L_x}$. Since $\tilde{f}_x$ is a linear isomorphism we conclude that $G|_{L_x} = \tilde{f}_x^{-1} \circ F|_{L_x}$. Now $F|_{L_x}$ is a linear map and hence $G|_{L_x}$ is the composition of linear maps and thus linear.

Now assume that $F|_{L_x}$ is an isomorphism for all $x \in N$. It then follows that F is bijective. The final statement thus follows from proposition 3.4.

$\square$

Now for a smooth section $s : M \to E$ of p we can associate a smooth section $N \to f^*(E)$ of q given by

$$x \mapsto (x, s(f(x))). \tag{3.1}$$

A few words on the above expression. First we notice that $(x, s(f(x))) \in f^*(E)$, further the above assignment is smooth, so the above rule indeed gives a smooth section. We now notice that $(x, s(f(x))) = \tilde{f}_x^{-1}s(f(x))$. It follows that f induces a map $\Gamma(E) \to \Gamma(f^*(E))$ given by $f^*(s)(x) = \tilde{f}_x^{-1}s(f(x))$. It clear that this is a linear map. We immediately see that if $s|_{f(N)} = s'|_{f(N)}$, then $f^*(s) = f^*(s')$. So f^* is injective in the case that f is surjective. We have thus proven the following lemma:

Lemma 3.7. *The map $f^* : \Gamma(E) \to \Gamma(f^*(E))$ as above is a linear map. Furthermore, if f is surjective, then f^* is injective.*

Chapter 4

Densities

In this chapter we will focus our attention on densities on a smooth manifold. Densities give us a way of integration, without needing an orientation on the considered manifold. This is in contrast to the integration of top-forms, where we need an orientation. Our use for them will be to introduce the generalized sections and the normalised induction procedure as will be discussed in Section 6.2 and Section 10.3 respectively. The material of this chapter is drawn from [1] Chapter 19, [2] Chapter 19 and [12] Chapter 16.

4.1 Densities

We shall first give a definition of an α-density on a finite dimensional vector space (over the real or complex numbers). After that we define the density bundle and densities of a smooth manifold.

Definition 4.1. Let V be a finite dimensional vector space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of dimension n and $\alpha > 0$. An α-density on V is a map $\omega : \underbrace{V \times \cdots \times V}_{\text{n times}} \to \mathbb{K}$ such that, for all $v_1, ..., v_n \in V$ and all $A \in \mathrm{End}(V)$, we have

$$\omega(Av_1, ..., Av_n) = |\det(A)|^{\alpha} \omega(v_1, ..., v_n). \tag{4.1}$$

We denote the space of densities of V by $D^{\alpha}V$. One easily verifies that $D^{\alpha}V$ is a linear space.

Let $b_1, ..., b_n$ a basis of V. For any $v_1, ..., v_n \in V$, there exists a unique $A \in \mathrm{End}(V)$ such that $A(b_i) = v_i$. So from (4.1) we conclude that $\omega \in D^{\alpha}V$ is completely determined by its value on $(b_1, ..., b_n)$. From this it follows that $D^{\alpha}V$ is a vector space of dimension at most one. On the other hand to any $\lambda \in \mathbb{K}$ we have a density $\omega : \underbrace{V \times \cdots \times V}_{\text{n times}} \to \mathbb{K}$, with $\omega(b_1, ..., b_n) = \lambda$, given by

$$\omega(v_1, ..., v_n) = |\det A|^{\alpha} \lambda,$$

where $A \in \mathrm{End}(V)$ is the unique linear transformation such that $A(b_j) = v_j$, for all $1 \leq j \leq n$. We thus see that $D^\alpha V$ is a one dimensional linear space over the field $\mathbb{K}$.

For a linear map $A : V \to W$ and $\omega : \underbrace{W \times \cdots \times W}_{\text{n times}} \to \mathbb{K}$, we define $A^*\omega : \underbrace{V \times \cdots \times V}_{\text{n times}} \to \mathbb{K}$ by

$$A^*\omega(v_1, ..., v_n) = \omega(Av_1, ..., Av_n), \qquad \text{for all } v_1, ..., v_n \in V.$$

Example 4.2. Let V be a n dimensional vector space over the field $\mathbb{K}$. Consider the space $\bigwedge^n V^*$ of alternating n-fold multilinear forms. If $\epsilon^1, ..., \epsilon^n$ is a basis of V^* then is $\epsilon^1 \wedge \cdots \wedge \epsilon^n$ a basis of $\bigwedge^n V^*$. We recall that for any linear transformation $A : V \to V$ we have that $A^*(\epsilon^1 \wedge \cdots \wedge \epsilon^n) = \det(A)\epsilon^1 \wedge \cdots \wedge \epsilon^n$, so we see that $|\epsilon^1 \wedge \cdots \wedge \epsilon^n|^\alpha \in D^\alpha V$.

Now let M be a smooth n dimensional manifold. Then we define $D^\alpha TM = \coprod_{x \in M} D^\alpha T_x M$. Let $\pi : D^\alpha TM \to M$ denote the projection. Given a coordinate chart (U, φ) of M a section $s : M \to D^\alpha TM$ is, on U, equal to

$$f_U |dx^1 \wedge \cdots \wedge \mathrm{d}x^n|$$

for some function $f : U \to \mathbb{K}$. We say that s is continuous if, for in coordinate charts (U, κ), the function f_U is continuous. The continuous sections, $\Gamma(D^\alpha TM)$, of π are referred to as densities.

For $F : M \to N$ a smooth map between manifolds M and N and μ a density we define the pullback $F^*\mu$ of μ under F by

$$(F^*\mu)_x(v_1, ..., v_n) := \mu_{F(x)}(T_x F v_1, ..., T_x F v_n), \qquad \text{for all } v_1, ..., v_n \in T_{F(x)}N.$$

As mentioned in the introduction there is a way to integrate densities. We give a definition. Let M be a smooth manifold of dimension n and let (U, κ) be a chart. For a density ω on M that is compactly supported in U. On U there is a unique $f : U \to \mathbb{K}$ such that $\omega = f|\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|$. We define

$$\int_U \omega := \int_{\kappa^{-1}(U)} (\kappa^{-1})^*\omega = \int_{\kappa^{-1}(U)} f(\kappa^{-1}(x))|\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|.$$

Now let $\mathcal{U} = \{(U_i, \kappa_i)\}_I$ be a collection of charts of which the chart domains cover M. Let $\{\psi_i\}$ be a partition of unity subordinate to $\mathcal{U}$. Then, for a density ω, we define

$$\int_M \omega := \sum_{i \in I} \int_{U_i} \psi_i \omega.$$

With the substitution of variables theorem, one can show that $\int_M \omega$ is independent on the chosen cover. For a more detailed discussion see [12] Chapter 16.

4.2 Invariant densities

On $\mathbb{R}^n$ we have the Lebesgue measure, which is translation invariant. In terms of Lie groups and densities that means that $l_g^* |\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n| = |\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|$, for all $g \in \mathbb{R}^n$. This can be formulated as that $|\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|$ is left invariant. This notion can be generalized to an arbitrary Lie Group. In this section we will discuss that every Lie group has a positive left invariant density. The results and proofs of this section are drawn from [1] Chapter 19.

Recall that $l_g : G \to G$, $x \mapsto gx$ and $r_g : G \to G$, $x \mapsto xg$.

Definition 4.3. Let G be a Lie group. We say that a density ω on G is left-invariant if $l_g^* \omega = \omega$, for all $g \in G$. We say that ω is right-invariant if $r_g^* \omega = \omega$, for all $g \in G$. A left-invariant density is also called a left Haar measure and a right-invariant density a right Haar measure.

Theorem 4.4 ([1], Lemma 19.6 and Corollary 19.7)**.** *Let G be a Lie group. Then G has an unique positive left-invariant density, up to a positive constant.*

Let dx be a left invariant density on the Lie group G. We notice that for all $g, h \in G$, we have that $l_h \circ r_g = r_g \circ l_h$, and thus $r_g^* l_h^* = l_h^* r_g^*$. It thus follows that $r_g^*(dx)$ is also a left invariant density. From theorem 4.4, we conclude that $r_g^*(\,\mathrm{d}x) = \Delta(g) dx$, for some $\Delta(g) \in \mathbb{C}$. We notice that if dx is a positive density then is $r_g^*(\,\mathrm{d}x)$ a positive density as well. It thus follows that $\Delta(g) \in \mathbb{R}_{>0}$. We thus have a function $\Delta : G \to \mathbb{R}_{>0}$ that assigns to g the value $\Delta(g)$. At last we notice that $r_{gg'} = r_{g'} \circ r_g$, so it follows that $\Delta(gg') = \Delta(g)\Delta(g')$. We thus conclude that the function $\Delta : G \to \mathbb{R}_{>0}$ is a group homomorphism. The homomorphism Δ is called the modular function of G. It turns out that $\Delta(g) = |\det(\mathrm{Ad}(g))|^{-1}$, as follows from the following lemma.

Lemma 4.5 ([1], lemma 19.12)**.** *Let dx be a left invariant density on the Lie group G. Then for every $g \in G$,*

$$r_g^*(dx) = |\det(\mathrm{Ad}(g))|^{-1}\,\mathrm{d}x.$$

Proof. We retain the notation of the above discussion. It follows from the above discussion that $\mathcal{C}_{g^{-1}}^*(dx) = l_{g^{-1}}^* r_g^*(dx) = r_g^*(dx) = \Delta(g)\,dx$. Evaluating in the unit element e we find that

$$\Delta(g)dx(e) = T_e(\mathcal{C}_{g^{-1}})^*\,dx(e) = \mathrm{Ad}(g^{-1})^*\,dx(e) = |\det(\mathrm{Ad}(g))|^{-1}\,dx(e).$$

We thus conclude that $\Delta(g) = |\det(\mathrm{Ad}(g))|^{-1}$. $\square$

Chapter 5
Locally convex vector space

In this chapter we will introduce a few notions of locally convex vector spaces. This has for us two applications. Firstly in chapter Chapter 6 we want to realize the space of distributions, on an open subset $U \subset \mathbb{R}^n$, as the topological dual of the space of the smooth compactly supported functions on U, of which the topology is locally convex. Secondly in Chapter 8 we will be considering continuous group actions of a Lie group on complete locally convex vector spaces.
In this chapter we will only state the necessary definitions and results for our proceedings in Chapter 6 and Chapter 8. We shall be following the structure of [4]. We first introduce the more general notion of a topological vector space and have a brief discussion about completeness of topological vector spaces. In the third section we will state the definition of a locally convex vector space and discuss the relation of locally convex vector spaces with collections of semi-norms. In the fourth section we will discuss to procedures to get a locally convex topology on a space given a collection of locally convex vector spaces. This chapter ends with a short discussion about the topology on the topological dual of a locally convex vector space.

5.1 Topological vector spaces

Let V be a vector space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\mathcal{T}$ be a topology on V

Definition 5.1. The space $(V, \mathcal{T})$ is called a topological vector space if the maps the $V \times V \to V$, given by $(v, w) \mapsto v + w$ and $\mathbb{K} \times V \to V$, given by $(\lambda, v) \mapsto \lambda v$, are continuous. In other words the addition and scalar multiplication maps are continuous with respect to the topology $\mathcal{T}$.

Since for every $x \in V$ the translation $T_x : y \mapsto y + x$ is an homeomorphism the topology is already determined by the collection of open neighbourhoods of 0, i.e. the set $\mathcal{T}(0) = \{D \in \mathcal{T} \mid 0 \in D\}$. So we have that $U \in \mathcal{T}$ if and only if for all $x \in U$ there is a $D \in \mathcal{T}(0)$ such that $x + D \subset U$. But this means that the topology is uniquely determined by any basis of neighbourhoods $\mathcal{B}(0)$. Conversely, a pair $(V, \mathcal{T})$, with V a vector space and $\mathcal{T}$ a translation invariant topology, does not need to be a topological vector space. One needs to put extra conditions on the topology to ensure that $(V, \mathcal{T})$ is a topological vector space. Introducing the necessary definitions will put us on a bit of a side track that has little further relevance for us, so I refer the interested reader to result 1.2 in chapter one of [14].

5.2 Sequences and completeness

Since the representation spaces that we will consider in chapter chapter 8 are complete locally convex vector spaces a short account on completeness seems in place. The material of this section is drawn from [13] and [4].

We start our discussion by saying something about convergence of sequences in a topological vector space an let $\mathcal{B}(0)$ be a basis of neighbourhoods of 0. We consider a topological vector space V. If $(v_n)_{\mathbb{N}}$ is a sequence that converges to $v \in V$, then $v_n - v \to 0$ as $n \to \infty$. Or expressed in terms of $\mathcal{B}(0)$, $(v_n)_{\mathbb{N}}$ converges to $v \in V$ if for all $B \in \mathcal{B}(0)$ there is an N such that for all $n \geq N$ we have that $v_n - v \in B$.

The basis $\mathcal{B}(0)$ gives us also a way to define a Cauchy sequence in V. Namely, a sequence $(v_n)_{\mathbb{N}}$ in V is called a Cauchy sequence if for ever $B \in \mathcal{B}(0)$ there is an N such that for all $n, m \geq N$ we have that $v_n - v_m \in B$. Now V is called sequentially complete if every Cauchy sequence converges.

Before we can say what it means for a topological vector space to be complete we will first need the following two definitions.

Definition 5.2. A directed set $(A, \leq)$ is a partially ordered set with the additional requirement that for all $\alpha, \beta \in A$ there is a $\gamma \in A$ such that $\alpha, \beta \leq \gamma$.

Definition 5.3. Let X be a topological space and A a directed set. A function $f : A \to X$, often denoted by (x_α), is called a net. A net is said to be convergent if there is a $x \in X$ such that for every $U \in \mathcal{T}(x)$ there is a $N \in A$ such that for all $\alpha \geq N$ we have that $x_\alpha \in U$.

Definition 5.4. A Cauchy net is a net such that for all $U \in \mathcal{B}(0)$ there is a $N \in A$ such that for all $\alpha, \beta \geq N$ we have that $x_\alpha - x_\beta \in U$. The space X is called complete if every Cauchy net converges.
In a metric space (X, d) a Cauchy net is a net such that for all $\epsilon > 0$ there is a $N \in A$ such that $d(x_\alpha, x_\beta) < \epsilon$, for all $\alpha, \beta \geq N$.

A complete space is always sequentially complete. We will state a partial converse.

Proposition 5.5. *A metric space is complete if and only if it is sequentially complete*

5.3 Locally convex vector spaces

We shall only be concerned with the topological vector spaces known as the locally convex vector spaces. Before we come to the definition, recall that a convex subset of a vector space is a subset C of V such that for all $x, y \in C$, $(1-t)x + ty \in C$, for all $t \in [0,1]$.

Definition 5.6. A locally convex vector space is a topological vector space $(V, \mathcal{T})$ such that every neighbourhood, of a point $x \in V$, contains a convex neighbourhood of x.

In all cases the topology on a locally convex vector space can be induced by a family of semi-norms. Before continuing our discussion let us recall that a semi-norm, on a vector space V, is a function $p : V \to \mathbb{R}_{>0}$ satisfying

1. $p(\lambda x) = |\lambda| p(x)$, for all $x \in V$ and $\lambda \in \mathbb{K}$
2. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in V$.

Let V be a vector space and let P be a collection of semi-norms on V. For $p \in P$, $x_0 \in V$ and $r > 0$ consider

$$B_p(x_0; r) := \{x \in V \mid p(x - x_0) < r\}.$$

The topology induced by the collection of semi-norms P is the smallest topology containing the collection $\{B_p(x; r) \mid p \in P,\ x \in V \text{ and } r > 0\}$. A basis of neighbourhoods of 0 is then given by all sets of the form

$$B_{p_1,\ldots,p_k}(0; r) = \{x \in V \mid p_1(x), \ldots, p_k(x) < r\},$$

for some $p_1, \ldots, p_k \in P$ and $r > 0$.

Notice that different sets of semi-norms may generate the same topology. Let P be a collection of semi-norms and $P_0 \subset P$. If for every $p \in P$ there exists a $p_0 \in P_0$ such that $p(v) \leq p_0(v)$ for all $v \in V$, then the topologies generated by P and P_0 coincide.

Example 5.7. Let Ω be an open subset $\mathbb{R}^n$. and consider the space $C^\infty(\Omega)$. For $K \subset \Omega$ compact and $k \in \mathbb{Z}_{\geq 0}$ we consider the semi-norms

$$\|f\|_{K,C^r} := \sup_{x \in K} \max_{|\alpha| \leq r} |\partial^\alpha f(x)|. \tag{5.1}$$

With the collection semi norms $\{\|\cdot\|_{K,C^k} \mid K \subset \Omega \text{ compact and } k \in \mathbb{Z}_{\geq 0}\}$, is $C^\infty(\Omega)$ a locally convex vector space.
Let $K \subset \mathbb{R}^n$ be a compact set. Consider $C^k_K(\Omega)$ of k times continuous differentiable functions on K. Equipping $C^k_K(\Omega)$ with the norm $\|\cdot\|_{K,C^k}$, as defined (5.1), makes it into a Banach space and thus in particular a locally convex space.

We have said that in a lot of cases the topology on a locally convex vector space is induced by a given family of semi-norms. We also have that for every locally convex vector space there is a collection of semi-norms that induces the topology on that space. We state this in a theorem.

Theorem 5.8 ([4])**.** *A topological vector space $(V, \mathcal{T})$ is locally convex if and only if there exists a collection of semi-norms that generate the topology $\mathcal{T}$.*

For a sketch of the proof of this theorem see [4]. The following result will be quite useful later on.

Proposition 5.9 ([4], Proposition 2.1.5)**.** *Let (V, P) and (W, Q) be locally convex vector spaces and let*

$$A : V \to W$$

be a linear map. Then A is continuous if and only if for every $q \in Q$ there are $p_1, ..., p_n \in P$ and a $C > 0$ such that

$$q(A(v)) \leq C \max\{p_1(v), ..., p_n(v)\} \qquad \textit{(for all } v \in V\textit{)}$$

5.3.1 Inductive and projective topology

Not all locally convex vector spaces come with an initial collection of semi-norms that induce the topology, as will be the case with the space $C^\infty_c(U)$, $U \subset \mathbb{R}^n$ open, which we shall consider in the next chapter.

Let X be a vector space and $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ a collection of locally convex vector spaces such that $X_\alpha \subset X$ and $X = \bigcup_\alpha X_\alpha$. We further assume that A is a partially ordered set and that $X_\alpha \subset X_\beta$, whenever $\alpha \leq \beta$. We endow X with the finest topology making it a locally convex vector space such that the inclusions $i_\alpha : X_\alpha \to X$ are continuous. This topology is called the inductive topology. A basis of neighbourhoods of 0 is given by the set

$$\mathcal{B}(0) = \{B \subset X \mid B \text{ convex and } B \cap X_\alpha \in \mathcal{T}_\alpha(0)\}.$$

We want to inquire about the nature of continuous maps and converging sequences. We following result is immediate.

Proposition 5.10 ([4], Proposition 2.1.10)**.** *Let Y be a locally convex vector space and let X be as above. A linear map $A : X \to Y$ is continuous if and only if $A_\alpha = A\big|_{X_\alpha} : X_\alpha \to Y$ is continuous, for all α.*

For the study of convergent sequences we will restrict to the the following setting: Let X be a vector space and $\{(X_i, \mathcal{T}_i) \mid i \in \mathbb{N}\}$ a collection locally convex vector space such that $X_\alpha \subset X$ and

$$X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \cdots .$$

Further we assume that X_i is closed in X_{i+1} and $\mathcal{T}_i = \mathcal{T}_{i+1}\big|_{X_i}$. In this setting we have the following result.

Proposition 5.11 ([4], Proposition 2.1.11)**.** *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X converging to a point $x \in X$. Then there is a $n_0 \in \mathbb{N}$ such that $x_n, x \in X_{n_0}$, for all $n \in \mathbb{N}$, and $x_n \to x$ in X_{n_0}.*

Proposition 5.12 ([14], Result 6.6)**.** *If $(X_i, \mathcal{T}_i)$ is complete for all $i \in \mathbb{N}$, then also X, equipped with the inductive topology, is complete.*

Now consider the situation that X is a vector space and $(X_\alpha, \mathcal{T}_\alpha)$ a collection of locally convex vector spaces such that $X \subset X_\alpha$ and $X = \bigcap_\alpha X_\alpha$. We assume again that A is a partially ordered set, but this time we assume that $X_\beta \subset X_\alpha$, whenever $\alpha \leq \beta$. We now endow X with the coarsest topology making it a locally convex vector space such that the inclusions $pr_\alpha : X \to X_\alpha$ are continuous. This topology is called the projective limit topology. We have a similar result as Theorem 5.10.

Proposition 5.13. *Let Y be a locally convex vector space and let X be as above. A linear map $A : Y \to X$ is continuous if and only if $pr_\alpha \circ A$ is continuous for all $\alpha \in A$.*

5.4 Topology on the dual space

Consider the topological dual V^* of a topological vector space V, where we recall that V^* is the space of continuous linear forms $u : V \to \mathbb{C}$. For our purposes later on we shall be interested in the topology on V^*. Now there is not a unique topology on V^*. We will only give the definition of the so called strong topology on V^*, since that will be the one we are interested in. We will only state the definition and then discuss this topology in the case that the topology on V is induced by a collection of semi-norms and state a result for the case that the topology on V is the inductive limit topology of some collection $\{(V_n, \mathcal{T}_n)\}$ of locally convex vector spaces.

Definition 5.14. A set $B \subset V$ is called bounded if for every $U \in \mathcal{T}(0)$ there is a $\lambda > 0$ such that $B \subset \lambda U$.

Definition 5.15. Let V be a locally convex vector space. For $U \subset V$ the polar is defined by

$$U^\circ : \{f \in V^* \mid |f(x)| \leq 1 \text{ for all } x \in U\}.$$

Definition 5.16. We define the strong topology β on V^* as the topology generated by the topology basis

$$\{B^\circ \mid B \subset V \text{ bounded}\}.$$

See [13] for a detailed account.

We discuss this topology in the case that the topology on V is induced by a collection of semi-norms. If the topology on V is generated by a collection of semi-norms P, we have that the ball $B_p(r) = \{v \in V \mid p(v) < r\}$, for $p \in P$, is an open neighbourhood of 0. So for any $p \in P$ there is a $r_p > 0$ such that $B \subset B_p(r_p)$. It then follows from proposition 5.9 that

$$p_B(u) := \sup\{|u(v)| \mid v \in B\} < \infty.$$

Then is $\{p_B \mid B \subset V \text{ bounded}\}$ a collection of semi-norms. Notice that

$$\overline{B}_{p_B}(0; 1) = B^\circ,$$

for all bounded sets B. We thus see that the topology induced by this collection of semi-norms is the same as the strong topology.

Now let X be a vector space and $\{(X_i, \mathcal{T}_i) \mid i \in \mathbb{N}\}$ a collection locally convex vector space such that $X_\alpha \subset X$ and $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \cdots$, such that X_i is closed in X_{i+1} and $\mathcal{T}_i = \mathcal{T}_{i+1}\big|_{X_i}$.

Proposition 5.17 ([16], Proposition 6.8)**.** *If $X = \lim_{\rightarrow} X_n$ is the strict inductive limit, then there is a natural topological isomorphism*

$$X^*{}_\beta \simeq \lim_{\leftarrow}(X_n^*)_\beta,$$

here $\lim_{\leftarrow}(X_n^*)_\beta$ denotes X^* equipped with the inductive limit of X_n^* equipped with the strong topology.

Chapter 6
Distributions

In this chapter we shall be concerned with distributions and generalized sections. Distributions will play a major role in the discussion of chapter 7. We shall start our discussion with the distributions on an open subset of $\mathbb{R}^n$ and after that we shall generalize the discussion to the generalized or distributional sections of a smooth vector bundle. In doing so we will follow [4].

6.1 Local theory

In this section we shall discuss distributions on an open subset $\Omega \subset \mathbb{R}^n$. We want to realize the space of distributions as the topological dual of the space $C_c^\infty(\Omega)$. It is thus natural to start our discussion with the topology on $C_c^\infty(\Omega)$, as we will do in subsection 6.1.1. Having done that we give a definition of the space of distributions $\mathcal{D}'(\Omega)$ on Ω, and we shall discuss briefly some operations on the space of distributions. On the whole, the given account will be brief. Fore a more detailed discussion we refer the reader to [4] and [7].

6.1.1 Test functions

Recall that for a continuous function $f : \Omega \to \mathbb{C}$ the support of f, $\mathrm{supp}(f)$ is defined as the closure of $\{x \in \Omega \mid f(x) \neq 0\}$. The space of compactly supported smooth functions, in $C^\infty(\Omega)$, is denoted by $C_c^\infty(\Omega)$ or in Schwartz' notation $\mathcal{D}(\Omega)$. The space $C_0^\infty(\Omega)$ is also referred to as the space of test functions. [1]

We now look into the topology on $C_c^\infty(\Omega)$. We let $C_K^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid \mathrm{supp}(f) \subset K\}$ the space of smooth functions with support contained in K. We shall consider $C_K^\infty(U)$ with the topology induced by the collection of semi-norms $\{\|\cdot\|_{K,C^r} \mid r \in \mathbb{Z}_{\geq 0}\}$, with $\|\cdot\|_{K,C^r}$ defined by (5.1).

[1] For the existence of compactly supported smooth functions see [7] Chapter 2.

For the moment we agree to write $\mathcal{E}_K(\Omega) = C_K^\infty(\Omega)$ and $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$. We notice that

$$\mathcal{D}(\Omega) = \bigcup_{\substack{K\subset\Omega,\\ K\text{compact}}} \mathcal{E}_K(\Omega).$$

We further notice that we have a partial order on the collection of compact subsets of Ω, namely the inclusion of sets. We can thus equip $\mathcal{D}(\Omega)$ with the inductive limit topology, as we indeed do.

6.1.2 Distributions

We define the space of distributions $\mathcal{D}'(U)$ on U as

$$\mathcal{D}'(U) := (C_c^\infty(U))^*.$$

We then have as a direct consequence from Theorem 5.9 and Theorem 5.10 that:

Corollary 6.1. *A linear form $u : C_c^\infty(\Omega) \to \mathbb{C}$ is a distribution if and only if for every compact subset $K \subset \Omega$ there is a constant $C > 0$ and an order of differentiation $k \in \mathbb{Z}_{\leq 0}$ such that for all $\phi \in C^\infty(K)$ we have*

$$|u(\phi)| \leq C\|\phi\|_{C^k,K}. \tag{6.1}$$

The equivalent condition in the above corollary is given as a definition in [9].

To any locally integrable [2] function $f : \Omega \to \mathbb{C}$ we can associate a distribution, namely

$$u_f(\phi) := \int_\Omega f\phi \,\mathrm{d}x.$$

This is indeed a distribution, since for every compact subset K of Ω we have that

$$|u_f(\phi)| \leq \int_K |f\phi| \,\mathrm{d}x = \|\phi\|_{K,0} \int_K |f| \,\mathrm{d}x, \qquad (\text{for all } \phi \in C_K^\infty(\Omega)).$$

It follows that mapping $f \mapsto u_f$ restricted to $C(\Omega)$ defines an continuous inclusion of $C(\Omega)$ into $\mathcal{D}'(\Omega)$. For a detailed discussion on the injectivity of this assignment, see [7] lemma 3.6.

[2]Lebesgue integrable

It then also follows that the map $f \mapsto u_f$ is a continuous inclusion of $C^\infty(U)$ into $\mathcal{D}'(\Omega)$. It turns out $C^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$. We shall not go into this, but refer the reader to chapter 11 of [7] and in particular corollary 11.7.

The space $C^\infty(U)$ is a ring. The $C^\infty(U)$-module structure of $C^\infty(U)$ can be extended to $\mathcal{D}'(U)$, by defining the multiplication of f with a distribution u as

$$(fu)(\phi) := u(f\phi), \qquad \text{for } \phi \in C_c^\infty(U).$$

We also have a notion of differentiability for distributions. We define

$$(\partial_i u)(\phi) := -u(\partial_i \phi), \qquad \text{for } \phi \in C_c^\infty(U).$$

Notice that this definition is compatible with the differentiation in $C^\infty(U)$, we namely see that

$$u_{\partial_i f}(\phi) = \int_U (\partial_i f)\phi \, \mathrm{d}x = -\int_U f \partial_i \phi \, \mathrm{d}x = (\partial_i u_f)(\phi),$$

where the second equality follows from integration by parts.

6.1.3 Sheaf property

Let $U \subset V \subset \mathbb{R}^n$ be open subsets. We notice that we have a continuous inclusion $i_{V,U} : C_c^\infty(U) \hookrightarrow C_c^\infty(V)$ by extending a function $f \in C_c^\infty(U)$ to $C_c^\infty(V)$, by setting f equal to 0 outside U. This inclusion induces a restriction $\mathrm{res}_{U,V} : \mathcal{D}'(V) \hookrightarrow \mathcal{D}'(U)$, by $\mathrm{res}_{U,V}(u)(\phi) := u(i_{V,U}(\phi))$, for $\phi \in C_c^\infty(U)$. This says that $\Omega \supset U \mapsto \mathcal{D}'(U)$, with the restrictions $\mathrm{res}_{U,V}$, is a presheaf on Ω. We will use the notation $s \mapsto s|_U$, for the restriction to U.

Lemma 6.2. *Let $\{U_i\}$ be an open cover of Ω. Then we have*

1. *(Locality) If $s, t \in \mathcal{D}'(U)$ such that $s\big|_{U_i} = t\big|_{U_i}$, then $s = t$.*
2. *(Gluing) If for every i we have a $s_i \in \mathcal{D}'(U_i)$ such that $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$, then there is a unique $s \in \mathcal{D}'(U)$ such that $s\big|_{U_i} = s_i$.*

Proof. The proof of both items follows from a partition of unity argument and is left for the reader. $\square$

From the above lemma it follows that the assignment $\Omega \supset U \mapsto \mathcal{D}'(U)$ is a sheaf on Ω.

6.1.4 Pullback of distributions

Let $\Phi : X \to Y$ be an diffeomrophism from open subsets $X, Y \subset \mathbb{R}^n$. We let $\Psi := \Phi^{-1}$. Now notice that for any $g \in C(Y)$ we have that

$$u_{\Phi^* g}(\phi) = \int_X g(\Phi(x))\phi(x)\,\mathrm{d}x = \int_Y g(y)\phi(\Psi(y))|\det D\Psi(y)|\,\mathrm{d}y = |\det D\Psi| u_g(\Psi^*\phi),$$

for all $\phi \in C_c^\infty(X)$. This suggests the following definition for the pullback of a distribution $u \in \mathcal{D}'(Y)$:

$$(\Phi^* u)(\phi) = j_\Psi u(\Psi^*\phi) \qquad \text{for all } \phi \in C_c^\infty(X).$$

Again I refer the reader for a more detailed account to chapter 10 of [7].

6.2 Global theory

We are now going to discuss generalized or distributional sections. We consider a smooth vector bundle $p : E \to M$. The space of distributional sections will be defined as the topological dual of the space $\Gamma_c^\infty(E^\vee)$, of compactly supported smooth sections of $p^\vee : E^\vee \to M$. To do this we first need a topology on $\Gamma_c^\infty(E)$. Just as in the local case we will equip $\Gamma_c^\infty(E)$ with the inductive topology originating from the spaces $\Gamma_K^\infty(E)$, of smooth sections with support in K. So before we can really make sense of this we first need to know about the topology on $\Gamma^\infty(E)$. So that is where we shall start.

6.2.1 Topology on the space of smooth sections

Let $\{U_i\}$ be an open cover of M such that U_i is both the domain of a chart $\kappa_i : U_i \to \kappa_i(U_i) \subset \mathbb{R}^m$ and a domain of a local trivialization $\tau_i : E_{U_i} \to U_i \times \mathbb{C}^k$. This data induces a linear isomorphism

$$\phi_i' : \Gamma(E\big|_{U_i}) \to C^\infty(\kappa_i(U_i), \mathbb{C}^k),$$

given by $s \mapsto \tau_i \circ s \circ \kappa_i^{-1}$. Now we define $\phi_i := \phi_i' \circ \mathrm{res}_{U_i}$, where $\mathrm{res}_{U_i} : \Gamma(E) \to \Gamma(E\big|_{U_i})$ is the restriction map. We now define

$$\phi : \Gamma(E) \to \prod_j C^\infty(\kappa_j(U_j), \mathbb{C}^k)$$

to be the map uniquely determined by the property that $pr_i \circ \phi = \phi_i$, where $pr_i : \prod_j C^\infty(\kappa_j(U_j), \mathbb{C}^k) \to C^\infty(\kappa_i(U_i), \mathbb{C}^k)$ is the projection map.
We now take the topology on $\Gamma(E)$ induced by ϕ, where we endow $\prod_j C^\infty(\kappa_j(U_j), \mathbb{C}^k)$ with the product topology. Now this topology is also generated by the collection of semi-norms given by

$$\|s\|_\gamma := \|\phi_i(s)\|_{K,r},$$

where $\gamma = (i, K, r)$, with $K \subset \kappa_i(U_i)$ compact and $r \in \mathbb{Z}_{\geq 0}$.
We observe that the just defined topology is independent of the choice of cover of total trivializations.

6.2.2 Generalized sections

Let $p : E \to M$ be a smooth vector bundle. We define

$$E^\vee := E^* \otimes DTM = \mathrm{Hom}(E, DTM).$$

As in the local case we notice that

$$\Gamma_c^\infty(E^\vee) = \bigcup_{K \subset M} \Gamma_K^\infty(E^\vee),$$

where $\Gamma_K^\infty(E^\vee)$ is the space of sections of $E^\vee \to M$ with their support contained in the compact set K. We equip $\Gamma_c^\infty(E^\vee)$ with the inductive limit topology. We now define

$$\Gamma^{-\infty}(M, E) := (\Gamma_c^\infty(E^\vee))^*.$$

In the case that $E = M \times \mathbb{C}$ we shall just write $\mathcal{D}'(M)$ for $\Gamma^{-\infty}(M, E)$, since we then have a canonical identification $\Gamma_c^\infty(E) \simeq C_c^\infty(M)$. Now the reason for defining $\Gamma^{-\infty}(M, E)$ in this way is that we have a canonical inclusion of $\Gamma(E)$ into $\Gamma^{-\infty}(M, E)$ as we shall now discuss.
We notice that we have a natural pairing

$$\langle \cdot, \cdot \rangle : \Gamma^\infty(E^\vee) \times \Gamma^\infty(E) \to \Gamma^\infty(DTM)$$

given by point wise evaluation. This gives us a natural bi-linear map

$$[\cdot,\cdot] : \Gamma_c^\infty(E^\vee) \times \Gamma^\infty(E) \to \mathbb{C}, \qquad (\omega, s) \mapsto \int_M \omega(s)\langle \omega, s\rangle.$$

We thus see that the map $\Gamma(E) \to \Gamma^{-\infty}(M, E)$, given by $s \mapsto \langle \cdot, s\rangle$, is a continuous linear embedding.

We notice that $\Gamma^\infty(E)$ is a $C^\infty(M)$-module. Hence, as in the local case, we can extend this module structure to $\Gamma^{-\infty}(M, E)$, by defining for $f \in C^\infty(M)$ and $u \in \Gamma^{-\infty}(M, E)$, the multiplication by the rule

$$fu(s) := u(fs), \qquad \text{for all } s \in \Gamma_c^\infty(E^\vee).$$

From this it follows that the discussion of subsection 6.1.3 can be generalized for $\Gamma^{-\infty}(M, E)$.

6.2.3 Invariance under vector bundle isomorphism

Let $p : E \to M$ and $q : L \to N$ be a smooth vector bundles. Let $F : M \to N$ be a vector bundle homomorphism over f. Let $F : M \to N$ be a vector bundle homomorphism over f. Now the homomorphism F induces an homomorphism $F^\vee : L^\vee \to E^\vee$ given by the rule

$$F^\vee(\omega)(v) := f^*\omega(F(v)), \qquad \text{for all } \omega \in {L^\vee}_{f(x)} \text{ and } v \in E_x.$$

This thus gives a homomorphism $\Gamma_c^\infty(L^\vee) \to \Gamma_c^\infty(E^\vee)$, also denoted by $F^\vee$. Hence $F^\vee$ induces a homomorphism $(F^\vee)^* : \Gamma^{-\infty}(M, E) \to \Gamma^{-\infty}(N, L)$, given by

$$((F^\vee)^*u)(s) := u(F^\vee(s)), \qquad \text{for all } s \in \Gamma_c^\infty(L^\vee).$$

In the case that F is an bundle isomorphism, $F^\vee$ will also be an isomorphism.
We can apply the above discussion to the restriction of a vector bundle $p : E \to M$, to the chart domain of the chart (U, κ) of M. We can then consider the pullback bundle $(\kappa^{-1})^*(E_U) \to \kappa(U)$. By the above discussion the corresponding bundle isomorphism $\widetilde{\kappa^{-1}}$ gives an isomorphism $\Gamma^{-\infty}(\kappa(U), (\kappa^{-1})^*(E_U)) \to \Gamma^{-\infty}(U, E)$.

Chapter 7

Boundary values of holomorphic functions on the upper half plane

In this chapter we will discuss the existence of distributional boundary values of holomorphic functions on the upper half plane. The first section will be concerned with the existence of distributional boundary values. In our discussion we follow Section 3.1 of [9]. In the second section we will apply the theory of the first section to define the, so-called, boundary value operator. (See also [7] Chapter 12). In the third section we want to relate the results of the first and second section to the natural action of $\mathrm{SL}(2,\mathbb{R})$ on $\widehat{\mathbb{C}}$.

7.1 Boundary values

Let I be an open interval and $\gamma > 0$. Let $Z := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in I \text{ and } 0 < \mathrm{Im}(z) < \gamma\}$ and $f : Z \to \mathbb{C}$ be a holomorphic function. For every $0 < y < \gamma$ we can associate a distribution $f(\cdot + iy)$ to f, given by

$$f(\cdot + iy)(\phi) := \int_I f(x+iy)\phi(x)\,\mathrm{d}x, \qquad \text{for all } \phi \in C_c^\infty(I).$$

We can now ask ourselves what happens if we let $y \to 0$. So if $\lim_{y\downarrow 0} f(\cdot + iy)(\phi)$ exists, for all $\phi \in C_c^\infty(I)$. In this case we will denote

$$f(\cdot + i0)(\phi) = \lim_{y\downarrow 0} f(\cdot + iy)(\phi).$$

Lemma 7.1. *Let $U \subset \mathbb{C}$ be an open subset with a C^1 boundary. Then for all $g \in C_c^1(\mathbb{C})$ we have:*

$$\int_{\partial U} g(z)\,\mathrm{d}z = 2i\int_U \partial_{\bar z} g(x+iy)\,\mathrm{d}x\,\mathrm{d}y \tag{7.1}$$

Proof. By Stoke's theorem we have that

$$\begin{aligned}\int_{\partial U} g(z)\,\mathrm{d}z &= \int_U \mathrm{d}(g(z)\,\mathrm{d}z) = \int_U (\partial_z g(z)\,\mathrm{d}z + \partial_{\bar z} g(z)\mathrm{d}\bar z) \wedge \mathrm{d}z \\ &= 2i \int_U \partial_{\bar z} g(z)\,\mathrm{d}x \wedge \mathrm{d}y,\end{aligned}$$

where we recall that $dz = dx + idy$ and $d\bar z = dx - idy$. □

Theorem 7.2. *Let $I \subset \mathbb{R}$ be an open interval and let $\gamma > 0$. Let $Z := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in I \text{ and } 0 < Im(z) < \gamma\}$. Assume that $f : Z \to \mathbb{C}$ is a holomorphic function, such that there is a $C > 0$ and a non negative integer N such that*

$$|f(x+iy)| \le Cy^{-N} \tag{7.2}$$

for all $x + iy = z \in Z$. Then for all $\phi \in C_c^\infty(I)$ the limit

$$\lim_{y \downarrow 0} \int_I f(x+iy)\phi(x)\,\mathrm{d}x$$

exists and $f(\cdot + i0) : C_c^\infty(I) \to \mathbb{C}$ defines a distribution of order at most $N + 1$.

Proof. We will follow the proof given by Hörmander in [9]. Let $\phi \in C_c^\infty(I)$. Define the function $\tilde\phi_N : Z \to \mathbb{C}$ by:

$$\tilde\phi_N(x+iy) := \sum_{k=0}^{N} \frac{\phi^{(k)}(x)}{k!}(iy)^k \tag{7.3}$$

Notice that this expression would be equal to the Nth-order Taylor polynomial of the analytic extension of ϕ to Z if such an extension would exist.
Now notice that

$$\begin{aligned}2\frac{\partial}{\partial \bar z}\tilde\phi_N(x+iy) &= \sum_{k=0}^{N} \frac{\phi^{(k+1)}(x)}{k!}(iy)^k + i\sum_{k=1}^{N} \frac{\phi^{(k)}(x)}{(k-1)!}(iy)^{k-1} i \\ &= \sum_{k=0}^{N} \frac{\phi^{(k+1)}(x)}{k!}(iy)^k - \sum_{k=0}^{N-1} \frac{\phi^{(k+1)}(x)}{k!}(iy)^k = \frac{\phi^{(N+1)}(x)}{(N+1)!}(iy)^N\end{aligned}$$

We write $w = u + iv$ and fix a $0 < Y < \gamma$. For any $0 < y < \gamma - Y$ we find, by applying lemma 7.1 to $f(w + iy)\tilde{\phi}_N(w)$ with $U = I \times i\,]0, Y[$, that

$$\int_I f(u + iy)\tilde{\phi}_N(u, 0)\,\mathrm{d}u - \int_I f(u + iy + iY)\tilde{\phi}_N(u, Y)\,\mathrm{d}u$$
$$= 2i \int_0^Y \int_I \partial_{\bar{w}}(f(w + iy)\tilde{\phi}_N(u, v))\,\mathrm{d}u\,\mathrm{d}v. \tag{7.4}$$

Notice that for any open $V \subset \mathbb{C}$, $\psi \in C^1(V)$ and $f \in \mathcal{O}(V)$, we have

$$\frac{\partial}{\partial \bar{z}}(\psi f) = \left(\frac{\partial}{\partial \bar{z}}\psi\right) f + \psi\left(\frac{\partial}{\partial \bar{z}} f\right) = \frac{\partial}{\partial \bar{z}}\psi,$$

since $\partial_{\bar{z}} f = 0$. So we find that

$$\begin{aligned}\int_I f(u + iy)\phi(u)\,\mathrm{d}u &= \int_I f(u + iy)\tilde{\phi}_N(u, 0)\,\mathrm{d}u \\ &= \int_I f(u + iy + iY)\tilde{\phi}_N(u, Y)\,\mathrm{d}u + \int_0^Y \int_I f(w + iy)\partial_{\bar{w}}\tilde{\phi}_N(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_I f(u + iy + iY)\tilde{\phi}_N(u, Y)\,\mathrm{d}u + \int_0^Y \int_I f(w + iy)\phi^{(N+1)}(u)\frac{(iv)^N}{N!}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Since $|v^N f(u + iv + iy)| \le C(y + v)^{-N} v^N \le C$, we have that the double integral is uniformly bounded as $y \downarrow 0$. So we have that both integrals on the left hand side converge as $y \to 0$. It thus follows that

$$\lim_{y \downarrow 0} \int_I f(x + iy)\phi(x)\,\mathrm{d}x$$
$$= \int_I f(u + iY)\tilde{\phi}_N(u, Y)\,\mathrm{d}u + \int_I \int_0^Y f(u + iv)\phi^{(N+1)}(u)\frac{(iv)^N}{N!}\,\mathrm{d}v\,\mathrm{d}u. \tag{7.5}$$

From this last expression one easily concludes that ... defines a distribution of order at most $N + 1$. □

7.2 Boundary value operator

We define the collection of semi-norms $\nu_{N,a,b,h} : \mathcal{O}(H_+) \to \mathbb{R}_{\ge 0}$, for $a < b$, $h > 0$ and $N \in \mathbb{N}$, by

$$\nu_{N,a,b,h}(f) := \sup\{y^N |f(x + iy)| \,|\, a \le x \le b,\ 0 < y \le h\} \tag{7.6}$$

We then define the space

$$\mathcal{O}(H_+)_N := \{f \in \mathcal{O}(H_+) \mid \nu_{N,a,b,h}(f) < \infty, \text{ for all } a < b, 0 < h\},$$

and equip it with the topology induced by the semi-norms $\nu_{N,a,b,h}$, for $a < b$ and $h > 0$.

Let $a < b$ and $0 < h$. Then by assumption there exists, for any $f \in \mathcal{O}(H_+)_N$, a $C > 0$ such that

$$|f(x + iy)| \leq Cy^{-N},$$

for all $a < x < b$ and $0 < y < h$. It follows from Theorem 7.2 that the map

$$\phi \mapsto \lim_{y \downarrow 0} \int_a^b f(x + iy)\phi(x)\,\mathrm{d}x$$

defines a distribution on $]a, b[$. For every $f \in \mathcal{O}(H_+)$, we thus have a distribution on $]a, b[$, which we denote by $\beta_{N,a,b,h}(f)$. It follows that the assignment $f \mapsto \beta_{N,a,b,h}(f)$ defines a map

$$\beta_{N,a,b,h} : \mathcal{O}(H_+)_N \to \mathcal{D}'(]a, b[).$$

Since the limit and integral are linear we conclude that $\beta_{N,a,b,h}$ is a linear operator.
We are now going to argue that this gives us a unique operator $\beta_N : \mathcal{O}(H_+)_N \to \mathcal{D}'(\mathbb{R})$. We first notice that for all $0 < h, h'$, $\beta_{N,a,b,h}(f) = \beta_{N,a,b,h'}(f)$, which is immediate of their definitions. Hence we can just talk about the map $\beta_{N,a,b}$. Now let $a' < b'$ and assume that $]a, b[\cap]a', b'[\neq \emptyset$. We then have, for every $\phi \in C_c^\infty(]a', b[)$, that $\beta_{N,a,b}(f)\big|_{]a',b[}(\phi) = \beta_{N,a',b'}(f)\big|_{]a',b[}(\phi)$. So $\beta_{N,a,b}(f)\big|_{]a',b[} = \beta_{N,a',b'}(f)\big|_{]a',b[}$. By Theorem 6.2 there is a unique distribution $\beta_N(f) \in \mathcal{D}'(\mathbb{R})$ such that $\beta_N(f)\big|_{]a,b[} = \beta_{N,a,b}(f)$, for all $a < b$. We thus have a unique operator

$$\beta_N : \mathcal{O}(H_+) \to \mathcal{D}'(\mathbb{R})$$

such that $\beta_N(f)\big|_{]a,b[} = \beta_{N,a,b}(f)$, for all $f \in \mathcal{O}(H_+)$, and all $a < b$.

Theorem 7.3. *The operator β_N is continuous.*

Proof. Recall that $C_c^\infty(\mathbb{R})$ is the inductive limit of the spaces $C_K^\infty(\mathbb{R})$, with $K \subset \mathbb{R}$ compact. So we have that the natural inclusions $i_K : C_K^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ are continuous. Now Proposition 5.17 says that $\mathcal{D}'(\mathbb{R})$, with the strong topology, is the projective limit of the spaces $(C_K^\infty(\mathbb{R}))^*$, equipped with the strong topology. This means that the projection maps $pr_K : \mathcal{D}'(\mathbb{R}) \to (C_K^\infty(\mathbb{R}))^*$, defined as $pr_K(u)(\phi) = u(i_K\phi)$, are continuous. By Proposition 5.17 and Proposition 5.13 it is thus sufficient to prove that

$$pr_K \circ \beta_N : \mathcal{O}(H_+)_N \to (C_K^\infty(\mathbb{R}))^*$$

is continuous for every compact subset $K \subset \mathbb{R}$.
We know that the topology on $C_K^\infty(\mathbb{R})$ is induced by the collection of semi-norms as defined in (5.1). From the discussion in Section 5.4 we know that the topology on $(C_K^\infty(\mathbb{R}))^*$ is induced by the collection of semi-norms given by

$$p_B(u) := \sup_{\phi \in B} |u(\phi)|,$$

for B a bounded subset of $C_K^\infty(\mathbb{R})$.
Let $B \subset C_K^\infty(\mathbb{R})$ be a bounded set. Then for every $\phi \in B$ we have that

$$\begin{aligned}
&|\lim_{y \downarrow 0} \int_{\mathbb{R}} f(x+iy)\phi(x)\,\mathrm{d}x| \\
&\leq |\int_{\mathbb{R}} f(u+iY)\tilde{\phi}_N(u,Y)\,\mathrm{d}u| + |\int_{\mathbb{R}}\int_0^Y f(u+iv)\phi^{(N+1)}(u)\frac{(iv)^N}{N!}\,\mathrm{d}v\,\mathrm{d}u| \\
&\leq \nu(f)_{N,a,b,h}\int_{\mathbb{R}} |\frac{\tilde{\phi}_N(u,Y)}{Y^N}|\,\mathrm{d}u + \nu(f)_{N,a,b,h}\int_{\mathbb{R}}\int_0^Y |\frac{\phi^{(N+1)}(u)}{N!}|\,\mathrm{d}v\,\mathrm{d}u \\
&\leq \nu(f)_{N,a,b,h}\|\phi\|_{K,N+1}C,
\end{aligned}$$

where

$$C = \frac{1}{N!}\int_K \int_0^Y \mathrm{d}v\,\mathrm{d}u + \sum_{k=0}^N \frac{Y^{k-N}}{k!}\int_K \mathrm{d}u.$$

Since B is a bounded set, we have that

$$\sup_{\phi \in B} \|\phi\|_{K,C^{N+1}} < \infty.$$

We thus conclude that

$$p_B(pr_K \circ \beta_N(f)) \leq C'\nu_{N,a,b,h}(f).$$

From Theorem 5.9 it then follows that $pr_K \circ \beta_N : \mathcal{O}(H_+) \to (C_K^\infty(\mathbb{R}))^*$ is continuous. $\square$

A holomorphic function g on $\overline{H_+}$ is in particular a smooth function, so $g\big|_{\mathbb{R}}\beta(f)$, for $f \in \mathcal{O}(H_+)_N$, is again a distribution on $\mathbb{R}$. On the other hand we have that $gf \in \mathcal{O}(H_+)_N$, so we have the distribution $\beta(gf)$. We can now ask ourselves whether $g\beta(f) = \beta(gf)$, or in other words whether multiplication with g commutes with β. The answer is yes.

Lemma 7.4. *Let $g : \overline{H_+} \to \mathbb{C}$ be a holomorphic function. Then*

$$\beta(f)(g\phi) = \beta(fg)(\phi), \qquad \textit{for all } \phi \in C_c^\infty(R) \textit{ and all } f \in \mathcal{O}(H_+)_N.$$

Proof. We notice that, since g is differentiable,

$$g(x+iy) = g(x) + i\frac{\partial g}{\partial y}(x)y + R(x,y),$$

where $R \in C^\infty\left(\overline{H}_+\right)$ and $\lim_{y\to 0} R(x,y) = 0$.

$$\begin{aligned}&\int_{\mathbb{R}} f(x+iy)g(x+iy)\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x+iy)g(x)\phi(x)\,\mathrm{d}x + iy\int_{\mathbb{R}} f(x+iy)\frac{\partial g}{\partial y}(x)\phi(x)\,\mathrm{d}x + \int_{\mathbb{R}} f(x+iy)\phi(x)R(x,y)\,\mathrm{d}x.\end{aligned}$$

Notice that $R(x,y) \to 0$, as $y \to 0$, uniformly on every compact subset of $\mathbb{R}$. So the last two terms on the right hand side go to zero when $y \to 0$, by theorem 7.2. We have thus proven the assertion. □

Lemma 7.5. *The map $\partial_z : \mathcal{O}(H_+)_N \to \mathcal{O}(H_+)_{N+1}$, given by $f \mapsto \partial_z f$, is well defined and continuous.*

Proof. Let $z = x+iy \in [a,b] \times i]0,d]$, for $a < b$ and $d > 0$. Let D be the disc with center z and radius $R = \frac{1}{2}y$. Then $D \subset [a-2d, b+2d] \times i]0,3d]$. We thus have, by the Cauchy integral formula, that

$$\begin{aligned}|\frac{\mathrm{d}f}{\mathrm{d}z}(z)| = \frac{1}{2\pi}|\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta| \leq \frac{1}{2\pi}\int_{\partial D}(\mathrm{Im}(\zeta))^{-N}\frac{|(\mathrm{Im}(\zeta))^N f(\zeta)|}{R}\,\mathrm{d}\zeta \\ \leq R^{-N-1}\nu_{N,a-d,b+d,3d}(f) = 2^{N+1}\nu_{N,a-d,b+d,3d}(f)y^{-N-1},\end{aligned}$$

from which the claim follows. □

We notice that $\mathcal{O}(H_+)_N \subset \mathcal{O}(H_+)_{N+1}$. We define

$$\mathcal{O}_*(H_+) = \bigcup_{N\geq 0} \mathcal{O}(H_+)_N$$

and equip it with the inductive limit topology. We notice that $\beta_N(f) = \beta_{N+1}(f)$, for all $f \in \mathcal{O}(H_+)_N$. There thus exists a unique operator

$$\beta : \mathcal{O}_*(H_+) \to \mathcal{D}'(\mathbb{R}),$$

such that $\beta\big|_{\mathcal{O}(H_+)_N} = \beta_N$.

The following result is an immediate consequence of Proposition 5.10 and Theorem 7.3.

Corollary 7.6. *The operator β is continuous.*

Lemma 7.7. *For all $f \in \mathcal{O}_*(H_+)$ we have that*

$$\partial\beta(f) = \beta(\partial_x f) = \beta(\partial_z f).$$

Proof. It follows from the Cauchy-Riemann equations that $\partial_z f = \partial_x f$. We thus find that

$$\int_{\mathbb{R}} \partial_z f(x+iy)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \partial_x f(x+iy)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} f(x+iy)\partial_x\phi(x)\,\mathrm{d}x.$$

The last equality follows by integrating by parts. The result now follows by letting $y \to 0$. □

Theorem 7.8 ([9], Theorem 3.1.15)**.** *Let $f \in \mathcal{O}_*(H_+)$. If $f(\cdot + i0) = 0$, then $f = 0$. In other words β is an injective operator.*

Proof. We will follow Hörmander ([9]). Fix $y > 0$. For $\phi \in C_c^\infty(\mathbb{R})$ we define

$$F_\phi(w) := \int_{\mathbb{R}} \phi(x) f(x+wy)\,\mathrm{d}x.$$

We notice that F_ϕ is an analytic function in w on H_+. Since $f(\cdot+i0) = 0$ we conclude, form Theorem 7.7, that $F_\phi(w) \to 0$ and $\frac{d^k}{dw^k}F_\phi(w) \to 0$ as $\mathrm{Im}(w) \to 0$. We thus conclude that the function G_ϕ, defined as $G_\phi = F_\phi$ on H_+ and $G_\phi = 0$ on $H_- \cup \mathbb{R}$, is an analytic function that extends F_ϕ. On the other hand is G_ϕ the analytic continuation of the zero function on $H_- \cup \mathbb{R}$. Since an analytic continuation on a simply connected domain is unique, we conclude that $F_\phi = 0$. Since this holds for any $\phi \in C_c^\infty(I)$ we conclude that $f = 0$. □

7.3 Boundary value operator revised

The action of $\mathrm{SL}(2,\mathbb{R})$ on H_+, by fractional linear transformations, induces an action of $\mathrm{SL}(2,\mathbb{R})$ on the space $C(H_+)$, namely the action given by

$$g\cdot f = {F_{g^{-1}}}^* f = f\circ F_{g^{-1}}, \qquad \text{for all } f\in C(H_+).$$

Since the maps F_g are bi-holomorphic this action restricts to an action on the space $\mathcal{O}(H_+)$. Now $\mathrm{SL}(2,\mathbb{R})$ also acts by fractional linear transformations on $\hat{\mathbb{R}}$. So we also have an action of $\mathrm{SL}(2,\mathbb{R})$ in $\mathcal{D}(\hat{\mathbb{R}}) = \mathcal{E}(\hat{\mathbb{R}})$, given by

$$g\cdot\omega := \left(F_{g^{-1}}\big|_{\widehat{\mathbb{R}}}\right)^* \omega.$$

Using the results of the previous sections we want to define a new operator from a subspace of $\mathcal{O}_*(H_+)$, that is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$, to $\mathcal{D}'(\widehat{\mathbb{R}})$. Before we come to the definition of this operator we first are going to have a look at "that" subspace of $\mathcal{O}_*(H_+)$.

Recall that $J:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is the fractional linear transformation corresponding to the matrix

$$\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}.$$

We define the space

$$\mathcal{O}(H_+)_N^J := \{f\in\mathcal{O}(H_+) \mid \nu_{N,a,b,h}(f)<\infty \text{ and } \nu_{N,a,b,h}(J^*f)<\infty,\ \text{for all } a<b, 0<h\}$$

We notice that a function $f\in\mathcal{O}(H_+)_N$ is in $\mathcal{O}(H_+)_N^J$ if and only if $J^*f\in\mathcal{O}(H_+)_N$.

Lemma 7.9. *The space $\mathcal{O}(H_+)_N^J$ is an invariant subspace of $\mathcal{O}(H_+)$ under the action of* $\mathrm{SL}(2,\mathbb{R})$.

Proof. By Theorem 1.14 it is sufficient to prove $\mathcal{O}(H_+)_N^J$ is invariant under the transformations J, T_β, M_α, for $\beta\in\mathbb{R}$ and $\alpha>0$. It is evident that $J^*f\in\mathcal{O}(H_+)_N^J$, when $f\in\mathcal{O}(H_+)_N^J$. We notice that, for $z\in[a,b]\times i]0,d]$, there are $C, C'>0$ such that $|f(x+iy)|<Cy^{-N}$ and $|f(\frac{-1}{x+iy})|<C'y^{-N}$. For such z we also have then that

$$|M_\alpha^* f(z)| = |f(\alpha z)| \le C\alpha^{-N}y^{-N}$$

and

$$|J^*M_\alpha^*f(z)| = |f\left(\alpha\frac{-1}{z}\right)| \leq C'\alpha^N y^{-N}$$

showing that $M_\alpha^* f \in \mathcal{O}(H_+)_N^J$.

We now go to the case of T_β. It is readily verified that $T_\beta^* f \in \mathcal{O}(H_+)_N$, whenever $f \in \mathcal{O}(H_+)_N^J$. Now to prove that $J^*T_\beta^* f \in \mathcal{O}(H_+)_N$ notice that it is sufficient to prove that $|T_\beta^* f(x+iy)| < C'y^{-N}$, for a collection $\{U_i\}_{i\in I}$, with U_i of the form $[a_i, b_i] \times i\,]0, d_i]$, $a_i < b_i$ and $d_i > 0$, such that the collection $\{U_i'\}_{i\in I}$ covers $\mathbb{R}$, where $U_i' =]a_i, b_i[\times i\,]-d_i, d_i[$.
We no notice that the map $T_\beta \circ J : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is continuous and bijective. Let $t \in \mathbb{R}$, then $r = (T_\beta \circ J)(t) \in \widehat{\mathbb{C}}$. Now take $\epsilon > 0$. Now there is a $\delta > 0$ such that

$$[t-\delta, t+\delta] \times i[-\delta, \delta] \subset (T_\beta \circ J)^{-1}(]r-\epsilon, r+\epsilon[\times i\,]-\epsilon, \epsilon[).$$

We choose $\delta > 0$, such that not both 0 and $\frac{1}{b}$ are contained in $[t-\delta, t+\delta]$. We now notice that for all $z \in [r-\epsilon, r+\epsilon] \times i]0, \epsilon]$ there are $C, C' > 0$ such that $|f(x+iy)| < Cy^{-N}$ and $|f(\frac{-1}{x+iy})| < C'y^{-N}$. Then for all $z \in [t-\delta, t+\delta] \times\, i\,]0, \delta]$ we have that

$$|f(\frac{-1}{z} + b)| = |f(\frac{bz-1}{z})| \leq C|z|^{2N} y^{-N}$$

and

$$|f(\frac{-1}{z} + b)| = |f\left(-1/\frac{-z}{bz-1}\right)| \leq C'|bz-1|^{2N} y^{-N}$$

The result thus follows. □

Lemma 7.10. *If for $f \in \mathcal{O}(H_+)_N$ there exists a $C > 0$ such that $|f(z)| \leq C(\operatorname{Im} z)^{-N}$, for all $z \in H_+$. Then $f \in \mathcal{O}(H_+)_N^J$.*

Proof. We just notice that $|f(\frac{-1}{z})| \leq C(\operatorname{Im}\left(\frac{-1}{z}\right))^{-N} = C|z|^{2N}(\operatorname{Im} z)^{-N}$. From which the result now easily follows. □

We define the semi-norms

$$\nu_{N,a,b,h}^J(f) := \nu_{N,a,b,h}(J^*f).$$

We equip $\mathcal{O}(H_+)^J$ with the topology induced by all semi-norms $\nu^J_{N,a,b,h}$ and $\nu_{N,a,b,h}$, for $a < b$ and $h > 0$. We then define

$$\mathcal{O}_*(H_+)^J := \bigcup_{N \in \mathbb{Z}_{\geq 0}} \mathcal{O}(H_+)^J_N,$$

and equip it with the inductive limit topology.

We now hope that we can define am operator $\beta : \mathcal{O}_*(H_+)^J \to \mathcal{D}'(\widehat{\mathbb{R}})$. To define an operator $\beta : \mathcal{O}_*(H_+)^J \to \mathcal{D}'(\widehat{\mathbb{R}})$ it is sufficient, given some open cover $\{U_i\}_{i \in I}$ of $\widehat{\mathbb{R}}$, to define operators $\beta_i : \mathcal{O}_*(H_+)^J \to \mathcal{D}'(U_i)$, such that, for all $f \in \mathcal{O}_*(H_+)^J$, we have that $\beta_i(f)\big|_{U_i \cap U_j} = \beta_j(f)\big|_{U_i \cap U_j}$, when $U_i \cap U_j \neq \emptyset$. Consider the charts (U_1, κ_1) and (U_2, κ_2), where $U_1 = \mathbb{R} \subset \hat{\mathbb{R}}$, $U_2 = \widehat{\mathbb{R}} \backslash \{0\}$, and $\kappa_1 : U_1 \to \mathbb{R}$ is given by the identity, and $\kappa_2 : U_2 \to \mathbb{R}$ is given by $\kappa_1 \circ J\big|_{\hat{\mathbb{R}}}$. We notice that $\mathscr{U} = \{U_1, U_2\}$ is an open cover of $\widehat{\mathbb{R}}$. We define

$$\begin{aligned}
\beta_1(f)(\phi) &:= \lim_{y \downarrow 0} \int_{\mathbb{R}} f(x+iy)(\kappa_1^{-1})^* \phi(x), \qquad && \phi \in \mathcal{D}(U_1) \\
\beta_2(f)(\phi) &:= \lim_{y \downarrow 0} \int_{\mathbb{R}} J^* f(x+iy)(\kappa_2^{-1})^* \psi(x) \qquad && \psi \in \mathcal{D}(U_2).
\end{aligned}$$

A priori it is not clear that $\beta_1(f)\big|_{U_1 \cap U_2} = \beta_2(f)\big|_{U_1 \cap U_2}$. In the next section we will develop the tools to show that this indeed holds. It then immediately follows from Corollary 7.6 that β is a continuous operator.

7.3.1 Proof that the boundary value operator is well defined

The method that is going to be presented was suggested to me by Erik van den Ban. We start by placing ourselves in a more flexible situation. The idea is based on the following observation. We notice that

$$\lim_{y \downarrow 0} \int_{\mathbb{R}} f(x+iy)\phi(x)\,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \eta_\epsilon^* f(t)\phi(t)\,dt$$

for $\phi \in C_c^\infty(\mathbb{R})$ and $f \in \mathcal{O}_*(H_+)$, were $\eta_\epsilon(t) := t + i\epsilon$. We are first going to show that the above equality still holds when we replace η_ϵ by a series of C^1-curves $\{\tilde{\eta}_j\}_{j \in \mathbb{N}}$ converging to the map $t \mapsto t$, with respect to the C^1 semi-norms on $C^\infty(\mathbb{R}, \mathbb{C})$. We shall first state more precisely the conditions on the curves $\{\tilde{\eta}\}_j$.

Assume that Ω is an open subset of H_+ such that $\overline{\Omega} \cap \mathbb{R}$ is an interval with non-empty interior. We let $\Omega_{\mathbb{R}} := \mathrm{int}(\bar{\Omega} \cap \mathbb{R})$. Let $(\eta_j)_{j \in \mathbb{N}}$ be a family C^1 curves, $\Omega_{\mathbb{R}} \to \overline{\Omega}$ satisfying

1. $\eta_j \to id_{\Omega_{1,\mathbb{R}}}$ with respect to the C^1 semi-norms on $C^1(\Omega_{\mathbb{R}}, \mathbb{C})$.
2. $\eta_j(\Omega_{\mathbb{R}}) \subset \Omega$, for all $j \in \mathbb{N}$.

In the proof of theorem 7.2 we defined for $\phi \in C_c^\infty(\Omega_{\mathbb{R}})$ an extension $\tilde{\phi}_N$. Now for a density $\phi \, \mathrm{d}x \in \mathcal{D}_c TI$ we define an extension to $\Omega_{\mathbb{R}} \times i\mathbb{R}$ as $\tilde{\phi}_N \, \mathrm{d}z$.

Lemma 7.11. *Let $(\eta_j)_{j \in \mathbb{R}}$ be a family of curves as above and $\phi \in C_c^\infty(\Omega_{\mathbb{R}})$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that for all $a, b \in \Omega_{\mathbb{R}}$, such that $a < b$, there is a $h > 0$ and a $C > 0$ such that $[a, b] \times i]0, h] \subset \Omega$ and*

$$|f(x + iy)| \leq Cy^N \qquad (\textit{for all } a \leq x \leq b, \textit{ and } 0 < y \leq h).$$

Then we have that

$$\lim_{j \to \infty} \int_{\Omega_{\mathbb{R}}} \eta_j^*(f \tilde{\phi} \, \mathrm{d}z) \qquad \textit{and} \qquad \lim_{y \downarrow 0} \int_{\Omega_{\mathbb{R}}} f(x + iy)\phi(x) \, \mathrm{d}x$$

exist and are equal.

Proof. We shall first prove the assertion under the additional assumption that $\mathrm{Re}(\eta_j(t)) = t$, for all $j \in \mathbb{N}$. Let $\phi \in C_c^\infty(\Omega_{\mathbb{R}})$ and assume $\mathrm{supp}(\phi) \subset [a, b]$, for $a, b \in \Omega_{\mathbb{R}}$, such that $a < b$. We notice that there is a $\delta > 0$ such that $K := [a - \delta, b + \delta] \subset \Omega_{\mathbb{R}}$. Now since K is compact, there is a $Y > 0$ such that the set

$$Z = \{z \in H_+ \mid a - \delta \leq \mathrm{Re}(z) \leq b - \delta \text{ and } 0 < \mathrm{Im}(z) < Y\}$$

is contained in Ω and such that there is a $C > 0$ such that (7.2) holds for all z in this set. From theorem 7.2 we get that $\lim_{y \downarrow 0} \int_{\Omega_{\mathbb{R}}} f(x + iy)\phi(x) \, \mathrm{d}x$ exists. Since $\eta_j \to \mathrm{id}_{\Omega_{\mathbb{R}}}$ in the C^1 norm on compact sets there is a $M \in \mathbb{N}$ such that for all $j \geq M$ we have that $\mathrm{Im}(\eta_j)(s) \leq Y$, for $s \in K$. Now, for $j \geq M$, we let $Z_j := \{z \in Z \mid \mathrm{Im}(z) > \mathrm{Im}(\eta_j(\mathrm{Re}(z)))\}$. Now applying lemma 7.1 we find that

$$\int_K \eta_j^*(f \tilde{\phi} \, \mathrm{d}z)$$

$$= \int_K f(u + iY)\tilde{\phi}_N(u, Y) \, \mathrm{d}u + \int \int_{Z_j} f(u + iv)\phi^{(N+1)}(u) \frac{(iv)^N}{N!} \, \mathrm{d}v \, \mathrm{d}u.$$

One readily confirms that

$$\lim_{j\to\infty}\int\int_{Z_j} f(u+iv)\phi^{(N+1)}(u)\frac{(iv)^N}{N!}\,\mathrm{d}v\,\mathrm{d}u = \int\int_{Z} f(u+iv)\phi^{(N+1)}(u)\frac{(iv)^N}{N!}\,\mathrm{d}v\,\mathrm{d}u$$

The claim thus follows from equation (7.5).
We shall now argue that given a $\phi \in C_c^\infty(\Omega_{\mathbb{R}})$, we can, without loss of generality, assume that the sequence $(\eta_j)_{j\in\mathbb{N}}$, restricted to suitable neighbourhood of the support of ϕ, consists of functions $\eta_j(t) = t + g(t)$, for some $g \in C^1(\Omega_{\mathbb{R}}, \mathbb{R}_{>0})$.
Let $\phi \in C_c^\infty(\Omega_{\mathbb{R}})$ and assume $\mathrm{supp}(\phi) \subset [a, b]$, for $a \leq b \in \Omega_{\mathbb{R}}$. We notice that there is a $\delta > 0$ such that $K := [a-\delta, b+\delta] \subset \Omega_{\mathbb{R}}$. Since $\eta_j \to id_{\Omega_{1,\mathbb{R}}}$ with respect to the C^1-norm on $C^1(\Omega_{\mathbb{R}}, \overline{\Omega})$, for large enough $j \in \mathbb{N}$ we have that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}(\eta_j) > 0$ and $\mathrm{int}(K) \subset \mathrm{Re}(\eta_j(\Omega_{\mathbb{R}}))$. It thus follows that $\mathrm{Re}(\eta_j)$ has a C^1 inverse on $\mathrm{int}(K)$. Replacing $(\eta_j|_{\mathrm{int}(K)})_{j\in\mathbb{N}}$ by a subsequence, we can thus assume without loss of generality that $\mathrm{Re}(\eta_j(t)) = t$, for all $t \in \mathrm{int}(K)$. Indeed we see that this sequence still converges to id, since $|\mathrm{Im}(\eta(s))| \leq |\eta(s) - \mathrm{id}| \leq \|\eta - \mathrm{id}\|_K$, for all $s \in \mathrm{int}(K)$. Let Z be as above and let $I = [a - \frac{1}{2}\delta,\, b - \frac{1}{2}\delta]$. We further let

$$Z' = \{z \in H_+ \mid \mathrm{Re}(z) \in I \text{ and } 0 < \mathrm{Im}(z) < Y\}.$$

The general result now follows by replacing Z with Z' in the definition of Z_j. $\square$

Lemma 7.12. *Let $X, \tilde{X} \subset \mathbb{R}^d$ and $Y, \tilde{Y} \subset \mathbb{R}^m$ be open. Let $\Phi : \tilde{X} \to X$ and $\Psi : Y \to \tilde{Y}$ be C^1 diffeomorphisms. Let $f_k : X \to Y$ be a sequence of C^1 functions converging to $f : X \to Y$ in the C^1 norm on every compact subset of X. Then the sequence $g_n := \Psi \circ f_n \circ \Phi$ converges to $\Psi \circ f \circ \Phi$ on compact subsets of $\tilde{X}$ with respect to the C^1 norms.*

Proof. We first prove the claim under the additional assumption that $\Psi = \mathrm{id}_Y$. Let $K \subset \tilde{X}$ be compact. Then is $\Phi(K)$ also a compact subset of X. Now

$$\|f \circ \Phi(x) - f_k \circ \Phi(x)\| \leq \|f - f_n\|_{\Phi(K)} \qquad (x \in K)$$

From the chain rule it follows that

$$\|\partial_{\tilde{x}^j}(f \circ \Phi)(\tilde{x}) - \partial_{\tilde{x}^j}(f_k \circ \Phi)(\tilde{x})\| \leq \sum \|\partial_{x^i} f(\Phi(\tilde{x})) - \partial_{x^i} f_k(\Phi(\tilde{x}))\| \|\partial_{\tilde{x}^j}\Phi(\tilde{x})\|.$$

We thus conclude that $f_k \circ \Phi \to f \circ \Phi$ on K with respect to the C^1-norm.
We now prove the claim under the assumption that $\Phi = \mathrm{id}$. Let $C \subset X$ be compact. Since the f is continuous $f(C)$ is compact. Then there is an $\epsilon' > 0$ such that $C' = \overline{f(C)_{\epsilon'}} \subset Y$ is compact, where $f(C)_{\epsilon'} = \{x \in \mathbb{R}^m \mid \inf_{x' \in f(C)} \|x - x'\| < \epsilon'\}$. Since Ψ is continuous it is

uniformly continuous on C'. So given an $\epsilon > 0$ there is a $\delta > 0$ such that if $y, y' \in \Psi(K)$, and $\|y - y'\| < \delta$, then $\|\Psi(y) - \Psi(y')\| < \epsilon$. Now since $f_n \to f$ uniformly on C, $\Psi \circ f_n$ goes to $\Psi \circ f$ on C with respect to the C^0-norm on C. Now notice that

$$\begin{aligned}
\|\partial_{x^j}(\Psi \circ f)(x) - \partial_{x^j}(\Psi \circ f)(x)\| &= \|\sum \partial_{y^i}\Psi(f(x))\partial_{x^j} f(x) - \sum \partial_{y^i}\Psi(f_k(x))\partial_{x^j} f_k(x)\| \\
&\leq \sum \|\partial_{y^i}\Psi(f(x))\partial_{x^j} f(x) - \partial_{y^i}\Psi(f_k(x))\partial_{x^j} f_k(x)\| \\
\leq \sum \|\partial_{y^i}\Psi(f(x))\partial_{x^j} f(x) - \partial_{y^i}\Psi(f(x))\partial_{x^j} f_k(x)\| &+ \|\partial_{y^i}\Psi(f_n(x))\partial_{x^j} f_k(x) - \partial_{y^i}\Psi(f(x))\partial_{x^j} f_k(x)\| \\
\leq \sum \|\partial_{y^i}\Psi(f(x))\|\|\partial_{x^j} f(x) - \partial_{x^j} f_k(x)\| &+ \|\partial_{y^i}\Psi(f_n(x)) - \partial_{y^i}\Psi(f(x))\|\|\partial_{x^j} f_k(x)\|.
\end{aligned}$$

Now, since C and C' are compact there is an $A > 0$ such that $\|\partial_{y^i}\Psi(f(x))\|, \|\partial_{x^j} f_k(x)\| \leq A$, for all $1 \leq i \leq m$, and $1 \leq j \leq n$. Invoking the uniform continuity of Ψ on C' and the uniform convergence of f_k to f on C, with respect to the C^1-norm on C, we conclude that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that

$$\|\partial_{x^j}(\Psi \circ f)(x) - \partial_{x^j}(\Psi \circ f)(x)\| < \epsilon.$$

We thus conclude that $\Psi \circ f_k \to \Psi \circ f$ uniformly on C with respect to the C^1-norm. $\square$

In the remaining of this section we will assume that Ω_1 and Ω_2 are open subset of H_+ such that $\overline{\Omega}_i \cap \mathbb{R}$ is a nonempty connected set with non-empty interior, for $i = 1, 2$. We let $\Omega_{i,\mathbb{R}} := \mathrm{int}(\bar{\Omega}_i \cap \mathbb{R})$. We further assume that $\Phi : \Omega_1 \to \Omega_2$ is a biholomorphic map that can be extended to $\Omega_1 \cup \Omega_{1,\mathbb{R}}$ in such a way that $\Phi|_{\Omega_{1,\mathbb{R}}}$ is a diffeomorphism from $\Omega_{1,\mathbb{R}}$ to $\Omega_{2,\mathbb{R}}$. Let $(\eta_j)_{j\in\mathbb{N}}$ be a sequence of curves as described in the beginning of this section. Then it follows from the above lemma that $\Phi^{-1} \circ \eta_j \circ \Phi|_{\Omega_{1,\mathbb{R}}}$ is again a sequence of curves satisfying the conditions. The following result thus makes sense.

Lemma 7.13. *Let $(\eta_j)_{j\in\mathbb{N}}$ be a sequence of C^1 curves as above, $f : \Omega_2 \to \mathbb{C}$ a function as in Theorem 7.11, with the additional assumption that $\Phi^* f$ satisfies the same conditions, and $\phi \in C_c^\infty(\Omega_{1,\mathbb{R}})$. Then*

$$\lim_{j\to\infty} \int_{\Omega_{2,\mathbb{R}}} \eta_j^*(f(\widetilde{\Phi_{\mathbb{R}}^{-1})^*(\phi}\, \mathrm{d}x)) = \lim_{j\to\infty} \int_{\Omega_{1,\mathbb{R}}} (\Phi^{-1} \circ \eta_j \circ \Phi|_{\mathbb{R}})^*(\Phi^*(f)\widetilde{\phi\, \mathrm{d}x}).$$

Corollary 7.14. *In the setting of Lemma 7.13*

$$\lim_{y\downarrow 0} \int_{\Omega_{2,\mathbb{R}}} f(x + iy)\phi(x) = \lim_{y\downarrow 0} \int_{\Omega_{1,\mathbb{R}}} \Phi^* f(x + iy) \left(\Phi\big|_{\mathbb{R}}\right)^* \phi(x).$$

Proof. This follows from combining Lemma 7.13 and Lemma 7.11. $\square$

It follows form corollary 7.14 that $\beta_1(f)(\phi) = \beta_2(f)(\phi)$, whenever $\phi \in \mathcal{D}(U_1 \cap U_2)$. It thus follows that there exists a distribution $\beta(f)$ on $\widehat{\mathbb{R}}$. Hence there is a unique operator $\beta : \mathcal{O}_*(H_+)^J \to \mathcal{D}'(\widehat{\mathbb{R}})$, such that $\beta\big|_{U_1} = \beta_1$ and $\beta\big|_{U_2} = \beta_2$.
The goal of the remainder of this section is to develop the tools to prove the above result.

Lemma 7.15. *Let $\phi \in C^\infty(\Omega_\mathbb{R})$ and $\psi \in C^\infty(\Omega_\mathbb{R} + i\mathbb{R})$ s.t. $\psi|_\mathbb{R} = \phi$ and $\partial_{\bar{z}}\psi(z) = \mathcal{O}(|y|^N)$, then*

$$\psi(z) = \sum_{k=0}^{N} \phi^{(k)}(x)\frac{(iy)^k}{k!} + \rho(x,y)y^{N+1}$$

with $\rho \in C^\infty(\mathbb{R}^2)$ such that $\rho(x,y) \to 0$ as $y \to 0$.

Proof. By Taylor's theorem we have that

$$\psi(z) = \sum_{k=0}^{N} c_k(x)\frac{(iy)^k}{k!} + \rho(x,y)y^{N+1}$$

with $c_k \in C^\infty(\mathbb{R})$, and $\rho \in C^\infty(\mathbb{R}^2)$ such that $\rho(x,y) \to 0$ as $y \to 0$. Since $\psi|_{\Omega_\mathbb{R}} = \phi$ we have that $c_0(x) = \phi(x)$. Now notice that

$$\begin{aligned}
\partial_{\bar{z}}\psi(z) &= \sum_{k=0}^{N} c_k'(x)\frac{(iy)^k}{k!} - \sum_{k=1}^{N} c_k(x)\frac{(iy)^{k-1}}{(k-1)!} + \partial_{\bar{z}}\left(\rho(x,y)y^{N+1}\right) \\
&= \sum_{k=0}^{N-1}[c_k'(x) - c_{k+1}(x)]\frac{(iy)^k}{k!} + c_N(x)\frac{(iy)^N}{N!} + \partial_{\bar{z}}\left(\rho(x,y)y^{N+1}\right)
\end{aligned}$$

Now since $\partial_{\bar{z}}\psi(z) = \mathcal{O}(|y|^N)$ we have that

$$\sum_{k=0}^{N-1}[c_k'(x) - c_{k+1}(x)]\frac{(iy)^k}{k!} = 0$$

So we conclude that $c_{k+1} = c_k'$. Since $c_0(x) = \phi(x)$, it follows that $c_k = \phi^{(k)}$. This concludes the proof. $\square$

Lemma 7.16. *Let $U, V \subset \mathbb{C}$ be open and let $T : U \to V$ be a biholomorphic map. Then*

$$\frac{\partial T}{\partial z}\frac{\partial \overline{T}}{\partial \bar{z}} = |\det DT(z)|,$$

where $DT(z)$ is the derivative of the associated vector field of T.

Lemma 7.17. *Let $U, V \subset \mathbb{C}$ be open and let $T : U \to V$ be a biholomorphic map and $\psi \in C^1(V, \mathbb{C})$. Then*

$$\frac{\partial \psi \circ T}{\partial \bar{w}}(w) = \frac{\partial \psi}{\partial \bar{z}}(T(w))\frac{\partial \overline{T}}{\partial \bar{w}}(w).$$

Lemma 7.18. *Let $U, V \subset \mathbb{C}$ be open and let $T : U \to V$ be a biholomorphic map. Then*

$$\bar{\partial} T^*(\psi \, \mathrm{d}z) = T^*(\bar{\partial}(\psi \, \mathrm{d}z))$$

Proof. First notice that $T^*(\psi \, \mathrm{d}z) = T^*(\psi) T' \, \mathrm{d}w$. Now

$$\begin{aligned} \bar{\partial} T^*(\psi \, \mathrm{d}z)(w) &= \frac{\partial T^*(\psi)}{\partial \bar{w}}(w) T'(w) \, \mathrm{d}\bar{w} \wedge \, \mathrm{d}w \\ = \frac{\partial \psi}{\partial \bar{z}}(T(w))\frac{\partial \bar{T}}{\partial \bar{w}}(w) T'(w) \, \mathrm{d}\bar{w} \wedge \, \mathrm{d}w &= \frac{\partial \psi}{\partial \bar{z}}(T(w)) |\det DT(w)| \, \mathrm{d}\bar{w} \wedge \, \mathrm{d}w, \end{aligned}$$

where we invoked lemma 7.17 to obtain the third equality and lemma 7.16 to obtain the fourth. Now notice that

$$\begin{aligned} T^*(\bar{\partial}(\psi \, \mathrm{d}z)) &= T^*(\partial_{\bar{z}} \psi \, \mathrm{d}\bar{z} \wedge \, \mathrm{d}z) = T^*(\partial_{\bar{z}} \psi \, 2i \, \mathrm{d}x \wedge \, \mathrm{d}y) \\ &= T^*(\partial_{\bar{z}} \psi) |\det DT| 2i \, \mathrm{d}u \wedge \, \mathrm{d}v = T^*(\partial_{\bar{z}} \psi) |\det DT| \, \mathrm{d}\bar{w} \wedge \, \mathrm{d}w. \end{aligned}$$

We have thus proven the lemma. □

Claim 1. *Let $\phi \in C^\infty(\Omega_{\mathbb{R}})$ and $\psi \in C^\infty(\Omega_{\mathbb{R}} + i\mathbb{R})$ as in lemma 7.15. Let $T : \Omega_1 \to \Omega_2$ be a biholomorphic map, then*

$$\partial_{\bar{w}} T^*(\psi \, \mathrm{d}z) = \mathcal{O}(|\mathrm{Im}(w)|^N) \, \mathrm{d}\bar{w} \wedge \, \mathrm{d}w.$$

Proof. Notice that we have $T(x + iy) = T(x) + \mathcal{O}(y)$, by Taylor's theorem. Since $T|_{\Omega_{1,\mathbb{R}}}$ was assumed to be a diffeomorphism from $\Omega_{1,\mathbb{R}}$ to $\Omega_{2,\mathbb{R}}$, we have that $\mathrm{Im}(T(x + iy)) = \mathcal{O}(y)$. By assumption we have that

$$\partial_{\bar{z}} \psi(z) = \mathcal{O}(\mathrm{Im}(z)^N).$$

So then

$$\partial_{\bar{z}}\psi(T(w)) = \mathcal{O}(\mathrm{Im}(T(w))^N) = \mathcal{O}(\mathcal{O}(\mathrm{Im}(w))^N).$$

Now notice that $\det DT(w) \geq 0$ and is defined on $\Omega_{\mathbb{R}}$. The claim thus follows from lemma 7.18. $\square$

Claim 2. *Let $\phi \in C^\infty(\Omega_{\mathbb{R}})$. Then*

$$T^*(\widetilde{\phi\,\mathrm{d}x}) - \widetilde{T|_{\mathbb{R}}^*(\phi\,\mathrm{d}x)} = \mathcal{O}(y^{N+1})\,\mathrm{d}w.$$

Proof. Notice that

$$T^*(\tilde{\phi}\,\mathrm{d}z)|_{\Omega_{1,\mathbb{R}}} = \tilde{\phi} \circ T|_{\Omega_{1,\mathbb{R}}} T'|_{\Omega_{1,\mathbb{R}}}\,\mathrm{d}u = \phi \circ T|_{\Omega_{1,\mathbb{R}}} T'|_{\Omega_{1,\mathbb{R}}}\,\mathrm{d}u$$

and

$$\widetilde{T|_{\mathbb{R}}^*(\phi\,\mathrm{d}x)}|_{\Omega_{1,\mathbb{R}}} = \phi \circ \widetilde{T|_{\Omega_{1,\mathbb{R}}}}|_{\Omega_{1,\mathbb{R}}} \widetilde{T'|_{\Omega_{1,\mathbb{R}}}\,\mathrm{d}u}|_{\Omega_{1,\mathbb{R}}} = \phi \circ T|_{\Omega_{1,\mathbb{R}}} T'|_{\Omega_{1,\mathbb{R}}}\,\mathrm{d}u.$$

It thus follows that

$$T^*(\widetilde{\phi\,\mathrm{d}x})|_{\Omega_{1,\mathbb{R}}} = \widetilde{T|_{\mathbb{R}}^*(\phi\,\mathrm{d}x)}|_{\Omega_{1,\mathbb{R}}}$$

From claim 1 follows that $\partial_{\bar{w}} T^*(\tilde{\phi}\,\mathrm{d}z)(w) = \mathcal{O}(|\mathrm{Im}(w)|^N)\,\mathrm{d}\bar{w} \wedge \mathrm{d}w$ and it follows from the definition of $\widetilde{T|_{\mathbb{R}}^*(\phi\,\mathrm{d}x)}$ that $\partial_{\bar{w}} \widetilde{T|_{\mathbb{R}}^*(\phi\,\mathrm{d}x)}(w) = \mathcal{O}(|\mathrm{Im}(w)|^N)\,\mathrm{d}\bar{w} \wedge \mathrm{d}w$. So from lemma 7.15 the claim follows. $\square$

We now have all the tools that are necessary to prove lemma 7.13.

Proof of lemma 7.13. Notice that by claim 2 we have

$$\int_{\mathbb{R}} \eta_j^*(f(\widetilde{\Phi_{\mathbb{R}}^{-1})^*(\phi\,\mathrm{d}x})) = \int_{\mathbb{R}} \eta_j^*(f(\Phi^{-1})^*(\widetilde{\phi\,\mathrm{d}x})) + \int_{\mathbb{R}} \eta_j^*(f\mathcal{O}(y^{N+1})\,\mathrm{d}w).$$

Now notice that

$$\begin{aligned}\int_{\mathbb{R}} \eta_j^*(f(\Phi^{-1})^*(\widetilde{\phi\,\mathrm{d}x})) = \int_{\mathbb{R}} \eta_j^*(\Phi^{-1})^*(\Phi^*(f)\widetilde{\phi\,\mathrm{d}x}) &= \int_{\mathbb{R}} (\Phi|_{\mathbb{R}})^* \eta_j^*(\Phi^{-1})^*(\Phi^*(f)\widetilde{\phi\,\mathrm{d}x}) \\ &= \int_{\mathbb{R}} (\Phi^{-1} \circ \eta_j \circ \Phi|_{\mathbb{R}})^*(\Phi^*(f)\widetilde{\phi\,\mathrm{d}x}),\end{aligned}$$

where the third equality holds by the change of variables formula. One readily confirms that

$$\lim_{j\to\infty} \int_{\mathbb{R}} \eta_j^*(f\mathcal{O}(y^{N+1})\,\mathrm{d}w) = 0.$$

The result thus follows. $\square$

7.3.2 Boundary value operator and $\mathrm{SL}(2,\mathbb{R})$

Theorem 7.19. *The operator β intertwines the actions of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{D}'(\widehat{\mathbb{R}})$ and $\mathcal{O}_*(H_+)^J$, i.e.*

$$g \cdot \beta(f) = \beta(g \cdot f),$$

for all $f \in \mathcal{O}_*(H_+)^J$ and $g \in \mathrm{SL}(2,\mathbb{R})$.

Proof. By Theorem 1.14 it is sufficient to show that $\beta(F_{g^{-1}}{}^* f)(\phi) = \beta(f)(F_g{}^*\phi)$, for $F_g = J, M_a, T_b$, $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$.
By definition we have that

$$\beta_1(J^* f)(\phi) = \beta_2(f)(\left(J\big|_{\widehat{\mathbb{R}}}\right)^* \phi), \qquad \text{for all } \phi \in \mathcal{D}(U_1),$$

and

$$\beta_2(J^* f)(\phi) = \beta_1(f)(\left(J\big|_{\widehat{\mathbb{R}}}\right)^* \phi), \qquad \text{for all } \phi \in \mathcal{D}(U_2).$$

Hence $\beta(J^* f)(\phi) = \beta(f)(\left(J\big|_{\widehat{\mathbb{R}}}\right)^* \phi)$.
In the case that $F_g = M_\alpha$ we can apply Corollary 7.14 with $\Phi = F_{g^{-1}}$ and $\Phi = J \circ F_{g^{-1}}$, to conclude that

$$\beta_1((F_{g^{-1}})^* f)(\phi) = \beta_1(f)(\left(F_g\big|_{\widehat{\mathbb{R}}}\right)^* \phi), \qquad \text{for all } \phi \in \mathcal{D}(U_1),$$

and

$$\beta_2((F_{g^{-1}})^* f)(\psi) = \beta_2(f)(\left(F_g\big|_{\widehat{\mathbb{R}}}\right)^* \psi), \qquad \text{for all } \psi \in \mathcal{D}(U_2).$$

Hence $\beta((F_{g^{-1}})^* f)(\phi) = \beta(f)(\left(F_g\big|_{\widehat{\mathbb{R}}}\right)^* \phi)$, for all $\phi \in \mathcal{D}(\widehat{\mathbb{R}})$.
Finally we consider the case that $F_g = T_b$. We consider the cases that $b > 0$ and $b < 0$. First assume that $b > 0$. Since $\widehat{\mathbb{R}} \supset U \to \mathcal{D}'(U)$ is a sheaf it is sufficient to prove that $\beta(T_{-b}^* f)\big|_{V_j}(\phi) = \beta(f)\big|_{V_j}(T_b^*\phi)$, for all $\phi \in \mathcal{D}(V_j)$, for some open cover $\{V_j\}$ of $\widehat{\mathbb{R}}$. We consider the cover consisting of $V_1 = U_1$ and $V_2 = (\kappa_2^{-1})^*]-\frac{1}{b}, \infty[$. It follows from the change of variables that

$$\beta_1((T_{-b})^* f)(\phi) = \beta_1(f)((T_b)^*\phi), \qquad \text{for all} \phi \in \mathcal{D}(U_1).$$

Now notice that

$$\beta(f)\big|_{V_2}(\phi) = \beta_2(f)\big|_{V_2}(\phi) = \lim_{y\downarrow 0}\int_{-\frac{1}{b}}^{\infty} J^* f(x+iy)(\kappa_2^{-1})^*\phi(x)$$

Also notice that $(J \circ T_{-b} \circ J)\big|_{\mathbb{R}}(]-\frac{1}{b},\infty[) =]-\frac{1}{b},\infty[$. So applying Corollary 7.14 with $\Phi = J \circ T_{-b} \circ J$ we find

$$\lim_{y\downarrow 0}\int_{-\frac{1}{b}}^{\infty} J^*(T_{-b})^* f(x+iy)(\kappa_2^{-1})^*\phi(x) = \lim_{y\downarrow 0}\int_{-\frac{1}{b}}^{\infty} J^* f(x+iy)(\Phi\big|_{\mathbb{R}})^*(\kappa_2^{-1})^*\phi(x).$$

Now notice that $(\Phi\big|_{\mathbb{R}})^*(\kappa_2^{-1})^* = (\kappa_2^{-1})^*(T_b)^*$. Hence we conclude that

$$\beta((T_{-b})^* f)\big|_{V_2}(\phi) = \beta(f)\big|_{V_2}((T_b)^*\phi).$$

The argumentation for the case that $b < 0$ is similar as the above. Now instead of taking $V_2 = (\kappa_2^{-1})^*]-\frac{1}{b},\infty[$ we take $V_2 = (\kappa_2^{-1})^*]-\infty,-\frac{1}{b}[$.

$\square$

Chapter 8

Representations

As mentioned earlier, we shall be interested in two particular representations of $\mathrm{SL}(2,\mathbb{R})$, namely holomorphic discrete series representations and the principal series representations. In this chapter we develop some theory along the lines of Chapter 20 of [1].

Definition 8.1. Let V be a locally convex vector space. With a continuous representation (π, V) of G we shall mean a continuous left action $\pi : G \times V \to V$ such that the map $\pi(x) : v \mapsto \pi(x, v)$ is a linear automorphism of V, for all $x \in G$. We say that the representation is finite dimensional if $\dim V < \infty$.

From now on when we talk about a representation, we will always assume that it is continuous.

Example 8.2. Let M be a manifold and G an Lie group that acts smoothly on M. We consider the space $C(M)$ of continuous complex valued functions. We equip $C(M)$ with the locally convex topology induced by the family of semi-norms

$$\|f\|_K := \sup_{x\in K} |f(x)|$$

with $K \subset M$ compact. Now G has a natural representation L in $C(M)$, known as the left regular representation, and is given by

$$[L(g)f](x) = f(g^{-1}x), \qquad \text{for all } g \in G \text{ and } f \in C(M).$$

To see that L is continuous notice that $\|L(g)f\|_K = \|f\|_{g^{-1}K}$, for all $g \in G$, $f \in C(M)$ and $K \subset M$ compact. So it is sufficient to show that L is continuous in (e, f_0), for any $f_0 \in C(M)$. Now notice that

$$\|L(g)f - f_0\|_K \leq \|L(g)f - L(g)f_0\|_K + \|L(g)f_0 - f_0\|_K \leq \|f - f_0\|_{g^{-1}K} + \|L(g)f_0 - f_0\|_K.$$

Since f_0 is uniformly continuous on K it follows that L is continuous.

Let (π, V) be a representation and W a linear subspace of V. We say that W is invariant if $\pi(x)W \subset W$, for all $x \in G$. The representation (π, V) is called irreducible, when $\dim V > 0$ and the only invariant subspaces of V are $\{0\}$ and V itself.

Definition 8.3. By an unitary representation we shall mean a representation (π, V), where V is a complex Hilbert space, such that $\pi(x)$ is an unitary operator, for all $x \in G$.

Definition 8.4. Let (π, V) and (ρ, W) be two representations of the group G. A continuous linear map $T : V \to W$ is said to be equivariant if $T \circ \pi(x) = \rho(x) \circ T$, for all $x \in G$. In other words, we have that the following diagram commutes, for every $x \in G$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi(x)"] & W \arrow[d, "\rho(x)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

With this definition we are able to state the following result.

Theorem 8.5 (Schur's lemma)**.** *Let (π, V) be a finite dimensional irreducible representation of the group G. Then* $\mathrm{End}_G(V) = \mathbb{C}I_V$.

The proof of Schur's lemma makes use of the following lemma and can be found in Chapter 20 of [1].

Lemma 8.6. *Let V be a linear space and $A, B : V \to V$ linear maps. If $A \circ B = B \circ A$ then are* $\ker A$*, $im\, A$ and the eigenspaces of A are B-invariant subspaces, i.e.* $B(\ker A) \subset \ker A$ *and $B(im\, A) \subset im\, A$.*

Corollary 8.7. *If G is a commutative Lie group and (π, V) is a finite dimensional representation of G, then* $\dim V = 1$.

As a application we can classify the finite dimensional irreducible representations of SO(2). This classification will be useful when introducing the holomorphic discrete series representations.
We first notice that SO(2) is a commutative group. From this it follows that all finite dimensional irreducible representations of SO(2) are one dimensional.
We now notice that SO(2) is compact, so for any continuous homomorphism $\varphi : \mathrm{SO}(2) \to \mathbb{C}^*$ we have that $\varphi(\mathrm{SO}(2)) \subset \mathbb{C}^*$ is a compact subgroup, so then $\varphi(\mathrm{SO}(2)) \subset S^1$. We now recall that SO(2) is isomorphic to S^1, so with the following lemma we have completed our classification:

Lemma 8.8. *Every continuous homomorphism $\varphi : S^1 \to S^1$ is of the form $z \mapsto z^n$, for some $n \in \mathbb{Z}$.*

Proof. By lemma ... φ is a Lie group homomorphism, so $\varphi_* = T_1\varphi$ is a Lie algebra homomorphism. We notice that $\mathrm{Lie}(S^1) = i\mathbb{R}$, since $S^1 \subset \mathbb{C}$. Since $\mathrm{End}(i\mathbb{R}) = \mathbb{R}$, we conclude that $\varphi_*(X) = cX$, for some $c \in \mathbb{R}$ and all $X \in i\mathbb{R}$. Now $\exp : i\mathbb{R} \to S^1$ is the usual exponential $it \mapsto e^{it}$. By theorem ... we have that $\varphi(\exp(X)) = \exp(\varphi_*(X)) = \exp(cX)$. Since $e^{i2\pi} = 1$, it follows that $\exp(\varphi_*(i2\pi)) = 1$. So $\exp(i2\pi c) = 1$, and hence $c \in \mathbb{Z}$. $\square$

Chapter 9

Iwasawa decomposition

In this chapter we shall consider the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$. The Iwasawa decomposition says that $\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to $N \times A \times \mathrm{SO}(2)$, via the map $(n, a, k) \mapsto nak$, where N is the subgroup of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$, with all diagonal elements equal to one, and A the subgroup with the off-diagonal elements equal to zero and positive diagonal entries. We shall start our discussion with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$, as [5], but we shall start somewhat ad hoc. After that we state and prove the Iwasawa decomposition.

Recall that $\mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathrm{Mat}(2,\mathbb{R}) \mid \mathrm{tr}(X) = 0\}$. One readily verifies that a linear basis for $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Notice that these satisfy the relations [1]

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

We set

$$\begin{aligned} \mathfrak{k} &= \mathbb{R}(Y - X) = \mathfrak{so}(2) \\ \mathfrak{p} &= \mathbb{R}H \oplus \mathbb{R}(X + Y) \\ \mathfrak{a} &= \mathbb{R}H \\ \mathfrak{n} &= \mathbb{R}X \\ \overline{\mathfrak{n}} &= \mathbb{R}Y. \end{aligned}$$

It is then immediate that, as a linear space, $\mathfrak{sl}(2,\mathbb{R})$ decomposes as $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. This composition

[1] This is a so called $\mathfrak{sl}_2$-triple.

of the Lie algebra is known as the infinitesimal Iwasawa decomposition. We now let

$$\begin{aligned}
K &= \exp(\mathbb{R}(Y - X)) = \{k_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \mid \varphi \in \mathbb{R}\} \\
A &= \exp(\mathbb{R}H) = \{a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid x \in \mathbb{R}\} \\
N &= \exp(\mathbb{R}X) = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\} \\
M &= \{\pm I\} \\
P &= MAN = \{\begin{pmatrix} \pm e^t & x \\ 0 & \pm e^{-t} \end{pmatrix} \mid x, t \in \mathbb{R}\}
\end{aligned}$$

We shall now state and prove the Iwasawa decomposition. The proof will make use of the action of $\mathrm{SL}(2,\mathbb{R})$ on H_+ as defined in Section 1.3. The first part of the proof I have from a lecture by Erik van den Ban and the second part of the proof comes from the proof of lemma 17.8 of [2].

Theorem 9.1 (Iwasawa decomposition for $\mathrm{SL}(2,\mathbb{R})$)**.** *The map* $\Psi : N \times A \times K \to \mathrm{SL}(2,\mathbb{R})$ *given by*

$$\Psi(n_x, a_t, k) = n_x a_t k \tag{9.1}$$

is a diffeomorphism.

Proof. We shall first prove that Ψ is bijective. Notice that the maps $\mathbb{R} \to A$, $t \mapsto a_t$ and $\mathbb{R} \to N$, $x \mapsto n_x$ are diffeomorphisms, so it suffices to show that the map

$$\psi : \mathbb{R} \times \mathbb{R} \times K \to \mathrm{SL}(2,\mathbb{R}), \qquad (x, t, k) \mapsto n_x a_t k$$

is a bijection. Therefore we consider the action of $\mathrm{SL}(2,\mathbb{R})$ on H_+ by fractional linear transformations. Notice that

$$\psi(x, t, k) \cdot i = n_x a_t k \cdot i = n_x a_t \cdot i = x + ie^{2t}.$$

From this we see that ψ is injective. Now let $g \in \mathrm{SL}(2,\mathbb{R})$. We write $g \cdot i = x + iy$. We notice that there is a $t \in \mathbb{R}$ such that $y = e^{2t}$, namely $t = \frac{1}{2}\log(y)$. We then have

that $\psi(x,t,e)\cdot i = g\cdot i$. We thus conclude that $g^{-1}\psi(x,t,e)$ stabilizes i and therefore $k^{-1} := g^{-1}\psi(x,t,e) \in K$. Thus $\psi(x,t,e) = gk^{-1}$ and hence $\psi(x,t,k) = g$. We thus conclude that ψ is bijective.
From the above discussion follows that the map $\Phi : K \times A \times N \to \mathrm{SL}(2,\mathbb{R})$, given by $(k,a,n) \mapsto kan$ is also a bijection, since $\Phi(k,a,n) = \Psi^{-1}(n^{-1},a^{-1},k^{-1})$. We shall show that Φ has a bijective derivative everywhere. It then follows from the inverse function theorem that Φ is a diffeomorphism and hence also Ψ.
We notice that it is sufficient to prove that Φ has a bijective derivative in the element (e,a,e), for $a \in A$. We indeed notice that he maps $l_k : g \mapsto kg$, the left translation by k, and $r_n : g \mapsto gn$, right translation by n, are diffeomorphisms and that

$$\Phi(k,a,n) = k\Phi(e,a,e)n = l_k \circ r_n \circ \Phi(e,a,e).$$

Let $U \in \mathfrak{k}$, $W \in \mathfrak{a}$ and $V \in \mathfrak{n}$. Then for $t \in \mathbb{R}$ we have that

$$\Phi(\exp(tU), \exp(tW)a, \exp(tV))a^{-1} = \exp(tU)\exp(tW)\exp(t\mathrm{Ad}(a)V)$$

Differentiating the above expression gives us

$$(T_a r_{a^{-1}} \circ T_{(e,a,e)}\Phi)(U + T_e r_a W + \mathrm{Ad}(a)V) = U + W + \mathrm{Ad}(a)V.$$

It thus suffices to show that the map $(U,W,V) \mapsto U + W + \mathrm{Ad}(a)V$ is a linear isomorphism. Since $[H,X] = 2X$ it follows that $\mathrm{Ad}(a)\mathfrak{n} \subset \mathfrak{n}$, for all $a \in A$. It thus follows from the infinitesimal Iwasawa decomposition that $(U,W,V) \mapsto U + W + \mathrm{Ad}(a)V$ is a linear isomorphism. $\square$

Lemma 9.2 ([2], Lemma 20.1)**.** *The subgroup P is closed and the map $M \times A \times N \to P$, given by $(m,a,n) \mapsto man$ is a diffeomorphism.*

Proof. That $(m,a,n) \mapsto man$ is a diffeomorphism follows from Theorem 9.1. We now notice that $P = pr_K^{-1}(M)$, where $pr_K := pr_1 \circ \Psi^{-1}$. Hence, since M is closed, P is closed. $\square$

From the proof of Theorem 9.1 we can conclude that the map

$$j : NA \to H_+$$

given by $na \mapsto na \cdot i$, is a diffeomorphism. Requiring j to be a bi-holomorphic map, gives us a complex structure on NA.
Now let

$$\begin{aligned}
K_{\mathbb{C}} &= \exp(\mathbb{C}(Y - X)) = \{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C},\, \alpha^2 + \beta^2 = 1\} \\
A_{\mathbb{C}} &= \exp(\mathbb{C}H) = \{\begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \mid z \in \mathbb{C}\} \\
N_{\mathbb{C}} &= \exp(\mathbb{C}X) = \{\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \mid w \in \mathbb{C}\} \\
\overline{N}_{\mathbb{C}} &= \exp(\mathbb{C}X) = \{\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \mid w \in \mathbb{C}\} \\
P_{\mathbb{C}} &= A_{\mathbb{C}} N_{\mathbb{C}} = \{\begin{pmatrix} e^z & w \\ 0 & e^{-z} \end{pmatrix} \mid w, z \in \mathbb{C}\} \\
\overline{P}_{\mathbb{C}} &= A_{\mathbb{C}} \overline{N}_{\mathbb{C}} = \{\begin{pmatrix} e^z & 0 \\ w & e^{-z} \end{pmatrix} \mid w, z \in \mathbb{C}\} \\
\overline{B}_{\mathbb{C}} &= (\mathrm{SL}(2, \mathbb{C}))_i = \{g \in \mathrm{SL}(2, \mathbb{C}) \mid g \cdot i = i\}.
\end{aligned}$$

Since $\mathrm{SL}(2, \mathbb{C})$ acts on $\widehat{\mathbb{C}}$. One readily verifies that this actions is transitive. Applying the orbit stabilizer theorem (Theorem 2.20), we see that the map $g \mapsto gi$ induces a diffeomorphism $\mathrm{SL}(2, \mathbb{C})/\overline{B}_{\mathbb{C}} \to \widehat{\mathbb{C}}$. One can show that this is actually a bi-holomorphic map.
We notice that $K = \mathrm{SL}(2, \mathbb{R}) \cap \overline{B}_{\mathbb{C}}$, so we have an inclusion $G/K \to G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$. We thus have the following commuting diagram

$$\begin{tikzcd}
G/K \arrow[r, hook] \arrow[d] & G_{\mathbb{C}}/\overline{B}_{\mathbb{C}} \arrow[d] \\
H_+ \arrow[r, hook] & \widehat{\mathbb{C}}
\end{tikzcd}$$

This implies in particular that G/K is an open subset of $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$.

Lemma 9.3 ([5], Lemma 1.1.11)**.** *There exists a* $g_0 \in \mathrm{SL}(2, \mathbb{C})$ *such that*

(1) $g_0 \cdot i = 0$.

(2) $K_{\mathbb{C}} = g_0 A_{\mathbb{C}} g_0^{-1}$

(3) $\overline{B}_{\mathbb{C}} \simeq K_{\mathbb{C}} \times g_0 \overline{N}_{\mathbb{C}} g_0^{-1}$

Proof. Notice that for any element $c \in \mathrm{SL}(2, \mathbb{C})$, such that $c \cdot i = 0$, we have

$$\overline{B}_{\mathbb{C}} = c\overline{P}_{\mathbb{C}}c^{-1} = cA_{\mathbb{C}}\overline{N}_{\mathbb{C}}c^{-1} \simeq cA_{\mathbb{C}}c^{-1} \times c\overline{N}_{\mathbb{C}}c^{-1}.$$

This means that $\mathrm{Lie}(\overline{B}_{\mathbb{C}}) \simeq \mathrm{Ad}(c)\mathfrak{a}_{\mathbb{C}} \oplus \mathrm{Ad}(c)\bar{\mathfrak{n}}_{\mathbb{C}}$. We notice that $K_{\mathbb{C}}$ and $A_{\mathbb{C}}$ are both the image of their respected Lie algebra's under the exponential map. So it is sufficient to find a c such that $\mathfrak{k}_{\mathbb{C}} = \mathrm{Ad}(c)\mathfrak{a}_{\mathbb{C}}$. We can find such a c by diagonalizing $Y - X$. We find that $Y - X = g_0(-iH)g_0^{-1}$, for

$$g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

One readily verifies that $g_0 \cdot i = 0$. $\square$

Chapter 10
Induced representation

Given a finite dimensional representation (π, V) of the Lie group G, we have a representation on any closed subgroup H, namely the restriction of π to H. In this chapter we consider the reversed case. Given a representation (ξ, V_ξ) of a closed subgroup H of G we give a representation on G, denoted by $\mathrm{ind}_H^G(\xi)$. This will be the so called induced representation of G. We shall give two different realizations of this induced representation, the so called induced picture and the vector bundle picture. We shall start with the induced picture, since this one is the most direct. In the final section we will discuss the process of normalized or unitary induction. This process ensures us that when we start with a unitary representation H, that the resulting representation of G is also unitary.
The material for this chapter is drawn from Chapter 19 of [2], and sections 1.1. and 1.2 of [3]

10.1 Induced picture

Let G be a Lie group and let H be a closed subgroup. Let (ξ, V_ξ) be a finite dimensional continuous representation of H. Consider the space of continuous functions $\varphi : G \to V_\xi$ that transform according to the rule

$$\varphi(xh) = \xi(h^{-1})\varphi(x), \qquad \text{(fore all } x \in G, \text{ and } h \in H).$$

We denote the space of these functions by $C(G : H : \xi)$. The induced representation on G is the pair $(L, C(G : H : \xi))$, where

$$[L(g)\varphi](x) = \varphi(g^{-1}x), \qquad \text{(fore all } g, x \in G).$$

10.2 Associated bundle

In the vector bundle picture the induced representation of G is realized in the space of continuous sections of the so called associated bundle. We shall thus firstly have to define what an associated bundle is. We shall do this for a general smooth manifold M.

Let M be a smooth manifold equipped with a smooth free and proper right action of the Lie group H. Let (ξ, V_ξ) be a finite dimensional representation of H, in the complex vector space V_ξ. Consider the right action of H on $M \times V_\xi$ given by

$$(g, v) \cdot h = (gh, \xi(h^{-1})v)$$

We notice that this action is also free and proper, so $M \times_H V_\xi := M \times V_\xi / H$ can be equipped with a smooth structure making the projection map $\tilde{\pi} : M \times V_\xi \to M \times_H V_\xi$ a smooth submersion. Let $pr : M \times V_\xi \to M$ be the natural projection. Now there is an unique map $p : M \times_H V_\xi \to M/H$ such that the following diagram commutes.

$$\begin{tikzcd} M \times V_\xi \arrow[r, "\tilde{\pi}"] \arrow[d, "pr"] & M \times_H V_\xi \arrow[d, "p"] \\ M \arrow[r, "\pi"] & M/H \end{tikzcd}$$

One readily verifies that this map is given by $[(m, v)] \mapsto mH$. Now the map $\varphi_m : V_\xi \to M \times_H V_\xi$ given by $v \mapsto [(m, v)]$, defines a diffeomorphism from V_ξ onto $p^{-1}(gH)$. Requiring that the maps φ_g be linear induces a vector space structure on $p^{-1}(gH)$ and thus a vector bundle structure on $p : M \times_H V_\xi \to G/H$. We notice that φ_m is a linear isomorphism and hence it follows that $\tilde{\pi}_m : \{m\} \times V_\xi \to (M \times_H V_\xi)_{\pi(m)}$ is a linear isomorphism. It thus follows from the universal property of the pull back bundle that the bundles $pr : M \times V_\xi \to M$ and $\pi^*(M \times_H V_\xi) \to M$ are isomorphic in a canonical way.

Lemma 10.1. *The map $s \mapsto \pi^*(s)$ defines a linear isomorphism from $\Gamma(M \times_H V_\xi)$ onto $\Gamma(M \times V_\xi)^H$, where*

$$\Gamma(M \times V_\xi)^H = \{t \in \Gamma(M \times V_\xi) \mid t(xh) \cdot h^{-1} = t(x), \textit{ for all } h \in H\}.$$

Furthermore, π^* restricted to $\Gamma^\infty(M \times_H V_\xi)$ is a linear isomorphism onto $\Gamma^\infty(M \times V_\xi)^H$

Proof. From lemma 3.7 we know that $\pi^* : \Gamma(M \times_H V_\xi) \to \Gamma(M \times V_\xi)$, given by $\pi^*(s)(x) = \tilde{\pi}_x^{-1} s(\pi(x))$, for $s \in \Gamma(M \times_H V)$, is an injective linear map.
We still have to prove that π^* is onto. We first prove that π^* maps $\Gamma(M \times_H V_\xi)$ into $\Gamma(M \times V_\xi)^H$. Let $s \in \Gamma(M \times_H V_\xi)$. Then $\pi^*(s)$ is the unique section such that the diagram

$$\begin{array}{ccc} M \times V_\xi & \xrightarrow{\tilde{\pi}} & M \times_H V_\xi \\ {\scriptstyle \pi^*(s)}\uparrow & & \uparrow{\scriptstyle s} \\ M & \xrightarrow{\pi} & M/H \end{array}$$

commutes. Given $h \in H$, let $t_h : M \to M \times V_\xi$ be defined by $t(x) = \pi^*(s)(xh) \cdot h^{-1}$. Notice that

$$\tilde{\pi}(t_h(x)) = \tilde{\pi}(\pi^*(s)(xh) \cdot h^{-1}) = \tilde{\pi}(\pi^*(s)(xh)) = s(\pi(xh)) = s(\pi(x)).$$

Hence $t_h = \pi^*(s)$, for all $h \in H$. We thus conclude that $\pi^*(s) \in \Gamma(M \times V_\xi)^H$.
We now prove that π^* maps $\Gamma(M \times_H V_\xi)$ onto $\Gamma(M \times V_\xi)^H$. Let $t \in \Gamma(M \times V_\xi)^H$. Since t commutes with the right action of H, t induces a section $\bar{t} : M/H \to M \times_H V_\xi$. It is clear that $t = \pi^*(\bar{t})$.
We are now left to prove that $\pi^*(s) \in \Gamma^\infty(M \times V_\xi)^H$, if $s \in \Gamma^\infty(M \times_H V_\xi)$. For this we just notice that $s \circ \pi$ is smooth and that $\tilde{\pi}$ is a smooth submersion. From which it follows that $\pi^*(s)$ is smooth. □

Now consider a Lie group G and a finite dimensional representation (ξ, V_ξ) of H. We have that G acts on itself by left translation l_g and on $G \times V_\xi$, by the rule $(x, v) \mapsto (gx, v)$. These actions induce actions of G on G/H and $G \times_H V_\xi$, which we shall denote by l_g and $\tilde{l_g}$ respectively, and we thus have the following commutative diagram:

$$\begin{array}{ccc} G \times_H V_\xi & \xrightarrow{\tilde{l_g}} & G \times_H V_\xi \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G/H & \xrightarrow{l_g} & G/H \end{array}$$

The group G has a natural representation π in $\Gamma(G \times_H V_\xi)$ given by

$$\Xi(g)(s) = l_g \circ s \circ \tilde{l}_g^{-1} \qquad (g \in G). \tag{10.1}$$

This representation of G is called the representation induced from the representation ξ of H.

We have thus far considered two representations of G that we both call the induced representation. The following lemma says that these representations are equivalent.

Lemma 10.2. *The representations* $(L, C(G : H : \xi))$ *and* $(\Xi, \Gamma(G \times_H V_\xi))$ *are equivalent.*

Proof. We first notice that there is a natural representation of G in $\Gamma(G \times V_\xi)$. Namely the one given by

$$i_\xi(g)s = l_g \circ s \circ l_g^{-1}, \qquad \text{for all } g \in G \text{ and } s \in \Gamma(G \times V_\xi).$$

The diagram

$$\begin{tikzcd} M \times V_\xi \arrow[r, "\tilde{\pi}"] \arrow[d, "pr"'] & M \times_H V_\xi \arrow[d, "p"] \\ M \arrow[r, "\pi"] & M/H \end{tikzcd}$$

is G-equivariant. So we have that the following diagram commutes

$$\begin{tikzcd} M \times V_\xi \arrow[r, "\tilde{\pi}"] & M \times_H V_\xi \\ M \arrow[r, "\pi"] \arrow[u, "l_g \circ \pi^*(s) \circ l_g^{-1}"] & M/H \arrow[u, "l_g \circ s \circ l_g^{-1}"'] \end{tikzcd}$$

Since $\pi^*(l_g \circ s \circ l_g^{-1})$ is the unique section such that

$$l_g \circ s \circ l_g^{-1} \circ \pi = \tilde{\pi} \circ \pi^*(l_g \circ s \circ l_g^{-1})$$

we conclude that $\pi^*(l_g \circ s \circ l_g^{-1}) = l_g \circ \pi^*(s) \circ l_g^{-1}$. Hence π^* is G-equivariant.
Now define $j : C(G, V_\xi) \to \Gamma(G \times V_\xi)$ by $j(f)(x) := (x, f(x))$. This is clearly a linear isomorphism. We notice that

$$[i_{xi}(g)j(f)](x) = (x, f(g^{-1}x)) = j(L(g)f)(x)$$

and $j(f)(xh) = j(f)(x) \cdot h$. So j induces a G-equivariant isomorphism $\bar{j} : C(G : H : \xi) \to \Gamma(G \times V_\xi)^H$. Our desired intertwining isomorphism is then given by $\bar{j}^{-1} \circ \pi^* : \Gamma(G \times_H V_\xi) \to C(G : H : \xi)$.

$\square$

10.3 Normalised induction

In this section we discuss an adapted induction process known as normalised or unitary induction. This process will guarantee that when we start whit a unitary representation (ξ, V_ξ) of H that the representation of G obtained in this process is also unitary. This need not be true for the ordinary induction process. See [2] Chapter 19 for a detailed discussion.

Let ξ be a representation of H in the Hilbert space V_ξ. Consider the modular function $\Delta : H \to \mathbb{R}_{>0}$ as discussed in section section 4.2. As discussed in section 4.2 is Δ a group homomorphism. Thence $\Delta^{1/2} : H \to \mathbb{R}_{>0}$, given by $\Delta^{1/2}(h) = \Delta(h)^{1/2}$, for $h \in H$, is also a group homomorphism. It follows that $(\Delta^{1/2}, \mathbb{C})$ is a representation of H. We now consider the representation $\xi \otimes \Delta^{1/2}$ in $V_\xi \otimes \mathbb{C}_{\Delta^{1/2}}$. This representation is naturally isomorphic to V_ξ with the representation $\xi \otimes \Delta^{1/2}$ given by

$$(\xi \otimes \Delta^{1/2})(h)v = \Delta(h)^{1/2}\xi(h)v, \qquad \text{(for all } v \in V_\xi \text{ and all } h \in H).$$

We agree to write $V_{\xi\otimes\Delta^{1/2}}$ for V_ξ equipped with the representation $\xi \otimes \Delta^{1/2}$. We define the representation

$$\mathrm{Ind}_H^G(\xi) := \mathrm{ind}_H^G(\xi \otimes \Delta^{1/2})$$

This adapted procedure is known as normalized or unitary induction.

Chapter 11

Holomorphic discrete series representation

In this chapter we will apply the induction process to define the discrete series representations of $\mathrm{SL}(2,\mathbb{R})$. This shall be done along the lines of [5] Section 1.4. In the second section of this chapter we will study complex line bundles over $\mathbb{P}^1(\mathbb{C})$ to gain some insights in the representation spaces. After that we will realize the holomorphic discrete series representations in $\mathcal{O}(H_+)$. The discussion is mainly based on explanations provided to me by Erik van den Ban in private meetings. For convenience we shall agree to write $G = \mathrm{SL}(2,\mathbb{R})$ and $G_{\mathbb{C}} = \mathrm{SL}(2, C)$.

We consider the characters $\hat{K} = \{\tau_n : K \to \mathbb{C} \mid \tau_n(k_\varphi) = e^{-in\varphi},\, n \in \mathbb{Z}\}$ of $\mathrm{SO}(2)$. By Lemma 9.3 these characters can be extended to characters on $\overline{B}_{\mathbb{C}}$, given by

$$\tau_n(\exp iz(Y - X) \exp w\mathrm{Ad}(g_0)Y) = e^{nz}$$

We will consider the the restriction of induced representation of $(\tau_n, \mathbb{C}_n)$ to the space of holomorphic sections $\Gamma_{hol}(G_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_n)$. Note that we are now considering the induced representation in the vector bundle picture, so we have the following commutative diagram

$$\begin{tikzcd} G_{\mathbb{C}} \times \mathbb{C}_n \arrow[r, "\tilde{\pi}"] \arrow[d, "pr"] & G_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_n \arrow[d, "p"] \\ G_{\mathbb{C}} \arrow[r, "\pi"] & G_{\mathbb{C}}/\overline{B}_{\mathbb{C}} \end{tikzcd}$$

It turns out that the space of holomorphic sections of $p : G_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_n \to G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ is trivial for $n \leq 1$. We must thus restrict ourselves to an open subset U_0 of $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ and induce locally to get a non trivial representation space. Let U be the preimage of U_0 in $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$, under the projection $\pi : G_{\mathbb{C}} \to G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$. With similar arguments as in the proof of Lemma 10.1, we can identify the space $\Gamma_{hol}(U_0, \mathcal{L}_n)$ with

$$\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n) = \{f \in \mathcal{O}(U) \mid R_{\bar{b}} f = \tau_n^{-1}(\bar{b}) f \text{ for all } \bar{b} \in \overline{B}_{\mathbb{C}}\}.$$

For our purposes it turns out that $U_0 = G/K$ will be a good choice. In Chapter 9 we indeed saw that $G/K \simeq NA$ is an open subset of $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$, and in particular a complex submanifold of $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$. In this case $U = G\overline{B}_{\mathbb{C}}$. We thus have the following commutative diagram

$$
\begin{array}{ccc}
G\overline{B}_{\mathbb{C}} \times \mathbb{C}_n & \longrightarrow & G\overline{B}_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_n \\
\Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle p} \\
G\overline{B}_{\mathbb{C}} & \longrightarrow & NA
\end{array}
$$

For any $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ we have that

$$f(nak\bar{b}) = \tau_n(\bar{b})^{-1}\tau_n(k)^{-1}f(na), \qquad \text{for all } n \in N,\, a \in A,\, k \in K,\, \bar{b} \in \overline{B}_{\mathbb{C}}.$$

The following theorem and its proof are drawn from [5] (Lemma 1.4.2).

Lemma 11.1. *The function $\sigma_n : U \to \mathbb{C}$ defined as*

$$\sigma_n(n_x a_t k_\varphi \bar{b}) := a^{-n\rho}\tau_n(\bar{b})^{-1}\tau_n(k)^{-1} = e^{in\rho(\log(a_t))}\tau_n(\bar{b})^{-1}\tau_n(k)^{-1}$$

defines a nowhere vanishing function in $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$. Further we have that $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n) = \sigma_n \pi^* \mathcal{O}(NA)$.

Proof. We consider the bi-linear map $\beta : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$, given by $(w, z) \mapsto w^t z$. One readily confirms that the function $\beta(w, \cdot)$, given by $z \mapsto w^t z$ is an holomorphic map. Now, the natural action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$ is holomorphic and therefore is the map $g \mapsto g \cdot w$ holomorphic, for all $w \in \mathbb{C}^2$. We thus have that the map $m_{w,z} : g \mapsto \beta(w, g \cdot z)$ is holomorphic, for all $w, z \in \mathbb{C}^2$. So if $\sigma_1 = m_{w,z}$, for some $w, z \in \mathbb{C}^2$, then we have that σ_1 is holomorphic. This is indeed possible. We define the function $m : U \to \mathbb{C}$ as

$$m(n_x a_t k_\varphi \bar{b}) := \beta(e_2, n_x a_t k_\varphi \bar{b}(ie_1 + e_2)).$$

We notice that $k_\varphi(ie_1 + e_2) = \tau_1(k)^{-1}(ie_1 + e_2)$ for all $k \in K_{\mathbb{C}}$. It follows from a direct computation that $g_0 \bar{n}_w g_0^{-1}(ie_1 + e_2) = ie_1 + e_2$, for all $\bar{n}_w \in \overline{N}_{\mathbb{C}}$. So it follows from Lemma 9.3 that $\bar{b}(ie_1 + e_2) = \tau_1(\bar{b})^{-1}(ie_1 + e_2)$, for all $\bar{b} \in \overline{B}_{\mathbb{C}}$. We further notice that ${n_x}^t e_2 = e_2$ and ${a_t}^t e^2 = {a_t}^{-\rho} e_2$. So we conclude that

$$m(n_x a_t k_\varphi \bar{b}) = {a_t}^{-\rho}\tau_1(\bar{b})^{-1}\tau_1(k_\varphi)^{-1}.$$

So we see that $\sigma_1 = m$, and thus by the above discussion we have that σ_1 is holomorphic. We shall now prove that $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n) = \sigma_n \pi^* \mathcal{O}(NA)$. Therefore recall that $\pi : G_{\mathbb{C}} \to G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ is a holomorphic submersion. So for every $f \in \mathcal{O}(NA)$ we have that $\pi^* f = f \circ \pi : G\overline{B}_{\mathbb{C}} \to \mathbb{C}$ is a holomorphic function. It is then clear that $\sigma_n \pi^* h \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$, for any $h \in \mathcal{O}(NA)$. For the other inclusion we observe That $\pi^*({\sigma_n}^{-1} f) \in \mathcal{O}(NA)$, for any $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$. $\square$

Since A and N are Lie groups there exist left Haar measures da and dn of A and N respectively. Now recall that $NA \simeq N \times A$, so $\mathrm{d}n\,\mathrm{d}a = \mathrm{d}n \times \mathrm{d}a$ is a measure on NA. We then also have that $a^{-2\rho}\,\mathrm{d}n\,\mathrm{d}a$ is a measure on NA. We can thus consider the space $L^2(NA,\, a^{-2\rho}\,\mathrm{d}a\,\mathrm{d}n)$. We now define the spaces X_n as the $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ such that $\sigma_n^{-1} f \in L^2(NA,\, a^{-2\rho}\,\mathrm{d}a\,\mathrm{d}n)$.

Definition 11.2. The discrete series representations of $\mathrm{SL}(2,\mathbb{R})$ are the G-modules (L, X_n), for $n \geq 2$.

11.1 Bundles over $\mathbb{P}^1(\mathbb{C})$

For $n \in \mathbb{Z}$ we consider the representation of $\mathbb{C}^*$ in $\mathbb{C}$ given by $\zeta \mapsto \zeta^{-n}$. We shall denote the representation space $\mathbb{C}$ in this case by $\mathbb{C}_n$. We define

$$\mathcal{L}_n := \mathbb{C}^2\backslash\{0\} \times_{\mathbb{C}^*} \mathbb{C}_n.$$

We consider the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2\backslash\{0\}$. This action is transitive and commutes with the action of $\mathbb{C}^*$ on $\mathbb{C}^2\backslash\{0\}$. This action of $\mathrm{SL}(2,\mathbb{C})$ induces an action on $\mathcal{L}_n$. We also saw that this action of $\mathrm{SL}(2,\mathbb{C})$ induces an action on $\mathbb{P}^1(\mathbb{C})$. We get the the following commutative diagram of $\mathrm{SL}(2,\mathbb{C})$-equivariant maps

$$\begin{tikzcd}
\mathbb{C}^2\backslash\{0\} \times \mathbb{C}_n \arrow[r, "\tilde{\pi}"] \arrow[d, "pr"] & \mathcal{L}_n \arrow[d, "p"] \\
\mathbb{C}^2\backslash\{0\} \arrow[r, "\pi"] & \mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

It follows that $\mathrm{SL}(2,\mathbb{C})$ acts on $\Gamma_{hol}(\mathcal{L}_n)$, $(n \geq 0)$, by

$$(g \cdot s)(z) = g \cdot s(g^{-1}z)$$

This defines a continuous representation of $\mathrm{SL}(2,\mathbb{C})$ in the space $\Gamma_{hol}(\mathcal{L}_n)$, which we shall denote by π_n.

The space of holomorphic sections of $\mathcal{L}_n \to \mathbb{P}^1\mathbb{C}$ can be identified with the space of holomorphic functions on $\mathbb{C}^2\backslash\{0\}$, which are H invariant, i.e. the space

$$\mathcal{O}(\mathbb{C}^2/\{0\})^{\mathbb{C}^*} = \{f \in \mathcal{O}(\mathbb{C}^2/\{0\}) \mid f(z\zeta) = \zeta^n f(z)\}.$$

We now have the following result.

Lemma 11.3. *Let $\mathcal{L}_n$ be as above. If $n < 0$, then $\Gamma_{hol}(\mathcal{L}_n) = \{0\}$. If $n \geq 0$, then $\Gamma_{hol}(\mathcal{L}_n) \simeq P_n(\mathbb{C}^2)$, where $P_n(\mathbb{C}^2)$ is the space of homogeneous polynomials on $\mathbb{C}^2$ of degree n.*

Proof. For $n = 0$ we have for $f \in \mathcal{O}(\mathbb{C}^2\backslash\{0\})$ that $f(\lambda z) = f(z)$, for all $z \in \mathbb{C}^2\backslash\{0\}$ and $\lambda \in \mathbb{C}$. So $f \in \mathcal{O}(\mathbb{C}^2\backslash\{0\}) \simeq \mathcal{O}(\mathbb{P}^1(\mathbb{C}))$. But by Louville's theorem $\mathcal{O}(\mathbb{P}^1(\mathbb{C}))$ is the space of constant functions.
Let $n > 0$. Let $f \in \mathcal{O}(\mathbb{C}^2\backslash\{0\} : \mathbb{C}^* : n)$. Since $f(\lambda z) = \lambda^n f(z)$ we have that $\lim_{z\to 0} f(z) = 0$. So we can extend f continuously to $\mathbb{C}^2$. One now also readily confirms that this extension is holomorphic. Now f is entire, so $f(z) = \sum_{k,l=n} a_{l,k} z_1^k z_2^l$. We conclude that $f(z) = \sum_{k+l=n} a_{k,l} z_1^k z_2^l$.
Now let $n < 0$ and $f \in \mathcal{O}(\mathbb{C}^2\backslash\{0\} : \mathbb{C}^* : n)$. We notice that for $g : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}$, given by $g(z) := z_1^n f(z)$, we have $g(\lambda z) = g(z)$, for all $\lambda \in \mathbb{C}\backslash\{0\}$. So this means we can interpret g as a holomorphic function on $\widehat{\mathbb{C}}$. It thus follows from Louvile's theorem that g is constant, and thus $f(z) = \frac{c}{{z_1}^n}$, for some $c \in \mathbb{C}$. But this function can only be defined on $\mathbb{C}^2\backslash\{0\}$ if $c = 0$. So we conclude that $\mathcal{O}(\mathbb{C}^2\backslash\{0\} : \mathbb{C}^* : n) = \{0\}$. $\square$

We now want to relate the spaces $\Gamma_{hol}(\mathcal{L}_n)$ and $\Gamma_{hol}(G_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_{\tau_n})$. Therefore let $\beta \in \mathbb{C}^2/\{0\}$. Then $[\beta] \in \mathbb{P}^1(\mathbb{C})$ and let $B_\beta = \mathrm{SL}(2,\mathbb{C})_{[\beta]}$. Now let $\alpha : \mathrm{SL}(2,\mathbb{C})/B_\beta \to \mathbb{P}^1(\mathbb{C})$ be the diffeomorphism induced by the map $\alpha_\beta : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, given by $g \mapsto g \cdot [\beta]$. It follows from the orbit stabilizer theorem that

$$\mathbb{P}^1(\mathbb{C}) \simeq \mathrm{SL}(2,\mathbb{C})/B_\beta.$$

It then follows that $q = p \circ \alpha^{-1} : \mathcal{L}_n \to \mathrm{SL}(2,\mathbb{C})/B_\beta$ is also a vector bundle. We claim that $q : \mathcal{L}_n \to \mathrm{SL}(2,\mathbb{C})/B_\beta$ is an equivariant vector bundle. To see this, let F_g be the fractional linear transformation corresponding to g. We notice that we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^2\backslash\{0\} \times_{\mathbb{C}^*} \mathbb{C}_n & \xrightarrow{g\cdot} & \mathbb{C}^2\backslash\{0\} \times_{\mathbb{C}^*} \mathbb{C}_n \\
\downarrow p & & \downarrow p \\
\mathbb{P}^1(\mathbb{C}) & \xrightarrow{F_g} & \mathbb{P}^1(\mathbb{C})
\end{array}$$

We also have the following commutative diagram

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C})/B_{[\beta]} \arrow[r, "l_g"] \arrow[d, "\alpha"] & \mathrm{SL}(2,\mathbb{C})/B_{[\beta]} \arrow[d, "\alpha"] \\ \mathbb{P}^1(\mathbb{C}) \arrow[r, "F_g"] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

From which we conclude that the diagram

$$\begin{tikzcd} \mathbb{C}^2\backslash\{0\} \times_{\mathbb{C}^*} \mathbb{C}_n \arrow[r, "g\cdot"] \arrow[d, "q"] & \mathbb{C}^2\backslash\{0\} \times_{\mathbb{C}^*} \mathbb{C}_n \arrow[d, "q"] \\ \mathrm{SL}(2,\mathbb{C})/B_\beta \arrow[r, "l_g"] & \mathrm{SL}(2,\mathbb{C})/B_\beta \end{tikzcd}$$

commutes. It is readily verified that $g\cdot : (\mathcal{L}_n)_x \to (\mathcal{L}_n)_{gx}$. So we conclude that $q : \mathcal{L}_n \to \mathrm{SL}(2,\mathbb{C})/B_\beta$ is an equivariant vector bundle. So $\mathcal{L}_n \simeq \mathrm{SL}(2,\mathbb{C}) \times_{B_\beta} V$, where $V := q^{-1}(eB_\beta)$. But one readily verifies that $V = (\mathcal{L}_n)_\beta$.

Now $\mathbb{C}^*\beta$ is invariant under the action of B_β, by definition of B_β. But this implies that $b \cdot x = \chi_\beta(b)x$, for all $x \in \mathbb{C}^*\beta$ and $b \in B_\beta$, for some homomorphism $\chi_\beta : B_\beta \to \mathbb{C}^*$. In other words χ_β is a character of B_β. We now conclude that

$$b \cdot [(\beta, w)] = [(b \cdot \beta, w)] = [(\chi_\beta(b)\beta, w)] = [(\beta, \chi_\beta(b)^{-n}w)].$$

So $(\mathcal{L}_n)_{[\beta]}$ is an invariant subspace of the action of B_β on $\mathcal{L}_n$. We thus conclude that $V \simeq \mathbb{C}_{\chi_\beta^{-n}}$, as representations of B_β. So we conclude $\mathcal{L}_n \simeq SL(2,\mathbb{C}) \times_{B_\beta} \mathbb{C}_{\chi_\beta^{-n}}$ as vector bundles.

11.1.1 Borel group

We now fix $\beta = \beta_0 = (0,1)$. Then one readily sees that

$$B_{\beta_0} = \{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \text{ and } c \in \mathbb{C}\}.$$

Now the character χ_{β_0} was determined by the equation

$$\chi_{\beta_0}(\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ a^{-1} \end{pmatrix} = a^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So

$$\chi_{\beta_0}(\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}) = a^{-1}.$$

We relate this to the Lie algebra structure of $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}Y \oplus \mathbb{C}H \oplus \mathbb{C}X$. We notice that $B_{\beta_0} = \overline{P}_{\mathbb{C}} = A_{\mathbb{C}}N_{\mathbb{C}}$, so $\mathrm{Lie}(B_{\beta_0}) = \mathfrak{a}_{\mathbb{C}} \oplus \bar{\mathfrak{n}}_{\mathbb{C}}$. The character χ_{β_0} is given by

$$\begin{pmatrix} e^z & 0 \\ e^{-z}w & e^{-z} \end{pmatrix} \mapsto e^{-z}.$$

Hence $\chi_{\beta_0}(\exp(zH)) = e^{-z}$ and $\chi_{\beta_0} = 1$ on $\bar{N}_{\mathbb{C}}$. It follows that

$$(\chi_{\beta_0})_*(H) = \frac{d}{dt}\bigg|_{t=0} \chi_{\beta_0}(\exp(tH)) = -1.$$

We agree to write $\chi_{\beta_0} = \chi_{-\rho}$ and $\chi_{\beta_0}^{-n} = \chi_{n\rho}$.

The results of this section and the previous section are summarized in the following commutative diagram

$$\begin{tikzcd}
\mathrm{SL}(2,\mathbb{C}) \times_{B_\beta} \mathbb{C}_{\chi_{n\rho}} \arrow[r] \arrow[d] & \mathcal{L}_n \arrow[d, "p"] \\
\mathrm{SL}(2,\mathbb{C})/B_{\beta_0} \arrow[r] & \mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

We thus have that $\Gamma_{hol}(\mathrm{SL}(2,\mathbb{C}) \times_{B_\beta} \mathbb{C}_{\chi_{n\rho}}) \simeq \Gamma_{hol}(\mathcal{L}_n) \simeq P_n(\mathbb{C}^2)$.

11.1.2 Conjugate of a Borel subgroup

Now let $\beta \in \mathbb{C}^2/\{0\}$ be arbitrary. Now there is a $\gamma \in \mathrm{SL}(2,\mathbb{C})$ such that $\beta = \gamma\beta_0$. It thus follows that $B_\beta = \gamma B_{\beta_0}\gamma^{-1}$. For a character χ of B_{β_0} we define

$$\gamma(\chi) := \chi \circ \mathcal{C}_{\gamma^{-1}}.$$

We then see that $\gamma : \hat{B}_{\beta_0} \to \hat{B}_\beta$ is a bijection.

For our purposes it will turn out that $\beta = \frac{1}{\sqrt{2}}(i, 1)$ is a good choice. Namely, we notice that $[\beta] = [i : 1]$, which corresponds to $i \in \hat{\mathbb{C}}$. We see that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So we take

$$\gamma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

We then have that $\mathrm{Ad}(\gamma_0)H = i(Y - X)$. We thus see that the functional $\gamma_0(\rho) : \mathfrak{k}_\mathbb{C} \oplus \mathrm{Ad}(\gamma_0)\bar{\mathfrak{n}}_\mathbb{C} \to \mathbb{C}$ is given by $\gamma_0(\rho)(i(Y-X)) = 1$. Accordingly we have that $\chi_{n\gamma_0(\rho)}(\exp(i\varphi(Y - X))) = e^{-in\varphi}$. We notice that this is an extension of the character $\exp(\varphi(X - Y)) \mapsto e^{-in\varphi}$ on $\mathrm{SO}(2)$.

11.2 Realisation in $C(H_+)$

We now want to realize the representations (L, X_n) in the space $C(H_+)$. We notice that we have an embedding $NA \hookrightarrow U$, so we we have a restriction map $\mathrm{res} : \mathcal{O}(U : \overline{B}_\mathbb{C} : \tau_n) \to C(NA)$.

Lemma 11.4. *For all $f \in \mathcal{O}(U : \overline{B}_\mathbb{C} : \tau_n)$ we have that*

$$f(g^{-1}j^{-1}(z)) = \left(\frac{cz + d}{|cz + d|}\right)^n f(j^{-1}(g^{-1}z)), \qquad \textit{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. By the Iwasawa decomposition there is a unique $\kappa(g^{-1}, z) \in K$ such that $g^{-1}j^{-1}(z) = j^{-1}(g^{-1}z)\kappa(g^{-1}, z)$. In particular we have that the Iwasawa decomposition implies that the map $\kappa : G \times H_+ \to K$, given by $(g, z) \mapsto \kappa(g, z)$ is a smooth map. We thus have that

$$f(g^{-1}j^{-1}(z)) = f(j^{-1}(g^{-1}z)\kappa(g^{-1}, z)) = \tau_n(\kappa(g^{-1}, z))^{-1} f(j^{-1}(g^{-1}z)).$$

We compute $\tau_n(\kappa(g^{-1}, z))$. We notice that $\tau_n(k_\varphi) = \tau_1(k_\varphi)^n$, for all $k_\varphi \in K$, so it is sufficient to compute $\tau_1(\kappa(g^{-1}, z))$. Therefore notice that $k_\varphi(ie_1 + e_2) = e^{i\varphi}(ie_1 + e_2) = \tau_1(k_\varphi)^{-1}(ie_1 +$

e_2). We further notice that $n_x{}^t e_2 = e_2$ and $a_t{}^t e_2 = e^{-t} e_2 = a_t{}^{-\rho} e_2$. So we have that $\langle e_2, n_x a_t k_\varphi (ie_1 + e_2)\rangle = a_t{}^{-\rho} \tau_1(k_\varphi)^{-1}$

$$\begin{aligned}\langle e_2, g^{-1} j^{-1}(z)(ie_1 + e_2)\rangle &= \langle e_2, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} (ie_1 + e_2)\rangle \\ &= \langle \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1} x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} (ie_1 + e_2)\rangle = \sqrt{y}^{-1}(cz + d).\end{aligned}$$

We thus conclude that $\tau_1(\kappa(g^{-1}, z))^{-1} = (a_{g^{-1}z})^\rho \sqrt{y}^{-1}(cz + d)$. We notice that

$$(a_{g^{-1}z})^\rho = \sqrt{\mathrm{Im}(g^{-1}z)} = \frac{\sqrt{y}}{|cz + d|},$$

so we conclude that

$$\tau_n(\kappa(g^{-1}, z))^{-1} = \left(\frac{cz + d}{|cz + d|}\right)^n.$$

□

To complete the picture we want to realize the representations .... in the space $\mathcal{O}(H_+)$. Therefore notice that image$((j^{-1})^* \circ \mathrm{res}) = ((j^{-1})^* \sigma_n)\mathcal{O}(H_+)$. It is then readily verified that ... is realized in $\mathcal{O}(H_+)$ by

$$[\pi_n(g) f](z) = (cz + d)^n f(g^{-1}z)$$

11.3 Another realization

For $m \geq 2$ we consider the measure $y^{m-2}\,\mathrm{d}x\,\mathrm{d}y$ on H_+. We now consider the space $H_m := \mathcal{O}(H_+) \cap L^2(H_+, y^{m-2}\,\mathrm{d}x\,\mathrm{d}y)$. For $f \in H_m$, $z \in H_+$, we define

$$D_m^+(g) f(z) = (a - cz)^{-m} f(g^{-1}z).$$

We notice that (D_m^+, H_m) is a representation of $\mathrm{SL}(2, \mathbb{R})$. It turns out that H_m, with the pairing

$$\langle \varphi, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \varphi(x + iy)\overline{\psi(x + iy)} y^{m-2}\,\mathrm{d}x\,\mathrm{d}y,$$

inherited from $L^2(H_+, y^{m-2}\,\mathrm{d}x\,\mathrm{d}y)$, is actually a closed subspace of $L^2(H_+, y^{m-2}\,\mathrm{d}x\,\mathrm{d}y)$ and therefore a Hilbert space.

Lemma 11.5. *The representations* (L, X_{-n}) *and* (D_n^+, H_n) *are equivalent. The map* $T : X_{-n} \to H_n$*, defined as*

$$T(f) = \frac{1}{\sqrt{2}} f\sigma_n^{-1}\big|_{NA} \circ j^{-1}, \qquad (f \in X_{-n}),$$

is a surjective linear isometry that intertwines the representations.

Proof. We notice that

$$2\|T(f)\|_{H_n}^2 = \int_0^\infty \int_{-\infty}^\infty |f(j^{-1}(z))|^2 y^{-n} y^{n-2}\,\mathrm{d}x\,\mathrm{d}y = \int_{NA} |f(na)|^2 j^*(y^{-2}\,\mathrm{d}x\,\mathrm{d}y).$$

We now have that

$$j^*(y^{-2}\,\mathrm{d}x\,\mathrm{d}y) = 2a^{-2\rho}\,\mathrm{d}a\,\mathrm{d}n.$$

So we have that

$$\int_{NA} |f(na)|^2 j^*(y^{-2}\,\mathrm{d}x\,\mathrm{d}y) = 2\int_{NA} |f(na)|^2 a^{-2\rho}\,\mathrm{d}a\,\mathrm{d}n.$$

We thus have that $\|T(f)\|_{H_n} = \|f\|_{X_{-n}}$, and hence we conclude that T is well defined and an isometry. $\square$

Chapter 12

Principal series representation

In the present chapter we will apply the Iwasawa decomposition and the induction process to define the, so called, principal series representations of $\mathrm{SL}(2, \mathbb{R})$. The material in this chapter is drawn from [2] Chapter 20.

Lemma 12.1. *The group MA normalizes N, i.e. $gN = Ng$, for all $g \in MA$, and N is a normal subgroup of P.*

Proof. The result follows from a direct computation and is left to the reader. $\square$

Let (ξ, V_ξ) and (λ, V_λ) finite dimensional representations of M and A respectively. Consider furthermore the representation $(1, \mathbb{C})$ on N, that is trivial. In the view of the Iwasawa decomposition and the above lemma we can define a representation of P on $V_\xi \otimes V_\lambda \otimes \mathbb{C} \simeq V_\xi \otimes V_\lambda$. Indeed, let $m, m' \in M$, $a, a' \in A$ and $n, n' \in N$. Since MA normalizes N, there is an $\tilde{n} \in N$ such that $nm'a' = m'a'\tilde{n}$. We further notice that M commutes with A and thence that the map $M \times A \to MA$ given by $(m, a) \mapsto ma$ a Lie group isomorphism. We thus see that $manm'a'n' = mm'aa'\tilde{n}n'$. So we see that the representation

$$(\xi \otimes \lambda \otimes 1)(man) = \xi(m) \otimes \lambda(a) \qquad ((m, a, n) \in M \times A \times N),$$

in $V_\xi \otimes V_\lambda$ is well defined.

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We have seen that $A = \exp(\mathfrak{a})$, and that $\exp : \mathfrak{a} \to A$ is a Lie group isomorphism. Therefore exp has a smooth inverse which we shall denote by log, for obvious reasons. We define $a^\lambda := e^{\lambda(\log a)}$. It is clear that the map $A \times \mathbb{C}$ defined by $(a, v) \mapsto a^\lambda v$ is a continuous representation. We have

$$(\xi \otimes \lambda \otimes 1)(man) = \xi(m) \otimes \lambda(a) = a^\lambda \xi(m) \qquad ((m, a, n) \in M \times A \times N). \tag{12.1}$$

Definition 12.2. The Principal series representation of $\mathrm{SL}(2, \mathbb{R})$ is the representation

$$\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1), \tag{12.2}$$

where (ξ, V_ξ) is an irreducible and unitary representation of M and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

We notice that $\widehat{M} = \{1, \epsilon\}$, where $\epsilon(-I) = -1$. We also notice that all irreducible representations of $M = \{\pm I\}$ are unitarizible, i.e. there exists a G invariant Hermitian inner product $\langle \cdot, \cdot \rangle_\xi$, namely

$$\langle v, w\rangle_\xi := \langle v, w\rangle + \langle \xi(-I)v, \xi(-I)w\rangle,$$

where $\langle \cdot, \cdot \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the Hermitian inner product $(v, w) \mapsto \bar{w}v$.
This implies that $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ is unitary if λ is a unitary representation of A.

Lemma 12.3 ([2], Lemma 20.3)**.** *The modular function of P is given by*

$$\Delta(man) = a^{2\rho}, \qquad (m \in M,\ a \in A \text{ and } n \in N).$$

It follows from the above lemma that

$$(\xi \otimes \lambda \otimes 1) \otimes \Delta^{1/2} = \xi \otimes (\lambda + \rho) \otimes 1,$$

so that

$$\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1) = \mathrm{ind}_P^G(\xi \otimes (\lambda + \rho) \otimes 1).$$

We shall from now on denote the representation space $C(G : P : (\xi \otimes \lambda \otimes 1) \otimes \Delta^{1/2})$ with $C(P : \xi : \lambda)$

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