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Bachelor Thesis

Dynamic Pricing in the Airline Industry

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Contents

1 Introduction

It is a common occurrence in air transport that two adjacent passengers bought tickets which differ largely in price. Markups and markdowns set by re-sellers only have a small impact on the price differences, whereas large price fluctuations are mainly caused by another phenomenon. It is revenue management what makes airline customers feel like they are in an unpredictable whirlpool of prices. According to Robert Cross, the man hailed by the *Wall Street Journal* as "the guru of revenue management", revenue management can most accurately be described as 'a selling technique of companies to sell the right product to the right customer at the right time for the right price' [1]. Major research fields of revenue management are forecasting, overbooking and pricing. Dynamic pricing (DP) is an area in revenue management (RM) what mainly deals with the price: how to determine the (optimal) price for different periods of time for different customers? The main objective of dynamic pricing is effectively managing a companys reservations inventory and thereby increasing (maximizing, if possible) revenue. Dynamic pricing is a legal practice and although most customers possess little to zero knowledge about the subject, they seem to accept the price fluctuations by considering it as given. The airline industry is an industry in which firms have to face the challenge of effectively managing their reservations inventory on a daily basis. They also compete in an environment of stiff price competition and due to the continuous developments in technology and the presence of *big data*, airlines are forced to use dynamic pricing if they want to avoid bankruptcy.

However, the airline industry is not unique to revenue management nor dynamic pricing in particular. According to Andersen [2], revenue management can be applied to any business if there exist:

- Perishable inventory
- Fixed capacity
- Advance purchase of products
- Dynamic demand
- High fixed costs and relatively low marginal costs of selling one additional unit

If we translate these conditions to the airline industry, we see that a seat on a flight is a typical perishable good and can be priced dynamically indeed [3]. It is therefore no surprise that revenue management has it roots in the airline branch. Dating back to the 70s, the Airline Deregulation Act of 1979 paved the way for airlines to choose their own set of prices instead of fixed prices set by the U.S. Civil Aviation Board. The consequence of this act was that low-cost carriers were formed, offering prices low enough to steal customers of the established airlines. New strategies were required in order for the established airlines to survive. It was American Airline who responded with *DINAMO* (Dynamic Inventory and Maintenance Optimizer), a system providing a flight specific analysis which made it possible to offer prices just as low or even lower than the budget airlines. In 1985, huge investments in *DINAMO* resulted into the introduction of Ultimate Super Saver Fares [4]. These were non-refundable fares that had to be purchased in advance and were capacity controlled. The system assigned these discount fares to only those flights where they had a surplus of empty seats. American Airline became a pioneer in revenue management as the systems net impact was enormous: additional revenues were estimated at 1.4 billion dollars over a three year period, while it also increased the productivity per revenue management- analyst by 30% [5].

As this is a Bachelor's thesis, the mathematical complexity of real-life decision-systems like *DINAMO* simply outgrow the mathematics we will treat here. However, we are able to simplify the pricing problems that real airlines face in such a manner that we can provide a mathematical background of how dynamic pricing works generally. This is done by proposing two model types; a monopoly model and a competitive model. Among the literature most articles concern monopoly problems and not so much the competitive cases. However, the major part of this thesis is devoted to the competitive model and the motivation for treating both types results from the desirability of possessing prior knowledge of the monopoly case.
It does not take much of an effort to conclude the existing literature contains many different approaches, models and objectives. Those who gained interest in a wide range of applications of RM are referred to McGill & Van Ryzin (1999) [6], who provide a clear research overview of the area of revenue management covering research done from 1958 until 1999.

Related literature

The amount of literature on RM and DP in the airline industry specifically took off as soon as the success of *DINAMO* became known. It was Belobaba (1987) [7] who examined the seat inventory control component of airline RM by laying an emphasis on the need for a practical solution of the problem. Until the beginning of the 90s most work focused on capacity management and overbooking while there was only few discussion of dynamic pricing policies available. Most models assumed fixed prices and it were people who were in charge of changing the fares [8].

In the 90s, dynamic pricing became one of the most prolifically investigated areas of revenue management. Generally, in the existing literature, pricing models can be decomposed into the following aspects: the arrival process of customers, the buying process, the optimization problem, assumptions and conditions belonging to the problem, and the solution to the problem. The arrival process of customers is mostly depicted as some type of Poisson point process, in which customers arrive random at a certain rate, and this will be the case in this thesis as well. One approach is to assume that the willingness-to-pay (the price a customer is willing to pay for a ticket) of arrived customers is unknown to the airline. Gallego & Van Ryzin (1994) [9] use this approach to study dynamic pricing under imperfect competition and they control the intensity of demand by changing prices. Their paper is most often referred to in the articles on dynamic pricing. It also acts as the major source for Dolgui & Proth (2009) [10], whose structure we follow in the monopolistic model.

The use of competitive models gradually increased among the literature since the millennial switch. Most articles on dynamic pricing under competition show the use of game theory, treating one airline as the dynamic fare operator and the other as the fixed fare operator. However, more realistically is when both airlines are using dynamic pricing policies, and this is what Li & Peng (2007) [11] investigated. Currie, Cheng & Smith (2008) [12] joined this discussion, while current research about competition rises in complexity as it regularly involves more than two airlines. Gallego & Hu (2014) [13] do this by considering an oligopolistic market with m airlines. Because of the popularity of the field of dynamic pricing, researchers follow each other quickly. The pricing models become increasingly realistic and the field has reached some kind of level of maturity. Partially because of this high level of research, this thesis has become more of an independent analysis of the existing literature rather than a piece of research.

Layout

We will start section 2 by introducing a dynamic pricing problem within the monopolistic environment. In section 3 we extend the problem by discussing a different model concerning competition. Finally, we will review this thesis in section 4.

2 The monopoly system

This section is devoted to a stochastic dynamic pricing model of a monopoly system. A system that is in our case resembled by an air travel market without rivalry between competitor airlines selling tickets for the same flight path. Here, we show a model that is based on Gallego & Van Ryzin (1994) [9], but adopt the mathematical structure and notation of Dolgui & Proth (2009) [10]. Although Dolgui & Proth (2009) [10] use the same approach as Gallego & Van Ryzin (1994) [9], they use more modern notation which benefits both legibility and comprehensibility. Before introducing the model we will make some assumptions for both economic and modeling reasons:

- Assume that the airline is in a market with imperfect competition, i.e., the airline has a monopoly on the product and so is able to influence demand by varying its price.
- There is an unlimited pool of potential customers that can buy a ticket.
- Customers are assumed to buy a ticket if the price is equal to or less than the price they are willing to pay for the ticket.
- Customers do not adjust their purchasing behaviour in response to the selling behaviour of the airline. More precisely, customers only respond to the current price and thus not act strategically. A strategic customer always tries to maximize its utility and that is not the case if they do not adjust their behaviour in response to price fluctuations.
- Tickets are the only product considered, and other selling activities by the airline do not affect ticket sales.
- We assume an unsold seat has no salvage value if the plane departs.

Throughout the section some additional assumptions concerning the mathematical model are made.

2.1 The model

Consider an airline selling N seats on a specific flight over a finite time horizon $[0, T]$. Potential customers arrive in the system at random and buy a ticket if its price is lower or equal to the price they are willing to pay for the ticket, and if it is higher they do not buy. The willingness-to-pay for a ticket depends on the customer. In turn, the airline seeks to maximize its expected revenue.

Potential customers arrive according to a Poisson process. A Poisson process is a common way to describe the stochastic process where a number of events occur with a constant average rate. In our case a Poisson process models demand by generating arrivals with rate or intensity λ, and because we have a monopoly, λ can be interpreted as the market demand. The airline controls this intensity by setting a price p over an infinitesimal time interval δ. During this time interval, the intensity λ will generate exactly one potential customer with probability $\lambda\delta$ and none with probability $1-\lambda\delta$. The probability of more than potential customer arriving during the time interval δ is $o(\delta)$ and is negligible small if δ becomes infinitely small. See Gallego & Van Ryzin (1994) [9] for this last claim.

In order to describe the process where a potential customer will buy a ticket or not, we denote the probability density function (pdf) as $f(p)$, which represents the probability that a customer is willing to pay the price p. Assume that $f(p)$ is non-increasing in price p, because of the obvious principle that the higher the price p of a product, the smaller the probability a customer is willing to pay p for it. Therefore, we can state the probability to buy a ticket at price p as follows:

$$P(p) = \int_{u=p}^{\infty} f(u)\, du = 1 - \int_{u=0}^{p} f(u)\, du = 1 - F(p)$$

Here, $F(p)$ is the distribution function of the price and also the probability to not buy a ticket at price p.

Also, it holds that:

- $\lim_{p\to\infty} P(p) = 0$
- $P(0) = 1$

Notice that the first bullet implies that when the price becomes infinitely large, the probability a customer is willing to pay that amount reaches 0. In reality, this price might of course be a real number. The second bullet shows that any potential customer becomes an actual customer, when ticket prices are equal to 0. Meanwhile, the airline tries to maximize expected revenue over the finite time horizon $[0, T]$ by selling as much seats as possible.

Definition 2.1.1. *Let $V(t, n)$ be the maximum expected revenue that can be earned by time T from n seats available at time t, with $t \in [0, T]$ and $n \in \{1, 2, ..., N\}$.*

Assumption 1. (Concavity and differentiability) *The maximum expected revenue function $V(t, n)$ is concave and continuously differentiable function with respect to t.*

Furthermore, we have

$$V(t, 0) = 0, \quad \forall t \in [0, T] \tag{1}$$

and

$$V(T, n) = 0, \quad \forall n \in \{1, 2, ..., N\} \tag{2}$$

respectively meaning that when the flight is sold out we do not generate any further revenue and when time is running out there is indeed no salvage value.

From now on, we will say that any pair (t, n), with $t \in [0, T]$ and $n \in \{1, 2, ..., N\}$, is a *state* under which the maximal expected revenue is determined. After any infinitely small increment of time δ the state corresponding to time t evolves to a new state. When this happens, there are three possible situations regarding to the expected revenue:

- No potential customer arrives in the time interval $[t, t + \delta)$. The probability of this happening is equal to $1 - \lambda\delta$, while the expected revenue under the new state $(t + \delta, n)$ is given by $V(t + \delta, n)$.
- A potential customer arrives in the time interval $[t, t + \delta)$, but does not buy a ticket. The first event has a probability of $\lambda\delta$ to occur, while the latter occurs

with a probability of $F(p)$. Therefore the joint probability is $\lambda\delta F(p)$ and the expected revenue under the new state $(t+\delta, n)$ is still given by $V(t+\delta, n)$.

- A potential customer arrives in the time interval $[t, t+\delta)$ and decides to buy a ticket. The joint probability of these two events to occur is $\lambda\delta P(p) = \lambda\delta[1-F(p)]$ and the expected revenue under the new state $(t+\delta, n-1)$ is given by $V(t+\delta, n-1) + p$ for $n \geq 1$, where p is the price of the ticket.

Because $V(t,n)$ is the maximum expected revenue corresponding to the state (t,n), we let $p^*(t,n)$ be the optimal price of one ticket at time t with inventory level n. Then, at time $t+\delta$ the maximum expected revenue can take on two possible values for $n \geq 1$: $V(t+\delta, n)$ with probability $1-\lambda\delta[1-F(p)]$ or $V(t+\delta, n-1)$ with probability $\lambda\delta[1-F(p)]$.

In the latter case, the airline has actually sold a ticket during the the infinitesimal time interval δ, therefore the value at time $t+\delta$ becomes p^* plus the maximum expected revenue for the remaining period $V(t+\delta, n-1)$. Now we are able to express the balance of maximum expected revenue for $n \geq 1$:

$$V(t,n) = [1 - \lambda\delta[1-F(p^*)]]V(t+\delta, n) + \lambda\delta[1-F(p^*)][V(t+\delta, n-1) + p^*]$$

The airline wants to maximize the right-hand-side of the equation, thus we have:

$$V(t,n) = \max_{p\geq 0}[V(t+\delta,n) - \lambda\delta[1-F(p)]V(t+\delta,n) + \delta\lambda[1-F(p)][V(t+\delta,n-1)+p]]$$

Then rewrite to obtain:

$$-\frac{V(t+\delta,n)-V(t,n)}{\delta} = \lambda\max_{p\geq 0}[-[1-F(p)]V(t+\delta,n)+[1-F(p)][V(t+\delta,n-1)+p]]$$

By basic calculus we know that if we let $\delta \to 0$ the LHS of the above equation is equivalent to the negative partial derivative with respect to t. Thus, we get:

$$\frac{\partial V(t,n)}{\partial t} = -\lambda\min_{p\geq 0}[[1-F(p)]V(t,n) - [1-F(p)][V(t,n-1)+p]] \tag{3}$$

for $n \geq 1$.

Assumption 2. *The probability density function is $f(p) = \mu e^{-\mu p}$ and so the distribution function of the price p is given by $F(p) = 1 - e^{-\mu p}$, with $\mu > 0$.*

Above assumption is made due to the fact that we want to analyze how the customer's behaviour, that is the willingness to pay a price p, affects the solution of the problem. This is not possible when we use a general function $F(p)$. Note that $f(p) = \mu e^{-\mu p}$ is indeed decreasing in p as required. Thus, when we substitute in equation (3), we get

$$\frac{\partial V(t,n)}{\partial t} = -\lambda \min_{p \geq 0} [e^{-\mu p}[V(t,n) - V(t,n-1) - p]] \tag{4}$$

for $n \geq 1$.

2.2 Solution

In order to maximize its expected revenue, the airline has to find the optimal price p^*, for each state (t,n), and does so by solving the minimization problem with regard to p within equation (4). That means setting the partial derivative of the second member of (4) equal to 0:

$$e^{-\mu p} \times [-\mu V(t,n) + \mu V(t,n-1) + \mu p - 1] = 0 \tag{5}$$

The solution of (5) is:

$$p^*(t,n) = V(t,n) - V(t.n-1) + \frac{1}{\mu} \tag{6}$$

for $n \geq 1$.
By substituting p for $p^*(t,n)$ inside equation (4) we derive:

$$\frac{\partial V(t,n)}{\partial t} = -\frac{\lambda}{\mu} e^{-\mu[V(t,n) - V(t,n-1) + \frac{1}{\mu}]} \tag{7}$$

Equation (7) is a differential equation that only holds for $n \geq 1$. Since we know that $V(t,0) = 0$ for all $t \in [0,T]$, it also holds that $\frac{\partial V(t,0)}{\partial t} = 0$ for all $t \in [0,T]$. For n=1, we get the following differential equation:

$$\frac{\partial V(t,1)}{\partial t} = -\frac{\lambda}{\mu} e^{-\mu[V(t,1) + \frac{1}{\mu}]}$$

The above differential equation is valid. However, solving a differential equation such as (7) algebraically is no easy task, if not impossible in some cases. As we will not be actually engaging in dynamic programming, we state an explicit solution through the following lemma.

Lemma 2.2.1. *The solution*

$$V(t,n) = \frac{1}{\mu}\ln\left(\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}\right) \qquad (*)$$

satisfies the differential equation (7) for $n \geq 1$.

Proof. Note that $(*)$ can be rewritten such that $e^{\mu V(t,n)} = \sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}$ $(\forall n)$, and likewise $e^{\mu V(t,n-1)} = \sum_{i=0}^{n-1}\frac{\lambda^i e^{-i}(T-t)^i}{i!}$ $(\forall n)$.
Thus, the RHS of (7) has the form

$$\begin{aligned}\frac{\lambda}{\mu}e^{-\mu[V(t,n)-V(t,n-1)+\frac{1}{\mu}]} &= \frac{\lambda}{\mu}e^{-\mu[V(t,n)-V(t,n-1)]-1} \\ &= \frac{\lambda}{\mu}e^{-1}\frac{\sum_{i=0}^{n-1}\frac{\lambda^i e^{-i}(T-t)^i}{i!}}{\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}}\end{aligned}$$

In case of the LHS of (7) we get

$$\begin{aligned}\frac{\partial V(t,n)}{\partial t} &= \frac{-1}{\mu}\frac{\sum_{i=1}^{n}\frac{\lambda^i e^{-i}(T-t)^{i-1}(-1)i}{i!}}{\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}} \\ &= \frac{1}{\mu}\frac{\sum_{i=1}^{n}\frac{\lambda^i e^{-i}(T-t)^{i-1}}{(i-1)!}}{\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}} \\ &= \frac{1}{\mu}\lambda e^{-1}\frac{\sum_{i=1}^{n}\frac{\lambda^{i-1}e^{-(i-1)}(T-t)^{i-1}}{(i-1)!}}{\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}} \\ &= \frac{\lambda}{\mu}e^{-1}\frac{\sum_{i=0}^{n-1}\frac{\lambda^i e^{-i}(T-t)^i}{i!}}{\sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}}\end{aligned}$$

Therefore the LHS equals RHS and $(*)$ indeed satisfies (7). □

Remember that the optimal price $p^*(t,n)$ depends on the maximum expected revenue associated with the state (t,n). By substituting $V(t,n)$ for the solution stated in Lemma 2.2.1., the formula for $p^*(t,n)$ becomes rather cumbersome to read. Therefore we let

$$K(t,n) = \sum_{i=0}^{n}\frac{\lambda^i e^{-i}(T-t)^i}{i!}$$

such that the solution becomes

$$V(t,n) \;=\; \frac{1}{\mu}\ln[K(t,n)] \tag{8}$$

and we substitute in equation (6) to get

$$p^*(t,n) = \frac{1}{\mu}[\ln\left(\frac{K(t,n)}{K(t,n-1)}\right) + 1] \tag{9}$$

for $n \geq 1$.

Thus, eventually, the airline can find the optimal price for any state (t,n) by determining the value of the recursively established term $\frac{K(t,n)}{K(t,n-1)}$ such that it maximizes expected revenue over the selling season $[0,T]$.

3 Dynamic pricing under competition

So far we have only considered dynamic pricing in a non competitive environment. In practice however, the airline industry is by no means monopolistic. In order to pursuit realism, this section treats a dynamic pricing model under competition. The objective remains the same, i.e., each airline seeks to maximize its revenue. The upcoming model is based on Li & Peng (2007) [8] and throughout this section we will follow their mathematical structure. We consider a continuous-time dynamic pricing model for two competitive flights, which uses stochastic control theory as well as game theory. Because its approach also assumes customer arrivals are generated by a Poisson process, it is a suitable extension of the monopolistic model. A key difference with the monopolistic model is that here only two prices levels are considered. The motivation for this contraction predominantly lies in the fact that we want to analyze optimal policies under effect of competition without making the game more complex than necessary. Another important feature of the competitive model is that we assume arriving customers will certainly buy a seat, either at airline 1 or at airline 2. This seems reasonable, because in the monopoly case customers did not have an alternative while in this case the customer can switch to the competitor for a cheaper ticket. Indeed, this results into demand being dependent on both ticket prices. Eventually, we discuss the sufficient optimality conditions for the model to support the optimal solutions.

3.1 Competitive model

Suppose two airlines operate a flight simultaneously from the same origin to the same destination with departure time T. This allows both airlines to sell tickets within the time horizon $[0, T]$. At $t = 0$, the number of available seats for flight 1 operated by airline 1 and flight 2 operated by airline 2 are given by N_1 and N_2 respectively. It is assumed that unsold seats have no salvage value at $t = T$. The airlines offer only high fare and low fare tickets, therefore the set of allowable prices for flight k $(k = 1, 2)$ is given by $P_k = \{\, p_i^k \mid p_1^k > p_2^k\}$ with $i = 1, 2$ representing the fare class. A Poisson process models the customer arrival rate with intensity $\lambda(t), t \in [0, T]$. We assume that demand of a flight is affected by the price of both airlines, thus demand is correlated. Taking this into account, we denote $\lambda_{i,j}^k(t)$ as the demand intensity of fare class $i = 1, 2$ for flight k at time t when price of flight 1 is p_i^1 $(i = 1, 2)$ and price of flight 2 is p_j^2 $(j = 1, 2)$. In contrast to the model in section 2 there is complete information; both airlines are aware of each others actual price as well as the number of unsold seats. Also, customers have an incentive to switch airlines if the rival airline offers a lower price. Therefore, we have

$$\lambda_{i,1}^1(t) > \lambda_{i,2}^1(t) \tag{10}$$

$$\lambda_{1,j}^2(t) > \lambda_{2,j}^2(t) \tag{11}$$

Assumption 1. *Revenue is a decreasing function in price, so that*

$$\lambda_{1,j}^1 p_1^1 < \lambda_{2,j}^1 p_2^1 \quad (j = 1, 2) \tag{12}$$

$$\lambda_{i,1}^2 p_1^2 < \lambda_{i,2}^2 p_2^2 \quad (i = 1, 2) \tag{13}$$

Also, the game is non-cooperative.

Inequalities (12) and (13) show that if an airline prices at high fare, then revenue is lower than it would be at low fare, regardless of what price the competitive airline uses. This would imply that in a cooperative game, which this is not, the airlines would cooperate by setting their prices at low fare from the start. If assumed otherwise, no airline will ever drop its price voluntarily.
For a random t in the sales season $[0, T]$, when remaining seats of both airlines are (n_1, n_2), airline k looks to maximize its total expected revenue by determining the price $p^k(s, n_1, n_2)$ at any point s in the remaining time horizon $[t, T]$. We can define the price of airline k under the state (s, n_1, n_2) as follows:

$$p^k(s, n_1, n_2) = \begin{cases} p_1^k & \text{if high fare at time } s \\ p_2^k & \text{if low fare at time } s \end{cases}$$

where $s \in [t, T]$.

The price determined by an airline therefore depends on the remaining sales time and the number of unsold seats, as well as the price set by the competitor airline. Intuitively one would argue that as unfilled capacity grows, the airline has an incentive to lower its price.

Assumption 2. *At any given time s, the optimal price of flight k decreases in the number of unsold seats (n_1, n_2).*

An alternative way to formulate this assumption is to say that when the remaining seats (n_1, n_2) of both flights are fixed, then the optimal price of each flight is decreased with time s. By interpreting it in this way, it is clear that the airline has an incentive to move from high fare to low fare at a certain moment in time, given that no additional seats are sold since time t. Thus, consider a random decision time t in the sales horizon $[0, T]$ when remaining seats are (n_1, n_2). In this state both airlines respectively decide their switching points c^1_{t,n_1,n_2} and c^2_{t,n_1,n_2}. That means before c^k_{t,n_1,n_2} airline k will price at high fare and it will price at low fare after c^k_{t,n_1,n_2}. Despite that revenue is expected to be higher if an airline prices at low fare (Assumption 2), both airlines start to price at high fare, otherwise there is no opportunity to draw more customers by switching to a lower fare if no additional seats are sold. In reality, however, this exact mechanism probably won't apply. Nowadays, when a flight's selling season starts, tickets are offered relatively cheap, then after a period of time the airlines start to increase their prices. During the period of high fares, the airline expects that on average most customers arrive and therefore it will try to maintain this price-level. It is most likely that after a while, when time T is relatively close, there is still a number of unsold seats left. This moment can be described as flight k's switching point c^k_{t,n_1,n_2}. In our model, airlines can maintain high price and take the risk of unfilled capacity or they set low fares hoping the flight leaves fully-packed by moving the switching point on the time horizon. Therefore, indirectly, these switching points can be interpreted as the policies set by the airlines. That means if no additional seat has been sold, the model describes the behaviour of a last-minute buying process.

Definition 3.1.1. *Let c^k_{t,n_1,n_2} be a point in time where a flight k switches its price from high to low fare. The position of the switching point is established at decision time t with remaining capacities (n_1, n_2).*

Therefore, the optimal price at time s, determined at time t through c^k_{t,n_1,n_2}, is given by

$$p^k_t(s,n_1,n_2) = \begin{cases} p^k_1 & t \leq s \leq c^k_{t,n_1,n_2} \\ p^k_2 & c^k_{t,n_1,n_2} < s \leq T \end{cases}$$

where $k \in 1, 2$.

Unless a new customer arrives at time s and buys a ticket, the airlines will stick to this policy. However, if a customer indeed buys a ticket at time s, we are in the next turn of the game and both airlines will have to update their inventory state (n_1, n_2) in order to decide new switching points. Prices will then restore to high fares. This process continues until the moment both airlines are out of stock, that is $(n_1, n_2) = (0, 0)$, or when both planes leave the ground, that is $t = T$.

Lemma 3.1.2. *Customers arrive according to a non-homogeneous Poisson process with an intensity λ. Then the inter-arrival times are independent and obey the exponential distribution with parameter λ.*

Although it is obvious, a proof of Lemma 3.1.1. can be found in Appendix A.1. Switching between high- and low fare means changing demand and thus switching parameters. Therefore the switching points c^k_{t,n_1,n_2} partitions the time horizon $[t, T]$. Assume the intensity is λ_1 at time interval $[t, c]$ and λ_2 at time interval $[c, T]$. Then, since the last customer arrival time t or from the start of the selling season, when $t = 0$, the next customer arrival time s follows the exponential distribution with pdf:

$$f_c(s) = \begin{cases} \lambda_1 e^{-\lambda_1(s-t)} & t \leq s \leq c \\ \lambda_2 e^{-\lambda_2(s-c)-\lambda_1(c-t)} & c \leq s \leq T \end{cases}$$

(*See Appendix A.2 for explanation*)

Since we have a symmetric game, we can assume $c^1_{t,n_1,n_2} \leq c^2_{t,n_1,n_2}$. This means the next customer arrival time s can lie in one of the intervals $[t, c^1_{t,n_1,n_2}]$, $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$ or $[c^2_{t,n_1,n_2}, T]$. In the interval $[t, c^1_{t,n_1,n_2}]$ both airlines are still using high fare. In $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$ airline 1 has switched to low fare, while airline 2 is still using high fare. The next customer arrival time s can also lie in $[c^2_{t,n_1,n_2}, T]$, where both airlines have switched to low fare. According to Lemma 3.1.1, from time t, flight k

will sell one seat at time s according to the following pdf:

$$f^k_{c^1_{t,n_1,n_2},c^2_{t,n_1,n_2}}(s) = \begin{cases} \lambda^k_{1,1} e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)} & t \leq s \leq c^1_{t,n_1,n_2} \\ \lambda^k_{2,1} e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})} \times \\ e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} & c^1_{t,n_1,n_2} \leq s \leq c^2_{t,n_1,n_2} \\ \lambda^k_{2,2} e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})} \times \\ e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} & c^2_{t,n_1,n_2} \leq s \leq T \end{cases} \tag{14}$$

(*See Appendix A.3 for additional explanation*)

Definition 3.1.3. *Let $V_k(t, n_1, n_2)$ be the expected revenue that can be earned by time T from respectively n_1 and n_2 seats available at time t, with $t \in [0, T]$ and $n_1 \in 1, 2, ..., N_1$, $n_2 \in, 1, 2, ..., N_2$.*

Every time an airline sells a ticket, the expected revenue for the upcoming period changes for both airlines. Suppose flight 1 sells a seat under the state (s, n_1, n_2), then it gathers either p^1_1 or p^1_2, depending on the position of c^k_{t,n_1,n_2}. The state changes to $(s, n_1 - 1, n_2)$, hence the expected revenue for the remaining period is $V_1(s, n_1 - 1, n_2)$. Therefore the total expected revenue of flight 1 becomes $p^1_i + V_1(s, n_1 - 1, n_2)$, and flight 2 becomes $V_2(s, n_1 - 1, n_2)$. The equivalent applies to flight 2 when it sells a seat at p^2_j; the expected revenue of flight 2 is $p^2_j + V_2(s, n_1, n_2 - 1)$ and flight 1 is $V_1(s, n_1, n_2 - 1)$.

The expected revenue for flight 1 can be expressed as follows:

$$\begin{aligned} V_1(t, n_1, n_2) = & \int_t^{c^1_{t,n_1,n_2}} [(V_1(s, n_1 - 1, n_2) + p^1_1)\lambda^1_{1,1} + V_1(s, n_1, n_2 - 1)\lambda^2_{1,1}] \times \\ & e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)} ds + \\ & \int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}} [(V_1(s, n_1 - 1, n_2) + p^1_2)\lambda^1_{2,1} + V_1(s, n_1, n_2 - 1)\lambda^2_{2,1}] \times \\ & e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} ds + \\ & \int_{c^2_{t,n_1,n_2}}^{T} [(V_1(s, n_1 - 1, n_2) + p^1_2)\lambda^1_{2,2} + V_1(s, n_1, n_2 - 1)\lambda^2_{2,2}] \times \\ & e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})} \times \\ & e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} ds \end{aligned} \tag{15}$$

The expected revenue for flight 2 is:

$$
\begin{aligned}
V_2(t,n_1,n_2) = &\int_t^{c^1_{t,n_1,n_2}} [(V_2(s,n_1,n_2-1)+p^2_1)\lambda^2_{1,1} + V_2(s,n_1-1,n_2)\lambda^1_{1,1}]\times \\
& e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)} ds + \\
& \int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}} [(V_2(s,n_1,n_2-1)+p^2_1)\lambda^2_{2,1} + V_2(s,n_1-1,n_2)\lambda^1_{2,1}]\times \\
& e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} ds + \\
& \int_{c^2_{t,n_1,n_2}}^{T} [(V_2(s,n_1,n_2-1)+p^2_2)\lambda^2_{2,2} + V_2(s,n_1-1,n_2)\lambda^1_{2,2}]\times \\
& e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})}\times \\
& e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} ds
\end{aligned}
\tag{16}
$$

Above definitions may need some explanation. If needed, first see Appendix A.2 to remember how all probabilities are established. Now, observe the first integral of the expected revenue for flight 1. It consists of the expected revenue values under corresponding state multiplied by all possible probabilities of what might happen during the interval $[t, c^1_{t,n_1,n_2}]$. For example: the term $(V_1(s,n_1-1,n_2)+p^1_1)$ is the expected value at time s if flight 1 sold a seat while flight 2 has not during the interval $[t, c^1_{t,n_1,n_2}]$. This value is multiplied by the probability of this actually happening, which is $\lambda^1_{1,1}e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)}$. On the other hand, the term $V_1(s,n_1,n_2-1)$ represents the expected value at time s when flight 2 sold a seat while flight 1 has not during the same interval. This value is also multiplied by the probability of this happening, which is $\lambda^2_{1,1}e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)}$. Consequently, the integral is taken of the average expected value at time s over the period $[t, c^1_{t,n_1,n_2}]$. The same procedure is executed with the other two integrals corresponding to the intervals $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$ and $[c^2_{t,n_1,n_2}, T]$. Because of the game's symmetry, the same reasoning applies to the expected revenue of flight 2. Now that we have expressed the expected revenue functions for both flights, we seek to maximize their values by determining equilibrium policies.

3.2 Equilibrium

Earlier we have shown that the switching moments c^1_{t,n_1,n_2} and c^2_{t,n_1,n_2} respectively act as the policies set by the two airlines. Expected revenue of an airline k ($k = 1, 2$) is affected by its own policy as well as the competitor's policy. In game theory, an equilibrium policy consists of a pair of those policies which are optimal policies for each other under the given policy of its competitor. An optimal policy is a policy that generates maximal expected revenue.

Assumption 3. (Concavity and differentiability) *The expected revenue function $V_k(t, n_1, n_2)$ is a concave function and continuously differentiable with respect to t.*

The optimal policy for airline 1, given the competitor's policy is c^2_{t,n_1,n_2}, is found by taking the partial derivative of $V_1(t, n_1, n_2)$ in its own policy c^1_{t,n_1,n_2} and setting it to 0. Therefore, given the competitor's policy is c^2_{t,n_1,n_2}, the partial derivative of $V_1(t, n_1, n_2)$ in c^1_{t,n_1,n_2} is

$$
\begin{aligned}
\frac{\partial V_1(t,n_1,n_2)}{\partial c^1_{t,n_1,n_2}} =& [V_1(c^1_{t,n_1,n_2}, n_1-1, n_2)(\lambda^1_{1,1} - \lambda^1_{2,1}) + p^1_1\lambda^1_{1,1} - p^1_2\lambda^1_{2,1} + \\
& V_1(c^1_{t,n_1,n_2}, n_1, n_2-1)(\lambda^2_{1,1} - \lambda^2_{2,1})] \times e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} + \\
& (\lambda^1_{2,1} + \lambda^2_{2,1} - \lambda^1_{1,1} - \lambda^2_{1,1}) \int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}} [(V_1(s, n_1-1, n_2) + p^1_2)\lambda^1_{2,1} + \\
& V_1(s, n_1, n_2-1)\lambda^2_{2,1}] \times e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} ds + \\
& (\lambda^1_{2,1} + \lambda^2_{2,1} - \lambda^1_{1,1} - \lambda^2_{1,1}) e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})} \times \\
& e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)} \times \int_{c^2_{t,n_1,n_2}}^{T} [(V_1(s, n_1-1, n_2) + p^1_2)\lambda^1_{2,2} + \\
& V_1(s, n_1, n_2-1)\lambda^2_{2,2}] e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})} ds
\end{aligned}
\tag{17}
$$

Similarly, given the competitor's policy is c^1_{t,n_1,n_2}, the partial derivative of $V_2(t, n_1, n_2)$ in c^2_{t,n_1,n_2} is

$$\begin{aligned}\frac{\partial V_2(t,n_1,n_2)}{\partial c^2_{t,n_1,n_2}} =&[V_2(c^2_{t,n_1,n_2},n_1,n_2-1)(\lambda^2_{2,1}-\lambda^2_{2,2})+p^2_1\lambda^2_{2,1}-p^2_2\lambda^2_{2,2}+\\ &V_2(c^2_{t,n_1,n_2},n_1-1,n_2)(\lambda^1_{2,1}-\lambda^1_{2,2})]\times e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})}\times\\ &e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}+(\lambda^1_{2,2}+\lambda^2_{2,2}-\lambda^1_{2,1}-\lambda^2_{2,1})\times\\ &\int_{c^2_{t,n_1,n_2}}^T[(V_2(s,n_1,n_2-1)+p^2_2)\lambda^2_{2,2}+V_2(s,n_1-1,n_2)\lambda^1_{2,2}]\times\\ &e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}ds\end{aligned} \tag{18}$$

(*See Appendix B.1 on how (17) and (18) are established*)

By the symmetry of the game the equilibrium policy $({c^1_{t,n_1,n_2}}^*, {c^2_{t,n_1,n_2}}^*)$ is given by the pair of policies that satisfies the following two equations:

$$\frac{\partial V_1(t,n_1,n_2)}{\partial c^1_{t,n_1,n_2}}=0 \tag{19}$$

and

$$\frac{\partial V_2(t,n_1,n_2)}{\partial c^2_{t,n_1,n_2}}=0 \tag{20}$$

Hence, the solution of (19) and (20) is an extreme value point. By assumption 3. we know that $V_k(t,n_1,n_2)$ is concave, which implies that second partial derivatives $\frac{\partial^2 V_k(t,n_1,n_2)}{\partial^2 c^k_{t,n_1,n_2}}<0$ and thus the solution is a maximum. Notice that basic calculus tells us that endpoints of a closed interval may behave as extreme value points as well. Therefore expected revenue will take on its maximum value if equilibrium policy $({c^1_{t,n_1,n_2}}^*, {c^2_{t,n_1,n_2}}^*)$ is the extreme value point which satisfies (19) and (20) or is at the endpoints. The following propositions concern a policy $(c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2})$.

Proposition 3.2.1. *Decision time t does not affect the position of equilibrium policy, that is, for $t,t'\in[0,T]$, if $({c^1_{t,n_1,n_2}}^*, {c^1_{t,n_1,n_2}}^*)$ is the equilibrium policy at time t, then at time t' $({c^1_{t,n_1,n_2}}^*, {c^1_{t,n_1,n_2}}^*)$ is still the equilibrium policy that makes both airlines gather maximal expected revenue in the remaining time interval $[t',T]$.*

Proof. Notice that in the partial derivative equations (17) and (18) at time t the only term affected by t is

$$e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}$$

and it always holds that $e^{f(t)} > 0$. Therefore, time t will not change the sign of the value of the partial derivative at time t. If $({c^1_{t,n_1,n_2}}^*, {c^2_{t,n_1,n_2}}^*)$ is the equilibrium policy, then optimality conditions (19) and (20) imply that the other terms are equal to 0. At time $t^{'}$ we still have that these terms are 0 as $e^{f(t^{'})} > 0$, therefore $({c^1_{t,n_1,n_2}}^*, {c^2_{t,n_1,n_2}}^*)$ still satisfies (19) and (20) and is still the equilibrium policy. $\square$

Proposition 3.2.2. *For a given policy* $(c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2})$*, expected revenue is decreasing in* t*.*

Proof. Assume that the optimal price of two flights are p^1_i and p^2_j respectively. Then by Proposition 3.2.1, for a sufficiently small positive δ, the optimal prices are still p^1_i and p^2_j at time $t-\delta$. For the time interval $[t-\delta, t]$ we can distinguish two cases: either a customer arrives and buys a ticket or no customer arrives at all. The expected revenue function of flight 1 at time $t-\delta$ can be formulated as follows:

$$\begin{aligned} V_1(t-\delta, n_1, n_2) = &\int_{t-\delta}^{t} [(V_1(s, n_1-1, n_2) + p^1_i)\lambda^1_{i,j} + V_1(s, n_1, n_2-1)\lambda^2_{i,j}] \times \\ & e^{-(\lambda^1_{i,j}+\lambda^2_{i,j})(s-t+\delta)} ds + V_1(t, n_1, n_2) e^{-(\lambda^1_{i,j}+\lambda^2_{i,j})\delta} \end{aligned}$$

Notice that the above expected revenue function at time $t-\delta$ consists of two terms, one with and one without an integral. The integral represents expected revenue if one seat is sold at time s by either one of the airlines during $[t-\delta, t]$, while the second term represents the expected revenue if both airlines have not sold a seat during $[t-\delta, t]$. This means the expected revenue at time $t-\delta$ is equal to that at time t and it is multiplied by the probability of both airlines not selling during $[t-\delta, t]$. It is unknown where $[t-\delta, t]$ is located in $[0, T]$ compared to the switching points, therefore the fare classes of the price and demand intensities are not defined. Now, if we let $\delta \to 0$, then we get by approximation:

$$\begin{aligned} V_1(t-\delta, n_1, n_2) \approx\ & [(V_1(t, n_1-1, n_2) + p^1_i)\lambda^1_{i,j} + V_1(t, n_1, n_2-1)\lambda^2_{i,j}] \times \\ & e^{-(\lambda^1_{i,j}+\lambda^2_{i,j})\delta} \times \delta + V_1(t, n_1, n_2) e^{-(\lambda^1_{i,j}+\lambda^2_{i,j})\delta} \qquad (21) \\ \geq\ & V_1(t, n_1, n_2) \end{aligned}$$

(*See Appendix B.2 on how to approximate integral term in order to derive (21)*)
Symmetrically, we get

$$\begin{aligned} V_2(t-\delta, n_1, n_2) &= \int_{t-\delta}^{t} [(V_2(s, n_1, n_2-1) + p_i^2)\lambda_{i,j}^2 + V_2(s, n_1-1, n_2)\lambda_{i,j}^1] \times \\ &\quad e^{-(\lambda_{i,j}^1+\lambda_{i,j}^2)(s-t+\delta)} ds \; + \; V_2(t, n_1, n_2) e^{-(\lambda_{i,j}^1+\lambda_{i,j}^2)\delta} \\ &\approx [(V_2(t, n_1, n_2-1) + p_i^2)\lambda_{i,j}^2 + V_2(t, n_1-1, n_2)\lambda_{i,j}^1] \times \\ &\quad e^{-(\lambda_{i,j}^1+\lambda_{i,j}^2)\delta} \times \delta + V_2(t, n_1, n_2) e^{-(\lambda_{i,j}^1+\lambda_{i,j}^2)\delta} \\ &\geq \; V_2(t, n_1, n_2) \end{aligned} \tag{22}$$

□

3.3 Solution

In order to solve the model, some boundary conditions have to be met. This model has the following:

(1) There is no salvage value at time $t = T$. In mathematical terms,

$$V_k(T, n_1, n_2) = 0 \qquad (k = 1, 2) \tag{23}$$

(2) Flights that have reached capacity do not gather additional revenue, thus

$$V_1(t, 0, n_2) = 0 \tag{24}$$

$$V_2(t, n_1, 0) = 0 \tag{25}$$

(3) If one of the flights has reached capacity, then the expected revenue of the other flight equals that of a flight acting in a monopolistic environment. Thus, we have

$$V_1(t, n_1, 0) = V_1^{monopolistic}(t, n_1) \tag{26}$$

$$V_2(t, 0, n_2) = V_2^{monopolistic}(t, n_2) \tag{27}$$

Monopoly case
The above stated $V_k^{monopolistic}(t, n_k)$ represents the value under the state (t, n_k) in a monopoly market. By equations (26) and (27) we mean that when a competitor airline has no seats left to sell, the model should be solved in a monopoly situation. Notice that in this case demand no longer depends on the competitor's price. Furthermore, if we follow the framework of this model, it makes no sense that in monopoly the airline would ever price at high fare (see Assumption 1.), because an arriving customer is guaranteed to make a purchase at the remaining airline and therefore there exist no switching point c^k_{t,n_k}. The expected revenue of the airline in this case would be *the expected number of sold seats* $\times$ *low fare*. Thus, in monopoly an airline will adopt a fixed price policy instead of a dynamic price policy. Remember that in section 2 we assumed that an arriving customer does not always buy depending on its willingness-to-pay and although this is more realistic, the solution of that model would not be directly applicable here. Therefore, see Appendix C on how to determine $V_k^{monopolistic}(t, n_k)$ within the framework of this section. We derived the following,

$$V_k^{monopolistic}(t, n_k) = \min\left(\lambda_2^k(T - t), n_k\right) \times p_2^k \tag{28}$$

Recursive procedure
The expected revenue functions are recursive formulas and therefore the model can be solved by undergoing a recursive process. We use a bottom-up approach, that is building up a complex system by combining individual problems:

- Let $n_1 = 1, n_2 = 1$. Consider inventory levels $(1, 0)$ and $(0, 1)$; using boundary condition (3) and a solution derived from the monopoly market to determine the values of $V_1(t, 1, 0) = V_1^{monopolistic}(t, 1)$ and $V_2(t, 0, 1) = V_2^{monopolistic}(t, 1)$.

- Next step is using equations (15) and (16), according to the inventory level $(1, 1)$, in order to determine the values of $V_1(t, 1, 1)$ and $V_2(t, 1, 1)$. Using boundary condition (2), within (15) and (16), we get that $V_1(s, 0, 1) = V_2(s, 1, 0) = 0$ and the other terms needed are known already. Therefore, $V_1(t, 1, 1)$ and $V_2(t, 1, 1)$ can be solved.

- Now check that $V_1(t, 2, 1)$ and $V_2(t, 1, 2)$ can be solved using equations (15) and (16) again, because all terms are already evaluated. Also from boundary condition (3) we know that $V_1(t, n_1, 0) = V_1^{monopolistic}(t, n_1)$ and $V_2(t, 0, n_2) = V_2^{monopolistic}(t, n_2)$.

- Therefore, for a given policy $(c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2})$, proceed this method to derive all values of $V_1(t, i, j)$ and $V_2(t, i, j)$ for all $i \leq N_1$ and $j \leq N_2$.

4 Conclusion

In this thesis, we have taken a closer look at dynamic pricing in the airline industry. The goal of this thesis was to introduce the reader to the mathematical background of dynamic pricing problems. By analyzing the existing literature and concluding there was a clear line between articles treating monopolistic models and those which treat competition, there was an incentive to show both situations. Therefore, we followed two specific articles, both discussing problems from the airline point of view with an objective of maximizing expected revenue. In Li & Peng (2007) [11] we discovered a few errors in determining the partial derivatives (17) and (18), therefore we derive different optimal policies, although the principles remain the same.

In the first model we assumed the airline acts in a market with imperfect competition and so is able to influence demand by varying its price. The set of allowable prices chosen by the airline has continuous values, but in reality airlines use discrete prices. Next to the assumption that potential customers arrive according to a Poisson process with an intensity λ, we also assumed a potential customer is not willing to pay for any price p set by the airline and let this price p be exponentially distributed.
Both concavity and differentiability of the maximum expected revenue function were necessary in order to allow a maximum. The optimization problem was defined through a differential equation and although it is particularly hard to solve these kind of differential equations without dynamic programming, we showed a solution through Lemma 2.2.1 that satisfies the optimality condition. Therefore, we are able to find the optimal price for any state (t, n).

Subsequently, we have extended the monopoly model to a competitive case. By adding an airline into the field and contracting the set of allowable prices down to two, namely, high- and low fare, we let arriving customers directly decide which airline offered the lowest fare in order to book a seat on their flight. We assumed that if no seats were to be sold, the optimal price decreased over time. More specifically, we introduced a switching point c^k_{t,n_1,n_2} after which airline k switched from high to low fare. Unless a new customer arrives and buys a ticket at any airline, both airlines will stick to this policy. However, if a customer does buy a ticket, the

game evolves to the next turn. This means both airlines renew their prices, but will always be p_1^1 and p_1^2 respectively. Hence, to future research, we consider the game where after any turn the set of allowable prices is able to change. Moreover, an increase of the set of allowable prices would be a favorable addition to the model as well.
Furthermore, it was inevitable to make the strong, but crucial assumption of concavity and differentiability once again. Finally, when we follow the recursive procedure, it is required to determine an explicit value of the monopolistic case. We see that within the framework of section 3, in monopoly the airline changes from dynamic to fixed pricing. The expected revenue is simply the expected number of arrivals multiplied by the low fare.

For further research, it would be interesting to investigate a model with n airlines, all of whom have the choice between different policies, including fixed price policies. Furthermore, we did not have enough incentives to show a numerical example. Although Li & Peng (2007) [11] have done this, the values of the intensity parameters and prices would be randomly chosen such that results might differ too much to conclude interesting statements. However, the availability of a realistic data set would be an incentive to do a numerical experiment.

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A Distributions

A.1 Proof of Lemma 3.1.1.

Proof. Let N_t denote the number of arrivals during a period of time t and X_t the inter-arrival time assuming the last arrival was at time t. We know by definition that:

$(X_t > h) \equiv (N_t = N_{t+h})$

Simply put, having no arrivals during the interval $[t, t+h]$ is equivalent with saying the number of arrivals at time $t + h$ is still equal to that at time t.

By definition,

$P(X_t \leq h) = 1 - P(X_t > h)$

thus, using the equivalence relation we get:

$P(X_t \leq h) = 1 - P(N_{t+h} - N_t = 0)$

Obviously,

$P(N_{t+h} - N_t = 0) = P(N_h = 0)$

As we have a Poisson distribution with parameter λ, which denotes the average number of arrivals per unit of time, and h is a number of time units, the probability mass function can be expressed as:

$P(N_{t+h} - N_t = 0) = \frac{(\lambda h)^0}{0!} e^{-\lambda h} = e^{-\lambda h}$

If we substitute this into the cumulative distribution function (cdf), we have:

$P(X_t \leq h) = 1 - e^{-\lambda h}$

Thus, we derived the cdf of an exponential distribution with parameter λ and by differentiation with respect to h, we derive its probability density function. $\square$

A.2 Exponential distribution: pdf with multiple parameters

A continuous random variable X is said to have an exponential distribution with parameter λ if it has a probability density function $f_X(t|\lambda) = \lambda e^{-\lambda t}$ for $t > 0$. Let X be the next arrival at time s in interval $[t, c]$, which is exponentially distributed with parameter λ_1. Then $f_X(s-t|\lambda_1) = \lambda_1 e^{-\lambda_1(s-t)}$ for $t \leq s \leq c$. Now, for the density function $f_c(s)$ of the next arrival lying in $[c, T]$, let Z be no arrival in interval $[t, c]$ and Y the next arrival at time s in interval $[c, T]$, which is exponentially distributed with parameter λ_2. The probability of having no arrival in $[t, c]$ is $f_Z(c-t|\lambda_1) = e^{-\lambda_1(c-t)}$ and $f_Y(s-c|\lambda_2) = \lambda_2 e^{-\lambda_2(s-c)}$ for $c \leq s \leq T$. Thus, finally we get $f_c(s) = f_Z(c-t|\lambda_1) f_Y(s-c|\lambda_2) = \lambda_2 e^{-\lambda_2(s-c)-\lambda_1(c-t)}$ for $c \leq s \leq T$.

A.3 Exponential distribution: pdf $f^k_{c^1_{t,n_1,n_2},c^2_{t,n_1,n_2}}(s)$

Notice that $f^k_{c^1_{t,n_1,n_2},c^2_{t,n_1,n_2}}(s)$ is established by using the general probability density function of an exponential distribution which is explained in '1. Probability density function of exponential distribution'. Now, the switching points c^k_{t,n_1,n_2} $(k = 1, 2)$ have partitioned the interval $[t, T]$ such that $f^k_{c^1_{t,n_1,n_2},c^2_{t,n_1,n_2}}(s)$ consists of three different density functions. Also the inter-arrival time is exponentially distributed with correlated demand intensity $\lambda^k_{i,j}$, with $(i = 1, 2)(j = 1, 2)$. The following reasoning applies to both flights k, because we have a symmetric game.

Suppose $k = 1$ and $s \in [t, c^1_{t,n_1,n_2}]$ such that demand is $\lambda^1_{1,1}$. Let X be the next arrival for airline 1 in interval $[t, c^1_{t,n_1,n_2}]$, then $f_X(s-t|\lambda^1_{1,1}) = \lambda^1_{1,1} e^{-\lambda^1_{1,1}(s-t)}$. Let Y be no arrival at airline 2 in interval $[t, c^1_{t,n_1,n_2}]$, then $f_Y(s-t|\lambda^2_{1,1}) = e^{-\lambda^2_{1,1}(s-t)}$. Therefore, we derive the first formula of $f^1_{c^1_{t,n_1,n_2},c^2_{t,n_1,n_2}}(s)$ which is $f_X(s-t|\lambda^1_{1,1}) f_Y(s-t|\lambda^2_{1,1}) = \lambda^1_{1,1} e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)}$ for $t \leq s \leq c^1_{t,n_1,n_2}$.
Similarly, let $s \in [c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$ such that demand is $\lambda^1_{2,1}$. Let X be the next arrival for airline 1 in interval $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$, then $f_X(s-c^1_{t,n_1,n_2}|\lambda^1_{2,1}) = \lambda^1_{2,1} e^{-\lambda^1_{2,1}(s-c^1_{t,n_1,n_2})}$. Now consider the joint event of no arrival at airline 2 in the intervals $[t, c^1_{t,n_1,n_2}]$ and $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$, and no arrival at airline 1 in interval $[t, c^1_{t,n_1,n_2}]$, then the joint probability of this event is given by $e^{-\lambda^2_{2,1}(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}$.
Therefore, we derive the second formula: $\lambda^1_{2,1} e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}$ for $c^1_{t,n_1,n_2} \leq s \leq c^2_{t,n_1,n_2}$.

Trivially, for $s \in [c^2_{t,n_1,n_2}, T]$ demand is $\lambda^1_{2,2}$. The probability of the next arrival for airline 1 to happen first in interval $[c^2_{t,n_1,n_2}, T]$ is given by $\lambda^1_{2,2} e^{-\lambda^1_{2,2}(s-c^2_{t,n_1,n_2})}$ and will be multiplied by the joint probability of no arrival for airline 2 to happen in the third interval $[c^2_{t,n_1,n_2}, T]$ and no arrivals for both airlines in the first two intervals, $[t, c^1_{t,n_1,n_2}]$ and $[c^1_{t,n_1,n_2}, c^2_{t,n_1,n_2}]$, in order to derive the third formula, which is $\lambda^1_{2,2} e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s-c^2_{t,n_1,n_2})-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2}-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}$.

B Calculus related issues

B.1 Partial derivatives

Using the first fundamental theorem of calculus:

Let f be a continuous function on a closed interval $[a,b]$ and $F(s)$ defined by

$F(x) = \int_a^x f(s)ds \quad$, for all $x \in [a,b]$.

Then F is uniformly continuous on $[a,b]$, differentiable on the open interval (a,b) and $F^{'}(x) = \frac{\partial}{\partial x}\int_a^x f(s)ds = f(x) \quad$ for all $x \in [a,b]$.
Similar for $F(x) = \int_x^a f(s)ds$, we get $F^{'}(x) = -\frac{\partial}{\partial x}\int_a^x f(s)ds = -f(x) \quad$ for all $x \in [a,b]$.
Also, by using the product rule, we have $\frac{\partial}{\partial x}\int_a^x f(s)g(x)ds = \int_a^x f(s)g^{'}(x)ds + f(x)g(x)$ and $\frac{\partial}{\partial x}\int_x^a f(s)g(x)ds = \int_x^a f(s)g^{'}(x)ds - f(x)g(x)$.

Therefore, for $V_1(t,n)$, we can evaluate the first two integrals as follows:

$$\frac{\partial}{\partial c^1_{t,n_1,n_2}}\int_t^{c^1_{t,n_1,n_2}}[(V_1(s,n_1-1,n_2)+p^1_1)\lambda^1_{1,1}+V_1(s,n_1,n_2-1)\lambda^2_{1,1}]\times e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(s-t)}ds = [(V_1(c^1_{t,n_1,n_2},n_1-1,n_2)+p^1_1)\lambda^1_{1,1}+V_1(c^1_{t,n_1,n_2},n_1,n_2-1)\lambda^2_{1,1}]\times e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}$$

and by using the product rule

$$\frac{\partial}{\partial c^1_{n_1,n_2}}\int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}}[(V_1(s,n_1-1,n_2)+p^1_2)\lambda^1_{2,1}+V_1(s,n_1,n_2-1)\lambda^2_{2,1}]\times e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}ds =$$
$$(\lambda^1_{2,1}+\lambda^2_{2,1}-\lambda^1_{1,1}-\lambda^2_{1,1})\int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}}[(V_1(s,n_1-1,n_2)+p^1_2)\lambda^1_{2,1}+V_1(s,n_1,n_2-1)\lambda^2_{2,1}]\times e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s-c^1_{t,n_1,n_2})-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}ds -$$
$$[(V_1(c^1_{t,n_1,n_2},n_1-1,n_2)+p^1_2)\lambda^1_{2,1}+V_1(c^1_{t,n_1,n_2},n_1,n_2-1)\lambda^2_{2,1}]\times e^{-(\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2}-t)}\ .$$

In case of the third integral, notice that it is of the form $\int_b^a f(s)g(x)ds$ with $g(x)$ representing the only term which contains c^1_{t,n_1,n_2}. Thus we have $\frac{\partial}{\partial x}[g(x)\times\int_b^a f(s)ds]$

where $\int_b^a f(s)ds$ is a constant. If we evaluate, we get the following:

$(\lambda^1_{2,1} + \lambda^2_{2,1} - \lambda^1_{1,1} - \lambda^2_{1,1}) e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2} - c^1_{t,n_1,n_2}) - (\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2} - t)} \times$
$\int_{c^2_{t,n_1,n_2}}^T [(V_1(s, n_1 - 1, n_2) + p^1_2)\lambda^1_{2,2} + V_1(s, n_1, n_2 - 1)\lambda^2_{2,2}] e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s - c^2_{t,n_1,n_2})} ds$

In a similar manner the partial derivative of $V_2(t, n_1, n_2)$ with respect to c^2_{t,n_1,n_2} is evaluated:

Obviously, the partial derivative of the first integral is equal to 0. In case of the second integral we get,

$\frac{\partial}{\partial c^2_{t,n_1,n_2}} \int_{c^1_{t,n_1,n_2}}^{c^2_{t,n_1,n_2}} [(V_2(s, n_1, n_2 - 1) + p^2_2)\lambda^2_{2,1} + V_2(s, n_1 - 1, n_2)\lambda^1_{2,1}] \times$
$e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(s - c^1_{t,n_1,n_2}) - (\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2} - t)} ds =$
$[(V_2(c^2_{t,n_1,n_2}, n_1, n_2 - 1) + p^2_2)\lambda^2_{2,1} + V_2(c^2_{t,n_1,n_2}, n_1 - 1, n_2)\lambda^1_{2,1}] \times$
$e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2} - c^1_{t,n_1,n_2}) - (\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2} - t)}$

In case of the third integral, by using the product rule, we have

$(\lambda^1_{2,2} + \lambda^2_{2,2} - \lambda^1_{2,1} - \lambda^2_{2,1}) \int_{c^2_{t,n_1,n_2}}^T [(V_2(s, n_1, n_2 - 1) + p^2_2)\lambda^2_{2,2} + V_2(s, n_1 - 1, n_2)\lambda^1_{2,2}] \times$
$e^{-(\lambda^1_{2,2}+\lambda^2_{2,2})(s - c^2_{t,n_1,n_2}) - (\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2} - c^1_{t,n_1,n_2}) - (\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2} - t)} ds$ -
$[(V_2(c^2_{t,n_1,n_2}, n_1, n_2 - 1) + p^2_2)\lambda^2_{2,2} + V_2(c^2_{t,n_1,n_2}, n_1 - 1, n_2)\lambda^1_{2,2}] \times$
$e^{-(\lambda^1_{2,1}+\lambda^2_{2,1})(c^2_{t,n_1,n_2} - c^1_{t,n_1,n_2}) - (\lambda^1_{1,1}+\lambda^2_{1,1})(c^1_{t,n_1,n_2} - t)}$.

B.2 Approximation of integral in Proposition 3.2.2

By letting $\delta \to 0$, we can state the behaviour of expected revenue in relation to infinitely small changes in t. The integral term of the expected revenue function of flight 1 at time $t - \delta$ is of the form $\int_{t-\delta}^t f(s)ds$. This represents the area under the curve $f(s)$ from $s = t - \delta$ to $s = t$. If we perceive the integral as a limit of a Riemann sum, then because δ is infinitesimal we can take the height to be constant equal to $f(t)$ with the width equal to δ such that the area is $f(t)\delta$. In mathematical terms, that is

$\int_{t-\delta}^t f(s)ds = F(s)|_{t-\delta}^t = F(t) - F(t - \delta) = F'(t)\delta = f(t)\delta,$

where F is the antiderivative of f.

C Monopoly model of section 3

Consider that at time t airline 2 sold all its seats. Then according to (26), the objective of airline 1 is to maximize $V_1^{monopolistic}(t, n_1)$ by selling tickets within the time horizon $[t, T]$. We assume airline will always choose to price at low fare p_2^1 as it wants to maximize its expected revenue. Therefore, conclude that we have switched to a fixed pricing policy if we are in a monopoly situation. Obviously, demand no longer depends on the price of airline 2, so we assume λ_2^1 denotes the demand intensity.

Therefore, from time t, the expected revenue of airline 1 is *the expected number of sold seats* $\times$ *low fare*. As the average number of arrivals (sold seats) per time-unit is given by the demand intensity λ_2^1, the expected number of sold seats in the remaining time horizon $[t, T]$ is then given by $\lambda_2^1(T - t)$. So, the expected revenue is given by

$$V_1^{monopolistic}(t, n_1) = \min\left(\lambda_2^1(T - t), n_1\right) \times p_2^1$$

Similar, the expected revenue of airline 2 when it is in monopoly, is given by

$$V_2^{monopolistic}(t, n_2) = \min\left(\lambda_2^2(T - t), n_2\right) \times p_2^2$$

Note that if t is close enough to 0, the expected number of sold seats might be higher than the number of available seats, depending on the value of the demand intensity. Thus, we take the minimum value of the two and multiply by low fare in order to calculate the expected revenue.