The monoidal category of D-branes in a Kazama-Suzuki model

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Abstract

A full description of the superstring demands the choice of an $N = 2$ superconformal field theory with central charge $c = 9$. Kazama-Suzuki models provide examples of such superconformal theories. We show that the D-branes of the most prolific Kazama-Suzuki model (the Grassmannian model) form a category finite on objects. We furthermore prove that this category admits a notion of tensor product and thus a monoidal structure. In the process we summarize the definitions and results from category theory, semisimple Lie algebra representation theory, Kac-Moody Lie algebras, Virasoro representation theory, coset models, superstring theory and boundary conformal field theory which are needed to understand the results. We assume knowledge of bosonic string theory at the level of a first graduate course in string theory and familiarity with the basic definitions of Lie algebra and representation theory.

Contents

1 Introduction

The classical bosonic string can be described by the so called Polyakov action, which is invariant under conformal transformations: conformal symmetry is a gauge symmetry of the system. The Polyakov action is the action of D free scalar fields, interpreted as the components of the embedding of the worldsheet on the D-dimensional spacetime (see *e.g.* the review [Ton]).

Denote by T the energy-momentum tensor of the Polyakov action. One can show that the modes L_n of T form a Virasoro algebra with central charge $c = D$ [BLT]. Demanding the quantum string theory to also be conformally invariant – and thus the Weyl anomaly to vanish – implies $D = c = 26$ [BBS].

Now, the equations of motion for the bosonic open string imply that each endpoint of a bosonic open string must satisfy either the Neumann (N) boundary condition – the endpoint can move freely – or the Dirichlet (D) boundary condition – the endpoint must lie on a certain spacetime hypersurface. The two endpoints of an open string can satisfy different boundary conditions, so that we are left with 4 possibilities: NN, RR, RN and NR boundary conditions.

A spacetime hypersurface to which an endpoint of an open string is restricted is called a *D-brane*. So a Dirichlet boundary condition is specified by a D-brane. Furthermore, a possible interpretation of a Neumann boundary condition is that there is a spacetime-filling D-brane, so that in fact the D-branes fully characterize the boundary conditions of the theory, even when Neumann boundary conditions are present [BBS].

The D-brane structure of a theory determines its gauge symmetries [Zwi]. Comparing these with the gauge symmetries of the Standard model one can conclude if the theory is realistic. This means that we can use D-branes to study string phenomenology.

Now, two possible generalizations of the Virasoro algebra are the $N = 1$ and the $N = 2$ superconformal algebras. The fermionic string – or superstring – can be described by the so called Ramond-Neveu-Schwarz (RNS) action, which is invariant under $N = 1$ superconformal transformations [BBS]. The RNS action adds D free fermionic fields to the Polyakov action. As in bosonic string theory, the equations of motion force the string to satisfy a number of boundary conditions. The bosonic fields still obey Neumann or Dirichlet boundary conditions, so that D-branes are still an integral part of the theory. The boundary conditions for the fermionic fields are discussed in §7.

In the case of the superstring, demanding the Weyl anomaly to vanish leaves us with $c = 15$. So the superstring must be a $c = 15$ CFT. A common *ansatz* is that we can separate the superstring in two conformal field theories (CFTs): the external and the internal CFTs, with central charges 6 and 9, respectively (see [Gre] and references thereof). One usually interprets the external CFT as describing a free string propagating in Minkowski space, and the internal CFT as corresponding to a non-linear sigma model on a (6-dimensional) Calabi-Yau manifold [BLT], so that in particular spacetime is 10-dimensional and the 6 non-observed dimensions are compactified.

However there are other (more general) possibilities. Namely, one may use other internal CFTs (which need not have a geometrical interpretation), as long as they are $N = 2$ superconformal theories with central charge $c = 9$ [Gep1] (this claim will be analyzed in §7). Kazama Suzuki models [KS2] are an example of such theories, and understanding them may allow us to investigate if taking them as internal CFTs is the right choice. In particular it is important to look into the D-branes in Kazama-Suzuki models.

In this text we will analyze the categorical structure of D-branes of the most common (and thus well-studied) Kazama-Suzuki model: the Grassmannian model. Namely, we will see how one can construct a category of D-branes of the Grassmannian model. Furthermore, we investigate how one can develop a notion of tensor product of D-branes, which gives a monoidal structure to the category of D-branes.

Although D-branes are important to understand by themselves, there is at least one other reason to study their categorical structure: there is a conjectured correspondence between D-branes of a CFT and the solitons of a Landau-Ginzburg model (see [BF, Noz, Cam] and references thereof). One can use category theory to try to prove this conjecture as an equivalence of categories. A first step towards this end in the case of the Grassmannian model is to construct the theory of D-branes of the Grassmannian model and study a few of its basic properties, so that this thesis can be seen as that first step.

A third motivation for the contents of this text is simply to see one instance of category theory being applied to physics (another interesting one being, for example, the categorical treatment of topological quantum field theories as in [Koc]).

This topic demands familiarity with many concepts which are not part of the standard curriculum of a Master's program in theoretical physics, both from mathematics, physics and mathematical physics. Hence most of the sections have the goal of introducing the necessary concepts and results to understand §9, where we construct the category of D-branes of a Grassmannian model and define a suitable monoidal structure – which is the main goal of this thesis.

The structure of this text is the following:

- §2 introduces the basic ideas from category theory which will be needed in the rest of the text. Since Master students of theoretical physics are usually not familiar with this topic, a clear and self-contained exposition is warranted, and that is what I aimed for.

- In contrast, physicists deal with semisimple representation theory since the bachelor days, albeit rather informally. For this reason, and so that the thesis is not overextended, I included §3 on semisimple representation theory in the form of a short review of the topic, and focusing on the concepts which are directly relevant to us.

- Kazama-Suzuki models are examples of the so-called coset models, of which any kind of mathematical understanding must include the study of untwisted affine Kac-Moody algebras and their unitary irreducible representations. This is discussed in §4, while in §5 we build up to coset models and their conformal weights (which will be essential when constructing the category of D-branes).

- §6 starts with a refresher on two-dimensional conformal field theory and follows with a discussion on $N = 1$ and $N - 2$ superconformality (also in two dimensions). The main goal of that discussion is to motivate the introduction of the $N = 1$ and $N = 2$ superconformal algebras. Full mathematical rigor was not aimed for.

- §7 gives an overview of superstring theory and connects it with Kazama-Suzuki models. Here we assume that the reader is familiar with bosonic string theory, at the level of a first course in string theory. I did not aim for mathematical rigor in this section, as that would be far beyond the scope of this thesis.

- The generalization of the concept of D-brane to Kazama Suzuki models (and to the so-called (two-dimensional) rational conformal field theories (RCFTs) in general) is discussed in §8. Again full rigor was not aimed for, for the same reasons as in §7.

- The construction of the monoidal category of D-branes is reserved to §9. It consists of two parts: first I define the category itself, and afterwards I come up with an appropriate notion of tensor product. A version of the Schur's lemma for the Virasoro algebra is derived in the process.

The contents of every proof environment are original. Furthermore, the entirety of §9 is original.

2 Category Theory

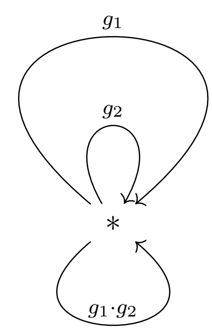
2.1 Categories, functors and natural transformations

Categories can be seen as a generalization of groups, and, like groups, them too are prolific in mathematics and physics. Roughly speaking, a category is a mathematical structure composed of a collection of "objects" connected by "arrows", which can be composed with each other. I will briefly discuss these concepts, define them and give examples. Much more can be found in [ML, Awo, Lei, AHS, Sim].

"Objects" can be mathematical objects such as sets, groups, vector spaces or topological spaces, with "arrows" being mathematical entities such as functions, group homomorphisms, linear maps or continuous maps, respectively.

In these examples, the arrows are functions preserving whatever structure the objects have. Although many categories have arrows of this nature, this need not be the case.

For example, a group G can be seen as a category with a single object $*$ (which can be anything) and whose arrows are the elements of G, their "composition" respecting the same rules as the group multiplication. In particular, the identity arrow plays the role of the neutral element of the group, and every arrow g has an inverse g^{-1} such that the composition of g with g^{-1} is the identity arrow.



It is in this way that the notion of category can be seen as a generalization of that of a group [Awo]. We will come back to this example after giving a more rigorous definition of category.

Remark 2.1. Since we want to have categories with large "collections" of objects, the objects of a category may not form a set. For example, we know that we cannot have the set of all sets due to Russel's paradox, so that in particular if we want to be able to speak of a category of sets we need some appropriate notion of "collection" of all sets That notion is called *class*, and generalizes the concept of set.

For our purposes, one can treat class exactly as one treats sets. For the curious reader, a short discussion on this can be found in the Appendix 11.1 and complemented with [AHS].

Definition 2.2. A *category* $\mathcal{C}$ consists of a class $\mathrm{Obj}(\mathcal{C})$ of *objects* (denoted $A, B, C, \dots$) and a class $\mathrm{Hom}(\mathcal{C})$ of *arrows* (denoted $f, g, h, \dots$) such that:

C1) To each arrow is assigned an object $\mathrm{dom}(f)$ (the *domain* of f) and an object $\mathrm{cod}(f)$ (the *codomain* of f). The arrow is then denoted $f\colon \mathrm{dom}(f) \to \mathrm{cod}(f)$. The class of all arrows with domain A and codomain B is called the *hom-set* between A and B, and is denoted $\mathrm{Hom}(A, B)$.

C2) For every two arrows f, g such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is an arrow

$$g \circ f\colon \mathrm{dom}(f) \to \mathrm{cod}(g)$$

called the *composition of f and g*. In other words, for all objects A, B, C there is a map $\circ\colon \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$

C3) For every object A there is a distinguished arrow $id_A \in \mathrm{Hom}(A, A)$ (called the *identity arrow of A*) such that $f \circ id_A = f$ for every arrow $f\colon A \to B$ and $id_A \circ g = g$ for every arrow $g\colon C \to A$.

C4) (Associativity:) For all $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark 2.3. Notice that the identity arrow of an object A is unique: if i and j are identity arrows of A, then in particular $i \circ j = j$, but also $i \circ j = i$, by definition of identity arrow.

Since an arrow is the categorical generalization of structure-preserving maps, then the categorical incarnation of isomorphism is not hard to guess:

Definition 2.4. Let $f\colon A \to B$ be an arrow in a category $\mathcal{C}$. We say that f is an *isomorphism* if there is an arrow $g\colon B \to A$ in $\mathcal{C}$ such that $g \circ f = id_A$ and $f \circ g = id_B$. Then g is called the[1] *inverse* of f and is denoted f^{-1}.

There is also a notion of "homomorphism between categories".

Definition 2.5. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair (F_0, F_1) of maps

$$F_0\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}), \quad F_1\colon \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{D})$$

such that for all arrows $f\colon A \to B$, $g\colon B \to C$ in $\mathcal{C}$:

F1) $F_1(f)\colon F_0(A) \to F_0(B)$

[1] If there is an inverse, it is unique.

F2) $F_1(g \circ f) = F_1(g) \circ F_1(f)$

F3) $F_1(id_A) = id_{F(A)}$

From now on we will omit the subscripts, *i.e* we will denote both F_0 and F_1 by F.

We can take these ideas further and consider "homomorphisms between functors". They are called natural transformations, and are defined such that they preserve the internal structure (*i.e.* the composition of arrows) of the categories involved.

Definition 2.6. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$. A *natural transformation* between F, G is a family $\mu = \{\mu_C\}_{C \in \mathrm{Obj}\,\mathcal{C}}$ of arrows of $\mathcal{D}$ such that for every arrow $f\colon C \to C'$ of $\mathcal{C}$ the diagram (called *naturality square*)

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \mu_C}\downarrow & & \downarrow{\scriptstyle \mu_{C'}} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

commutes (in $\mathcal{D}$). We write $\mu\colon F \Rightarrow G$, and μ_C is called the *component at* C of μ. A *natural isomorphism* is a natural transformation whose components are isomorphisms.

2.2 Subcategories

It is common to come across categories which are embedded in other categories. Coming back to the example of a group G as a category, then subgroups of G can be seen as categories themselves. Another example is the category **Ab** of abelian groups which sits inside the category **Grp** of groups. Let's formalize this concept:

Definition 2.7. A category $\mathcal{D}$ is a *subcategory* of a category $\mathcal{C}$ if the objects of $\mathcal{D}$ are objects of $\mathcal{C}$, the arrows of $\mathcal{D}$ are arrows of $\mathcal{C}$, the identity arrows of the objects in $\mathcal{D}$ are the identity arrows of those objects in $\mathcal{C}$, and the composition of arrows in $\mathcal{D}$ is just the restriction of the composition of arrows in $\mathcal{C}$.

Given any subclass S of the class $\mathrm{Obj}(\mathcal{C})$ of all objects in a category $\mathcal{C}$, there is a natural subcategory of $\mathcal{C}$ whose objects are precisely the elements of S. Its arrows are simply all the arrows in $\mathcal{C}$ between the objects in S. Such a subcategory is called a full subcategory:

Definition 2.8. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *full subcategory* if for all $A, B \in \mathrm{Obj}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.

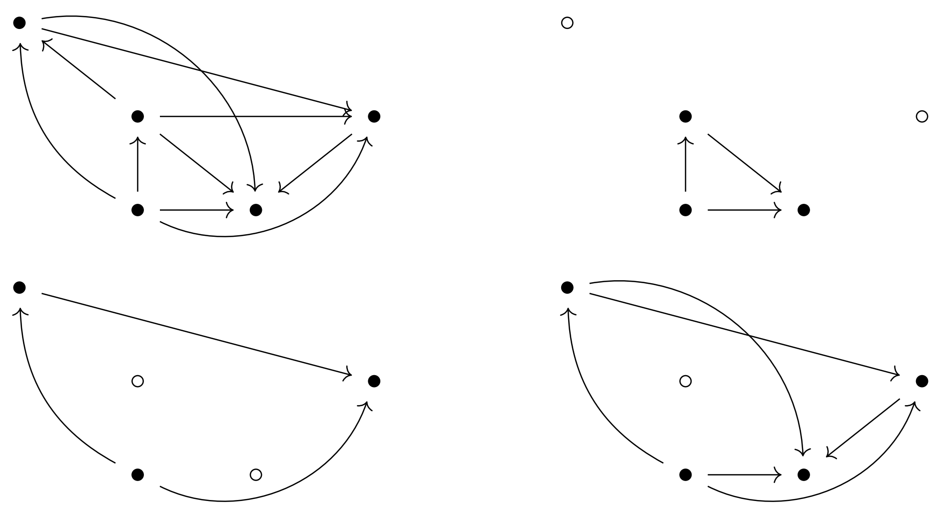


Figure 1: Illustration of the usual process to construct the full subcategories of a category.

Given its importance in the rest of the text, we will illustrate the procedure described above of obtaining full subcategories of a subcategory:

The original category is depicted in the first diagram, and the others are some of its full subcategories. Black dots represent the objects that we keep from the original category, while white dots are the objects that we do not include in the full subcategory. Notice that each full subcategory is fully specified by simply saying what objects we want to keep from the original category.

2.3 Some important Categories

We shall list some categories to illustrate the definitions above. Some will be relevant for our discussion of monoidal categories and the like.

(a) **Set** is the category whose:

- objects are sets.
- arrows are maps.

A full subcategory of **Set** is the category **FinSet** of finite sets.

(b) $\mathbf{Vect}_{\mathbb{K}}$ is the category whose:

- objects are vector spaces over $\mathbb{K}$.
- arrows are linear maps.

Sometimes we omit $\mathbb{K}$ if it is clear from context.
An important full subcategory of **Vect** is the category **FinVect** of finite vector spaces.

(c) If $\mathfrak{g}$ is a Lie algebra, $\mathbf{Rep}(\mathfrak{g})$ is the category whose:

- objects are $\mathfrak{g}$-modules.
- arrows are intertwiners ($\mathfrak{g}$-module homomorphisms).

This category will show up later in this text.

2.4 Products

We must define the product of two categories [Sim], since it will play a role in the definition of monoidal category:

Definition 2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is the category whose objects are the ordered pairs (C, D), with $C \in \mathrm{Obj}(\mathcal{C}),\ D \in \mathrm{Obj}(\mathcal{D})$ and whose arrows are the pairs of arrows (f, g) where $f : A \to B$ is an arrow of $\mathcal{C}$, $g : R \to S$ is an arrow of $\mathcal{D}$ and:

$$(A, R) \xrightarrow{(f,g)} (B, S)$$

The composition of arrows is the obvious one: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$.

Remark 2.10. We can take ordered pairs since the product of classes is well-defined, just like with sets.

Remark 2.11. Notice that, given an object (A, B) of the product category $\mathcal{C} \times \mathcal{D}$, the arrow $id_A \times id_B \in \mathrm{End}((A, B))$ is an identity arrow for (A, B) (trivial to check). By the uniqueness of the identity arrow (see Remark 2.3) we have $id_{(A,B)} = id_A \times id_B$.

We now turn to the product of two objects of a category. This can be defined by generalizing the Cartesian product of objects of **Set** (for details, see [Lei]), resulting in the following definition:

Definition 2.12. Let $\mathcal{D}$ be a category and $X, Y \in \mathrm{Obj}(\mathcal{D})$. A *product* of X and Y is a tuple (P, p_1, p_2) where $P \in \mathrm{Obj}(\mathcal{D})$ and

$$P \xrightarrow{p_1} X \quad P \xrightarrow{p_2} Y$$

such that, for any diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

in $\mathcal{D}$ there is an unique arrow $A \xrightarrow{u} P$ of $\mathcal{D}$ such that

$$\begin{tikzcd} & A \arrow[ddl, "f"'] \arrow[d, dashed, "u"] \arrow[ddr, "g"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

commutes. p_1, p_2 are called *projections*.

Remark 2.13. Products are unique up to isomorphism. We thus speak of *the* product of X and Y.

2.5 Monoidal Categories

We now want to give some extra structure to categories. For example, we can give our category some kind of multiplication that mimics the multiplication in a *monoid*, which is a set M together with an associative map $\cdot : M \times M \to M$ and an (identity) element $e \in M$ such that $me = em = m$ $(\forall m \in M)$. In other words, a monoid is a group whose elements need not have inverses. Roughly speaking, a monoidal category is a category with a monoid-like structure. Hence we can think of a monoidal category as a categorification of the notion of monoid [TV]: there is a product $\otimes$ acting on the objects of the monoidal category (as opposed to the elements of a monoid) with properties which are analogous to the properties of the monoid multiplication.

This type of category is ubiquitous in physics and mathematics, with the usual tensor products (*e.g.* between vector spaces, Hilbert spaces or representations) often constituting the monoidal structure of the corresponding categories. This justifies why we use the notation $\otimes$ for the "monoidal product", and indeed we call it tensor product.

Definition 2.14. A *bifunctor on the category* $\mathcal{C}$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where $\times$ is the product of categories from Definition 2.9.

Remark 2.15. If we fix one argument of a bifunctor $F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ we obtain a functor $\mathcal{C} \to \mathcal{C}$ [ML], *i.e.* if A is an object of $\mathcal{C}$ then $F(A, -)$ and $F(-, A)$ are functors.

Definition 2.16. A *monoidal category* is a category M together with a bifunctor $\otimes : M \times M \to M$ on M called *tensor product* and an object $\mathbb{1} \in M$ called the *identity object* such that:

M1) There is a natural isomorphism

$$a : (id_M \otimes id_M) \otimes id_M \Longrightarrow id_M \otimes (id_M \otimes id_M)$$

called the *associator*.

M2) There are natural isomorphisms

$$r : (-) \otimes \mathbb{1} \Longrightarrow id_M$$

and

$$l : \mathbb{1} \otimes (-) \Longrightarrow id_M$$

called *right unitor* and *left unitor* respectively[2].

M3) The *pentagon identity* holds. *i.e.* the diagram

$$\begin{tikzcd}
 & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "a_{A\otimes B,C,D}"'] \arrow[dr, "a_{A,B,C}\otimes id_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "a_{A,B,C\otimes D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "a_{A,B\otimes C,D}"] \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "id_A\otimes a_{B,C,D}"]
\end{tikzcd}$$

[2]The notation $(-) \otimes \mathbb{1}$ has an hopefully obvious meaning: it denotes the functor $\otimes(-, \mathbb{1})\colon M \to M$ induced by the tensor product (see Remark 2.15).

commutes in M.

M4) The *triangle identity* holds. *i.e.* the diagram

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "a_{A,\mathbb{1},B}"] \arrow[dr, "r_A \otimes id_B"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "id_A \otimes l_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes in M.

Remark 2.17. The pentagon and triangle identities may seem mysterious. Before we move on, let us just remark that the pentagon identity comes from demanding compatibility between the associator and the tensor product, while the triangle identities arise from demanding the associator and the right and left unitors to be compatible. For more details, see [ML].

Example 2.18. Set with the Cartesian product

The tensor product on arrows is the Cartesian product of maps: $(f, g)(x) = (f(x), g(x))$. The identity object can be any chosen singleton, which we denote $\{*\}$. The components of the associator are the "obvious" bijections (isomorphisms in **Set**) $a_{A,B,C} \colon (A \times B) \times C \to A \times (B \times C)$, $((a, b), c) \mapsto (a, (b, c))$. Similarly, the right and left unitors have as components the bijections $r_A \colon A \times \{*\}$, $(a, *) \mapsto a$ and $l_A \colon \{*\} \times A$, $(*, a) \mapsto a$, respectively. (It is not hard to show that M1) and M2) are satisfied by checking that a, r and l are natural isomorphisms (by checking that they obey the naturality square in Definition 2.6), and it is also a trivial matter to check that both the pentagon and the triangle identities hold).

Example 2.19. Vect with the Vector Space Tensor Product

The tensor product on arrows is the usual tensor product of linear maps. The identity object is the vector space $\mathbb{K}$. Similarly to **Set**, the associator components are the "obvious" vector space isomorphisms given by $a_{U,V,W}((u \otimes v) \otimes w) = u \otimes (v \otimes w)$, with u, v, w basis elements of U, V, W, respectively. The unitors are also the obvious ones, again analogously to the case of **Set**.

Example 2.20. Rep($\mathfrak{g}$) with the Tensor Product of $\mathfrak{g}$-modules

The identity object is the trivial representation $\mathbb{K}$. The associator and the unitors are analogous to the ones in **Vect**.

3 Semisimple Lie algebras

In this section, we will summarize the main results from the representation theory of (finite dimensional) semisimple Lie algebras that we will need. For more details, see for example [Hum2, FS2, FH, BDK, Coo, Hal1, EW].

Recall that a *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a bilinear antisymmetric map $[\cdot,\cdot]\colon\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ which satisfies the Jacobi identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$. This map is called the *Lie bracket*. The Lie algebra is said to be *real* if the underlying vector space is real, and *complex* if the underlying vector space is complex[3].

Example 3.1. Let W be a complex vector space with a basis $\{L_m,\ m\in\mathbb{Z}\}$. Equipped with the Lie bracket given by

$$[L_m,L_n]=(m-n)L_{m+n} \tag{1}$$

W is a (infinite-dimensional) Lie algebra called the Witt algebra [BP].

Let $\mathbb{C}\,K$ be a complex one-dimensional vector space. The vector space $W\oplus\mathbb{C}\,K$ together with the Lie bracket given by

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{m(m^2-1)}{12}\delta_{m+n}K \tag{2}$$

is the Lie algebra Vir known as the Virasoro algebra [BP].

Every Lie group G has an associated Lie algebra $\mathfrak{g}$, whose vector space is the tangent space at the identity T_eG and with Lie bracket $[X,Y]=ad(X)(Y)=Ad_{*,e}(X)(Y)$. The elements of a chosen basis $\{T^a\}$ of $\mathfrak{g}$ are called its *generators*. The Lie bracket is completely determined by its action on the generators, which can be written $[T^a,T^b]=f^{ab}_cT^c$. The coefficients f^{ab}_c are called the *structure constants* of the Lie algebra with respect to the chosen basis.

The *direct sum* $\mathfrak{g}_1\oplus\mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ is the Lie algebra whose underlying vector space is the direct sum of the vector spaces of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and with Lie bracket given by $[X_1\oplus X_2,Y_1\oplus Y_2]=[X_1,Y_1]_1\oplus[X_2,Y_2]_2$, where $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ denote the Lie brackets of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively.

The *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of elements $X\in\mathfrak{g}$ which commute with all elements of $\mathfrak{g}$. In particular, if $[\mathfrak{g},\mathfrak{g}]=0$ then $Z(\mathfrak{g})=\mathfrak{g}$ and $\mathfrak{g}$ is said to be *abelian*.

[3]The underlying vector space may be a vector space over a field other than $\mathbb{R}$ or $\mathbb{C}$, but those cases will not be relevant for us.

A *Lie subalgebra* of $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ that is also a Lie algebra when equipped with the same Lie bracket as $\mathfrak{g}$, which holds iff $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If furthermore $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is said to be an *ideal* of $\mathfrak{g}$. The *trivial ideals* of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$ itself. Non-trivial ideals are said to be *proper*. A non-abelian Lie algebra with no proper ideals is said to be *simple*. A Lie algebra is *semisimple* if it is finite-dimensional and is a direct sum of simple Lie algebras. Semisimple Lie algebras can also be characterized by the fact that any element $X \in \mathfrak{g}$ can be written as $X = [Y, Z]$ for some $Y, Z \in \mathfrak{g}$, *i.e.* the Lie bracket is surjective.

Example 3.2. Some of the Lie algebras familiar to physicists are semisimple. For example $\mathfrak{sl}_n(\mathbb{C})$ is semisimple for $n \geq 2$ and $\mathfrak{so}_n(\mathbb{C})$ is semisimple when $n \geq 3$. On the other hand, $\mathfrak{gl}_n(\mathbb{C})$ is not semisimple for any value of n. [Hal1].

For Lie algebras there are standard bases (the Cartan-Weyl bases), meaning that there is also a standard form for the structure constants. Working on one of these bases turns out to be very useful to analyse the structure of semisimple Lie algebras. A Cartan-Weyl basis for $\mathfrak{g}$ is composed by generators of the *Cartan subalgebra* $\mathfrak{g}_0$ and *step operators*, and there is one of these step operators for every *root* of $\mathfrak{g}$. We will briefly review what each of these all-important concepts are.

3.1 Cartan subalgebra

Until the end of this section, $\mathfrak{g}$ is a complex semisimple Lie algebra.

Definition 3.3. An element X of $\mathfrak{g}$ is *ad-diagonalizable* or *semisimple* if the map $ad_X = [X, -] \in End(\mathfrak{g})$ is diagonalizable.

It turns out that, if the base field of $\mathfrak{g}$ is algebraically closed (like $\mathbb{C}$), then there are ad-diagonalizable elements in $\mathfrak{g}$. By finite dimensionality, there is a maximal set of linearly independent ad-diagonalizable elements. If those elements commute, then by definition they span what we call a Cartan subalgebra of $\mathfrak{g}$:

Definition 3.4. A *Cartan subalgebra* of $\mathfrak{g}$ is a Lie subalgebra of the form

$$\mathfrak{g}_0 := \mathrm{span}_{\mathbb{C}}\{H^i | i = 1..., r\} \tag{3}$$

where $\{H^i\}$ is a maximal set of linearly independent, commuting, ad-diagonalizable elements of $\mathfrak{g}$. The integer $r = \dim \mathfrak{g}_0$ is the *rank* of $\mathfrak{g}$.

All Cartan subalgebras are conjugated[4], and in particular have the same dimension. We thus talk of *the* Cartan subalgebra.

[4] Two Lie subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ of the Lie algebra $\mathfrak{g}$ of a Lie group G are *conjugated* if there is $g \in G$ such that $\mathrm{Ad}(g)(\mathfrak{h}_1) = \mathfrak{h}_2$. Recall that $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$, and thus preserves dimension.

3.2 Roots and the Cartan-Weyl basis

Recall from your courses in Quantum Mechanics and Linear Algebra that, if two operators commute, then they are simultaneously diagonalizable. Also, if $A, B \in \mathfrak{g}$ and $[A, B] = 0$, then $[\mathrm{ad}_A, \mathrm{ad}_B] = 0$. This will be important for what comes next.

By definition, for every $h_1, h_2 \in \mathfrak{g}_0$ we have $[h_1, h_2] = 0$. Then $[\mathrm{ad}_{h_1}, \mathrm{ad}_{h_2}] = 0$, so that $\mathfrak{g}$ is spanned by $n = \dim \mathfrak{g}$ elements y_j which are simultaneous eigenvectors for all $\mathrm{ad}_h, h \in \mathfrak{g}_0$, *i.e.*

$$\forall i = 1 \dots n, \forall h \in \mathfrak{g}_0, \mathrm{ad}_h(y_j) = [h, y_j] = \alpha_{y_j}(h) y \tag{4}$$

for some $\alpha_{y_j} \colon \mathfrak{g}_0 \to \mathbb{C}$. This map turns out to be linear, so that $\alpha_{y_j} \in \mathfrak{g}_0^*$.

Definition 3.5. Each nonzero α_{y_j} is called a *root* of $\mathfrak{g}$. The set of all roots (with respect to a chosen basis $\{y_j\}$ of $\mathfrak{g}$ of simultaneous eigenvectors of all $\mathrm{ad}_h, h \in \mathfrak{g}_0$) is denoted Φ and is called a *root system*.

We can write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{5}$$

with $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,|\forall h \in \mathfrak{g}_0, [h, x] = \alpha(h)x\}$: since α runs through all the roots, then all the elements of the basis $\{y_j\}$ of $\mathfrak{g}$ are in some $\mathfrak{g}_\alpha$, namely each y_j is in $\mathfrak{g}_{\alpha_j}$.

Notice that $\mathfrak{g}_0$ in this notation coincides with the Cartan subalgebra, by maximality. Equation (5) is called the *root space decomposition* of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{g}_0$. This decomposition tells us that there is a basis $\mathcal{B}$ of $\mathfrak{g}$ composed of the $\{H^i\}$ that span $\mathfrak{g}_0$ together with the elements $E^\alpha \in \mathfrak{g}_\alpha$ which satisfy $[H^i, E^\alpha] = \alpha(H^i)E^\alpha$. These operators E^α are called *ladder operators*. So we write

$$\mathcal{B} = \{H^i\}_{i=1,\dots,r} \cup \{E^\alpha\}_{\alpha \in \Phi} \tag{6}$$

This is called the *Weyl-Cartan basis*.

The name "root" is also used for something slightly different:

Definition 3.6. The r-dimensional vector $(\alpha^i)_{i=1,\dots,r}$ of eigenvalues of the eigenvector E^α of ad_{H^i} is called a *root vector* (or simply *root*) of $\mathfrak{g}$ relative to the basis $\{H^i\}$ of $\mathfrak{g}_0$.

Note that a root vector is simply the coordinate representation of $\alpha \in \mathfrak{g}_0^*$ with respect to the basis $\{H^i\}$ of $\mathfrak{g}_0$.

The Killing form, long roots and positive roots

An *inner product* is a symmetric, non-degenerate[5] bilinear map. It is useful to define an inner product on $\mathfrak{g}$. The *Killing form* $\kappa\colon \mathfrak{g}\times\mathfrak{g} \mapsto \mathbb{C}$, defined by $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$, is an inner product. Thus it is in particular non-degenerate. It turns out that its restriction to $\mathfrak{g}_0$ is also non-degenerate. But we know that a non-degenerate bilinear map acting on a vector space can be used to construct an isomorphism between the vector space and its dual.

In particular, the Killing form induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_0^*$, which in turn means we have a notion of *length* of a root. A root with maximal length is called a *long root*, and is usually denoted ψ.

If the Lie algebra $\mathfrak{g}$ is the Lie algebra of a compact group (we say that $\mathfrak{g}$ is a *compact Lie algebra*), then its Killing form is negative. This has the important consequence that in the case of a compact algebra we can use the Gram-Schmidt procedure to obtain a basis $\{T^a\}$ which is orthonormal with respect to the Killing form (*i.e.* $\kappa(T^a,T^b) = \delta^{a,b}$).

It turns out that we can take the real span $\mathrm{span}_{\mathbb{R}}(\Phi)$ of the roots to be isomorphic to $\mathbb{R}^r$. Since $\mathfrak{g}$ is finite dimensional and the roots span $\mathfrak{g}_0^* \cong \mathfrak{g}_0 \leq \mathfrak{g}$, then there are only finitely many roots. Thus one can construct a hyperplane in the root space $\mathrm{span}_{\mathbb{R}}(\Phi)$ which contains no root. Fixing such a plane divides the root system into two parts, which we label $\Phi_\pm$. The roots in Φ_+ and Φ_- are said to be *positive* and *negative*, respectively, and the corresponding ladder operators E_α are called *raising* and *lowering operators*, respectively. We can thus rewrite the root space decomposition (5) as

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \tag{7}$$

where $\mathfrak{g}_\pm = \mathrm{span}\{E_\alpha | \alpha \in \Phi_\pm\}$. This is called the *triangular decomposition* of $\mathfrak{g}$.

3.3 The Cartan matrix

A *simple root* is a positive root which cannot be obtained by linear combination of other positive roots with positive real coefficients. They will be denoted by $\alpha^{(i)}$ and their set by Φ_s. It turns out that there are exactly r simple roots, and Φ_s spans the root space: $\mathrm{span}_{\mathbb{R}}(\Phi_s) = \mathrm{span}_{\mathbb{R}}(\Phi)$. Furthermore, $\mathrm{span}_{\mathbb{N}}(\Phi_s) \supseteq \Phi_+$.

It is natural to ask if the set Φ_s of simple roots is an orthogonal basis of the root space. The answer is no [FS2], and how much it strays away from orthogonality is

[5] *i.e.* for all $x \in \mathfrak{g}$, the kernel of $\kappa(x,-)$ is trivial.

captured by the *Cartan matrix*: an $r \times r$ matrix A with entries

$$A^{ij} = 2\frac{\langle \alpha^{(i)}, \alpha^{(j)} \rangle}{\langle \alpha^{(j)}, \alpha^{(j)} \rangle} \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced by the Killing form. It turns out that the entries of the Cartan matrix are actually integers, sometimes called *Cartan integers*.

These matrices have some characteristic properties, and any matrix with such properties will be called a Cartan matrix [BDK].

The classification of simple Lie algebras reduces to the classification of Cartan matrices, because from a Cartan matrix we can obtain the corresponding semisimple Lie algebra by *Serre's construction* (see, for example, [BDK]).

A generalization of the Cartan matrices are used to construct infinite-dimensional generalizations of simple Lie algebras, called *Kac-Moody algebras*. We will discuss this later.

3.4 Highest weight representations

We will briefly review how the highest weight representations[6]of semisimple Lie algebras are defined, starting with the case of $\mathfrak{sl}(2)$ for motivation. More details can be found in [FS2].

Highest weight representations of $\mathfrak{sl}(2)$

Recall from your Quantum Mechanics courses that $\mathfrak{sl}(2) = \mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of 2×2 traceless matrices with complex entries, and it is spanned by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9}$$

so that

$$[H, E^+] = 2E^+, \quad [H, E^-] = -2E^-, \quad [E^+, E^-] = H \tag{10}$$

Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2)$. By finite dimensionality, there is a maximum of the spectrum (*i.e.* the set of eigenvalues) of $\pi(H)$, which we denote by Λ. The eigenvalues of $\pi(H)$ are called the *weights* of π, and accordingly Λ is the *highest weight*, and the corresponding eigenvectors are the *highest weight vectors*.

[6] In Appendix 11.2 I included a refresher on the definitions of representations, modules and related basic concepts.

Let v_Λ be a highest weight vector for this representation. We can obtain all other eigenvectors of $\pi(H)$ with different eigenvalues (up to rescaling) by acting on v_Λ with $\pi(E^-)^n$ [FH]. This string of eigenvectors must of course be finite, and its length is equal to the dimension of the representation. It turns out (see [FS2]) that $\{\text{length of the string}\} = \dim V = \Lambda + 1$ and $\Lambda \in \mathbb{N}$, the spectrum of $\pi(H)$ thus being

$$\{-\Lambda, -\Lambda + 2, \ldots, \Lambda - 2, \Lambda\} \tag{11}$$

Since π is irreducible, then $\pi(H)$ is diagonalizable [FH]. Thus we can write

$$V = \bigoplus_{\substack{\lambda=-\Lambda \\ \Lambda-\lambda\in 2\,\mathbb{Z}}}^{\Lambda} V_\lambda \tag{12}$$

where each V_λ is the eigenspace of the eigenvalue λ, and is one-dimensional. The V_λ are called *weight spaces*.

Remark 3.7. For the case of angular momentum in Quantum Mechanics, one can identify $E^\pm = J^\pm$ and $H = J^0$, and also $j = \frac{\Lambda}{2}$ and $m = \frac{\lambda}{2}$. Notice that this means that $m \in \{-j, -j+1, \ldots, j-1, j\}$, as we already know from Quantum Mechanics.

Highest weight representations of semisimple Lie algebras

Let (π, V) be a representation of $\mathfrak{g}$. A *weight space* of the module V with *weight* $\lambda \in \mathfrak{g}_0^*$ is a subspace of the form

$$V_\lambda := \{v \in V | \forall H \in \mathfrak{g}_0, \pi(H)(v) = \lambda(H)v\} \tag{13}$$

Notice that this generalizes the notions of weight and weight space from the $\mathfrak{sl}(2)$ theory (it is easy to see that the Cartan algebra of $\mathfrak{sl}(2)$ is $\mathbb{C}H$).

A *highest weight* is a weight Λ such that

$$\forall v \in V_\Lambda, \forall \alpha \in \Phi_+, \pi(E_\alpha)(v) = 0 \tag{14}$$

the elements of V_Λ are called *highest weight vectors*. If V has a highest weight, then (π, V) is said to be a *highest weight representation*.

Remark 3.8. Highest weight representations of simple Lie algebras are important in part because all finite-dimensional representations of simple Lie algebras are in fact highest weight representations.

Physicists have one extra reason to study highest weight representations. Namely, even when one wants infinite-dimensional representations of simple Lie algebras or of infinite-dimensional representations of generalizations of simple Lie algebras (Kac-Moody algebras), one is usually interested in the highest weight representations because in most physical applications we can take $-H$ to generate a (one-dimensional) Cartan subalgebra, where H is the Hamiltonian of the physical system. Hence demanding finite positive energy is equivalent to demanding the existence of a highest weight.

4 Kac-Moody Algebras

Remark 4.1. It can be terribly confusing to try to understand what is a Kac-Moody algebra by looking at the literature, as it seems each writer is talking about different things. There is a unifying general concept behind it all, which comes from the theory of Cartan matrices and their generalizations – I will briefly state the main ideas here, for context. If you do not care about that, just skip to §4.1.

In string theory physicists only care about a particular type of Kac-Moody algebras (the so-called untwisted affine algebras) which has a concrete construction, with no mention to Cartan matrices. These are the focus of this section. More details are to be found in the references cited along the section.

We mentioned in §3 that to a semisimple Lie algebra $\mathfrak{g}$ corresponds a Cartan matrix, and that $\mathfrak{g}$ can be recovered from the Cartan matrix by the so-called Serre's construction.

The *generalized Cartan matrices* are defined by relaxing some of the properties of the Cartan matrices. *Kac-Moody algebras* generalize semisimple Lie algebras, and are obtained by applying Serre's construction to a generalized Cartan matrix [Kac].

The generalized Cartan matrices can be separated in three disjoint classes, depending on certain properties they can have [BDK], [FS2]. This means that Kac-Moody algebras are also separated in three classes, namely: *finite-dimensional Kac-Moody algebras, affine Kac-Moody algebras* and *indefinite Kac-Moody algebras*. The finite-dimensional Kac-Moody algebras are old news: they are the semisimple Lie algebras. The indefinite Kac-Moody algebras are not very common in physics. The important ones for us are a subset of the affine Kac-Moody algebras, called the *untwisted* affine algebras, since they are prolific in theoretical physics, particularly in String theory and Conformal field theory, and will play a key role in the coset construction discussed in §5.

$$
\begin{tikzcd}
 & \text{Generalized Cartan matrices} \arrow[d, "\text{Serre's construction}"] & \\
 & \text{Kac-Moody algebras} \arrow[dl] \arrow[d] \arrow[dr, "\text{Three types}"] & \\
\text{Finite} & \text{Affine} \arrow[dl] \arrow[dr, "\text{Two types}"] & \text{Indefinite} \\
\text{Twisted} & & \boxed{\text{Untwisted}}
\end{tikzcd}
$$

There is an explicit way to obtain an untwisted affine algebra from a simple Lie algebra, and it is in this form that these algebras appear in physics. We will now summarize how this explicit realization of untwisted affine algebras is constructed, and what their highest weight unitary irreducible representations look like. These representations will be crucial in the study of the Kazama-Suzuki models.

4.1 Untwisted affine Kac Moody algebras and loop algebras

In order to construct an untwisted affine algebra $\hat{\mathfrak{g}}$ explicitly (without recurring to Cartan matrices), we first take a semisimple Lie algebra $\mathfrak{g}$ and construct its loop algebra[7] $\mathring{\mathfrak{g}}$. Then, we perform a central extension, obtaining a Lie algebra $\bar{\mathfrak{g}} = \mathring{\mathfrak{g}} \oplus \mathbb{C}\, K$. Finally, the extension of $\bar{\mathfrak{g}}$ by a derivation δ leaves us with an untwisted affine algebra $\hat{\mathfrak{g}} = \bar{\mathfrak{g}} \oplus \mathbb{C}\, \delta$.

$$
\begin{tikzcd}
\text{Simple Lie algebra } \mathfrak{g} \arrow[d] \\
\text{Loop algebra } \mathring{\mathfrak{g}} \arrow[d, "\text{Central extension}"] \\
\text{Current algebra } \bar{\mathfrak{g}} \arrow[d, "\text{Derivation extension}"] \\
\text{Untwisted affine Kac-Moody algebra } \hat{\mathfrak{g}}
\end{tikzcd}
$$

(We will come back to this diagram later in this section).

It will be important to understand what these steps are comprised of, so that one can discuss the representation theory of untwisted affine algebras later on.

[7] In the physics literature, it is common for writers to say “Kac-Moody algebra” when they really mean loop algebra, untwisted affine algebra or current algebra (defined in §4.1.3).

4.1.1 Loop algebras

We start by defining the ring of Laurent polynomials using the terminology of [MPP] and [Hal2].

Definition 4.2. The *ring of complex Laurent polynomials over one indeterminate* is the ring $(\mathbb{C}[t,t^{-1}],+,\cdot)$ where $\mathbb{C}[t,t^{-1}]$ is the set of formal sums

$$\mathbb{C}[t,t^{-1}] = \{\sum_{j\in\mathbb{Z}} a_j t^j \mid a_j \in \mathbb{C}\,\text{and only a finite number of } a_j \text{ are nonzero}\}, \tag{15}$$

where t is an indeterminate[8]. The addition is given by $\sum_{j\in\mathbb{Z}} a_j t^j + \sum_j^n b_j t^j = \sum_{j\in\mathbb{Z}}(a_j + b_j)t^j$ and the multiplication by $\sum_{j\in\mathbb{Z}} a_j t^j \cdot \sum_j^n b_j t^j = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(a_j + b_{k-j})t^k$.

We can now construct the loop algebra associated with a simple Lie algebra, which can be seen as a simple Lie algebra whose coefficients are in the ring of Laurent polynomials, as we will see shortly. It is not hard to check that this is indeed a Lie algebra [FMS].

Definition 4.3. Let $\mathfrak{g}$ be a simple Lie algebra. Its *loop algebra* is the Lie algebra whose underlying vector space is $\mathring{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}$, and with Lie brackets given by $[t^n \otimes T^a, t^m \otimes T^b] = t^{n+m} \otimes [T^a, T^b]$, where the T^a are the generators of $\mathfrak{g}$.

Example 4.4. Recall the standard basis $\{H, E^{\pm}\}$ of $\mathfrak{sl}_2(\mathbb{C})$ in (9). The matrices

$$t^m H = \begin{pmatrix} t^m & 0 \\ 0 & -t^m \end{pmatrix}, \quad t^m E^+ = \begin{pmatrix} 0 & t^m \\ 0 & 0 \end{pmatrix}, \quad t^m E^- = \begin{pmatrix} 0 & 0 \\ t^m & 0 \end{pmatrix} \tag{16}$$

form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C}[t,t^{-1}])$. There is an obvious isomorphism $\mathfrak{sl}_2(\mathbb{C}[t,t^{-1}]) \cong \mathfrak{sl}_2(\mathbb{C})$ given by $t^m X \mapsto t^m \otimes X$ for all $X \in \mathfrak{sl}_2(\mathbb{C})$ [MPP].

This illustrates the fact that we can think of a loop algebra of a Lie algebra $\mathfrak{g}$ as the Lie algebra $\mathfrak{g}$ with Laurent polynomials as coefficients.

Remark 4.5. The Cartan subalgebra of $\mathring{\mathfrak{g}}$ is $\mathbb{C}[t,t^{-1}] \otimes \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ [Sen].

Notation 4.6. Following the notation in the physics literature, we often write $T^a_n := t^n \otimes T^a$.

[8] For more on formal sums and indeterminates see [Hal2]. For us, it suffices to think of t as a symbol or placeholder which can be summed and respects the power rules of complex numbers.

4.1.2 Central extensions

Loosely speaking, an extension of a Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{k}$ is a Lie algebra $\tilde{\mathfrak{g}}$ such that $\mathfrak{g}$ and $\mathfrak{k}$ are both subalgebras of $\tilde{\mathfrak{g}}$. More concretely [DKBTK]:

Definition 4.7. An *extension* of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{k}$ is a Lie algebra $\tilde{\mathfrak{g}}$ together with Lie algebra homomorphisms k and λ such that the sequence

$$0 \to \mathfrak{k} \xrightarrow{k} \tilde{\mathfrak{g}} \xrightarrow{\lambda} \mathfrak{g} \to 0 \tag{17}$$

is exact[9]. We usually simply say that $\tilde{\mathfrak{g}}$ is the extension, and omit the homorphisms. The extension is *central* if $\operatorname{im} k \subseteq Z(\tilde{\mathfrak{g}})$.

The central extensions which will be relevant for our purposes are one-dimensional central extensions:

Definition 4.8. A *one-dimensional central extension* of a Lie algebra $\mathfrak{g}$ is a central extension (17) such that $\mathfrak{k}$ is one-dimensional and $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k}$.

Of course in this case we can write $\mathfrak{k} = \mathbb{C}\, K$ where K is a nonzero element of $\mathfrak{k}$. It turns out that we can construct a one dimensional central extension by using a "2-cocycle" of $\mathfrak{g}$ [DKBTK]:

Definition 4.9. A 2-*cocycle* of a Lie algebra $\mathfrak{g}$ is a bilinear antisymmetric map $\phi\colon\ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\phi(x, [y, z]) + \phi(y, [z, x]) + \phi(z, [x, y]) = 0$.

Proposition 4.10. *Let $\mathfrak{g}$ be a Lie algebra with Lie brackets $[\cdot,\cdot]_{\mathfrak{g}}$. If there is a 2-cocycle ϕ of $\mathfrak{g}$, then there is a one-dimensional central extension $\mathfrak{g} \oplus \mathbb{C}\, K$ of $\mathfrak{g}$, with the Lie bracket given by*

$$[x \oplus \mu K, y \oplus \nu K] := [x, y]_{\mathfrak{g}} \oplus \phi(x, y)K \tag{18}$$

Remark 4.11. In this case the exact sequence (17) reads

$$0 \to \mathbb{C}\, K \xrightarrow{k} \mathfrak{g} \oplus \mathbb{C}\, K \xrightarrow{\lambda} \mathfrak{g} \to 0 \tag{19}$$

where $k(\mu K) = 0 \oplus \mu K$ and $\lambda(x, \mu K) = x$, where $\mu \in \mathbb{C}$ and $x \in \mathfrak{g}$. It is easy to check that this sequence is a central extension of $\mathfrak{g}$. More details can be found on §18.2 of [DKBTK].

[9] A sequence $A \xrightarrow{a_1} B \xrightarrow{a_2} C \xrightarrow{a_3} D \xrightarrow{a_4} E$ is exact if $\operatorname{im} a_i = \ker a_{i+1}$ for all $i = 1, \ldots, 4$.

4.1.3 The universal central extension of a loop algebra

Let $\mathfrak{g}$ be a Lie algebra. There is a *universal* one-dimensional central extension $\mathfrak{k} \to \bar{\mathfrak{g}} \to \mathring{\mathfrak{g}}$ of the loop algebra $\mathring{\mathfrak{g}}$, meaning that for any other one-dimensional central extension $\mathfrak{k}' \to \bar{\mathfrak{g}}' \to \mathring{\mathfrak{g}}$ there are maps Φ, Ψ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{k} \arrow[r] \arrow[d, "\Phi"] & \bar{\mathfrak{g}} \arrow[r] \arrow[d, "\Psi"] & \mathring{\mathfrak{g}} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathfrak{k}' \arrow[r] & \bar{\mathfrak{g}}' \arrow[r] & \mathring{\mathfrak{g}} \arrow[r] & 0
\end{tikzcd}$$

commutes [DKBTK]. In this sense $\mathring{\mathfrak{g}}$ has a unique central extension.

Recall from §4.1.1 that a one-dimensional central extension of the loop algebra $\mathring{\mathfrak{g}}$ can be constructed using a specific 2-cocycle ϕ of $\mathring{\mathfrak{g}}$. From [DKBTK]:

Proposition 4.12. *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathring{\mathfrak{g}}$ its loop algebra. Define the map $\phi\colon\ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by*

$$\phi(t^m \otimes T^a, t^n \otimes T^b) = m\delta_{m+n,0}\kappa(T^a, T^b) \tag{20}$$

where the $t^m \otimes T^a$ are generators of $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, $x, y \in \mathfrak{g}$ and κ is the Killing form in $\mathfrak{g}$. Then ϕ is a 2-cocycle of $\mathring{\mathfrak{g}}$.

Notation 4.13. We will often write simply T^a_n for $T^a_n \oplus K$ and K for $0 \oplus K$.

We know from §3 that we can choose a basis $\{T^a\}$ of $\mathfrak{g}$ such that $\kappa(T^a, T^b) = \delta^{a,b}$. Hence using (18) and (20) one sees that the central extension $\bar{\mathfrak{g}}$ of $\mathring{\mathfrak{g}}$ is the Lie algebra with underlying vector space $\bar{\mathfrak{g}} = \mathring{\mathfrak{g}} \oplus \mathbb{C}\,K$ and with commutation relations

$$\begin{aligned}
&[T^a_m, T^b_n] = f^{ab}{}_c T^c_{m+n} \oplus mK\delta^{ab}\delta_{m,-n} \\
&[T^a_m, K] = 0
\end{aligned} \tag{21}$$

We will follow the physics literature and call $\bar{\mathfrak{g}}$ the *current algebra* of $\mathfrak{g}$.

Remark 4.14. The Cartan subalgebra of $\bar{\mathfrak{g}}$ is $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus K$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ [Her]. Notice that this Cartan subalgebra[10] is infinite-dimensional, which creates difficulties for the study of the representation theory of $\bar{\mathfrak{g}}$. This is the motivation for physicists to extend $\bar{\mathfrak{g}}$ by a derivation [GO], ending up, as we will see, with a finite dimensional Cartan subalgebra.

[10] We have not defined Cartan subalgebras for non-semisimple Lie algebras. This will be done in 4.2. See also footnote 11.

4.1.4 The explicit realization of an untwisted affine Kac-Moody algebra

We still need to perform another extension, called an "extension by a derivation". First of all, we must say what we mean by derivation [DKBTK]:

Definition 4.15. Let $\mathfrak{g}$ be a Lie algebra. A *derivation of* $\mathfrak{g}$ is a linear map $\delta\colon\ \mathfrak{g} \to \mathfrak{g}$ such that $\delta[x,y] = [\delta x, y] + [x, \delta y]$ for every $x, y \in \mathfrak{g}$.

Definition 4.16. Let $\mathfrak{g}$ be a Lie algebra with Lie brackets $[\cdot,\cdot]_{\mathfrak{g}}$ and δ a derivation of $\mathfrak{g}$. The *extension of* $\mathfrak{g}$ *by the derivation* δ is the Lie algebra with underlying vector space is $\mathfrak{g} \oplus \mathbb{K}\,\delta$ and Lie brackets given by $[x \oplus \mu\delta, y \oplus \nu\delta] = [x,y]_{\mathfrak{g}} + \mu\delta(y) - \nu\delta(x)$.

This is indeed a Lie algebra, and is also an extension in the sense of §4.7 [CC]. For more on extensions by derivations see [DKBTK].

Now, if we want to extend the central extension $\bar{\mathfrak{g}}$ of the loop algebra $\mathring{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$ by a derivation, we must first of all find a derivation of $\bar{\mathfrak{g}}$. From [DKBTK, FMS]:

Proposition 4.17. *Let* $\mathfrak{g}$ *be a simple Lie algebra and* $\bar{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}\,K$ *the central extension of its loop algebra. Define the map* $\delta\colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ *by*

$$\delta(P(t) \otimes x \oplus \alpha K) = \frac{\mathrm{d}P(t)}{\mathrm{d}t} \otimes x \oplus 0 \tag{22}$$

where $P(t) \in \mathbb{C}[t,t^{-1}], x \in \mathfrak{g}$ *and* $\alpha \in \mathbb{C}$. *Then* δ *is a derivation of* $\bar{\mathfrak{g}}$.

It turns out that the derivation extension by δ of the central extension of $\bar{\mathfrak{g}}$ (where $\mathfrak{g}$ is a simple Lie algebra) is an untwisted Kac-Moody algebra [CC], which we will denote by $\hat{\mathfrak{g}}$. Schematically:

$$\begin{array}{c}
\text{Simple Lie algebra } \mathfrak{g} \\
\big\downarrow \\
\text{Loop algebra } \mathring{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \\
\big\downarrow{\scriptstyle \text{Central extension}} \\
\text{Current algebra } \bar{\mathfrak{g}} = \mathring{\mathfrak{g}} \oplus \mathbb{C}\,K \\
\big\downarrow{\scriptstyle \text{Derivation extension}} \\
\text{Untwisted affine Kac-Moody algebra } \hat{\mathfrak{g}} = \bar{\mathfrak{g}} \oplus \mathbb{C}\,\delta
\end{array}$$

Remark 4.18. The derivation δ does not commute with the generators $t^n \otimes T^a$ for $n \neq 0$ (in fact $[\delta, t^n \otimes T^a] = nt^n \otimes T^a$ [FMS]), and this has the effect of shrinking the Cartan subalgebra of $\bar{g}$: the Cartan subalgebra of $\hat{\mathfrak{g}}$ is $\hat{\mathfrak{h}} := (1 \otimes \mathfrak{h}) \oplus \mathbb{C}\, K \oplus \mathbb{C}\, \delta$ [FMS]. Compare with Remark 4.14.

Since by definition the weights are just the eigenvalues of the generators of the Cartan subalgebra, from Remark 4.18 we have the following:

Conclusion 4.19. *The weights of a representation $\hat{\pi}$ of $\hat{\mathfrak{g}}$ can be written (Λ, k, n), where Λ is a $\mathfrak{g}$-weight, k is the eigenvalue of $\hat{\pi}(K)$ and n is the eigenvalue of $\hat{\pi}(\delta)$ [Fuc2, FMS, CC].*

4.2 Unitary irreducible representations of Affine Kac-Moody algebras

Notation 4.20. In this section, $\mathfrak{k}$ is a Kac-Moody algebra and $\hat{\mathfrak{g}}$ is an untwisted affine algebra, with $\mathfrak{g}$ a semisimple Lie algebra.

Just like semisimple Lie algebras, every Kac-Moody algebra has a so-called triangular decomposition $n_- \oplus \mathfrak{h} \oplus n_+$ [Kac]. In the case of $\hat{\mathfrak{g}}$ we write $\hat{n}_- \oplus \hat{\mathfrak{h}} \oplus \hat{n}_+$ instead.

Such decompositions are defined in full generality in [MPP]. We will not delve into such deep waters, since understanding the proper general definition of triangular decomposition is irrelevant for our purposes: we just need to know that there is a special decomposition $n_- \oplus \mathfrak{h} \oplus n_+$ of $\mathfrak{k}$.

The concept of Cartan subalgebra which we found in the semisimple case can also be broadened to general Lie algebras. We include the definition of Cartan subalgebra of a general Lie algebra for completeness[11] [Hum2]:

Definition 4.21. Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{l}$.

$\mathfrak{h}$ is *nilpotent* if $L^n = 0$ for some $n \in \mathbb{N}$[12].

Furthermore, the *normalizer* of $\mathfrak{h}$ in $\mathfrak{l}$ is a subalgebra $N_{\mathfrak{l}}(\mathfrak{h})$ of $\mathfrak{l}$ defined by $N_{\mathfrak{l}}(\mathfrak{h}) = \{x \in \mathfrak{l} \,|\, [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$.

A *Cartan subalgebra* of $\mathfrak{l}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ which is nilpotent and is its own normalizer in $\mathfrak{l}$.

[11] In the physics literature one rarely if ever sees the Cartan subalgebra defined for non-semisimple Lie algebras. This does not bring problems because usually one can get away with thinking of Cartan subalgebras as they were defined for the semisimple Lie algebras.

[12] By definition, $L^0 = L$ and $L^i = [L, L^{i-1}]$ for $i > 0$.

Now, Lie algebras with triangular decomposition $n_- \oplus \mathfrak{h} \oplus n_+$ always have the $\mathfrak{h}$ as the Cartan subalgebra [MPP]. We are therefore allowed to talk of weights in the case of a Kac-Moody algebra, which are defined as in the semisimple case. More information about the general picture can be found in [MPP].

One more concept from the theory of Lie algebras is necessary before we tackle the representation theory of Kac-Moody algebras [Lor, Bek]. We will introduce it as notation:

Notation 4.22. Let $\mathfrak{l}$ be a Lie algebra with a basis $\{T^i\}_{i \in I \subseteq \mathbb{N}}$. Let π be a representation of $\mathfrak{l}$. Denote

$$U(\mathfrak{l}) := \text{span}\{T^{i_1} T^{i_2} \dots T^{i_k} \mid i_1 \leq i_2 \leq \dots i_k \text{ and } k \in \mathbb{N}^+\} \tag{23}$$

(where as is current practice we omit π, so that $T^i := \pi(T^i)$).

This is a particular instance of a much more general object called the universal enveloping algebra [Hum1, Hum2]. However, to simplify the discussion, we take $U(\mathfrak{l})$ to be just a notational artifact.

We are finally in a position to define highest weight modules in the context of Kac-Moody algebras. The following follows the conventions of [Kac] and [MPP].

Definition 4.23. A *(Kac-Moody) highest weight $\mathfrak{k}$-module* with *highest weight* $\Lambda \in \mathfrak{h}^*$ is a $\mathfrak{k}$-module V containing a nonzero vector $v \in V$ such that:

$$\begin{aligned} &n_+ \cdot v = 0 \\ &H \cdot v = \Lambda(H) v \quad (\forall H \in \mathfrak{h}) \\ &U(n_-) \cdot v = V \end{aligned} \tag{24}$$

The vector v is called a *highest weight vector.*

Definition 4.24. Let $M(\Lambda)$ be a highest weight $\mathfrak{k}$-module. $M(\Lambda)$ is said to be a *Verma module* if every highest weight $\mathfrak{k}$-module with highest weight Λ is a quotient[13] of $M(\Lambda)$.

Remark 4.25. For each $\Lambda \in \mathfrak{h}^*$ we can define a Verma module $M(\Lambda)$, and it is the unique Verma module with highest weight Λ up to isomorphism [Kac]. Every Verma module $M(\Lambda)$ has a unique proper maximal submodule $J(\Lambda)$, and the quotient $L(\Lambda) := M(\Lambda)/J(\Lambda)$ is the unique irreducible highest weight $\mathfrak{k}$-module with highest weight Λ [Kac]. Notice that this implies in particular that all irreducible highest weight $\mathfrak{k}$-modules are of the form $L(\Lambda)$ for some $\Lambda \in \mathfrak{h}^*$. This means that *there is a bijection between $\mathfrak{h}^*$ and the set of all irreducible highest weight $\mathfrak{k}$-modules.*

[13] Quotient modules are defined in Appendix 11.2.

Remark 4.26. Recall from Conclusion 4.19 that for an untwisted affine algebra $\hat{\mathfrak{g}}$ we can denote the elements $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ by $\hat{\lambda} = (\lambda, \hat{\lambda}(K), \hat{\lambda}(d))$, where λ is an element of $\mathfrak{g}_0^*$. Hence we can label irreducible highest weight $\hat{\mathfrak{g}}$-modules with such triples. It turns out [Fuc1] that we can set $n = 0$. Hence we can label the $\hat{\mathfrak{g}}$-weights (and thus also the irreducible $\hat{\mathfrak{g}}$-modules) by pairs (Λ, k).

We will only want to use unitary representations. There is a simple condition for a highest weight irreducible $\hat{\mathfrak{g}}$-module (Λ, k) to be unitary [GO]:

Proposition 4.27. *Let $\mathfrak{g}$ be a simple Lie algebra. The irreducible representation (Λ, k) of $\hat{\mathfrak{g}} = \mathring{\mathfrak{g}} \oplus \mathbb{C}\,K \oplus \mathbb{C}\,\delta$ is unitary if and only if $\frac{2k}{\psi^2} \in \mathbb{Z}$ and $k \geq \psi \cdot \Lambda \geq 0$, where ψ is a long root of $\mathfrak{g}$.*

Notation 4.28. A unitary representation as in Proposition 4.27 is usually denoted $(\Lambda, \frac{2k}{\psi^2})$. $\frac{2k}{\psi^2}$ is called the *level* of the representation, and is often also denoted by k.

Notation 4.29. We are frequently interested in the unitary irreducible highest weight representations with a certain fixed level. Denote by $\hat{\mathfrak{g}}_k$ the family of unitary irreducible highest weight $\hat{\mathfrak{g}}$-modules with level k.

5 Highest weight irreducible representations of the Virasoro algebra

Coset models are unitary highest weight irreducible Vir-modules. The Grassmannian model is a Kazama-Suzuki model, which in turn is a coset model. In this section we go through the necessary concepts from representation theory of the Virasoro algebra which are necessary to understand highest weight irreducible Vir-modules and coset models in particular.

5.1 Highest weight representations of the Virasoro algebra

Just like semisimple Lie algebras and Kac-Moody algebras, the Virasoro algebra has a so-called triangular decomposition $n_- \oplus \mathfrak{h} \oplus n_+$. We already mentioned in §4 that Lie algebras with a triangular decomposition have the Cartan subalgebra $\mathfrak{h}$, so that we can talk of weights of a representation of the Virasoro algebra, which are defined as in the semisimple and the Kac-Moody cases.

Remark 5.1. The fact that the Virasoro algebra has a triangular decomposition turns out to be the reason why its representation theory resembles the representation theory of semisimple Lie algebras and Kac Moody algebras [FS2], since one can still define concepts like highest weight representations and Verma modules [MPP].

The exact form of the triangular decomposition of the Virasoro algebra turns out to be important, so we will define it here. Again, more details about the general picture can be found in [MPP].

Definition 5.2. The *triangular decomposition of the Virasoro algebra* is the triple $(n_-, \mathfrak{h}, n_+)$, where $n_- = \mathrm{span}_{\mathbb{C}}\{L_m, m < 0\}$, $n_+ = \mathrm{span}_{\mathbb{C}}\{L_m, m > 0\}$ and $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{L_0 \oplus 0, 0 \oplus C\}$. Notice that

$$Vir = n_- \oplus \mathfrak{h} \oplus n_+ \tag{25}$$

Definition 5.3. A Vir-module V is called a *highest weight (Virasoro) module with highest weight* $(c, h) \in \mathbb{C}$ if there is a nonzero vector $c \in V$ (called *highest weight vector*) such that:

$$L_0 \cdot v = hv, \quad C \cdot v = cv, \quad U(n_-)(v) = V \tag{26}$$

Remark 5.4. Notice that $U(n_-) = \mathrm{span}\{L_{-i_k} \dots L_{-i_1}, k \in \mathbb{N}^+\}$ (see 4.22).

Remark 5.5. Of course the pair (c, h) is not, strictly speaking, a weight in the usual sense. But we can identify it with the weight $\Lambda = \hat{h} \oplus 0 + 0 \oplus \hat{c} \in \mathfrak{h}^*$, and define $\hat{h}(L_0) = h$ and $\hat{c}(C) = c$. We can use (26) to see that this Λ is a highest weight:

$$\begin{aligned}
&(L_0 \oplus 0) \cdot v =: L_0 \cdot v = hv = \hat{h}(L_0) + \hat{c}(0) = \Lambda(L_0 \oplus 0)v, \\
&(0 \oplus C) \cdot v =: C \cdot v = cv = \Lambda(0 \oplus C)v, \\
&U(n_-)(v) = V
\end{aligned} \tag{27}$$

Remark 5.6. In other contexts (as for example for Kac-Moody algebras [Kac]) one would usually also demand in the definition of highest weight representation that $n_+ \cdot v = 0$. However for Vir this is unnecessary since the two last conditions in (26) already imply $n_+ \cdot v = 0$ (see (3.12d) of [KRR]).

5.2 Virasoro Verma modules

There are different (but analogous) definitions of Verma module in different contexts. Verma modules are crucial to understand the representation theory of semisimple Lie algebras [Hal1], Kac-Moody algebras [Kac] and the Virasoro algebra [KRR], and take slightly different forms in each case.

Physicists are usually introduced to Verma modules in the context of conformal field theory [BP], where one wants to understand the representation theory of the Virasoro algebra. This is also the context which interests us, although we will follow the definitions and conventions of the mathematics literature, instead of the physics literature, since the former better serves our purposes. More concretely, we will follow the conventions in [KRR].

Definition 5.7. A *(Virasoro) Verma module* $M(\Lambda)$ is a highest weight module of Vir with highest weight Λ and highest weight vector v such that the vectors of the form

$$L_{-i_k} \dots L_{-i_1}(v), \quad 0 < i_1 \leq \dots \leq i_k \tag{28}$$

are linearly independent.

Remark 5.8. In particular, this definition implies that Verma modules are infinite dimensional.

Remark 5.9. In the physics literature, the highest weight vector of a Verma module is called a *primary state*, and the elements of the type (28) are said to be its descendants [FMS] (pp157, 177). The name primary state comes from the fact that primary states are related to the primary fields (whose definition we will recall in §6.1) through the state-operator map of CFT [FMS].

Remark 5.10. Let $M(\Lambda)$ be a Verma module. Then every highest weight Vir-module with highest weight Λ is a quotient of $M(\Lambda)$ (see page 23 of [KRR]). This property is actually the defining property of Verma modules in other contexts (*e.g.* in Definition 4.24).

The existence of Verma module was shown on page 23 of [KRR].

Lemma 5.11. *For every pair $(c, h) \in \mathbb{C}$, there exists a Verma module $M(c, h)$.*

Using Remark 5.10, we can adapt the proof of uniqueness of Kac-Moody Verma modules in [Kac] to the Virasoro algebra case. We start with a useful Lemma.

Lemma 5.12. *Let $\mathfrak{g}$ be a Lie algebra and V, W be $\mathfrak{g}$-modules of the form $V = \oplus_{\lambda \in \mathfrak{g}_0^*} V_\lambda$, $W = \oplus_{\lambda \in \mathfrak{g}_0^*} W_\lambda$ (*i.e.* V and W are $\mathfrak{g}_0$-diagonalizable). If $\psi : V \to W$ is a $\mathfrak{g}$-module homomorphism, then $\forall \lambda \in \mathfrak{g}_0^*, \psi(V_\lambda) \subseteq W_\lambda$.*

Proof. Let $\lambda \in \mathfrak{g}_0$, $v \in V_\lambda$ and $H \in \mathfrak{g}_0$. Then $H \cdot v = \lambda v$. By the properties of $\mathfrak{g}$-module homomorphisms (intertwinners), we have $H \cdot \psi(v) = \psi(H \cdot v) = \psi(\lambda v) = \lambda \psi(v)$. Since H was arbitrary, then $\psi(v) \in W_\lambda$. This shows that $\psi(V_\lambda) \subseteq W_\lambda$. $\square$

Proposition 5.13. *For every pair (c, h), there is a unique Verma module $M(c, h)$ up to isomorphism.*

Proof. Existence was asserted in Lemma 5.11. Let us show uniqueness. Let $M_1(c, h)$ and $M_2(c, h)$ be two Verma modules. In particular, both are highest weight Vir-modules with highest weight (c, h).

Using Remark 5.10, $M_1(c, h)$ is a quotient of $M_2(c, h)$. The quotient map $q_{21}\colon M_2(c, h) \to M_1(c, h)$ is a surjective homomorphism of Vir-modules. Indeed: surjectivity is obvious by the definition of quotient map and quotient space. Linearity comes from how scalar multiplication and addition are defined on the quotient space $M_1(c, h)$:

$$\alpha[v] := [\alpha v], \quad [v] + [w] := [v + w], \quad \text{with } \alpha \in \mathbb{C} \text{ and } v, w \in M_2(c, h)$$

and compatibility with the action is established by the definition of quotient module, for which: $T \cdot [v] := [T \cdot v]$ for all $T \in Vir$. This shows that q_{21} is a surjective homomorphism of Vir-modules. Since Verma modules are Vir_0-diagonalizable [KRR] we can use Lemma 5.12 to conclude that $q_{21}(M_{2\lambda}) \subseteq M_{1\lambda}$. But Verma modules have finite-dimensional weight spaces [KRR], so that we can use the rank-nullity theorem to see that $\dim M_1(c, h)_\lambda = \dim M_2(c, h_\lambda) - \dim \ker q_{21} \leq \dim M_2(c, h)_\lambda$.
Now, we have another quotient map $q_{12}\colon M_1(c, h) \to M_2(c, h)$. Repeating the argument above for this map we see that $\dim M_2(c, h)_\lambda \leq \dim M_1(c, h)_\lambda$.
In conclusion, $\dim M_2(c, h)_\lambda = \dim M_1(c, h)_\lambda$ for every λ and thus also $M_2(c, h)_\lambda \cong M_1(c, h)_\lambda$. Hence $M_2(c, h) \cong M_1(c, h)$. ☐

Remark 5.14. From Proposition 3.3(c) of [KRR], the Verma module $M(c, h)$ has a unique proper submodule $J(c, h)$, and the unique irreducible highest weight module with highest weight (c, h) is precisely the quotient

$$V(c, h) := \frac{M(c, h)}{J(c, h)} \tag{29}$$

This means in particular that there is a one-to-one relationship between the pairs (c, h) and the irreducible highest weight Vir-modules.

5.3 The GKO (or coset) construction

In physics, one is usually interested in representations of the Virasoro algebra which are not only highest weight and irreducible, but also unitary. For example in string theory this ensures that the Hamiltonian is hermitian [Ton].

It is easy to see that unitarity implies that c and h are non-negative real numbers [KRR]. Now, there is an unitary irreducible highest-weight representation of Vir for each pair (c, h) with $c \geq 1$ and $h \geq 0$, but the only values of c and h for which we *may* have a unitary irreducible highest weight representation when $c < 1$ are [FQS]:

$$c = 1 - \frac{6}{m(m-1)}, \quad m \in \mathbb{Z}_{\geq 2} \tag{30}$$

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \quad p = 1, ..., m-1; \quad q = 1, ..., p \tag{31}$$

(In particular, for each fixed value of the central charge there are finitely many unitary irreducible highest-weight representations, each corresponding to a different value of h). These results come from imposing the non-existence of ghosts. See [FQS] and [BP] for more details.

To construct all the possible $(c < 1, h \geq 0)$ unitary highest weight irreducible representations of the Virasoro algebra, we do the following [GKO]:

- Consider a level N and a level 1 irreducible highest weight families of representations of the untwisted affine algebra $\hat{\mathfrak{su}}(2)$ with the same representation space. Set $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and take the representation of the untwisted affine algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{su}}(2)$ to be the direct sum of the aforementioned representations (see the Appendix 11.2). We denote such representation by $\hat{\mathfrak{su}}(2)_N \oplus \hat{\mathfrak{su}}(2)_1$. This induces [GKO] a level $N+1$ representation of the untwisted affine algebra $\hat{\mathfrak{h}}$ of the diagonal subalgebra $\mathfrak{h} = \{x \oplus x \mid x \in \mathfrak{su}(2)\} \cong \mathfrak{su}(2)$ of $\mathfrak{g}$.

- Now, every unitary highest weight representation π of an untwisted affine algebra $\hat{\mathfrak{l}}$ (for a simple Lie algebra $\mathfrak{l}$) induces a unitary highest weight representation $Vir(\mathfrak{l})$ of the Virasoro algebra through the so-called *Sugawara construction* [FMS]. The resulting representation is called a *WZW model* [FMS]. Its representation space is the same as the one of the representation π. We denote the generators of such representation by $L_m^{\mathfrak{l}}$, and they are constructed using the generators T_m^a of $\hat{\mathfrak{l}}$ as follows:

  $$L_n^{\mathfrak{l}} = \frac{1}{2\beta} \sum_{a=1}^{\dim \mathfrak{l}} \sum_{m \in \mathbb{Z}} : T_{m+n}^a T_{-m}^a : \tag{32}$$

  where β is a normalization factor [GKO].

  If $\mathfrak{l}$ is actually semisimple, the Sugawara construction still holds, now with generators [GO]

  $$L_n^{\mathfrak{l}} = \bigoplus_{i=1} L_n^{\mathfrak{l}_i} \tag{33}$$

where the $\mathfrak{l}_i$ are the simple components of $\mathfrak{l}$ and $L_n^{\mathfrak{l}_i}$ is given by (32). The representation space in this case is the direct sum of the representations spaces V_i of the representations π_i of the simple components[14].

- Therefore in our case we have two representations of the Virasoro algebra: $Vir(\mathfrak{g})$ and $Vir(\mathfrak{h})$. We now construct a third unitary highest weight irreducible representation of the Virasoro algebra, by defining its generators K_n by

$$K_n := \; L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}} \tag{34}$$

This is the so-called *coset representation* of $\mathfrak{g}$ and $\mathfrak{h}$. This construction (the *coset* or *GKO construction*) can be carried out for any simple compact Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$. We denote it by $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$, $\mathfrak{g}/\mathfrak{h}$ or ${}^{G}/_{H}$.

Notation 5.15. We can also make the levels of the representations used more explicit. For example, back to the $\mathfrak{su}(2)$ case, the coset construction yields the representation $\frac{\hat{\mathfrak{su}}(2)_N \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{N+1}}$.

- We chose specifically these $\mathfrak{g}$ and $\mathfrak{h}$ because these coset representations turn out to cover all possible values (30) of the central element $c < 1$, thus exhausting the list of all unitary highest weight irreducible representations of the Virasoro algebra with central element $c < 1$ [GKO]. Interestingly, these also cover all possible highest weights $h(c < 1)$ in (31). Hence *the GKO construction provides us with unitary highest weight irreducible representations of the Virasoro algebra*, and if we choose $\mathfrak{g}$ and $\mathfrak{h}$ appropriately we actually obtain *all* the unitary highest weight irreducible representations.

Remark 5.16. Notice that a coset model $\frac{\hat{\mathfrak{su}}(2)_N \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{N+1}}$ has a certain central charge c and a conformal dimension h which belongs to the *finite* series of possible values (31). Conformal field theories with a finite number of unitary highest weight irreducible representations (for each value of c) are called *Rational Conformal field theories* (RCFTs)[15] [BP]. Hence $\frac{\hat{\mathfrak{su}}(2)_N \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{N+1}}$ is a RCFT. It turns out that all coset models are RCFTs [FMS].

[14] See the Appendix 11.2 for the definition of the direct sum of representations of a Lie algebra.

[15] It is known that CFTs with this property must have rational central charges, thus justifying the name "Rational" [BP].

5.4 The conformal weights in a coset model

It will prove useful to know how the conformal weights of a coset model $\mathfrak{g} \,/\, \mathfrak{h}$ can be written in terms of the conformal weights of the WZW models constructed from (the chosen representations of) $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$, and how the latter are themselves expressed in terms of (the chosen representations of) $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$.

Recall from §4 that the highest weight irreducible representations of an affine Kac-Moody algebra are labelled by pairs (λ, k), where λ is the highest weight of a highest weight $\mathfrak{g}$-module and k is an integer called the level.

Also recall from §5.3 that the Sugawara construction still makes sense when one starts from semisimple Lie algebras $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ (not only simple Lie algebras). So the first thing we have to do is see how the conformal weight of the $Vir(\hat{g})$ WZW model is written in terms of the conformal weights of the WZW models $Vir(\hat{g}_i)$.

Proposition 5.17. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be simple Lie algebras. Let h_1 be the highest weight of $Vir(\mathfrak{g}_1)$ and h_2 be the highest weight of $Vir(\mathfrak{g}_2)$. Then the WZW model $Vir(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ has highest weight $h = h_1 + h_2$.*

Proof. We have $L_0^{\mathfrak{g}} \,|h\rangle = h\,|h\rangle$, where $|h\rangle$ is the highest weight state of $Vir(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$. But $L_0^{\mathfrak{g}} \,|h\rangle = (L_0^{\mathfrak{g}_1} + L_0^{\mathfrak{g}_2})\,|h\rangle = (h_1 + h_2)\,|h\rangle$, where the last equality comes from the fact that the representation space of $Vir(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ is the tensor product of the representation spaces of $Vir(\mathfrak{g}_1)$ and $Vir(\mathfrak{g}_2)$ and thus have highest weight state $|h\rangle = |h_1\rangle \otimes |h_2\rangle$. $\square$

The way that the conformal weight $h_{\mathfrak{g}}^{\Lambda,k}$ of a WZW model $Vir(\mathfrak{g})$ of an affine untwisted algebra $\hat{\mathfrak{g}}$ with a chosen highest weight representations (Λ, k) can be written in terms of Λ, k and $\mathfrak{g}$ is well known [FMS, DJ]:

$$h_{\mathfrak{g}}^{\Lambda,k} = \frac{\Lambda^2 + 2\Lambda \cdot \rho_{\mathfrak{g}}}{2(k+g)} \tag{35}$$

where $\rho_{\mathfrak{g}}$ is the *Weyl vector* defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \tag{36}$$

(where Φ_+ is the set of positive roots, as usual), and g is the so-called *dual Coxeter number*, which can be written $g = \frac{1}{2} c_{\mathfrak{g}}$, where $c_{\mathfrak{g}}$ is the eigenvalue for the *quadratic Casimir operator* in the adjoint representations. The definitions and properties of these objects will not be relevant for this text, so we will say nothing else about them. More can be found in [FS2] or [FMS].

Equation (35) together with Proposition 5.17 gives us a recipe to obtain the expression of the conformal weight of WZW models of the form $Vir(\mathfrak{g} = \bigoplus_i \mathfrak{g}_i)$, where the $\mathfrak{g}_i$ are simple Lie algebras. We still have to connect this with the coset construction. Again we simply present a well-known result [FMS], [DJ]:

Proposition 5.18. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Denote by $h_{\mathfrak{g}\,/\,\mathfrak{h}}$ the conformal dimension of the coset model $\mathfrak{g}\,/\,\mathfrak{h}$, by $h_{\mathfrak{g}}$ the conformal dimension of the WZW model $Vir(\mathfrak{g})$ and by $h_{\mathfrak{h}}$ the conformal dimension of the WZW model $Vir(\mathfrak{h})$. There is a unique $n \in \mathbb{Z}$ such that $h_{\mathfrak{g}\,/\,\mathfrak{h}} = h_{\mathfrak{g}} - h_{\mathfrak{h}} + n$.*

If $\mathfrak{g}$ and $\mathfrak{h}$ are simple Lie algebras, then $h_{\mathfrak{g}}$ and $h_{\mathfrak{h}}$ are given by (35). If not, one uses Proposition 5.17 to reduce the problem to the simple case.

Notation 5.19. It is often useful to keep the levels k and r of the representation of $\mathfrak{g}$ and $\mathfrak{h}$ fixed, while letting Λ and λ take on any possible value (respecting the usual restrictions – as discussed in §4). We denote (the family of) coset models of this type by $\mathfrak{g}_k\,/\,\mathfrak{h}_r$. It is customary to refer to $\mathfrak{g}_k\,/\,\mathfrak{h}_r$ as a coset model, although this is technically incorrect. We will adopt this convention, keeping in mind that we are actually talking about a family of cosets, not a single coset.

Notice that Proposition 5.18 means that there is a bijection j between the set S of all conformal weights of the coset model $\mathfrak{g}\,/\,\mathfrak{h}$ and the quotient $S/\,\mathbb{Z}$. The elements of $S/\,\mathbb{Z}$ can be written $h \mod 1$, where $h \in S$. From now on we will say that the elements of $j(S)$ are the conformal weights, so that using Proposition 5.18 we can write

$$h_{\mathfrak{g}\,/\,\mathfrak{h}} = h_{\mathfrak{g}} - h_{\mathfrak{h}} \mod 1 \tag{37}$$

6 Conformal and superconformal field theory in two dimensions

In this section, we will briefly review the main concepts of two-dimensional conformal field theory (CFT) that are relevant for this text, and follow with a quick look into superconformality as defined in [BLT]. §6.1 is just a refresher of some results from CFT which are discussed in a first string theory course, so that familiarity with the topic is assumed. More details can be found for example in [BP]. In contrast, I do not expect the reader to be familiar with the ideas in §6.2 and §6.3. Nonetheless, the topic of superconformality is much deeper than what is discussed here, and these sections only have the function to motivate the definitions of the $N = 1$ and $N = 2$ superconformal algebras.

6.1 Conformal field theory in two dimensions

Conformal transformations and the Virasoro algebra

A *conformal transformation* is a coordinate transformation $x \mapsto x'$ which preserves the metric up to a positive scalar factor, *i.e.* $g' = \Lambda g$, where $\Lambda(x) > 0$ for all x.

In the two dimensional case (with coordinates (x^0, x^1)), and after introducing complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix_1$, it turns out that an infinitesimal transformation $z \mapsto z + \epsilon(z, \bar{z})$ in two dimensions is an infinitesimal conformal transformation exactly when $\epsilon(z, \bar{z}) = \epsilon(z)$ is a holomorphic function (in some neighborhood of z). Similarly for $\bar{z}$.

Under the assumption that $\epsilon(z)$ is a meromorphic function (only has isolated singularities), one may write down a Laurent expansion around any point – and one conveniently chooses that point to be $z = 0$. The algebra of the infinitesimal two-dimensional conformal transformations is then extracted in the usual way, and one concludes that it is a Lie algebra with the generators $l_n = -z^{n+1}\partial_z$, $(n \in \mathbb{Z})$ and Lie bracket $[l_m, l_n] = (m - n)l_{m+n}$. This is precisely the Witt algebra from Example 3.1, which is an infinite-dimensional Lie algebra. Again, the same can be done for $\bar{z}$, leaving us with another copy of the Witt algebra whose generators are denoted $\bar{l}_n$, and furthermore $[l_m, \bar{l}_n] = 0$.

Now, one should actually expect the Hilbert space of a conformally invariant system to be in a representation of the central extension of the Witt algebra – *i.e.* the Virasoro algebra from Example 3.1 – instead of simply being in a representation of the Witt algebra.

To explain the exact reasoning behind this assertion is out of the scope of this thesis, but the main idea is that the state space of the system is not really a Hilbert space, but rather a (complex) Hilbert space modulo $\mathbb{C}\setminus\{0\}$ – a *projective Hilbert space* – because multiplying a Hilbert space vector $|\psi\rangle$ by a complex number $\lambda \neq 0$ does not change the physical state, by the basic principles of Quantum Mechanics. This means that what we actually want is a so-called projective representation of the Witt algebra on this projective Hilbert space.

However, it is much nicer to work with Hilbert spaces than with projective Hilbert spaces, and fortunately this is still possible, since to work with projective representations of the Witt algebra turns out to be equivalent to working with (standard) representations of its central extension: the Virasoro algebra. For more on this the reader is advised to consult for example [DKBTK].

The energy-momentum tensor

Now, back to the case of general arbitrary dimension D, given a theory with an infinitesimal conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$ one must have a Noether current j_μ. We can define[16] the *energy-momentum tensor* of the theory by $j_\mu = T_{\mu\nu}\epsilon^\nu$, and it is straightforward to check that T is traceless *i.e.* $T^\mu_\mu = 0$.

In the two dimensional case (again using the complex coordinates $z, \bar{z}$), tracelessness actually implies that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and the non-vanishing components of the stress-energy tensor are $T_{zz}(z,\bar{z}) = T_{zz}(z) =: T(z)$ and $T_{\bar{z}\bar{z}}(z,\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}) =: \bar{T}(\bar{z})$. $T(z)$ is said to be the *chiral* or *holomorphic* part of the stress-energy tensor, while $\bar{T}(\bar{z})$ is called the *anti-chiral* or *anti-holomorphic* part of the stress-energy tensor. One usually only discusses the holomorphic part, since anti-holomorphic part is identical.

It is possible to perform a Laurent expansion of $T(z)$ as follows:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{38}$$

or to invert it and write

$$L_n = \frac{1}{2\pi i}\oint \mathrm{d}z z^{n+1} T(z) \tag{39}$$

and the L_n generate a (representation of the) Virasoro algebra with some central charge $c \in \mathbb{C}$.

Primary fields

There is a special type of field which transforms under conformal transformations as a tensor would. These are the *primary fields*. Concretely, a field $\phi(z,\bar{z})$ is a primary field if it transforms under a conformal transformation $z \mapsto f(z)$ according to the transformation rule

$$\phi(z,\bar{z}) \mapsto \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{40}$$

The primary fields are intimately related (through the so-called state-operator map) to the primary states – and thus to the Verma modules – of the Virasoro algebra of the CFT (see §5).

[16] The usual way of defining the energy-momentum tensor as proportional to the variation of the action with respect to the metric turns out to agree with this definition – see for example [Ton]

Concrete connection with string theory

Finally, to make a concrete connection with bosonic string theory, let us mention that the modes of the energy-momentum tensor coming from the action of one free boson generate a Virasoro algebra with central charge $c = 1$, so that the Polyakov action (which can be seen as the action for D free bosons [BLT]) gives rise to a Virasoro algebra with central charge $c = D$.

On the other hand, the action of one free fermion, which (as we will see in §7) plays an important role in fermionic string theory, is a CFT with $c = \frac{1}{2}$.

6.2 $N = 1$ superconformal transformations in two dimensions

The most natural way to define superconformal transformations is using (m, n) supermanifolds, which are essentially manifolds with m real coordinates and n Grassmannian coordinates [Ali]. N is the number of Grassmannian coordinates that we introduce for each real coordinate. So for the $N = 1$ case we extend the coordinate space of the worldsheet from a two-dimensional space to a $(2, 2)$ superspace, so that the coordinates are not $(z, \bar{z})$ anymore, but we have instead $(z, \bar{z}, \theta, \bar{\theta})$. Concepts like derivation still make sense in superspace [Ali], so that we have super-derivatives $D, \bar{D}$ (defined by $D = \partial_\theta + \theta\partial_z$ and $\bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$).

We defined conformal transformations in §6.1. Recall from bosonic string theory that a conformal transformation on a 2-dimensional manifold with coordinates $(z, \bar{z})$ can also be defined to be a map $\phi : U \to \phi(U), (z, \bar{z}) \mapsto (z', \bar{z}')$ such that $\partial\bar{z} = 0$ and $\bar{\partial}z = 0$ (*i.e.* the first component of ϕ is holomorphic and the second one is anti-holomorphic) [BP]. In particular $\partial = \frac{\partial z'}{\partial z}\partial'$ and $\partial' = \frac{\partial z}{\partial z'}\partial$.

Similarly, a superconformal transformation is a map in a $(2, 2)$ supermanifold is a map ϕ whose "unbared" components are holomorphic, while the "bared" ones are anti-holomorphic, and the super-derivatives $D, \bar{D}$ transform similarly to the derivatives of the conformal transformation [BLT]:

$$\phi(z, \theta, \bar{z}, \bar{\theta}) = (z'(z, \theta), \theta'(z, \theta), \bar{z}'(\bar{z}, \bar{\theta}), \bar{\theta}'(\bar{z}, \bar{\theta})) \quad \text{and} \quad \begin{cases} D = (D\theta')D' \\ \bar{D} = (\bar{D}\bar{\theta}')\bar{D}' \end{cases} \tag{41}$$

Recall from §6.1 that in CFT the energy-momentum tensor is the 2-dimensional 2-tensor whose components $T_{\alpha\beta}$ are such that the Noether current j^α corresponding to the translational infinitesimal symmetry $x \mapsto x + \epsilon(x)$ can be written $j_\alpha = T_{\alpha\beta}\epsilon^\beta$. The Noether current of an infinitesimal conformal transformation $z \mapsto z + \epsilon(z)$ can, surprisingly enough, be written in terms of the energy-momentum tensor as well: $j_z = 0$ and $j_{\bar{z}} = T(z)\epsilon(z)$ (and similarly for the anti-holomorphic current $\bar{j}$) [Ton]. For this reason, the energy-momentum tensor is often said to be the conformal current.

Something similar happens in the superconformal case. Suppose we have a system with 2-dimensional superconformal symmetry. In this case, an infinitesimal superconformal transformation has two infinitesimal parameters ξ, ϵ. If we set ϵ to zero we are left with an infinitesimal conformal transformation, while if we set ξ to zero then we obtain an infinitesimal transformation which we call a supersymmetry transformation [BLT]. Similarly to the conformal case described above, the Noether current corresponding to the conformal symmetry is the energy-momentum tensor $T(z)$, while the Noether current for the supersymmetry is denoted $G(z)$. Naturally, the modes L_m of $T(z)$ are defined as in CFT and again generate a Virasoro algebra. The modes G_r of $G(z)$ come from its expansion and can be written

$$G_r = 2 \oint \frac{\mathrm{d}z}{2\pi i} G(z) z^{r+\frac{1}{2}} \tag{42}$$

All together, these modes, together with the modes of $T(z)$, generate the $N = 1$ *Super Virasoro algebra*, characterized by the commutation relations [BS, BLT]:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{\hat{c}}{8}\left(m^3 - m\right)\delta_{m+n} \\ [L_m, G_r] &= \left(\tfrac{1}{2}m - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \tfrac{c}{12}\left(4r^2 - 1\right)\delta_{r+s} \end{aligned} \tag{43}$$

6.3 $N = 2$ superconformal transformations in two dimensions

The advantage of using the superspace formulation to define $N = 1$ superconformal symmetry is that generalizing the definition of superconformal transformation for $N = 2$ is straightforward. In this case we extend the coordinate space of the worldsheet to a $(2, 4)$ superspace, so that the coordinates are now $(z, \theta, \bar{\theta}, \bar{z}, \xi, \bar{\xi})$. We focus on the holomorphic part $(z, \theta, \bar{\theta})$. There are two super-derivatives D and $\bar{D}$, corresponding to θ and $\bar{\theta}$, respectively. An $N = 2$ superconformal transformation ϕ is a natural generalization of the $N = 1$ superconformal transformation:

$$\phi(z, \theta, \bar{\theta}) = (z'(z, \theta, \bar{\theta}), \theta(z, \theta, \bar{\theta}), \bar{\theta}(z, \theta, \bar{\theta})) \quad \text{and} \quad \begin{cases} D = (D\theta')D' \\ \bar{D} = (\bar{D}\bar{\theta}')\bar{D}' \end{cases} \tag{44}$$

and similarly for the anti-holomorphic coordinates.

It turns out [BLT] that the infinitesimal $N=2$ superconformal transformations (again ignoring the anti-holomorphic coordinates) have four infinitesimal parameters $(\xi, \epsilon, \bar{\epsilon}, \alpha)$, corresponding to one infinitesimal conformal transformation, two infinitesimal supersymmetric transformations and one infinitesimal $U(1)$ transformation, respectively. The Noether currents of the symmetries corresponding to the first three transformations are $T(z), G^{\pm}(z)$ whose modes are:

$$L_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z), \quad G^{\pm}_{n\pm a} = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+\frac{1}{2}\pm a} G^{\pm}(z) \tag{45}$$

so that

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad G(z) = \frac{1}{2} \sum_{r\in\mathbb{Z}+a} z^{-\frac{3}{2}-r} G_r \tag{46}$$

(where a depends on the monodromy of $G^{\pm}$ [GSW1]) and the current $J(z)$ of the $U(1)$ symmetry has the modes:

$$J_n = \oint \frac{\mathrm{d}z}{2\pi i} z^n J(z) \tag{47}$$

Together they form an algebra called the $N=2$ *Super Virasoro algebra*:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n} \\
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n} \\
\left[J_m, G^{\pm}_{n\pm a}\right] &= \pm G^{\pm}_{m+n\pm a} \\
\left[L_m, G^{\pm}_{n\pm a}\right] &= \left(\frac{1}{2}m - n \mp a\right) G^{\pm}_{m+n\pm a} \\
\left\{G^{+}_{m+a}, G^{-}_{n-a}\right\} &= 2L_{m+n} + (m-n+2a)J_{m+n} + \frac{c}{3}\left[(m+a)^2 - \frac{1}{4}\right]\delta_{m+n} \\
\left\{G^{+}_{m+a}, G^{+}_{n+a}\right\} &= \left\{G^{-}_{m-a}, G^{-}_{n-a}\right\} = 0
\end{aligned}$$

The last four equations can be rewritten in another useful way by setting $r, s \in \mathbb{Z} + a$:

$$\begin{aligned}
\left[J_n, G_r^{\pm}\right] &= \pm G_{r+n}^{\pm} \\
\left[L_m, G_r^{\pm}\right] &= \left(\frac{m}{2} - r\right) G_{m+r}^{\pm} \\
\left\{G_r^{+}, G_s^{-}\right\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\
\left\{G_r^{\pm}, G_s^{\pm}\right\} &= 0
\end{aligned} \tag{48}$$

Remark 6.1. Every $N = 2$ superconformal algebra A with generators $\{L_m, G_r^{\pm}, J_n\}$ has an $N = 1$ subalgebra generated by $\{L_m, G_r = \frac{G_r^{+} + G_r^{-}}{\sqrt{2}}\}$ with the same central charge as A[17]. We say that this is the *standard* $N = 1$ subalgebra of A.

7 Superstrings and Kazama Suzuki models

Bosonic string theory is not a realistic theory because of at least two reasons: it contains tachyons (particles with imaginary mass) and it has no fermions. One way to fix these two problems at once is to demand our theory to be supersymmetric by introducing fermionic fields in an appropriate manner. This procedure extends the conformal symmetry of the bosonic string to a superconformal symmetry, and the resulting string theory is called *superstring theory*. In what follows, we will briefly review the basics of superstring theory, with the ultimate goal of understanding why Kazama-Suzuki models are important to analyze. More details can be found for example in [GSW1, GSW2, BBS, Pol, BLT].

There are different (and equivalent) approaches to superstring theory. We will only discuss one of them: the commonly named Ramond-Neveu-Schwarz (RNS) superstring. We start with a summary of the RNS superstring, and briefly discuss how after the so-called GSO projection we are left with theories with no tachyons and $\mathcal{N} = 1$ spacetime supersymmetry (see the Appendix 11.3). These are the Type IIA and Type IIB superstrings, which are the ones which concern us[18]. Finally, we include a short discussion on why these theories must include a $c = 9$ $N = 2$ superconformal field theory (SCFT) on the worldsheet, and how so-called minimal models and Kazama-Suzuki models are candidates for this SCFT.

[17] This simple and yet important result is alluded to in [BS] but I did not find a proof. So I decided to show it in the Appendix 6.3

[18] It turns out that there are five superstring theories: Type I, Type IIA, Type IIB and two heterotic theories. The heterotic superstring theories cannot have D-branes [BP]. Type I superstring theory does have D-branes [BBS], but including a discussion of this theory would extend this text too much.

7.1 The RNS superstring

There are two common ways to integrate supersymmetry in the string: the Ramond-Neveu-Schwarz (RNS) formalism and the Green-Schwarz (GS) formalism, which turn out to be equivalent in ten-dimensional Minkowski spacetime [BBS]. In the RNS superstring there is manifest worldsheet supersymmetry (*i.e.* two-dimensional superconformal symmetry in the sense of §6.2, while in the GS superstring there is manifest spacetime supersymmetry. We will summarize the RNS superstring.

The RNS superstring action

From bosonic string theory we know that the Polyakov action S_p of the bosonic string (in conformal gauge) is the action of a free field theory in two dimensions with D scalar fields $X^\mu(\sigma,\tau)$, where $\mu = 0, 1, \ldots, D-1$ and D is the dimension of spacetime [BLT, GSW1, Ton]. These scalar fields are the components of the embedding $X\colon \Sigma \to M$ of the string worldsheet Σ into the spacetime M.

The most straightforward way to include such fermionic fields in the string action is by keeping it a free theory. Following this rationale, we add the action[19] [BP][GSW1]

$$S_F = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma(-i)\overline{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu \tag{49}$$

of D free fermionic fields $\Psi^\mu(\sigma,\tau)$ in 2 dimensions to the Polyakov action. The ρ^α are 2-dimensional matrices satisfying the 2-dimensional Clifford algebra $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha,\beta}$.

More precisely, the fields $\Psi^m u$ are 2-dimensional Majorana spinors, *i.e.* we can write [Wei]

$$\Psi^\mu = \begin{pmatrix}\psi^\mu_+\\ \psi^\mu_-\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\psi^\mu_+\\ \psi^\mu_-\end{pmatrix}^* = \begin{pmatrix}\psi^\mu_+\\ \psi^\mu_-\end{pmatrix} \tag{50}$$

and one furthermore assumes that $(\Psi^\mu)_{\mu=0,\ldots,D-1}$ transforms in the vector representation of $SO(1, D-1)$.

We end up with the RNS superstring action

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu - i\overline{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu\right) \tag{51}$$

[19] Following the conventions of [GSW1], we use units in which the string tension T is set to $\frac{1}{\pi}$.

Boundary conditions

Just like in bosonic open string theory, also in fermionic open string theory setting $\delta S = 0$ forces some boundary terms to vanish, leading to boundary conditions on the fields (Neumann or Dirichlet boundary conditions in the bosonic case). In the open superstring case, the bosonic fields X^μ still obey the same boundary conditions – thus leaving us with Neumann and Dirichlet boundary conditions, just like in bosonic string theory – while the fermionic fields must satisfy [GSW1]

$$\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \quad \text{and} \quad \psi_+^\mu(\pi,\tau) = \pm\psi_-^\mu(\pi,\tau) \tag{52}$$

The choice $\psi_+^\mu(\pi,\tau) = +\psi_-^\mu(\pi,\tau)$ is called the Ramond (R) boundary condition, while the choice $\psi_+^\mu(\pi,\tau) = +\psi_-^\mu(\pi,\tau)$ is the Neveu-Schwarz (NS) boundary condition. It turns out [Zwi, GSW1] that this leads to the mode expansions

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+a} b_r^\mu e^{-ir(\tau-\sigma)} \quad \text{and} \quad \psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+a} b_r^\mu e^{-ir(\tau+\sigma)} \tag{53}$$

where $a = 0$ in the R case and $a = 1/2$ in the NS case.

For the closed string the vanishing of the boundary terms implies instead

$$\psi_-^\mu(\sigma,\tau) = \pm\psi_-^\mu(\sigma+\pi,\tau) \quad \text{and} \quad \psi_+^\mu(\sigma,\tau) = \pm\psi_+^\mu(\sigma+\pi,\tau) \tag{54}$$

meaning that we must have parity (Ramond (R) boundary condition) or anti-parity (Neveu-Schwarz (NS) boundary condition) in both components of the fermionic fields Ψ^μ. Hence we have 4 different boundary conditions for the closed superstring, denoted R-R, NS-NS, NS-R and R-NS.

$N = 1$ superconformal symmetry in the superstring

The RNS superstring (51) is invariant under the infinitesimal transformations [GSW1]

$$\delta X^\mu = \bar{\epsilon}\Psi^\mu, \quad \delta\Psi^\mu = -\rho^\alpha\partial^\alpha X^\mu\epsilon \tag{55}$$

where ϵ is a constant infinitesimal Majorana 2-dimensional spinor. These are called *infinitesimal supersymmetry transformations*, and indeed the modes $F_n, n \in \mathbb{Z}$ ($G_r, r \in \mathbb{Z} + \frac{1}{2}$) of the R sector (NS sector) of the Noether current J of this infinitesimal symmetry, together with the modes L_m of the energy momentum tensor T form an $N = 1$ superconformal algebra (43) with generators $\{L_m, F_n\}$ ($\{L_m, G_r\}$).

Spectrum

After choosing the boundary conditions, finding the mode expansions of the fermionic fields and quantizing the theory, one can apply the operators coming from the quantization of those modes to the vacuum state to find the spectrum of the theory: the states and their masses. It turns out that the excited states in the NS sector are bosons, while the ones in the R sector are fermions.

Furthermore, the ground state of the open string of the NS sector is a tachyon, which we must get rid of [BBS]. There are also gravitinos [IU], [BBS], which being the gauge particles of supersymmetry demand by consistency that the theory be spacetime supersymmetric.

GSO projection

In order to make the spectrum realistic, we truncate it in a controlled (and consistent with modular invariance at one and two loops) way through the GSO projection, which consists of keeping only the states of the RNS open superstring spectrum containing an odd number of fermions in the NS sector and similarly (with some technical caveats) for states in the R sector, although in the R sector one can choose to truncate the states in two distinct ways [BBS] – the details do not concern us here. The GSO projection thus "projects out" the open superstring tachyon, since it is precisely the ground state of the NS sector, which has no fermionic excitations. After the GSO projection we are left with an equal number of bosons and fermions at each mass level [BBS]. This is good news, since it is a necessary condition for spacetime supersymmetry. To actually prove spacetime supersymmetry one usually uses the GS formalism. We will leave it at that.

For the closed string spectrum, the GSO projection is done for both left and right movers. On R sectors we can again choose if we want to project out the positive or the negative G-parity states. We get different theories depending on these choices: types IIA and IIB, whose NS-NS sectors coincide, while the NS-R, R-NS, R-R sectors differ.

7.2 Why do we care about representations of the $N = 2$ superconformal algebra?

The $N = 2$ superconformal algebra in the RNS superstring

As described in [BS] the RNS string has a hidden $N = 2$ superconformal algebra with generators $\{L_m, G_r^{\pm}, J_n\}$, and its standard $N = 1$ subalgebra (see Remark 6.1) is precisely the $N = 1$ superconformal algebra generated by the modes of the currents of (55). The value of $a \in \mathbb{R}$ in the $N = 2$ superconformal algebra (48) dictates the boundary conditions of the currents, and in particular the case $a = 0$ corresponds to the R sector while the case $a = \frac{1}{2}$ corresponds to the NS sector [BLT].

Denote $\eta := a - \frac{1}{2}$. The $N = 2$ superconformal algebras (48) for different values of $\eta \in \mathbb{R}$ are all isomorphic [Gre]. In particular the Ramond and the Neveu-Schwarz superalgebras are isomorphic. This induces a map (called the *spectral flow*) from the states in a representation of the $\eta = 0$ superalgebra onto the states in a representation of the η superalgebra. This map is parametrized by η.

Since bosons live in the NS sector and fermions live in the R sector, then the spectral flow by $\eta = \frac{1}{2}$ is a candidate for spacetime supersymmetry operator. It turns out [Gre] that this is exactly the case, meaning that *the existence of $N = 2$ worldsheet supersymmetry implies the existence of $\mathcal{N} = 1$ spacetime supersymmetry.* This is the "algebraic counterpart" of the GSO projection [BS].

The usual argument leading to compactification in bosonic string theory

In bosonic string theory, when gauge fixing in the Polyakov path integral the Fadeev-Popov method makes it so that the full action must have a ghost action added the Polyakov action [GSW1, Ton]. The ghost action forms a CFT with central charge -26. Since the Polyakov action forms a CFT with central charge D (with each boson contributing with 1 for the central charge) and the total central charge must me zero to cancel the Weyl anomaly, then we must have $D = 26$. From these dimensions, 22 are not observed, and thus must be "compactified". This is where the compactification formalism comes about.

Generalizing the argument[20]

There is an important caveat here: although one originally obtains the ghost action by applying the Fadeev-Popov method to the Polyakov path integral [GSW1][Ton]

$$Z = \int \mathcal{D}g\mathcal{D}X e^{-S_{\text{Poly}}[X,g]} \tag{56}$$

the exact form of the Polyakov action is not used in determining the ghost action. The only properties of the Polyakov action that one used in the Fadeev-Popov method

[20] The rest of this section (§7.2) expands on the arguments in [Ath, Gep2, Gep1] in favor of the necessity to look for $N = 2$ CFTs. These arguments seemed too poor in details to me, so I try here to provide what I think to be a more complete story.

is the Weyl invariance and the reparametrization invariance, since the Fadeev-Popov method aims precisely at gauge fixing these gauge symmetries[Ton]. Therefore[21], if instead of the Polyakov action we chose to start with some other string action S_{string} with Weyl and diffeomorphism invariance, we would end up with the same ghost action after gauge fixing, and thus with a contribution of -26 to the total central charge[22].

Thus in this more general case one does not necessarily have $D = 26$. In fact, we can have for example a 4-dimensional Polyakov action with central charge $c = 4$ and some other $c = 22$ conformally symmetric action whose fields need not be embeddings as in the Polyakov action. So in this case the bosonic string action can be written

$$S_{bosonic} = S_P^{c=4} + S^{c=22} \tag{57}$$

where $S_P^{c=4}$ is the 4-dimensional version of the Polyakov action.

Back to the superstring

In fermionic string theory, the ghost action is an $N = 2$ superconformal field theory with central charge $c_{gh} = -15$ [BS]. Again, instead of the RNS action we could have started with any $N = 2$ superconformal superstring action S, and the value of c_{gh} would be the same[21] since the Fadeev-Popov determinant only depends on the symmetries it is fixing.

Now we want S to have at least 4 embeddings (corresponding to the four observed spacetime dimensions), so again we live part of the RNS untouched. We thus set

$$S = S_{RNS}^{c=6} + S^{c=9} \tag{58}$$

where $S_{RNS}^{c=6}$ is the 4-dimensional version of the RNS superstring, which has $c = 6$ since each boson-fermion supermultiplet contributes with $3/2$ to the total central charge. I should stress that $N = 1$ and the $N = 2$ superconformal algebras coming from the RNS action have the same central charge (see Remark 6.1). $S^{c=9}$ (the *internal* action) can be any action with $N = 2$, $c = 9$ superconformal symmetry. Therefore, if we want to understand the internal action, we should study $c = 9$, $N = 2$ SCFTs.

[21] This is the justification of why we can start with a different action - and thus different CFTs - and still assume the same central charge, which is missing in the references in the footnote 20.

[22] S_{string} also has conformal symmetry since, just as in the Polyakov action, it is a consequence of Weyl invariance together with diffeomorphism invariance: if the action is invariant under rescalings of the metric (Weyl transformations) and under reparametrizations, then it is also invariant under reparametrizations which rescale the metric (conformal transformations).

Remark 7.1. We demand the entire action to be $N = 2$ supersymmetric in order to ensure $\mathcal{N} = 1$ spacetime supersymmetry[Gep1].

Remark 7.2. The compactification procedures (and the Calabi-Yau compactification in particular) arise from the specific case where one assumes that the internal action is a non-linear sigma model [BLT].

The simplest examples of $N = 2$ SCAs are the so-called $N = 2$ minimal models [BLT][Gep1], which have $0 < c < 3$. Gepner [Gep2] constructed $c = 9$ representations of the $N = 2$ SCA by tensoring $N = 2$ minimal models.

There are also models with $c > 3$, the Kazama Suzuki models being one (family of) example(s). Some Kazama Suzuki models are $c = 9$, $N = 2$ SCAs and are therefore good candidates for internal $N = 2$ SCFT of the superstring.

7.3 Minimal models

Recall from §5 that, for unitary irreducible highest weight Vir-modules, the possible central charges smaller than 1 are in the discrete series (30). Similarly, in $N = 2$ superconformal representation theory we have that the possible central charges for the unitary irreducible highest weight representations with $0 < c < 3$ are in the discrete series [DVPYZ][BP][23]

$$c = \frac{3k}{k+2}, \quad k \geq 1 \tag{59}$$

Using the GKO method with $\mathfrak{g} = \mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2$ and $\mathfrak{h} = \mathfrak{u}(1)_{k+2}$ one gets exactly these central charges (and also the correct conformal dimensions) [DVPYZ][BP], meaning that these coset models provide explicit constructions for all possible unitary representations of the $N = 2$ SCA with $c < 3$. These are the so-called minimal models.

Gepner [Gep2] constructed $c = 9$, $N = 2$ superconformal representations by taking the tensor product of r minimal models with central charge c_i in such a way that [BP]

$$\sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 9 \tag{60}$$

[23] In [DVPYZ], $c = 3c_2$.

7.4 Kazama-Suzuki models

As we discussed above, in Gepner models one superposes many minimal models to form a $c = 9$, $N = 2$ SCA. Besides being aesthetically unappealing, the reducibility of these models causes technical problems in the heterotic string [KS2].

In [KS2] and [KS1], Kazama and Suzuki find other (*i.e.* non-minimal) $N = 2$ SCAs, some of which have $c = 9$ (so that one does not have to superpose models). To do this, they use a variation of the GKO method applied to so-called super-Kac Moody algebras [KT], obtaining $N = 1$ SCAs. Some of these are actually $N = 2$ SCAs. They find the conditions for which one of these $N = 1$ SCA is a $N = 2$ SCA. It turns out that these conditions mean exactly that the coset ${}^{G}/_{H}$ is a Kahler manifold [KS1, Noz].

We will not need to delve into these concepts because there is a way to reduce the construction of Kazama-Suzuki models to the already familiar GKO construction (without the need to deal with super Kac-Moody algebras). Concretely, a "'Kazama-Suzuki coset" $\frac{G}{H}$ can be written as a "GKO coset" $\frac{G\times SO(2d:=\dim\mathfrak{g}-\dim\mathfrak{h})_1}{H}$ [BF, Noz, KS2]. We will adopt this last convention for the remainder of this text.

7.5 Primary states in KS models

As explained in the end of §7.4, a Kazama-Suzuki model can be seen as a GKO coset of the type $\frac{G_k\times SO(2d)}{H}_1$.

From now on we shall focus on a specific Kazama Suzuki model called the (n,k) *Grassmannian model*:

$$\mathbf{Grass}(n,k) := \frac{\widehat{SU}(n+1)_k\times\widehat{SO}(2n)_1}{\widehat{SU}(n)_{k+1}\times\widehat{U}(1)_{n(n+1)(k+n+1)}} \tag{61}$$

Where $n,k\in\mathbb{Z}^+$.

Remark 7.3. The central charge of the Grassmannian model $\mathbf{Grass}(n,k)$ is $c = \frac{3nk}{n+k+1} \geq 1$ [Noz]. It is easy to see that $\frac{3nk}{n+k+1} = 9 \iff n = \frac{3(k+1)}{k-3}$. Hence for example Grassmannian models $\mathbf{Grass}(15,4)$ and $\mathbf{Grass}(9,5)$ have $c = 9$. This means that there are Grassmannian models which are candidates for internal CFT of the superstring.

Recall from Remark 5.14 that to each Verma module corresponds a unique irreducible highest weight Vir-module. Since from Remark 5.9 there is a bijection between Verma modules and primary states, the trivial but crucial conclusion is that *there is a one-to-one relation between primary states and irreducible highest weight Vir-modules.*

Now, from section 5 we can label uniquely an irreducible highest weight Vir-module my a pair (c, h). This is of course still true for the representations in a coset model $\mathfrak{g}_k \,/\, \mathfrak{h}_r$ (with $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ and $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$ semisimple Lie algebras, as usual). But from the coset construction, c and h can be written in terms of the highest weights Λ_i and λ_i of the simple components of $\mathfrak{g}$ and $\mathfrak{h}$. Hence *we can label the primary states by* (Λ_i, λ_i).

In the specific case of the Grassmannian model $\mathbf{Grass}(n, k)$, one denotes by Λ, λ, m and s the highest weights of $\mathfrak{su}(n+1)_k$, $\mathfrak{su}(n)_{k+1}$, $\mathfrak{u}(1)_{n(n+1)(k+n+1)}$ and $\mathfrak{so}(2n)_1$, respectively. For more details on these highest weights and the conditions they must satisfy to be acceptable labels, see for example [Noz], [BF] and references thereof. These details will not be relevant for our discussion.

8 D-branes in a general RCFT

We want to start by generalizing boundary conditions which arise in open bosonic string theory (Neumann and Dirichlet) to general RCFTs (and thus in particular for coset models). These may not have an explicit lagrangian.

The search for a more intrinsic way to describe boundary conditions gives rise to *boundary conformal field theory* and to the *boundary state formalism* [BP]. Boundary states are the generalization of D-branes to a general RCFT [BP], although the former need not have the geometrical interpretation of the latter. We can use the names boundary state and D-brane interchangeably, but we will mostly stick with the first.

In open string theory the worldsheet Σ is not an infinitely long cylinder but an infinitely long strip, hence it is a two dimensional manifold with boundary[24] which we can actually cover with a single chart $\Sigma \to \mathbb{H}^2$ taking one of the edges of the strip to the negative real axis and the other edge to the positive real axis (see Figure 2).

[24]The definition of an n-dimensional manifold with boundary is similar to the definition of an n-dimensional manifold, with the crucial difference that the latter is locally $\mathbb{H}^n$ (where $\mathbb{H}^n$ is the n-dimensional upper half space) as opposed to being locally $\mathbb{R}^n$ [Tu].

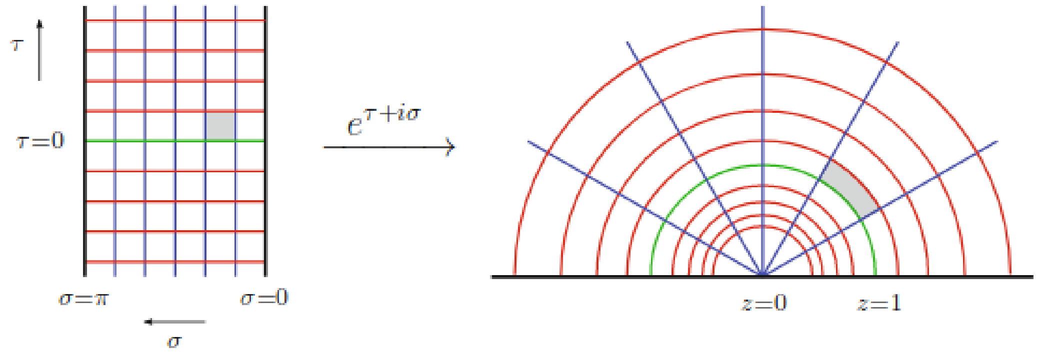


Figure 2: Chart from the infinite strip (the open string worldsheets) to the upper half-plane $\mathbb{H}^2$. Taken from [BP].

8.1 Boundary conditions in terms of the bosonic currents in the free boson CFT

Recall[25] that the free boson CFT has an extended ($i.e$ non-conformal) $U(1)$ symmetry with currents $j(z) = i\partial X(z,\bar{z})$, $\bar{j}(\bar{z}) = i\bar{\partial}X(z,\bar{z})$, called bosonic currents. One can use the map in Figure 2 to determine how the Neumann (N) and Dirichlet (D) conditions look like in terms of the charges $j_n, \bar{j}_n$ of the bosonic currents. It turns out that [BP]:

$$(NN) \quad \partial_\sigma X|_{\sigma=0} = 0, \partial_\sigma X|_{\sigma=\pi} = 0 \iff j_n - \bar{j}_n = 0, n \in \mathbb{Z} \tag{62}$$

$$(ND) \quad \partial_\sigma X|_{\sigma=0} = 0, \partial_\tau X|_{\sigma=\pi} = 0 \iff j_n - \bar{j}_n = 0, n \in \mathbb{Z} + \frac{1}{2} \tag{63}$$

$$(DN) \quad \partial_\tau X|_{\sigma=0} = 0, \partial_\sigma X|_{\sigma=\pi} = 0 \iff j_n + \bar{j}_n = 0, n \in \mathbb{Z} + \frac{1}{2} \tag{64}$$

$$(DD) \quad \partial_\tau X|_{\sigma=0} = 0, \partial_\tau X|_{\sigma=\pi} = 0 \iff j_n + \bar{j}_n = 0, n \in \mathbb{Z} \tag{65}$$

and this implies (writing T and $\bar{T}$ in terms of the j, $\bar{j}$) [BP]:

$$L_n - \bar{L}_n = 0 \tag{66}$$

Notice that the equation (66) implies that the conformal symmetry is broken from two Virasoro algebras to one, and equations (62)-(65) mean that the $U(1)$ current symmetry (the extended symmetry for the free boson) is similarly broken.

[25]This is usually covered in an introductory string theory course. You can also see this derived in [BP].

8.2 The open-closed string duality and Boundary states

The one-loop open string worldsheet can be represented as follows:

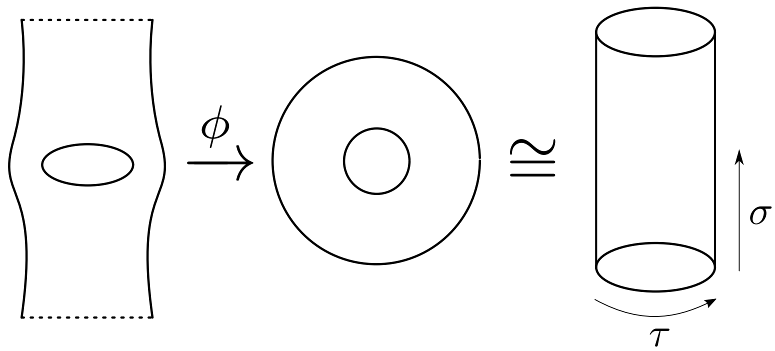


Figure 3: The one-loop open string worldsheet (on the left) can be conformally mapped to a disk, which is diffeomorphic to the cylinder. A detailed discussion of this can be found on §23.2 of [Zwi].

where the dashed lines represent the incoming and the outgoing open strings.

Since string theory is invariant under both conformal and diffeomorphic transformations of the worldsheet, we can simply take the one-loop open string worldsheet to be the cylinder, which is usually parametrized as in Figure 3, so that we can thing of an open string propagating along the cylinder.

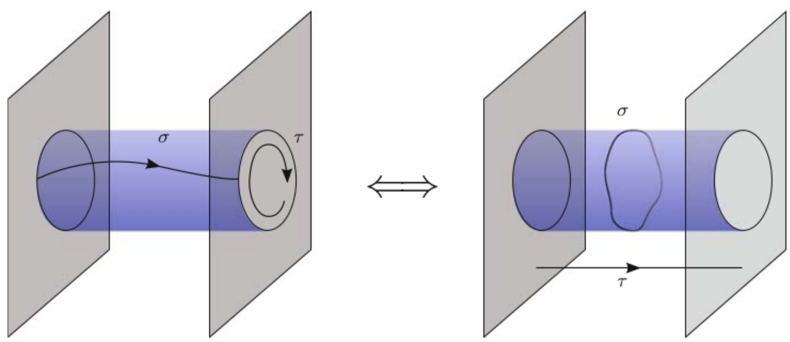


Figure 4: Depiction of the open-closed worldsheet duality. Taken from [BP].

To gain insight into boundary conditions, it is useful to use the *open-closed worldsheet duality* (also called *loop-channel-tree-channel equivalence*), which relates the worldsheet of the one-loop worldsheet for the open string (a disk with a disk cut out, hence a cylinder) with the tree-level worldsheet for the closed string by switching the coordinates σ and τ. See Figure 4.

This allows us to use the *boundary state formalism*, where D-branes are seen as states of the Hilbert space of the closed string, satisfying some conditions called *gluing conditions*, and by inspecting Figure 4 we see that they can be interpreted as geometrical objects which emit and absorb closed strings [BLT](§4.3, §6.5), [BP].

The intuition is that the information about how the boundary of the open string propagates (in the open string picture) is recorded (in the closed string picture) in certain closed string states $|\alpha\rangle$, $|\beta\rangle$ with the string lying on the first and second D-branes, respectively[26]. The open-closed string duality provides the mathematical rigour. See [BP] for more details.

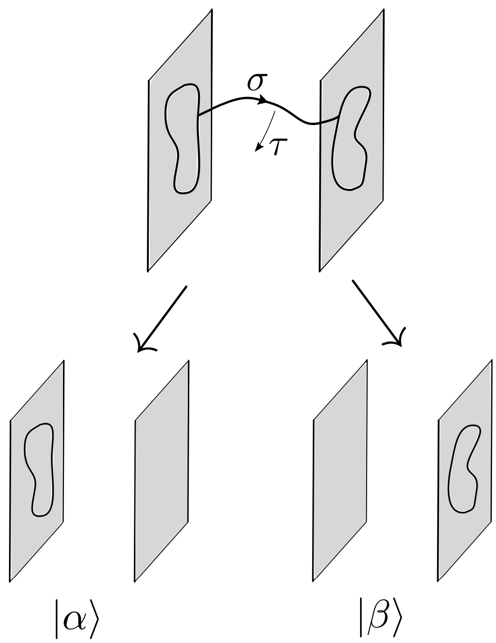


Figure 5: Depiction of the boundary states coming from the open-closed worldsheet duality.

In summary, the boundary states are the subset of states of the closed string respecting the so called *gluing conditions* (which are the result of expressing the open string boundary conditions in the closed string picture using the open-closed string duality) and the *Cardy condition*, which ensures that the open-closed worldsheet

[26] This is just the picture to have in mind.

duality holds. We leave the latter for later, and discuss the boundary conditions now.

8.3 Gluing conditions in the free boson CFT

In the bosonic open string, a Neumann condition at $\sigma = 0$ in the open string translates to

$$\partial_\tau X|_{\tau=0} \, |B_N\rangle = 0 \tag{67}$$

on the closed string, while a Dirichlet condition at $\sigma = 0$ on the open string translates to

$$\partial_\sigma X|_{\tau=0} \, |B_D\rangle = 0 \tag{68}$$

on the closed string.

If we again want to rewrite this in terms of current charges (Laurent modes), we have to expand the currents in these new coordinates (of the closed string). One easily obtains (§6.2.1 of [BP]):

$$(j_n + \bar{j}_{-n}) \, |B_N\rangle = 0, \quad n \in \mathbb{Z} \tag{69}$$

and

$$(j_n - \bar{j}_n) \, |B_D\rangle = 0, \quad n \in \mathbb{Z} \tag{70}$$

for the Neumann and Dirichlet boundary conditions (at $\sigma_{open} = 0$, *i.e.* $\tau_{closed} = 0$), respectively. These conditions describing how the current modes act on the boundary states are the so-called *gluing conditions*.
Furthermore, again using the fact that we can write the energy-momentum tensor in terms of the currents, one can prove that

$$(L_n - \bar{L}_{-n}) \, |B_{D,N}\rangle = 0 \tag{71}$$

So from (69) and (70) we see that the two $U(1)$ symmetries break to a diagonal $U(1)$, and from (71) the two conformal symmetries also break, leaving us with a diagonal $U(1)$ symmetry and a diagonal conformal symmetry.

The next step is to generalize these results.

8.4 Boundary conditions and boundary states in a general RCFT

To generalize what we saw in §8.1 for the conformal symmetry breaking in the bosonic open string to a general BCFT, we impose the boundary condition that there is no energy-momentum flowing through the real axis, *i.e.* $T_{10} = 0$ at the boundary. This implies (see (2.21) of [BP]) that $T(z) = \bar{T}(\bar{z})$ whenever $z = \bar{z}$, and it generalizes equation (66). Hence we demand $L_n = \bar{L}_n$ at $z = \bar{z}$, meaning that at the boundary the conformal symmetry is not two copies of the Virasoro algebra anymore, but a single Virasoro algebra (the diagonal subalgebra generated by $\{L_n \oplus L_n\}$). In the case of the bosonic open string, demanding $T(z) = \bar{T}(\bar{z})$ when $z = \bar{z}$ is equivalent to demanding that the bosonic fields obey Neumann or Dirichlet boundary conditions - see §6.1 of [BP].

The symmetry breaking that we saw in §8.3 for the boundary states of the bosonic CFT is generalized to a general RCFT case by using the gluing conditions [Zub, BP]:

$$\left(L_n - \overline{L}_{-n}\right)|B\rangle = 0 \tag{72}$$

$$\left(W_n^i - (-1)^{h_i}\Omega(\overline{W}^i_{-n})\right)|B\rangle = 0 \tag{73}$$

where the W_n^i, $\bar{W}_n^i$ are the Laurent modes of the *extended symmetry* (the non-conformal part of the total symmetry algebra $\mathcal{A} \oplus \bar{\mathcal{A}}$), with conformal weight h_i. The map Ω is an automorphism of $\mathcal{A}$, called a *gluing automorphism.*

Remark 8.1. In the bosonic case $i = 1$ and $W_n = j_n$. Also, $\Omega = id_{\mathcal{A}}$ for Neumann boundary conditions, while for Dirichlet boundary conditions $\Omega(j_n) = -j_n$ (compare (73) with (69) and (70)).

8.5 Gluing conditions for the $N = 2$ superconformal algebra

As we saw in 6.3, in the case of $N = 2$ superconformal algebra the chiral symmetry algebra is generated by $\{L_n, J_n, G_r^+, G_r^-\}$, so that the extended symmetry is generated by $\{J_n, G_r^+, G_r^-\}$. Using the identity as the gluing automorphism and the fact that the supergenerators are primary fields with conformal dimension $\frac{3}{2}$ [BP], we get the boundary conditions [BLT]:

$$\begin{aligned}
\left(L_n - \overline{L}_{-n}\right)|B\rangle &= 0\\
\left(J_n + \overline{J}_{-n}\right)|B\rangle &= 0\\
\left(G_r^+ - i\eta\overline{G}^+_{-r}\right)|B\rangle &= 0\\
\left(G_r^- - i\eta\overline{G}^-_{-r}\right)|B\rangle &= 0
\end{aligned} \tag{74}$$

where $\eta = +1$ or $\eta = -1$. These are called B-type boundary conditions. Setting instead Ω to be the usual outer automorphism of the $N = 2$ super Virasoro algebra ((12.108) of [BLT]), we get:

$$
\begin{aligned}
\left(L_n - \overline{L}_{-n}\right)|B\rangle &= 0 \\
\left(J_n - \overline{J}_{-n}\right)|B\rangle &= 0 \\
\left(G_r^+ - i\eta\overline{G}_{-r}^-\right)|B\rangle &= 0 \\
\left(G_r^- - i\eta\overline{G}_{-r}^+\right)|B\rangle &= 0
\end{aligned}
\tag{75}
$$

These are the A-type boundary conditions. We will only consider the B-type boundary conditions from now on[27].

8.6 Ishibashi states and the Cardy condition

The next natural question is: what states satisfy equations (74)? This has been answered (for the particular case of RCFTs[28]) by Ishibashi [Ish1] for the B-type boundary conditions: the solution are the *Ishibashi states* $|\mathcal{B}_i\rangle\rangle$ (and their linear combinations), where $i \in I$ labels the highest weight representations of the symmetry algebra $\mathcal{A}$, which, since we are restricting ourselves to RCFTs, has a finite number of highest weight representations ($|I| < \infty$) [BP]. So there are finitely many linearly independent Ishibashi states (solutions of the gluing conditions). A proof that they indeed satisfy the gluing conditions can be found in [BLT].

The form of the Ishibashi states $|\mathcal{B}_i\rangle\rangle$ depends on the automorphism Ω. So, for example, in the bosonic string we have different Ishibashi states for the Dirichlet and for the Neumann conditions. This is well illustrated in §6.2.1 of [BLT].

The boundary states must be linear combinations of Ishibashi states. But not all linear combinations of Ishibashi states are boundary states. As was said above, boundary states not only have to satisfy the gluing conditions, but also have to ensure the *loop-channel-tree-channel equivalence*. This leads to a restriction on the coefficients of the linear combination of Ishibashi states known as the *Cardy condition*. The details of this condition are of no consequence for our argument, so the curious

[27] The reason why I decided to leave out the A-type boundary conditions is because their treatment is more involved than the one for B-type boundary conditions. More concretely, although the Ishibashi states of §8.6 can be used in the A-type case (see [Noz] and references thereof), it is not clear to me how. In contrast, the original construction of the Ishibashi states [Ish1] is clearly aimed at the B-type boundary conditions.

[28] This does not work for every RCFT [Gab, BP], but only for a special class of CFTs of which the KazamaSuzuki models are a part of (see [BF, LW] and references thereof). The details of this discussion would lead us far astray, and are inconsequential to our analysis.

reader is advised to consult [Car] and [Zub] for more. For us, it is the following remark that matters:

Remark 8.2. Consider a Grassmannian model $\mathbf{Grass}(n, k)$. Being a coset model, it is in particular a RCFT. Its Ishibashi states can be labelled as the primary states [BP, Ish1, Ish2, FS1], so that we write them as $|\Lambda, \lambda, m, s\rangle\rangle$. Also, these labels obey the same selection rules and identification as the primary state labels. It turns out [BP, BLT] that the boundary states are also labelled like the primary states. Hence *there is a one-to-one correspondence between primary states and boundary states.*

9 Categorical structures in boundary states

Recall that our goal is to understand the categorical structure of the boundary states of the Grassmannian Kazama-Suzuki model (61). So the question is: what category can we construct whose objects are precisely the boundary states of the Grassmannian model? And what properties does this category have?

9.1 The category BSG

We saw in §8.6 that there is a bijection between the set of primary states of the Grassmannian model and the set of its boundary states. We also saw that the primary states of the Grassmannian model are in a one-to-one correspondence with the irreducible highest weight (HW) Vir-modules *which arise from the Grassmannian model*. Schematically, then:

$$\begin{aligned}\{\text{Boundary states of } \mathbf{Grass}(n,k)\} &\cong \{\text{Primary states of } \mathbf{Grass}(n,k)\} \\ &\cong \{\text{Irreducible HW } Vir\text{-modules from } \mathbf{Grass}(n,k)\} \\ &\subset \{\text{Irreducible HW } Vir\text{-modules}\} \\ &\subset \{Vir\text{-modules}\} = \mathbf{Obj}(\mathbf{Rep}(Vir))\end{aligned} \tag{76}$$

This means that we can identify the boundary states with a subset of the irreducible HW representations of the Virasoro algebra. We know what subset this is: from the discussion in §5 and Remark 7.3 we see that the irreducible HW representations of the Virasoro algebra from $\mathbf{Grass}(n, k)$ are precisely the $V(c, h)$ with $c = \frac{3nk}{k+n+1}$ and h given by (37) adapted to the Grassmannian case (more on this later).

In this view, the natural arrows of the category we want to construct are clear: they must be Vir-module homomorphisms (also called intertwinners). Thus we will take our category to be the full subcategory of $\mathbf{Rep}(Vir)$ whose objects are precisely the irreducible HW Vir-modules coming from the Grassmannian model, so that the arrows will be inherited from the category $\mathbf{Rep}(Vir)$.

Hence it will help to answer the following question: *what intertwinners are there between HW irreducible Vir-modules?* The arrows of our category will be of this form. If the HW irreducible Vir-modules were all finite-dimensional, then we could use Schur's Lemma as usually stated (see below) to deduce which intertwinners are there between them. However, we know these modules are infinite-dimensional (see Remark 5.8), so this is not an option.

But we can try to show a result similar to Schur's lemma for the case of the Virasoro algebra. The "usual" Schur's lemma is (adapted from [Hal1]):

Theorem 9.1. (Schur's lemma) *Let V and W be finite-dimensional, irreducible complex $\mathfrak{g}$-modules, with $\mathfrak{g}$ a Lie algebra, and $\phi\colon V \to W$ a $\mathfrak{g}-$module homomorphism. Then:*

(i) *If V and W are not isomorphic as $\mathfrak{g}$-modules, then ϕ is the zero map.*

(ii) *If V and W are isomorphic as $\mathfrak{g}$-modules, then $\phi = \alpha I$ where α is a complex number and I is the identity map.*

The proof of the first statement does not use the fact that the modules are finite dimensional or that the field is $\mathbb{C}$ (see the first part of the proof of the Schur's lemma on page 95 of [Hal1]), so that it is also valid in our case. On the other hand, the second statement hinges on these assumptions (again, see [Hal1]). We can write a slightly more general version of the Schur's Lemma (by altering the second statement in Theorem 9.1 using the Schur's lemma on page 33 of [MPP]) which will be useful for us:

Theorem 9.2. (Generalized Schur's lemma) *Let V and W be irreducible $\mathfrak{g}$-modules over a field $\mathbb{K}$, with $\mathfrak{g}$ a Lie algebra, and $\phi\colon V \to W$ a $\mathfrak{g}-$module homomorphism. Then:*

(i) *If V and W are not isomorphic as $\mathfrak{g}$-modules, then ϕ is the zero map.*

(ii) *Suppose V and W are isomorphic as $\mathfrak{g}$-modules and let σ be a vector space endomorphism on V. If σ commutes with $\pi_V(\mathfrak{g})$ and it has an eigenvector, then $\sigma = \alpha I$ for some $\alpha \in \mathbb{C}$.*

Since ϕ commutes with π_V by definition of $\mathfrak{g}$-module, this implies:

Corollary 9.3. *In the conditions of Theorem 9.2, If ϕ has an eigenvector in V, then ϕ is a complex multiple of the identity map.*

Remark 9.4. Theorem 9.1 follows from Theorem 9.2 because in the finite dimensional case and with $\mathbb{K}$ algebraically closed ϕ always has an eigenvector in V [MPP].

Notice that, if we manage to show that the intertwinners between Virasoro highest weight representations have eigenvectors, then we can use the above results to write a version of the Schur's lemma for the Virasoro algebra.

Lemma 9.5. *Let V be a highest weight Vir-module with highest weight vector v and highest weight $\Lambda = (h, c)$, and $\phi\colon V \to V$ be a Vir-module endomorphism. Then v is an eigenvector of ϕ.*

Proof. Recall from Definition 5.3 that $U(n_-)(v) = V$ and $(L_0 \oplus C)(v) = (hv) \oplus (cv) = \Lambda v$. Furthermore, it is clear that all the elements in $U(n_-)(v)$ non-collinear with v have weights smaller than h (and so in particular different from h), since the elements of n_- are precisely the lowering operators. Since Vir-module homomorphisms preserve weight spaces (Lemma 5.12), then $\phi(v) \in V_\Lambda$, so that $\phi(v) = \alpha v$ for some $\alpha \in \mathbb{C}$. □

We arrive at a "Schur's lemma for Vir":

Theorem 9.6. (Schur's Lemma for the Virasoro algebra) Let V and W be two irreducible Vir-modules and $\phi\colon V \to W$ a Vir-module homomorphism. Then:

(i) If V and W are not isomorphic as Vir-modules, then ϕ is the zero map.

(ii) If V and W are isomorphic as Vir-modules then $\phi = \alpha I$ for some $\alpha \in \mathbb{C}$.

Proof. Immediate using Lemma 9.5, Theorem 9.2 and Corollary 9.3. □

In other words: $\mathrm{Hom}(V, W) = \{0_{VW}\}$ for $V \ncong W$, and $End(V) = \mathbb{C}\, I$. Since from Remark 5.14 for each pair (c, h) there is a unique irreducible highest weight representation $V(c, h)$ of Vir (and every irreducible highest weight representation is of this form), then we can conclude that $V \cong W$ iff $V = W$. This gives us a complete characterization of the arrows of our category.

Hence we can finally define the category explicitly:

Definition 9.7. The *Category of boundary states of the Grassmannian model* $\mathbf{Grass}(n,k)$ (denoted $\mathbf{BSG}(n,k)$ or simply $\mathbf{BSG}$) is the category whose objects are the irreducible highest weight Vir-modules $V_{(c,h)}$ with $c = \frac{3nk}{n+k+1}$ and h of the form (37), and whose hom-sets are:

$$\mathrm{Hom}(V_{(c,h)}, V_{(c,h')}) = \begin{cases} \{0_{VW}\}, & (c,h) \neq (c,h') \\ \mathrm{End}(V(c,h)) = \mathbb{C}\, I, & (c,h) = (c,h') \end{cases} \tag{77}$$

We will frequently write (c,h) instead of $V_{(c,h)}$ from now on. We will also often omit the subscript of the zero maps.

We cannot draw the entire category in general since there is one object for each h and different Grassmannian models will have different numbers of possible values for h, but let's draw the arrows between 3 selected objects, to visualize the category.

$$\begin{tikzcd}
(c,h) \arrow[loop above, "\lambda I"] \arrow[loop left, "\lambda' I"] \arrow[rr, bend left, "0"] \arrow[dr, "0"] & & (c,h') \arrow[ll, "0"] \arrow[dl, bend left, "0"] \arrow[loop above, "\lambda I"] \arrow[loop right, "\lambda' I"] \\
& (c,h'') \arrow[ul, bend left, "0"] \arrow[ur, "0"] \arrow[loop below, "\lambda I"] \arrow[loop, out=290, in=250, distance=4em, "\lambda' I"] &
\end{tikzcd}$$

Remark 9.8. Coset models have a finite number of primary fields (page 806 of [FMS]), and thus also a finite number of primary states. Since a Grassmannian model $\mathbf{Grass}(n,k)$ is a coset model, then it has a finite number of primary states, and therefore the category $\mathbf{BSG}(n,k)$ has a finite number of objects.

9.2 BSG is a monoidal category

9.2.1 The altered Cartan product of objects

An apparently obvious choice for a tensor product in **BSG** is the usual tensor product $\otimes$ of Lie algebra modules (see Appendix 11.2). But this immediately presents a difficulty: we know that the tensor product of irreducible modules is in general not irreducible [FH], and thus in particular Obj(**BSG**) is not closed under $\otimes$. There is another product which avoids this problem and shares many of the properties of $\otimes$ called the *Cartan product*[E+].

Definition 9.9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Furthermore, let V_h and V_r be irreducible highest weight $\mathfrak{g}$-modules. Their *Cartan product* is $V_h \circledcirc V_r = V_{h+r}$, where V_{h+r} is the unique highest weight $\mathfrak{g}$-module with highest weight $h + r$ in the weight space decomposition of the tensor product $V_h \otimes V_r$.

We see that the Cartan product was originally defined with finite-dimensional representations in mind. Unfortunately, although its definition can be immediately extended to infinite-dimensional Lie algebras, the resulting product does not preserve irreducibility anymore. To see why, note that the definition of the Cartan product uses the following facts about representations of finite-dimensional Lie algebras [FS2]:

(i) Given two irreducible highest weight representations V_h and V_r of a finite-dimensional Lie algebra, the weights of $V_h \otimes V_r$ are of the form $h' + r'$ where h' is a weight of V_h and r' is a weight of V_r.

(ii) In the conditions of (i), the multiplicity of $r' + h'$ is $\mathrm{mult}_{V_h}(h')\mathrm{mult}_{V_r}(r')$. In particular, $\mathrm{mult}_{V_h \otimes V_r}(h + r) = 1$.

And irreducibility is preserved by the Cartan product due to [FS2]:

(iii) In the conditions of (i), the highest weight representations $V_{h'+r'}$ are all irreducible. In particular, V_{h+r} is irreducible.

Remark 9.10. We see that (iii) implies that $V_h \circledcirc V_r$ is irreducible, so that indeed the Cartan product takes irreducible highest weight representations into irreducible highest weight representations (in the finite-dimensional case).

Unfortunately, while (i) and (ii) are still true for (possibly infinite-dimensional) Lie algebras which, like Vir (see §5), have a triangular decomposition [FS2], (iii) is not guaranteed to hold. So in order to define a Cartan product of irreducible highest weight Virasoro modules which preserves irreducibility, we instead use the fact that there is a unique irreducible highest weight Virasoro module $V(c,h)$ for each pair $(c,h) \in \mathbb{C}^2$.

This allows us to define a product of irreducible highest weight Vir-modules analogous to the Cartan product.

Definition 9.11. Let $V(c,h)$ and $V(c,r)$ be irreducible highest weight Vir-modules. Their *pseudo-Cartan product* is $V(c,h) \circledcirc V(c,r) = V(c,h+r)$.

Remark 9.12. Note that we do not add the central charges because our goal is to come up with a product that is to be used in a specific Grassmannian model, in which all representations have the same central charge.

Notation 9.13. We will often write h or V_h instead of $V(c,h)$ when it is clear from the context.

By construction, the pseudo-Cartan product maps each pair of irreducible highest weight Vir-modules to an irreducible highest weight Vir-module. But this is not enough. We must check if it maps each pair of irreducible highest weight Grassmannian Vir-modules to an irreducible highest weight Grassmannian Vir-module. If it does, then Obj(**BSG**) is closed under the pseudo-Cartan product.

Recall from (37) that the highest weight of a coset model $\mathfrak{g}\,/\,\mathfrak{h}$ is

$$h_{\mathfrak{g}\,/\,\mathfrak{h}} = h_{\mathfrak{g}} - h_{\mathfrak{h}} \mod 1 \tag{78}$$

From now on we omit the “mod 1”.

In the case of a Grassmannian model $\mathbf{Grass}(n,k)$, we have $\mathfrak{g} = \mathfrak{su}(n+1) \oplus \mathfrak{so}(2n)$, $\mathfrak{h} = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. From Proposition 5.17 we know that $h_{\mathfrak{g}} = h^{\Lambda,k}_{\mathfrak{su}(n+1)} + h^{s,1}_{\mathfrak{so}(2n)}$ and $h_{\mathfrak{h}} = h^{\lambda,k+1}_{\mathfrak{su}(n)} + h^{m,n(n+1)(k+n+1)}_{\mathfrak{u}(1)}$.

Also recall from (35) that

$$h^{\Lambda,k}_{\mathfrak{g}} = \frac{\Lambda^2 + 2\Lambda \cdot \rho_{\mathfrak{g}}}{2(k+g)} \tag{79}$$

for $\mathfrak{g}$ semisimple, with Λ a highest weight of the $\mathfrak{g}$-module and k the chosen level.

Definition 9.14. For a chosen Grassmannian model $\mathbf{Grass}(n,k)$, an *admissible conformal dimension* is a conformal dimension h such that $V(c = \frac{3nk}{n+k+1}, h)$ is an irreducible highest weight Vir-module $\mathbf{Grass}(n,k)$.

We see that a conformal dimension of the coset model $\mathbf{Grass}(n,k)$ (and thus any admissible conformal dimension) must be of the form:

$$\begin{aligned} h &= h^{\Lambda,k}_{SU(n+1)} + h^{s,1}_{SO(2n)} - h^{\lambda,k+1}_{SU(n)} - h^{m,n(n+1)(k+n+1)}_{U(1)} \\ &= \frac{\Lambda^2 + 2\Lambda\cdot\rho_{SU(n+1)}}{2(k+c_{\mathfrak{su}(n+1)})} + \frac{s^2+2s\cdot\rho_{SO(2n)}}{2(1+c_{\mathfrak{so}(2n)})} \\ &\quad - \frac{\lambda^2+2\lambda\cdot\rho_{SU(n)}}{2(k+1+c_{\mathfrak{su}(n)})} - \frac{m^2+2m\cdot\rho_{U(1)}}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]} \end{aligned} \tag{80}$$

Now notice that $h = h(\Lambda,\lambda,s,m)$. Hence the sum of two weights can be written

$$\begin{aligned} h(\Lambda,\lambda,s,m) + h(\Lambda',\lambda',s',m') &= \frac{(\Lambda+\Lambda')^2 + 2(\Lambda+\Lambda')\cdot\rho_{SU(n+1)}}{2(k+c_{\mathfrak{su}(n+1)})} \\ &+ \frac{(s+s')^2+2(s+s')\cdot\rho_{SO(2n)}}{2(1+c_{\mathfrak{so}(2n)})} - \frac{(\lambda+\lambda')^2+2(\lambda+\lambda')\cdot\rho_{SU(n)}}{2(k+1+c_{\mathfrak{su}(n)})} \\ &- \frac{(m+m')^2+2(m+m')\cdot\rho_{U(1)}}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]} + e(\Lambda,\lambda,m,s,\Lambda',\lambda',m',s') \\ &= h(\Lambda+\Lambda',\lambda+\lambda',s+s',m+m') + e(\Lambda,\lambda,s,m,\Lambda',\lambda',s',m') \end{aligned} \tag{81}$$

where the *extra term* $e(\Lambda,\lambda,s,m,\Lambda',\lambda',s',m')$ is given by

$$\begin{aligned} e(\Lambda,\lambda,s,m,\Lambda',\lambda',s',m') &:= -\frac{\Lambda\cdot\Lambda'}{k+c_{\mathfrak{su}(n+1)}} - \frac{s\cdot s'}{1+c_{\mathfrak{so}(2n)}} \\ &+ \frac{\lambda\cdot\lambda'}{2(k+1+c_{\mathfrak{su}(n)})} + \frac{m\cdot m'}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]} \end{aligned} \tag{82}$$

Therefore $h(\Lambda,\lambda,s,m) + h(\Lambda',\lambda',s',m')$ is not an admissible conformal weight, because of the extra term. If we subtract the extra term away, however, we simply get the admissible conformal weight $h(\Lambda+\Lambda',\lambda+\lambda',s+s',m+m')$.

Notation 9.15. From now on, we often write $\bar\Lambda$ and $\bar\Lambda'$ for (Λ,λ,s,m) and $(\Lambda',\lambda',s',m')$, respectively. Furthermore, we often denote the extra term by[29] $e(h,r)$ instead of $e(\bar\Lambda,\bar\Lambda')$, whenever $h := h(\bar\Lambda)$ and $r := h(\bar\Lambda')$.

This discussion suggests the following definition for an altered Cartan product:

[29] In general there can be $\bar\Lambda \neq \bar{\bar\Lambda}$ such that $h(\bar\Lambda) = h(\bar{\bar\Lambda})$, so when writing $e(h,r)$ we actually make a choice. But the argument that follows is independent of this choice.

Definition 9.16. Let V_h and V_r be irreducible highest weight Vir-modules. Their *altered Cartan product* is $V_h \boxtimes V_r = V_{h+r-e(h,r)}$.

Remark 9.17. We know that Obj(**BSG**) is closed under the altered Cartan product because, from the discussion above, if h and r are admissible conformal dimensions, so is $h + r - e(h, r)$.

9.2.2 The altered Cartan product of arrows

If we want to use the altered Cartan product to give a monoidal structure to **BSG**, we must extend it to a bifunctor. In particular, the altered Cartan product must be such that for any two arrows $h \xrightarrow{f} r$, $h' \xrightarrow{f'} r'$ we have:

$$\begin{aligned}(h \xrightarrow{f} r) \boxtimes (h' \xrightarrow{f'} r') &= h \boxtimes h' \xrightarrow{f \boxtimes f'} r \boxtimes r' \\ &= h + h' - e(h, h') \xrightarrow{f \boxtimes f'} r + r' - e(r, r')\end{aligned} \tag{83}$$

There are so little arrows in **BSG** that this restriction almost completely determines the altered Cartan product of arrows. Indeed, it is clear from (83) and 9.7 that:

$$f \boxtimes f' = \begin{cases} 0, & h + h' - e(h, h') \neq r + r' - e(r, r') \\ \alpha I, & h + h' - e(h, h') = r + r' - e(r, r') \end{cases} \tag{84}$$

for some $\alpha \in \mathbb{C}$.

We make the simplest choice: $\alpha = 1$:

Definition 9.18. Let $h \xrightarrow{f} r$ and $h' \xrightarrow{f'} r'$ be two arrows of **BSG**. Their *altered Cartan product* is given by (84) with $\alpha = 1$.

9.2.3 The identity object

We just saw that we have a good candidate for tensor product of objects in a putative monoidal structure on **BSG**. Such a structure needs another ingredient: the identity object. There is a natural candidate.

Lemma 9.19. *$V(c = \frac{3nk}{n+k+1}, h = 0) =: V_0$ is an object of* **BSG**. *Furthermore, if $h \in \mathrm{Obj}(\mathbf{BSG})$, then $e(h, 0) = e(0, h) = 0$.*

Proof. From (80) it is clear that $h(0, 0, 0, 0) = 0$. This shows that $h = 0$ is an admissible weight, or in other words that V_0 is in Obj(**BSG**). We now use (82) to see that $e(h, 0) = e(\bar{\Lambda}, 0, 0, 0, 0) = 0$ and $e(0, h) = e(0, 0, 0, 0, \bar{\Lambda})$ hold trivially. $\square$

Looking at the Definition 9.16 of the altered Cartan product, it is now clear that the unit object should be V_0. Namely, since the Cartan product of V_h and V_r is simply $V_{h+r-e(h,r)}$, $e(0,h) = e(h,0) = 0$ and furthermore $V_h \cong V_{h'}$ iff $h = h'$, then the unit object can be taken to be V_0.

9.2.4 Properties of the altered Cartan product

Before proving that $(\mathbf{BSG}, \boxtimes, V_0)$ is a monoidal category, we must show that the altered Cartan product is a bifunctor, and also prove some properties of the altered Cartan product which will make our life much easier.

Lemma 9.20. *The altered Cartan product is a bifunctor.*

Proof. Let $f \times f' \colon (h, h') \to (r, r')$ and $g \times g' \colon (r, r') \to (s, s')$ be two arrows of the product category $\mathbf{BSG} \times \mathbf{BSG}$. We want to show that F1), F2) and F3) from 2.5 are satisfied.

F1) $\boxtimes(f \times f') =: f \boxtimes f' \colon h \boxtimes h' \to r \boxtimes r'$ by construction of $\boxtimes$.

F2) $\boxtimes\left((g \times g') \circ (f \times f')\right) = \boxtimes\left((g \circ f) \times (g' \circ f')\right) =: (g \circ f) \boxtimes (g' \circ f')$ and from the Definition 9.18 we have

$$(g \circ f) \boxtimes (g' \circ f') = \begin{cases} 0, & h + h' - e(h,h') \neq s + s' - e(s,s') \\ I, & h + h' - e(h,h') = s + s' - e(s,s') \end{cases} \tag{85}$$

On the other hand:

$$\begin{aligned} \boxtimes(g \times g') \circ \boxtimes (f \times f') &=: (g \boxtimes g') \circ (f \boxtimes f') \\ &= \begin{cases} 0 \circ 0, & r + r' - e(r,r') \neq s + s' - e(s,s') \\ & \quad \text{and } h + h' - e(h,h') \neq r + r' - e(r,r') \\ 0 \circ I, & r + r' - e(r,r') \neq s + s' - e(s,s') \\ & \quad \text{and } h + h' - e(h,h') = r + r' - e(r,r') \\ I \circ 0, & r + r' - e(r,r') = s + s' - e(s,s') \\ & \quad \text{and } h + h' - e(h,h') \neq r + r' - e(r,r') \\ I \circ 0, & r + r' - e(r,r') = s + s' - e(s,s') \\ & \quad \text{and } h + h' - e(h,h') = r + r' - e(r,r') \end{cases} \\ &= \begin{cases} 0, & h + h' - e(h,h') \neq s + s' - e(s,s') \\ I, & h + h' - e(h,h') = s + s' - e(s,s') \end{cases} \end{aligned} \tag{86}$$

Hence $\boxtimes\left((g \times g') \circ (f \times f')\right) = \boxtimes(g \times g') \circ \boxtimes(f \times f')$

F3) Using Remark 2.11 we see that $\boxtimes(id_{(h,h')}) = \boxtimes(id_h \times id_{h'}) =: id_h \boxtimes id_{h'}$. From Definition 9.18 we have $id_h \boxtimes id_{h'} = I := idh + h' + e(h,h') = id_{h \boxtimes h'}$. $\square$

Lemma 9.21. *The altered Cartan product is associative on objects.*

Proof. Let $h, h', h'' \in \mathrm{Obj}(\mathbf{BSG})$. We want to show that $(h \boxtimes h') \boxtimes h'' = h \boxtimes (h' \boxtimes h'')$. First, notice that

$$\begin{aligned}(h \boxtimes h') \boxtimes h'' &= (h + h' - e(h,h')) \boxtimes h'' \\ &= h + h' + h'' - e(h,h') - e(h + h' - e(h,h'), h'')\end{aligned} \tag{87}$$

On the other hand:

$$\begin{aligned}h \boxtimes (h' \boxtimes h'') &= h \boxtimes (h' + h'' - e(h',h'')) \\ &= h + h' + h'' - e(h',h'') - e(h, h' + h'' - e(h',h''))\end{aligned} \tag{88}$$

But

$$e(h,h') + e(h + h' - e(h,h'), h'') = e(h',h'') + e(h, h' + h'' - e(h',h'')) \tag{89}$$

Indeed: (denoting $r := h(\bar{\Lambda} + \bar{\Lambda}'))$

$$\begin{aligned}e(h,h') + e(h + h' - e(h,h'), h'') &= e(h,h') + e(r,h'') \\ &= \frac{\Lambda \cdot \Lambda'}{k + c_{\mathfrak{su}(n+1)}} - \frac{s \cdot s'}{1 + c_{\mathfrak{so}(2n)}} + \frac{\lambda \cdot \lambda'}{2(k + 1 + c_{\mathfrak{su}(n)})} \\ &+ \frac{m \cdot m'}{2[n(n+1)(k+n+1) + c_{\mathfrak{u}(1)}]} \\ &+ \frac{(\Lambda + \Lambda') \cdot \Lambda''}{k + c_{\mathfrak{su}(n+1)}} - \frac{(s + s') \cdot s''}{1 + c_{\mathfrak{so}(2n)}} + \frac{(\lambda + \lambda') \cdot \lambda''}{2(k + 1 + c_{\mathfrak{su}(n)})} \\ &+ \frac{(m + m') \cdot m''}{2[n(n+1)(k+n+1) + c_{\mathfrak{u}(1)}]}\end{aligned} \tag{90}$$

where in the first equality we used that $h + h' - e(h,h') = r$ by construction of $\boxtimes$.

Also: (denoting $r' := h(\bar{\Lambda}' + \bar{\Lambda}''))$

$$
\begin{aligned}
e(h',h'')+e(h,h'+h''-e(h,h'')) &= e(h,h'')+e(h,r') \\
&= \frac{\Lambda\cdot\Lambda''}{k+c_{\mathfrak{su}(n+1)}} - \frac{s\cdot s''}{1+c_{\mathfrak{so}(2n)}} + \frac{\lambda\cdot\lambda''}{2(k+1+c_{\mathfrak{su}(n)})} \\
&+ \frac{m\cdot m''}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]} \\
&+ \frac{\Lambda\cdot(\Lambda'+\Lambda'')}{k+c_{\mathfrak{su}(n+1)}} - \frac{s\cdot(s'+s'')}{1+c_{\mathfrak{so}(2n)}} + \frac{\lambda\cdot(\lambda'+\lambda'')}{2(k+1+c_{\mathfrak{su}(n)})} \\
&+ \frac{m\cdot(m'+m'')}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]}
\end{aligned}
\tag{91}
$$

By inspection, (90) and (9.2.4) coincide. This concludes the proof. $\square$

Lemma 9.22. *$h \boxtimes 0 = h = 0 \boxtimes h$.*

Proof. By definition, $h\boxtimes 0 = h+0-e(h,0)$. But $e(h,0) = e(h(\bar{\Lambda}), h(0,0,0,0))$ from the proof of Lemma 9.19. Hence

$$
\begin{aligned}
e(h,0) = &-\frac{\Lambda\cdot 0}{k+c_{\mathfrak{su}(n+1)}} - \frac{s\cdot 0}{1+c_{\mathfrak{so}(2n)}} \\
&+ \frac{\lambda\cdot 0}{2(k+1+c_{\mathfrak{su}(n)})} + \frac{m\cdot 0}{2[n(n+1)(k+n+1)+c_{\mathfrak{u}(1)}]} = 0
\end{aligned}
\tag{92}
$$

Thus $h\boxtimes 0 = h$. The argument for $0\boxtimes h = h$ is identical. $\square$

Because of these properties (which tell us that the altered Cartan product satisfies the characteristics typical of a monoid "on the nose" (and not just up to isomorphism) the associator, the left unitor and the right unitor are trivial, so that we are now in a position to easily prove the most important result of this section.

Before doing that, we will show that the altered Cartan product is also associative on arrows. This will be helpful for the proof that the **BSG** theorem is monoidal.

Lemma 9.23. *The altered Cartan product is associative on arrows.*

Proof. Let $f\colon h\to h'$, $g\colon r\to r'$, $k\colon s\to s'$ be three arrows in **BSG**. Notice that $f\boxtimes g\colon h\boxtimes r\to h'\boxtimes r'$ and $g\boxtimes k\colon r\boxtimes s\to r'\boxtimes s'$. Thus

$$
(f\boxtimes g)\boxtimes k = \begin{cases} 0, & (h\boxtimes r)\boxtimes s \neq (h'\boxtimes r')\boxtimes s' \\ I, & (h\boxtimes r)\boxtimes s = (h'\boxtimes r')\boxtimes s' \end{cases}
\tag{93}
$$

and

$$f \boxtimes (g \boxtimes k) = \begin{cases} 0, & h \boxtimes (r \boxtimes s) \neq h' \boxtimes (r' \boxtimes s') \\ I, & h \boxtimes (r \boxtimes s) = h' \boxtimes (r' \boxtimes s') \end{cases} \tag{94}$$

Since $h \boxtimes (r \boxtimes s) = (h \boxtimes r) \boxtimes s$, then (93) and (94) coincide. □

9.2.5 BSG is monoidal

We are finally ready to prove the main result of this text.

Theorem 9.24. *The tuple* $(\mathbf{BSG}, \boxtimes, \mathbb{1} := 0 := V_0)$ *is a monoidal category.*

Proof. From Lemma 9.20 the altered Cartan product $\boxtimes$ is a bifunctor on **BSG**. We now have to show that M1), M2), M3) and M4) of 2.16 are satisfied.

M1) The associator must have components of the form

$$a_{h,r,s} \colon (h \boxtimes r) \boxtimes s \to h \boxtimes (r \boxtimes s), \quad h, r, s \in \mathrm{Obj}(\mathbf{BSG}) \tag{95}$$

From Lemma 9.20 we have $(h \boxtimes r) \boxtimes s = h \boxtimes (r \boxtimes s) =: h \boxtimes r \boxtimes s$. Set $a_{h,r,s} = id_{h \boxtimes r \boxtimes s}$ for every $h, r, s \in \mathrm{Obj}(\mathbf{BSG})$. We must check that $a = (a_{h,r,s})_{h,r,s \in \mathrm{Obj}(\mathbf{BSG})}$ is a natural isomorphism. Since its components are clearly isomorphisms (with inverses equal to themselves), then we just need to show that the naturality square

$$\begin{tikzcd} h \boxtimes r \boxtimes s \arrow[r, "f \boxtimes g \boxtimes k"] \arrow[d, "a_{h,r,s}"'] & h' \boxtimes r' \boxtimes s' \arrow[d, "a_{h',r',s'}"] \\ h \boxtimes r \boxtimes s \arrow[r, "f \boxtimes g \boxtimes k"'] & h' \boxtimes r' \boxtimes s' \end{tikzcd}$$

commutes for all arrows $f \colon h \to h'$, $g \colon r \to r'$, $k \colon s \to s'$. (Notice that we used the associativity of the altered Cartan product on both objects and arrows to write the naturality square). Since the components of a are just the identities, it is immediate that the naturality square commutes, meaning that a is a natural transformation.

M2) We want to define appropriate components for the right and left unitors. They must be of the form $r_h \colon h \boxtimes 0 \to h$, $l_h \colon 0 \boxtimes h \to h$. Lemma 9.22 suggests that the identity maps can be used as components of the right and left unitors. Define $r_h = id_h$ and $l_h = id_h$ for every $h \in \mathrm{Obj}(\mathbf{BSG})$. Again these are isomorphisms (trivially). Because of Lemma 9.22, the naturality squares for r and l are identical, and are simply:

$$\begin{tikzcd} h \arrow[r, "f"] \arrow[d, "r_h"'] & h' \arrow[d, "r_{h'}"] \\ h \arrow[r, "f"'] & h' \end{tikzcd}$$

where $f\colon h \to h'$ is an arrow in **BSG**. It is again trivial to see that these natural squares commute, meaning that r and l are natural transformations, and thus natural isomorphisms.

M3) Because the associator components and the (left and right) unitor components are just identity arrows, the pentagon identity holds trivially. To see this, first notice that the pentagon identity in this case is given by the diagram

$$\begin{tikzcd}
 & h \boxtimes r \boxtimes s \boxtimes t \arrow[dl, "a_{h\boxtimes r,s,t}"'] \arrow[dr, "a_{h,r,s}\boxtimes id_t"] & \\
h \boxtimes r \boxtimes s \boxtimes t \arrow[d, "a_{h,r,s\boxtimes t}"'] & & h \boxtimes r \boxtimes s \boxtimes t \arrow[d, "a_{h,r\boxtimes s,t}"] \\
h \boxtimes r \boxtimes s \boxtimes t & & h \boxtimes r \boxtimes s \boxtimes t \arrow[ll, "id_h\boxtimes a_{r,s,t}"]
\end{tikzcd}$$

which indeed commutes:

$$\begin{aligned}
&(id_h \boxtimes a_{r,s,t}) \circ a_{h,r\boxtimes s,t} \circ (a_{h,r,s} \boxtimes id_t) \\
&\qquad = (id_h \boxtimes id_{r\boxtimes s\boxtimes t}) \circ id_{h\boxtimes r\boxtimes s\boxtimes t} \circ (id_{h\boxtimes r\boxtimes s} \boxtimes id_t) \\
&\qquad = (id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t) \circ (id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t) \\
&\qquad\qquad \circ (id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t) \\
&\qquad = id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t
\end{aligned} \tag{96}$$

where in the second and third equalities the bifunctoriality of $\boxtimes$. On the other hand:

$$\begin{aligned}
a_{h,r,s\boxtimes t} \circ a_{h\boxtimes r,s,t} &= id_{h\boxtimes r\boxtimes s\boxtimes t} \circ id_{h\boxtimes r\boxtimes s\boxtimes t} \\
&= (id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t) \circ (id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t) \\
&= id_h \boxtimes id_r \boxtimes id_s \boxtimes id_t
\end{aligned} \tag{97}$$

This shows that the pentagon identity holds.

M4) The only thing left to show is that the triangle identity holds *i.e.* that the diagram

$$\begin{tikzcd}
(h \boxtimes \mathbb{1}) \boxtimes s \arrow[rr, "a_{h,\mathbb{1},s}"] \arrow[dr, "r_h\boxtimes id_s"'] & & h \boxtimes (\mathbb{1} \boxtimes s) \arrow[dl, "id_h\boxtimes l_s"] \\
 & h \boxtimes s &
\end{tikzcd}$$

commutes for all $r, s \in \mathrm{Obj}(\mathbf{BSG})$. Just notice that

$$
\begin{aligned}
(id_h \boxtimes l_r) \circ a_{h,\mathbb{1},r} &= (id_h \boxtimes id_r) \circ id_{h \boxtimes r} \\
&= (id_h \boxtimes id_r) \circ (id_h \boxtimes id_r) \qquad (98) \\
&= (id_h \boxtimes id_r) = r_h \boxtimes id_r \qquad (99)
\end{aligned}
$$

This completes the proof $\square$

10 Conclusion and Outlook

We were able to construct a (finite) category **BSG** of D-branes of a Grassmannian model by using the fact that they can be seen as the representations arising from said model, and taking the arrows to be all the intertwiners between those representations. We then proved a generalization of the Schur's Lemma for Virasoro modules, which revealed the surprising simplicity of the category **BSG**. Finally, we were able to construct a tensor product in this category by altering the Cartan product, which serves as tensor product of other representation categories. Thus we ended up with a monoidal category **BSG**.

Further properties of this category could have been explored, if time had allowed, and are left for possible future work. Some of these seem quite immediate, although a formal thorough proof would still be necessary. Namely[30] the category **BSG** seems to be braided, semisimple and **Ab**-enriched. It may also be worth checking if **BSG** can be made into a tensor category (and thus in particular also a ribbon, $\mathbb{K}$-linear, abelian, rigid category). One can even attempt to keep going and try to show that **BSG** is a fusion category, or even a pre-modular category. Besides its mathematical interest, exploring these properties give us a better understanding of the D-branes, and furthermore can bring us closer to showing the correspondence with a Landau-Ginzburg model [Cam]. The roadmap one can follow if one decides to go down this path is in the Appendix 11.5.

Another possible continuation of this work is to extend the treatment to other Kazama-Suzuki models. In fact, it seems that the strategy we adopted in constructing the category **BSG** would be effective for other Kazama-Suzuki models, since we did not use anything specific to the Grassmannian model. Also the formulation of the altered Cartan product seems to invite generalization to other Kazama Suzuki models, if only one adjusts the extra term e to each particular case.

[30] Skip the rest of this paragraph if you are not familiar with the category theoretical concepts referred to. It would be impractical to include all their definitions here. All of them can be found in [EGNO].

It is also worth noting that we only treated B-type boundary conditions, and one could try to extend this treatment to A-type boundary conditions.

From the geometrical point of view, there are a few interesting natural questions which I was not able to answer, to wit:

- What is the geometrical interpretation of the arrows of **BSG**?
- What is the geometrical interpretation of the altered Cartan product of D-branes?

Although both questions only make sense if the boundary states of the Grassmannian model have a geometrical meaning (which in principle needs not be the case), there seem to be some ways to attribute geometrical meaning [Sta] to boundary states of Kazama-Suzuki model. This is still an unexplored topic for me, so again I simply leave these questions for future work.

Finally, one can use the results of this thesis as a starting point to an eventual proof of the CFT/LG correspondence in the case of the Grassmannian models, as mentioned in the introduction. This correspondence – illustrated *e.g.* in [BF] – can be used by string theorists to obtain information about a CFT (in our case a Grassmannian model) by studying the corresponding Landau-Ginzburg model.

11 Appendix

11.1 A foundational remark

A *set* can be defined [Hal3] as a mathematical object which satisfies a list of axioms called the ZFC axioms, so a generalization of set can be done accordingly. Let's see how without going into the details.

A *class* is defined by the axioms it satisfies, which are a modification of the ZFC axioms and mimic these when possible, so that one can often deal with classes as if they were sets. In particular it still makes sense to form products, intersections and unions of classes. A set can then be seen as a class that is an element of some other class. A *proper class* is a class that does not belong to any other class.

One of the neat things about classes is that we do not get a paradox by considering the class of all sets. This class must clearly be proper though, otherwise we would get Russel's paradox again.

Now, the origin of Russel's paradox is the ZFC axiom of specification, which allows us to define a set as a subset of a pre-existing set by specifying a property that all the elements of the new set must have. One of the axioms in the definition of class (which generalizes the axiom of specification from set theory) says that statements defining a class by some common property of its elements are not allowed to have quantifiers running over proper classes, only over sets. In particular, we cannot define the class of all classes, and this implies that we do not run into Russel's paradox for classes.

We see that we can take the category of all sets, whose objects form a proper class Obj(**Set**).

Taking this one step further, notice that we still can not define a category of all categories: we can not speak of the class of all categories, since some of these categories will be proper classes. The notion of a *conglomerate* comes to the rescue: it is a generalization of class, just like class is a generalization of set.

We can keep going and generalize conglomerates, and then generalize that, and so on. In this text we do not deal with the category of categories or any other category whose objects do not form a class, so our definitions will use classes (as opposed to conglomerates or other more general concepts).

The above is of course a very incomplete version of the story, but it is already more than what I found in most introductory category theory textbooks. To know more, see for example [AHS].

11.2 Modules and representations

Let's start by quickly reviewing basic definitions which should have been encountered by physics Master's students. More can be found in [Hum2, FS2].

A linear map $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ between Lie algebras is a *Lie algebra homomorphism* if $\phi([x,y]) = [\phi(x), \phi(y)]$.

A *representation* of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is a pair (π, V), where $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism and V is a vector space over $\mathbb{K}$. It is common practice to call π itself the representation. An important example is the *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, given by $ad(x)(y) = [x,y]$. The vector space V is also called representation in many texts, although a better name for it is *representation space.*

Let $\pi_1\colon \mathfrak{g} \to \mathrm{End}(V_1)$ and $\pi_2\colon \mathfrak{g} \to \mathrm{End}(V_2)$ be two representations. A linear map $\phi\colon V_1 \to V_2$ between Lie algebras is an *intertwiner* (or *homomorphism of representations*) between π_1 and π_2 if $\phi(\phi_1(x)(v_1)) = \phi_2(x)(\phi(v_1))$ for all $x \in \mathfrak{g}$ and $v_1 \in V_1$.

Consider two representations $\pi_1\colon \mathfrak{g} \to \mathrm{End}(V_1)$ and $\pi_2\colon \mathfrak{g} \to \mathrm{End}(V_2)$ of a Lie algebra $\mathfrak{g}$. Their *direct sum* is the representation $\pi_1 \oplus \pi_2\colon \mathfrak{g} \to \mathrm{End}(V_1 \oplus V_2)$ given by $\left((\pi_1 \oplus \pi_2)(x)\right)(v_1 \oplus v_2) = \pi_1(x)(v_1) \oplus \pi_2(x)(v_2)$. Their *tensor product* is the representation $\pi_1 \otimes \pi_2\colon \mathfrak{g} \to \mathrm{End}(V_1 \times V_2)$ given by $((\pi_1 \otimes \pi_2)(x))(v_1 \otimes v_2) =$

$\pi_1(x)(v_1) \otimes v_2 + v_1 \otimes \pi_2(x)(v_2)$.

There is another way to look at representations, which is common to find in the mathematical literature and is often more natural and useful. We look briefly at it now, following [Hum2, EW].

Let $\mathfrak{g}$ be a Lie algebra. A vector space V together with a bilinear map $\cdot\colon\ \mathfrak{g} \times V \to V$ is a $\mathfrak{g}$-*module* if $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$. The map $\cdot$ is called the *(left) action of* $\mathfrak{g}$ *on* V. A *submodule* of a $\mathfrak{g}$-module V is a vector subspace W of V which is invariant under the action of $\mathfrak{g}$.

An important construction is the quotient module. Let V be a $\mathfrak{g}$-module and W be a submodule of V. There is an action of $\mathfrak{g}$ on the quotient vector space V/W given by

$$x \cdot (v + W) = (x \cdot v) + W, \quad \forall x \in \mathfrak{g}, \forall v \in V$$

making V/W a $\mathfrak{g}$-module called a *quotient module.*

It is easy to see the connection between $\mathfrak{g}$-modules and representations of $\mathfrak{g}$: if $\pi\colon \pi \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$, then $(x, v) \mapsto x \cdot v := \pi(x)(v)$ is defines an action on V, *i.e.* V is a $\mathfrak{g}$-module. Conversely, if V is a $\mathfrak{g}$-module with an action $\cdot$, then we can define a representation π of $\mathfrak{g}$ by setting $\pi(x)(v) = x \cdot v$.

Hence we can speak of representations and modules of a Lie algebra interchangeably, and one can think of a $\mathfrak{g}$-module simply as the representation space of a representation of $\mathfrak{g}$.

All concepts defined for representations of $\mathfrak{g}$ carry over to the $\mathfrak{g}$-modules. In particular, homomorphisms of $\mathfrak{g}$-modules can be seen as intertwiners of representations of $\mathfrak{g}$, and one can take direct sums and tensor products of $\mathfrak{g}$-modules, which can also be interpreted as direct sums and tensor products of representations of $\mathfrak{g}$.

11.3 Spacetime supersymmetry

Recall from Quantum Field Theory that the Lorentz algebra is generated by the operators $M^{\mu\nu}$ [Sre] (the angular momentum operator J is given by $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ and the boost operator K by $K_i = M_{i0}$). The Poincaré algebra adds translations to the mix through the momentum operator P^μ, and has the commutation relations [BLOT]:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
\frac{1}{i}[M_{\mu\nu}, P_\rho] &= \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu \\
\frac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}
\end{aligned} \tag{100}
$$

It is natural to ask what are the most general possible spacetime symmetries of a quantum field theory: maybe Nature realizes these symmetries.

The so called Coleman-Mandula theorem [CM] states that in a generic quantum field theory the most general spacetime continuous symmetry of the S-matrix is the Poincaré algebra – assuming the S-matrix is non-trivial, *i.e.* the theory has interactions. However, Haag, Lopuszanski and Sohnius [PH] proved that it is possible to extend it to a super Lie algebra[31]in a non-trivial way (see [Ber] for a discussion on this).

The simplest such extension is the $\mathcal{N} = 1$ super-Poincaré algebra which extends (100) by [FVP]:

$$\left\{Q_\alpha, \bar{Q}^\beta\right\} = -\frac{1}{2}\left(\gamma_\mu\right)_\alpha^\beta P^\mu \tag{101}$$

$$\left[M_{[\mu\nu]}, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha^\beta Q_\beta \tag{102}$$

$$[P_\mu, Q_\alpha] = 0 \tag{103}$$

In a $\mathcal{N} = 1$ supersymmetric theory the supersymmetry generator spinor components Q_α do not commute with J_3, and thus change the spin of the states they act on. It turns out that Q takes bosons of spin j to fermions with spin $j - \frac{1}{2}$ [Ait]. Such pairs of bosons and fermions are said to be *superpartners*.

11.4 The $N = 1$ superconformal subalgebra of an $N = 2$ superconformal algebra

Claim 11.1. *Every $N = 2$ superconformal algebra A with generators $\{L_m, G_r^\pm, J_n\}$ has an $N = 1$ subalgebra generated by $\{L_m, G_r = \frac{G_r^+ + G_r^-}{\sqrt{2}}\}$ with the same central charge as A. We say that this is the* standard $N = 1$ *subalgebra of A.*

[31] Roughly speaking, a super Lie algebra is a vector space V which decomposes into two subspaces V_0 and V_1 whose elements are said to be *even* and *odd* respectively, together with an operation $[\cdot, \cdot]$ respecting a modified version of the Lie bracket axioms, in such a way that even elements commute while odd elements anti-commute. The details of this construction will not matter for us. See [Ali] for more.

Proof. We just have to show that (43) holds. We have:

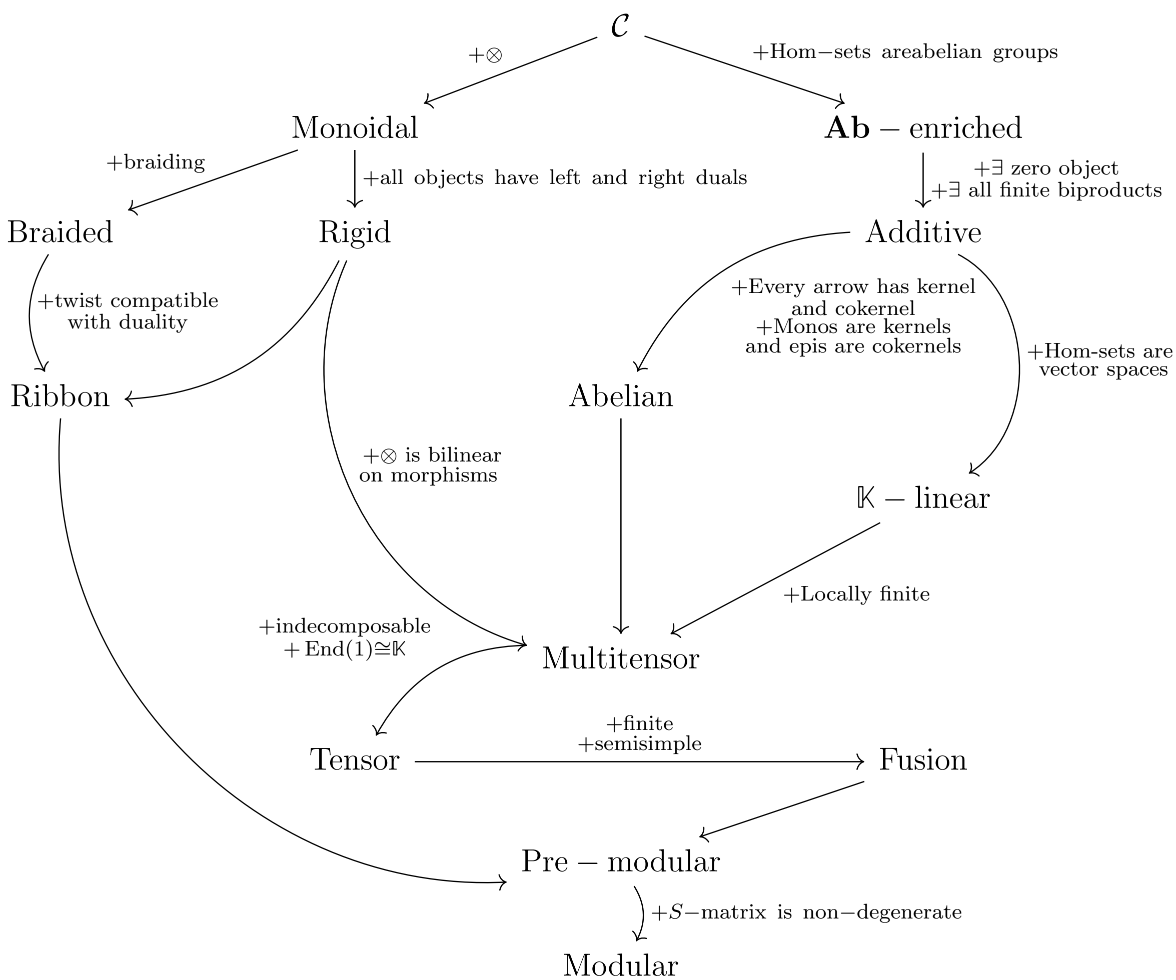
$$
\begin{aligned}
\{G_r, G_s\} &= \{\frac{G_r^+ + G_r^-}{\sqrt{2}}, \frac{G_s^+ + G_s^-}{\sqrt{2}}\} \\
&= \frac{1}{2}\left(\{G_r^+, G_s^-\} + \{G_r^-, G_s^+\}\right) \\
&= \frac{1}{2}\left(2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}\right) \\
+\frac{1}{2}&\left(2L_{s+r} + (s-r)J_{s+r} + \frac{c}{3}(s^2 - \frac{1}{4})\delta_{s+r,0}\right) \\
&= 2L^{r+s} + \frac{c}{6}\delta_{r,-s}(r^2 + s^2 - \frac{1}{2}) \\
&= 2L^{r+s} + \frac{c}{12}\delta_{r,-s}(4r^2 - 1)
\end{aligned}
\tag{104}
$$

Furthermore:

$$
\begin{aligned}
[L_m, G_r] &= \frac{1}{2}[L_m, G_r^+ + G_r^-] \\
&= \frac{1}{2}\left((\frac{m}{2} - r)G_{m+r}^+ + (\frac{m}{2} - r)G_{m+r}^-\right) \\
&= (\frac{m}{2} - r)G_{m+r}
\end{aligned}
\tag{105}
$$

This proves the claim. □

11.5 Categorical Roadmap



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