UTRECHT UNIVERSITY

GRADUATE SCHOOL OF NATURAL SCIENCES

From Logic to Mathematical Logic

ON THE HISTORY OF LOGIC IN THE 19TH CENTURY

MATHEMATICAL SCIENCES

MASTER'S THESIS

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Utrecht, 10th December 2019

To the many small people in many small places doing many small things,
navigating this ever-changing world.

— Based on a mural on the East Side Gallery in Berlin.

Preface

Here is a very common belief: the mathematics we learn nowadays, the notation and the language and the techniques we use in education and when we do research –surely that must be *the* way mathematics is, and we just happened to discover it. How else could the work of Euclid, written down more than 2000 years ago, still be respected and accepted as valid nowadays? Which other scientific disciplines can say the same about their historical legacy? Mathematics must certainly be some sort of eternal truth, of divine gift that we humans happen to stumble upon in our intellectual explorations.

As a mathematics student myself with an interest in the history of my discipline, I have pondered much about this issue throughout my life, only to come to the conclusion that this mystical perspective on the story misses a very important actor: mathematicians themselves. By idealizing the eternal validity and other-worldliness of mathematics –think for example about Erdős and his Book[1]– the tale undermines or plainly ignores the hard work of many people. The way we do mathematics now is the direct legacy of the generations of mathematicians that came before us. And this is just as true for mathematical logic, one of the newest branches of the mathematical tree.

I can only describe my delight, then, when I came in contact with some nineteenth century writings on logic and mathematics while following the course on the history and philosophy of mathematics taught by professors Dr. Gerard Alberts and Dr. Danny Beckers. The debates we had in that course showed me that, just like mathematics was the result of hard human work, so was the study of its history: generations of historians trying to make sense of the historical sources. Yet, at many moments I was left with doubts, questioning some aspects of the certain historical interpretations and wondering if it would be possible to look for more convincing answers. And as the course was approaching its end, the situation had not improved: I had even more questions than when the course had started!

The present thesis is my attempt at navigating the current literature on the history of math-

[1]'Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book.' (Martin Aigner and Ziegler, Günther H., *Proofs from THE BOOK*, 4th ed. (Berlin: Springer, 2013), p. V)

ematics and logic, at directly confronting the sources, and at suggesting connections and new conclusions that might be extracted from them. The end result may not be exhaustive or complete, but I do believe that it at least constitutes a strong case for maintaining a critical approach to the history and historiography of mathematics and logic: there is still much to be done, much to be improved. Time to get to work!

Acknowledgments

I would like to thank Gerard for his constant support and trust. For many an insightful conversation, and for making sure that passion was always driving my research. Thank you for your time, your patience, your feedback, your gentle listening, and your weirdly inspiring vague metaphors: you have been such a source of wisdom to me. Doing my master's thesis under your supervision has taught me *a lot*, not only in history, but also in the broader sense of what being a historian is like and how being an academic is more than just doing research. I am very thankful for that.

Thanks to Viktor and Jaap for your time, for your honest criticism, and for not being surprised that I want to try my luck in academia. Thank you for challenging me. Thanks to Jaap for suggesting that I read the first article that lead me to choosing the topic of this thesis.

Thanks to Guillem and Jorge (and Tomek!) for proofreading.

Big thanks to the participants of the Mastermath course *History and Philosophy of Mathematics* in the spring semester 2019 and the research seminar in History of Mathematics in the fall semester 2019 for the stimulating discussions and for making historical and mathematical inquiry seem so natural and important. Especially to Cas, Christo, Danny, Marco and Pieter.

Thank you to my university family for being friends who understand what it takes to go through academia: Ana, Álvaro, Gisela, Itziar, Joana, Jorge, Leo, Loe, Mar, Matteo, Menno, Pili, Tom, Renae, and Xareni. For being some of the biggest nerds and most peculiar people I know. For teaching me tolerance and understanding and widening my views of the world in so many ways. A special shout-out to Guillem, for being such a unique presence in my life. Thank you for all I have learned from you, for being open to learning from me, and for asking such f*cking annoying questions.

And, of course, thank you to my family and my nonacademic friends for putting up with my questioning, my doubting, my planning, and my need for independence. Thank you for being the ultimate therapist: Claire, Dominika, Esther, Gea, mama, Nunu, Núria, papa, Paula, Tali, Torunn. Thank you for loving me even when I know it looks like I am doing everything in my power to move further and further away from you: I literally never felt closer to any of you than now that I am away.

Utrecht, December 4, 2019

Contents

Chapter 1

Introduction

According to the standard narrative, the nineteenth century was a remarkable period in the history of logic because it witnessed the emergence of mathematical logic at the hands of two main figures: George Boole and Gottlob Frege.

Allegedly, interest in logic had faded at the end of the fifteenth century[1] and, after three centuries of little to no development in the field, 'during the first half of the nineteenth century the Aristotelian syllogism was still regarded as the ultimate form of all reasoning.'[2] The publication in 1847 of George Boole's *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning* brought the field back to life and started a new chapter in formal logic. Boole 'proposed a calculus that brought up the algebraic analogies between propositions and classes,'[3] but he still regarded logic as the 'the fundamental laws of those operations of the mind by which reasoning is performed,'[4] meaning 'his work lay on the boundary of philosophy, psychology, and mathematics.'[5] Some logicians quickly joined the new trend and started using and expanding Boole's methods and notation, for example Charles Sanders Peirce[6] and later Ernst Schröder.[7]

Some time later, in 1879, Jena professor and logician Gottlob Frege published his first work in logic: the *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinenen Denkens*. This was a milestone event which has been considered by many to mark the birth of

[1]Jean van Heijenoort, 'Historical Development of Modern Logic', *Logica Universalis* 6, no. 3 (2012): p.327.

[2]Gregory H. Moore, 'The Emergence of First-Order Logic', in *History and Philosophy of Modern Mathematics*, ed. Philip Kitcher William Aspray, vol. 11, Minnesota Studies in the Philosophy of Science (Minneapolis: University of Minnesota Press, 1988), p.96.

[3]van Heijenoort, 'Historical Development of Modern Logic', p.329.

[4]George Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities* (Dover Publications, 1854), p. 3.

[5]Moore, 'The Emergence of First-Order Logic', p.96.

[6]Ibid., p.98.

[7]Ibid., p.102.

modern, mathematical logic.[8] In the words of van Heijenoort: '[The *Begriffsschrift*] is perhaps the most important single work ever written in logic,' and 'Frege's contribution marks one of the sharpest breaks that ever occurred in the development of a science.'[9] Similarly, according to Pedriali: 'The modern way of *doing* logic [...] was undoubtedly born with that slim book.'[10]

Nidditch wrote that 'only when logic was married to mathematics did it become fertile.'[11] If one accepts this along with the previous interpretation, the story seems clear: the fate of logic changed forever with the work of Gottlob Frege, because he transformed logic into mathematical logic and opened the doors for mathematicians and logicians to a new discipline.

By looking at that narrative with a skeptical eye, however, a question quickly surfaces:

"Did mathematical logic really emerge with Frege's *Begriffsschrift*?"

This thesis is mainly an attempt at answering this question. And not only that, but it is also an attempt at understanding how the question gets posed in the first place. Hence the title of this thesis, 'From Logic to Mathematical Logic': the goal of the following pages is ultimately to gain a better grasp of the metamorphosis —if any— of logic into mathematical logic during the nineteenth century.

The title itself was inspired by other books in the history of mathematical logic whose titles follow the same pattern. Namely, *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s* by Paolo Mancosu, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879 – 1931* by Jean van Heijenoort, and *From Peirce to Skolem: A Neglected Chapter in the History of Logic* by Geraldine Brady. The parallelism should be obvious. However, the aforementioned books placed a lot of focus on the protagonists of the story, building a narrative which was structured around some main figures. By not mentioning names of people in the title, it is hoped that the reader of this theses will be encouraged to focus their attention on the ideas, on their transmission and their modification, on their expansion and evolution, rather than on the authors themselves.

The way to approach the motivating question draws inspiration from diverse sources, mostly from critical takes on the mainstream views on different historical topics. A quick mention of some of the sources of inspiration for this endeavor, in no particular order: Gregory H. Moore's 'The Emergence of First-Order Logic,'[12] Suzanne Bobzien's 'The Development of

[8] William Kneale and Kneale, Martha, *The Development of Logic* (Oxford: Clarendon Press, 1962).

[9] van Heijenoort, 'Historical Development of Modern Logic', p.327.

[10] Walter B Pedriali, 'Frege', chap. 8 in *The History of Philosophical and Formal Logic: From Aristotle to Tarski*, ed. Alex Malpass and Marianna Antonutti Marfori (London: Bloomsbury, 2017), p.184.

[11] Peter H. Nidditch, *The Development of Mathematical Logic*, 3rd ed., Monographs in Modern Logic (London: Routledge / Kegan Paul, Dover Editions, 1966), p.9.

[12] Moore, 'The Emergence of First-Order Logic'.

Modus Ponens in Antiquity: From Aristotle to the 2nd century AD,'[13] James Van Evra's 'The Development of Logic as Reflected in the Fate of the Syllogism 1600-1900,'[14] José Ferreirós' 'The Road to Modern Logic —An Interpretation,'[15] Joan Bertan-San Millán's dissertation *La Lógica de Gottlob Frege: 1879-1903*,[16] and Gerard Alberts and Danny Beckers' approach to modernity in mathematics as explained in the course 'History and Philosophy of Mathematics' —which in turn was greatly inspired by Jeremy Gray's *Plato's Ghost*.[17] These are beautiful instances of how a respectful yet critical approach to periods of history, no matter how widely studied, can deliver new insight and improve our understanding of those times and those historical changes. Especially Bobzien's sharp eye and ability to interpret widely known texts in new and thought-provoking ways, as well as Van Evra's storytelling skills, served as a motor for the preparation and redaction of this thesis.

The text is organized along the following structure: chapter 2 contains a historiographical overview, a survey of reactions to the standard narrative in the form of criticisms, and an explanation of the stance taken in this thesis.

Chapter 3 compiles introductions to three relevant works in logic which were published in the nineteenth century and have been acknowledged as influential in the history of logic. That chapter seeks to get the reader quickly acquainted with the theoretical setup of each author and the form of their logical systems.

Chapter 4 offers a different take on the developments in logic during the 1800s, structured around four main threads or lines of transformation: the use of mathematical notation in logical texts, the subsequent process of generalization of logical concepts, the distinction between language and metalanguage, and a broader trend of redefinition in and of fields of knowledge.

The thesis closes with a summary of the findings and some suggestions for further research.

Three small remarks about notation, citations, and translations. First: *Begriffsschrift*, in italics, refers here to the article written by Gottlob Frege and published in 1879; Begriffsschrift, in plain text, refers to the formal language which Frege presented within the aforementioned article. Second: references to specific passages in the literature have been given as *chapter.paragraph* whenever possible, and any other references are to the page of the source, if the corresponding text was not divided into paragraphs. Third: English translations of

[13] Suzanne Bobzien, 'The Development of Modus Ponens in Antiquity: From Aristotle to the 2nd Century AD', *Phronesis* 72, no. 4 (2002): 359–394.

[14] James W. van Evra, 'The Development of Logic as Reflected in the Fate of the Syllogism 1600-1900', *History and Philosophy of Logic* 21, no. 2 (2000): 115–134.

[15] José Ferreirós, 'The Road to Modern Logic – An Interpretation', *Bulletin of Symbolic Logic* 7, no. 4 (2001): 441–484.

[16] Joan Bertran San Millán, 'La Lógica de Gottlob Frege: 1879-1903' (PhD diss., Universitat de Barcelona, 2015).

[17] Jeremy Gray, *Plato's Ghost: the Modernist Transformation of Mathematics* (Princeton: Princeton University Press, 2008).

quotes from the *Begriffsschrift* are taken from Gottlob Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort (Cambridge, Massachusetts: Harvard University Press, 1967), 1–82 unless stated otherwise. English translations from Gottlob Frege, 'Über den Zweck der Begriffsschrift', in *Begriffsschrift und andere Aufsätze*, ed. Ignacio Angelelli (Hildesheim: Georg Olms Verlag, 1993), 97–106 and from Bertran San Millán, 'La Lógica de Gottlob Frege: 1879-1903' are ours.

Chapter 2

19th century logic: mathematics yet?

2.1 Establishment of a narrative

In 1957, Józef M. Bocheński published the book *Formale Logik*, which was later translated into English by Ivo Thomas and published under the title *A History of Formal Logic* in 1961. The English title proved to be a bit misleading, for the aim of the text was not to present a history of formal logic but a selection of texts about relevant problems in logic, accompanied by commentary by Bocheński and other logicians and historians of logic from Warsaw (the school of Łukasiewicz) and Münster (the school of Scholz).[1] The book was well received, being qualified as 'un instrument de travail remarquable'[2] and a 'useful addition to [the logician's] library.'[3] The second reviewer even wrote with admiration that '[i]t is the only extant book of its kind, and it sets such a high standard of excellence that one may doubt whether it will have any serious rivals for a long time to come.'[4] In his *Formale Logik*, Bocheński acknowledged an earlier book on history of logic written by Carl Prantl in 1855, only to describe it as little more than a complete disappointment. Time had shown Prantl's *Geschichte der Logik* to be basically a compilation of prejudices and biased opinions, and Bocheński was clear on his stance upon the usefulness of the book: 'It is better to disregard [Prantl] entirely. He must, unhappily, be treated as non-existent by a modern historian of logic.'[5]

Bocheński's interpretation was that logic, although a unity, had not followed a continuous growth, but rather that it had appeared throughout history as a series of emergences followed by periods of decadence. This view, according to him, 'markedly diverges not only from all previous conceptions of the history of logic, but also from opinions that are still widespread

[1] I. M. Bocheński, *A History of Formal Logic*, trans. Ivo Thomas (University of Notre Dame Press, 1961), p. v.

[2] Marie-Louise Roure, review of *Formale Logik*, by I. M. Bocheński, *Les Études philosophiques (Nouvelle Série)*, no. 4 (1957): p. 395.

[3] Benson Mates, review of *Formale Logik*, by I. M. Bocheński, *The Journal of Symbolic Logic* 25, no. 1 (1960): p. 57.

[4] Ibid.

[5] Bocheński, *A History of Formal Logic*, p. 8.

about the general history of thought,' and was 'a position adopted in accordance with empirical findings.'[6] This last comment pointed at the fact that Bocheński was not really developing a history of logic in a strict sense, but rather presenting a compilation of textual evidence from different moments in time and interpolating from those the ways in which logic had changed through the ages. The sixteenth to the nineteenth centuries, labeled 'the older period of modern "classical" logic,' were seen by Bocheński as one of the decadence periods, with mathematical logic being the flourishing period afterwards. Remarkably, the chapter on this period received the title of 'The mathematical variety of Logic.' At the beginning of said chapter, Bocheński described four characteristic features of mathematical logic: the presence of a formalistic method, the use of an abstractive method, the use of an artificial language, and the formulation of theorems in an object language.[7] He mentioned a great number of authors as belonging to this period, such as Boole, Peirce, Peano... 'But of all mathematical logicians,' Bocheński wrote, '[Gottlob Frege] is undoubtedly the most important.'[8] His reasons for this praise were many: Frege had formulated the distinction between constants and variables for the first time, as well as the concept of the quantifier, he had introduced a clear distinction between language and metalanguage, etc.

When a couple of years later *The Development of Logic* by William and Martha Kneale was published, in 1962, it was celebrated as a great accomplishment. The book managed to compile, within one volume, a detailed history of the development logic from the ancient Greeks to the first half of the twentieth century, and contained philosophical and technical remarks alongside the historical text. The book was received with enthusiasm: a reviewer qualified it as 'epoch-making'[9] and another reviewer stated that '[t]o review this book is to assist at the unveiling of a monument.'[10] The general feeling was thus that a work of exceptional quality had just been made available: 'there is no doubt that this treatise will be a standard work for years to come.'[11]

As many reviewers made sure to point out, a remarkable feature of the book was its explicit aim to provide 'an account of the growth of logic, rather than an attempt to chronicle all that past scholars, good and bad, have said about the science.'[12] That is, the goal of the book was not

[6] Bocheński, *A History of Formal Logic*, p. 10.

[7] Ibid., pp. 266–267.

[8] Ibid., p. 269.

[9] Roland Hall, review of *The Development of Mathematical Logic*, by P. H. Nidditch, *The Philosophical Quarterly (1950-)* 13, no. 53 (1963): 379.

[10] Renford Bambrough, 'The Growth of Logic', review of *The Development of Logic*, by William Kneale and Martha Kneale, *The Classical Review* 13, no. 2 (1963): 186–188.

[11] Benson Mates, review of *The Development of Logic*, by William Kneale and Martha Kneale, *The Journal of Symbolic Logic* 27, no. 2 (1962): 213–217. Interestingly enough, this is the same reviewer who had had high words of praise for Bocheński a few years earlier, yet the review contained no mention to the *Formale Logik*.

[12] Kneale and Kneale, Martha, *The Development of Logic*, p. v.

to present a chronological account of historical events related to logic, but rather to select those authors or ideas who had made contributions which had had an impact in later developments within the field. Some of the reviews focused exclusively on the first chapters of the book and congratulated the authors on their excellent and thorough treatment Greek logic, while making no comment on the rest of the text. But the book did more besides a praiseworthy coverage of Greek logic. One reviewer noted an interesting characteristic of the narrative presented by Kneale and Kneale:

> Naturally enough the development of logic in the nineteenth century is shown to have been influenced by the radical reconstruction of mathematics initiated by such men as Cauchy and Weierstrass, and continued by Dedekind and Cantor. However, the place of honor in the revival of logical studies in modern times is rightly accorded to Gottlob Frege. It is significant that Frege's name appears in the titles of four consecutive chapters.[13]

Indeed, the chapters dedicated to logic after the Renaissance are as follows: chapter V, adequately titled 'Logic after the Renaissance'; chapter VI, 'Mathematical Abstraction'; chapter VII, 'Numbers, sets, and series'. The book continues then with chapter VIII titled 'Frege's general logic'; chapter IX, 'Formal developments after Frege'; chapter X, 'The philosophy of logic after Frege'; and chapter XI on 'The philosophy of mathematics after Frege'. It seems, thus, as if for Kneale and Kneale the work of Gottlob Frege was the point of reference for an understanding of contemporary logic. And the reviewers must have agreed, for there seem to be no reviews mentioning that peculiarity in the way the book was structured. If anything, more words of praise: 'Among parts of the book that I can find no fault with at all (and there are many such) I should single out at least the section on the *Begriffsschrift*, which is a model of clear exposition.'[14]

However, why should logic have developed 'naturally enough' in this way? Why should the honor of reviving logical studies be 'rightly' accorded to Frege? Did logic need to be 'revived' at all in the first place? The comments of the reviewer hint at the amount of unstated assumptions that the interpretation suggested by Kneale and Kneale relied on. The expectation that it was just 'natural' for logic in the nineteenth century to develop in the way that Kneale and Kneale presented it was dangerously teleological: the recognition in Frege's of features present in the mathematical logic being done mid-twentieth century made the association seem immediate.

The year after the publication of Kneale and Kneale's celebrated treatise, Peter H. Nidditch

[13]Czeslaw Lejevski, review of *The Development of Logic*, by William Kneale and Martha Kneale, *Philosophy* 40, no. 151 (1965): p. 83.

[14]Robert Barret, review of *The Development of Logic*, by William Kneale and Martha Kneale, 62, no. 2 (1965): p. 55.

published a little booklet titled *The Development of Mathematical Logic*. Just like Kneale and Kneale had endeavored to do, the purpose of the book was 'to give such an account of Mathematical Logic as will make clear in the framework of its history some of the chief directions of its ideas and teachings.' Nidditch suggested four main lines of thought which, according to him, produced mathematical logic: 'the old logic, the invention of Aristotle; the idea of a complete and automatic language for reasoning; the new developments in algebra and geometry which took place after 1825; and the idea of the parts of mathematics as being systems of *deductions*.'[15] This was thus yet a different interpretation of the history of mathematical logic but, despite its attractive conceptual setup, Nidditch's book was not very well received. His decision to write using Odgen's Basic English[16] was criticized by reviewers as an unnecessary complication, disagreeing with Nidditch's claims that Basic English would 'certainly not make things harder for any.'[17]

In the following years, works by other influential historians helped reaffirm the view that Frege had been a major figure in the history of logic. In the introduction to the English translation of the *Begriffsschrift*, published in 1967 as part of the source compilation *From Frege to Gödel*, van Heijenoort described Frege's book as 'perhaps the most important single work ever written in logic.'[18] According to him, '[Frege's] analysis of the proposition into function and argument, rather than subject and predicate, and quantification theory, which became possible only after such an analysis, are the very foundations of modern logic.'[19] Later, Michael Dummett's influential books *Frege: Philosophy of Language*, published in 1973, and *Frege: Philosophy of Mathematics*, published in 1991 also played an important part in establishing Frege as an almost legendary figure. In them, Dummett consolidated Frege's reception not only as a crucial figure within the history of mathematical logic, but also as the father of analytic philosophy.

Thus, by 1980 the narrative as presented in Kneale and Kneale had become the standard interpretation of the development of logic during the nineteenth century[20] —and with it their interpretation of Frege as a pivotal figure in the development of mathematical logic.

[15]Nidditch, *The Development of Mathematical Logic*, p. 3.

[16]Odgen's Basic English was a simplified version of English created by linguist Charles K. Odgen. Its basic feature was a list of 850 root words which could be then expanded by means of affixes and other forms of the words.

[17]Nidditch, *The Development of Mathematical Logic*, p. 2.

[18]Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', p. 1.

[19]Ibid., p. 3.

[20]At the time these lines were written (November 2019), Kneale and Kneale's book had been cited over 2500 times according to Google Scholar, whereas Nidditch's booklet did not even reach 50 citations.

2.2 Criticism

However, just as that narrative became the mainstream view on logic in the nineteenth century, it didn't take long until other authors started coming forth and pointing out inconsistencies which followed from that interpretation of the story.

2.2.1 The role and recognition of Frege

One of the most striking cases of divergence was Ivor Grattan-Guinness' article 'Living Together and Living Apart. On the Interactions Between Mathematics and Logics from the French Revolution to the First World War,' which appeared in 1988. In that article, Grattan-Guinness set forth to 'survey . . . the connections made between branches of mathematics and types of logic during the period 1800 – 1914' and he did so by highlighting two main streams: that of algebraic logic, which according to him culminated in Peirce and Schröder, and that of mathematical logic, culminating in the work of Russell. The way Grattan-Guinness presented his survey was provocative to say the least, for the two following reasons: on the one side, Boole was mentioned without much ado and as a transitional figure; on the other side, Frege was remarkably omitted from the list of 'principal figures.' His role in the story was simply relegated to a clarification about the extent of Russell's innovations,[21] as well a note in which Grattan-Guinness made some 'puzzled remarks' about Frege:

> While [Frege] spoused a version of logicism before Russell, and gained some circulation among a few mathematicians and philosophers between the later 1880s and the early 1900s, he then sunk into neglect (partly due to eclipse by Russell, I suspect), and the extent of his influence at that time is hard to judge.[22]

This 'eclipse' Grattan-Guinness talked about could refer not only to Russell's fame but to the latter's own view of Frege. According to Moore, the perception that Frege had received little to no attention was largely due to some claims Russell made in his 1919 book *Introduction to Mathematical Philosophy*,[23] where in the fourth footnote of chapter 3 he had written:

> These definitions, and the generalized theory of induction, are due to Frege, and were published so long ago as 1987 in his *Begriffsschrift*. In spite of the great value

[21] Ivor Grattan-Guinness, 'Living Together and Living Apart. On the Interactions Between Mathematics and Logics from the French Revolution to the First World War', *South African Journal of Philosophy* 7, no. 2 (1988): p. 77, sec. 8.

[22] Ibid., pp. 79–80, note 9.

[23] Moore, 'The Emergence of First-Order Logic', p.129, note 4: 'There is a widespread misconception, due to Russell (1919, 25n), that Frege's *Begriffsschrift* was unknown before Russell publicized it.'

> of this work, I was, I believe, the first person who ever read it —-more than twenty years after its publication.[24]

Why Russell made such bold claims can only be left for speculation and falls out of the scope of this thesis, although it does hint that the *Begriffsschrift* had certainly not become a popular text among educated circles even by the beginning of the twentieth century. This was then a major weak point of the interpretation presented by Kneale and Kneale, since their framing would suggest a wide reception and recognition of Frege by his contemporaries — how could he have been so influential otherwise?— and yet, historical evidence of this fact was scarce and debatable. Kneale and Kneale did acknowledge this in passing:

> Unfortunately, Frege's epoch-making little book was neglected by mathematicians and philosophers alike.[25]

but that did not lead them to question the consistency of their interpretation anyway.

Much has been speculated about reasons for this early neglect. Some commentators have defended the view that Frege's ideas were not successful at the time of publication because Boole's approach had become the mainstream theory by then. For example, according to Moore the *Begriffsschrift* 'failed to persuade other logicians to adopt Frege's approach to logic because most of them (Schröder and Venn, for example) were already working in the Boolean tradition.'[26] Another reason for this might be the complicated two-dimensional notation that Frege proposed, which was 'unanimously declared hard to read, and counterintuitive in the extreme.'[27]

Hans Sluga had some words about this unhistorical approach to the figure of Frege. His book *Gottlob Frege* was an attempt at a better understanding of Frege as a historical figure and of his ideas. A former student of Dummett, he was clear about his opinion on the way the latter had approached Frege: 'Michael Dummett's extensive discussions of Frege and the philosophy of language can serve as a paradigm for the failure of analytic philosophers to come to grips with the actual, historical Frege.' He quoted Dummett's statement that Frege's logic should be

[24]Bertrand Russell, *Introduction to Mathematical Philosophy* (London: George Allen / Unwin, 1919), p. 25.

[25]Kneale and Kneale, Martha, *The Development of Logic*, p. 436.

[26]Moore, 'The Emergence of First-Order Logic', p.129, note 4.

[27]Pedriali, 'Frege', p.187. Bocheński had already speculated along these lines, although he appealed mostly to the difficulty of humans to change habits: 'It is not true that it is particularly difficult to read, . . but it is certainly too original, and contrary to the age-old habits of mankind, to be acceptable.' (Bocheński, *A History of Formal Logic*, p. 268) Other authors have been much more harsh in their assessment of the aesthetic appeal of the Begriffsschrift. For example, Florian Cajori described it like this in his *History of Mathematical Notations*: 'This early neglect has been attributed to Frege's repulsive symbolism.' (Florian Cajori, *A History of Mathematical Notations. Two Volumes Bound As One* (New York: Dover Publications, 2011), p.295)

assumed 'to have been born from Frege's brain unfertilized by external influences.'[28]

A more recent account of Frege which was critical with the interpretation given by Kneale and Kneale was Walter Pedriali's chapter on Frege in the book *The History of Philosophical and Formal Logic: from Aristotle to Tarksi*. He pointed out the simplicity of the historical narrative around Frege, and accused it of overlooking 'the importance of the work of De Morgan, Boole, Jevons, McColl, Schröder and Peirce (the *modern turn* in logic actually predates 1879). It is also too neat because it glosses over the fact that modern logic differs in important respects from that of Frege (logic became truly 'modern' a lot later than 1879).'[29] However, a few lines later he wrote that 'the modern way of *doing* logic . . . was undoubtedly born with that slim book.'[30] So, even though the chapter opened with the promise of a critical reassessment of the role of Frege in modern logic, Pedriali failed to escape the veneration of Frege and his influence and the chapter ended with these dramatic words:

> In assessing Frege's unique contribution to the history of logic . . . his unremitting dedication to the cause of truth is perhaps even more remarkable and praiseworthy than his technical achievements. . . . Frege, then, didn't just show us *how* to do logic. He also showed us hot to *be* a logician, how to put truth first, at all times, no matter what the costs may be.[31]

2.2.2 Frege's logic in the *Begriffsschrift*

There was yet another aspect of the story which was recently questioned, regarding the common assumption that Frege's work in the *Begriffsschrift* was a formal system of second-order logic. This perception came from the use Frege made of the 'Höhlung' or 'Allgemeinheit,' a feature of his system which allowed him to make universal statements (see section 3.3 for a description of the setup of the Begriffsschrift).

In his recent doctoral dissertation titled *La Lógica de Gottlob Frege: 1879-1903*, Joan Bertran-San Millán recalled how modern glances at Frege's use of the Allgemeinheit were eager to interpret it as a universal quantifier that ranged over arguments as well as over functions, thus leading to the tempting conclusion that Frege's logic was indeed second-order. Yet, Bertran-San Millán argued that, for instance, 'Si prestamos atención a la distinción desarrollada en Begriffsschrift, la cuantificación sobre funciones, propiamente, no tiene lugar.'[32] This was due

[28]Michael Dummett, *Frege. The Philosophy of Language* (Harvard University Press, 1981), p. xvii, quoted in Hans D. Sluga, *Frege* (London: Routledge, 1980), p. 5

[29]Pedriali, 'Frege', p.183.

[30]Ibid., p.184.

[31]Ibid., p.213.

[32]'If we pay attention to the distinction developed in the Begriffsschrift, quantification over functions, properly speaking, does not take place.' [Our translation.] (Bertran San Millán, 'La Lógica de Gottlob Frege: 1879-1903', p. 61)

to the fact that using expressions of the kind $f(a)$ in the *Begriffsschrift* did not automatically imply, for Frege, that f was meant to be the function and a its argument. In fact, 'tanto "f" como "a" pueden ser argumento. Tal es la flexibilidad de esta estructura, y la ambigüedad que expresan las letras "f" y "a", que puede decirse que toda proposición de la conceptografía es una instancia de "$f(a)$."'[33]

Bertan-San Millán was thus advocating for a strict textual interpretation of the contents of the *Begriffsschrift*. In doing so, he highlighted the distorted, anachronistic manner in which that work had been usually approached: a reading of the *Begriffsschrift* which sees in it a system of second order logic can only be done from a moment in history where second order logic has been distilled as a differentiated logical construct. Bertan-San Millán's analysis convincingly showed that Frege's logic in the *Begriffsschrift* was not yet as sophisticated as some mathematicians and logicians have wanted it to be.

2.2.3 Fatherhood of logic

Another inconsistency stood out which concerned the attribution of the founding role of mathematical logic, disputed mainly between Boole and Frege: some authors gave the honor to the one, some to the other. A third contestant to the title would have been Leibniz, but the lack of strong results authored by him or a complete theory, as well as the very late acknowledgment of his contributions, kept him mostly out of the competition. Bocheński did highlight Leibniz's work and expressed his opinion that he had been the first mathematical logician, but still noted that if Leibniz 'cannot count as the founder of mathematical logic it is because his logical words were for the most part published after his death.'[34]

Although Frege was generally perceived as the most important figure in mathematical logic, many historians defended Boole as the father of the discipline. This was the opinion of C. I. Lewis, for example, who claimed for Boole the role of 'second founder of the discipline'[35] in his *Survey of Symbolic Logic* from 1918 —the first founder being Leibniz, even though according to Lewis his works on the field were 'prophetic but otherwise without value.'[36] This early positioning in favor of Boole was noteworthy because it gave insight as to the way things were seen back in the beginning of the twentieth century, already after Frege but before the advent of the foundational crisis in mathematics and logic. In Corcoran's words,

> Lewis's judgment that Boole was the founder of mathematical logic, the person

[33]'both "f" and "a" can be the argument. Such is the flexibility of this structure, and the ambiguity that the letters "f" and "a" express, that it could be argued that any proposition in the idea-language is an instance of "$f(a)$." [Our translation.] (Bertran San Millán, 'La Lógica de Gottlob Frege: 1879-1903', p. 37)

[34]Bocheński, *A History of Formal Logic*, p. 267.

[35]Clarence I. Lewis, *A Survey of Symbolic Logic* (Berkeley: University of California Press, 1918), p. 4.

[36]Ibid.

> whose work began the continuous development of the subject, stands as a massive obstacle to revisionists whose philosophical or nationalistic commitments render this fact inconvenient.[37]

Statements defending Boole as the father of mathematical logic abound indeed, for example in Nidditch's *Development of Mathematical Logic*:

> What was the start of Mathematical Logic? The shortest and simplest answer is George Boole's *The Mathematical Analysis of Logic*. . . . There is nothing completely new under the sun. Every birth is the outcome of earlier events. So Boole's little book of 82 pages was only marking a stage in an unbroken line of thought from the past. However, thought it certainly had connections with what some others had done, it was still different enough from the rest for it rightly to be seen as starting a quite new theory —the theory of Mathematical Logic— and not as being another step in an old one.[38]

So Nidditch held the view that Boole's ideas were new enough to justify viewing them as the start of a new period in logic. Notably, though, he recognized the broader context of logic and the developments therein as 'an unbroken line of thought from the past.'

A more cautious stance was taken by Volker Pekhaus in the essay 'Was George Boole Really the "Father" of Modern Logic?' It contained a detailed historiography on the debate and explored the subtleties of claiming 'fatherhood' of a current of ideas: was an alleged 'father' to be seen as an 'initiator' or as an 'originator'? —an initiator being the instigator of a new development, and an originator being the first formulator and creator of seminal ideas.[39] Peckhaus answered his own question in a soft positive: 'I have the feeling Boole was the first only by historical accident,' and he indeed closed his essay with the words:

> Multiple creation would mean multiple fatherhood. Terms like 'multiple fatherhood' or 'collective fatherhood ' indicate that the metaphor of fatherhood gives a distorted picture when applied to the history of logic. We should therefore stop searching for the father of modern logic![40]

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[37]John Corcoran, 'C. I. Lewis: History and Philosophy of Logic', *Transactions of the Charles S. Peirce Society* 42, no. 1 (2006): p. 1.

[38]Nidditch, *The Development of Mathematical Logic*, p.34.

[39]Volker Peckhaus, 'Was George Boole Really the 'Father' of Modern Logic?', in *A Boole Anthology: Recent and Classical Studies in the Logic of George Boole*, ed. James Gasser, vol. 291, Synthese Library (Springer-Science+Business Media, 2000), p. 275.

[40]Ibid., p. 282.

2.2.4 Logic vs. mathematical logic

The lack of consensus on the 'fatherhood' of logic was yet another indicator of the difficulty in pinning down the moment which marked the clean cut ('before this there was logic, from now on there is mathematical logic'), thus suggesting that the transformation of (a part of) logic into mathematical logic occurred gradually through the nineteenth century —and well into the twentieth century!

Note Bocheński's own words in the introduction to the section of *A history of formal logic* dedicated to mathematical logic, which were probably written in the original German sometime around 1956 or 1957: 'The development of *mathematical variety of logic* is *not yet complete*, and discussions still go on about its characteristic scope'.[41] Bocheński's rendering was hereby acknowledging the subordination of mathematical logic to a broader understanding of logic, that is, seeing mathematical logic as yet another sort of logic, an approach to the more general science that was distinguished by its strong connections with mathematics.

In contrast, Kneale and Kneale's interpretation relied on the unstated assumption that modern logic was logic in itself, i.e. that mathematical logic was not just a subspecies of logic, but the state logic had reached after many centuries of development. The difference between these two stances may not have been consciously acknowledged at the time. For instance, a review of *The Development of Logic* contained both descriptions: it stated that mathematical logic is 'logic into which mathematical ideas have been infused,' and also that 'mathematical logic is simply logic, in the form that is appropriate to the present.'[42] The last one was the interpretation which followed from Kneale and Kneale's text, in which the history of logic from the nineteenth century onward had been merged with the history of mathematical logic and there was no telling them apart. Indeed, the reviewer also wrote that, upon accepting Kneale and Kneale's rendering, 'the distinction between two modes of logic, the one philosophical and the other mathematical, must now be deemed obsolete.'[43] According to this, to the modern scholar there was no difference between 'logic' and 'mathematical logic.'

2.3 An alternative interpretation

The previous sections explained how the standard interpretation of the history of logic in the nineteenth century was established, and some of the points of criticism which were raised against it in the subsequent years. One could now ask: what is the narrative nowadays? Did historians manage to modify their retelling of the story, taking into consideration the problem-

[41] Bocheński, *A History of Formal Logic*, p. 266, italics are ours.

[42] G. T. Kneebone, review of *The Development of Logic*, by William Kneale and Martha Kneale, *The British Journal for the Philosophy of Science* 16, no. 61 (1965): p. 64.

[43] Ibid.

atic aspects which had been identified in the 1960s rendering?

Consider for instance Gray's succinct narration of the development of logic in the nineteenth century, which was written as part of the introduction to his 2008 book *Plato's Ghost* on modernism in mathematics:

> Unlike the developments in classical mathematics, contemporary changes in logic grew out of a moribund field. . . . [Investigations into logic] were revived by a remarkable British school, prominent among whom were de Morgan and Boole. After them, but independently, Gottlob Frege recast the traditional theory of logic insofar as it was an analysis of mathematical language, and the period of modern logic began. The nature of Frege's contribution and its significance (then or since) remains controversial, and so too does the work of his contemporaries. Some [historians] . . . see a bifurcation into algebraic logic, associated with George Boole, Charles Sanders Peirce, and Ernst Schröder, and the alternative offered by Frege and Peano. An early exponent of this view was Russell, who, it is often said, was influential in marginalizing the algebraic viewpoint. Others . . . find the algebraic school more active in its day, and more substantial in their points, than Russell appreciated, and consequently differ in their historical assessment of the period.[44]

Granted that this was meant to be a very short recollection of the developments in nineteenth century logic, but the general casting of the narrative was quite close to the version discussed earlier. Gray did acknowledge, however, the doubtful status of Frege as a main character, and incorporated the view that different lines of development took place at the same time.

Yet, comments like 'Boole had seen his mathematical logic as a new form of mathematics that was not restricted to the study of quantity'[45] imply an understanding that Boole's work could be already considered to belong to mathematical logic, or at least that his system was a mathematical creation. Further, within the broader picture of the history of science, the inclusion of a section dedicated to Boole and the 'Algebra of Logic' in a book about modernism in mathematics published as late as 2008, as well as recurring references to logicians and debates happening in logic as part of its introduction, supports the claim that the understanding of the relationship between mathematics and logic during the nineteenth century is still being improved. Gray did recognize the complexity of this relationship during that period of time, and how 'many of the earliest analyses of the relation between logic and mathematics are bedeviled by obscurities.'[46] He also noted the complicated relationship of logic with itself

[44] Gray, *Plato's Ghost: the Modernist Transformation of Mathematics*, pp. 21–22.
[45] Ibid., p. 106.
[46] Ibid., p. 22.

and the dramatic changes it underwent leading up to and through modernity: 'the modernist transformation of logic itself meant that even logic no longer has a straightforward connection with simple clear thinking.'[47]

That being said, it does seem that contemporary historians of logic and mathematics are more self-aware of the biases that exist in their discipline. For instance, in the enthusiastic introduction to the book *The Architecture of Modern Mathematics: Essays in History and Philosophy of Mathematics*, published in 2006, Gray and Ferreirós pointed one of the peculiar, subtle ways in which the standard narrative still pervades, despite the abundance of evidence suggesting that Frege's importance in it should be reassessed:

> One often finds authors who seem to believe that we are living in the year 126 a. F.,

with the footnote: 'That is, after Frege's *Begriffsschrift*, the locus classicus where the Method was born.'[48] The Method referred to that of 'formulating and studying philosophical questions by formal logic.' That comment evidenced how the Fregean myth is still a reality among mathematicians, logicians, philosophers, and historians of these disciplines —even after the turn of the twenty-first century!

In that same introduction, Ferreirós and Grey worded nicely what could be seen as one of the fundamental problems displayed by the historiography in the previous sections. The quoted paragraph was meant to be a critical self-reflection abut the historiography of mathematics, but it could just as well be applied to logic:

> The process of research was truncated because of the easy consensus about what needed to be said, which was the acknowledged highlights – the famous names, the great theorems, the exceptional biographies. Books of this kind tend to confirm what their authors set out to discover. They may turn up unexpected details, usually to do with priorities, but they go looking for the key developments that 'must' have been there in the past or mathematics could not be as it is today.[49]

In fact, it is not surprising that self-reflection in mathematics applies to logic so well in this context, since the history of logic in the nineteenth century has mostly been approached from the mathematical perspective, i.e. with an understanding that those theories would eventually crystallize in modern and present-day mathematical logic. The challenge for contemporary historians, then, is to find a way to approach the nineteenth century with an open mind, trying to describe the events that were taking place as if it was not known what those theories would eventually turn into.

[47]Gray, *Plato's Ghost: the Modernist Transformation of Mathematics*, p. 22.

[48]José Ferreirós and Gray, Jeremy, eds., *The Architecture of Modern Mathematics: Essays in History and Philosophy* (Oxford: Oxford University Press, 2006), p. 8.

[49]Ibid., p. 22.

In the introduction of this thesis, the motivating question for was posed: 'did mathematical logic really emerge with Frege's *Begriffsschrift*?' Taking into consideration the observations made so far, it becomes clear that the question in itself is ill-posed because it relies on unstated assumptions and vaguely defined terms: what does 'mathematical logic' actually refer to — specifically when studying the nineteenth century? Does it make sense to talk about it going through a process of 'emergence' at all? Were Frege and the *Begriffsschrift* really that important in this story? These are the questions that are going to be examined in the rest of this work.

Part of the confusion in telling the story comes from the terminology itself, from the loose use of the expressions 'mathematical logic,' 'modern logic,' 'formal logic,' 'symbolic logic,' 'algebraic logic'... Depending on the author these were sometimes taken to be synonyms, sometimes they referred to different branches of logic, sometimes they were meant to indicate a hierarchy of concepts in which some term was part of another.[50] The distinction (if there ever was one) was anything but clear already throughout the nineteenth century. The variety of titles given to treatises and books covering topics which are nowadays considered to belong to a unique tradition of knowledge attested to this. Consider, for example, the books *Pure Logic* by Jevons, *Symbolic Logic* by Venn, *Formal Logic* by De Morgan, a later *Symbolic Logic* by Lewis... All of these books were treating —modifications and discrepancies among them aside— similar issues, yet it seems as if their authors could not decide which adjective suited best the kind of logic they were working out. [51]

Historians have tried to identify earlier and earlier occurrences of the expression 'mathematical logic.' Nidditch, in 1966, thought that

> Peano was the first to give the new logic the name of 'Mathematical Logic,' because of his view of it as an instrument for mathematics.[52]

Grattan-Guinness, twenty-two years later, knew better:

> It is worth noting that the expression 'mathematical logic' is due to De Morgan, in the third of his papers on the syllogism.[53]

However, in what ways could this information be enlightening towards the questions which strive to be elucidated? If anything, it only points out that some sort of connection

[50]Grattan-Guinness, for instance, stated that he was using the expression 'symbolic logics to refer both to the algebraic and the mathematical traditions.' (Ivor Grattan-Guinness, 'Mathematics and Symbolic Logics: Some Notes on an Uneasy Relationship', *History and Philosophy of Logic* 20, nos. 3–4 (1999): p. 159)

[51]The multiplicity of expressions seemingly referring to the same concept was pointed out already by Bocheński, who noted that modern logic was 'simultaneously called "mathematical logic," "symbolic logic," and "logistic" . . . and is sometimes simply called "theoretical logic."' (Bocheński, *A History of Formal Logic*, p. 266)

[52]Nidditch, *The Development of Mathematical Logic*, p.73.

[53]Grattan-Guinness, 'Living Together and Living Apart. On the Interactions Between Mathematics and Logics from the French Revolution to the First World War', p.77, footnote.

or relation between mathematics and logic had been noticed. The underlying questions are, rather: what are really the concepts that need to be traced? And which sort of phenomena might be encountered?

In the following pages, the focus will be brought back to the events taking place during the nineteenth century, pre-Frege, in an attempt to understand the debates that were taking place before the publication of the *Begriffsschrift*. It will follow that Boole's innovations did not emerge from nothing, but were a product of the time. Similarly, it will be shown that Frege's ideas did not arise in a vacuum either. Further, the kind of logic that was being developed during the beginning and middle of the nineteenth century will be explored.

A full analysis of the state of affairs and an attempt at a deconstruction of the 'Fregean myth,' while very attractive, would be too big an endeavor to undertake within the scope of this master's thesis. However, there are some steps which can be taken on the path to such a reassessment of the history of logic: reaching again to the sources, to the writings that are perceived as important, and trying to understand what the motivations of their authors where at the time. We undertake this quest with the hope that such an analysis might shed some light on the prevalent questions concerning the history of logic during the nineteenth century: first, is the history of mathematical logic one of continuity or of break? That is, is it possible to place the history of mathematical logic within the broader tradition of logic, or is its appearance an instance of emergence? Secondly, in case continuity is the best explanation for this phenomenon, can we identify some lines of change within logic that can explain for its transformation into mathematical logic? And thirdly, can these considerations shed some light on the bigger debate of the mutual roles between logic and mathematics? Which one was viewed as the foundation of the other?

Chapter 3

Three important works in logic

This chapter describes three important works in logic which were published during the nineteenth century. First, Richard Whately's *Elements of Logic*, published in 1826 and conceived as a textbook in logic for university students. Second, George Boole's *Mathematical Analysis of Logic* from 1847, in which he condensed his ideas about a calculus which would represent the laws of reasoning and the deductive processes of the mind. Lastly, Gottlob Frege's *Begriffsschrift*, a proposal for a formula-language that would serve the purpose of unambiguously representing chains of deductive reasoning.

The descriptions are meant to give the reader an idea of the logical systems that the authors were putting forth in each of the works, and the ways in which these diverged in form and purpose from the previously known systems of logic while still retaining a common theoretical core. For this reason, the contents of each piece are described only up to their treatment of the syllogism or an equivalent feature, since this is what will be needed for the analysis in the next chapter.

3.1 The *Elements of Logic*

The book *Elements of Logic* by Richard Whately, published in 1826, was conceived as an extension of the popular article 'Logic' by the same author, which had appeared in 1823 as part of the *Encyclopaedia Metropolitana*. The *Elements of Logic* became one of the most popular textbooks in logic in Great Britain, being in print from the moment of its first publication until the first decade of the twentieth century[1] —it should not be surprising, then, that it was acknowledged by Boole in his *Mathematical Analysis of Logic* as the main reference for his treatment of logic. The book set the stage for a reappraisal of logic in the nineteenth century in two ways. On the one hand, Whately's presentation of the syllogism was the way in which Boole received the

[1] James W. van Evra, 'Richard Whately and the Rise of Modern Logic', *History and Philosophy of Logic* 5 (1984): p. 2.

tradition and the structure he paralleled when developing his own system. On the other hand, Whately did not only present the theory of the syllogism as he had in turn received it (mainly from Aldrich):[2] Whately's exposition of the topic constituted a reform in itself and contained remarks and ideas which influenced the way logic was studied and conceived after him.

In the first book of his *Elements* Whately lay down his views on logic, how it should be structured, and how it should be used. That book was a defense of logic as a science, a view that was quite extraordinary at the time because logic was mostly considered an art. Whately argued that logic was a science in the most formal sense of the word, and his goal was to compile and present the basic concepts and methods of the discipline viewed under this light. Whately also drew numerous analogies between logic and other sciences, especially mathematics and chemistry.

The second book contained Whately's take on Aristotelian logic, the theory that Boole reworked in his *Mathematical Analysis of Logic*. A proposition,[3] according to Whately, was a sentence which affirmed or denied something. It contained two terms: the subject, 'that which is spoken of,' and the predicate, 'that which is said of [the subject].'[4] Subject and predicate were connected by the verb *is* or *is not*, which was the only allowed copula. For Whately any other verb had to be resolved into an expression of the form *is + adjective*: '*e. g.* "the Romans conquered;" the word conquered is both copula and predicate, being equivalent to "*were* (Cop.) *victorious*" (Pred.)'[5]

Propositions could be classified according to two criteria, their *Quality* —being affirmative and negative— and their *Quantity* —universal or particular. A proposition was affirmative or negative depending on whether the copula was the verb *is* or *is not*, respectively. A proposition was universal 'if the Predicate is said of the *whole* of the Subject'. Similarly, a proposition was particular if the predicate is said 'of part of [the Subject] only.'[6] For example, 'No birds are mammals' and 'Socrates is a man' were instances of universal propositions; 'Some triangles are equilateral' and 'Some birds are not carnivorous,' of particular ones. Thus Whately discerned four kinds of propositions: universal affirmative ('All X is Y'), represented by A; universal negative ('No X is Y'), by E; particular affirmative ('Some X is Y'), by I; and particular negative ('Some X is not Y'), by O.[7]

[2]Richard Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, 9th ed. (Boston: James Munroe / Company, 1855), p. xiv.

[3]The propositions treated in this section are, to be precise, the 'pure categorical propositions' of Whately's classification. His treatise considers other sorts of propositions as well, such as hypothetical and mixed categorical propositions, but since this section will only contain extracts corresponding to the pure categorical ones, they will be referred to simply as 'propositions.'

[4]Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 64.

[5]Ibid., p. 65.

[6]Ibid., p. 72.

[7]Ibid.

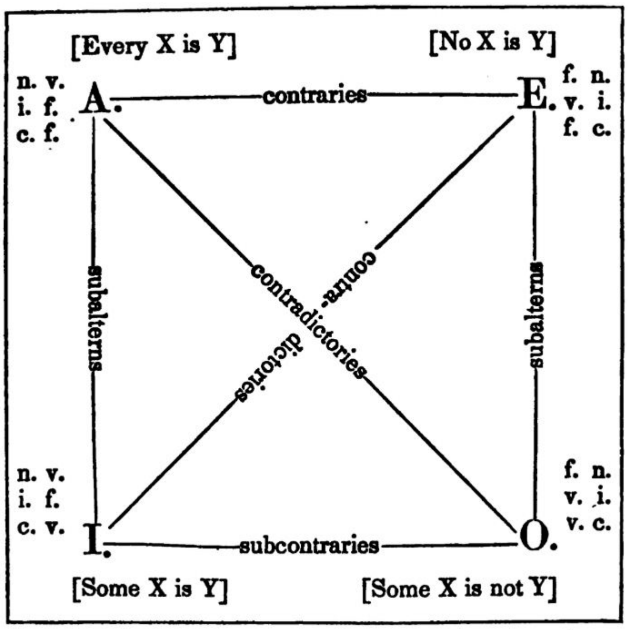


Figure 3.1: Whately's Opposition Scheme [8]

Propositions were said to be opposed to each other 'when, having the same Subject and Predicate, they differ, in *quantity*, or *quality*, or *both*.'[9] According to this, there were four sorts of opposition. That is, two propositions could be contraries, subcontraries, contradictories, or subalterns, depending on the two kinds of propositions that were being compared (see figure 3.1). The opposition square gave information about the truth or falsity of certain propositions in relation to the truth or falsity of their opposed statements.

On this basis, Whately introduced syllogisms. In his view, a syllogism was an argument 'stated *at full length* and in its *regular form*.'[10] It contained three propositions, set up in a specific order: the first two were the premises, and the last one was the conclusion. So the basic structure of the syllogism was like this:

Premise:
Premise;
therefore Conclusion.

Denoting the order of the propositions within a syllogism by listing their corresponding symbols or letters in order determined the *mood* of the syllogism. Thus, for example the mood

[8]ibid., p. 78. Note the lowercase letters next to the capital letters denoting the kind of proposition. According to Whately, 'v' and 'f' stood for 'verum' and 'falsum' respectively. Similarly, 'n' stood for 'necessary matter', 'i' for 'impossible,' and 'c' for 'contingent.' See ibid., p. 77 for a more extended explanation on how to interpret the opposition scheme.

[9]Ibid., p. 76.

[10]Ibid., p. 86.

AEI corresponded to a syllogism of the form:

> Universal affirmative premise:
> Universal negative premise;
> *therefore* Particular affirmative conclusion.

Since there were four types of pure categorical propositions, and each syllogism contained three propositions, once the order of those propositions had been fixed there were 64 possible moods. However, most of the combinations were 'inadmissible' since they violated some of the canons for the syllogism Whately had established earlier on, and so 'there remain but eleven Moods which can be used in legitimate syllogism.'[11]

Further, the premises and the conclusion were related to each other within a syllogism by the fact that exactly three terms appeared as their subjects and predicates.[12] The subject of the conclusion was called the *minor term* and was always one of the two terms of the second premise, which was accordingly called the *minor premise*. The predicate of the conclusion was the *major term*, and it was one of the terms of the first premise, therefore called the *major premise*. The remaining term, common to both premises but not present in the conclusion, was called the *middle term*. Depending on how the terms were placed within the syllogism, one could differentiate four different *figures*: '[t]he Figure of a syllogism consists in the situation of the Middle-term with respect to the Extremes of the Conclusion [*i.e. the major and minor term.*]'[13] Figure 3.2 lists the four possible syllogistic figures according to Whately.

Let X represent the Major term, Z the Minor, Y the Middle.

1st Fig.	2d Fig.	3d Fig.	4th Fig.
Y, X,	X, Y,	Y, X,	X, Y,
Z, Y,	Z, Y,	Y, Z,	Y, Z,
Z, X,	Z, X,	Z, X,	Z, X.

Figure 3.2: The four Figures of the Syllogism[14]

An analysis of all possible combinations of moods and figures showed that, if adhering to the canons of the syllogisms, each figure admitted only six moods. Out of those, many of them were be 'useless' since they had 'a particular Conclusion, when a *universal* might have been drawn.'[15]

So in practice there remained nineteen relevant syllogisms, which were identified by their

[11] Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 95.
[12] As required by Whately's first canon of the syllogisms. See ibid., p. 91
[13] Ibid., p. 96.
[14] ibid., p. 97
[15] Ibid.

mood and figure and for which 'logicians have devised names to distinguish both the Mood itself, and the Figure in which it is found' (see figure 3.3).

Fig. 1. { **bArbArA, cElArEnt, dArII, fErIOque prioris.**
Fig. 2. { **cEsArE, cAmEstrEs, fEstInO, bArOkO,* secundæ.**
Fig. 3. { **tertia, dArAptI, dIsAmIs, dAtIsI, fElAptOn, bOkArdO,† fErIsO, habet: quarta insuper addit.**
Fig. 4. { **brAmAntIp, cAmEnEs, dImArIs, fEsApO, frEsIsOn.**

Figure 3.3: The Names of the Syllogism[16]

One could refer thus to a particular kind of syllogism by specifying its mood and figure, or just by giving the corresponding name. For example, the classic syllogism

All men are mortal beings:	*(All Y are X)*
Socrates is a man;	*(All Z are Y)*
therefore, Socrates is a mortal being.	*(All Z are X)*

would turn out to be an instance of Barbara, i.e. the mood AAA in the first figure.[17] Similarly, Celarent always referred to EAE in the 1st figure, whereas EAE in the 2nd figure was denoted by Cesare.[18]

3.2 The *Mathematical Analysis of Logic* and the *Laws of Thought*

Around mid nineteenth century two books were published: *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning* and *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, published in 1847 and 1854 respectively and both written by the Irish mathematician George Boole. The

[16] ibid., p. 98

[17] The proposition 'Socrates is a mortal being' was considered to be universal because the subject is an instance of a single individual. Hence, in a way, it would be like saying 'All men-who-are-Socrates are mortal beings.' In this way it becomes clearer that the propositions in the example are indeed universal propositions.

[18] In the mnemonic verses, not just the vowels in the names provided information about the syllogism they referred to: in the sections 'Ostensive Reduction' and 'Reductio ad impossibile' of the *Elements* Whately showed that the syllogisms in the second, third, and fourth figure could be 'reduced' to one of the syllogisms in the first figure, to which Aristotle's Dictum *de omni et nullo* could be immediately applied. The consonants in the names of the syllogisms provided information about the sort of conversion which had to be performed on the syllogism in order to reduced it to the suitable syllogism in the first figure. See section 7 (Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, pp. 106–107) for Whately's explanation.

two books presented more or less the same theory: the *Mathematical Analysis of Logic* was a first compilation of Boole's ideas on the topic, and the *Laws of Thought* consisted of a more structured and elaborated presentation of those same ideas.

In the *Mathematical Analysis of Logic*, Boole presented a new approach to the logic he had learned from Whately. He realized that it was possible to express propositions by means of algebraic expressions, and that manipulation of these expressions by mathematical rules delivered expressions which could be interpreted back into propositions in a way that was logically sound according to traditional syllogistic.

To begin, Boole let capital letters X, Y, Z denote classes and 1 the Universe, which he understood as something 'comprehending every conceivable class of objects whether actually existing or not.'[19] The symbols x, y, z, which he called *elective* symbols,[20] had the property of selecting, from a given class, the elements which were X, Y, Z, respectively.[21] This process of 'selection' was indicated by concatenation of the corresponding symbols. Consecutive application of elective symbols, such as xy, represented thus the operation of first selecting the class of Ys, and then selecting from that class the elements which were both Xs and Ys. A single elective symbol was assumed as operating on the Universe, thus resulting in the expression

$$x1 = x.$$

Boole gave three laws governing the operation represented by elective symbols: first, he noted that it made no difference to separate a class into parts, select from each part those elements which were for example Xs, and then group these selections back into a single class, or to directly select the elements which were Xs from the class as whole. He expressed this first law with elective symbols by means of the equality

$$x(y + z) = xy + xz,$$

where $y + z$ represented 'the undivided subject' and y and z 'the component parts of it.'[22]

In the second place, Boole argued that, if x represented 'horned' and y represented 'sheep,' then the expression xy indicated the operation of selecting, from the class of horned things, those which were sheep, hence referring to the class of 'horned sheep.' Similarly, the expression yx referred to 'sheep which are horned' or just 'horned sheep' again. Therefore, xy and yx were actually the same, implying that the equality

$$xy = yx$$

[19] George Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning* (Cambridge: Macmillan, Barclay / Macmillan, 1847), p. 15.

[20] ibid., p.16. He called them *literal* in Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, p.27.

[21] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 15.

[22] Ibid., p. 17.

held for any two elective symbols x and y. This was the second law.

Next, Boole noted how performing the operation of selecting x from the class x did not modify anything since one obtained the totality of x again. Expressing this fact in elective signs gave Boole the equality

$$xx = x \quad \text{or} \quad x^2 = x.$$

He remarked that this operation could be performed an arbitrary number of times and still deliver the same result, i.e.

$$x^n = x,$$

an equality which Boole identified as the third law of the science of reasoning.

He concluded:

> From the first of these [rules], it appears that elective symbols are *distributive*, from the second that they are *commutative*; properties which they possess in common with symbols of *quantity*, and in virtue of which, all the processes of common algebra are applicable to the present system.

This was him acknowledging the clear similarities between the 'laws governing the mind' and the rules of operation in arithmetic. He added:

> The third law we shall denominate the index law. It is peculiar to elective symbols, and will be found of great importance in enabling us to reduce our results to forms meet for interpretation.[23].

That is, the index law was actually unique to the science of the mind and had no corresponding counterpart in common arithmetic.

There were some differences in Boole's presentation between the *Mathematical Analysis of Logic* and the *Laws of Thought*. For example, in the *Mathematical Analysis of Logic*, the symbol 0 was introduced as the result of selecting from the class of not-Ys those elements which are Xs, given the assumption that all Xs are Ys. Boole expressed this as follows:

$$x(1 - y) = 0$$

This implicitly defined 0 as the empty class (in contrast with the class 1 representing the Universe) and Boole proceeded to use the symbol without further consideration.

However, in the *Laws of Thought*, the introduction of 0 and 1 as the symbols for Nothing and the Universe, respectively, was given as a derivation, in Proposition II of Chapter III *Derivation*

[23]Grattan-Guinness noted in his 1988 paper that these laws run almost parallel to the rules Boole had proposed for some operators in a mathematical paper from 1844. Only the index rule was different, but it was given the same name. See Grattan-Guinness, 'Living Together and Living Apart. On the Interactions Between Mathematics and Logics from the French Revolution to the First World War', p. 75.

of the Laws: 'To determine the logical value and significance of the symbols 0 and 1.'[24] By noting that

$$0y = 0$$

and

$$1y = y$$

were true equalities for the symbols 0 and 1 in algebra for any value of y, and inquiring for an interpretation in the system of logic such that the same laws hold there too, Boole concluded once more that 0 must mean Nothing (the empty class) and 1 must in fact be the Universe.

Boole also introduce the class $1 - x$ 'including all individuals that are not Xs.'[25] That is, if x represented 'men,' then $1 - x$ stood for 'not-men.' In this manner, it followed for example that

$$x(1 - x) = 0,$$

i.e. the class x and the class $1 - x$ had no elements in common, a consequence that was consistent with the way the classes had been defined. Boole noted correctly that this fact could be derived from the index law as well, since from the equation

$$x^2 = x$$

it followed by algebraic manipulation that

$$x - x^2 = 0,$$

which was just

$$x(1 - x) = 0.$$[26]

These considerations were enough to provide equations of elective symbols which expressed all four kinds of pure categorical propositions (see section 3.1, page 20):[27] Boole showed that A propositions, of the form 'All Xs are Ys,' became

$$xy = x$$

[24]Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, III.13.

[25]Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p.20.

[26]This is the content of Proposition IV of Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, III.15. This also exemplified something for which Boole would be criticized later on, and that was the fact that not all symbolic expressions which appeared in his chains of derivations could be interpreted back in a way that made sense within the context of logic. This would be the case, for example, of the expression $x - x^2 = 0$, which, written like that, had not direct logical interpretation.

[27]Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, pp. 20–25.

because if all members of Xs were in Ys already, then selecting from Y those elements which were X just gave X itself. Another way to express this was as $x(1-y) = 0$. Similarly, E propositions, of the form 'No Xs are Ys,' became

$$xy = 0,$$

meaning that selecting from Y those elements which were X resulted in the empty class, i.e. there were no Xs which were Ys (and vice versa, note the symmetry of the expression).

Problems would have arisen, however, when trying to express particular propositions: how could a symbolic equality capture the notion of 'some'? Boole's solution was to introduce a separate elective symbol, v, which could operate on other elective symbols and was 'indefinite in every aspect but this, viz., that some of its members are [members of the class it is operating on].'[28] He gave the following example: 'let x stand for "mortal beings," then will vx represent "some mortal beings."' Both in the *Mathematical Analysis of Logic* and in the *Laws of Thought*, Boole did not go into much more detail about the way this sign was defined, and he used it more or less freely. The way he expressed particular propositions was as follows: I propositions, represented by 'Some Xs are Ys' became

$$v = xy.$$

And similarly, O propositions, expressed generally by 'Some Xs are not Ys,' became

$$v = x(1-y).$$

With this setup in place, it became possible to express syllogistic reasoning by means of systems of two equations in three variables, each equation representing one premise and each variable representing one term. By eliminating the variable corresponding to the middle term, Boole obtained an equation in two variables which represented the conclusion.[29] So, for instance, the syllogism

All Xs are Ys,
No Zs are Ys,
$\therefore$ No Zs are Xs.

The premises here are AE in the second figure, which in Boole's notation were expressed as

$$x(1-y) = 0 \quad \text{or} \quad x = xy,$$
$$\text{and} \quad zy = 0.$$

Multiplying the equations and eliminating the y symbol yielded the equation $zx = 0$, which correctly corresponded to the expected conclusion.[30]

[28] Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, IV.11.

[29] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 32.

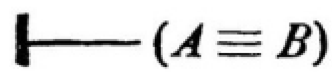
[30] This example is given by Boole on p. 35 of the *Mathematical Analysis of Logic*.

3.3 The *Begriffsschrift*

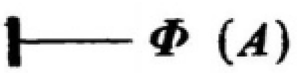
The *Begriffsschrift, eine der arithmetischen nachgebildeten Formelsprache des reinen Denkens* was written by Gottlob Frege and first published in 1879 as a little booklet of 88 pages, organised in three chapters and preceded by a foreword. The preface contained Frege's justification for the need of such a formula-language.[31] In the first chapter, called 'Erklärung der Bezeichnungen,' Frege introduced his new language and explained how it was built up from a few basic symbols. In the second chapter, 'Darstellung und Ableitung einiger Urtheile des reinen Denkens,'[32] he showed how inferential reasoning could be represented within his system, gave its basic rules, and showed how it could be used to deduce a number of judgments. The third part, 'Einiges aus einer allgemeinen Reihenlehre,'[33] culminated in his giving a formal definition of sequence using the Begriffsschrift as he had introduced it in the previous chapters.

(a) Die Bedingtheit
(Conditionality)
'If B, then A.'

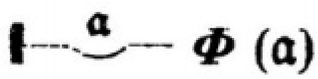
(b) Die Verneinung
(Negation)
'Not A.'



(c) Die Inhaltsgleichheit
(Identity of content)
'A and B represent the same content.'



(d) Die Funktion
(Function)
'Φ holds of A.'



(e) Die Allgemeinheit
(Generality)
'Φ holds of any A.'

Figure 3.4: Frege's basic notation

Frege opened his exposition by explaining the 'Urtheilsstrich' (judgment stroke) and the 'Inhaltsstrich' (content stroke), the basic and most easily recognizable features of the Begriffsschrift. Whereas the horizontal content stroke was just used to indicate the conception that the corresponding judgment might hold, appending to it a vertical judgment stroke transformed

[31] See section 4.4 for further explanation on the 'Vorwort' and Frege's justification of the *Begriffsschrift*.

[31] 'Definition of the symbols.' All English translations in this chapter are taken from Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought' unless otherwise stated.

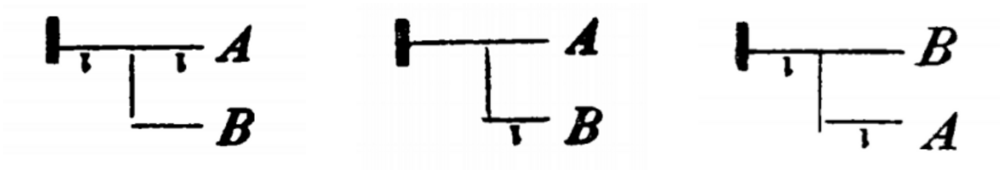
[32] 'Representation and derivation of some judgments of pure thought.'

[33] 'Some topics from a general theory of sequences.'

the whole expression into the assertion that the judgment did in fact hold. Therefore, only judgeable content could be appended to a content stroke.[34]

He then moved on to explaining the building blocks of his system. The first one was 'die Bedingtheit' or implication (figure 3.4a). Frege defined it in a truth-table manner: he stated the four possible combinations of truth values given two judgments A and B and then defined his implication as being true whenever the possibility that A be true and B be false was excluded.

Next he introduced a symbol for negation or 'Verneinung' (figure 3.4b), and explored the possibilities it offered in combination with the implication symbol. In particular, he showed how to express the connectives 'and' and 'or' (figures 3.5a and 3.5b, respectively) and a connective that could be read as 'neither ... nor' (figure 3.5c).



(a) 'B and A' (b) 'B or A' (c) 'neither B nor A'

Figure 3.5: Some combinations of 'Bedingtheit' and 'Verneinung'

Then came the symbol to express identity of content or 'Inhaltsgleichheit' (figure 3.4c). In the *Begriffsschrift*, such a symbol was introduced without much hassle, but it seems that later in his life Frege raised some concerns regarding its use.[35]

[34] Gottlob Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', in *Begriffsschrift und andere Aufsätze*, ed. Ignacio Angelelli (Hildesheim: Georg Olms Verlag, 1993), I.2.

[35] The issue here concerns the problem that arises from considering expressions such as '$a = a$' versus '$a = b$', that is, the subtle difference between identity and equality. In the first case, no matter which content a refers to, the equality $a = a$ will hold vacuously. However in the second case further investigation is necessary. In Frege's own words at the beginning of his paper 'Über Sinn und Bedeutung,' ('On meaning and reference')

> Die Gleichheit* fordert das Nachdenken heraus durch Fragen, die sich daran knüpfen und nicht ganz leicht zu beantworten sind. Ist sie eine Beziehung? eine Beziehung zwischen Gegenständen? oder zwischen Namen oder Zeichen für Gegenstände? Das letzte hatte ich in meiner Begriffsschrift angenommen. Die Gründe, die dafür zu sprechen scheinen, sind folgende: $a = a$ und $a = b$ sind offenbar Sätze von verschiedenem Erkenntniswert: $a = a$ gilt a priori und ist nach Kant analytisch zu nennen, während Sätze von der Form $a = b$ oft sehr wertvolle Erweiterungen unserer Erkenntnis enthalten und a priori nicht immer zu begründen sind.
>
> *) Ich brauche dies Wort im Sinne von Identität und verstehe "$a = b$" in dem Sinne von "a" ist dasselbe wie "b" oder "a und b fallen zusammen". [Frege's footnote]

(Gottlob Frege, 'Über Sinn und Bedeutung', *Zeitschrift für Philosophie und philosophische Kritik* 100, no. 1 (1892): p.37)

In fact, this paper became very influential in linguistics and modal logic: the famous example of determining the truth value of a statement like 'Venus = morning star' was precisely the one given by Frege in the aforementioned article.

Next, he discussed his distinction between function and argument and introduced his notation for these concepts (figure 3.4d). He pointed out that the difference between the roles of argument and function was merely an issue of perspective: it was not relevant how a specific content was expressed as a function in certain arguments as long as both function and argument were determined at the moment of judgment. However, once one of the arguments was left undetermined, then the distinction was relevant because that generalized the meaning of the expression and therefore introduced differences at the level of content.[36]

The fact that a function could have more than one argument was also mentioned: by considering a function and letting one of its determined constituents become undetermined, one obtained a new function which had precisely one more argument than the original function.

There was yet another subtlety in the function-argument dichotomy that Frege promptly pointed out. It had to do with the use of quantified expressions as arguments. His example was as follows: he considered the statements 'the number 20 can be expressed as the sum of four squares' and 'every positive integer number can be expressed as the sum of four squares.' Then one might have been tempted to view them as one same function 'Being able to be expressed as the sum of four squares' with different argument: 'the number 20' in one case and 'every positive integer number' in the other. However, as Frege noted, these two concepts were not of the same kind ('Begriffe gleichen Ranges'), i.e. what could be predicated of 'the number 20' could not be said of 'every positive integer number' but possibly of each positive integer number (one-at-a-time).[37]

[36]'Für uns haben die verschiedenen Weisen, wie derselbe begriffliche Inhalt als Function dieses oder jenes Arguments aufgefasst werden kann, keine Wichtigkeit, solange Function und Argument bestimmt sind. Wenn aber das Argument *unbestimmt* wird . . . so gewinnt die Unterscheidung von Function und Argument eine *inhaltliche* Bedeutung.' In Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p.17. (''For us the fact that there are various ways in which the same conceptual content can be regarded as a function of this or that argument has no importance so long as function and argument are completely determinate. But, if the argument becomes *indeterminate* . . . then the distinction between function and argument takes on a *substantive* significance.' In Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', p. 23

[37]This explanation paraphrases Frege quite directly from Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', I.9:

> Wenn man die beiden Sätze: 'die Zahl 20 ist als Summe von vier Quadratzahlen darstellbar' und 'jede positive ganze Zahl ist als Summe von vier Quadratzahlen darstellbar' vergleicht, so scheint es möglich zu sein, 'als Summe von vier Quadratzahlen darstellbar zu sein' als Function aufzufassen, die einmal als Argument 'die Zahl 20,' das andre Mal 'jede positive ganze Zahl' hat. Die Irrigkeit dieser Auffassung erkennt man durch die Bemerkung, dass 'die Zahl 20' und 'jede positive ganze Zahl' nicht Begriffe gleichen Ranges sind. Was von der Zahl 20 ausgesagt wird, kann nicht in demselben Sinne von 'jede positive ganze Zahl,' allerdings aber unter Umständen von jeder positiven ganzen Zahl ausgesagt werden.

The last section in the first chapter of the *Begriffsschrift* was dedicated to 'die Allgemeinheit' (figure 3.4e), that is, the generality (equivalent to what we nowadays would call 'universal quantifier'). Frege's 'Allgemeinheit' was a basic building block of his system and had no limitation as to the kind of expressions it could act upon: it could refer to terms but also to functions themselves. This aspect of the 'Allgemeinheit' would be key later on in the text (in chapter 3 of the *Begriffsschrift*) as it allowed Frege to define, using only the tools he had developed within the Begriffsschrift, the concept of number succession.

Finally, the first chapter ended with the traditional square of opposition but written with the notation of the Begriffsschrift (see figure 3.6).

In chapter 2, Frege showed the Begriffsschrift in action. A total of 133 propositions were shown, from which he singled out nine propositions which built the core of his system and were not derived from any other proposition: these were 1, 2, 8, 28, 31, 41, 52, 54, 58 following Frege's numbering in the *Begriffsschrift*. For example, proposition 28 is just *modus tollens*,[38] and proposition 31 was Frege's admitting of elimination of double negation: 'Die Verneinung der Verneinung ist Bejahung.'[39]

As a final remark, it is true that Frege did not specifically treat syllogisms nor did he dedicate a part of the text to explicitly how they could be expressed with the notation of the Begriffsschrift. However, he did acknowledge this last fact in some remarks throughout the text, which are discussed below in section 4.3, page 44.

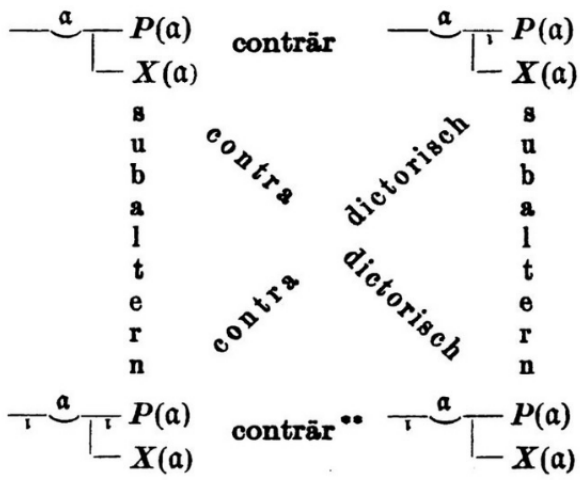


Figure 3.6: Die logische Gegensätze.[40]

[38] *Modus ponens* is an intrinsic characteristic of the Begriffsschrift since that was the only rule of inference Frege accepted. See ibid., I.6.

[39] ibid., II.18. 'The denial of the denial is affirmation.'

[40] Compare to the square of opposition in Whately's *Elements of Logic*. Note the double asterisk next to the word 'conträr': the footnote in the original text reads 'Sollte offenbar 'subconträr' sein. Anm. d. Hrsgs.' (ibid., p.24)

Chapter 4

Logic and Mathematics: four threads

After having discussed the historiography of logic in the nineteenth century and taken a look at some of remarkable texts on logic of that time, time has come to offer an interpretation of the period that accounts for what was pointed out at the end of chapter 2. The purpose of this chapter is to make a compelling argument that logic was going through a process of change from long before the publication of the *Begriffsschrift*; and not only this, but that the ways in which the *Begriffsschrift* was innovative nicely match patterns of change which were taking place throughout the century and not exclusively in the work of Frege. If this is shown convincingly, then it will follow that the question posed in the introduction of this thesis was indeed ill-posed and therefore inherently misleading.

The chapter opens with a description of two pieces of relevant historical context, namely the debate on symbolical algebra, exemplified in the work of George Peacock, and the question of the legitimacy of a generalized version of arithmetical algebra called symbolical algebra. It continues by analyzing some works on logic published during the nineteenth century up to and including the *Begriffsschrift*. These observations are organized around four threads, each outlining a different aspect of the transformation taking place in logic during that period of time: changes in the use of mathematical symbols for logical purposes, a widespread process of generalization of the concepts that logic was concerned with, the recognition of the distinction between language and metalanguage, and a general tendency to redefine the objects of study and thus implicitly —sometimes explicitly— the corresponding fields of knowledge.

4.1 Symbolical algebra and a modern approach to logic

Both logic and mathematics were going through important debates at the turn of the nineteenth century. On the one hand, discussions on the usefulness of logic and its place in the academic curricula of English universities was taking place, in particular Oxford and Cambridge. On the other hand, there was an ongoing debate about the legitimacy of certain mathematical constructions such as the negative and the complex numbers, and about the consequences of their existence for the classical understanding of symbolic manipulation in arithmetic.

Logic had been building up a bad reputation since the seventeenth century. Big figures like Kant and Locke had made it clear that they thought there was not much to be done in logic: the criticism of logic had grown excessively and the whole field was being dismissed as useless.[1] Doubts were being raised about the purpose of teaching logic at English universities: was it not archaic knowledge? What was the point of teaching a discipline which was of questionable usefulness and was complicated to explain? Logicians had allegedly been unable to improve the Aristotelian theory for a couple hundred years. Hence, critics were sure that there was simply nothing else to be discovered, just like Kant had famously remarked.

It was in this context that Richard Whately wrote the *Elements of Logic*, a textbook on logic which compiled the classic theory and based on older treatises (especially Herny Aldrich's *Artis Logic Compendium* from 1756). He did so with the aim to 'give logic a sound foundation, i.e. one which would accurately fix the value and point of the subject in a manner avoiding both sorts of excess'[2] —the two sorts 'excesses' being blind optimism about the power of logic on the one side, and pervasive skepticism about its value on the other. He attempted this by vigorously defending the status of logic as a science and explaining its foundations. His renewed presentation of logic became very popular, so much so that he himself talked about the raised interest in the subject as a 'revival' in the preface to the ninth edition of the *Elements*.[3] While the *Elements of Logic* was mainly a syllogistic text like the treatises on logic that had been published before him, the way in which Whately presented the theory and the aspects on which he put emphasis reflected a modern mentality. What he did was more than just simply restructuring of the older ways of presenting logic:[4] Whately chose to fully focus on the syllogism. Yet, he did not just re-explain the syllogism the way Aldrich and his predecessors had done; his conception thereof was novel in itself. To Whately, the syllogism had to be seen not anymore as a tool to produce true knowledge or as a specific kind of argument, but as a formal device for determining correct inference in reasoning, a basic structure to which any other argument could be reduced. Consider the following quotation:

> One or both of the Premisses of a perfectly valid Syllogism may be utterly false and absurd: and then, the Conclusion, though inevitably following from them, may be

[1] van Evra, 'Richard Whately and the Rise of Modern Logic', pp. 7–8.

[2] Ibid., p. 3.

[3] 'The *revival* of a study which had for a long time been regarded as an obsolete theory, would probably have appeared to many persons, thirty years ago as an undertaking far more difficult than the introduction of some *new* study; —as resembling rather the attempt to restore life to one of the antediluvian fossil-plants, than the rearing of a young seedling into a tree.' In Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. xvii.

[4] van Evra, 'Richard Whately and the Rise of Modern Logic', p. 3.

> either true or false . . . the *conclusiveness*, —that is, the connection between the Premisses and the Conclusion— is perfectly certain.[5]

This is an example of the way Whately was framing the syllogism: its usefulness was not based on the fact that it could be used to detect truth, but just on the fact that by means of the syllogism it was possible to determine instances in which inference had been performed properly.

Meanwhile, the turn of the century witnessed an intense debate on the validity and desirability of certain developments in algebra, such as the use of negative and imaginary numbers. As a consequence, the validity of algebra as a science of its own —and not simply as an extension of arithmetic— was under scrutiny. A representative figure in this debate was the English mathematician George Peacock, fellow of the Trinity College in Cambridge.[6] According to Peacock, algebra was to be considered 'the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws.'[7] This general approach to algebra he called 'Symbolical Algebra' in contrast to 'Arithmetical Algebra,' which was the 'most general form'[8] of arithmetic, the 'mode of exhibiting its laws in their most general form'.[9] In his *Treatise on Algebra*, Peacock presented a thorough justification of the use of 'arbitrary signs' and laws as a sufficient foundation of symbolical algebra. He wrote:

> we assume [the laws and operations of Arithmetic] as the guide for those *assumptions* in Symbolical Algebra, which constitute its real foundation, and which alone can give it the dignity and the character of a demonstrative science.[10]

This was Peacock's main point indeed: the laws upon which symbolical algebra was built were 'chosen, not deduced,'[11] i.e. they were rules that were fixed beforehand and independent of the particular system. That was in contrast with the laws of arithmetical algebra, which were deduced from the nature of numbers and the different operations that could be performed with them. Accordingly, rules in arithmetical algebra were basically formulas, placeholders for actual numerical computations. This, in turn, restricted the variety of the manipulations that could be done on strings of symbols, since, for example, expressions like $a - b$ became meaningless in the case where $b > a$. Hence, the emphasis on the arbitrariness of the laws of

[5] Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 88.

[6] For a nice presentation of the state of affairs in this debate at the turn of the nineteenth century as well as an exposition of Peacock's contributions to it, see Helena M. Pycior, 'George Peacock and the British origins of symbolical algebra', *Historia Mathematica* 8, no. 1 (1981): 23–45.

[7] George Peacock, *A Treatise on Algebra*, 2nd ed. (Cambridge: J. / J. J. Deyton, 1830), III.78.

[8] Ibid., III.79.

[9] Ibid., III.83.

[10] ibid., italics in the original.

[11] Ibid.

symbolical algebra, because in that context a sound numerical interpretation to each symbolical expression was not required anymore. Therefore, the possibilities of algebraic manipulation were much broader and the meaning of the operations became abstract.

The two influences —Whately's defense of logic as a science based on a strong parallelism with arithmetic, and the conceptual changes happening to logic which Peacock exemplified—contribute crucially to a historical understanding of Boole's system of logic in the *Mathematical Analysis of Logic*, later expanded in the *Laws of Thought*. There is explicit as well as implicit acknowledgment of both of them already in the *Mathematical Analysis of Logic*. To begin with, the opening sentence of the book directly placed it in the context of the debate in symbolical algebra:

> They who are acquainted with the present state of the theory of Symbolical Algebra are aware that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed but solely upon the laws of their combination.[12]

This implies, in particular, that Boole accepted the positive answer to the debate. Consequently, the whole building of the *Mathematical Analysis of Logic* was based upon the very premise that it was valid to study any system via the arbitrary rules that lay at its core. For this reason, Boole, in the *Mathematical Analysis of Logic*, first introduced the symbols and the laws of their combination, making sure at each step that the rules really captured the actual laws of reasoning. Only then did he proceed to study the remaining system on its own on the basis of said rules, almost detached to the interpretation for which it had been conceived in the first place.

At the same time, Boole explicitly acknowledged Whately's presentation of logic as being the source for his own treatment of the topic: he stated that '[t]he above is taken, with little variation, from the Treatises of Aldrich and Whately.'[13]

In this way, all innovations introduced by Whately (both at the of the manner in which he structured and delivered his logical theory, and also at the level of his conscious use of variables, the analogies with mathematics, and the remarks about the role of the syllogism within logic) were inherited and further explored by Boole. Therefore, it is reasonable to conclude that, as much as Boole's contributions were original and creative, the role of his particular historical context in his ideas and writings is undeniable: he was a child of his time.

[12] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 3.

[13] ibid., p. 20. Note that, in turn, Whately had followed Aldrich —both as a source of information and as an example of the ways in which his own *Elements of Logic*presented an improved theory of logic.

4.2 The use of mathematical symbols

When trying to understand the transformation of logic into mathematical logic, the incorporation of mathematical symbols in logic texts cannot be ignored. This concerns not only the presence of mathematical symbols in logical texts, but especially the way in which they were used. The role those symbols played in logic was changing: influenced by the shift in mentality brought about by symbolical algebra and combined with a revised perspective on logic itself, which had put the syllogism at the forefront and re-framed it as a purely formal device, mathematical notation became the way of approaching this modernized version of logic instead of simply being the auxiliary notation it had been in the past. Furthermore, the symbols started being used operationally and not as simple abbreviations.

The use of letters to denote propositions, as well as the use of some mathematical symbols such as $=$ or $+$ to express connectives was not a novelty of the nineteenth century: the Greeks had already used capital letters to denote concepts, and Leibniz experimented with using the sum sign $+$ to denote the combination of concepts (instead of concatenating the corresponding letters as it was commonly done). He then accepted expressions of the form $A + B$, provided that A and B didn't have anything in common.[14] Leibniz also wrote about an ideal *lingua characterica* or *calculus ratiotinator*, motivated by the hope that 'a scientifically designed language would help men to think clearly.'[15] However, he never fully explored the consequences of those ideas and certainly did not develop them into an actual program or a concrete language. As much as the use of letters in syllogistic texts was commonplace, it never went further than being short-hand for longer expressions, a mere placeholder for terms within propositions.

A slightly different take on the usage of mathematical symbols in logic appeared in George Bentham's *Outline of a New System of Logic*, published in 1827 shortly after the *Elements of Logic* had come out. The book was a detailed commentary on Whately's treatise, in which Whately's ideas were compared to those of Bentham and of his uncle, the philosopher Jeremy Bentham. The book also contained words of admiration towards the contents of Whately's exposition of logic, for Bentham considered the *Elements* to be 'the last and most improved edition of the Aristotelian system.'[16] Bentham was very thorough with his commentary on Whately, going through the *Elements of Logic* almost point by point and presenting an alternative approach to logic alongside his comments. The *Outline of a New System of Logic* contained mathematical notation as means of expressing the general structure of the different kinds of propositions.

In Bentham's notation, the usual equality sign was used as the copula 'is' and the same sign

[14]Kneale and Kneale, Martha, *The Development of Logic*, p. 340.

[15]Ibid., p. 326.

[16]George Bentham, *Outline of a New System of Logic, with a Critical Examination of Dr. Whately's "Elements of Logic"* (London: Hunt / Clarke, 1827), p. vii.

[17]ibid., p. 134

1.	X in toto	=	Y	in toto	or	t X	=	t Y
2.	X in toto	=	Y	ex parte	or	t X	=	p Y
3.	X in toto	||	Y	{ in toto or ex parte	or	t X	||	t/p Y
4.	X ex parte	=	Y	ex parte	or	p X	=	p Y
5.	X ex parte	||	Y	ex parte	or	p X	||	p Y

Figure 4.1: Bentham's classification of propositions.[17]

rotated 90 degrees represented the copula 'is not.' Besides, distributed terms were indicated by preceding their corresponding symbol by the letter *t* (from 'in toto'). Analogously, terms which not were distributed were indicated by placing the letter *p* before the corresponding symbol (from 'ex parte,' see figure 4.1).

Still, the best known author for introducing mathematical notation in logical reasoning was George Boole. Boole's use of mathematical symbols in logic was different than that of his contemporaries (most notably William Hamilton and Augustus De Morgan) because it pushed the analogy between logic and mathematics one step further. He did not limit his system to a mere method of translation of expressions from natural language to mathematical notation, but went beyond that by exploiting the resulting equations with techniques and ideas coming from symbolical algebra. That is, he did not just rewrite natural language in the form of equations, but actually manipulated them algebraically *as if* they were actual equations. This proved to be a fruitful and attractive approach, as it retained enough structure from common logic to still be recognizable and familiar, but at the same time it was new and different enough to deliver results and insights that had not been available before.

An explicit instance of the sort of new insight that was gained by using mathematical notation in logic and employing algebraic rules to manipulate the expressions would be the following. In the chapter on syllogisms, Boole did a thorough analysis of all possible combinations of premises or systems of equations that one could encounter and derived, on general terms, the conclusion that could be reached by elimination of the variable corresponding to the middle term (see example at the end of section 3.2, page 23). He listed some 'lawful cases,' in which an interpretable equation was obtained as the conclusion, but also other cases in which the equation obtained could not be interpreted back into a meaningful proposition.[18] After analyzing this issue in some depth, Boole came to the realization that there was indeed something in common among those forms which delivered no inference, something which could be expressed as a 'mathematical condition':

> The mathematical condition in question, therefore, —the irreducibility of the final

[18] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 40.

> equation to the form $0 = 0$,— adequately represents the logical condition of there being no middle term, or common medium of comparison, in the given premises. I am not aware that the distinction occasioned by the presence or absence of a middle term, in the strict sense here understood, has been noticed by logicians before. The distinction, though real and deserving attention, is indeed by no means an obvious one, and it would have been unnoticed in the present instance but for the peculiarity of its mathematical expression.[19]

Thus, to Boole it was clear that his approach was enlightening inasmuch as it allowed to reach logical insight which would have not been discernible by just using natural language or the techniques from classical logic.

However, the issue of the introduction of mathematical symbols in logic and the way they should be interpreted was not settled with the publication of Boole's work. Jevons wrote that Boole's system 'is, perhaps, one of the most marvelous and admirable pieces of reasoning ever put together'[20] in his essay *Pure Logic* from 1864, but that same text contained Jevons' objections to some aspects of Boole's creation. For instance, he rejected Boole's exclusive or disjunctive interpretation of the particles 'or' and 'and'[21] as well as the legitimacy of the operation of addition within pure logic.[22] Nevertheless, Jevons did recognize the peculiarity of the method which Boole had described. The quotation is fascinating enough to make it worth presenting in full:

> Supposing it prove true that Professor Boole's Calculus of 1 and 0 has no real logical force and meaning, it cannot be denied that there is still something highly remarkable, something highly mysterious in the fact, that logical forms can be turned into numeral forms, and while treated as numbers, still possess formal logical truth. It proves that there is a certain identity of logical and numerical reasoning. Logic and mathematics are certainly not independent. And the clue to their connection seems to consist in distinct logical terms forming the units of mathematics.[23]

These instances show how the use of mathematical notation was far from the obvious or natural way to proceed in logic. It had indeed many supporters, and it did open many doors

[19]ibid., p. 41. In the postscript to that same piece, he recognized that the aforementioned discovery followed from the theory presented by De Morgan, and that he would therefore 'relinquish all claim to a discovery.' Still, he was aware of the peculiarity of his presentation: 'The mode in which it appears in this treatise is, however, remarkable.'

[20]W. Stanley Jevons, *Pure Logic, or the Logic of Quality apart from Quantity, with remarks on Boole's system and the relation between Logic and Mathematics* (Edward Stanford, 1864), XV.174.

[21]Ibid., XV.178.

[22]Ibid., XV.184.

[23]Ibid., XV.204.

to new problems and new questions, but that practice would still need some more time to become the norm.

4.3 Generalization

The use of mathematical notation in logical texts brought deeper transformations into the realm of logic. The expression in mathematical symbols of certain aspects of the classical logical theory made their underlying structure clearer to logicians. Some of these logicians were mathematicians as well, and so the next step could only have been natural for them: to explore to what extent the symbolical expressions obtained from logic could be generalized while still making sense from the logical standpoint. It seems that this kind of transformation was taking place since the beginnings of the nineteenth century, noticeable in the *Elements of Logic*. As Van Evra put it, 'Whately's contribution . . . is a good example of the type of abstractive ascent typical of significant changes in scientific theory generally.'[24] In other words: Whately's work was setting a trend of abstraction —and subsequent generalization— in logic.

Bentham, for example, was one of the first to remark that the rules around the issue of 'distributing the predicate' were unnecessarily rigid. Classically, the possibility of distributed predicates was ruled out or only allowed in very specific circumstances.[25] However, Bentham wrote:

> Logicians in general make no mention of the first form,[26] which they consider as useless, and they say that the predicate (or second term of a proposition) is *never distributed* (that is, *universal*). I should think however, that this assertion can scarcely be logical. Many fallacies arise from the considering terms as synonymous which are not so in reality; and it may be found as advantageous to reduce perfect identity to a logical form; as partial identity, or perfect or partial diversity.[27]

[24]van Evra, 'Richard Whately and the Rise of Modern Logic', p. 16.

[25]The concept of 'distribution of terms' is a technical aspect of logic which will not be covered here; however let it be an illustrative remark the 'two practical rules to be observed respecting distribution' that Whately had given in the *Elements of Logic*:

> 1st. All universal propositions (and no particular) distribute the *subject*.
>
> 2d. All negative (and no affirmative) the predicate.

(Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 75)

[26]The first form was 'X in toto $=$ Y in toto' (see figure 4.1 on page 38).

[27]Bentham, *Outline of a New System of Logic, with a Critical Examination of Dr. Whately's "Elements of Logic"*, p. 135. The resemblance of this last sentence to Frege's concerns in his article *Über Sinn und Bedeutung* about the validity of his sign for equivalence are remarkable to say the least (see footnote on page 29): Bentham wrote these lines more than 50 years before Frege published the *Begriffsschrift*, yet he did not become a widely recognized protagonist in the history of modern logic. Remarks like this one show how certain questions were bigger than the authors

These remarks by Bentham make the *Outline* one of the first texts where the issue of the distribution or 'quantification' of the predicate was considered, predating Boole and De Morgan's better known contributions.

As described earlier, Boole's system was based on the logic he had learned from Whately's *Elements of Logic*. Via his algebraic rendering of logic, Boole was led to recognize that there were certain aspects of syllogistic logic that were mere conventions. Firstly, he identified the order of the major and the minor terms in the conclusion of a syllogism as being an instance of such a convention.[28] Boole noticed that there was not a strong reason to restrict the study of syllogism just to those formal arguments that abode by the rule that the major term be the subject and the minor term be the predicate of the conclusion of a syllogism:

> Between the forms about to be developed, and the Aristotelian canons, some points of difference will occasionally be observed. . . . [T]he essential structure of a Syllogism is, in some measure, arbitrary. Supposing the order of the premises to be fixed, and the distinction of the major and the minor term to be thereby determined, it is purely a matter of choice which of the two shall have precedence in the Conclusion. . . . [T]his is a convention. Had it been agreed that the major term should have the first place in the conclusion, a logical scheme might have been constructed, less convenient in some cases than the existing one, but superior in others. What is lost in *barbara*, it would gain in *bramantip*.[29]

His remark that 'what is lost in *barbara* it would gain in *bramantip*' was meant to illustrate the fact that, should the order of the terms in the conclusion have been different, then the classification of valid moods of the syllogism would be different. In particular, and assuming that the major term was the subject of the conclusion, then Barbara would not be valid in the first figure anymore, because

All men are mortal,	*(All Y are X)*
Socrates is a man;	*(All Z are Y)*
therefore, All mortal entities are Socrates.	*(All X are Z)*

would clearly not be not a valid inference. Conversely, Bramantip would be a valid mood in the first figure, although it was not according to the traditional Aristotelian classification (see section 3.1):

All men are mortal,	*(All Y are X)*
Socrates is a man;	*(All Z are Y)*
therefore, Some mortal entity is Socrates.	*(Some X are Z)*

themselves: they belonged to the discipline, to the problems that were relevant at that time.

[28] As pointed out in chapter three, traditional logic required that the conclusion be a proposition of the form 'All/Some *Z* are/are not *X*' where *Z* was the minor term and *X* was the major term of the syllogism.

[29] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 33.

which would indeed be valid. The fact that Bramantip would make up for Barbara in such a scheme was due to the fact that the proposition 'Socrates is mortal' became 'Some mortal (entity) is Socrates' by conversion *per accidens*, so they expressed the same inference.[30]

Boole was therefore realizing that there was no need to restrict the study of the syllogism to the traditionally accepted forms. He saw that if one wanted to treat logic as a science and uncover the laws that governed it, it would be necessary to follow the consequences of this general approach to its last consequences. Hence, he was acknowledging the possibility of expanding the scope of what Aristotelian logic was capable of accounting for —aided by the abstraction which the use of mathematical notation in logical reasoning provided and influenced by the similar processes going on in symbolical algebra.[31]

To sum up, consider the following quotation from the *Mathematical Analysis of Logic*:

> Every process will represent deduction, every mathematical consequence will express a logical inference. The generality of the method will even permit us to express arbitrary operations of the intellect, and this lead to the demonstration of general theorems in logic analogous, in no slight degree, to the general terms of ordinary mathematics.[32]

This exemplifies almost word by word the points made so far: Boole's perception of his system was that he had devised a method for logical analysis, which was so general that it could be used to even express 'arbitrary operations of the intellect.' This sort of extrapolation represents the generalized view of logic that followed from combining Whately's modern recasting of logic with the abstract techniques of symbolical algebra. And it also shows that Boole still saw his system as chiefly concerned with logic and the faculties of the intellect, for even if he was able to recognize the parallels with mathematics and mathematical reasoning, that's all it was: a similarity which reinforced the legitimacy of approaching logic as a science.

Similar reflections on the conventional nature of certain teachings of logic, as well as remarks about syllogism can be found in the *Begriffsschrift*. Frege's celebrated rejection of the subject-predicate division and the introduction of the function-argument dichotomy introduced a different sort of generality into the traditional structure of syllogistic logic. The division into subject and predicate lay at the basic understanding of the structure of any proposition and was consequently tied to the classification of syllogisms into four figures, because that classification was in turn based on the relative positions of the minor, middle, and major terms

[30] See Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 83.

[31] A study of the syllogism as a means to detect changes happening in logic mid-nineteenth century was done by Van Evra in his paper van Evra, 'The Development of Logic as Reflected in the Fate of the Syllogism 1600-1900'. Especially relevant to our text are sections 3 and 4 of the article, in which Van Evra made very similar remarks to the ones made here.

[32] Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 6.

within the syllogism.[33] Frege clearly advocated for a different approach: he suggested viewing propositions as instances of 'functions' with different 'arguments.' Under this view, the subject of a proposition was just the main ('hauptsächlich') argument and usually the second most important argument was the object.[34] Nonetheless, the definition of what constituted a function and an argument was very lax, and in fact according to Frege's setup any symbol appearing in a judgment could be seen as an argument.[35]

Hence, Frege's suggestion to abandon the subject-predicate division and to focus instead on the function-argument structure of propositions was allowing for a more inclusive understanding of what constituted a proposition and the kinds of logical operations that could be performed on it. This then once more implied that the traditional classification of valid syllogistic moods was obsolete, as Frege noticed.

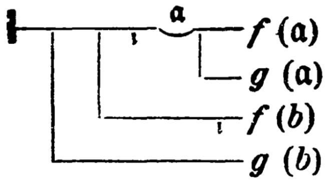


Figure 4.2: Proposition 59 in the *Begriffsschrift*.

For instance, after deriving Proposition 59 in the *Begriffsschrift* Frege presented the following example. Letting b be 'einen Vogel Strauss, nämlich einen einzelnes zu dieser Art gehörendes Thier,'[36] $g(A)$ the proposition "A ist ein Vogel,"[37] and $f(A)$ the proposition "A kann fliegen,"[38] then by application of Proposition 59 (see figure 4.2) it followed that "wenn dieser Strauss ein Vogel ist und nicht fliegen kann, so ist daraus zu schliessen, dass einige Vögel nicht fliegen können." And then he remarked:

> Man sieht, wie dieses Urtheil eine Schlussart ersetzt, nämlich Felapton oder Fesapo, zwischen denen hier kein Unterschied gemacht wird, *weil die Hervorhebung eines Subjects wegfallt*.[39]

[33]See 3.1.

[34]Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. 18.

[35]Bertran San Millán, 'La Lógica de Gottlob Frege: 1879-1903', p. 61.

[36]Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. 51. 'an ostrich, that is, an individual animal belonging to the species.' This and the following translations from the *Begriffsschrift* are taken from Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought'.

[37]'A is a bird'

[38]'A can fly'

[38]'If this ostrich is a bird and cannot fly, then it can be inferred from this that some birds cannot fly.'

[39]Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. 51, italics are ours.

With this remark Frege was highlighting the fact that, since there was no special distinction being made to the subject of a proposition, it became irrelevant to differentiate between the moods Felapton (EAO in the third figure) and Fesapo (EAO in the fourth figure). Indeed, let X stand for 'entities which can fly', let Y stand for 'ostriches,' and let Z stand for 'birds.' Then Felapton would yield

No ostrich can fly,	*(No Y are X)*
All ostriches are birds;	*(All Y are Z)*
therefore, Some birds cannot fly.	*(Some Z are X)*

and Fesapo would be

No entities which can fly are ostriches,	*(No X are Y)*
All ostriches are birds;	*(All Y are Z)*
therefore, Some birds cannot fly.	*(Some Z are X)*

That is, the two syllogisms would express the exact same bit of reasoning and therefore would not need to be considered separately.

A bit later in the text Frege noted that Barbara could be substituted by Proposition 65 in a similar fashion. That his system had enough complexity to encapsulate the traditional syllogisms seemed to be clear to Frege, since those were all the comments which he was willing to dedicate to that topic in the *Begriffsschrift*:

> Der Leser, der sich in die Ableitungsart der Begriffsschrift hineingedacht hat, wird im Stande sein, auch die Urtheile herzuleiten, welche den andern Schlussweisen entsprechen. Hier mögen diese ald Beispiele genügen.[40]

Frege was challenging the reader to find expressions for all possible syllogisms, which was an indicator that the Begriffsschrift was powerful enough not only to express classical syllogistic logic, but to do so in an even more general manner.

Thus, the traditional study of the syllogism had become obsolete: it was not relevant for an accurate logical analysis of an argument. The key now, after Frege, had become to identify which parts of a proposition constituted the function, which the argument, and the logical relations between premises. This would provide the necessary information about which sort of argument one was dealing with and whether it was valid or not. Frege's new approach was general enough to even encompass other teachings of traditional Aristotelian logic, such as the square of opposition (see figure 3.6). Conveniently enough, Frege took note of this at the

[40] Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. 53. 'The reader who has familiarized himself with the way derivations are carried out in the ideography will be in a position to derive also the judgments that answer to the other modes of inference. These should suffice as examples here.' (Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', p. 54)

end of the first chapter of the *Begriffsschrift*, after having shown how all four basic forms of propositions could be expressed with his notation.[41] It is remarkable how the depiction of the square in the notation of the Begriffsschrift really highlighted some of the inner symmetries of the square. By expressing each kind of proposition with the language of the Begriffsschrift, it was clear at first sight, for example,a that the diagonals expressed a relation of contradiction, since they related expressions which were the negation of the other (remember that, for Frege, double negation was assertion). It also became clear that the universal propositions were the ones on the top row, and the particular ones were on the bottom row, because they had the negation symbol on the main content stroke, before the generality symbol. Further, the subaltern relation appeared to require that negation marks be placed where there was none, and removed where there was one.[42] On the other hand, Frege's square did not contain any mention to the different relations of truth depending on the matter of the propositions (necessary, impossible, contingent). This was an aspect of Whately's presentation that had been dropped: that part of the theory was not interesting anymore because it was not related to the form of the syllogism, but to specific instances of its potential meanings.

Therefore, both Boole and Frege were using the traditional structure of logic as a starting point for more abstract and thereby more general versions of those older systems. Note the frequent use of the words 'arbitrary,' 'general,' 'convention' when reflecting about the structure demanded by classic logic. These authors were not quite rejecting Aristotle, but pointing out in which regards their new ways of approaching logic were, in themselves, stimulating enough to identify aspects of logic could be studied fruitfully. Some of those aspects had not been even noticed up to that point, and became only obvious or visible after setting an adequately generalized framework: for example, the algebraic expression of logical propositions in the case of Boole and the classification of propositions in function-argument structure in the case of Frege.

4.4 Language and metalanguage

There is a quote from the *Mathematical Analysis of Logic* which seems to hint at a distinction between two levels of logic:

> If it was lawful to regard [Logic] from *without*, as connecting itself through the medium of Number with the intuitions of Space and Time, it was lawful also to regard it from *within*, as based upon facts of another order which have their abode

[41] Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. 24.

[42] This is consistent with the fact that negating a universal statement gives a particular one, and in this case indeed the truth of a universal proposition in the upper row implies the truth of the corresponding particular ('negated universal') proposition in the lower row.

> in the constitution of the Mind,[43]

Although this is a peculiar early remark from Boole in the preface to the *Mathematical Analysis of Logic*, it seems like he did not devote any other moment to ponder about these two facets of logic, the external and the internal. Therefore, it is not strong enough a reason to say that Boole put any thought into the difference between logic itself and the language used to study it.

As it stands, Frege is considered to be one of the first —if not the first— people to consciously reflect on the two different levels of logical discourse. Frege's ultimate goal when developing the Begriffsschrift was to be able to present relations of dependence between judgments, and not just outline logical formulas that could be applied to reasoning. Hence, it was important that the signs of his language be completely different from those which probably would come up as part of the judgments that he wanted to analyze by means of his Begriffsschrift.

In the preface of the *Begriffsschrift*, he explained that he viewed his language as an approximation to the ideal *calculus ratiotinator* that Leibniz had envisioned, a formula-language that lay somewhere in the middle of the symbolical languages used in arithmetic, geometry, and chemistry.[44] Yet, he was convinced that the Begriffsschrift would not only be beneficial for mathematicians or chemists: also physicists, philosophers, and logicians themselves would benefit from embracing his new writing system because it allowed for clearer, more illuminating expression of ideas and relations of inference in reasoning.[45] He himself endeavored to first apply the Begriffsschrift to arithmetic, since that was the science that had triggered the considerations that had lead to developing the new language. For these reasons Frege was aware from the very beginning about the difference between the language expressing the judgments he wanted to analyze, and the language in which such an analysis would actually be performed.

Precisely for these reasons, Frege had complained about Boole's use of mathematical symbols as part of the notation of his logical language. To him, this was a fatal mistake because it introduced ambiguity into the reasoning: in a situation where Boole's language would be used to reason about arithmetic, the same symbols would come up as part of what was being studied and the language used to do said study. Therefore, according to Frege, the only acceptable use of Boole's notation was as a logical language that could be used to reason about anything *but* arithmetic:

> es geht nicht an, dass in derselben Formel beispielsweise das + Zeichen theils im logischen theils im arithmetischen Sinne vorkomme. Die Analogie zwischen

[43]Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 1.
[44]Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. V.
[45]Ibid., p. VI.

> den logischen und arithmetischen Rechnungsarten, die für Boole wertvoll ist, kann nur verwirrend wirken, wenn beide in Verbindung mit einander gesetzt werden. Booles Zeichensprache ist nur denkbar in gänzlicher Trennung von der Arithmetik.[46]

Frege's disagreement went even further: he considered Boole's system to be a mere costume, a way to present just regular logical language in a different guise. In Frege's own words,

> Ueberblicken wir die boolesche Formelsprache im Ganzen, so erkennen wir, dass sie eine Einkleidung der abstracten Logik in das Gewand algebraischer Zeichen ist; zur Wiedergabe eines Inhalts ist sie nicht geeignet, und das ist auch nicht ihr Zweck. Und dies ist grade meine Absicht. Ich will die wenigen Zeichen, die ich einführe, mit den schon vorhandenen Zeichen der Mathematik zu einer einzigen Formelsprache verschmelzen. Dabei entsprechen die bestehenden Zeichen ungefähr den Stämmen der Wortsprache, während die von mir hinzugefügten Zeichen den Endungen und Formwörtern zu vergleichen sind, welche die in den Stämmen liegenden Inhalte in logische Beziehungen setzen.[47]

Thus it could be argued that probably this difference in approach, and in the goal of the theoretical construct, made it easier for Frege to realize the need for a clean separation between the language of mathematics and the symbols used to indicate logical reasoning outside the system of study. Especially because his goal was to use the Begriffsschrift to study relations of inference within arithmetic,[48] so he needed to be especially careful from the very start with this distinction between the languages at play.

Similarly, and as much as Jevons celebrated and admired Boole for the system he set forth in the *Mathematical Analysis of Logic* and later the *Laws of Thought*, one of the main criticisms

[46] Frege, 'Über den Zweck der Begriffsschrift', p.100 Our translation: 'It is not acceptable that in the same formula the sign $+$, for instance, should appear partly in its logical and partly in its arithmetical sense. The analogy between the logical and the arithmetical calculi, while valuable for Boole, can only cause confusion when they are put in contact with one another. Boole's sign language is only conceivable completely separately from arithmetic.'

[47] ibid. Our translation: 'If we oversee the boolean formula-language in its totality, we recognize that it is abstract logic fitted out with the garment of algebraic signs; it is not suited for representing content, and that is also not its purpose. And this is precisely my aim. I want to merge the few signs that I introduce with the ones which are already available in mathematics into a single formula-language. In doing so, the existing signs correspond approximately to the roots of the natural language, while the signs added by me are comparable to the endings and function words, which establish the logical relations between the content matter contained in the roots.'

[48] 'Die Arithmetik . . . ist der Ausgangspunkt des Gedankenganges gewesen, der mich zu meiner Begriftsschrift geleitet hat. Auf diese Wissenschaft denke ich sie daher auch zuerst anzuwenden, indem ich ihre Begriffe weiter zu zergliedern und ihre Sätze tiefer zu begründen suche.' (Frege, 'Begriffsschrift, eine der artihmetischen nachgebildete Formelsprache des reinen Denkens', p. XIV) English translation: 'arithmetic was the point of departure that led me to my ideography. And that is why I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems.' (Frege, 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', p. 8)

Jevons had was the abuse, according to him, that Boole had made of the mathematical notation. In Jevons' view, Boole had taken the analogy too far and that had left him with a system that was '[p]ure Logic fettered with a condition which converts it from a purely logical into a numerical system.'[49] Hence, for Jevons the use of mathematical symbols had exceeded the mere functional aspect and taken over the actual purpose of the system as a logical one.

4.5 Redefinition

The last thread of transformation was something affecting all areas of human knowledge during the nineteenth century. Logic, just like mathematics and any other discipline, was going through modernity. One of the main characteristics of modernity was, precisely, a process of purification in the sciences, humanities, arts... This broader historical context shaped in a decisive manner the transformations taking place in the way logic was studied and viewed. One of the most obvious modernisms of the period was a tendency to re-define concepts within disciplines, or even to re-think disciplines themselves.[50] Logic was no exception to this, and the debate on what logic was supposed to be was an active one during the nineteenth century. Was logic a formal representation of the laws of reasoning which governed thought within the human mind? Was it a way to attain truth? Was it merely a tool for reasoning with accuracy, a compilation of rules for correct reasoning? Was it the underlying structure of all constructs of the mind, in particular meaning that mathematics was based on logic?

In the *Elements of Logic*, Whately wrote that

> Definition is another metaphorical word, which literally signifies, "laying down a boundary;" and is used in Logic to signify "an expression which explains any term, so as to *separate* it from every thing else," as a boundary separates fields.[51]

In a way, this is precisely what he himself was doing: laying down clear boundaries in an attempt to settle for once the extent of what logic was supposed to be and do. That was an explicit choice:

> a large portion of what is usually introduced into Logical treatises, relative to the finding of Arguments, —the different kinds of them, &c., I have referred to the head of *Rethoric*, and treated of in a work on the Elements of that Art.[52]

[49] Jevons, *Pure Logic, or the Logic of Quality apart from Quantity, with remarks on Boole's system and the relation between Logic and Mathematics*, XV.177.

[50] This is the view defended by Dr. Gerard Alberts and Dr. Danny Beckers in the course on History and Philosophy of Mathematics, taught during the Spring semester 2019 at the Vrije Universiteit Amsterdam. See also Gray, *Plato's Ghost: the Modernist Transformation of Mathematics* for an instance of this interpretation applied to the history of mathematics.

[51] Whately, *Elements of Logic. Comprising the Substance of the Article in the Encyclopaedia Metropolitana*, p. 71.

[52] Ibid., p. xxii.

In other words, Whately was consciously determining that certain parts of the theory were not to be considered as purely belonging to logic anymore, but as a part of rhetoric. Therefore, the way in which Whately was presenting logic in the *Elements of Logic* was a statement in itself, for he was actively recasting the perception of which aspects of the theory belonged to logic and which not. Comparing Whately's and Bentham's approaches, Van Evra wrote: 'Where Whately drew modern limits around the legitimate domain of logic, Bentham, while agreeing with Whately's basic ideas, sought to temper them in the direction of earlier, broader, more inclusive conceptions of logic.[53]' This contrast is understandable because Whately's perception of logic as a science motivated him to focus on describing the features of the theory leading to formal,[54] 'objective' results and leave the analysis of those aspects of logic which were more subjective of related to meaning for other studies, e.g. rhetoric.

Boole, in turn, suggested a new reduction of the scope of logic: the *Mathematical Analysis of Logic* was his defense that logic could be seen as a calculus, and that, as such, it could be considered as belonging to mathematics:

> I purpose to establish the Calculus of Logic, and . . . I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must at present stand alone.[55]

So Boole had taken Whately's suggestion of viewing logic as a science and reached the quite radical conclusion that certain features of logic were in fact part of mathematical analysis. Nonetheless, this did not imply that logic in its entirety had to be assimilated into mathematics. This can be seen in his insistence that logic was the science of the mind, the study of the laws of thought, and that the laws of reasoning were mathematical in form. He was much more articulate about this in the *Laws of Thought*—understandably so, because in that book he did take the time and space to develop his version of logic in a consistent and 'scientific' manner. Consider for example the following two extracts from that book:

> those laws [of the operations of the mind] are mathematical in their form, and . . . they are actually developed in the essential laws of human language. Wherefore the laws of the symbols of Logic are deducible from a consideration of the operations of the mind in reasoning.[56]

and

[53]James W. van Evra, review of *Outline of a New System of Logic*, by George Bentham, *Modern Logic* 2, no. 40 (1992): 406.

[54]In the most literal sense of the word: the focus on the syllogisms was placed on their properties as a structure, as a shape of reasoning which was common to most logical arguments.

[55]Boole, *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*, p. 4.

[56]Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, p.45.

> whether we regard signs as the representatives of things and of their relations, or as the representatives of the conceptions and operations of the human intellect, in studying the laws of signs, we are in effect studying the manifested laws of reasoning.[57]

Here, one can see that Boole's view of the relation between logic and mathematics was similar to, for example, the relation between physics and mathematics, insofar as mathematics provided an appropriate language to express the underlying laws of the other field as a science. Compare this view with that of Jevons, who, upon consideration of Boole's system and development of his own in the book *Pure Logic*, reached the opposite conclusion. To Jevons, the 'mysterious connection' between mathematics and logic was to be explained by the fact that logic underlay mathematics (see the full quote in page 39). To be precise, that '[n]umber . . . and the science of number, arise out of logic, and the conditions of number are defined by logic.' This was the case, according to Jevons, because '[u]nits are units inasmuch as they are logically contrary,' that is, because of the recognition that an abstract unit (and in particular, unit as the number 1) was 'something only known as logically distinct from or contrary to other things.'[58] With this, Jevons was placing logic at the foundation of mathematics because to him the very concept of number was based on the ability of the intellect to discern things which were the same in some aspects, but different in some other —thus 'logically distinct' inasmuch as they were not perfectly similar.

[57] Boole, *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, p.24.

[58] Jevons, *Pure Logic, or the Logic of Quality apart from Quantity, with remarks on Boole's system and the relation between Logic and Mathematics*, XV.185.

Chapter 5

Conclusions

Enough was said in the previous pages to finally attempt at providing answers to the motivating question: did 'mathematical logic' really 'emerge' with 'Frege's *Begriffsschrift*'?

No.

It depends.

It should be clear by now that the expression 'mathematical logic' is an ambiguous one: it meant different things at different times in history. If 'mathematical logic' refers to the current field of knowledge, which generally speaking encompasses proof theory, model theory, set theory, and recursion theory, then it would be a mistake to say that this is what Boole and De Morgan, even Frege, were working on. It is undeniable that, even if they were ever referred to by the same name, the mathematical logics of Boole, of Russell and of Tarski are very different things. If, however 'mathematical logic' refers to an approach to traditional logic which uses mathematical notation and ideas coming from mathematics, then from chapter 4 above it follows that this field of knowledge indeed started during the first half of the nineteenth century at the heart of some lively debates, at the hand of a multitude of authors. This second interpretation also would justify the act of using the same expression to refer to the seemingly disparate works of Leibniz, Boole, Jevons, and Frege, just to mention some names.

Secondly, the observations collected in this thesis provide a lot of evidence against the use of the word 'emergence' to describe any part of the history of logic during the nineteenth century. The process which molded a part of logic into mathematical logic was not one of emergence, but of continuous transformation. Chapter 4 showed how certain transforming threads could be traced throughout the century, connecting the work of different authors and evidencing a narrative of continuity rather than of break. Logic and the way it was studied at the end of the 1800s had changed a lot compared to that same field of knowledge at the end of the century, from the problems it was concerned with and down to its very appearance. Yet, this change was not a discrete event; it was the result of a century of conversation and debate within and without logic circles about the nature of very diverse issues.

And this leads to a third conclusion: Frege's contributions, as well as Boole's, did indeed shape the process of change logic was going through. Nonetheless, the converse was also true: the work and ideas of those authors were deeply rooted in that very same process, immersed in their historical context and explicitly influenced by the work of their immediate and not-so-immediate predecessors.

Chapter 2 analyzed the historiography of nineteenth century logic in order to understand where the standard narrative came from, and which criticisms it faced upon its establishment. The chapter concluded that pre-Fregean developments in logic had to be analyzed in more detail, both as a way to appropriately contextualize the work of Frege, but also in order to better understand the nature and origin of 'mathematical logic' and the assertion that it was born in the nineteenth century.

Next, chapter 3 contained a short overview of the logical setups in three main works in logic, namely Whately's *Elements of Logic*, Boole's *Mathematical Analysis of Logic*, and Frege's *Begriffsschrift*. Placing these works next to each other brought forth the striking differences in presentation between them, while also making evident a certain continuity from one to the other. This could be traced through the consistent reference to syllogistic logic and the use of fundamental aspects of that theory (classification of propositions, classification of syllogisms, the square of opposition) as units of measurement of the extent to which each new theory was similar and different from the previous ones.

Finally, in chapter 4 an interpretation was offered which showed that in fact the features of mathematical logic at the beginning of the twentieth century were a direct legacy of the ways in which logic had been transformed during the previous century. This transformation was presented in four main threads: the use of mathematical notation, the generalization of concepts in logic following from that use, a better understanding of the differences between language and metalanguage, and the modernization of the field.

Notably, the resemblance of the first three threads explored there and the four features of mathematical logic listed by Bocheński in his *History of Formal Logic* is remarkable. Recall that these features were: the presence of a formalistic method, the use of an abstractive method, the use of an artificial language, and the formulation of theorems in an object language.[1] Indeed, the discussions in chapter 4 provide some insight into Bocheński's singling out those features as being characteristic of mathematical logic, because they could be used to argue that precisely *those* were the aspects in which a part of logic transformed and finally crystallized in what became known as mathematical logic. Therefore, the historical evidence presented in that chapter constitutes a detailed reasoning about why mathematical logic was different to

[1] See page 6 above.

common logic precisely in the ways Bocheński had suggested in 1957.

All in all, the evidence and historical analysis presented in this thesis make a compelling case for the need of a more contextualized, nuanced, source-based retelling of the history of logic during the nineteenth century. Both the historiographical overview and the proposed interpretation hint at the view that the conclusions reached by historians during the mid-twentieth century were slightly biased towards wanting to find a clear history of logic. Maybe mathematical logicians were too eager to delineate and describe their own discipline. This would help explain that they were so prone to identifying certain historical works and ideas as already belonging to mathematical logic, when the arguments given in the present work suggest that they were mere predecessors.

The project which has been started in this thesis is, nonetheless, far from complete. The present work can be seen as a first approximation to a richer understanding of the relation between logic and mathematical logic around the time they became differentiated. Accordingly, there are a number of ways in which the explorations started here can be taken further.

First, the historiographical overview should be expanded and improved. The amount of secondary literature on nineteenth century logic is so abundant and varied that any extension of the list of works taken into consideration can only be positive and enriching to the overall assessment of the historical and historiographical situation.

Second, it is imperative that the picture painted here be improved with the addition of evidence coming from the vast pool of other authors in logic during the nineteenth century: De Morgan, Hamilton, Peirce, Schröder, Peano, further input from Jevons... Only by contrasting the above conclusions with a greater diversity of works will it be possible to determine their validity and historical accuracy.

Third, the suggested interpretation for the history of logic would also benefit greatly from a side-to-side comparison with the changes taking place in mathematics during the nineteenth century. The stance taken for example by Gray[2] and other historians[3] on modernity in mathematics is very attractive, and it would certainly be beneficial to attempt at a study of logic along the same lines: let the work done here be a piece of evidence for that. Besides, there is abundant evidence of a recurring appeal from mathematics to logic and from logic to mathematics as a foundation or to offer an unambiguous language, and therefore it is to be expected that, from a certain point onward, the changes happening in one of the fields will be mirrored in the other, and viceversa.

[2]Gray, *Plato's Ghost: the Modernist Transformation of Mathematics*.

[3]Ferreirós and Gray, Jeremy, *The Architecture of Modern Mathematics: Essays in History and Philosophy*; Moore, 'The Emergence of First-Order Logic'.

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