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Master Thesis

Scalar Fields In Rindler Spacetime And The Near Horizon Black Hole Entropy

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Conventions

Throughout this thesis, natural units are used such that $\hbar = c = G = k_B = 1$, unless otherwise stated. The signature $(-, +, ..., +)$ is also employed.

Abstract

One constructs the Feynman propagator for real scalar fields in both Minkowski and Rindler spacetimes. This is done using the Poincaré invariant vacuum state in Minkowski space and a thermal state in the Rindler case. One shows that both results are a function of the invariant distance, and therefore one can be obtained from the other by a coordinate transformation. Since the near horizon limit of a Schwarzschild black hole has the same geometry as a Rindler observer, the Rindler frame is used to study the near horizon limit. The observer at constant acceleration in the Rindler frame is similar to an observer of constant radial distance to the black hole horizon and has no access to information inside the horizon. One constructs the reduced density matrix by tracing out the degrees of freedom inside the black hole. For a Rindler observer, this corresponds to tracing out the left Rindler wedge. One also constructs a projected Statistical function in order to express the reduced density matrix in terms of spacetime coordinates. One uses these calculations to make important steps towards calculating Remyi entropy in the near horizon limit of a black hole, however this calculation is not yet completed.

Contents

Chapter 1

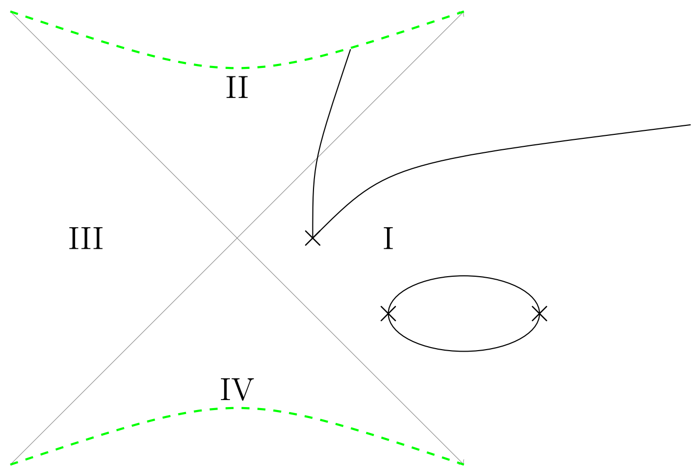
Introduction

To begin this thesis, one will start with a broad outline of the black hole information problem. It is a fundamental concept of physics that information must be preserved in any process. Should this not be the case, one would have to deal with non unitary evolution of states. The unitarity condition is that the time evolution of a quantum state according to the Schrödinger equation is mathematically represented by a unitary operator. This is taken as a basic postulate of quantum mechanics. There is a fundamental problem how to describe a system which loses information through black hole evaporation. One requires a theory preserving unitarity here. Although there is a historic story to tell, the modern consensus is that information may enter a black hole via matter falling past an event horizon, as in figure 1.1. This information is no longer accessible to the outside universe, but is at least still held by the black hole in question. It was later shown that black holes evaporate through emitting Hawking radiation [1]. It may be that the emmitted radiation makes information available to the outside observer. It is not however clear how to use the information stored in the Hawking radiation to reconstruct all information present at the black holes formation. This leaves the question of where the information is carried to or whether it is even destroyed.

1.1 Black Hole Entropy

To provide further context to this project, one will give a brief description of the topic of black hole thermodynamics and its relation to information contained by a black hole. Starting with the discussion of black holes and entropy as presented in [2]. The relation between black hole mechanics and thermodynamics was initially motivated by the irreducible mass of the black hole. that is, the black hole's mass cannot become smaller, meaning the area of its horizon cannot be reduced. It was believed that almost all processes involving black holes could only result in an increase of its mass. The only exception were reversible processes which leave the mass unchanged. This is strikingly similar to the second law of thermodynamics, which states that entropy can only increase. Without considering Hawking radiation, other physical processes involving black holes had to be examined

Figure 1.1: Pair creation near a horizon where one particle falls into the singularity and the other escapes to asymptotic infinity. The singularity is represented by dashed green lines while the horizon is represented by the solid diagonal lines. Region I represents the outside universe. Region II is the region past the event horizon of the black hole. Region IV is described as unphysical as it describes a white hole, or a singularity that is not covered by a horizon. Region III is the result of analytic continuation of the diagram.



to see if they were consistent with the thermodynamics law. For example, the process of extracting energy from a rotating black hole. It was however shown that a rotating black hole has a irreducible mass related to the surface area of its horizon which can be thought of as its inert energy, which cannot be extracted [3] [4]. Another situation is the merger of two Schwarzschild black holes in which energy is released in the form of gravitational waves. This is however analogous to the situation of having two systems in equilibrium interact and being able to extract work.

One now makes the link between entropy and information, which is stored by matter falling into a black hole. Supposing a system may have a number of internal configurations with a probability of being in the n^{th} state p_n. Then the entropy associated with the system is given by Shannon's formula: [5]

$$S = -\sum_n p_n \log(p_n).$$

Although Beckenstein could not find the precise coefficient, which he admits in [2], he was able to show that the entropy of a black hole was proportional to its area. $S_{bh} = \eta \hbar^{-1} A$ [2], where η is a dimensionless constant and A is the area of the horizon. This was in alignment with the second law of thermodynamics as information was lost from the observable universe by matter falling past an event horizon. The information was in fact still stored by the black hole as an increase in its irreducible mass. Although this was later complicated by the discovery that black holes in fact lose mass through the process of Hawking radiation. The total entropy of the universe was shown not to decrease as the loss of entropy from the reduction of the surface area of the black hole was compensated by the entropy held in the outgoing Hawking radiation [6]. The question later posed is the information of the in going particles saved by some mechanism involving the outgoing particles. This again is discussed in depth by Gerard 't Hooft in numerous papers, such as [7].

1.2 Current status of the problem

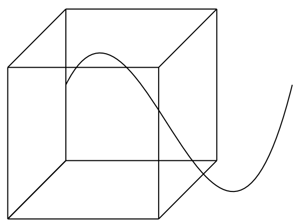
It is a problem that has been tackled through many different approaches as well, which fundamentally challenge ones understanding of physics. There are many ways of dealing with the problem, involving string theory and introduction of new degrees of freedom at the Planck scale [8]. Avoiding these means using more classical physics and few assumptions [9]. Here, one obtains an S-matrix to describe the system which preserves unitarity. Since unitarity is a fundamental part of ones understanding, previous attempts were willing to make serious concessions to preserve this. Including theories that involve non-locality [10]. This too is difficult to integrate into ones current understanding of physics.

The previous method of calculating entropy near the black hole was named the "Brick Wall" method [11]. In this approach, it is argued that any attempt to calculate entropy on the horizon of a black hole is doomed to failure. Instead, one must consider the region that is of order of the Planck length away from the horizon. It most importantly shows that Hawking radiation can be seen to be compatible with quantum mechanical purity. This model has now been recently replaced by a method of boundary conditions where $|in\rangle$ states are replaced by $|out\rangle$ states [9]. Without going into details of calculating the scattering matrix. The solution comes about by finding boundary conditions for a quantum field at the horizon, by doing this, one is not "missing" any information. This is the situation in figure 1.2 What is important to comment on, is that this procedure leads to a unitary evolution law. Now quoting Gerard 't Hooft and using figure 1.1"relating region III to region I by demanding that the angular coordinates are antipodes, means that now the mapping from Schwarzschild coordinates to Kruskal Szekeres coordinates is one-to-one." [9] This is the result of choosing the boundary condition that identifies one point on the surface of the black hole with a point on the opposite side of the black hole and will become relevant later in this thesis when one performs a coordinate transformation from the Minkowski frame to the Rindler frame, in the hope of making comparisons between the Rindler metric and the near horizon limit of the Schwarzschild metric. It is in this way that one identifies region I of figure 1.1 with the right Rindler wedge.

1.3 Objective of the project

The question posed in this project is how to calculate the entropy from the entanglement between degrees of freedom inside and outside the horizon. Should a horizon form, then a quantum field's

Figure 1.2: Example of wave function in a box. This is a simple example of having no boundary conditions for a wave function in a box. An observer inside the box does not have any information about the outside, analogous to the situation of horizons where an outside observer does not have any information about the interior. Should there be boundary conditions preventing the wave function to exist outside the box, there would be no missing information.



degrees of freedom outside the horizon should be entangled with those inside of it. An external observer cannot access the interior of the horizon, so the meaningful state, from the point of view of the external observer, is one in which the internal degrees of freedom are traced out. This results in a mixed state, which has associated entropy. It has been shown that the entanglement entropy is indeed proportional to the horizon area [12]. Although this was done by using a system of coupled harmonic oscillators. Now, one wishes to do this using the Rindler frame since the Rindler observer is equivalent to the near horizon geometry of a Schwarzschild black hole.

In this project, it is hoped to compare the simplest possible black hole, where the charge and angular momentum are zero, to the case of the accelerated observer. This is the case in which the observer experiences a constant uniform acceleration, which translates to an observer that hovers at constant radial distance away from the event horizon. Thus, it is very important to consider the case of the accelerated observer very carefully, since these two physical situations, on the surface, appear very different. Yet one hopes that it is still possible to make comparisons.

The accelerated observer may describe their metric using a coordinate transformation of the flat Minkowski space. It is also possible to describe the Schwarzschild metric in the same form. Thus, in this project, one will first study the D-dimensional Minkowski case and then perform the coordinate transformation for the accelerated observer or the Rindler observer. The results of the Minkowski and Rindler cases are then compared and their relation is as to be expected from a coordinate transformation. It is reasonable to state that physical quantities such as the stress-energy-momentum tensor should transform in a way such that performing coordinate transformations does not change anything physical and the original form can always be recovered by performing the inverse of the coordinate transformation. Likewise if a two point function has a singularity where the spacetime coordinates are the same, i.e. the coincidence limit, one would expect this in any other coordinate frame used.

Using Rindler coordinates, spacetime is split into two sections which will now be referred to as the left and right Rindler wedge, these are bounded by the apparent horizons. The physics of these wedges is of interest in this thesis and so techniques on how to work on them as opposed to the whole of spacetime will be examined.

So in this project, one will begin by finding Wightman functions in Minkowski space and then taking time ordering to find the Feynman propagator. This is then used to calculate the stress-energy-momentum tensor. Having outlined this procedure, the same calculation will then be performed using Rindler coordinates. This can be done for massive scalar fields in D-dimensional space.

Next, the density matrix will be used to describe the statistical state of the system, but only over half of the space. This is because working in the Rindler frame, one does not consider the full space to be accessible. In fact one becomes restricted to the boundaries of the Rindler wedges that are marked by the presence of apparent horizons as a result of the acceleration, this will be described further in chapter 3. Thus care must be taken when defining this reduced density matrix as it describes two different regions, described by two different Hilbert spaces.

Now to use this reduced density matrix, one must take the previously time ordered two point functions for the entire space, and project them onto the regions to which the reduced density matrix applies, i.e. if the reduced density matrix is defined by tracing out the left side of the space, the two point functions must be projected onto only the right side. All projections are outlined later in this thesis.

1.4 Rindler observer

Now to motivate the use of Rindler coordinates to study black holes, the relation between the Rindler and Schwarzschild metrics will be illustrated in the near horizon limit. The general form of the box operator in curved space time is given by:

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi).$$

Working with the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin\theta d\phi^2.$$

Taking the near horizon limit ($r \to r_s$) and letting $\xi = r - r_s$

$$ds^2 = -\frac{\xi}{2m}dt^2 + \frac{2m}{\xi}d\xi^2 + (2m)^2 d\Omega^2,$$

replacing $\xi = \frac{\rho'^2}{8m}$ then yields the metric

$$ds^2 = -\frac{\rho'^2}{16m^2}dt^2 + d\rho'^2 + (2m)^2 d\Omega^2 \tag{1.1}$$

whose dt^2 and $d\rho'^2$ part is of the same form as the Rindler metric.

$$ds^2 = -(\alpha\rho)^2 d\tau^2 + d\rho^2 + d\vec{x}^2. \tag{1.2}$$

Provided the acceleration parameter, α, is identified with $1/4m$. This is though comparing Cartesian and spherical coordinates. So to make comparisons, the Schwarzschild metric will be written as

$$ds^2 = -\frac{\rho'^2}{4r_s^2}dt^2 + d\rho'^2 + r_s^2 d\Omega^2. \tag{1.3}$$

Now writing the Rindler metric in spherical coordinates, one must note that the ρ will carry some angular dependence, but all other angles may be absorbed in the $d\Omega$.

$$ds^2 = -\alpha^2\rho^2 d\tau^2 + dr^2 + r^2 d\Omega^2 \tag{1.4}$$

Indeed, using the Kruskal coordinates for the Schwarzschild metric involves using hyperbolic functions, just as in Rindler coordinates. Thus, the diagram for the coordinates are analogous to figure 1.

1.5 Unruh effect

The Unruh effect [13] is a quantum field theory result and is insightful to the particle content of a field theory. It implies that there exists some level of observer dependency. This is the troublesome case of different observers seeing states differently. To summarise the Unruh effect, it is that uniformly accelerated observers in Minkowski spacetime, also known as Rindler observers, associate a thermal bath of Rindler particles to the vacuum state of inertial observers, i.e. the Minkowski vacuum. It is the equivalence between a Minkowski vacuum and a thermal bath of Rindler particles. Indeed, it has been checked that there are no correlations between these observed particles, meaning they are truly thermal [14]. the relation between states in Minkowski and Rindler spacetimes is further discussed in chapter three.

It is important to note that, a quantum field expanded by the Rindler modes can be used in the whole of Minkowski space time. Although observations of this claim are disputed, [15] the predicted temperature can be expressed as

$$T = \frac{\hbar a}{2\pi c k_{\mathrm{B}}}$$

this is of the same form as the Hawking temperature of a black hole, which is expressed as

$$T_{\mathrm{H}} = \frac{\hbar g}{2\pi c k_{\mathrm{B}}}$$

[1] where g is the surface gravity.

Surface gravity

Although surface gravity is gravitational acceleration, measured in m/s^2, this concept must be thought of slightly differently on the horizon of a black hole. While in a non-relativistic setting, the acceleration experienced by a test particle can be called the surface gravity, this quantity diverges on the horizon of a black hole. However, an observer at infinity will detect the acceleration to be redshifted by a factor "V". The surface gravity is then the product of the acceleration times the redshift factor. $\kappa = Va$. Which is typically finite and is so for the Schwarzschild case considered. The formal expression for the surface gravity is expressed in terms of Killing vectors that obey the geodesic equation such that

$$\chi^{\mu}\nabla_{\mu}\chi^{\nu} = -\kappa\chi^{\nu}$$

where χ^a is a killing vector normal to its associated Killing horizon and κ is the surface gravity [14]. An example of this can be given by the Schwarzschild metric above. Where the Killing vector is given by ∂_t. The four-acceleration is given in terms of the redshift factor as $a_{\mu} = \nabla_{\mu} \ln(V)$. Combing these and evaluating at horizon, so that $r = r_s$, the surface gravity is then:

$$\kappa = \frac{1}{2r_s} \tag{1.5}$$

vacuum state

In the current description of quantum field theory, space is filled with quantised fields. The definition of the vacuum is not then that of empty space, but of the lowest possible energy state of these fields.

Thus, the vacuum depends on the path of the observer through spacetime. It becomes more intuitive to see this considering that the energy of states is given by a Hamiltonian which may have some time dependence. Special relativity shows the necessity for observers to define a proper time. Thus for the case of accelerating observers, there may be no shared coordinate system. Hence, different quantum states will be observed.

The link to make here is that the accelerated observer will note an apparent horizon forming, the Rindler horizon. Closely analogous to that of a Schwarzschild black hole. To this end, it is speculated that the Unruh effect is the near horizon form of Hawking radiation.

Density matrix

In this thesis, one considers Gaussian density matrices centered at the origin. This is appropriate for situations where the evolution of a system is described by a quadratic Hamiltonian. It is noted that higher order interactions are usually weak and thus their non-Gaussianities introduced are neglected. The density matrix can thereby be described using two general states $|\phi\rangle$ and $|\phi'\rangle$, and the unity operator that is $\int \mathcal{D}\phi \, |\phi\rangle \langle\phi| = \mathbb{1}$, such that:

$$\rho_{\mathrm{E}} = \int \mathcal{D}\phi \int \mathcal{D}\phi' |\phi\rangle \hat{\rho}\,[\phi, \phi'; t) \,\langle\phi'|$$
$$\rho[\phi, \phi'; t) = \langle\phi'|\, \hat{\rho}(t)\, |\phi\rangle$$

where

$$\hat{\rho}\,[\phi, \phi'; t] = N \exp\left(-\phi^T \cdot A \cdot \phi - \phi'^T \cdot B \cdot \phi' + 2\phi^T \cdot C \cdot \phi'\right). \tag{1.6}$$

[16] In the above, the dot product notation was used. this is important for the correct treatment of these objects as matrices. In matrix multiplication, it is the repeated index that is summed over. Here it is the common variable of the functions that is integrated over in what is now referred to as a convolution integral. For example, $[A \cdot B](x, z) = \int dy A(x, y) B(y, z)$ or $[\phi^T \cdot A \cdot \phi](t) = \int dx dy \phi(x) A(t; x, y) \phi(y)$. The density operator for a mixed state is simply,

$$\hat{\rho} = \sum_i c_i \,|\phi_i\rangle \langle\phi_i|$$

so that taking its trace will yield unity, this is always the case. The normalisation, N can be found by the requirement that the trace of the density matrix is one. There is also the property that the density matrix is hermitian, i.e. that $\hat{\rho}^\dagger = \hat{\rho}$. This implies that the coefficients, c_i are real and also that $A^T = B$ and that $C^T = C$. The mixed density operator is, however, useful for when one has incomplete knowledge of the system [17].

Chapter 2

Minkowski

2.1 Overview

In this section, one will calculate the Feynman propagator for massive scalar fields in a Minkowski space time. This is a preliminary to when this will later be done in Rindler spacetime, but it is important to outline the procedure first for the simplest case and then to compare it to later results.

To begin, the field can be defined in terms of creation and annihilation operators with the commutation rules for these operators defined in a usual manner. Then the momentum operator can also be defined and its commutation relation with the field as well. Then the Wronskian can be used to find a solution for the field in position space and from there, the Feynman propagator is given by the time ordered correlation function, as outlined below.

Firstly, the vacuum state for this space can be defined:

$$\hat{a}\,|0\rangle = 0$$

such that the annihilation operator, $\hat{a}$, acting on the vacuum state, $|0\rangle$, annihilates the vacuum. Next, the counting operator, gives the number of particles in a state and is expressed as an eigenvalue equation where $|n\rangle$ is the eigenstate of $\hat{N}$, which is a shorthand for $\hat{a}^\dagger \hat{a}$.

$$\hat{N}\,|n\rangle = n\,|n\rangle \tag{2.1}$$

where n is the number of particles and $|n\rangle$ spans a basis of the Hilbert space. Then it is obvious to see that $\hat{N}$ applied to the vacuum yields zero. In later sections, when dealing with different operators, this will not be the case. Then, the box operator has the following properties when operating on the field and the Feynman propagator, respectively,

$$(\Box + m^2)\hat{\phi} = (-\partial_t^2 + \nabla^2 + m^2)\hat{\phi} = 0, \tag{2.2}$$

$$(\Box + m^2)i\Delta(x;x') = i\hbar\delta^D(x-x'). \tag{2.3}$$

In momentum space, the field can be written in terms of creation and annihilation operators:

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left(\phi(t,\vec{k})e^{ikx}\hat{a}(\vec{k}) + h.c\right)$$

where kx is the vector product, yielding $-k_0x^0 + k_1x^1 + ...$ .The momentum operator is given by:

$$\hat{\Pi}(t,\vec{x}) = \dot{\hat{\phi}}(t,\vec{x})$$

Using this to find the same relation between ϕ and Π:

$$\left[\hat{\phi}(t,\vec{x}), \hat{\Pi}(t',\vec{x}')\right] = i\hbar\delta^{D-1}(\vec{x}-\vec{x}').$$

One demands that the creation and annihilation operators obey the commutation relation:

$$\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right] = (2\pi)^{D-1}\delta(\vec{k}-\vec{k}').$$

Setting $\hbar$ to one, one now uses the Wronskian to find a solution for ϕ:

$$W[\phi,\phi^*] = i$$

$$\phi = \frac{1}{\sqrt{2k^0}}e^{-ik^0t}.$$

2.2 Two point functions in 1+1 dimensions

Since it is very important to choose the correct $i\epsilon$ prescription to perform the coming integrals. It is worth focusing on solely that, briefly. One will take the $1+1$ dimensional, massless case for simplicity. In doing this, one will use the shorthand, $\Delta t = t - t'$ and $r_1 = x_1 - x_1'$.

$$
\begin{aligned}
& i\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-x')}}{k^2 - i\epsilon} = i\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik^0\Delta t}e^{ik_1r_1}}{-k^{0\,2} + k_1^2 - i\epsilon} \\
&= -i\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik^0\Delta t}e^{ik_1r_1}}{(k^0 - k_1 + i\bar{\epsilon})(k^0 + k_1 - i\bar{\epsilon})} \\
&= -i\int \frac{(-2\pi i)}{(2\pi)^2}\frac{\Theta(\Delta t)e^{-i|k_1|\Delta t}e^{ik_1r_1}}{2|k_1|} - i\int \frac{(2\pi i)}{(2\pi)^2}\frac{\Theta(-\Delta t)e^{i|k_1|\Delta t}e^{ik_1r_1}}{-2|k_1|} \\
&= \frac{-1}{4\pi}\int \frac{dk_1 e^{ik_1r_1}}{|k_1|}\Big\{\Theta(\Delta t)e^{-i|k_1|\Delta t} + \Theta(-\Delta t)e^{i|k_1|\Delta t}\Big\} \\
&\equiv i\Delta^+(x,x')\Theta(\Delta t) + i\Delta^-(x,x')\Theta(-\Delta t)
\end{aligned}
$$

This is now in the form of the positive and negative Wightman functions:

$$
i\Delta^+ = \frac{-1}{4\pi}\Big\{\int_0^\infty \frac{e^{ik_1(r_1-\Delta t)}}{k_1} + \int_{-\infty}^0 \frac{e^{ik_1(r_1+\Delta t)}}{-k_1}\Big\}.
$$

This integral is infrared divergent and requires regularisation. To do this, the lower limit of integration must be taken to be δ where the limit of δ tending to zero will yield a finite answer.

$$
i\Delta^+ = \frac{-1}{4\pi}\Big\{\int_\delta^\infty \frac{e^{ik_1(r_1-\Delta t+i\epsilon)}}{k_1} + \int_\delta^\infty \frac{e^{ik_1(-r_1-\Delta t+i\epsilon)}}{k_1}\Big\}.
$$

As δ becomes small, one considers leading order in δ:

$$
\begin{aligned}
&= \frac{-1}{4\pi}\Big\{-\gamma_E - \ln[\delta(r_1 - \Delta t + i\epsilon)] - \gamma_E - \ln[\delta(r_1 + \Delta t - i\epsilon)]\Big\} \\
i\Delta^+(\Delta t, r_1) &= \frac{1}{4\pi}\ln[\mu^2(r_1^2 - (\Delta t - i\epsilon)^2)] + \phi(\mu)^2. \qquad (2.4)
\end{aligned}
$$

Where a condensate term, $\phi(\mu)^2$ has been added. This removes the δ dependence from the Wightman funtion. One may also note the dependence of the function on the spatial distance. The magnitude of the function in fact grows with r_1, corresponding to a strongly interacting system. The negative frequency Wightman function is

$$
\begin{aligned}
i\Delta^-(x,x') &= \frac{-1}{4\pi}\Big\{\int_0^\infty \frac{e^{ik_1(r_1+\Delta t)}}{k_1} + \int_{-\infty}^0 \frac{e^{ik_1(r_1-\Delta t)}}{-k_1}\Big\}\\
&= \frac{-1}{4\pi}\Big\{\int_\delta^\infty \frac{e^{ik_1(r_1+\Delta t+i\epsilon)}}{k_1} + \int_\delta^\infty \frac{e^{ik_1(-r_1+\Delta t+i\epsilon)}}{k_1}\Big\}
\end{aligned}
$$

again, as δ becomes small, one considers leading order:

$$
\begin{aligned}
&= \frac{-1}{4\pi}\Big\{-\gamma_E - \ln[\delta(r_1+\Delta t+i\epsilon)] - \gamma_E - \ln[\delta(-r_1+\Delta t+i\epsilon)]\Big\}\\
i\Delta^-(\Delta t, r_1) &= \frac{1}{4\pi}\ln[\mu^2(r_1^2-(\Delta t+i\epsilon)^2)] + \phi(\mu)^2.
\end{aligned} \tag{2.5}
$$

What is interesting to ask is whether the choice of metric signature matters. To check that, one will repeat the calculation for the opposite choice of signs in the metric.

$$
\begin{aligned}
i&\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik(x-x')}}{k^2-i\epsilon} = i\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik^0\Delta t}e^{-ik_1r_1}}{k^{0 2}-k_1^2-i\epsilon}\\
&= i\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik^0\Delta t}e^{-ik_1r_1}}{k^{0 2}-k_1^2+i\epsilon} = i\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik^0\Delta t}e^{-ik_1r_1}}{(k^0-k_1+i\bar\epsilon)(k^0+k_1-i\bar\epsilon)}\\
&= i\int \frac{(-2\pi i)}{(2\pi)^2}\frac{\Theta(-\Delta t)e^{i|k_1|\Delta t}e^{-ik_1r_1}}{2|k_1|} + i\int \frac{(2\pi i)}{(2\pi)^2}\frac{\Theta(\Delta t)e^{-i|k_1|\Delta t}e^{-ik_1r_1}}{-2|k_1|}\\
&= \frac{1}{4\pi}\int \frac{dk_1 e^{-ik_1r_1}}{|k_1|}\Big\{\Theta(\Delta t)e^{-i|k_1|\Delta t} + \Theta(-\Delta t)e^{i|k_1|\Delta t}\Big\}.
\end{aligned}
$$

This is now in the form of the positive and negative Wightman functions.

$$
\begin{aligned}
i\Delta^+(x,x') &= \frac{1}{4\pi}\Big\{\int_0^\infty \frac{e^{ik_1(-r_1-\Delta t)}}{k_1} + \int_{-\infty}^0 \frac{e^{ik_1(-r_1+\Delta t)}}{-k_1}\Big\}\\
&= \frac{1}{4\pi}\Big\{\int_\delta^\infty \frac{e^{ik_1(-r_1-\Delta t+i\epsilon)}}{k_1} + \int_\delta^\infty \frac{e^{ik_1(r_1-\Delta t+i\epsilon)}}{k_1}\Big\}
\end{aligned}
$$

considering leading order:

$$= \frac{1}{4\pi}\Big\{ -\gamma_E - \ln[\delta(-r_1 - \Delta t + i\epsilon)] - \gamma_E - \ln[\delta(r_1 - \Delta t + i\epsilon)]\Big\}$$
$$i\Delta^+(\Delta t, r_1) = \frac{1}{4\pi}\ln[\mu^2(r_1^2 - (\Delta t - i\epsilon)^2)] + \phi(\mu)^2. \tag{2.6}$$

The negative frequency Wightman function is

$$i\Delta^-(x,x') = \frac{1}{4\pi}\Big\{\int_0^\infty \frac{e^{ik_1(r_1+\Delta t)}}{k_1} + \int_{-\infty}^0 \frac{e^{ik_1(r_1-\Delta t)}}{-k_1}\Big\}$$
$$= \frac{1}{4\pi}\Big\{\int_\delta^\infty \frac{e^{ik_1(r_1+\Delta t+i\epsilon)}}{k_1} + \int_\delta^\infty \frac{e^{ik_1(-r_1+\Delta t+i\epsilon)}}{k_1}\Big\}.$$

As before, this can be written as:

$$= \frac{1}{4\pi}\Big\{ -\gamma_E - \ln[\delta(r_1 + \Delta t + i\epsilon)] - \gamma_E - \ln[\delta(-r_1 + \Delta t + i\epsilon)]\Big\}$$
$$i\Delta^-(\Delta t, r_1) = -\frac{1}{4\pi}\ln[\mu^2(r_1^2 - (\Delta t + i\epsilon)^2)] + \phi(\mu)^2. \tag{2.7}$$

This however is the same as the first case by an overall sign change. Of course, the order of integration should not matter. Therefore, here it will be outlined how to recover the same structure as before while carrying out the k_1 integral first.

$$i\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik^0\Delta t}e^{ik_1r_1}}{k^2 - i\epsilon} = i\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik^0\Delta t}e^{ik_1r_1}}{-k^{0\,2} + k_1^2 - i\epsilon}$$
$$= i\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik^0\Delta t}e^{ik_1r_1}}{(k_1 - k^0 + i\epsilon)(k_1 + k^0 - i\epsilon)}$$
$$= i\int \frac{(-2\pi i)}{(2\pi)^2}\frac{\Theta(-r_1)e^{-ik^0\Delta t}e^{i|k^0|r_1}}{2|k^0|} + i\int \frac{(2\pi i)}{(2\pi)^2}\frac{\Theta(r_1)e^{-ik^0\Delta t}e^{-i|k^0|r_1}}{-2|k^0|}.$$

One can note the similar structure where before one had the positive and negative Wightman functions. Here however, each integral contains a mixture of these previous functions. These integrals can be computed separately and, remarkably, be recombined to give the positive and negative Wightman functions exactly the way they are in equations 2.6 and 2.7.

$$\int dk^0 \frac{e^{-ik^0\Delta t}e^{-i|k^0|r_1}}{|k^0|} = \int_0^\infty \frac{e^{ik^0(-\Delta t - r_1 + i\epsilon)}}{k^0} + \int_{-\infty}^0 \frac{e^{ik^0(-\Delta t + r_1 - i\epsilon)}}{-k^0}$$
$$= \int_0^\infty \frac{e^{ik^0(-\Delta t - r_1 + i\epsilon)}}{k^0} + \int_0^\infty \frac{e^{ik^0(\Delta t - r_1 + i\epsilon)}}{k^0}$$
$$= -\gamma_E - \ln[-\delta(\Delta t + r_1 - i\epsilon)] - \gamma_E - \ln[-\delta(-\Delta t + r_1 - i\epsilon)] + i\pi$$
$$= -\ln[\mu^2((r_1 - i\epsilon)^2 - \Delta t^2)] + \phi(\mu)^2,$$

where δ is small and only leading order is considered. Similarly, the other integral yields

$$-\ln[\mu^2((r_1 + i\epsilon)^2 - \Delta t^2)] + \phi(\mu)^2, \tag{2.8}$$

so that the full expression is now

$$-\frac{1}{4\pi}\Big\{\Theta(r_1)\ln[\mu^2((r_1 - i\epsilon)^2 - \Delta t^2)] + \Theta(-r_1)\ln[\mu^2((r_1 + i\epsilon)^2 - \Delta t^2)]\Big\} + \phi(\mu)^2.$$

Now to put it into the previous form, a simple identity is inserted

$$-\frac{1}{4\pi}\Big\{\Theta(r_1)[\Theta(\Delta t) + \Theta(-\Delta t)]\ln[\mu^2((r_1 - i\epsilon)^2 - \Delta t^2)] + \Theta(-r_1)[\Theta(\Delta t) + \Theta(-\Delta t)]\ln[\mu^2((r_1 + i\epsilon)^2 - \Delta t^2)]\Big\}$$
$$+ \phi(\mu)^2.$$

Remembering the role of the Theta functions for positive and negative r_1, the $i\epsilon$ prescription within the logs can be re-written as,

$$\frac{-1}{4\pi}\Big\{\Theta(\Delta t)[\Theta(r_1) + \Theta(-r_1)]\ln[\mu^2(r_1^2 - \Delta t^2 - i\epsilon)] + \Theta(-\Delta t)[\Theta(r_1) + \Theta(-r_1)]\ln[\mu^2(r_1^2 - \Delta t^2 + i\epsilon)]\Big\} + \phi(\mu)^2$$
$$= \frac{-1}{4\pi}\Big\{\Theta(\Delta t)\ln[\mu^2(r_1^2 - (\Delta t - i\epsilon)^2)] + \Theta(-\Delta t)\ln[\mu^2(r_1^2 - (\Delta t + i\epsilon)^2)]\Big\} + \phi(\mu)^2,$$

which is exactly the positive and negative Wightman functions as in equations 2.6 and 2.7. Together, they yield the Feynmann propagator as in equation 2.8.

2.3 Two-point functions in D dimensions

One will now find correlation functions for D dimensions. This means finding the expectation value for the product of two field operators at different positions in space and time. Here, time ordering is not included and will be dealt with later. In scalar field theory, this quantity is the amplitude for a particle to propagate from the coordinates x to x'.

$$\langle\phi(x)\phi(x')\rangle = \int \frac{d^{D-1}k}{(2\pi)^D}\phi(t,\vec{k})\phi^*(t',\vec{k})e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\langle\phi(x)\phi(x')\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{2\omega}e^{-i\omega(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\langle\phi(x)\phi(x')\rangle = \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{1}{2E_{\vec{p}}}e^{-i(E_{\vec{p}}t-\vec{p}\cdot\vec{x})}.$$

Using spherical coordinates:

$$\langle\phi(x)\phi(x')\rangle = \int_0^\infty dp\frac{p^{D-2}e^{-iE(t-t')}}{(2\pi)^{D-1}2E}\int d\Omega e^{ipr\cos\theta}.$$

One can do this calculation using the volume element of the D-2 dimensional sphere and associating the last angle to the one found in the exponential.

$$d\Omega = \sin^{D-3}(\theta_1)...\sin(\theta_{D-3})d\theta_1...d\theta_{D-3}d\phi$$

$$\begin{aligned}P = \int d\Omega e^{ipr\cos\theta_1} &= \int d\theta_1 \sin^{D-3}\theta_1 e^{ipr\cos\theta_1}\int d\Omega_{D-3} \\ &= \int_0^\pi d\theta_1\sin^{D-3}\theta_1 e^{ipr\cos\theta_1}\frac{2\pi^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)}.\end{aligned}$$

Now using (8.411.7) from [18]

$$\int_0^\pi e^{\pm iz\cos\phi}\sin^{2v}\phi d\phi = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(v+\frac{1}{2}\right)}{\left(\frac{z}{2}\right)^v}J_v(z)$$

$$P = (2\pi)^{\frac{D-1}{2}}(pr)^{\frac{3-D}{2}}J_{\frac{D-3}{2}}(pr).$$

This leaves one with a final integral over p such that:

$$\langle\phi(x)\phi(x')\rangle = \int_0^\infty dp \frac{p^{\frac{D-1}{2}}}{(2\pi)^{\frac{D-1}{2}}}(r)^{\frac{3-D}{2}}\frac{e^{-iE(t-t')}}{2E}J_{\frac{D-3}{2}}(pr),$$

which can be dealt with by firstly making a substitution.

$$x = \frac{E}{m} = \frac{\sqrt{p^2+m^2}}{m}$$

$$\langle\phi(x)\phi(x')\rangle = \frac{m^{\frac{D-1}{2}}r^{\frac{3-D}{2}}}{2(2\pi)^{\frac{D-1}{2}}}\int_1^\infty dx(x^2-1)^{\frac{D-3}{4}}e^{-im(t-t')x}J_{\frac{D-3}{2}}(mr\sqrt{x^2-1}).$$

Relation 6.645 of [18] states:

$$\int_1^\infty (x^2-1)^{\frac{1}{2}\nu}e^{-\alpha x}J_\nu(\beta\sqrt{x^2-1})dx = \sqrt{\frac{2}{\pi}}\beta^\nu(\alpha^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\nu+\frac{1}{2}}(\sqrt{\alpha^2+\beta^2}).$$

Taking an analytical extension of this one obtains:

$$\begin{aligned}&\int_1^\infty (x^2-1)^{\frac{1}{2}\nu}e^{-iax}J_\nu(b\sqrt{x^2-1})dx \\ &= \begin{cases}\sqrt{\frac{2}{\pi}}b^\nu(-a^2+b^2)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\nu+\frac{1}{2}}(\sqrt{-a^2+b^2}), & \text{if } b>a>0. \\ \sqrt{\frac{\pi}{2}}b^\nu\frac{(-i)^{2(\nu+1)}}{(\sqrt{a^2-b^2-i\epsilon})^{\nu+\frac{1}{2}}}H^{(2)}_{\nu+\frac{1}{2}}(\sqrt{a^2-b^2-i\epsilon}), & \text{if } a>b>0.\end{cases}\end{aligned} \tag{2.9}$$

Using this, the full Feynman propagator is given by:

$$\begin{aligned}\Delta_F(x,x') &= \langle T[\phi(x)\phi(x')]\rangle \\ &= \theta(\Delta t^2-r^2)\frac{(-i)^{D-1}m^{\frac{D-2}{2}}}{2^{\frac{D+2}{2}}\pi^{\frac{D-2}{2}}((|\Delta t|-i\epsilon)^2-r^2)^{\frac{D-2}{4}}}H^{(2)}_{\frac{D-2}{2}}(m\sqrt{(|\Delta t|-i\epsilon)^2-r^2}) \\ &+ \theta(-\Delta t^2+r^2)\frac{m^{\frac{D-2}{2}}}{(2\pi)^{\frac{D}{2}}(-(|\Delta t|-i\epsilon)^2+r^2)^{\frac{D-2}{4}}}K_{\frac{D-2}{2}}(m\sqrt{-(|\Delta t|-i\epsilon)^2+r^2})\end{aligned} \tag{2.10}$$

where the step function, θ, was introduced to separate the expression into two parts. To briefly examine the $i\epsilon$ prescription more carefully, one has the following shift in time:

$$
\begin{aligned}
-(|\Delta t| - i\epsilon)^2 + r^2 &= -\Delta t^2 + 2i|\Delta t|\epsilon + r^2 + \epsilon^2 \\
&= \Delta X^2(x, x').
\end{aligned}
$$

This is the invariant distance and will be used to express equation 2.10 in a more compact manner. This is the propagator for a free scalar field theory. This expression specifies the probability amplitude for a particle to travel from one point in space to another in a given time. This is true for the entire space, but what will become interesting in later sections is its restriction to different parts of the space. Although it is Lorentz invariant, one will consider scenarios when the space is split into separate sections. One may also write the limit of $D \to 2$, in this limit, one may write an expansion of the Bessel function of the second kind and obtain a log dependence. In the end, one can recover equation 2.8 in this manner.

2.4 Stress-energy tensor

This is a quantity that describes the density and flux of energy and momentum in a space time. It is the source of the gravitational field in Einstein's field equations for general relativity and is thus an important quantity to calculate, especially if one wishes to consider back reaction effects.

The tensor is written in the form $T^{\mu\nu}$ and gives the flux of the μth component of the momentum vector across a surface with constant X^ν coordinate. Here, the tensor was given in its contravariant form, with raised indices. In the following, it will be given in its covariant form, which is easily related by applying the metric twice.

$$
T_{\mu\nu} = g_{\alpha\mu} g_{\beta\nu} T^{\alpha\beta}. \tag{2.11}
$$

The stress-energy tensor is a conserved quantity, this means that its divergence must be zero. In flat space time, this means taking normal derivatives of the tensor gives zero. i.e.

$$
\partial_\mu T^{\mu\nu} = 0. \tag{2.12}
$$

More generally though, one will be working with curved spacetimes, such as the spherically symmetric black hole which will be discussed later. In this case, one must use a coordinate-free definition of the divergence. This makes use of the covariant derivative. This will be denoted as ∇_μ. Note that the stress-energy tensor, when performing coordinate transformations, x to x', it transforms as

$$
T^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} T_{\alpha\beta}(x).
$$

Thus finding the stress-energy tensor in the simple, flat Minkowski space will be useful when examining other spaces that are related by a coordinate transformation. This is the motivation for what follows. The action for a massive, non-minimally coupled, real scalar field in a curved background is given by:

$$S = -\frac{1}{2}\int d^D x\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2 + \xi R\phi^2\right]$$

[19] where there is a coupling ξ, which is a dimensionless quantity that accounts for the possible coupling between the scalar field and the gravitational background. The field equations are found as:

$$\frac{\delta S}{\delta\phi(x)} = \nabla^2\phi - m^2\phi - \xi R\phi = 0.$$

The stress-energy-momentum tensor is defined as the variation of the action with respect to the metric. One has:

$$\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} \\
\delta R &= -R^{\alpha\beta}\delta g_{\alpha\beta} + \nabla^\alpha\nabla^\beta(\delta g_{\alpha\beta}) - \nabla^\rho\nabla_\rho(g^{\alpha\beta}\delta g_{\alpha\beta}).
\end{aligned}$$

The stress-energy-momentum tensor is defined as [20]:

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}}.$$

Using the relations:

$$\begin{aligned}
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho_{\rho\mu}) \\
\delta\Gamma^\sigma_{\mu\nu} &= -\frac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right]
\end{aligned}$$

$$\begin{aligned}
T_{\mu\nu} =& (1-2\xi)\nabla_\mu\phi\nabla_\nu\phi + (2\xi - \frac{1}{2})g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - 2\xi\phi\nabla_\mu\nabla_\nu\phi \\
&+ 2\xi g_{\mu\nu}\phi\nabla^2\phi + \xi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\phi^2 - \frac{1}{2}g_{\mu\nu}m^2\phi^2.
\end{aligned}$$

Looking at the Minkowski case, this simplifies to:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 - \frac{1}{2}\eta_{\mu\nu}m^2\phi^2.$$

Now to take expectation value:

$$\langle T_{\mu\nu}\rangle = \langle T[\partial_\mu\phi(x)\partial_\nu\phi(x')]\rangle - \frac{1}{2}\eta_{\mu\nu}\langle T[(\partial\phi)^2]\rangle - \frac{1}{2}\eta_{\mu\nu}m^2\langle T[\phi(x)\phi(x')]\rangle,$$

where T is the time ordering operator. To take out the derivatives, one must define a new ordering operator that commutes with the derivatives, T^*.

$$\langle T_{\mu\nu}(x)\rangle = \lim_{x'\to x}\partial_\mu\partial'_\nu\langle T^*[\hat{\phi}(x)\hat{\phi}(x')]\rangle - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial'_\beta\langle T^*[\hat{\phi}(x)\hat{\phi}(x')]\rangle - \frac{1}{2}\eta_{\mu\nu}m^2\langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle.$$

Now one can use the following relation for the two different orderings of the fields:

$$\langle\phi(x)\phi(x')\rangle = \langle\phi(x')\phi(x)\rangle^*.$$

So now only the Feynman propagator is needed.

$$\langle T_{\mu\nu}\rangle = (\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta})\langle T[\partial_\alpha\phi(x)\partial_\beta\phi(x')]\rangle - \frac{1}{2}m^2\eta_{\mu\nu}\langle T[\phi(x)\phi(x')]\rangle.$$

Although neglected, one can include terms for non-minimal coupling of the field, here they are given by:

$$\langle T_{\mu\nu}\rangle_\xi = \left(-2\xi\delta^\rho_\mu\delta^\sigma_\nu + 2\xi\eta_{\mu\nu}\eta^{\rho\sigma}\right)\left(\langle\partial_\rho\phi\partial_\sigma\phi\rangle + \langle\phi\partial_\rho\partial_\sigma\phi\rangle\right).$$

Now using equation 2.10 for the propagator in D dimensions:

$$\Delta(x,x') = \frac{m^{\frac{D-2}{2}}}{(2\pi)^{\frac{D}{2}}(\Delta X^2)^{\frac{D-2}{4}}}K_{\frac{D-2}{2}}(m\sqrt{\Delta X^2})$$

Now, performing an expansion in order to isolate constant terms and terms of second order, as required for $\langle T_{\mu\nu}(x)\rangle$: $y = m\sqrt{\Delta X^2}$

$$\Delta(x,x') \approx \frac{m^{D-2}}{(2\pi)^{\frac{D}{2}}}\Big((m\sqrt{\Delta X^2})^{-D}(...)$$
$$+ 2^{-\frac{D}{2}}\Gamma\left(\frac{2-D}{2}\right) + \frac{2^{\frac{-2-D}{2}}}{D}\Gamma\left(\frac{2-D}{2}\right) m^2(\sqrt{\Delta X^2})^2 + O[(m\sqrt{\Delta X^2})^4]\Big)$$
$$= \frac{m^{D-2}}{(2\pi)^{\frac{D}{2}}}2^{-\frac{D}{2}}\Gamma\left(\frac{2-D}{2}\right) + \frac{m^D}{(2\pi)^{\frac{D}{2}}}2^{\frac{-D-2}{2}}\frac{1}{D}(\sqrt{\Delta X^2})^2\Gamma\left(\frac{2-D}{2}\right) + O[(m\sqrt{\Delta X^2})^4].$$

Since one is only interested in the limit of $X \to X'$ these are the only terms that matter.

$$\partial_\mu\partial'_\nu\langle T^*[\phi(x)\phi(x')]\rangle = -\frac{m^D}{\pi^{\frac{D}{2}}}2^{-D+2}\frac{1}{(D)(2-D)(4-D)}\Gamma\left(\frac{6-D}{2}\right)\eta_{\mu\nu}$$

$$\langle T_{\mu\nu}\rangle = -(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta})\frac{m^D}{\pi^{\frac{D}{2}}}2^{-D}\frac{1}{D}\Gamma\left(\frac{2-D}{2}\right)\eta_{\alpha\beta} - \frac{1}{2}m^2\eta_{\mu\nu}\frac{m^{D-2}}{\pi^{\frac{D}{2}}}2^{-D}\Gamma\left(\frac{2-D}{2}\right)$$
$$= -\eta_{\mu\nu}(1-\frac{1}{2}D)\frac{m^D}{\pi^{\frac{D}{2}}}2^{-D+2}\frac{1}{(D)(2-D)(4-D)}\Gamma\left(\frac{6-D}{2}\right) - \eta_{\mu\nu}m^D\frac{2^{-D+1}}{\pi^{\frac{D}{2}}}\frac{1}{(2-D)(4-D)}\Gamma\left(\frac{6-D}{2}\right),$$

where the Gamma functions were divergent for $D = 4$, but can be re-written using:

$$z\Gamma(z) = \Gamma(z+1),$$

now the divergences appear in the denominator for $D = 4$.

$$\langle T_{\mu\nu}\rangle = -\eta_{\mu\nu}m^D\pi^{\frac{-D}{2}}2^{-D+1}\frac{2}{D(2-D)(4-D)}\Gamma\left(\frac{6-D}{2}\right).$$

Performing an expansion about $D = 4$ yields a finite term and a divergent term. In the expansion, terms of the form $(D-4)^n$ where $n > 0$ are in fact zero.

$$\langle T_{\mu\nu}\rangle = -2\eta_{\mu\nu}m^D\pi^{\frac{-D}{2}}2^{-D+1}\left(-\frac{D+2}{8D(D-2)} + \frac{1}{8(D-4)}\right)\Gamma\left(\frac{6-D}{2}\right) + \mathcal{O}(D-4).$$

So there is a divergent term when $D = 4$ and some finite terms. Now, a counter term can be added to cancel this divergent part using minimal subtraction:

$$\langle T_{\mu\nu} \rangle + \langle T_{\mu\nu} \rangle_{C.T.} = -\eta_{\mu\nu} m^4 \pi^{-2} 2^{-6} \ln\left(\frac{m^2}{\mu^2}\right).$$

One can note that the non-minimal coupling terms would have canceled each other, as each of them involved taking derivatives, but in the coincidence limit, these terms would become the same, thus the previous expression for $\langle T_{\mu\nu} \rangle_\xi$ would vanish. As this is a general result for flat space time, one can transform this result using any coordinate transformation by the properties of the tensor outlined by equation 3.1. One may also comment on the type of counter term added, namely it is one that shifts the action as a cosmological constant. Meaning the action takes a counter term such that taking its variation with respect to the metric will yield a term with the opposite sign of the divergent term in the stress-energy tensor, thus cancelling. Even if this counter term is formally infinite.

$$\frac{2}{\sqrt{-g}} \frac{\delta S_{C.T}}{\delta g^{\mu\nu}} = g_{\mu\nu} \delta\Big(\frac{\Lambda}{8\pi G}\Big) = -2\eta_{\mu\nu} m^D \pi^{\frac{-D}{2}} 2^{-D+1} \left(\frac{1}{8(D-4)}\right) \Gamma\left(\frac{6-D}{2}\right)$$

$$S_{C.T} = \delta\Big(\frac{\Lambda}{8\pi G}\Big) = \eta_{\mu\nu} m^4 \pi^{-2} 2^{-5} \frac{1}{(D-4)}.$$

Chapter 3

Rindler

3.1 Massless case

One can now consider the massless scalar field in a new coordinate system. It is only the time and first spatial coordinate being transformed, such that all other coordinates are mutually orthogonal. This still has the effect of splitting the space into two different regions. Although it is not possible to picture this in D-dimensions, one can see it in 1+1 and 1+2 dimensions, see figure 5.1. For the case of 1+2 dimensions, only a cross section is given, but one can imagine placing this anywhere on the z axis to generate the full space. The important thing to note is that in the D-dimensional system, performing the Rindler transformation will always split the space into left and right parts, so that the entire Hilbert space will be the product of the left and right spaces. Taking D-dimensional space now and applying a uniform acceleration along a single axis:

$$t = \frac{\rho}{c}\sinh\left(\frac{\alpha\tau}{c}\right), \quad x^1 = \rho\cosh\left(\frac{\alpha\tau}{c}\right), \quad \vec{x} = (x^2, ..., x^{D-1}), \tag{3.1}$$

so that the massless wave equation is:

$$(\partial_\rho^2 + \rho^{-1}\partial_\rho + \vec{\partial}^2 - (\alpha\rho)^{-2}\partial_\tau^2)\hat{\phi} = 0.$$

As this is first the simple case, α is set to one, but is later left in the more general case. Solutions to this equation are given in the right and left Rindler wedge as:

$$\hat{\phi}^R(\rho, \tau, x) = \int_0^\infty d\omega \int d^{D-2}k N_{\omega k}[f_{\omega k}(\rho, \tau, x)\hat{a}^R_{\omega k} + h.c.]$$

and

$$\hat{\phi}^L(\rho, \tau, x) = \int_0^\infty d\omega \int d^{D-2}k N_{\omega k}[\tilde{f}_{\omega k}(\rho, \tau, x)\hat{a}^L_{\omega k} + h.c.]$$

with mode functions given by

$$f_{\omega k}(\rho,\tau,x) = K_{i\omega}(|k|\rho)e^{i(kx-\omega t)}, \quad \tilde{f}_{\omega k}(\rho,\tau,x) = K_{i\omega}(|k|\rho)e^{i(kx+\omega t)}$$

and normalisation of

$$N_{\omega k}^2 = (2\pi)^{D-2}\frac{\sinh(\pi\omega)}{\pi^2}.$$

For the purpose of computing correlation functions, it is useful to separate the product of fields into the commutator and anti commutator parts

$$\phi(\Xi)\phi(\Xi') = \frac{1}{2}[\phi(\Xi),\phi(\Xi')] + \frac{1}{2}\{\phi(\Xi),\phi(\Xi')\}.$$

The commutator is independent of the state, thus the anti-commutator is the interesting piece for computing the correlation functions. The coordinates of τ, ρ and x here are collectively denoted by Ξ.

$$\begin{aligned}\langle\Omega|\,\{\phi_R(\Xi),\phi_R(\Xi')\}\,|\Omega\rangle = \langle\Omega|&\int_0^\infty d\omega d\omega' \int d^{D-2}k d^{D-2}k' N_{\omega k}N_{\omega' k'}\left(f_{\omega k}(\Xi)a^R_{\omega k} + f^*_{\omega k}(\Xi)(a^R_{\omega k})^\dagger\right)\\ &\times\left(f_{\omega' k'}(\Xi')a^R_{\omega' k'} + f^*_{\omega' k'}(\Xi')(a^R_{\omega' k'})^\dagger\right)|\Omega\rangle\\ &+(\Xi\leftrightarrow\Xi').\end{aligned} \tag{3.2}$$

The terms which involve the product of creation or the product of annihilation operators do not contribute as the left and right occupation numbers must match. Now the mixed terms can be found using properties of the Bose-Einstein distribution, since $|\Omega\rangle$ is a thermal state in ω with $\beta = 2\pi$

$$\langle\Omega|\,a^\dagger_{k\omega}a_{k'\omega'}\,|\Omega\rangle = \frac{1}{e^{2\pi\omega}-1}\delta_{k,k'}\delta_{\omega,\omega'}, \quad \langle\Omega|\,a_{k\omega}a^\dagger_{k'\omega'}\,|\Omega\rangle = \frac{e^{2\pi\omega}}{e^{2\pi\omega}-1}\delta_{k,k'}\delta_{\omega,\omega'},$$

[21] which sum up to be $\coth(\pi\omega)\delta_{k,k'}\delta_{\omega,\omega'}$

$$\begin{aligned}&\langle\Omega|\,\{\phi_R(\Xi),\phi_R(\Xi')\}\,|\Omega\rangle\\ &=\int_0^\infty d\omega\int d^{D-2}k\coth(\pi\omega)N^2_{\omega k}[f_{\omega k}(\Xi)f^*_{\omega' k'}(\Xi') + f^*_{\omega k}(\Xi)f_{\omega' k'}(\Xi')] + (\Xi\leftrightarrow\Xi')\\ &=\int_0^\infty d\omega\int d^{D-2}k\frac{\cosh(\pi\omega)}{\pi^2(2\pi)^{D-2}}\left[e^{i[k(x-x')-\omega(\tau-\tau')]}K_{i\omega}(|k|\rho)K^*_{i\omega}(|k|\rho')\right] + (\Xi\leftrightarrow\Xi').\end{aligned}$$

[21] To evaluate this integral, one first considers it without the cosh function

$$\frac{1}{\pi^2}\int_0^\infty d\omega\int\frac{d^{D-2}k}{(2\pi)^{D-2}}\left[e^{i(k\Delta x-\omega\Delta\tau)}K_{i\omega}(|k|\rho)K^*_{i\omega}(|k|\rho')+(\Xi\leftrightarrow\Xi')\right]$$
$$\frac{1}{2\pi^2}\int_{-\infty}^\infty d\omega\int\frac{d^{D-2}k}{(2\pi)^{D-2}}\left[e^{i(k\Delta x-\omega\Delta\tau)}K_{i\omega}(|k|\rho)K^*_{i\omega}(|k|\rho')+(\Xi\leftrightarrow\Xi')\right].$$

Now, using an integral representation for the modified bessel function of the second kind, 8.432.6 [18]

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty dt\frac{1}{t^{\nu+1}}e^{-t-\frac{z^2}{4t}},\quad |\arg(z)|<\frac{\pi}{2},\quad Re(z^2)>0.$$

Using this, the product of the Bessels becomes

$$K_{i\omega}(|k|\rho)K^*_{i\omega}(|k|\rho')=\frac{1}{4}\int_0^\infty dudu'\left(\frac{\rho u'}{\rho' u}\right)^{i\omega}\frac{1}{uu'}e^{-(u+u')}e^{-\frac{k^2}{4}\left(\frac{\rho^2}{u}+\frac{\rho'^2}{u'}\right)}.$$

So now the relevant integral to be evaluated is given by

$$\begin{aligned}
&\frac{1}{8\pi^2}\int_{-\infty}^\infty d\omega\int\frac{d^{D-2}k}{(2\pi)^{D-2}}e^{i(k\Delta x-\omega\Delta\tau)}\int_0^\infty dudu'\left(\frac{\rho u'}{\rho' u}\right)^{i\omega}\frac{1}{uu'}e^{-(u+u')}e^{-\frac{k^2}{4}\left(\frac{\rho^2}{u}+\frac{\rho'^2}{u'}\right)}\\
&=\frac{1}{8\pi^2}\int\frac{d^{D-2}k}{(2\pi)^{D-3}}e^{ik\Delta x}\int_0^\infty dudu'\frac{1}{uu'}e^{-(u+u')}e^{-\frac{k^2}{4}\left(\frac{\rho^2}{u}+\frac{\rho'^2}{u'}\right)}\delta\left(\log\frac{\rho u'}{\rho' u}-\Delta\tau\right)\\
&=\frac{1}{8\pi^2}\int\frac{d^{D-2}k}{(2\pi)^{D-3}}\int_0^\infty du\frac{\rho}{\rho' u^2}e^{-\Delta\tau}\frac{\rho' u}{\rho}e^{\Delta\tau}e^{-u\left(1+\frac{\rho'}{\rho}e^{\Delta\tau}\right)}e^{-\frac{\rho^2k^2}{4u}\left(1+\frac{\rho'}{\rho}e^{-\Delta\tau}\right)}e^{ik\Delta x}\\
&=\frac{1}{8\pi^2}\int\frac{d^{D-2}k}{(2\pi)^{D-3}}\int_0^\infty du\frac{1}{u}e^{-u\left(1+\frac{\rho'}{\rho}e^{\Delta\tau}\right)}e^{-\frac{\rho^2k^2}{4u}\left(1+\frac{\rho'}{\rho}e^{-\Delta\tau}\right)}e^{ik\Delta x}\\
&=\frac{1}{8\pi^2}\frac{(2\pi)^{\frac{D-2}{2}}}{(2\pi)^{D-3}}\int_0^\infty du\frac{1}{u}e^{-u\left(1+\frac{\rho'}{\rho}e^{\Delta\tau}\right)}e^{\frac{-u\Delta x^2}{\rho^2+\rho\rho' e^{-\Delta\tau}}}\left(\frac{2u}{\rho^2+\rho\rho' e^{-\Delta\tau}}\right)^{\frac{D-2}{2}}\\
&=\frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{\frac{D}{2}}}(\rho^2+\rho'^2-2\rho\rho'\cosh(\Delta\tau)+\Delta x^2)^{\frac{2-D}{2}}.
\end{aligned}$$

To perform this integral, one first computes the integral over ω. Then, using properties of the delta function, the integral over u'. This then leaves a $D-2$ dimensional Gaussian integral over k. Finally, the integral over u can be found.

The two-sided correlator is given by

$$\langle\Omega|\,\phi_R(\Xi)\phi_L(\Xi')\,|\Omega\rangle = \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{\frac{D}{2}}}(\rho^2+\rho'^2+2\rho\rho'\cosh(\Delta\tau)+\Delta x^2)^{\frac{2-D}{2}} \tag{3.3}$$

while the one-sided Wightman function is given by

$$\begin{aligned} & i\Delta_{RR}(x,x') = \\ & \langle\Omega|\,\phi_R(\Xi)\phi_R(\Xi')\,|\Omega\rangle = \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{\frac{D}{2}}}(\rho^2+\rho'^2-2\rho\rho'\cosh(\Delta\tau-i\epsilon)+\Delta x^2)^{\frac{2-D}{2}}. \end{aligned} \tag{3.4}$$

3.2 Massive scalar fields

The previous calculation may now be done including a mass for the scalar field. Starting with the wave equation, which is given by the expression:

$$(\Box+m^2)\phi = 0.$$

Doing the Rindler transformations for the time and first spatial component, the differential operators are given in the form of:

$$\partial_x^2 = \frac{\partial^2\rho}{\partial x^2}\partial_\rho + \left(\frac{\partial\rho}{\partial x}\right)^2\partial_\rho^2 + \frac{\partial^2\tau}{\partial x^2}\partial_\tau + \left(\frac{\partial\tau}{\partial x}\right)^2\partial_\tau^2 + 2\frac{\partial\rho}{\partial x}\frac{\partial\tau}{\partial x}\partial_\rho\partial_\tau$$

$$\partial_t^2 = \frac{\partial^2\rho}{\partial t^2}\partial_\rho + \left(\frac{\partial\rho}{\partial t}\right)^2\partial_\rho^2 + \frac{\partial^2\tau}{\partial t^2}\partial_\tau + \left(\frac{\partial\tau}{\partial t}\right)^2\partial_\tau^2 + 2\frac{\partial\rho}{\partial t}\frac{\partial\tau}{\partial t}\partial_\rho\partial_\tau.$$

This gives the equation,

$$\left[\frac{1}{\rho}\partial_\rho + \partial_\rho^2 - \frac{1}{\alpha^2\rho^2}\partial_\tau^2 + \vec{\partial}^2 + m^2\right]\phi = 0. \tag{3.5}$$

Next, one must Fourier transform all but the ρ coordinate so the differential operators may be re-written.

$$\phi(\tau,\rho,x_i) = \int e^{i(k^i x_i-\omega\tau)}\phi(\omega,\rho,k_i)d\omega d^{D-2}k$$

$$\left[\rho^2\partial_\rho^2 + \rho\partial_\rho + \frac{1}{\alpha^2}\omega^2 + \rho^2(-k_ik^i + m^2)\right]\phi = 0.$$

Once again, substituting to obtain the form of the modified Bessel equation

$$x = i\rho\sqrt{-k_ik^i + m^2}$$

$$\left[x^2\partial_x^2 + x\partial_x - \left(x^2 - \frac{\omega^2}{\alpha^2}\right)\right]\phi = 0$$

the solutions to this are given by

$$\phi = c_1 I_n(x) + c_2 K_n(x)$$

where n is ω/α.

$$\phi = c_1 I_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i - m^2}) + c_2 K_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i - m^2}).$$

One considers the latter linearly independent solution as it is finite as $\omega \to \infty$. Transforming back to the real space

$$\phi(\tau, \rho, x_i) = \int d\omega \int d^{D-2}k e^{i(k_ix^i - \omega\tau)} K_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i - m^2}).$$

Solutions of the massive wave equation are of the form

$$f_{R/L\omega k} = e^{-i\omega\tau_{R/L}} e^{i\vec{k}\cdot\vec{y}} K_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i - m^2})$$

where $\omega > 0$.

The field may now be expanded as

$$\hat{\phi} = \sum_{k,\omega}(f_{R\omega k}\hat{a}_{R\omega k} + f_{L\omega k}\hat{a}_{L\omega k} + f^*_{R\omega k}\hat{a}^\dagger_{R\omega k} + f^*_{L\omega k}\hat{a}^\dagger_{L\omega k})$$

where the right and left modes for the Rindler wedge can be written separately.

$$
\begin{aligned}
\hat{\phi}^R &= \int_0^\infty d\omega \int d^{D-2}k N_{\omega k}[f_{R\omega k}\hat{a}_{R\omega k} + f^*_{R\omega k}\hat{a}^\dagger_{R\omega k}] \\
\hat{\phi}^L &= \int_0^\infty d\omega \int d^{D-2}k N_{\omega k}[f_{L\omega k}\hat{a}_{L\omega k} + f^*_{L\omega k}\hat{a}^\dagger_{L\omega k}]
\end{aligned} \tag{3.6}
$$

where $N_{\omega k}$ is a normalisation constant such that the mode functions are orthonormal.

$$
N^2_{\omega k} = (2\pi)^{d-2}\frac{\sinh\left(\pi\frac{\omega}{\alpha}\right)}{\pi^2}.
$$

First start with the one sided Wightman function, i.e. the correlation function for the right Rindler wedge.

$$
\begin{aligned}
\langle\Omega|\{\phi_R(\Xi),\phi_R(\Xi')\}|\Omega\rangle = \langle\Omega| &\int_0^\infty d\omega d\omega' \int d^{D-2}k d^{D-2}k' N_{\omega k}N_{\omega' k'}\left(f_{R\omega k}(\Xi)\hat{a}_{R\omega k} + f^*_{R\omega k}(\Xi)(\hat{a}_{R\omega k})^\dagger\right) \\
&\times \left(f_{R\omega' k'}(\Xi')\hat{a}_{R\omega' k'} + f^*_{R\omega' k'}(\Xi')(\hat{a}_{R\omega' k'})^\dagger\right)|\Omega\rangle \\
&+ (\Xi \leftrightarrow \Xi').
\end{aligned} \tag{3.7}
$$

The aa and $a^\dagger a^\dagger$ terms do not contribute as the left and right occupation numbers will not match, thus this will yield zero. While the $aa^\dagger$ and $a^\dagger a$ terms will yield similar expressions as equation 3.3:

$$
\langle\Omega|\hat{a}^\dagger_{\omega k}\hat{a}_{\omega' k'}|\Omega\rangle = \frac{1}{e^{\beta\omega}-1}\delta_{k,k'}\delta_{\omega\omega'}
$$

$$
\langle\Omega|\hat{a}_{\omega k}\hat{a}^\dagger_{\omega' k'}|\Omega\rangle = \frac{e^{\beta\omega}}{e^{\beta\omega}-1}\delta_{k,k'}\delta_{\omega\omega'}
$$

where β is the inverse temperature, but can be expressed in terms of the acceleration as $\beta = 2\pi/\alpha$. $\langle\Omega|$ denotes the Minkowski thermal state. This can be summed to give

$$
\coth\left(\frac{\beta\omega}{2}\right) = \coth\left(\pi\frac{\omega}{\alpha}\right)
$$

so that this can be used to express the one sided Wightman function as

$$
\begin{aligned}
\langle\Omega|\{\phi_R(\Xi),\phi_R(\Xi')\}|\Omega\rangle = &\int_0^\infty d\omega \int \frac{d^{D-2}k}{(2\pi)^{D-2}}\frac{\cosh\left(\pi\frac{\omega}{\alpha}\right)}{\pi^2}e^{i(k_i\Delta x^i - \omega\Delta\tau)} \\
&\times K_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i - m^2})K^*_{i\frac{\omega}{\alpha}}(\rho'\sqrt{k_ik^i - m^2}) + (\Xi \leftrightarrow \Xi').
\end{aligned} \tag{3.8}
$$

The cosh function can be expressed in terms of exponentials, which can then be grouped with $\Delta\tau$. The modified Bessel functions can then be written in their integral representations so that

$$\langle\Omega|\left\{\phi_R(\Xi),\phi_R(\Xi')\right\}|\Omega\rangle = \int_0^\infty d\omega \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \frac{e^{\pi\frac{\omega}{\alpha}}+e^{-\pi\frac{\omega}{\alpha}}}{2\pi^2} e^{i(k_i\Delta x^i-\omega\Delta\tau)}$$
$$\times \int du du' \frac{1}{4}\Big(\frac{\rho u'}{\rho' u}\Big)^{i\frac{\omega}{\alpha}} \frac{e^{-(u+u')}}{uu'} e^{-\frac{(k_ik^i-m^2)}{4}\left(\frac{\rho^2}{u}+\frac{\rho'^2}{u'}\right)} + (\Xi\leftrightarrow\Xi'). \tag{3.9}$$

This is similar to the integral computed before, but now has a factor of $e^{\pm\pi\frac{\omega}{\alpha}}$. This will be regulated with a factor of $i\epsilon$ to make it convergent. The integral is in fact similar to the massless case until the last integral where now one is left with

$$\frac{1}{4\pi^2}(\rho^2-\rho\rho' e^{-\alpha(\Delta\tau-i\epsilon)})^{\frac{2-D}{2}} \int_0^\infty du u^{\frac{D-4}{2}} e^{uA+\frac{B}{u}}$$

$$A = \frac{-\rho^2-\rho'^2-\Delta x^2+2\rho\rho'\cosh(\alpha(\Delta\tau-i\epsilon))}{\rho^2-\rho\rho' e^{-\alpha(\Delta\tau-i\epsilon)}}$$

$$B = \frac{m^2}{4}(\rho^2-\rho\rho' e^{-\alpha(\Delta\tau-i\epsilon)}).$$

One can now compute this integral to find

$$\langle\Omega|\,\phi_R(\Xi),\phi_R(\Xi')\,|\Omega\rangle =$$
$$\frac{1}{(2\pi)^{\frac{D}{2}}}\Big(\rho^2+\rho'^2+\Delta x^2-2\rho\rho'\cosh\Big(\alpha(\Delta\tau-i\epsilon)\Big)\Big)^{\frac{2-D}{4}} i m^{\frac{D-2}{2}}$$
$$\times K_{\frac{2-D}{2}}\Big(m\Big[-\rho^2-\rho'^2-\Delta x^2+2\rho\rho'\cosh\Big(\alpha(\Delta\tau-i\epsilon)\Big)\Big]^{\frac{1}{2}}\Big). \tag{3.10}$$

Or, written in terms of the temperature

$$\langle\Omega|\,\phi_R(\Xi),\phi_R(\Xi')\,|\Omega\rangle =$$
$$\frac{1}{(2\pi)^{\frac{D}{2}}}\Big(\rho^2+\rho'^2+\Delta x^2-2\rho\rho'\cosh\Big(\frac{2\pi}{\beta}(\Delta\tau-i\epsilon)\Big)\Big)^{\frac{2-D}{4}} i m^{\frac{D-2}{2}}$$
$$\times K_{\frac{2-D}{2}}\Big(m\Big[-\rho^2-\rho'^2-\Delta x^2+2\rho\rho'\cosh\Big(\frac{2\pi}{\beta}(\Delta\tau-i\epsilon)\Big)\Big]^{\frac{1}{2}}\Big).$$

This is how things perform for half of the space, but the comparrison to the Minkowski case is when the entire space is considered.

3.3 Whole space

Now to compute the correlation function for the entire space, one uses the solutions for the left and right Rindler wedges.

$$\begin{aligned}\langle\Omega|\left\{\phi_R(\Xi),\phi_L(\Xi')\right\}|\Omega\rangle = \langle\Omega|\int_0^\infty d\omega d\omega'\int d^{D-2}kd^{D-2}k'\left(f_{R\omega k}(\Xi)\hat{a}_{R\omega k}+f^*_{R\omega k}(\Xi)(\hat{a}_{R\omega k})^\dagger\right)\\ \times\left(f_{L\omega'k'}(\Xi')\hat{a}_{L\omega'k'}+f^*_{L\omega'k'}(\Xi')(\hat{a}_{L\omega'k'})^\dagger\right)|\Omega\rangle\\ +(\Xi\leftrightarrow\Xi').\end{aligned}\tag{3.11}$$

From the appendix of [21], the operators between the states are given by

$$\langle\Omega|\hat{a}_{R\omega k}\hat{a}_{L\omega'k'}|\Omega\rangle=\frac{1}{2}\operatorname{csch}\left(\pi\frac{\omega}{\alpha}\right)\delta_{\omega\omega'}\delta_{k(-k)'}=\Big(\langle\Omega|\hat{a}^\dagger_{R\omega k}\hat{a}^\dagger_{L\omega'k'}|\Omega\rangle\Big)^\dagger.$$

So the two sided anticommutator correlator is given by

$$\begin{aligned}\langle\Omega|\left\{\phi_R(\Xi),\phi_L(\Xi')\right\}|\Omega\rangle=\int_0^\infty d\omega\int\frac{d^{D-2}k}{(2\pi)^{D-2}}\frac{1}{\pi^2}e^{i(k\Delta x-\omega(\Delta\tau-i\epsilon))}\\ \times K_{i\frac{\omega}{\alpha}}(\rho\sqrt{k_ik^i-m^2})K^*_{i\frac{\omega}{\alpha}}(\rho'\sqrt{k_ik^i-m^2})+(\Xi\leftrightarrow\Xi')\end{aligned}\tag{3.12}$$

$$\begin{aligned}&\langle\Omega|\,\phi_R(\Xi),\phi_L(\Xi')\,|\Omega\rangle=\\ &\frac{1}{(2\pi)^{\frac{D}{2}}}(\rho^2+\rho'^2+\Delta x^2+2\rho\rho'\cosh(\alpha(\Delta\tau-i\epsilon)))^{\frac{2-D}{4}}m^{\frac{D-2}{2}}\\ &\times K_{\frac{2-D}{2}}\left(m(\rho^2+\rho'^2+\Delta x^2+2\rho\rho'\cosh(\alpha(\Delta\tau-i\epsilon)))^{\frac{1}{2}}\right)\end{aligned}\tag{3.13}$$

which is indeed the same as taking the correlator from the Minkowski case and performing the coordinate transformation to Rindler coordinates. This is expected from using scalar fields as they are invariant under coordinate transformations. Indeed, the two point function is a bi-scalar and a function of the invariant distance. One could now compute the stress energy tensor and find that it is what to expected from transforming coordinates, namely, $\eta_{\mu\nu}\to g_{\mu\nu}$. As this is a bi-scalar, insertions in different wedges are spacelike separate. This also implies that the corresponding commutator vanishes as required by causality.

3.4 Left wedge

Finally, an analogous calculation on the left wedge,

$$\begin{aligned}\langle 0|_M \hat{a}^\dagger_{R\omega k}\hat{a}_{R\omega' k'}|0\rangle_M &= \langle 0|_M \hat{a}^\dagger_{L\omega k}\hat{a}_{L\omega' k'}|0\rangle_M \\ &= (e^{\beta\omega}-1)^{-1}\delta_{kk'}\delta_{\omega\omega'}.\end{aligned} \tag{3.14}$$

These relations are most easily shown by expressing the Rindler operators in terms of the Minkowski ones such as

$$\hat{a}_{R\omega k} = \frac{\hat{b}_{-\omega k} + e^{-\pi\frac{\omega}{\alpha}}\hat{b}^\dagger_{\omega -k}}{\sqrt{1-e^{-2\pi\frac{\omega}{\alpha}}}}, \quad \hat{a}_{L\omega k} = \frac{\hat{b}_{\omega k} + e^{-\pi\frac{\omega}{\alpha}}\hat{b}^\dagger_{-\omega -k}}{\sqrt{1-e^{-2\pi\frac{\omega}{\alpha}}}}.$$

This ultimately yields precisely the same expression as the case of the right Rindler wedge.

3.5 Operators

This section is to show how the creation and annihilation operators act on different states and how the the Minkowski and Rindler operators are related by Bogoliubov transformations following [22] and [23].

Firstly, the free field operator can be written in the Minkowski and Rindler frame such that

$$\hat{\phi}(t,x,x_i) = \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega t+ikx+ik_ix^i}\hat{a}_\omega + e^{i\omega t-ikx-ik_ix^i}\hat{a}^\dagger_\omega\right]$$

$$\hat{\phi}(\tau,\rho,x_i) = \int_{-\infty}^{\infty}\frac{d\Omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\Omega}}\left[e^{-i\Omega\tau+ik\rho+ik_ix^i}\hat{b}_\Omega + e^{i\Omega\tau-ik\rho-ik_ix^i}\hat{b}^\dagger_\Omega\right]$$

re-written in light-cone coordinates, one can make the useful relation

$$\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega\bar{u}(u)}\hat{a}_\omega + e^{i\omega\bar{u}(u)}\hat{a}^\dagger_\omega\right] = \int_{-\infty}^{\infty}\frac{d\Omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\Omega}}\left[e^{-i\Omega u}\hat{b}_\Omega + e^{i\Omega u}\hat{b}^\dagger_\Omega\right].$$

Fourier transforming the left and right sides of this equation yields the Bogoliubov transformation,

$$\hat{b}_\Omega = \int_0^\infty d\omega[\alpha_{\omega\Omega}\hat{a}_\omega + \beta_{\omega\Omega}\hat{a}^\dagger_\omega] \quad \text{where} \begin{cases}\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}F(\omega,\Omega) \\ \beta\omega\Omega = \sqrt{\frac{\Omega}{\omega}}F(-\omega,\Omega)\end{cases} \quad \omega,\Omega > 0$$

with

$$F(\omega,\Omega) = \int_{\infty}^{\infty} \frac{du}{2\pi} e^{i\Omega u + i\frac{\omega}{a}e^{-au}}.$$

Now recalling the number operator

$$\begin{aligned}
\langle \hat{N} \rangle &= \langle 0|_M \hat{b}^\dagger_\Omega \hat{b}_\Omega |0\rangle_M \\
&= \langle 0|_M \int d\omega d\omega' (\alpha^*_{\omega'\Omega}\hat{a}^\dagger_{\omega'} + \beta^*_{\omega'\Omega}\hat{a}_{\omega'})(\alpha_{\omega\Omega}\hat{a}_\omega + \beta_{\omega\Omega}\hat{a}^\dagger_{\omega'}) |0\rangle_M \\
&= \int d\omega |\beta_{\omega\Omega}|^2 \\
&= \int d\omega e^{\frac{2\pi}{a}\Omega} \\
&= (e^{\beta\omega} - 1)^{-1}.
\end{aligned}$$

Also, the relation can be seen from how the Rindler creation and annihilation operators act on the Minkowski vacuum:

$$(\hat{a}_{R\omega} - e^{-\pi\frac{\omega}{\alpha}}\hat{a}^\dagger_{L\omega}) |0\rangle_M = 0$$

$$(\hat{a}_{L\omega} - e^{-\pi\frac{\omega}{\alpha}}\hat{a}^\dagger_{R\omega}) |0\rangle_M = 0$$

where $\hat{a}_{R\omega}$ and $\hat{a}_{L\omega}$ obey the commutation relations

$$[\hat{a}^\dagger_{R\omega}, \hat{a}_{R\omega'}] = [\hat{a}^\dagger_{L\omega}, \hat{a}_{L\omega'}] = \delta_{\omega\omega'}.$$

With all other commutations vanishing, this relation and how the the Minkowski vacuum are annihilated leads to

$$\langle 0|_M \hat{a}^\dagger_{R\omega}\hat{a}_{R\omega} |0\rangle_M = e^{-2\pi\frac{\omega}{\alpha}} \langle 0|_M \hat{a}^\dagger_{L\omega}\hat{a}_{L\omega} |0\rangle_M + e^{-2\pi\frac{\omega}{\alpha}}$$

and similarly for $R \leftrightarrow L$, this then leads to

$$\langle 0|_M \hat{a}^\dagger_{R\omega}\hat{a}_{R\omega} |0\rangle_M = \langle 0|_M \hat{a}^\dagger_{L\omega}\hat{a}_{L\omega} |0\rangle_M = (e^{2\pi\frac{\omega}{\alpha}} - 1)^{-1}.$$

[13] From the previous expressions, one can write the Rindler operators in terms of the Minkowski ones as follows

$$\hat{a}_{R\omega k} = \frac{\hat{b}_{-\omega k} + e^{-\pi\frac{\omega}{\alpha}}\hat{b}^{\dagger}_{\omega -k}}{\sqrt{1-e^{-2\pi\frac{\omega}{\alpha}}}}, \quad \hat{a}_{L\omega k} = \frac{\hat{b}_{\omega k} + e^{-\pi\frac{\omega}{\alpha}}\hat{b}^{\dagger}_{-\omega -k}}{\sqrt{1-e^{-2\pi\frac{\omega}{\alpha}}}},$$

where the operator $\hat{b}_{\pm\omega k}$ annihilates the Minkowski vacuum and obeys standard commutation relations with $\hat{b}^{\dagger}_{\pm\omega k}$.

Thus, in this chapter one has shown the equivalence between the Minkowski and Rindler coordinates. Namely that the two point function is a bi-scalar and is a function of the invariant distance between two points in spacetime. One may perform a simple coordinate transformation and this quantity transforms trivially as a scalar. However, this section demonstrates how this is done by performing explicit calculations. That is to say starting with the wave equation and finding solutions for this, while showing how the Minkowski vacuum state transforms to a thermal state in the Rindler frame.

Chapter 4

Density Matrix

Firstly, one will provide some motivation for studying the density matrix. Later, one will attempt to calculate physical quanties in a mixed system, this is what the Gaussian density matrix is useful for. For example, a physical observable $\hat{\mathcal{O}}$ is given by:

$$\langle \hat{\mathcal{O}} \rangle = Tr[\hat{\rho}\hat{\mathcal{O}}]$$

thus, one will use this property to write a density matrix in terms of the previous two point functions. It is only then that one may consider the case of region I, or the right Rindler wedge.

Now one will find the three Gaussian correlators in terms of the functions that compose the density matrix. This will be useful later; it is these same functions that compose the reduced density matrix.

To find the Gaussian correlator relations for fields, one must take care in the manipulation of the operators.

It must be noted that the correlators here are in fact functions of distance such that they are functions of $|\vec{x} - \vec{y}|$, not of $\vec{x}$ and $\vec{y}$ explicitly. This means that the functions A, B and C are functions of this variable as well. Hence there is a symmetry between $\vec{x}$ and $\vec{y}$. This is used below, but not written in this form.

4.1 Position Position correlator

The first relation calculated will be for the expectation value of the the fields at two different positions in space at the same point in time. Working in the Schrödinger picture, one writes

$$
\begin{aligned}
&\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle = tr(\hat{\rho}\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})) \\
&= \int \mathcal{D}\psi \langle\psi|\hat{\rho}\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}a \int \mathcal{D}b \langle\psi|a\rangle\rho(a,b;t)\langle b|\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}b\rho(t;\psi,b)\langle b|\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}b\rho(t;\psi,b)b^{\dagger}(\vec{x})\psi(\vec{y})\langle b|\psi\rangle \\
&= \int \mathcal{D}\psi\rho(\psi,\psi;t)\psi^{\dagger}(\vec{x})\psi(\vec{y}).
\end{aligned} \tag{4.1}
$$

This is now the form of a Gaussian integral and will be dealt with by introducing source terms.

$$
\int \mathcal{D}x e^{-x\cdot A\cdot x + J\cdot x} = (\det(A))^{-\frac{1}{2}} e^{\frac{1}{4}J\cdot A^{-1}\cdot J}.
$$

Taking two variations of this with respect to the source leads to

$$
\begin{aligned}
&\int \mathcal{D}\psi\psi(\vec{x})\psi(\vec{y})e^{-\int \psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int J(\vec{x}'\psi(\vec{x}')} \\
&= (\det(D))^{-\frac{1}{2}}\frac{1}{2}D^{-1}(t;\vec{x},\vec{y})e^{\frac{1}{4}\int J(\vec{x}')D^{-1}(t;\vec{x}',\vec{y}')J(\vec{y}')} + \dots \quad .
\end{aligned}
$$

This means that the two point function is given by

$$
\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle = \frac{1}{2}(A+B-2C)^{-1}(t;\vec{x},\vec{y}) \equiv \frac{1}{2}D^{-1}(t;\vec{x},\vec{y}).
$$

4.2 Momentum-Position correlator

Considering two different orthogonal states for the density matrix to have, as it represents a mixed state.

$$
\begin{aligned}
&\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle = tr(\hat{\rho}\hat{\Pi}(\vec{x})\hat{\phi}(\vec{y})) \\
&= \int \mathcal{D}\psi \langle\psi|\hat{\rho}\hat{\Pi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}a \int \mathcal{D}b \langle\psi|a\rangle \rho(a,b;t)\langle b|\hat{\Pi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}b \rho(t;\psi,b)\langle b|\hat{\Pi}(\vec{x})\hat{\phi}(\vec{y})|\psi\rangle \\
&= \int \mathcal{D}\psi \int \mathcal{D}b \rho(t;\psi,b)\psi(\vec{y})\langle b|\hat{\Pi}(\vec{x})|\psi\rangle \\
&= -i\hbar \int \mathcal{D}\psi \int \mathcal{D}b \rho(t;\psi,b)\psi(\vec{y})\frac{\delta}{\delta b(t,\vec{x})}\langle b|\psi\rangle \\
&= i\hbar \int \mathcal{D}\psi \int \mathcal{D}b \frac{\delta}{\delta b(t,\vec{x})}\Big(\rho(t;\psi,b)\psi(\vec{y})\Big)\langle b|\psi\rangle \\
&= i\hbar N(t) \int \mathcal{D}\psi \int \mathcal{D}b \frac{\delta}{\delta b(t,\vec{x})}\Big(- \int dx''dy''\psi(\vec{x}'')A(t;\vec{x}'',\vec{y}'')\psi(\vec{y}'') \\
&- \int dx''dy'' b(\vec{x}'')B(t;\vec{x}'',\vec{y}'')b(\vec{y}'') + 2\int dx''dy''\psi(\vec{x}'')C(t;\vec{x}'',\vec{y}'')b(\vec{y}'')\Big) \\
&\times \psi(\vec{y})e^{-\int dx'dy'\psi(\vec{x}')A(t;\vec{x}',\vec{y}')\psi(\vec{y}') - \int dx'dy' b(\vec{x}')B(t;\vec{x}',\vec{y}')b(\vec{y}') + 2\int dx'dy'\psi(\vec{x}')C(t;\vec{x}',\vec{y}')b(\vec{y}')}\langle b|\psi\rangle \\
&= i\hbar N(t) \int \mathcal{D}\psi \int \mathcal{D}b\psi(\vec{y})\Big(- \int dy'' B(t;\vec{x},\vec{y}'')b(\vec{y}'') - \int dx'' b(\vec{x}'')B(t;\vec{x}'',\vec{x}) + 2\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{x})\Big) \\
&\times e^{-\int dx'dy'\psi(\vec{x}')A(t;\vec{x}',\vec{y}')\psi(\vec{y}') - \int dx'dy' b(\vec{x}')B(t;\vec{x}',\vec{y}')b(\vec{y}') + 2\int dx'dy'\psi(\vec{x}')C(t;\vec{x}',\vec{y}')b(\vec{y}')}\langle b|\psi\rangle \\
&= i\hbar N(t) \int \mathcal{D}\psi\psi(\vec{y})\Big(- \int dy'' B(t;\vec{x},\vec{y}'')\psi(\vec{y}'') - \int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{x}) + 2\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{x})\Big) \\
&\times e^{-\int dx'dy'\psi(\vec{x}')A(t;\vec{x}',\vec{y}')\psi(\vec{y}') - \int dx'dy'\psi(\vec{x}')B(t;\vec{x}',\vec{y}')\psi(\vec{y}') + 2\int dx'dy'\psi(\vec{x}')C(t;\vec{x}',\vec{y}')\psi(\vec{y}')}.
\end{aligned}
$$

Here, one has used that one state acting on another will yield a product of delta functions for the fields at every point, this can then be easily integrated. This expression is an awkward integral to take, but will be dealt with by introducing source terms and setting them to zero in the final expression. Now the exponential itself is an issue as there are three distinct terms, however it is possible to utilise the fact that all the matrices are symmetric.

$$
\begin{aligned}
&\int dx'dy'\psi(\vec{x}')A(t;\vec{x}',\vec{y}')\psi(\vec{y}') = \psi^T \cdot A \cdot \psi \\
&= \sum_j \lambda_{Aj} y_j^2,
\end{aligned}
$$

where $y = \mathcal{O}^T\psi$. This means the exponential can be written in the following way.

$$
\begin{aligned}
&e^{-\int dx'dy'\psi(\vec{x}')A(t;\vec{x}',\vec{y}')\psi(\vec{y}')-\int dx'dy'\psi(\vec{x}')B(t;\vec{x}',\vec{y}')\psi(\vec{y}')+2\int dx'dy'\psi(\vec{x}')C(t;\vec{x}',\vec{y}')\psi(\vec{y}')} = e^{\sum_j(-\lambda_{Aj}-\lambda_{Bj}+2\lambda_{Cj})y_j^2} \\
&= e^{-\sum_j \lambda_j y_j^2} \\
&= e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}.
\end{aligned}
$$

Now one has the expression

$$
\begin{aligned}
&\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle = \\
&i\hbar N(t)\int \mathcal{D}\psi\psi(\vec{y})\Big(-\int dy''B(t;\vec{x},\vec{y}'')\psi(\vec{y}'')-\int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{x})+2\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{x})\Big) \\
&\times e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}
\end{aligned}
$$

Each of these terms can be dealt with separately and then recombined in the end.

$$
\begin{aligned}
&-\int \mathcal{D}\psi\psi(\vec{y})\int dy''B(t;\vec{x},\vec{y}'')\psi(\vec{y}'')e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')} \\
&-\int \mathcal{D}\psi\psi(\vec{y})\int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{x})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')} \\
&2\int \mathcal{D}\psi\psi(\vec{y})\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{x})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}.
\end{aligned}
$$

These integrals can all be done by performing integrals of a gaussian form, taking variations and then multiplying the result by the relevant matrix:

$$
\begin{aligned}
&\frac{\delta}{\delta J(\vec{y})}\Big\{\int \mathcal{D}\psi e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int dx'J(\vec{x}')\psi(\vec{x}')}\Big\} \\
&= \int \mathcal{D}\psi\psi(\vec{y})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int dx'J(\vec{x}')\psi(\vec{x}')}
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&\frac{\delta^2}{\delta J(\vec{y})\delta J(\vec{z})}\Big\{\int \mathcal{D}\psi e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int dx'J(\vec{x}')\psi(\vec{x}')}\Big\} \\
&= \int \mathcal{D}\psi\psi(\vec{y})\psi(\vec{z})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int dx'J(\vec{x}')\psi(\vec{x}')} + \ldots
\end{aligned}
\tag{4.3}
$$

where the omitted terms will be zero when the source is taken to be zero.

$$\int d^n x e^{-a_i D^{ij} a_j + J_i a^i} = \int d^n x e^{-\int dx' dy' a(\vec{x}') D(t;\vec{x}',\vec{y}') a(\vec{y}') + \int dx' J(\vec{x}') a(\vec{x}')}$$
$$= (\det D)^{-\frac{1}{2}} e^{\frac{1}{4}\int dx' dy' J(\vec{x}') D^{-1}(t;\vec{x}',\vec{y}') J(\vec{y}')}. \tag{4.4}$$

Taking two variations of this leads to

$$\frac{\delta^2}{\delta J(\vec{y})\delta J(\vec{z})}\Big\{e^{\frac{1}{4}\int dx' dy' J(\vec{x}') D^{-1}(t;\vec{x}',\vec{y}') J(\vec{y}')}\Big\}$$
$$= \frac{1}{4}(D^{-1}(t;\vec{y},\vec{z}) + D^{-1}(t;\vec{z},\vec{y}))e^{\int dx' dy' J(\vec{x}') D^{-1}(t;\vec{x}',\vec{y}') J(\vec{y}')} + ...$$
$$= \frac{1}{2}D^{-1}(\vec{y},\vec{z};t)e^{\int dx' dy' J(\vec{x}') D^{-1}(t;\vec{x}',\vec{y}') J(\vec{y}')} + ... \tag{4.5}$$

as D is also a symmetric matrix, meaning one now has an expression

$$-\int \mathcal{D}\psi\psi(\vec{y})\int dy'' B(t;\vec{x},\vec{y}'')\psi(\vec{y}'')e^{-\int dx' dy' \psi(\vec{x}') D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$= -\frac{1}{4}\int dy'' B(t;\vec{x},\vec{y}'')(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{y}'') + D^{-1}(t;\vec{y}'',\vec{y})) \tag{4.6}$$

$$-\int \mathcal{D}\psi\psi(\vec{y})\int dx'' \psi(\vec{x}'')B(t;\vec{x}'',\vec{x})e^{-\int dx' dy' \psi(\vec{x}') D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$= -\frac{1}{4}\int dx'' B(t;\vec{x}'',\vec{x})(\det D)^{-\frac{1}{2}}(D^{-1}(\vec{y},\vec{x}'';t) + D^{-1}(t;\vec{x}'',\vec{y})) \tag{4.7}$$

$$2\int \mathcal{D}\psi\psi(\vec{y})\int dx'' \psi(\vec{x}'')C(t;\vec{x}'',\vec{x})e^{-\int dx' dy' \psi(\vec{x}') D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$= \frac{1}{2}\int dx'' C(t;\vec{x}'',\vec{x})(\det D)^{-\frac{1}{2}}(D^{-1}(\vec{y},\vec{x}'';t) + D^{-1}(t;\vec{x}'',\vec{y})) \tag{4.8}$$

$$\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle = \frac{i\hbar}{4}N(t)\Big\{-\int dy'' B(t;\vec{x},\vec{y}'')(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{y}'') + D^{-1}(t;\vec{y}'',\vec{y}))$$
$$-\int dx'' B(t;\vec{x}'',\vec{x})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{x}'') + D^{-1}(t;\vec{x}'',\vec{y}))$$
$$+ 2\int dx'' C(t;\vec{x}'',\vec{x})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{x}'') + D^{-1}(t;\vec{x}'',\vec{y}))\Big\}$$

$$\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle = i\hbar\Big\{ - BD^{-1}(t;\vec{x},\vec{y}) + CD^{-1}(t;\vec{x},\vec{y})\Big\}. \tag{4.9}$$

A similar computation can be done for the opposite order of these operators. This case is analogous and the details are presented in the appendix.

$$\langle \phi(t,\vec{x})\Pi(t,\vec{y})\rangle = i\hbar\Big\{\delta(\vec{x}-\vec{y}) - B(t;\vec{x},\vec{y})D^{-1}(t;\vec{x},\vec{y}) + C(t;\vec{x},\vec{y})D^{-1}(t;\vec{x},\vec{y})\Big\}. \tag{4.10}$$

It is interesting to see that this is the same structure as before. It is straightforward to show that a diagonal matrix commutes with a symmetric one. Then one has the commutation of these relations as

$$\langle [\phi(t,\vec{x}),\Pi(t,\vec{y})]\rangle = i\hbar\delta(\vec{x}-\vec{y})$$

The computation for the momentum-momentum two point function is also similar as before, so the details are left to the appendix and the result is:

$$\langle \hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y})\rangle = 2\hbar^2[B - BD^{-1}B + 2BD^{-1}C - CD^{-1}C](t;\vec{x},\vec{y}). \tag{4.11}$$

So one has now achieved the objective of this section, relations between all the relevant two point functions and the time dependent coefficients of the density matrix. These relations may also be inverted and will be examined later for different uses.

Chapter 5

Reduced Density Matrix

as stated in the introduction, one is interested in examining the right Rindler wedge. In chapter 4, one had worked with a density matrix describing the entire space, but the amazing fact is, when considering physical observables only defined on the right wedge, one cannot distinguish between using the full density matrix on this "projected operator", or using a "reduced density matrix" on the full operator. These are terms that will be discussed in detail in this chapter.

5.1 Trace over the left part

The density matrix can be reduced to just one of the Rindler wedges by tracing out the other half. One may see this by recalling that the Rindler coordinate transformation takes the entire flat, Minkowski, space and reduces it to the two Rindler wedges where the right wedge was illustrated in figure 5.1. Therefore, considering the space where the spatial coordinate can only be positive and then making the Rindler transformation corresponds to then being on the right Rindler wedge. This means having no access to the left wedge.

One can also note some properties of what happens when considering operators that have support on the right wedge only.

$$ {}_R\langle \hat{\mathcal{O}} \rangle_R = Tr[\hat{\rho}_{red}\hat{\mathcal{O}}] = Tr[\hat{\rho}({}_R\hat{\mathcal{O}}_R)]. $$

One would then like to trace over the left part and be left with a reduced density matrix that represents the right Rindler wedge. Recall, there are wave modes describing the left and right Rindler wedges and may be written as ϕ_R and ϕ_L. These modes can be used to give the density matrix as

$$ \rho_{red} = \int \mathcal{D}\phi_L N(t) e^{-\phi_R^\dagger D\phi_R - \phi_L^\dagger D\phi_L + \phi_R^\dagger D\phi_L + \phi_L^\dagger D\phi_R}, $$

where again, $D = A + B - 2C$. This is an integral which can be dealt with using a shift of variables, such that it assumes the form of the normal gaussian.

$$\phi_L(t,\vec{x}) = \chi_L(t,\vec{x}) + \phi_R(t,\vec{x})$$

$$\rho_{red} = \int \mathcal{D}\chi_L N(t) e^{-\phi_R^\dagger D\phi_R - (\chi_L^\dagger(t,\vec{x})+\phi_R^\dagger(t,\vec{x}))D(\vec{x},\vec{y})(\chi_L(\vec{y})+\phi_R(\vec{y}))+\phi_R^\dagger D(\chi_L(\vec{y})+\phi_R(\vec{y}))+(\chi_L^\dagger(t,\vec{x})+\phi_R^\dagger(t,\vec{x}))D\phi_R}$$
$$\rho_{red} = \int \mathcal{D}\chi_L N(t) e^{-\phi_R^\dagger D\phi_R + \phi_R^\dagger D\phi_R - \chi_L^\dagger(t,\vec{x})D(\vec{x},\vec{y})\chi_L(\vec{y})}.$$

Recall

$$\int \mathcal{D}x e^{-x\cdot A\cdot x + J\cdot x} = (\det(A))^{-\frac{1}{2}} e^{\frac{1}{4}J\cdot A^{-1}\cdot J}$$

meaning the reduced density matrix goes to unity.

$$\begin{aligned}\rho_{red} &= (\det(D))^{\frac{1}{2}}(\det(D))^{-\frac{1}{2}} e^{-\phi_R^\dagger D\phi_R + \phi_R^\dagger D\phi_R} \\ &= \mathbb{1},\end{aligned} \tag{5.1}$$

where the relations for A, B and C are given in the previous subsection. The problem is doing the shift of integration variables here. The left and right wedges require two separate Hilbert spaces, so this method of finding ρ_{red} is incorrect. Although a Hilbert space extends the methods of vector algebra and calculus to n dimensional spaces, the left and right wedges are separate spaces. One must, therefore, keep this in mind for finding the reduced density matrix.

5.2 Constructing the reduced density matrix

The purpose of haveing a reduced density matrix is to have a density matrix that describes the system by tracing over the environmental degrees of freedom that are observable. In this case, being in one Rindler wedge will make the other unobservable.

$$\hat{\rho}_{red} = tr_L(\hat{\rho}).$$

Say that $\phi = \phi_L + \phi_R$. It is the combination of the left and right.

$$\rho(t;\bar{\phi},\phi') = N e^{-(\bar{\phi}_L+\bar{\phi}_R)^\dagger\cdot A\cdot(\bar{\phi}_L+\bar{\phi}_R) - (\phi'_L+\phi'_R)^\dagger\cdot B\cdot(\phi'_L+\phi'_R) + 2(\bar{\phi}_L+\bar{\phi}_R)^\dagger\cdot C\cdot(\phi'_L+\phi'_R)}$$

$$\hat{\rho} = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi' \left|\bar{\phi}\right\rangle \rho(t;\bar{\phi},\phi') \left\langle\phi'\right| .$$

The environment that one wants to trace over is the left wedge. Using that the trace over the left part means doing the integral of $\int \mathcal{D}\phi_L \left\langle\phi_L\right| ... \left|\phi_L\right\rangle$ one has:

$$\begin{aligned}
\hat{\rho}_{red} &= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}\mathcal{D}\phi' \langle\phi_L \left|\bar{\phi}\right\rangle \rho(t;\bar{\phi},\phi') \left\langle\phi'\right| \phi_L\rangle \\
&= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}\mathcal{D}\phi' \langle\phi_L|\otimes\mathbb{1} \left|\bar{\phi}_L\right\rangle \otimes \left|\bar{\phi}_R\right\rangle \rho(t;\bar{\phi},\phi') \left\langle\phi'_L\right| \otimes \left\langle\phi'_R\right| \mathbb{1} \otimes \left|\phi_L\right\rangle \\
&= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}\mathcal{D}\phi' \langle\phi_L|\bar{\phi}_L\rangle \otimes \left|\bar{\phi}_R\right\rangle \rho(t;\bar{\phi},\phi')\langle\phi'_L|\phi_L\rangle \otimes \left\langle\phi'_R\right| \\
&= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}_L\mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_L\mathcal{D}\phi'_R\langle\phi_L|\bar{\phi}_L\rangle \left|\bar{\phi}_R\right\rangle \rho(t;\bar{\phi},\phi')\langle\phi'_L|\phi_L\rangle \left\langle\phi'_R\right| \\
&= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle \rho(t;\phi_L+\bar{\phi}_R,\phi_L+\phi'_R) \left\langle\phi'_R\right| . \qquad (5.2)
\end{aligned}$$

Here, one could now take the trace over the right wedge as well. This will clearly be equivalent to taking the trace of the full density matrix and will thus yield unity. At this stage though, it is still possible to take the $\mathcal{D}\phi_L$ integral and be left with a density matrix that solely depends on ϕ_R.

$$\begin{aligned}
\hat{\rho}_{red} &= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle \rho(t;\phi_L+\bar{\phi}_R,\phi_L+\phi'_R) \left\langle\phi'_R\right| \\
&= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle N(t)e^{-\bar{\phi}_R^\dagger A\bar{\phi}_R-{\phi'_R}^\dagger B\phi'_R+2\bar{\phi}_R^\dagger C\phi'_R}e^{-\phi_L^\dagger(A+B-2C)\phi_L-2[\bar{\phi}_R^\dagger(A-C)+{\phi'_R}^\dagger(B-C)]\phi_L} \left\langle\phi'_R\right| .
\end{aligned}$$

Recall that:

$$\int \mathcal{D}x e^{-x\cdot A\cdot x+J\cdot x} = (\det(A))^{-\frac{1}{2}}e^{\frac{1}{4}J\cdot A^{-1}\cdot J},$$

which implies:

$$\begin{aligned}
\hat{\rho}_{red} = &\int \mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle N(t)(\det\{_L D_L\})^{-\frac{1}{2}} \\
&\times e^{-\bar{\phi}_R^\dagger[A-(A-C)_L D_L^{-1}(A-C)]\bar{\phi}_R-{\phi'_R}^\dagger[B-(B-C)_L D_L^{-1}(B-C)]\phi'_R+2\bar{\phi}_R^\dagger[C+(A-C)_L D_L^{-1}(B-C)]\phi'_R} \left\langle\phi'_R\right| .
\end{aligned}$$

5.3 Projection

Care must be taken when dealing with the functions A, B and C in that the normalisation for the density matrix takes these functions over the entire space. In the reduced density matrix though, these functions may be projected onto different parts of the space and one must identify what this means.

$$\phi \cdot A \cdot \phi = \int dx'dy'\phi(\vec{x}')A(t;\vec{x}',\vec{y}')\phi(\vec{y}').$$

This refers to the integrals being done over all space, however in the previous section, one may recall the gaussian integral over $\mathcal{D}\phi_L$. In this case, one had terms of the form $\phi_L^\dagger(...)\phi_L$ and $\phi_R^\dagger(...)\phi_L$. In these cases, the integrals over the positions will not be over the entire space.

$$[(A-C)D^{-1}(A-C)](\vec{x},\vec{y}) = \int dx'dy'(A-C)(\vec{x},\vec{x}')D^{-1}(\vec{x}',\vec{y}')(A-C)(\vec{y}',\vec{y}).$$

This was an expression that appeared between different modes and was not properly projected onto the left and right Rindler wedges. What the initial terms in the exponential should have been are

$$\begin{aligned}
&\bar{\phi}_R^\dagger(A-C)\phi_L + {\phi'_R}^\dagger(B-C)\phi_L \\
&= \int dx'_R dy'_L \bar{\phi}_R^\dagger(\vec{x}'_R)(A-C)(\vec{x}'_R,\vec{y}'_L)\phi_L(\vec{y}'_L) + \int dx'_R dy'_L {\phi'_R}^\dagger(\vec{x}'_R)(B-C)(\vec{x}'_R,\vec{y}'_L)\phi_L(\vec{y}'_L)\phi_L.
\end{aligned}$$

This will now be given the shorthand expression of P_L and P_R for projection onto the left and right spaces respectively. Finding the reduced density matrix now goes as

$$\begin{aligned}
\rho_{red} &= \int \mathcal{D}\phi_L\mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle N(t)e^{-\bar{\phi}_R^\dagger A\bar{\phi}_R - {\phi'_R}^\dagger B\phi'_R + 2\bar{\phi}_R^\dagger C\phi'_R} e^{-\phi_L^\dagger(A+B-2C)\phi_L - 2[\bar{\phi}_R^\dagger(A-C)+{\phi'_R}^\dagger(B-C)]\phi_L} \left\langle\phi'_R\right| \\
&= \int \mathcal{D}\bar{\phi}_R\mathcal{D}\phi'_R \left|\bar{\phi}_R\right\rangle N(t)[\det(P_L D P_L)]^{-\frac{1}{2}} \\
&\times e^{-\bar{\phi}_R^\dagger[A-P_R(A-C)P_LP_LD^{-1}P_LP_L(A-C)P_R]\bar{\phi}_R} \\
&\times e^{-{\phi'_R}^\dagger[B-P_R(B-C)P_LP_LD^{-1}P_LP_L(B-C)P_R]\phi'_R} \\
&\times e^{2\bar{\phi}_R^\dagger[C+P_R(A-C)P_LP_LD^{-1}P_LP_L(B-C)P_R]\phi'_R} \left\langle\phi'_R\right|.
\end{aligned}$$

Where the projection operator has been introduced and has the following properties.

$$
\begin{aligned}
&P_R^2 = P_R, \quad P_L^2 = P_L \\
&P_R + P_L = \mathbb{1} \\
&P_L P_R = 0.
\end{aligned} \tag{5.3}
$$

Using this, one has

$$
\begin{aligned}
\rho_{red} = &\int \mathcal{D}\bar{\phi}_R \mathcal{D}\phi'_R \, |\bar{\phi}_R\rangle \, N(t) [\det(P_L D P_L)]^{-\frac{1}{2}} \\
&\times e^{-\bar{\phi}_R^\dagger [A - P_R(A-C)P_L D^{-1} P_L (A-C) P_R] \bar{\phi}_R} \\
&\times e^{-\phi_R'^\dagger [B - P_R(B-C)P_L D^{-1} P_L (B-C) P_R] \phi'_R} \\
&\times e^{2\bar{\phi}_R^\dagger [C + P_R(A-C)P_L D^{-1} P_L (B-C) P_R] \phi'_R} \, \langle \phi'_R | \, .
\end{aligned} \tag{5.4}
$$

Performing the trace over the right part now yields,

$$
\begin{aligned}
&N(t)(\det P_L D P_L)^{-\frac{1}{2}} [\det\big(P_R D P_R - P_R D P_L D^{-1} P_L D P_R\big)]^{-\frac{1}{2}} \\
&= [\det(P_R + P_L) D (P_R + P_L)]^{\frac{1}{2}} (\det P_L D P_L)^{-\frac{1}{2}} [\det\big(P_R (D - D P_L D^{-1} P_L D) P_R\big)]^{-\frac{1}{2}} \\
&= [\det(P_R + P_L) D (P_R + P_L)]^{\frac{1}{2}} [\det\big(P_L D P_L P_R (D - D P_L D^{-1} P_L D) P_R\big)]^{-\frac{1}{2}}.
\end{aligned}
$$

Looking at these terms carefully, one can see the structure of these matrices in terms of the coordinates of which they are functions of.

$$
\begin{aligned}
P_R D P_L D^{-1} P_L D P_R &= \int_{-\infty}^{\infty} dx' dy' P_R D(\vec{x}, \vec{x}') P_L D^{-1}(\vec{x}', \vec{y}') P_L D(\vec{y}', \vec{y}) P_R \\
&= \int_{-\infty}^{\infty} dx'_L dy'_L D(\vec{x}_R, \vec{x}'_L) D^{-1}(\vec{x}'_L, \vec{y}'_L) D(\vec{y}'_L, \vec{y}_R) \\
&= \int_{-\infty}^{0} dx' dy' D(\vec{x}_R, \vec{x}') D^{-1}(\vec{x}', \vec{y}') D(\vec{y}', \vec{y}_R).
\end{aligned}
$$

At this stage, it may be logical to split the position vectors into two parts. The spatial coordinate that is transformed and the $D-2$ orthogonal spatial coordinates.

5.4 Projection operators

Here, the projection operators will be looked at explicitly in terms of step functions, so that the wave modes ϕ_L and ϕ_R will only exist in one half of the space each.

$$\phi_L(\vec{x}) = P_L(\vec{x})\phi(\vec{x})$$

where the projection operators could be given as

$$P_L(x_1) = \Theta(-x_1) \quad \text{and} \quad P_R(x_1) = \Theta(x_1).$$

In this way, the sum of the operators would be the identity operator as one would desire. Working in momentum space, one can transform these operators as

$$\begin{aligned}\tilde{P}_L(k) &= \int_{-\infty}^{\infty} \Theta(-x)e^{ikx}dx = \int_{-\infty}^{0} e^{ikx+\epsilon x}dx \\ &= \frac{-i}{k-i\epsilon} = \mathcal{P}\frac{-i}{k} + \pi\delta(k)\end{aligned}$$

$$\begin{aligned}\tilde{P}_R(k) &= \int_{-\infty}^{\infty} \Theta(x)e^{ikx}dx = \int_{0}^{\infty} e^{ikx-\epsilon x}dx \\ &= \frac{i}{k+i\epsilon} = \mathcal{P}\frac{i}{k} + \pi\delta(k).\end{aligned}$$

This is easily generalised to D-dimensions by breaking kx into $k_1x^1 + \vec{k}_\perp \cdot \vec{x}_\perp$, where the first term represents the axis to which the acceleration is applied. This has the effect of brining a coefficient of $(2\pi)^{D-2}\delta^{D-2}(\vec{k}_\perp)$

$$\tilde{\phi}_L(k_x) = \int_{-\infty}^{\infty} \frac{dk'_x}{2\pi}\frac{-i}{(k_x-k'_x)-i\epsilon}\tilde{\phi}(k'_x).$$

So again, one has achieved the goal here of finding a density matrix for only the right Rindler wedge, which is now defined with time dependent coefficients that only exist on the right wedge. One introduced the necessary projecting operators as step functions. This implies a sharp cut off at the horizon. One might comment on the use of smooth functions with some with as the projectors. Although harder to deal with mathematically, these might have better physical motivation as the horizon no longer needs to be considered sharp. Indeed, many descriptions of black holes treat the horizon as something that is influenced by the matter around it. Namely, a particle falling into the horizon will cause it to grow by a small amount. This is discussed in the old "Brick Wall" model. [11]

Figure 5.1: The 1+1 dimensional right Rindler wedge for values of the acceleration of 1, 0.5 and 0.1 respectively

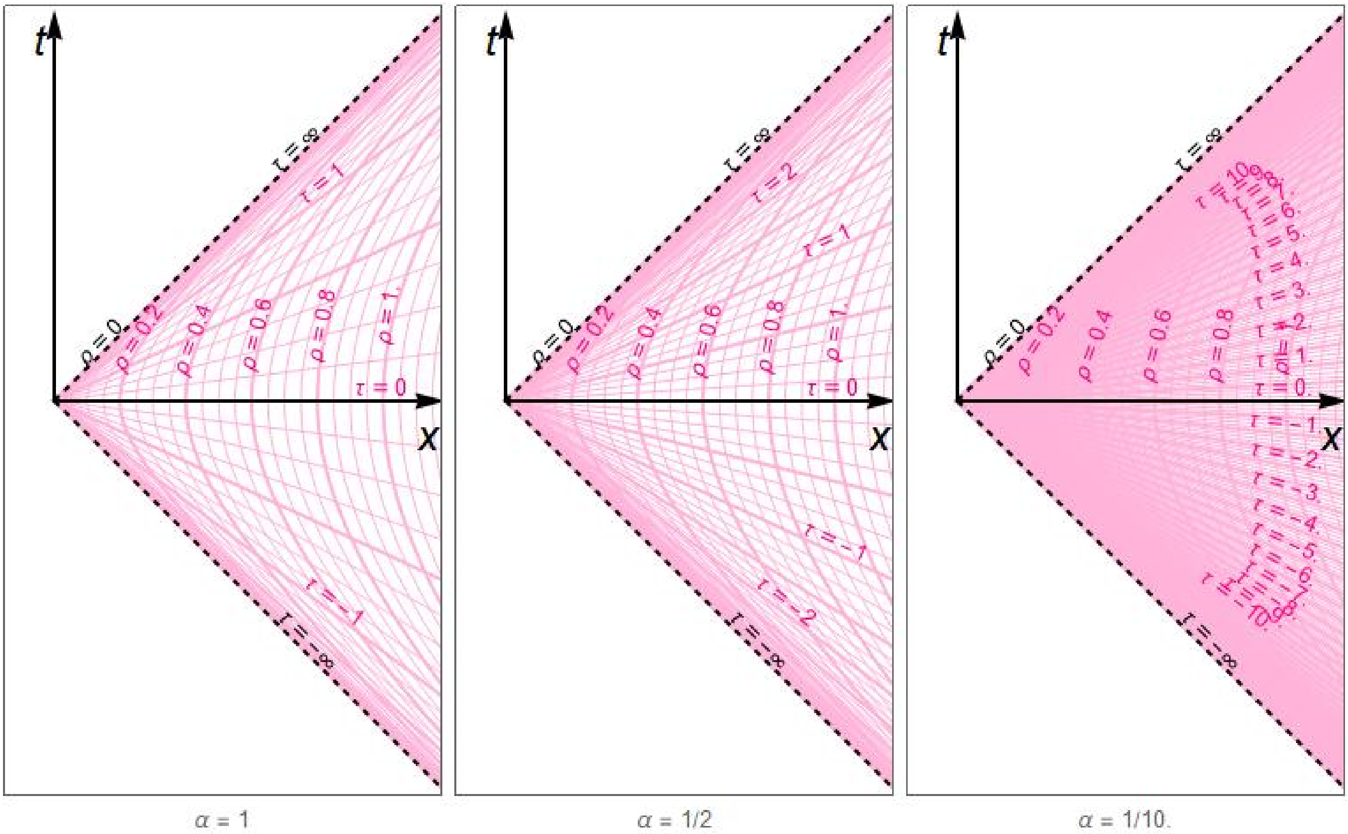
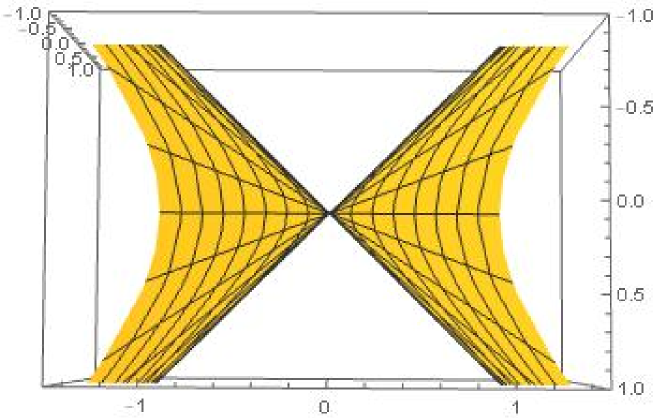


Figure 5.2: The 1+2 dimensional left and right Rindler wedges for acceleration set to 1



Chapter 6

Projected Propagators

The ultimate goal is to calculate measurable, physical quantities on the Rindler wedge, for this the reduced density matrix requires all possible projections of the two point functions, this can be seen from equation 5.4. In this section, propagators will be considered in mixed space and the projection operators will be applied. This will lead to complicated expressions, but is still preferable to working in position space. The case of the massless free scalar in Minkowski $1+1$ dimensions is taken. Once the procedure for this is outlined, one can then begin to return to more general cases.

The reason for applying projection operators is to be able to define the two point functions in different subsections of position space. Namely one can take the entire space and divide it into four subsections and call them left left, left right, right left and right right. One can already imagine some aspects of the these reduced propagators. Such as that one projected only on the left side should have nothing to do with the left side of space and vice-versa. While one projected on the left and right side will have components of both sides of the space. This will later be observed through the appearance of step functions in the projected propagators.

Applying the Theta function first

In position space, the propagator is given by

$$
\begin{aligned}
i\Delta_F(x,x') &= \frac{i}{(2\pi)^2}\int d^2k \frac{e^{-ik(x-x')}}{k^2+i\epsilon'} \\
&= \frac{i}{(2\pi)^2}\int d^2k \frac{e^{ik^0(t-t')-ik_1(x_1-x_1')}}{-k^{0^2}+k_1^2+i\epsilon'}.
\end{aligned}
\tag{6.1}
$$

Note the $i\epsilon$ prescription as $(k^0 - i\bar{\epsilon}\mathrm{sgn}(k^0))^2 = k^{0^2} - i\epsilon'$.

$$
\begin{aligned}
&P_L(x_1) i\Delta_F(x,x')P_L(x_1') \\
&= i\int \frac{dk_1'}{(2\pi)}\frac{(-i)e^{-ik_1'x_1}}{k_1'-i\epsilon}\int \frac{d^2k}{(2\pi)^2}\Big[\frac{e^{-ik(x-x')}}{k^2+i\epsilon'}\Big]\int \frac{dk_1''}{(2\pi)}\frac{(-i)e^{-ik_1''x_1}}{k_1''-i\epsilon}.
\end{aligned}
$$

Performing a coordinate substitution for average position and distance and using the previous expressions for the projectors.

$$
\begin{aligned}
&r_1 = x_1 - x_1', \quad X_1 = \frac{x_1+x_1'}{2} \\
&x_1 = X_1 + \frac{r}{2}, \quad x_1' = X_1 - \frac{r}{2}, \quad T = t-t'.
\end{aligned}
\tag{6.2}
$$

Using this particular substitution means that one can now easily transform to mixed space by integrating over the coordinates $x - x'$.

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X) \\
&= i\int d^2(x-x')e^{ip(x-x')}\int \frac{dk_1'}{(2\pi)}\frac{ie^{-ik_1'x_1}}{k_1'-i\epsilon}\int \frac{d^2k}{(2\pi)^2}\Big[\frac{e^{-ik(x-x')}}{k^2+i\epsilon'}\Big]\int \frac{dk_1''}{(2\pi)}\frac{ie^{-ik_1''x_1}}{k_1''-i\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X) \\
&= i\int dTdr_1 e^{ip_1r_1-ip^0T}\int \frac{dk_1'}{(2\pi)}e^{-ik_1'x_1}\frac{i}{k_1'-i\epsilon}\int \frac{d^2k}{(2\pi)^2}\Big[\frac{e^{-ik(x-x')}}{k^2+i\epsilon'}\Big]\int \frac{dk_1''}{(2\pi)}e^{-ik_1''x_1'}\frac{i}{k_1''-i\epsilon} \\
&= -i\int \frac{dTdr_1dk_1'd^2kdk_1''}{(2\pi)^4(k_1'-i\epsilon)(k_1''-i\epsilon)}e^{ip_1r_1}e^{-ik_1'x_1}\Big[\frac{e^{-ik_1(x_1-x_1')+i(k^0-p^0)(t-t')}}{k^2+i\epsilon'}\Big]e^{-ik_1''x_1'} \\
&= -i\int \frac{dTdr_1dk_1'd^2kdk_1''}{(2\pi)^4(k_1'-i\epsilon)(k_1''-i\epsilon)}e^{ip_1r_1}e^{-\frac{i}{2}(k_1'-k_1'')r_1}e^{-i(k_1'+k_1'')X_1}\Big[\frac{e^{-ik_1r+i(k^0-p^0)T}}{k^2+i\epsilon'}\Big].
\end{aligned}
$$

The delta functions here can be used to simplify things immediately. Then the first two integrals are dealt with straightforwardly as

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X) \\
&= -i\int \frac{dk_1' d^2k dk_1''}{(2\pi)^4(k_1'-i\epsilon)(k_1''-i\epsilon)}(2\pi)^2\delta(p^0-k^0)\delta(k_1-p_1+\frac{k_1'-k_1''}{2})e^{-i(k_1'+k_1'')X_1} \\
&\times \Big[\frac{1}{k^2+i\epsilon'}\Big] \\
&= -i\int \frac{dk_1' d^2k dk_1''}{(2\pi)^2(k_1'-i\epsilon)(k_1''-i\epsilon)}\delta(p^0-k^0)\delta(k_1-p_1+\frac{k_1'-k_1''}{2})e^{-i(k_1'+k_1'')X_1} \\
&\times \Big[\frac{1}{-k^{0\,2}+k_1^2+i\epsilon'}\Big].
\end{aligned}
$$

The integrals over the unprimed variables can then be performed using these delta functions to yield

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X) \\
&= -i\int \frac{dk_1' dk_1''}{(2\pi)^2(k_1'-i\epsilon)(k_1''-i\epsilon)}\Big[\frac{e^{-i(k_1'+k_1'')X_1}}{-p^{0\,2}+(p_1+\frac{k_1''-k_1'}{2})^2+i\epsilon'}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X) \\
&= -4i\int \frac{dk_1' dk_1''}{(2\pi)^2(k_1'-i\epsilon)(k_1''-i\epsilon)} \\
&\times \Big[\frac{e^{-i(k_1'+k_1'')X_1}}{\Big((2p_1+k_1''-k_1')-(2(p^0-i\bar\epsilon\,\mathrm{sign}(p^0)\Big)\Big((2p_1+k_1''-k_1')+2(p^0-i\bar\epsilon\,\mathrm{sign}(p^0))\Big)}\Big].
\end{aligned}
$$

This can now be calculated using contour integration. If one attempts to do this immediately, it can be seen that there will be problems identifying the locations of poles.

6.1 The problem of the poles with different prescriptions for the imaginary component

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X)\\
&= 4i\int \frac{dk_1' dk_1''}{(2\pi)^2(k_1'-i\epsilon)(k_1''-i\epsilon)}\\
&\times \frac{e^{-i(k_1'+k_1'')X_1}}{\Big(2p_1+k_1''-k_1'-2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))\Big)\Big(2p_1+k_1''-k_1'+2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))\Big)}.
\end{aligned}
$$

There are in fact two different epsilon prescriptions in this expression. When evaluating the contour integrals, one must choose conditions to determine the position of the poles. Although two conditions are physical, the sign of p^0 and of X_1, there arises conditions between the different epsilons. This is not physical. To avoid this, another transformation must be performed to decouple ϵ from $\bar{\epsilon}$. This will be examined in the simplest case first for clarity. i.e. $1+1$ dimensional and massless case.

$$
\begin{aligned}
q_1 &= k_1' - k_1''\\
s_1 &= k_1' + k_1''.
\end{aligned}
$$

One first aims to perform the integral over k_1'

$$
\begin{aligned}
&{}_L i\Delta_{FL}(p,X)\\
&= -2i\int \frac{dq_1 ds_1}{(2\pi)^2(\frac{q_1}{2}+\frac{s_1}{2}-i\epsilon)(\frac{s_1}{2}-\frac{q_1}{2}-i\epsilon)}\\
&\times \frac{e^{-is_1X_1}}{\Big(2p_1-q_1-2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))\Big)\Big(2p_1-q_1+2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))\Big)}\\
&= -8i\int \frac{dq_1 ds_1}{(2\pi)^2(q_1+s_1-2i\epsilon)(s_1-q_1-2i\epsilon)}\\
&\times \frac{e^{-is_1X_1}}{\Big(2p_1-2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))-q_1\Big)\Big(2p_1+2(p^0-i\bar{\epsilon}\mathrm{sign}(p^0))-q_1\Big)}.
\end{aligned}
$$

So there are four poles initially for q_1 and two for s_1.

6.2 Looking at the "s" variable first

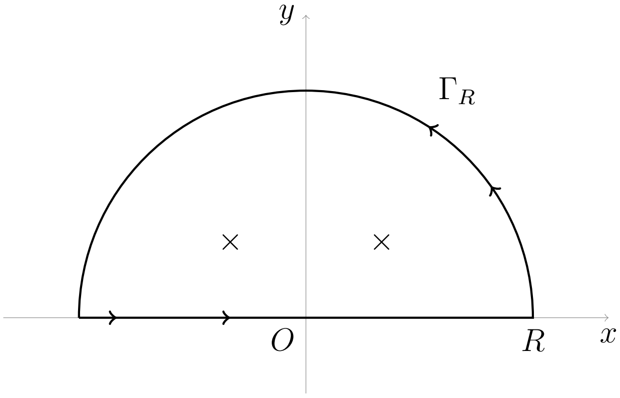
The two poles for s_1 are

$$s_1 = -q_1 + 2i\epsilon, \quad s_1 = q_1 + 2i\epsilon.$$

These evaluate to

$$
\begin{aligned}
&1) - i\frac{(2\pi i)8}{(2\pi)^2(-2q_1)} \\
&\times \frac{e^{-i(-q_1+2i\epsilon)X_1}}{\Big(2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)\Big(-2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)} \\
&2) - i\frac{(2\pi i)8}{(2\pi)^2(2q_1)} \\
&\times \frac{e^{-i(q_1+2i\epsilon)X_1}}{\Big(2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)\Big(-2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)}.
\end{aligned}
$$

Both of the poles are in the lower half of the s_1 plane, so the contour must be closed from below, this means that taking $X_1 > 0$ encompasses no poles, thus it does not contribute.



$$
\begin{aligned}
&{}_L\Delta_{FL}(p, X) \\
&= \int dq_1 \frac{8e^{-2\epsilon X_1}\Theta(-X_1)}{(2\pi)(2q_1)\Big(2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)\Big(-2(p^0 - i\bar{\epsilon}\mathrm{sign}(p^0)) - 2p_1 + q_1\Big)} \\
&\times \Big\{e^{-iq_1X_1} - e^{iq_1X_1}\Big\}.
\end{aligned}
$$

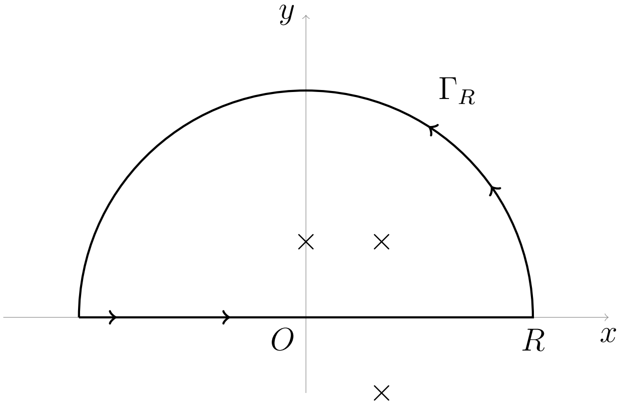
Now that there is only one more integral to compute, one can see that there are three poles, but only one physical condition to choose to determine their location. That is the sign of p^0.

Taking the "q" integral secondly

One has already chosen the sign of X_1 so the first expontential will encompass the poles above the plane, while the second takes the ones below the plane. The problem now is that the first pole lies on the real axis. This initially seems that the integral is divergent. Physically however, this makes little sense since all one has done is project the normal propagator onto a region. Since the normal propagator is not divergent due to its original $i\epsilon$ prescription, one should not have a divergent answer here. Thus one should perhaps consider the principle part of the integral.

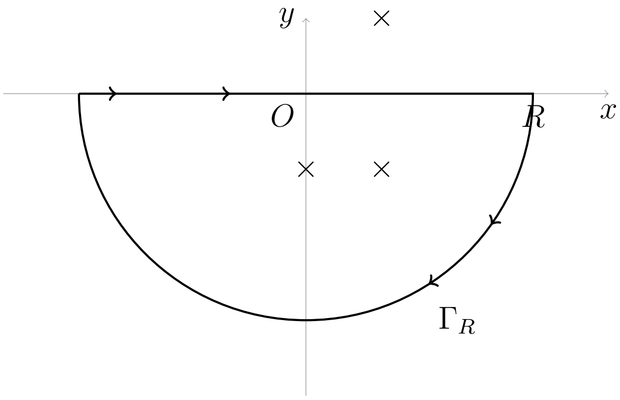
So now the only physical choice left if the sign of p^0. Taking first the positive case. Closing this contour from above yields

$$i\Theta(-X_1)\Theta(p^0)\Big\{\frac{e^{-2\epsilon X_1}}{2[p_1 - p^0 + i\bar{\epsilon}][p_1 + p^0 - i\bar{\epsilon}]} - \frac{e^{-2\epsilon X_1}e^{-2i(p_1 - p^0 + i\bar{\epsilon})X_1}}{2[p_1 - p^0 + i\bar{\epsilon}][p^0 - i\bar{\epsilon}]}\Big\}.$$



Whilst closing it from below gives

$$i\Theta(-X_1)\Theta(p^0)\Big\{\frac{2e^{-2\epsilon X_1}}{2[p_1 - p^0 + i\bar{\epsilon}][p_1 + p^0 - i\bar{\epsilon}]} + \frac{e^{-2\epsilon X_1}e^{2i(p_1 + p^0 - i\bar{\epsilon})X_1}}{2[p_1 + p^0 - i\bar{\epsilon}][p^0 - i\bar{\epsilon}]}\Big\}.$$



So adding theses contributions, one has the full expression for ${}_L\Delta_L(p,X)$

Next, one can take the opposite value of p^0, this swaps the positions of the poles.

$$
\begin{aligned}
&{}_Li\Delta_{FL}(p,X)\\
&= i\Theta(-X_1)\Theta(p^0)\Big\{\frac{1}{[p_1-p^0+i\bar{\epsilon}][p_1+p^0-i\bar{\epsilon}]}-\frac{1}{2(p^0-i\bar{\epsilon})}\Big[-\frac{e^{2i(p_1+p^0-i\bar{\epsilon})X_1}}{(p_1+p^0-i\bar{\epsilon})}+\frac{e^{-2i(p_1-p^0+i\bar{\epsilon})X_1}}{(p_1-p^0+i\bar{\epsilon})}\Big]\Big\}\\
&+ i\Theta(-X_1)\Theta(-p^0)\Big\{\frac{1}{[p_1-p^0-i\bar{\epsilon}][p_1+p^0+i\bar{\epsilon}]}+\frac{1}{2(p^0+i\bar{\epsilon})}\Big[-\frac{e^{2i(p_1-p^0-i\bar{\epsilon})X_1}}{(p_1-p^0-i\bar{\epsilon})}+\frac{e^{-2i(p_1+p^0+i\bar{\epsilon})X_1}}{(p_1+p^0+i\bar{\epsilon})}\Big]\Big\}.
\end{aligned}
$$

Taking two right projectors is almost the same calculation. The difference is that for the s_1 integral, the poles will both be in the opposite half of the plane. This restricts one to the case of $\Theta(X_1)$, but all else remains the same. The different scenario is when one considers mixed projectors.

The other projected propagator is

$$
\begin{aligned}
&{}_Ri\Delta_{FR}(p,X_1)\\
&= i\Theta(X_1)\Theta(p^0)\Bigg\{\frac{1}{[p_1+p^0-i\bar{\epsilon}][p_1-p^0+i\bar{\epsilon}]}+\frac{1}{2(p^0-i\bar{\epsilon})}\Big\{-\frac{e^{2i(p_1-p^0+i\bar{\epsilon})X_1}}{[p_1-p^0+i\bar{\epsilon}]}+\frac{e^{-2i(p_1+p^0-i\bar{\epsilon})X_1}}{[p_1+p^0-i\bar{\epsilon}]}\Big\}\Bigg\}\\
&+ i\Theta(X_1)\Theta(-p^0)\Bigg\{\frac{1}{[p_1-p^0-i\bar{\epsilon}][p_1+p^0+i\bar{\epsilon}]}+\frac{1}{2(p^0+i\bar{\epsilon})}\Big\{-\frac{e^{-2i(p_1-p^0-i\bar{\epsilon})X_1}}{[p_1-p^0-i\bar{\epsilon}]}+\frac{e^{2i(p_1+p^0+i\bar{\epsilon})X_1}}{[p_1+p^0+i\bar{\epsilon}]}\Big\}\Bigg\}.
\end{aligned}
\tag{6.3}
$$

6.3 Mixed projectors

The case of having a right and left projector acting on the propagator is different in that the initial s_1 integral has one pole above and one pole below the plane. This leads to more cases as in addition to having $p^0 < 0$ and $p^0 > 0$, one also must consider the two cases for X_1.

Note that the expression for having the projectors the other way around would swap the positions of the s_1 poles

The left right case is then

$$
\begin{aligned}
{}_Li\Delta_{FR}(p,X_1) = &-\frac{ie^{-2i(p_1+p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2[p_1+p^0-i\bar{\epsilon}-i\epsilon](p^0-i\bar{\epsilon})}\Theta(X_1)\Theta(p^0)+\frac{ie^{-2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2[p_1-p^0-i\bar{\epsilon}-i\epsilon](p^0+i\bar{\epsilon})}\Theta(X_1)\Theta(-p^0)\\
&-\frac{ie^{2i(p_1+p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2[p_1+p^0-i\bar{\epsilon}-i\epsilon](p^0-i\bar{\epsilon})}\Theta(-X_1)\Theta(p^0)+\frac{ie^{2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2[p_1-p^0-i\bar{\epsilon}-i\epsilon](p^0+i\bar{\epsilon})}\Theta(-X_1)\Theta(-p^0).
\end{aligned}
$$

The right left case is

$$
\begin{aligned}
{}_{R}i\Delta_{FL}(p,X_1) &= \frac{ie^{2i(p_1-p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2[p_1-p^0+i\bar{\epsilon}+i\epsilon](p^0-i\bar{\epsilon})}\Theta(X_1)\Theta(p^0) - \frac{ie^{2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2[p_1+p^0+i\bar{\epsilon}+i\epsilon](p^0+i\bar{\epsilon})}\Theta(X_1)\Theta(-p^0) \\
&+ \frac{ie^{-2i(p_1-p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2[p_1-p^0+i\bar{\epsilon}+i\epsilon](p^0-i\bar{\epsilon})}\Theta(-X_1)\Theta(p^0) - \frac{ie^{-2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2[p_1+p^0+i\bar{\epsilon}+i\epsilon](p^0+i\bar{\epsilon})}\Theta(-X_1)\Theta(-p^0).
\end{aligned}
$$

6.4 Relation to Wightman functions

Using the Theta function on projected propagators

In order to see how the positive and negative Wightman functions are projected, one will firstly attempt to apply the step function directly to the known expressions. To do this, one will need to write the positive and negative Wightman functions in terms of the Feynmann propagator.

$$\begin{aligned}
&i\Delta_F(x,x') = \Theta(t-t')\langle\phi(x)\phi(x')\rangle + \Theta(t'-t)\langle\phi(x')\phi(x)\rangle \\
&\Theta(t-t')i\Delta_F(x,x') = \langle\phi(x)\phi(x')\rangle = i\Delta^+ \\
&\Theta(t'-t)i\Delta_F(x,x') = \langle\phi(x')\phi(x)\rangle = i\Delta^-
\end{aligned}$$

$$F(x,x') = \langle\frac{1}{2}\{\phi(x),\phi(x')\}\rangle = \frac{1}{2}(i\Delta^+ + i\Delta^-).$$

For the simplest case, the Feynman propagator of a massless scalar field is

$$i\Delta_F(x,x') = i\int\frac{d^2k}{(2\pi)^2}\frac{e^{-ik(x-x')}}{k^2+i\epsilon}$$

and the positive frequency Wightman function is:

$$i\Delta^+(x,x') = \Theta(t-t')i\Delta_F(x,x').$$

But this would also hold for the projected propagators as well, which have already been calculated.

$$\begin{aligned}
{}_Li\Delta_L^+(x,x') &= \Theta(t-t')\,{}_Li\Delta_L(x,x') \\
&= i\int\frac{dk'^0}{2\pi}\frac{ie^{-ik'^0T}}{k'^0+i\epsilon}\int\frac{d^2k}{(2\pi)^2}e^{-ik(x-x')}\,{}_Li\Delta_L(k,X).
\end{aligned}$$

Expressing this in momentum space, one gets:

$$\begin{aligned}
{}_Li\Delta_L^+(p,X) &= -\int\frac{dT\,dr_1\,dk'^0d^2k}{(2\pi)^3}\frac{e^{ip(x-x')}e^{i(-k'^0+k^0)T}e^{-ik_1r_1}}{k'^0+i\epsilon}\,{}_L\Delta_L(k,X) \\
&= -\int\frac{dk'^0}{2\pi}\frac{{}_L\Delta_L(p^0+k'^0,p_1;X)}{(k'^0+i\epsilon)} \\
&= \int\frac{d\tilde{p}^0}{2\pi}\frac{{}_L\Delta_L(\tilde{p}^0,p_1;X)}{(-p^0+\tilde{p}^0+i\epsilon)}.
\end{aligned}$$

This also applies for any of the projected propagators. The problem now is that this final integral is from minus infinity to positive infinity. The projected propagator is different however for $\tilde{p}^0$ being less than or greater than zero. This is not exactly ideal for this calculation.

Projecting the propagator last

The Feynman and Dyson propagators can be expressed in terms of the positive and negative frequency Wightman functions as

$$\begin{aligned} i\Delta_F(x,x') &= \Theta(T)i\Delta^+(x,x') + \Theta(-T)i\Delta^-(x,x') \\ i\Delta_D(x,x') &= \Theta(T)i\Delta^-(x,x') + \Theta(-T)i\Delta^+(x,x'), \end{aligned}$$

so that

$$i\Delta_F(x,x') + i\Delta_D(x,x') = i\Delta^+(x,x') + i\Delta^-(x,x') = 2F(x,x'),$$

where $F(x,x')$ denotes the statistical two point function. Since one already has the Feynman propagator projected onto all half-spaces, one simply has to find the same for the Dyson propagator

$${}_Li\Delta_{FL}(p,X) + {}_Li\Delta_{DL}(p,X) = 2{}_LF_L(p,X),$$

where

$${}_Li\Delta_L^+(p,X) = i\int \frac{dTdr_1dk_1'dl^0d^2kdk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-il^0T}e^{-ik_1'x_1}e^{-ik_1''x_1'}}{(l^0+i\epsilon)(k_1'-i\epsilon)(k_1''-i\epsilon)}i\Delta_F(x,x').$$

6.5 Projecting the positive Wightman function

A more convenient way of finding the projected Wightman functions is by staring with the functions in real space such that:

$$i\Delta^+(x,x') = i\int \frac{dl^0}{2\pi}\frac{d^2k}{(2\pi)^2}\frac{ie^{-il^0T}}{l^0+i\epsilon}\frac{e^{-ik(x-x')}}{k^2+i\epsilon}.$$

Left Left

To perform the projection so that that the positive Wightman function only has support on the left side. Meaning only negative x_1 values are considered, one applies the step functions such that:

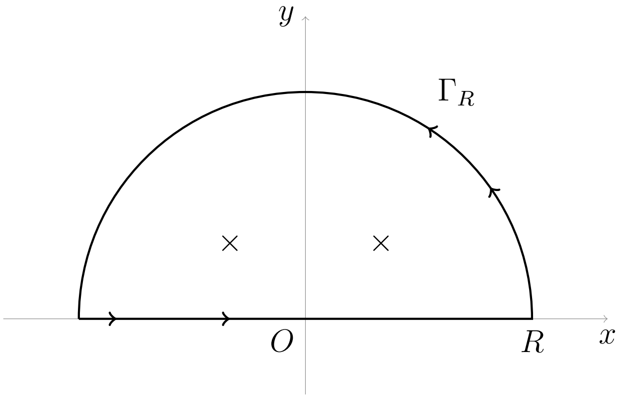
$$
{}_L i\Delta_L^+(x,x') = i\int \frac{dl^0}{2\pi}\frac{dk_1' d^2k dk_1''}{(2\pi)^4}\frac{ie^{-ik_1'x_1}}{k_1'-i\epsilon}\frac{ie^{-il^0T}}{l^0+i\epsilon}\frac{e^{-ik(x-x')}}{k^2+i\epsilon}\frac{ie^{-ik_1''x_1'}}{k_1''-i\epsilon}.
$$

This is similar to the previous calculation, but carries the extra variable of l^0, which can be integrated out in the end. Again, there are four cases for the different values of X_1 and p^0. In addition to the four possibilities of the projections.

$$
\begin{aligned}
&{}_L i\Delta_L^+(p,X) = \\
&i\int \frac{dl^0 dk_1' d^2k dk_1''}{(2\pi)^5}\frac{-ie^{-i(k_1'+k_1'')X_1}\delta(k^0-p^0-l^0)\delta(p_1-k_1-\frac{1}{2}(k_1'-k_1''))}{(l^0+i\epsilon)(k_1'-i\epsilon)[-\bar{k}^{02}+k_1^2](k_1''-i\epsilon)}
\end{aligned}
$$

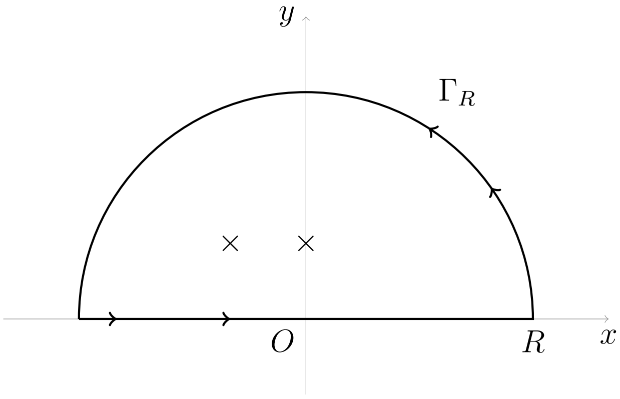
$$
\begin{aligned}
&{}_L i\Delta_L^+(p,X) = \\
&\frac{1}{2}\int \frac{dl^0 ds_1 dq_1}{(2\pi)^3}\frac{e^{-is_1X_1}}{(l^0+i\epsilon)(\frac{1}{2}(s_1+q_1)-i\epsilon)(\frac{1}{2}(s_1-q_1)-i\epsilon)} \\
&\times \frac{1}{[p_1-\frac{1}{2}q_1-p^0-l^0+i\bar{\epsilon}\mathrm{sign}(p^0+l^0)][p_1-\frac{1}{2}q_1+p^0+l^0-i\bar{\epsilon}\mathrm{sign}(p^0+l^0)]}.
\end{aligned}
$$

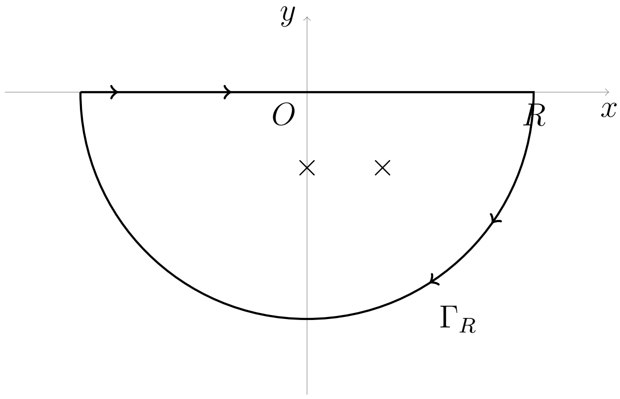
The s_1 integral requires the contour to be closed from above, encompassing two poles.



$$
\begin{aligned}
&{}_L i\Delta_L^+(p,X) \\
&= i\int \frac{dl^0 dq_1}{4(2\pi)^2}\frac{\Theta(-X_1)16e^{-2\epsilon X_1}}{(l^0+i\epsilon)(q_1)(-2p_1+2\bar{p}^0+2l^0+q_1)(-2p_1-2\bar{p}^0-2l^0+q_1)}\Big(e^{-iq_1X_1}-e^{iq_1X_1}\Big).
\end{aligned}
$$

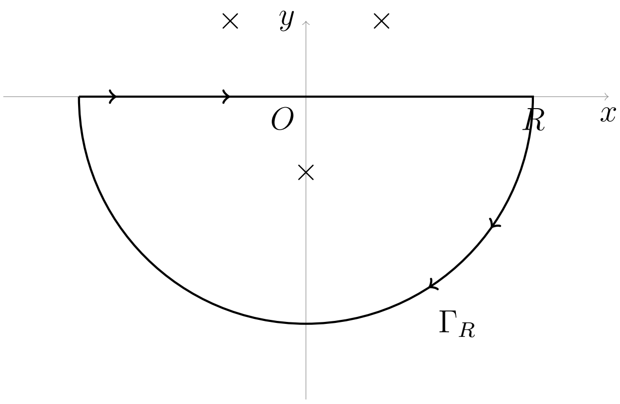
The q_1 requires the contour to be closed from above and below.





$$
\begin{aligned}
&{}_L i\Delta^+_L(p,X) = \\
&\frac{1}{(2\pi)}\int \frac{dl^0}{l^0+i\epsilon}\frac{\Theta(-X_1)\Theta(p^0)e^{-2\epsilon X_1}}{(-p_1+p^0+l^0-i\bar{\epsilon})(p_1+p^0+l^0-i\bar{\epsilon})} \\
&-\frac{1}{2(2\pi)}\int \frac{dl^0}{l^0+i\epsilon}\frac{\Theta(-X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{-2i(p_1-p^0+i\bar{\epsilon}-l^0)X_1}}{(p^0+l^0-i\bar{\epsilon})(-p_1+p^0-i\bar{\epsilon}+l^0)} \\
&-\frac{1}{2(2\pi)}\int \frac{dl^0}{l^0+i\epsilon}\frac{\Theta(-X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{2i(p_1+p^0-i\bar{\epsilon}+l^0)X_1}}{(p^0+l^0-i\bar{\epsilon})(p_1+p^0-i\bar{\epsilon}+l^0)}.
\end{aligned}
$$

This has one pole below the real axis and two above. The contour can be closed in either direction in the first term, but it is easiest to close it around one pole.



$$
{}_L i\Delta^+_L(p,X) = i\Theta(-X_1)\Theta(p^0)\Bigg\{
$$

$$
-\frac{e^{-2\epsilon X_1}}{(-p_1+p^0-i\tilde{\epsilon})(p_1+p^0-i\tilde{\epsilon})} + \frac{e^{-2\epsilon X_1}e^{-2i(p_1-p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(-p_1+p^0-i\bar{\epsilon}-i\epsilon)(p^0-i\bar{\epsilon}-i\epsilon)} + \frac{e^{-2\epsilon X_1}e^{2i(p_1+p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(p_1+p^0-i\bar{\epsilon}-i\epsilon)(p^0-i\bar{\epsilon}-i\epsilon)}\Bigg\}.
$$

The other case to consider is $p^0 < 0$. This puts all poles on the same side of the real axis. Thus this integral yields zero.

$$ {}_L i\Delta^+_L(p,X) = i\Theta(X_1)\Theta(-p^0)(0). $$

To check this computation, one can do it again, but doing the integrals in another order

Right Right

$$ {}_R i\Delta^+_R(x,x') = i\int \frac{dk_1' dl^0 d^2k dk_1''}{(2\pi)^5} \frac{ie^{-ik_1'x_1}}{k_1'+i\epsilon} \frac{ie^{-il^0T}}{l^0+i\epsilon} \frac{e^{-ik(x-x')}}{k^2+i\epsilon} \frac{ie^{-ik_1''x_1'}}{k_1''+i\epsilon}. $$

Transforming, one has

$$ {}_R i\Delta^+_R(p,X) = \int \frac{dT dr_1 dk_1' dl^0 d^2k dk_1''}{(2\pi)^5} e^{ip(x-x')} \frac{e^{-ik_1'x_1}}{k_1'+i\epsilon} \frac{e^{-il^0T}}{l^0+i\epsilon} \frac{e^{-ik(x-x')}}{k^2+i\epsilon} \frac{e^{-ik_1''x_1'}}{k_1''+i\epsilon} $$

$$ = \int \frac{dk_1' dl^0 d^2k dk_1''}{(2\pi)^5} \frac{1}{k_1'+i\epsilon} \frac{\delta(k^0-p^0-l^0)}{l^0+i\epsilon} \frac{e^{-i(k_1'+k_1'')X_1}}{(k_1-k^0+i\bar{\epsilon}\mathrm{sign}())(k_1+k^0-i\bar{\epsilon}\mathrm{sign}())} \frac{\delta(p_1-k_1-\frac{1}{2}(k_1'-k_1''))}{k_1''+i\epsilon} $$

$$ \begin{aligned} &{}_R i\Delta^+_R(p,X) \\ &= \int \frac{dk_1' dl^0 dk_1''}{(2\pi)^3} \frac{e^{-i(k_1'+k_1'')X_1}}{(k_1'+i\epsilon)(l^0+i\epsilon)(k_1''+i\epsilon)} \\ &\times \frac{1}{[p_1-\frac{1}{2}(k_1'-k_1'')+p^0+l^0-i\bar{\epsilon}\mathrm{sign}(p^0+l^0)][p_1-\frac{1}{2}(k_1'-k_1'')-p^0-l^0+i\bar{\epsilon}\mathrm{sign}(p^0+l^0)]} \end{aligned} $$

$$ \begin{aligned} &{}_R i\Delta^+_R(p,X) \\ &= \int \frac{ds_1 dl^0 dq_1}{(2\pi)^3} \frac{8e^{-is_1X_1}}{(s_1+q_1+2i\epsilon)(s_1-q_1+2i\epsilon)(l^0+i\epsilon)} \\ &\times \frac{1}{[2p_1+2p^0+2l^0-2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)-q_1][2p_1-2p^0-2l^0+2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)-q_1]} \end{aligned} $$

$$ \begin{aligned} &{}_R i\Delta^+_R(p,X) \\ &= i\int \frac{dl^0 dq_1}{(2\pi)^2} \frac{4\Theta(X_1)e^{2\epsilon X_1}}{(q_1)(l^0+i\epsilon)} \\ &\times \frac{e^{iq_1X_1}-e^{-iq_1X_1}}{[2p_1+2p^0+2l^0-2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)-q_1][2p_1-2p^0-2l^0+2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)-q_1]} \end{aligned} $$

$$
{}_R i\Delta_R^+(p,X)
$$
$$
= i\int \frac{dl^0 dq_1}{(2\pi)^2}\frac{4\Theta(X_1)e^{2\epsilon X_1}}{(q_1)(l^0+i\epsilon)}
$$
$$
\times \frac{e^{iq_1X_1}-e^{-iq_1X_1}}{[-2p_1-2p^0-2l^0+2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)+q_1][-2p_1+2p^0+2l^0-2i\bar{\epsilon}\mathrm{sign}(p^0+l^0)+q_1]}
$$

$$
\begin{aligned}
{}_R i\Delta_R^+(p,X) = i\Theta(X_1)\Theta(p^0)\Bigg\{ \\
&-\frac{i}{2\pi}\int\frac{dl^0}{(l^0+i\epsilon)}\frac{e^{2\epsilon X_1}}{(p_1+p^0+l^0-i\bar{\epsilon})(-p_1+p^0+l^0-i\bar{\epsilon})} \\
&+\frac{i}{2}\int\frac{dl^0}{2\pi}\frac{e^{2\epsilon X_1}e^{-2i(p_1+p^0+l^0-i\bar{\epsilon})X_1}}{(p_1+p^0+l^0-i\bar{\epsilon})(l^0+i\epsilon)(p^0+l^0-i\bar{\epsilon})} \\
&+\frac{i}{2}\int\frac{dl^0}{2\pi}\frac{e^{2\epsilon X_1}e^{2i(p_1-p^0-l^0+i\bar{\epsilon})X_1}}{(-p_1+p^0+l^0-i\bar{\epsilon})(l^0+i\epsilon)(p^0+l^0-i\bar{\epsilon})}\Bigg\}.
\end{aligned}
$$

These three integrals must be computed separately. Each has three poles.

$$
{}_R i\Delta_R^+(p,X) = i\Theta(X_1)\Theta(p^0)\Bigg\{
$$
$$
-\frac{e^{2\epsilon X_1}}{(p_1+p^0-i\tilde{\epsilon})(-p_1+p^0-i\tilde{\epsilon})}+\frac{1}{2}\frac{e^{2\epsilon X_1}e^{-2i(p_1+p^0-i\bar{\epsilon})X_1}}{(p^0-i\tilde{\epsilon})(p_1+p^0-i\tilde{\epsilon})}+\frac{1}{2}\frac{e^{2\epsilon X_1}e^{2i(p_1-p^0+i\bar{\epsilon})X_1}}{(p^0-i\tilde{\epsilon})(-p_1+p^0-i\tilde{\epsilon})}\Bigg\}. \quad (6.4)
$$

The other case of the opposite sign for p^0 is zero.

$$
{}_R i\Delta_R^+(p,X) = i\Theta(X_1)\Theta(-p^0)\{0\} = 0.
$$

Left Right

$$
{}_L i\Delta_R^+(p,X) =
$$
$$
-\int\frac{dT\,dr_1 dk_1' dl^0 d^2k\,dk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-ik_1'x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik_1''x_1'}}{(l^0+i\epsilon)(k_1'-i\epsilon)[-(k^0-i\bar{\epsilon}\mathrm{sign}(k^0))^2+k_1^2](k_1''+i\epsilon)}
$$

$$
{}_L i\Delta_R^+(p,X) =
$$
$$
-\frac{i\Theta(X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{-2i(p_1+p^0-i\epsilon-i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(p_1+p^0-i\bar{\epsilon}-2i\epsilon)}-\frac{i\Theta(-X_1)\Theta(p^0)e^{2\epsilon X_1}e^{2i(p_1+p^0-i\epsilon-i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(p_1+p^0-i\bar{\epsilon}-2i\epsilon)}.
$$

Right Left

$$
{}_R i\Delta^+_L(p,X) = \\
-\int \frac{dT dr_1 dk'_1 dl^0 d^2k dk''_1}{(2\pi)^5} \frac{e^{ip(x-x')}e^{-ik'_1x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik''_1x'_1}}{(l^0+i\epsilon)(k'_1+i\epsilon)[-(k^0-i\bar{\epsilon}\mathrm{sign}(k^0))^2+k_1^2](k''_1-i\epsilon)}
$$

$$
{}_R i\Delta^+_L(p,X) = \\
-\frac{i\Theta(X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{2i(p_1-p^0+i\epsilon+i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(-p_1+p^0-i\bar{\epsilon}-2i\epsilon)} - \frac{i\Theta(-X_1)\Theta(p^0)e^{2\epsilon X_1}e^{-2i(p_1-p^0+i\epsilon+i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(-p_1+p^0-i\bar{\epsilon}-2i\epsilon)}.
$$

Firstly, it is reassuring to note that the positive Wightman function has zero value for the range of negative frequency. The opposite is true for the negative Wightman function. Secondly, a similar statement can be made about position. Indeed, the right right projected Wightman functions only have value for positive x_1 coordinates with the opposite true for the left left case. It is only the left right and right left case that have contributions from each side. One will now justify calculating all possible projections as the previously calculated density matrix involved coefficients A, B and C projected on both the left and right. Since these coefficients have relations to the two point functions in the entire space, one will need to project these on the left and right as well. At this stage, one can now see that the reduced density matrix will now be a complicated expression. Regardless, one now has all the components of the reduced density matrix and will thus be able to make all expressions in terms of the familiar variables of time and position.

Chapter 7

Statistical Function

Here, it is attempted to make the connection between the previous calculations and physical quantities. Firstly, previous expressions will be transformed to real space. Doing this allows one to recover the original Feynman propagator, validating the calculations thus far. Then relations made be made for the two point functions and quantities such as the Gaussian invariant and density matrix.

The statistical two point point function will be useful for dealing with time dependent problems and is given in terms of the Wightman functions in the following way

$$F(x, x') = \langle \frac{1}{2}\{\phi(x), \phi(x')\}\rangle = \frac{1}{2}(i\Delta^{+} + i\Delta^{-}).$$

[17] This leads to the previous two point functions as

$$\langle \hat{\phi}(\vec{x})\hat{\phi}(\vec{y})\rangle = F(\vec{x}, \vec{y}; t) = \mathrm{Tr}[\hat{\rho}(t)\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})].$$

7.1 Right Right

The right projected positive and negative Wightman functions together give the projected statistical function.

$$
{}_RF_R(p,X) = \frac{i}{2}e^{2\epsilon X_1}\times
$$
$$
\Bigg\{ -\frac{\Theta(X_1)\Theta(p^0)}{(p_1+p^0-i\tilde{\epsilon})(-p_1+p^0-i\tilde{\epsilon})} + \frac{\Theta(X_1)\Theta(p^0)e^{-2i(p_1+p^0-i\tilde{\epsilon})X_1}}{2(p^0-i\tilde{\epsilon})(p_1+p^0-i\tilde{\epsilon})} + \frac{\Theta(X_1)\Theta(p^0)e^{2i(p_1-p^0+i\tilde{\epsilon})X_1}}{2(p^0-i\tilde{\epsilon})(-p_1+p^0-i\tilde{\epsilon})}
$$
$$
-\frac{\Theta(X_1)\Theta(-p^0)}{(-p_1+p^0+i\tilde{\epsilon})(p_1+p^0+i\tilde{\epsilon})} + \frac{\Theta(X_1)\Theta(-p^0)e^{-2i(p_1-p^0-i\tilde{\epsilon})X_1}}{2(-p_1+p^0+i\tilde{\epsilon})(p^0+i\tilde{\epsilon})} + \frac{\Theta(X_1)\Theta(-p^0)e^{2i(p_1+p^0+i\tilde{\epsilon})X_1}}{2(p_1+p^0+i\tilde{\epsilon})(p^0+i\tilde{\epsilon})}\Bigg\}.
$$

One now computes derivatives with respect to time to find relations.

$$
{}_RF_R(x,x') = \int \frac{d^2p}{(2\pi)^2}e^{-ip(x-x')}{}_RF_R(p,X)
$$

$$
\begin{aligned}
\frac{\partial_R F_R(x,x')}{\partial t} &= \int \frac{d^2p}{(2\pi)^2} ip^0 e^{-ip(x-x')}{}_RF_R(p,X) \\
\frac{\partial_R F_R(x,x')}{\partial t'} &= \int \frac{d^2p}{(2\pi)^2} (-ip^0) e^{-ip(x-x')}{}_RF_R(p,X) \\
\frac{\partial^2{}_R F_R(x,x')}{\partial t \partial t'} &= \int \frac{d^2p}{(2\pi)^2} {p^0}^2 e^{-ip(x-x')}{}_RF_R(p,X).
\end{aligned}
$$

It is hoped to compute these integrals by means of using the Theta function to evaluate from zero to infinity. Then take the coincidence time limit. This will lead to infinities however.
The six terms above, without and derivatives, may be combined to give the projected propagator.

$$
\begin{aligned}
&{}_RF_R(T,X_1,r_1) = {}_R\langle\phi\phi\rangle_R \\
&= -\frac{\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T-r_1+i\epsilon)\tilde{\epsilon})+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T+r_1+i\epsilon)\tilde{\epsilon})+...\Big]\Big\} \\
&+\frac{\Theta(X_1)\Theta(-2X_1-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-i\pi+\gamma_E+\ln((T+r_1+i\epsilon)\tilde{\epsilon})+...\Big] \\
&+\frac{\Theta(X_1)\Theta(r_1-2X_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-i\pi+\gamma_E+\ln((T-r_1+i\epsilon)\tilde{\epsilon})+...\Big] \\
&-\frac{\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T+r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T+r_1-i\epsilon)\tilde{\epsilon})+...\Big] \\
&+e^{\tilde{\epsilon}(T-r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T-r_1-i\epsilon)\tilde{\epsilon})+...\Big]\Big\} \\
&+\frac{\Theta(X_1)\Theta(-2X_1-r_1)e^{\tilde{\epsilon}r_1}e^{(T-r_1)\tilde{\epsilon}}}{2(2\pi)}[-i\pi+\gamma_E+\ln((T-r_1-i\epsilon)\tilde{\epsilon})+...] \\
&+\frac{\Theta(X_1)\Theta(r_1-2X_1)e^{-\tilde{\epsilon}r_1}e^{(T+r_1)\tilde{\epsilon}}}{2(2\pi)}[-i\pi+\gamma_E+\ln((T+r_1-i\epsilon)\tilde{\epsilon})+...].
\end{aligned}
$$

In the interest of expressing this in terms of Theta functions of T, as this highlights the positive and negative Wightman functions, extra Theta functions of r_1 may be added to the terms that do not already have them, for free by what is implied by the already existing Theta functions.

$$
\begin{aligned}
&-\frac{\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)+...\Big]\Big\} \\
&+\frac{\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)+...\Big] \\
&+\frac{\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)+...\Big].
\end{aligned}
$$

Using the identity for the step functions here, this is

$$
\begin{aligned}
&-\frac{\Theta(T)\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\Big\}\\
&-\frac{\Theta(-T)\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\Big\}\\
&+\frac{\Theta(T)\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\\
&+\frac{\Theta(T)\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\\
&+\frac{\Theta(-T)\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\\
&+\frac{\Theta(-T)\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]
\end{aligned}
$$

this can be combined to give:

$$
\begin{aligned}
&-\frac{\Theta(T)\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T+i\epsilon)^2-r_1^2)\big)+...\Big]\\
&-\frac{\Theta(-T)\Theta(X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T-i\epsilon)^2-r_1^2)\big)+...\Big]\Big\}\\
&+\frac{\Theta(T)\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T+i\epsilon)^2-r_1^2)\big)+...\Big]\Big\}\\
&+\frac{\Theta(T)\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T+i\epsilon)^2-r_1^2)\big)+...\Big]\\
&+\frac{\Theta(-T)\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-r_1)e^{\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T-i\epsilon)^2-r_1^2)\big)+...\Big]\\
&+\frac{\Theta(-T)\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\bar{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(\mu^2((T-i\epsilon)^2-r_1^2)\big)+...\Big]+\phi(\mu)^2.
\end{aligned}
$$

One may attempt further compactification of the expression, however this would require assumptions that r_1 and X_1 are always positive. This is not true in general, it is indeed the reason that these step functions are here to define the function at given intervals. Had this been true in general, all step functions involving r_1 could be combined to $(\Theta(r_1+2X_1)-\Theta(r_1-2X_1)$. At this stage, one can see the original $1+1$ dimensional, massless propagator that carries the correct prefactor of $\frac{1}{4\pi}$. The other terms may be dealt with by considering left right and right left projected propagators.

$_R\langle\phi\pi\rangle_R$

The six terms above, where one derivative has been taken, can be recombined to

$$\frac{\Theta(X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T-r_1+i\epsilon)}+\frac{\Theta(-r_1)}{(T+r_1+i\epsilon)}\Big]-\frac{\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T+r_1+i\epsilon)}-\frac{\Theta(X_1)\Theta(r_1-2X_1)}{2(2\pi)(T-r_1+i\epsilon)}$$
$$+\frac{\Theta(X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T+r_1-i\epsilon)}+\frac{\Theta(-r_1)}{(T-r_1-i\epsilon)}\Big]-\frac{\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)}-\frac{\Theta(X_1)\Theta(r_1-2X_1)}{2(2\pi)(T+r_1-i\epsilon)}$$

$$\frac{2T\Theta(X_1)}{4\pi}\Big[\frac{\Theta(T)}{((T+i\epsilon)^2-r_1^2)}\Big]-\frac{2T\Theta(X_1)\Theta(-2X_1-r_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}-\frac{2T\Theta(X_1)\Theta(r_1-2X_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}$$
$$+\frac{2T\Theta(X_1)}{4\pi}\Big[\frac{\Theta(-T)}{((T-i\epsilon)^2-r_1^2)}\Big]-\frac{2T\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}-\frac{\Theta(X_1)\Theta(r_1-2X_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}$$

$\pi\phi$ is similar, but with an overall sign change. One can recall that the properties of the Theta function are such that $\Theta(-x)=1-\Theta(x)$.

$_R\langle\pi\pi\rangle_R$

The six terms where two derivatives were taken are

$$\frac{\Theta(X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T-r_1+i\epsilon)^2}+\frac{\Theta(-r_1)}{(T+r_1+i\epsilon)^2}\Big]-\frac{\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T+r_1+i\epsilon)^2}-\frac{\Theta(X_1)\Theta(r_1-2X_1)}{2(2\pi)(T-r_1+i\epsilon)^2}$$
$$+\frac{\Theta(X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T+r_1-i\epsilon)^2}+\frac{\Theta(-r_1)}{(T-r_1-i\epsilon)^2}\Big]-\frac{\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)^2}-\frac{\Theta(X_1)\Theta(r_1-2X_1)}{2(2\pi)(T+r_1-i\epsilon)^2}.$$

The reduced density matrix, from tracing out the left side, is

$$\rho_{red}=N(t)(\det D_{LL})^{-\frac{1}{2}}e^{-\bar{\phi}_R^{\dagger}[A-(A-C)D^{-1}(A-C)]\bar{\phi}_R-\phi_R'^{\dagger}[B-(B-C)D^{-1}(B-C)]\phi_R'+2\bar{\phi}_R^{\dagger}[C+(A-C)D^{-1}(B-C)]\phi_R'}$$

7.2 Recover the Feynmann propagator

As a check, one can recombine all components of all projected propagators to recover the Feynman propagator.

$$i\Delta_F = i_L\Delta_L + i_L\Delta_R + i_R\Delta_L + i_R\Delta_R. \tag{7.1}$$

The surviving terms are

$$\begin{aligned}
&\frac{\Theta(X_1)\Theta(r_1)}{4\pi}\Big[C + \ln[(T - r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T + r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big] \\
&+ \frac{\Theta(X_1)\Theta(-r_1)}{4\pi}\Big[C + \ln[(T + r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T - r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big] \\
&+ \frac{\Theta(-X_1)\Theta(r_1)}{4\pi}\Big[C + \ln[(T - r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T + r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big] \\
&+ \frac{\Theta(-X_1)\Theta(-r_1)}{4\pi}\Big[C + \ln[(T + r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T - r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big].
\end{aligned}$$

Firstly, the terms that are of positive and negative X_1 can be combined to simplify the expression.

$$\begin{aligned}
&\frac{\Theta(r_1)}{4\pi}\Big[C + \ln[(T - r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T + r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big] \\
&+ \frac{\Theta(-r_1)}{4\pi}\Big[C + \ln[(T + r_1 + i\epsilon)\tilde{\epsilon}] + ln[(T - r_1 - i\epsilon)\tilde{\epsilon}] + ...\Big]
\end{aligned}$$

where C is a constant $(-i\pi + 2\gamma_E)$. All these terms can now be combined and the scaling μ can be used to store the constants. This then returns the Feynman propagator as before, using the identity $1 = \Theta(T) + \Theta(-T)$ one has the answer in terms of the positive and negative Wightman functions.

$$\begin{aligned}
&\frac{\Theta(T)}{4\pi}\Big[\ln[\mu^2((T + i\epsilon)^2 - r_1^2)] + \phi(\mu)^2\Big] \\
&+ \frac{\Theta(-T)}{4\pi}\Big[\ln[\mu^2((T - i\epsilon)^2 - r_1^2)] + \phi(\mu)^2\Big].
\end{aligned}$$

Or as it was in an earlier section

$$\begin{aligned}
&\frac{\Theta(T)}{4\pi}\Big[\ln[\mu^2(r_1^2 - (T - i\epsilon)^2)] + \phi(\mu)^2\Big] \\
&+ \frac{\Theta(-T)}{4\pi}\Big[\ln[\mu^2(r_1^2 - (T + i\epsilon)^2)] + \phi(\mu)^2\Big].
\end{aligned}$$

Chapter 8

Gaussian Invariant

8.1 General Gaussian Invariant

The Gaussian invariant is a conserved quantity contained by two Heisenberg equations for the position and momentum operators, given by a general quadratic Hamiltonian.

$\hat{H}(t) = \frac{1}{2}\left[A(t)\hat{Q}^2 + B(t)\hat{P}^2 + C(t)\{\hat{Q},\hat{P}\} + 2D(t)\hat{Q} + 2E(t)\hat{P}(t)\right]$ [17], where $\hat{Q}$ and $\hat{P}$ can be shifted to the familiar operators of $\hat{\phi}$ and $\hat{\Pi}$. It is a quantity that can parameterise the purity of a state. One can therefore state that the entropy should be a function of this quantity. In terms of the two point functions, it is expressed as:

$$\begin{aligned}
&\Delta^2(t;\vec{x},\vec{y}) = \frac{4}{\hbar^2}\left[\langle\hat{\phi}\hat{\phi}\rangle \cdot \langle\hat{\Pi}\hat{\Pi}\rangle - \frac{1}{4}\langle\{\hat{\phi},\hat{\Pi}\}\rangle \cdot \langle\{\hat{\phi},\hat{\Pi}\}\rangle\right](t;\vec{x},\vec{y})\\
&= \frac{4}{\hbar^2}\Big[\frac{\hbar^2}{2}(2D^{-1}B - 2D^{-1}BD^{-1}B + 4D^{-1}BD^{-1}C - 2D^{-1}CD^{-1}C)\\
&+ \frac{\hbar^2}{4}(4((C-B)D^{-1})^2 + \delta + 2\delta(C-B)D + 2(C-B))D^{-1}\delta\Big](t;\vec{x},\vec{y})\\
&= \delta(\vec{x}-\vec{y}) + 4CD^{-1}(t;\vec{x},\vec{y}).
\end{aligned} \tag{8.1}$$

Putting this in terms of the reduced density matrix, one has

$$
\begin{aligned}
A_{red} &= P_R[A - P_R(A-C)P_L D^{-1} P_L(A-C)P_R]P_R \\
B_{red} &= P_R[B - P_R(B-C)P_L D^{-1} P_L(B-C)P_R]P_R \\
C_{red} &= P_R[C + P_R(A-C)P_L D^{-1} P_L(B-C)P_R]P_R \\
D_{red} &= P_R[D - (A+B)P_L D^{-1} P_L(A+B-4C)P_R - 4CP_L D^{-1} P_L C]P_R \\
D_{red} &= P_R[D - (A+B)P_L D^{-1} P_L(A+B)P_R + 4(A+B-C)P_L D^{-1} P_L C]P_R
\end{aligned}
$$

finding the inverse of D_{red} and acting on it with C would be a complicated task and will involve the inverse of the two point functions. This is not ideal, as resulting convolution integrals will appear divergent and one can not regulate these integrals as one has already chosen an $i\epsilon$ prescription for the propagator. Taking the inverse of the two point functions will change this choice of $i\epsilon$.

It can also be expressed in terms of $F(t,t')$

$$
\Delta^2 = 4[F(t,t')\partial_t\partial_t' F(t,t') - \{\partial_t F(t,t')\}^2]_{t=t'}.
$$

One can also apply projection operators to this object such that,

$$
\begin{aligned}
&{}_R\Delta^2_R(t;\vec{x},\vec{y}) \\
&= \frac{4}{\hbar^2}\Big[{}_R\langle\hat{\phi}\hat{\phi}\rangle_R \cdot_R \langle\hat{\Pi}\hat{\Pi}\rangle_R - \frac{1}{4}{}_R\langle\{\hat{\phi},\hat{\Pi}\}\rangle_R \cdot_R \langle\{\hat{\phi},\hat{\Pi}\}\rangle_R\Big](t;\vec{x},\vec{y})
\end{aligned} \tag{8.2}
$$

this is a simpler expression and can be read as:

$$
{}_R\Delta_R^2(t;\vec{x},\vec{x'}) = \int dy
$$

$$
\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\Bigg\{4
$$

$$
\times\Bigg[-\theta(-x)\log\left(t^2-(x-y+i\epsilon)^2\right)+\theta(x-y)\log\left(t^2-(-x+y+i\epsilon)^2\right)
$$

$$
+\theta(-y)\log\left(t^2-(-x+y+i\epsilon)^2\right)+\theta(y-x)\log\left(t^2-(x-y+i\epsilon)^2\right)\Bigg]
$$

$$
\times\Bigg[\frac{\theta(-y)}{(t+y-x'+i\epsilon)^2}+\frac{\theta(-y)}{(t-y+x'-i\epsilon)^2}-\frac{\theta(y-x')}{(t-y+x'+i\epsilon)^2}-\frac{\theta(y-x')}{(t+y-x'-i\epsilon)^2}
$$

$$
+\frac{\theta(-x')}{(t+y-x'-i\epsilon)^2}+\frac{\theta(-x')}{(t-y+x'+i\epsilon)^2}-\frac{\theta(x'-y)}{(t-y+x'-i\epsilon)^2}-\frac{\theta(x'-y)}{(t+y-x'+i\epsilon)^2}\Bigg]
$$

$$
+\Bigg[\frac{\theta(-x)}{t+x-y+i\epsilon}+\frac{\theta(-x)}{t-x+y-i\epsilon}-\frac{\theta(x-y)}{t+x-y-i\epsilon}-\frac{\theta(x-y)}{t-x+y+i\epsilon}
$$

$$
+\frac{\theta(-y)}{t+x-y-i\epsilon}+\frac{\theta(-y)}{t-x+y+i\epsilon}-\frac{\theta(y-x)}{t-x+y-i\epsilon}-\frac{\theta(y-x)}{t+x-y+i\epsilon}\Bigg]
$$

$$
\times\Bigg[\frac{\theta(-y)}{t+y-x'+i\epsilon}+\frac{\theta(-y)}{t-y+x'-i\epsilon}-\frac{\theta(y-x')}{t+y-x'-i\epsilon}-\frac{\theta(y-x')}{t-y+x'+i\epsilon}
$$

$$
+\frac{\theta(-x')}{t+y-x'-i\epsilon}+\frac{\theta(-x')}{t-y+x'+i\epsilon}-\frac{\theta(x'-y)}{t-y+x'-i\epsilon}-\frac{\theta(x'-y)}{t+y-x'+i\epsilon}\Bigg]\Bigg\}
$$

factorising:

$$
\begin{aligned}
&{}_R\Delta_R^2(t;\vec{x},\vec{x'}) = \int dy \\
&\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\Bigg\{ \\
&\Bigg[-\theta(-x)\log\left(t^2-(x-y+i\epsilon)^2\right)+\theta(x-y)\log\left(t^2-(x-y-i\epsilon)^2\right) \\
&-\theta(-y)\log\left(t^2-(x-y-i\epsilon)^2\right)+\theta(y-x)\log\left(t^2-(x-y+i\epsilon)^2\right)\Bigg] \\
&\times 4\Bigg[+\frac{\theta(-y)2[t^2+(y-x'+i\epsilon)^2]}{(t+y-x'+i\epsilon)^2(t-y+x'-i\epsilon)^2}-\frac{\theta(y-x')2[t^2+(y-x'-i\epsilon)^2]}{(t-y+x'+i\epsilon)^2(t+y-x'-i\epsilon)^2} \\
&+\frac{\theta(-x')2[t^2+(y-x'-i\epsilon)^2]}{(t+y-x'-i\epsilon)^2(t-y+x'+i\epsilon)^2}-\frac{\theta(x'-y)2[t^2+(y-x'+i\epsilon)^2]}{(t-y+x'-i\epsilon)^2(t+y-x'+i\epsilon)^2}\Bigg] \\
&+\Bigg[\frac{\theta(-x)2t}{(t+x-y+i\epsilon)(t-x+y-i\epsilon)}-\frac{\theta(x-y)2t}{(t+x-y-i\epsilon)(t-x+y+i\epsilon)} \\
&+\frac{\theta(-y)2t}{(t+x-y-i\epsilon)(t-x+y+i\epsilon)}-\frac{\theta(y-x)2t}{(t-x+y-i\epsilon)(t+x-y+i\epsilon)}\Bigg] \\
&\times\Bigg[\frac{\theta(-y)2t}{(t+y-x'+i\epsilon)(t-y+x'-i\epsilon)}-\frac{\theta(y-x')2t}{(t+y-x'-i\epsilon)(t-y+x'+i\epsilon)} \\
&+\frac{\theta(-x')2t}{(t+y-x'-i\epsilon)(t-y+x'+i\epsilon)}-\frac{\theta(x'-y)2t}{(t-y+x'-i\epsilon)(t+y-x'+i\epsilon)}\Bigg]\Bigg\}.
\end{aligned}
$$

Here there are a total of 32 terms, each containing 4 theta functions, which effect the limits of integration. Eight of these terms are in fact immediately zero because of this, though the rest are not. The terms may be sectioned to the 16 different combinations of Theta functions, the first non-zero one, by multiplying terms chronologically, being:

$$
\begin{aligned}
&\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\theta(-x)\theta(y-x')\Bigg\{\\
&2\Big(\frac{(-x')\log[(t-(x+i\epsilon))(t+(x+i\epsilon))]}{(i\epsilon-t-x')(i\epsilon+t-x')}-\frac{(t+x-x')\log(-i\epsilon-t-x)}{(2i\epsilon+x-x')(2i\epsilon+2t+x-x')}\\
&-\frac{(-t+x-x')(\log(-i\epsilon+t-x)-\log(i\epsilon-t-x'))}{(2i\epsilon+x-x')(2i\epsilon-2t+x-x')}+\frac{(t+x-x')\log(i\epsilon+t-x')}{(2i\epsilon+x-x')(2i\epsilon+2t+x-x')}\Big)\\
&+2t\left(\frac{\log(i\epsilon+t+x)-\log(i\epsilon-t+x')}{(2t+x-x')(x-x')}+\frac{\log(i\epsilon-t+x)-\log(i\epsilon+t+x')}{(2t-x+x')(x-x')}\right)\Bigg\}.
\end{aligned}
$$

One has already performed the integral over y, here, it is now just an arbitrary variable. setting it to zero in the theta functions now implies x and x' must be zero. This makes the imaginary parts divergent. There are although four terms like this, where two of them come with a positive sign and two with a negative sign. So in the case where both x and x' are zero, the Gaussian invariant is zero.

$x > 0$ **and** $x' > 0$

The next term in the expression has theta functions, such that it is implied that both x and x' are greater than zero. There are four of these terms in total.
All four terms together simplifies to:

$$
\begin{aligned}
&\Theta(x_1)\Theta(x_1')\Theta(x_1-x_1')\\
&2\Bigg(-\frac{x'\log\left[(t-(x-i\epsilon))(t+(x-i\epsilon))\right]}{[t-(x'-i\epsilon)][t+(x'-i\epsilon)]}-\frac{x'\log\left(t^2-i\epsilon^2\right)}{(i\epsilon-t+x-x')(i\epsilon+t+x-x')}+\frac{x\log\left(t^2-i\epsilon^2\right)}{(i\epsilon-t+x-x')(i\epsilon+t+x-x')}\\
&-\frac{\log(i\epsilon-t)}{2t+x-x'}+\frac{\log(i\epsilon+t)}{2t-x+x'}-\frac{x'\log(i\epsilon-t-x)}{(x-x')(2t+x-x')}+\frac{x\log(i\epsilon-t-x)}{(x-x')(2t+x-x')}+\frac{x\log(i\epsilon+t-x)}{(x-x')(-2t+x-x')}\\
&+\frac{x'\log(i\epsilon+t-x)}{(x-x')(2t-x+x')}+\frac{x'\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')}+\frac{t\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')}-\frac{x\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')}\\
&+\frac{t\log(i\epsilon-t-x')}{(x-x')(2t-x+x')}+\frac{x'\log(i\epsilon+t-x')}{(x-x')(2t+x-x')}-\frac{x\log(i\epsilon+t-x')}{(x-x')(2t+x-x')}-\frac{x'\log(i\epsilon-t+x-x')}{(x-x')(-2t+x-x')}\\
&+\frac{x\log(i\epsilon-t+x-x')}{(x-x')(-2t+x-x')}-\frac{x'\log(i\epsilon+t+x-x')}{(x-x')(2t+x-x')}+\frac{x\log(i\epsilon+t+x-x')}{(x-x')(2t+x-x')}\Bigg).
\end{aligned}
$$

This is also under the condition that x is greater than x'. Otherwise the expression reads:

$$\Theta(x_1)\Theta(x_1')\Theta(x_1'-x_1)$$
$$2\Bigg(-\frac{x'\log\left[(t-(x'-i\epsilon))(t+(x'-i\epsilon))\right]}{[t-(x'-i\epsilon)][t+(x'-i\epsilon)]} + \frac{\log(i\epsilon-t)}{-2t+x-x'} + \frac{\log(i\epsilon+t)}{2t+x-x'} - \frac{x'\log(i\epsilon-t-x)}{(x-x')(2t+x-x')}$$
$$+\frac{x\log(i\epsilon-t-x)}{(x-x')(2t+x-x')} + \frac{x\log(i\epsilon+t-x)}{(x-x')(-2t+x-x')} + \frac{x'\log(i\epsilon+t-x)}{(x-x')(2t-x+x')} + \frac{x'\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')}$$
$$+\frac{t\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')} - \frac{x\log(i\epsilon-t-x')}{(x-x')(-2t+x-x')} + \frac{t\log(i\epsilon-t-x')}{(x-x')(2t-x+x')} + \frac{x'\log(i\epsilon+t-x')}{(x-x')(2t+x-x')}$$
$$-\frac{x\log(i\epsilon+t-x')}{(x-x')(2t+x-x')} + \frac{x'\log(i\epsilon-t-x+x')}{(x-x')(2t+x-x')} - \frac{x\log(i\epsilon-t-x+x')}{(x-x')(2t+x-x')} + \frac{x'\log(i\epsilon+t-x+x')}{(x-x')(-2t+x-x')}$$
$$+\frac{x\log(i\epsilon+t-x+x')}{(x-x')(2t-x+x')}\Bigg).$$

One would like to express this in momentum space and then sum over modes. This quantity is what would compose the entropy of the system. Although doing this for every term is heterogeneous, one would at least like to state if such a thing is finite or not.

To do this, one will examine the transform of each term and then the sum over the modes. If every term is finite, then the finite sum of finite terms is finite.

Writing this in terms of r and X, one has:

$$\Theta(x_1)\Theta(x_1')\Theta(x_1-x_1')\Bigg\{$$
$$\frac{(X_1-\frac{r_1}{2})\log\left[t-(X_1+\frac{r_1}{2}-i\epsilon)\right]}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]} + \frac{(X_1-\frac{r_1}{2})\log\left[t+(X_1+\frac{r_1}{2}-i\epsilon)\right]}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]}$$
$$+\frac{(r_1)\log\left(t^2-i\epsilon^2\right)}{(i\epsilon-t+r_1)(i\epsilon+t+r_1)} - \frac{\log(i\epsilon-t)}{2t+r_1} + \frac{\log(i\epsilon+t)}{2t-r_1}$$
$$+\frac{\log\big(i\epsilon-t-(X_1+\frac{r_1}{2})\big)}{(2t+r_1)} - \frac{\log\big(i\epsilon+t-(X_1+\frac{r_1}{2})\big)}{(2t-r_1)} - \frac{\log\big(i\epsilon-t-(X_1-\frac{r_1}{2})\big)}{(-2t+r_1)}$$
$$-\frac{\log\big(i\epsilon+t-(X_1-\frac{r_1}{2})\big)}{(2t+r_1)} + \frac{\log(i\epsilon-t+r_1)}{(-2t+r_1)} + \frac{\log(i\epsilon+t+r_1)}{(2t+r_1)}\Bigg\}.$$

One may now note the limits of the integration, for the transformation, are between $-2X_1$ and $2X_1$.

$$
\Delta^2(t;k,X_1) = \int_{-2X_1}^{2X_1} dr_1 e^{ikr_1}\Theta(x_1)\Theta(x_1')\Theta(x_1-x_1')\Bigg\{
$$

$$
\frac{(X_1-\frac{r_1}{2})\log\left[t-(X_1+\frac{r_1}{2}-i\epsilon)\right]}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]} + \frac{(X_1-\frac{r_1}{2})\log\left[t+(X_1+\frac{r_1}{2}-i\epsilon)\right]}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]}
$$

$$
+\frac{(r_1)\log(t^2-i\epsilon^2)}{(i\epsilon-t+r_1)(i\epsilon+t+r_1)} - \frac{\log(i\epsilon-t)}{2t+r_1} + \frac{\log(i\epsilon+t)}{2t-r_1}
$$

$$
+\frac{\log\big(i\epsilon-t-(X_1+\frac{r_1}{2})\big)}{(2t+r_1)} - \frac{\log\big(i\epsilon+t-(X_1+\frac{r_1}{2})\big)}{(2t-r_1)} - \frac{\log\big(i\epsilon-t-(X_1-\frac{r_1}{2})\big)}{(-2t+r_1)}
$$

$$
-\frac{\log\big(i\epsilon+t-(X_1-\frac{r_1}{2})\big)}{(2t+r_1)} + \frac{\log(i\epsilon-t+r_1)}{(-2t+r_1)} + \frac{\log(i\epsilon+t+r_1)}{(2t+r_1)}\Bigg\}.
$$

The first half of the above expression has poles below the real axis, thus they can be computed by closing a contour from below. There are difficulties however in dealing with the branch cut from the log in the expression. Also note the expression for the case of $x' > x$, in this case the signs of these two variables are reversed. One can then relabel variables and combine with the first case to eliminate the step function for $x - x'$.

To calculate the entropy, one then needs the sum over the modes, this means integrating over k. Then the easiest way to do this would be to do the k integral first and then use the resulting delta function to perform the original transform integral, but there is a problem commuting integrals with the branch cut from the Log functions. Making an approximation for the Logs, one has:

$$
\Delta^2(t;k,X_1) = \lim_{\delta\to 0}\int_{-2X_1}^{2X_1} dr_1 e^{ikr_1}\Theta(x_1)\Theta(x_1')\Bigg\{
$$

$$
\frac{(X_1-\frac{r_1}{2})\left\{\frac{\left[t-(X_1+\frac{r_1}{2}-i\epsilon)\right]^\delta-1}{\delta}\right\}}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]} + \frac{(X_1-\frac{r_1}{2})\left\{\frac{\left[t+(X_1+\frac{r_1}{2}-i\epsilon)\right]^\delta-1}{\delta}\right\}}{[X_1-\frac{r_1}{2}-i\epsilon+t][X_1-\frac{r_1}{2}-i\epsilon-t]}
$$

$$
+\frac{(r_1)\log(t^2-i\epsilon^2)}{(i\epsilon-t+r_1)(i\epsilon+t+r_1)} - \frac{\log(i\epsilon-t)}{2t+r_1} + \frac{\log(i\epsilon+t)}{2t-r_1}
$$

$$
+\frac{\left\{\frac{\left[i\epsilon-t-(X_1+\frac{r_1}{2})\right]^\delta-1}{\delta}\right\}}{(2t+r_1)} - \frac{\left\{\frac{\left[i\epsilon+t-(X_1+\frac{r_1}{2})\right]^\delta-1}{\delta}\right\}}{(2t-r_1)} - \frac{\left\{\frac{\left[i\epsilon-t-(X_1-\frac{r_1}{2})\right]^\delta-1}{\delta}\right\}}{(-2t+r_1)}
$$

$$
-\frac{\left\{\frac{\left[i\epsilon+t-(X_1-\frac{r_1}{2})\right]^\delta-1}{\delta}\right\}}{(2t+r_1)} + \frac{\left\{\frac{[i\epsilon-t+r_1]^\delta-1}{\delta}\right\}}{(-2t+r_1)} + \frac{\left\{\frac{[i\epsilon+t+r_1]^\delta-1}{\delta}\right\}}{(2t+r_1)}\Bigg\}.
$$

Now one claims, $\Delta^2(t,X) = \int dk\Delta^2(t;k,X_1)$. Remembering that in a definite domain, $\int_b^c f(x)\delta(x-a)dx = f(a)$ for $b < a < c$. Doing this integral means the entropy is written as:

$$\Delta^2(t;X_1) = \lim_{\delta\to 0}\Theta(X_1)\Bigg\{$$
$$\frac{(X_1)\left\{\frac{[t-(X_1-i\epsilon)]^\delta-1}{\delta}\right\}}{[X_1-i\epsilon+t][X_1-i\epsilon-t]} + \frac{(X_1)\left\{\frac{[t+(X_1-i\epsilon)]^\delta-1}{\delta}\right\}}{[X_1-i\epsilon+t][X_1-i\epsilon-t]}$$
$$-\frac{\log(i\epsilon-t)}{2t} + \frac{\log(i\epsilon+t)}{2t}$$
$$+\frac{\left\{\frac{[i\epsilon-t-X_1]^\delta-1}{\delta}\right\}}{(2t)} - \frac{\left\{\frac{[i\epsilon+t-(X_1)]^\delta-1}{\delta}\right\}}{(2t)} - \frac{\left\{\frac{[i\epsilon-t-(X_1)]^\delta-1}{\delta}\right\}}{(-2t)}$$
$$-\frac{\left\{\frac{[i\epsilon+t-(X_1)]^\delta-1}{\delta}\right\}}{(2t)} + \frac{\left\{\frac{[i\epsilon-t]^\delta-1}{\delta}\right\}}{(-2t)} + \frac{\left\{\frac{[i\epsilon+t]^\delta-1}{\delta}\right\}}{(2t)}\Bigg\}$$

$$\Delta^2(t;X_1) = \Theta(X_1)\Bigg\{\frac{(X_1)\log[t-(X_1-i\epsilon)]}{[X_1-i\epsilon+t][X_1-i\epsilon-t]} + \frac{(X_1)\log[t+(X_1-i\epsilon)]}{[X_1-i\epsilon+t][X_1-i\epsilon-t]}$$
$$-\frac{\log(i\epsilon-t)}{t} + \frac{\log(i\epsilon+t)}{t} + \frac{\log[i\epsilon-t-X_1]}{t} - \frac{\log[i\epsilon+t-(X_1)]}{t}\Bigg\}.$$

To comment on scaling symmetry in the above result. There are terms which are X_1 independent, so this is a symmetry which has been broken by the projection. One can also note that the quantity is unbounded.

One might have attempted this integral using known special functions, this means doing the transformation integral first and looking at the case of large X_1. Using the integral $\int_0^u (x+\beta)^\nu(u-x)^{\mu-1}dx = \frac{\beta^\nu u^\mu}{\mu}{}_2F_1(1,-\nu;1+\mu;-u/\beta)$ [18]. Taking an expansion for the Hypergeometric function for large u yields: $a(1/u) + u^\nu(b + c/u + ...) + O(1/u^2)$. The problem is that in order to do the integral, one must write the exponential as a sum, then one has the sum from $\mu = 0$ to infinity. So each term of the sum corresponds to each term of the expansion for the X_1 independent terms. In addition to this, there are issues with the singularities, as the integral is undefined there. Thus, this method does not appear to be useful.

8.2 Renyi Entropy

In the first subsection of this chapter, one attempted to calculate the Von Neumann entropy using the Gaussian invariant. This seemed like a good idea as the expression for the reduced density matrix was much more complicated. The problem faced was the inability to write the Gaussian invariant in a diagonal form. Namely, there was no clear way to solve the eigenvalue equation. One speculates what might have happened had one used different projectors. In that, the step functions used are really distributions, not functions, perhaps these problems would be avoided if a well defined function, such as *tanh*, had been used for the projections. Regardless, one will still attempt to calculate a different type of entropy using the existing relations.

$$\begin{aligned} S_R^{(n)} &= -\frac{1}{n-1}\log[Tr(\hat{\rho}^n)] \\ S_R^{(2)} &= -\log[Tr(\hat{\rho}^2)] \qquad n = 2, 3, ... \end{aligned}$$

So now, one will calculate ρ^2 in a general case.

$$\begin{aligned} Tr[\rho^2] &= \int \mathcal{D}\phi\mathcal{D}\phi' \langle\phi|\,\hat{\rho}\,|\phi'\rangle \langle\phi'|\,\hat{\rho}\,|\phi\rangle \\ &= \int \mathcal{D}\phi\mathcal{D}\phi' N^2 e^{-\phi(A+B)\phi-\phi'(A+B)\phi'+2\phi C\phi'+2\phi' C\phi} \\ &= \int \mathcal{D}\phi\mathcal{D}\phi' N^2 e^{-[\phi-2\phi' C(A+B)^{-1}][A+B][\phi-2(A+B)^{-1}C\phi']+4\phi' C(A+B)^{-1}C\phi'-\phi'(A+B)\phi'} \\ &= \int \mathcal{D}\tilde{\phi}\mathcal{D}\phi' N^2 e^{-\tilde{\phi}[A+B]\tilde{\phi}-\phi'[(A+B)-4C(A+B)^{-1}C]\phi'}. \end{aligned}$$

Now that the familiar Gaussian form has been recovered, the integrals are easily calculated.

$$\begin{aligned} Tr[\rho^2] &= N^2\left(\frac{\pi}{\det(A+B)}\right)^{\frac{1}{2}}\left(\frac{\pi}{\det(A+B-4C(A+B)^{-1}C)}\right)^{\frac{1}{2}} \\ &= \frac{\det(A+B-2C)}{[\det(A+B)]^{\frac{1}{2}}[\det(A+B-4C(A+B)^{-1}C)]^{\frac{1}{2}}} \\ &= \frac{\det(D)}{\sqrt{\det(A+B)\det(D+2C[1-(A+B)^{-1}2C)]}} \\ &= \frac{\det(D)}{\sqrt{\det[(A+B)D+2(A+B)C(A+B)^{-1}D]}} \\ &= \frac{\sqrt{\det(A+B-2C)}}{\sqrt{\det[A+B+2C]}} \end{aligned}$$

The question is how to compute this, one may now return to the definition of the entropy and write:

$$S_R = -\frac{1}{2} Tr \left[\log \left(\frac{A + B - 2C}{A + B + 2C} \right) \right].$$

Using properties of the trace, one could compute this by integrating over the variables x and y:

$$S_R = -\frac{1}{2} \int dxdy \log \left(\frac{[A + B - 2C](t; \vec{x}, \vec{y})}{[A + B + 2C](t; \vec{x}, \vec{y})} \right) \tag{8.3}$$

Here, one again encounters the problem of having the inverse of the two point functions from the relations of A, B and C. One may now speculate how things may look had smooth well defined functions been used for the projection. The theta functions appear here from performing contour integrations. It was very useful to us contour integration otherwise the integrals would have been too hard to solve. It is hard to imagine these being easier should one use more complicated functions for projecting, but this is for future work.

Chapter 9

Conclusion

To highlight what has been achieved in this project, one has been able to construct the Feynman propagator in D-dimension for a free massive scalar field. This was done using the Poincaré invariant vacuum state. Although this result is well known, it was a useful exercise in preparation for the next result, that is performing the same calculation in the Rindler frame using a thermal state. This again leads to the two point function, as given by equation 3.13. This is in fact a function of the invariant distance, as was the Minkowski case. This shows how this bi-scalar quantity transforms and that one may change from one frame to the other by the simple coordinate transformation as outlined in chapter 3.

Returning to the problem introduced at the start of this thesis, that is the one of calculating entanglement entropy in the near horizon limit. One then introduced the Gaussian density matrix in equation 1.6 for free scalar fields. This represented the entire space of figure 1.1, but one needed a reduced density matrix to describe region I of the figure. This was achieved by tracing out the left hand side of the space, or rather the region behind the horizon. The resulting reduced density matrix was given by equation 5.3. In which, all components were projected onto the left and right parts of the space. The important relation between the full and reduced density matrices was that physical observables in region I do not distinguish between using the full or reduced density matrix. As was outlined by equation 5.1.

This motivated the next part of the project, that was to define projection operators to project the relevant two point functions onto the left and right sides. Although a tedious calculation, one was able to perform all projections and all relations to the density matrix. The expression for the right right two point function is given by equation 7.1. While the other results are found in the appendix.

The ultimate goal here was to calculate the entanglement entropy and to see whether it was divergent or finite. Although an expression was written for the entropy in the form of equation 8.3. One was unable to calculate the integrals here. Another attempt was made by using the Gaussian invariant as given by equation 8.1. Frustratingly though, this method also led too calculations which could not be performed.

Having not been able to make the final step here, one can still note that much work has been done in the right direction. This project was perhaps ambitious, but one would now like to use hindsight to direct further research. The choice of projectors in equation 6.1 involved sharp, step functions. Perhaps it would be interesting to use smooth functions with a finite width. Physically, this could be related to the idea of the "Brick Wall" model discussed in the introduction, where one is a small distance away from the event horizon and the horizon can grow due to in falling matter. This would also be useful when considering backreation effects. In addition to this, some of the calculation difficulties may have resulted from using what is really a distribution for the projection. One may hope that well behaved functions would lead to simpler expressions.

Chapter 10

Appendix

10.1 Position Momentum Correlator

$$\langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle$$
$$= i\hbar N(t)\int\mathcal{D}\psi\Big\{\delta(\vec{x}-\vec{y})$$
$$+\psi(t,\vec{x})\Big[-\int dy''B(t;\vec{y},\vec{y}'')\psi(\vec{y}'')-\int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{y})+2\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{y})\Big]$$
$$\times\, e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}\Big\}$$

This is very similar to before, but now the difficult integrals are

$$-\int\mathcal{D}\psi\psi(t,\vec{x})\int dy''B(t;\vec{y},\vec{y}'')\psi(\vec{y}'')e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$-\int\mathcal{D}\psi\psi(t,\vec{x})\int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{y})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$2\int\mathcal{D}\psi\psi(t,\vec{x})\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{y})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$

then doing the same steps as before

$$-\int\mathcal{D}\psi\psi(t,\vec{x})\int dy''B(t;\vec{y},\vec{y}'')\psi(\vec{y}'')e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}$$
$$=-\frac{1}{4}\int dy''B(t;\vec{y},\vec{y}'')(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{x},\vec{y}')+D^{-1}(\vec{y}',\vec{x};t)) \qquad (10.1)$$

$$
\begin{aligned}
&-\int \mathcal{D}\psi\psi(t,\vec{x})\int dx''\psi(\vec{x}'')B(t;\vec{x}'',\vec{y})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}\\
&=-\frac{1}{4}\int dx''B(t;\vec{x}'',\vec{y})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{x},\vec{x}')+D^{-1}(t;\vec{x}',\vec{x}))
\end{aligned}
\tag{10.2}
$$

$$
\begin{aligned}
&2\int \mathcal{D}\psi\psi(t,\vec{x})\int dx''\psi(\vec{x}'')C(t;\vec{x}'',\vec{y})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')}\\
&=\frac{1}{2}\int dx''C(t;\vec{x}'',\vec{y})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{x}')+D^{-1}(t;\vec{x}',\vec{y}))
\end{aligned}
\tag{10.3}
$$

Putting these together

$$
\begin{aligned}
&\langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle=i\hbar\Big\{\delta(\vec{x}-\vec{y})-\frac{1}{4}\int dy''B(t;\vec{y},\vec{y}'')(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{x},\vec{y}')+D^{-1}(\vec{y}',\vec{x};t))\\
&-\frac{1}{4}\int dx''B(t;\vec{x}'',\vec{y})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{x},\vec{x}')+D^{-1}(t;\vec{x}',\vec{x}))\\
&+\frac{1}{2}\int dx''C(t;\vec{x}'',\vec{y})(\det D)^{-\frac{1}{2}}(D^{-1}(t;\vec{y},\vec{x}')+D^{-1}(t;\vec{x}',\vec{y}))\Big\}
\end{aligned}
\tag{10.4}
$$

10.2 Momentum-Momentum correlator

Finally, the last relation will be computed.

$$
\begin{aligned}
&\langle\hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y})\rangle=tr(\hat{\rho}_f\hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y}))\\
&=\int \mathcal{D}\psi dadb\langle\psi|a\rangle\rho_f(a,b;t)\langle b|\hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y})|\psi\rangle
\end{aligned}
\tag{10.5}
$$

$$
\begin{aligned}
&=\int \mathcal{D}\psi db\rho_f(\psi,b;t)\langle b|\hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y})|\psi\rangle\\
&=\int \mathcal{D}\psi db\rho_f(\psi,b;t)(-i\hbar)\frac{\delta}{\delta b(\vec{x})}\langle b|\hat{\Pi}(\vec{y})|\psi\rangle\\
&=i\hbar\int \mathcal{D}\psi db\frac{\delta}{\delta b(\vec{x})}\{\rho_f(\psi,b;t)\}\langle b|\hat{\Pi}(\vec{y})|\psi\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= i\hbar \int \mathcal{D}\psi db \\
&\times \Big\{ - \int dy'' B(\vec{x}, \vec{y}'') b(\vec{y}'') - \int dx'' b(\vec{x}'') B(\vec{x}'', \vec{x}) + \int dy'' C(\vec{x}, \vec{y}'') \psi(\vec{y}'') + \int dx'' \psi(\vec{x}'') C(\vec{x}'', \vec{x}) \Big\} \\
&\times \rho_f(\psi, b; t) \langle b | \hat{\Pi}(\vec{y}) | \psi \rangle \\
&= -\hbar^2 \int \mathcal{D}\psi db \\
&\times \frac{\delta}{\delta b(\vec{y})} \Big\{ \Big[- \int dy'' B(\vec{x}, \vec{y}'') b(\vec{y}'') - \int dx'' b(\vec{x}'') B(\vec{x}'', \vec{x}) + \int dy'' C(\vec{x}, \vec{y}'') \psi(\vec{y}'') + \int dx'' \psi(\vec{x}'') C(\vec{x}'', \vec{x}) \Big] \\
&\times \rho_f(\psi, b; t) \Big\} \langle b | \psi \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\hbar^2 \int \mathcal{D}\psi db \\
&\times \Big[- B(\vec{y}, \vec{x}) - B(\vec{x}, \vec{y}) \Big] \rho_f(\psi, b; t) \langle b | \psi \rangle \\
&- \hbar^2 \int \mathcal{D}\psi db \\
&\times \Big[- \int dy'' B(\vec{x}, \vec{y}'') b(\vec{y}'') - \int dx'' b(\vec{x}'') B(\vec{x}'', \vec{x}) + \int dy'' C(\vec{x}, \vec{y}'') \psi(\vec{y}'') + \int dx'' \psi(\vec{x}'') C(\vec{x}'', \vec{x}) \Big] \\
&\times \Big[- \int dy''' B(\vec{y}, \vec{y}''') b(\vec{y}''') - \int dx''' b(\vec{x}''') B(\vec{x}''', \vec{y}) + \int dy''' C(\vec{y}, \vec{y}''') \psi(\vec{y}''') + \int dx''' \psi(\vec{x}''') C(\vec{x}''', \vec{y}) \Big] \\
&\times \rho_f(\psi, b; t) \} \langle b | \psi \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\hbar^2 \int \mathcal{D}\psi db \Big\{ - 2B(\vec{x}, \vec{y}) + \\
&\times \Big[- \int dy'' B(\vec{x}, \vec{y}'') b(\vec{y}'') - \int dx'' b(\vec{x}'') B(\vec{x}'', \vec{x}) + \int dy'' C(\vec{x}, \vec{y}'') \psi(\vec{y}'') + \int dx'' \psi(\vec{x}'') C(\vec{x}'', \vec{x}) \Big] \\
&\times \Big[- \int dy''' B(\vec{y}, \vec{y}''') b(\vec{y}''') - \int dx''' b(\vec{x}''') B(\vec{x}''', \vec{y}) + \int dy''' C(\vec{y}, \vec{y}''') \psi(\vec{y}''') + \int dx''' \psi(\vec{x}''') C(\vec{x}''', \vec{y}) \Big] \\
&\times \rho_f(\psi, b; t) \} \langle b | \psi \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\hbar^2 \int \mathcal{D}\psi \Big\{ -2B(\vec{x},\vec{y}) + \\
&\times \Big[-\int dy'' B(\vec{x},\vec{y}'')\psi(\vec{y}'') - \int dx'' \psi(\vec{x}'')B(\vec{x}'',\vec{x}) + \int dy'' C(\vec{x},\vec{y}'')\psi(\vec{y}'') + \int dx'' \psi(\vec{x}'')C(\vec{x}'',\vec{x}) \Big] \\
&\times \Big[-\int dy''' B(\vec{y},\vec{y}''')\psi(\vec{y}''') - \int dx''' \psi(\vec{x}''')B(\vec{x}''',\vec{y}) + \int dy''' C(\vec{y},\vec{y}''')\psi(\vec{y}''') + \int dx''' \psi(\vec{x}''')C(\vec{x}''',\vec{y}) \Big] \\
&\times \rho_f(\psi,\psi;t)\}
\end{aligned}
$$

Using the relation that

$$
\begin{aligned}
&\int \mathcal{D}\psi \int dvdw E(\vec{y},\vec{v};t)\psi(\vec{v})E(\vec{x},\vec{w};t)\psi(\vec{w})e^{-\int dx'dy'\psi(\vec{x}')D(t;\vec{x}',\vec{y}')\psi(\vec{y}')+\int J(\vec{x}')E(\vec{x}',\vec{u})\psi(\vec{u})} \\
&= (\det(D))^{-\frac{1}{2}}\frac{1}{2}ED^{-1}E(\vec{x},\vec{y};t)e^{\frac{1}{4}\int dx'dy' J(\vec{x}')D^{-1}(t;\vec{x}',\vec{y}')J(\vec{y}')} + \ldots
\end{aligned}
$$

the result may be given as

$$
\langle \hat{\Pi}(\vec{x})\hat{\Pi}(\vec{y})\rangle = -2\hbar^2 N(t)(\det D)^{-\frac{1}{2}}[-B + BD^{-1}B - BD^{-1}C - CD^{-1}B + CD^{-1}C](t;\vec{x},\vec{y})
$$

10.3 Projecting the negative Wightman function

What is good to note, that in the last subsection, one saw that the case of $p^0 < 0$ did not contribute. Physically this is because one was dealing with the positive frequency Wightman functions, hence the $^+$ notation. Now, one will do the same for the negative Wightman functions and find that only negative frequency terms contribute. Once this is done, every component will have been found and may be re summed to give the Feynman propagator. This is just a check. Really one is interested in working further with the projected components.

Left Left

$$
\begin{aligned}
&i_L\Delta_L^-(p,X) = \\
&i\int \frac{dTdr_1dk_1'dl^0d^2kdk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-ik_1'x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik_1''x_1'}}{(k_1'-i\epsilon)(l^0-i\epsilon)[-(k^0-i\bar{\epsilon})^2+k_1^2](k_1''-i\epsilon)}.
\end{aligned}
$$

Performing the integrals yields

$$i_L\Delta_L^-(p,X) = \Theta(-X_1)\Theta(-p^0)$$
$$-\frac{e^{-2\epsilon X_1}}{(-p_1+p^0+i\bar{\epsilon}+i\epsilon)(p_1+p^0+i\bar{\epsilon}+i\epsilon)} + \frac{e^{-2\epsilon X_1}e^{2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)} + \frac{e^{-2\epsilon X_1}e^{-2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X}}{2(p_1+p^0+i\bar{\epsilon}+i\epsilon)(p^0+i\bar{\epsilon}}$$

Left Right

$$i_L\Delta_R^-(p,X) =$$
$$-i\int\frac{dTdr_1dk_1'dl^0d^2kdk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-ik_1'x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik_1''x_1'}}{(k_1'-i\epsilon)(l^0-i\epsilon)[-(k^0-i\bar{\epsilon})^2+k_1^2](k_1''+i\epsilon)}.$$

Performing the integrals yields

$$i_L\Delta_R^-(p,X) =$$
$$-\frac{\Theta(X_1)\Theta(-p^0)e^{-2\epsilon X_1}e^{-2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)} - \frac{\Theta(-X_1)\Theta(-p^0)e^{2\epsilon X_1}e^{2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}.$$

Right Left

$$i_R\Delta_L^-(p,X) =$$
$$-i\int\frac{dTdr_1dk_1'dl^0d^2kdk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-ik_1'x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik_1''x_1'}}{(k_1'+i\epsilon)(l^0-i\epsilon)[-(k^0-i\bar{\epsilon})^2+k_1^2](k_1''-i\epsilon)}.$$

Performing the integrals yields

$$i_R\Delta_L^-(p,X) =$$
$$-\frac{\Theta(X_1)\Theta(-p^0)e^{-2\epsilon X_1}e^{2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)} - \frac{\Theta(-X_1)\Theta(-p^0)e^{2\epsilon X_1}e^{-2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}.$$

Right Right

$$i_R\Delta_R^-(p,X) =$$
$$i\int\frac{dTdr_1dk_1'dl^0d^2kdk_1''}{(2\pi)^5}\frac{e^{ip(x-x')}e^{-ik_1'x_1}e^{-il^0T}e^{-ik(x-x')}e^{-ik_1''x_1'}}{(k_1'+i\epsilon)(l^0-i\epsilon)[-(k^0-i\bar{\epsilon})^2+k_1^2](k_1''+i\epsilon)}.$$

Performing the integrals yields

$$i_R\Delta_R^-(p,X) = \Theta(X_1)\Theta(-p^0)$$
$$-\frac{e^{2\epsilon X_1}}{(-p_1+p^0+i\bar{\epsilon}+i\epsilon)(p_1+p^0+i\bar{\epsilon}+i\epsilon)} + \frac{e^{2\epsilon X_1}e^{-2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)} + \frac{e^{2\epsilon X_1}e^{2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(p_1+p^0+i\bar{\epsilon}+i\epsilon)(p^0+i\bar{\epsilon}} \tag{10.6}$$

10.4 Left Left

The left projected positive and negative Wightman functions together give the projected statistical function.

$$_LF_L(p,X) = \frac{1}{2}e^{-2\epsilon X_1}$$
$$\Bigg\{-\frac{\Theta(-X_1)\Theta(p^0)}{(-p_1+p^0-i\tilde{\epsilon})(p_1+p^0-i\tilde{\epsilon})} + \frac{\Theta(-X_1)\Theta(p^0)e^{-2i(p_1-p^0+i\tilde{\epsilon})X_1}}{2(-p_1+p^0-i\tilde{\epsilon})(p^0-i\tilde{\epsilon})} + \frac{\Theta(-X_1)\Theta(p^0)e^{2i(p_1+p^0-i\tilde{\epsilon})X_1}}{2(p_1+p^0-i\tilde{\epsilon})(p^0-i\tilde{\epsilon})}$$
$$-\frac{\Theta(-X_1)\Theta(-p^0)}{(-p_1+p^0+i\tilde{\epsilon})(p_1+p^0+i\tilde{\epsilon})} + \frac{\Theta(-X_1)\Theta(-p^0)e^{2i(p_1-p^0-i\tilde{\epsilon})X_1}}{2(-p_1+p^0+i\tilde{\epsilon})(p^0+i\tilde{\epsilon})} + \frac{\Theta(-X_1)\Theta(-p^0)e^{-2i(p_1+p^0+i\tilde{\epsilon})X_1}}{2(p_1+p^0+i\tilde{\epsilon})(p^0+i\tilde{\epsilon})}\Bigg\}.$$

One now computes derivatives with respect to time to find relations.

$$_LF_L(x,x') = \int \frac{d^2p}{(2\pi)^2}e^{-ip(x-x')}{}_LF_L(p,X)$$

$$\begin{aligned}
\frac{\partial_L F_L(x,x')}{\partial t} &= \int \frac{d^2p}{(2\pi)^2} ip^0 e^{-ip(x-x')}{}_LF_L(p,X) \\
\frac{\partial_L F_L(x,x')}{\partial t'} &= \int \frac{d^2p}{(2\pi)^2} (-ip^0) e^{-ip(x-x')}{}_LF_L(p,X) \\
\frac{\partial^2{}_L F_L(x,x')}{\partial t \partial t'} &= \int \frac{d^2p}{(2\pi)^2} p^{0\,2} e^{-ip(x-x')}{}_LF_L(p,X).
\end{aligned}$$

Together, this yields

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_L = {}_LF_L(x,x') \\
&= \frac{i\Theta(-X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T-r_1+i\epsilon)\tilde{\epsilon})+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T+r_1+i\epsilon)\tilde{\epsilon})+...\Big]\Big\} \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[\gamma_E+\ln(-i(T-r_1+i\epsilon)\tilde{\epsilon})+i\mathrm{si}(-i(T-r_1)\tilde{\epsilon})\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)e^{-i\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[\gamma_E+\ln(-i(T+r_1+i\epsilon)\tilde{\epsilon})+i\mathrm{si}(-i(T+r_1)\tilde{\epsilon})\Big] \\
&+\frac{i\Theta(-X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T+r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T+r_1-i\epsilon)\tilde{\epsilon})+...\Big] \\
&+e^{\tilde{\epsilon}(T-r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-i\pi+\gamma_E+\ln((T-r_1-i\epsilon)\tilde{\epsilon})+...\Big]\Big\} \\
&-\frac{i\Theta(-X_1)\Theta(2X_1-r_1)e^{\tilde{\epsilon}r_1}}{2(2\pi)}e^{(T-r_1)\tilde{\epsilon}}[\gamma_E+\ln(-i(T-r_1-i\epsilon)\tilde{\epsilon})+i\mathrm{si}(i(T-r_1)\tilde{\epsilon})+...] \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{(T+r_1)\tilde{\epsilon}}[\gamma_E+\ln(-i(T+r_1-i\epsilon)\tilde{\epsilon})+i\mathrm{si}(i(T+r_1)\tilde{\epsilon})+...].
\end{aligned}
$$

This is rewritten as,

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_L = {}_LF_L(x,x') \\
&= \frac{i\Theta(-X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)+...\Big]\Big\} \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(r_1)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(-r_1)e^{-i\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)\Big].
\end{aligned}
$$

Inserting an identity,

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_L = {}_LF_L(x,x') \\
&= \frac{i\Theta(-X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)\Theta(T)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T-r_1)}\Theta(r_1)\Theta(-T)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)\Theta(T)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T+r_1)}\Theta(-r_1)\Theta(-T)e^{\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big]\Big\} \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(r_1)\Theta(T)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(r_1)\Theta(-T)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(-r_1)\Theta(T)e^{-i\bar{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(-r_1)\Theta(-T)e^{-i\bar{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big].
\end{aligned}
$$

Some of this can be combined,

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_L = {}_LF_L(x,x') \\
&= \frac{i\Theta(-X_1)}{2(2\pi)}\Big\{e^{\tilde{\epsilon}(T-r_1)}\Theta(T)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big] \\
&+e^{\tilde{\epsilon}(T-r_1)}\Theta(-T)e^{-\tilde{\epsilon}r_1}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...\Big] \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(T)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(-T)e^{\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(T)e^{-i\bar{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big] \\
&-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(-T)e^{-i\bar{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}\Big[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\phi\rangle_L\\
&\frac{i\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T-r_1+i\epsilon)}+\frac{\Theta(-r_1)}{(T+r_1+i\epsilon)}\Big]-\frac{i\Theta(-X_1)\Theta(2X_1+r_1)}{2(2\pi)(T-r_1+i\epsilon)}-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T+r_1+i\epsilon)}\\
&+\frac{i\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T+r_1-i\epsilon)}+\frac{\Theta(-r_1)}{(T-r_1-i\epsilon)}\Big]+\frac{i\Theta(-X_1)\Theta(2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)}-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T+r_1-i\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\phi\rangle_L\\
&\frac{2iT\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(T)}{((T+i\epsilon)^2-r_1^2)}\Big]-\frac{2iT\Theta(-X_1)\Theta(2X_1+r_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}-\frac{2iT\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}\\
&+\frac{2iT\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(-T)}{((T-i\epsilon)^2-r_1^2)}\Big]+\frac{2iT\Theta(-X_1)\Theta(2X_1-r_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}-\frac{2iT\Theta(-X_1)\Theta(r_1+2X_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\pi\rangle_L\\
&\frac{i\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T-r_1+i\epsilon)^2}+\frac{\Theta(-r_1)}{(T+r_1+i\epsilon)^2}\Big]-\frac{i\Theta(-X_1)\Theta(2X_1+r_1)}{2(2\pi)(T-r_1+i\epsilon)^2}-\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T+r_1+i\epsilon)^2}\\
&+\frac{i\Theta(-X_1)}{4\pi}\Big[\frac{\Theta(r_1)}{(T+r_1-i\epsilon)^2}+\frac{\Theta(-r_1)}{(T-r_1-i\epsilon)^2}\Big]+\frac{i\Theta(-X_1)\Theta(2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)^2}-\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T+r_1-i\epsilon)^2}.
\end{aligned}
$$

10.5 Right Left

The right left projected positive and negative Wightman functions together give the projected statistical function.

$$
\begin{aligned}
&{}_RF_L(p,X)=\\
&\frac{1}{2}\Bigg\{-\frac{\Theta(X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{2i(p_1-p^0+i\epsilon+i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(-p_1+p^0-i\bar{\epsilon}-2i\epsilon)}-\frac{\Theta(-X_1)\Theta(p^0)e^{2\epsilon X_1}e^{-2i(p_1-p^0+i\epsilon+i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(-p_1+p^0-i\bar{\epsilon}-2i\epsilon)}\\
&-\frac{\Theta(X_1)\Theta(-p^0)e^{-2\epsilon X_1}e^{2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}-\frac{\Theta(-X_1)\Theta(-p^0)e^{2\epsilon X_1}e^{-2i(p_1+p^0+i\bar{\epsilon}+i\epsilon)X_1}}{2(p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}\Bigg\}.
\end{aligned}
$$

One now computes derivatives with respect to time to find relations.

$$ {}_L F_R(x,x') = \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-x')} {}_L F_R(p,X) $$

$$ \begin{aligned} \frac{\partial {}_L F_R(x,x')}{\partial t} &= \int \frac{d^2p}{(2\pi)^2} ip^0 e^{-ip(x-x')} {}_L F_R(p,X) \\ \frac{\partial {}_L F_R(x,x')}{\partial t'} &= \int \frac{d^2p}{(2\pi)^2} (-ip^0) e^{-ip(x-x')} {}_L F_R(p,X) \\ \frac{\partial^2 {}_L F_R(x,x')}{\partial t \partial t'} &= \int \frac{d^2p}{(2\pi)^2} p^{0\,2} e^{-ip(x-x')} {}_L F_R(p,X). \end{aligned} $$

Together, this yields

$$ \begin{aligned} &{}_R\langle\phi\phi\rangle_L \\ &= i\frac{\Theta(X_1)\Theta(r_1-2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)} e^{-(T-r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T-r_1+i\epsilon)\tilde{\epsilon})+...] \\ &+ \frac{i\Theta(-X_1)\Theta(2X_1+r_1)e^{-\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}[-i\pi+\gamma_E+\ln((T-r_1+i\epsilon)\tilde{\epsilon})] \\ &+ i\frac{\Theta(X_1)\Theta(r_1-2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)} e^{(T+r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T+r_1-i\epsilon)\tilde{\epsilon})] \\ &+ i\frac{\Theta(-X_1)\Theta(r_1+2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)} e^{(T+r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T+r_1-i\epsilon)\tilde{\epsilon})+...]. \end{aligned} $$

Expressing this otherwise

$$ \begin{aligned} &{}_R\langle\phi\phi\rangle_L \\ &= i\frac{\Theta(X_1)\Theta(r_1-2X_1)\Theta(r_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)} e^{-(T-r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)+...] \\ &+ \frac{i\Theta(-X_1)\Theta(2X_1+r_1)\Theta(r_1)e^{-\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1-i\epsilon)^2)\tilde{\epsilon}\big)]. \end{aligned} $$

Inserting the identitiy,

$$
\begin{aligned}
&= i\frac{\Theta(X_1)\Theta(r_1-2X_1)\Theta(T)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T-r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...] \\
&+ i\frac{\Theta(X_1)\Theta(r_1-2X_1)\Theta(-T)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T-r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)+...] \\
&+ \frac{i\Theta(-X_1)\Theta(2X_1+r_1)\Theta(T)e^{-\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)] \\
&+ \frac{i\Theta(-X_1)\Theta(2X_1+r_1)\Theta(-T)e^{-\tilde{\epsilon}r_1}e^{-(T-r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)]
\end{aligned}
$$

$$
{}_R\langle\pi\phi\rangle_L
$$

$$
\frac{i\Theta(X_1)\Theta(-2X_1+r_1)}{2(2\pi)(T-r_1+i\epsilon)}+\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T-r_1+i\epsilon)}+\frac{i\Theta(X_1)\Theta(-2X_1+r_1)}{2(2\pi)(T+r_1-i\epsilon)}+\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T+r_1-i\epsilon)}
$$

$$
{}_R\langle\pi\phi\rangle_L = 2T
$$

$$
\frac{i\Theta(X_1)\Theta(-2X_1+r_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}+\frac{i\Theta(-X_1)\Theta(r_1+2X_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}+\frac{i\Theta(X_1)\Theta(-2X_1+r_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}+\frac{i\Theta(-X_1)\Theta(r_1+}{2(2\pi)((T-i}
$$

$$
{}_R\langle\pi\pi\rangle_L
$$

$$
\frac{i\Theta(X_1)\Theta(-2X_1+r_1)}{2(2\pi)(T-r_1+i\epsilon)^2}+\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T-r_1+i\epsilon)^2}+\frac{i\Theta(X_1)\Theta(-2X_1+r_1)}{2(2\pi)(T+r_1-i\epsilon)^2}+\frac{i\Theta(-X_1)\Theta(r_1+2X_1)}{2(2\pi)(T+r_1-i\epsilon)^2}.
$$

10.6 Left Right

The left right projected positive and negative Wightman functions together give the projected statistical function.

$$
\begin{aligned}
&{}_LF_R(p,X) = \\
&\frac{1}{2}\Bigg\{-\frac{\Theta(X_1)\Theta(p^0)e^{-2\epsilon X_1}e^{-2i(p_1+p^0-i\epsilon-i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(p_1+p^0-i\bar{\epsilon}-2i\epsilon)}-\frac{\Theta(-X_1)\Theta(p^0)e^{2\epsilon X_1}e^{2i(p_1+p^0-i\epsilon-i\bar{\epsilon})X_1}}{2(p^0-i\epsilon-i\bar{\epsilon})(p_1+p^0-i\bar{\epsilon}-2i\epsilon)} \\
&-\frac{\Theta(X_1)\Theta(-p^0)e^{-2\epsilon X_1}e^{-2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}-\frac{\Theta(-X_1)\Theta(-p^0)e^{2\epsilon X_1}e^{2i(p_1-p^0-i\bar{\epsilon}-i\epsilon)X_1}}{2(-p_1+p^0+i\bar{\epsilon}+2i\epsilon)(p^0+i\bar{\epsilon}+i\epsilon)}\Bigg\}.
\end{aligned}
$$

One now computes derivatives with respect to time to find relations.

$$
{}_LF_R(x,x') = \int \frac{d^2p}{(2\pi)^2}e^{-ip(x-x')}{}_LF_R(p,X)
$$

$$
\begin{aligned}
\frac{\partial_L F_R(x,x')}{\partial t} &= \int \frac{d^2p}{(2\pi)^2}ip^0e^{-ip(x-x')}{}_LF_R(p,X) \\
\frac{\partial_L F_R(x,x')}{\partial t'} &= \int \frac{d^2p}{(2\pi)^2}(-ip^0)e^{-ip(x-x')}{}_LF_R(p,X) \\
\frac{\partial^2{}_L F_R(x,x')}{\partial t\partial t'} &= \int \frac{d^2p}{(2\pi)^2}p^{0\,2}e^{-ip(x-x')}{}_LF_R(p,X).
\end{aligned}
$$

Together, this yields

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_R \\
&\quad= i\frac{\Theta(X_1)\Theta(-r_1-2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T+r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T+r_1+i\epsilon)\tilde{\epsilon})+...] \\
&\quad+\frac{i\Theta(-X_1)\Theta(2X_1-r_1)e^{-\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}[-i\pi+\gamma_E+\ln((T+r_1+i\epsilon)\tilde{\epsilon})] \\
&\quad+i\frac{\Theta(X_1)\Theta(-r_1-2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{(T-r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T-r_1-i\epsilon)\tilde{\epsilon})] \\
&\quad+i\frac{\Theta(-X_1)\Theta(-r_1+2X_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{(T-r_1)\tilde{\epsilon}}[-i\pi+\gamma_E+\ln((T-r_1-i\epsilon)\tilde{\epsilon})].
\end{aligned}
$$

Expressing this otherwise as,

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_R \\
&\quad = i\frac{\Theta(X_1)\Theta(-r_1-2X_1)\Theta(-r_1)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T+r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)] \\
&\quad + \frac{i\Theta(-X_1)\Theta(2X_1-r_1)\Theta(-r_1)e^{-\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big((T^2-(r_1+i\epsilon)^2)\tilde{\epsilon}\big)].
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
&{}_L\langle\phi\phi\rangle_R \\
&\quad = i\frac{\Theta(X_1)\Theta(-r_1-2X_1)\Theta(T)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T+r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)] \\
&\quad + i\frac{\Theta(X_1)\Theta(-r_1-2X_1)\Theta(-T)e^{-\tilde{\epsilon}r_1}}{2(2\pi)}e^{-(T+r_1)\tilde{\epsilon}}[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)] \\
&\quad + \frac{i\Theta(-X_1)\Theta(2X_1-r_1)\Theta(T)e^{-\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big(((T+i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)] \\
&\quad + \frac{i\Theta(-X_1)\Theta(2X_1-r_1)\Theta(-T)e^{-\tilde{\epsilon}r_1}e^{-(T+r_1)\tilde{\epsilon}}}{2(2\pi)}[-2i\pi+2\gamma_E+\ln\big(((T-i\epsilon)^2-r_1^2)\tilde{\epsilon}\big)]
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\phi\rangle_R \\
&\frac{i\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T+r_1+i\epsilon)}+\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T+r_1+i\epsilon)}+\frac{i\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)}+\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T-r_1-i\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\phi\rangle_R = 2T \\
&\frac{i\Theta(X_1)\Theta(-2X_1-r_1)\Theta(T)}{2(2\pi)((T+i\epsilon)^2-r_1^2)}+\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)\Theta(T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}+\frac{i\Theta(X_1)\Theta(-2X_1-r_1)\Theta(-T)}{2(2\pi)((T-i\epsilon)^2-r_1^2)}+\frac{i\Theta(-X_1)\Theta(-r}{2(2\pi)((T}
\end{aligned}
$$

$$
\begin{aligned}
&{}_L\langle\pi\pi\rangle_R \\
&\frac{i\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T+r_1+i\epsilon)^2}+\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T+r_1+i\epsilon)^2}+\frac{i\Theta(X_1)\Theta(-2X_1-r_1)}{2(2\pi)(T-r_1-i\epsilon)^2}+\frac{i\Theta(-X_1)\Theta(-r_1+2X_1)}{2(2\pi)(T-r_1-i\epsilon)^2}.
\end{aligned}
$$

10.7 Gaussian invariant for $x > 0$ and $x' = 0$

The case in which x is greater than zero while x' is zero occurs in two terms:

$$\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\theta(x-y)\theta(y-x')\Bigg\{$$
$$4t^2\left(\frac{(x-2t)\log(i\epsilon-t)+(-2t-x)\log(i\epsilon+t)+(2t+x)\log(-i\epsilon-t+x)+(2t-x)\log(-i\epsilon+t+x)}{2t(2i\epsilon-x)(2i\epsilon+2t-x)(-2i\epsilon+2t+x)}\right.$$
$$\left.-\frac{(2t+x)\log(-i\epsilon-t)+(2t-x)\log(t-i\epsilon)-(2t-x)\log(i\epsilon-t-x)+(-2t-x)\log(i\epsilon+t-x)}{2t(2i\epsilon-x)(2i\epsilon+2t-x)(-2i\epsilon+2t+x)}\right)$$
$$-2\left(-\frac{x\log(t^2-i\epsilon^2)}{(i\epsilon-t-x)(i\epsilon+t-x)}-\frac{(t-x)\log(-i\epsilon-t)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(t+x)\log(i\epsilon-t)}{(2i\epsilon-x)(2i\epsilon-2t-x)}\right.$$
$$-\frac{(-t-x)\log(t-i\epsilon)}{(2i\epsilon-x)(2i\epsilon-2t-x)}-\frac{(t-x)\log(i\epsilon+t)}{(2i\epsilon-x)(2i\epsilon+2t-x)}-\frac{(t+x)\log(i\epsilon-t-x)}{(2i\epsilon-x)(2i\epsilon-2t-x)}$$
$$\left.+\frac{(t-x)\log(i\epsilon+t-x)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(t-x)\log(-i\epsilon-t+x)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(-t-x)\log(-i\epsilon+t+x)}{(2i\epsilon-x)(2i\epsilon-2t-x)}\right)\Bigg\}$$

and

$$\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\theta(x-y)\theta(-x')\Bigg\{$$
$$-4t^2\left(\frac{(x-2t)\log(i\epsilon-t)+(-2t-x)\log(i\epsilon+t)+(2t+x)\log(-i\epsilon-t+x)+(2t-x)\log(-i\epsilon+t+x)}{2t(2i\epsilon-x)(2i\epsilon+2t-x)(-2i\epsilon+2t+x)}\right.$$
$$\left.-\frac{(2t+x)\log(-i\epsilon-t)+(2t-x)\log(t-i\epsilon)-(2t-x)\log(i\epsilon-t-x)+(-2t-x)\log(i\epsilon+t-x)}{2t(2i\epsilon-x)(2i\epsilon+2t-x)(-2i\epsilon+2t+x)}\right)$$
$$+2\left(-\frac{x\log(t^2-i\epsilon^2)}{(i\epsilon-t-x)(i\epsilon+t-x)}-\frac{(t-x)\log(-i\epsilon-t)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(t+x)\log(i\epsilon-t)}{(2i\epsilon-x)(2i\epsilon-2t-x)}\right.$$
$$-\frac{(-t-x)\log(t-i\epsilon)}{(2i\epsilon-x)(2i\epsilon-2t-x)}-\frac{(t-x)\log(i\epsilon+t)}{(2i\epsilon-x)(2i\epsilon+2t-x)}-\frac{(t+x)\log(i\epsilon-t-x)}{(2i\epsilon-x)(2i\epsilon-2t-x)}$$
$$\left.+\frac{(t-x)\log(i\epsilon+t-x)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(t-x)\log(-i\epsilon-t+x)}{(2i\epsilon-x)(2i\epsilon+2t-x)}+\frac{(-t-x)\log(-i\epsilon+t+x)}{(2i\epsilon-x)(2i\epsilon-2t-x)}\right)\Bigg\}$$

10.8 Gaussian invariant for $x = 0$ and $x' > 0$

The case in which x' is greater than zero while x is zero also occurs in two terms:

$$
\begin{aligned}
&\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\theta(-x)\theta(x'-y)\Bigg\{ \\
&2\left(\frac{(-x')\log(t^2-i\epsilon^2)}{(i\epsilon-t-x')(i\epsilon+t-x')} - \frac{(t-x')\log(-i\epsilon-t)}{(2i\epsilon-x')(2i\epsilon+2t-x')} + \frac{(-t-x')\log(i\epsilon-t)}{(2i\epsilon-x')(-2i\epsilon+2t+x')}\right. \\
&- \frac{(t+x')\log(t-i\epsilon)}{(2i\epsilon-x')(-2i\epsilon+2t+x')} - \frac{(t-x')\log(i\epsilon+t)}{(2i\epsilon-x')(2i\epsilon+2t-x')} - \frac{(-t-x')\log(i\epsilon-t-x')}{(2i\epsilon-x')(-2i\epsilon+2t+x')} \\
&\left.+ \frac{(t-x')\log(i\epsilon+t-x')}{(2i\epsilon-x')(2i\epsilon+2t-x')} + \frac{(t-x')\log(-i\epsilon-t+x')}{(2i\epsilon-x')(2i\epsilon+2t-x')} + \frac{(t+x')\log(-i\epsilon+t+x')}{(2i\epsilon-x')(-2i\epsilon+2t+x')}\right) \\
&+ 4t^2\left(\frac{(-2t-x')\log(i\epsilon+t) + (2t+x')\log(-i\epsilon-t+x') + (2t-x')(\log(-i\epsilon+t+x')-\log(i\epsilon-t))}{2t(2i\epsilon-x')(2i\epsilon+2t-x')(-2i\epsilon+2t+x')}\right. \\
&\left.\left. - \frac{(2t+x')\log(-i\epsilon-t) + (2t-x')(\log(t-i\epsilon)-\log(i\epsilon-t-x')) + (-2t-x')\log(i\epsilon+t-x')}{2t(2i\epsilon-x')(2i\epsilon+2t-x')(-2i\epsilon+2t+x')}\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{4}\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{y+x'}{2}\right)\theta(y-x)\theta(x'-y)\Bigg\{ \\
&4t^2\left(\frac{(-2t-x')\log(i\epsilon+t) + (2t+x')\log(-i\epsilon-t+x') + (2t-x')(\log(-i\epsilon+t+x')-\log(i\epsilon-t))}{2t(2i\epsilon-x')(2i\epsilon+2t-x')(-2i\epsilon+2t+x')}\right. \\
&\left. - \frac{(2t+x')\log(-i\epsilon-t) + (2t-x')(\log(t-i\epsilon)-\log(i\epsilon-t-x')) + (-2t-x')\log(i\epsilon+t-x')}{2t(2i\epsilon-x')(2i\epsilon+2t-x')(-2i\epsilon+2t+x')}\right) \\
&- 2\left(\frac{(-x')\log(t^2-i\epsilon^2)}{(i\epsilon-t-x')(i\epsilon+t-x')} - \frac{(t-x')\log(-i\epsilon-t)}{(2i\epsilon-x')(2i\epsilon+2t-x')} - \frac{(t-x')\log(i\epsilon+t)}{(2i\epsilon-x')(2i\epsilon+2t-x')}\right. \\
&- \frac{(-t-x')(\log(t-i\epsilon)-\log(i\epsilon-t-x'))}{(2i\epsilon-x')(2i\epsilon-2t-x')} + \frac{(t-x')\log(i\epsilon+t-x')}{(2i\epsilon-x')(2i\epsilon+2t-x')} + \frac{(t-x')\log(-i\epsilon-t+x')}{(2i\epsilon-x')(2i\epsilon+2t-x')} \\
&\left.\left. + \frac{(-t-x')(\log(-i\epsilon+t+x')-\log(i\epsilon-t))}{(2i\epsilon-x')(2i\epsilon-2t-x')}\right)\right\}.
\end{aligned}
$$

Here, it is implied by the thetas that x' is greater than zero. So the previous divergence does not occur. Transforming involves several difficult integrals.

The transformation integrals of note are:

$$\int_0^\infty dx' \frac{x' e^{ik'x'}}{(x'+t-i\epsilon)(x'-t-i\epsilon)} \qquad \int_0^\infty dx' \frac{x' e^{ik'x'}\log(i\epsilon - t - x')}{(2i\epsilon - x')(-2i\epsilon + 2t + x')}$$
$$\int_0^\infty dx' \frac{x' e^{ik'x'}}{(2i\epsilon - x')(2i\epsilon + 2t - x')(-2i\epsilon + 2t + x')} \qquad \int_0^\infty dx' \frac{x' e^{ik'x'}\log(-i\epsilon + t + x')}{(2i\epsilon - x')(2i\epsilon + 2t - x')(-2i\epsilon + 2t + x')}$$
$$\int_0^\infty dx' \frac{e^{ik'x'}\log(-i\epsilon + t + x')}{(2i\epsilon - x')(2i\epsilon + 2t - x')(-2i\epsilon + 2t + x')} \qquad \int_0^\infty dx' \frac{e^{ik'x'}\log(-i\epsilon + t - x')}{(2i\epsilon - x')(2i\epsilon + 2t - x')(-2i\epsilon + 2t + x')}.$$

10.9 Relations

Using the results of previous sections, one can now attempt to find expressions for the time dependent matrices A, B and C.

$$\langle \phi(t,\vec{x})\phi(t,\vec{y})\rangle = \frac{1}{2}(A + B - 2C)^{-1}(t;\vec{x},\vec{y})$$

$$\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle = i\hbar\Big(C - B\Big)D^{-1}(t;\vec{x},\vec{y})$$

$$\langle \phi(t,\vec{x})\Pi(t,\vec{y})\rangle = i\hbar\Big(\delta(\vec{x}-\vec{y}) + (C-B)D^{-1}(t;\vec{x},\vec{y})\Big)$$

$$\langle \Pi(t,\vec{x})\Pi(t,\vec{y})\rangle = 2\hbar^2\Big(B - BD^{-1}B + 2BD^{-1}C - CD^{-1}C\Big)(t;\vec{x},\vec{y}). \tag{10.7}$$

Using that the transpose of a symmetric matrix is the matrix itself, these relations can be rearranged for

$$C(t;\vec{x},\vec{y}) = \frac{1}{2\hbar^2}\langle \Pi(t,\vec{x})\Pi(t,\vec{y})\rangle - \frac{i}{2\hbar}\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle \phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1} - \frac{1}{2\hbar^2}\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle \phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$

$$B(t;\vec{x},\vec{y}) = \frac{1}{2\hbar^2}\langle \Pi(t,\vec{x})\Pi(t,\vec{y})\rangle - \frac{1}{2\hbar^2}\langle \Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle \phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$

$$A(t;\vec{x},\vec{y}) = \frac{1}{2\hbar^2}\langle\Pi(t,\vec{x})\Pi(t,\vec{y})\rangle - \frac{1}{2\hbar^2}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$
$$- \frac{i}{\hbar}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1} + \frac{1}{2}\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}. \quad (10.8)$$

Note that these can also be expressed in terms of the Gaussian invariant:

$$A(t;\vec{x},\vec{y}) = \frac{1}{2}\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1} - \frac{1}{2\hbar^2}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1} - \frac{i}{\hbar}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$
$$+ \frac{1}{8\hbar^2}[\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2 + 2\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle + \langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle^2 + \hbar^2\Delta^2(t;\vec{x},\vec{y})]\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$

$$B(t;\vec{x},\vec{y}) = -\frac{1}{2\hbar^2}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$
$$+ \frac{1}{8\hbar^2}[\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2 + 2\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle + \langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle^2 + \hbar^2\Delta^2(t;\vec{x},\vec{y})]\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$

$$C(t;\vec{x},\vec{y}) = -\frac{1}{2\hbar^2}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1} - \frac{i}{\hbar}\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$
$$+ \frac{1}{8\hbar^2}[\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle^2 + 2\langle\Pi(t,\vec{x})\phi(t,\vec{y})\rangle\langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle + \langle\phi(t,\vec{x})\Pi(t,\vec{y})\rangle^2 + \hbar^2\Delta^2(t;\vec{x},\vec{y})]\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle^{-1}$$

These can now be reinserted into the density matrix

$$\rho_{red} = N(t)[\det(P_L D P_L)]^{-\frac{1}{2}}$$
$$\times\, e^{-\bar{\phi}_R^\dagger[A-P_R(A-C)P_L D^{-1}P_L(A-C)P_R]\bar{\phi}_R} e^{-\phi_R'^\dagger[B-P_R(B-C)P_L D^{-1}P_L(B-C)P_R]\phi'_R} e^{2\bar{\phi}_R^\dagger[C+P_R(A-C)P_L D^{-1}P_L(B-C)P_R]\phi'_R}.$$

It is true here that this density matrix is only valid for the right side, thus A, B and C should be distinguished here as they too can only apply to the right side here. From the equations above however, the reduced A, B and C can be written by simply taking the projected two point functions.

$$D^{-1} = 2\langle\phi\phi\rangle$$
$$(A-C) = -\frac{i}{2\hbar}\langle\pi\phi\rangle\langle\phi\phi\rangle^{-1} + \frac{1}{2}\langle\phi\phi\rangle^{-1}$$
$$(B-C) = \frac{i}{2\hbar}\langle\pi\phi\rangle\langle\phi\phi\rangle^{-1}.$$

$$(A+B) = \frac{1}{\hbar^2}\langle \Pi\Pi\rangle - \frac{1}{\hbar^2}\langle \Pi\phi\rangle^2\langle\phi\phi\rangle^{-1} - \frac{i}{\hbar}\langle\Pi\phi\rangle\langle\phi\phi\rangle^{-1} + \frac{1}{2}\langle\phi\phi\rangle^{-1}. \tag{10.9}$$

Now one wants to see these relations with the projectors acting on them.

$$_LD_L^{-1} = 2_L\langle\phi\phi\rangle_L$$

$$\begin{aligned}
{}_R(A-C)_L &= -\frac{i}{2\hbar}{}_R\langle\pi\phi\rangle(P_R+P_L)\langle\phi\phi\rangle_L^{-1} + \frac{1}{2}{}_R\langle\phi\phi\rangle_L^{-1} \\
&= -\frac{i}{2\hbar}{}_R\langle\pi\phi\rangle_{RR}\langle\phi\phi\rangle_L^{-1} - \frac{i}{2\hbar}{}_R\langle\pi\phi\rangle_{LL}\langle\phi\phi\rangle_L^{-1} + \frac{1}{2}{}_R\langle\phi\phi\rangle_L^{-1} \\
{}_L(A-C)_R &= -\frac{i}{2\hbar}{}_L\langle\pi\phi\rangle(P_R+P_L)\langle\phi\phi\rangle_R^{-1} + \frac{1}{2}{}_L\langle\phi\phi\rangle_R^{-1} \\
&= -\frac{i}{2\hbar}{}_L\langle\pi\phi\rangle_{RR}\langle\phi\phi\rangle_R^{-1} - \frac{i}{2\hbar}{}_L\langle\pi\phi\rangle_{LL}\langle\phi\phi\rangle_R^{-1} + \frac{1}{2}{}_L\langle\phi\phi\rangle_R^{-1}
\end{aligned}$$

$$\begin{aligned}
{}_R(B-C)_L &= \frac{i}{2\hbar}{}_R\langle\pi\phi\rangle(P_R+P_L)\langle\phi\phi\rangle_L^{-1} \\
&= \frac{i}{2\hbar}{}_R\langle\pi\phi\rangle_{RR}\langle\phi\phi\rangle_L^{-1} + \frac{i}{2\hbar}{}_R\langle\pi\phi\rangle_{LL}\langle\phi\phi\rangle_L^{-1} \\
{}_L(B-C)_R &= \frac{i}{2\hbar}{}_L\langle\pi\phi\rangle(P_R+P_L)\langle\phi\phi\rangle_R^{-1} \\
&= \frac{i}{2\hbar}{}_L\langle\pi\phi\rangle_{RR}\langle\phi\phi\rangle_R^{-1} + \frac{i}{2\hbar}{}_L\langle\pi\phi\rangle_{LL}\langle\phi\phi\rangle_R^{-1}.
\end{aligned}$$

This is indeed a very complicated expression. worse than this, is the appearance of the inverse of the two point function. These two point functions are defined in terms of step functions, however these are not really functions, rather they are distributions. there is in fact no inverse to these functions and thus any expression that involves inverses of two point functions is not useful.

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