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Size and Infinite Sets

An analysis of an unorthodox paper

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Abstract

This thesis revolves around the unorthodox paper of Whittle (2015). Whittle's paper is unorthodox because it does not conform with the Cantorian orthodox position regarding set cardinality and set size. We will explain some key concepts necessary to understand Whittle's paper and the analysis thereof. In this analysis we will offer some criticism on his arguments. We will conclude among other things that finite concepts are always at play in the human mind and this should be taken into account when researching beyond finite concepts.

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Introduction

This thesis revolves around the unorthodox paper of Whittle (2015). Whittle's paper is unorthodox because it does not follow the Cantorian orthodox position. We will elaborate on this orthodox position further in this thesis. The subject of Whittle's paper is about set theory and includes infinite sets. We assume the reader knows about the basics of set theory, often referred to as naïve set theory.

As written by Bagaria (2019), "Both aspects of set theory, namely, as the mathematical science of the infinite, and as the foundation of mathematics, are of philosophical importance.". Set theory is important due to it being the foundation of mathematics. Almost anything mathematical that we deem true can be rewritten in set theoretic language and in turn be proven. For example, "1 + 1 = 2"can be proven within their own system that is number theory but also in set theoretic language. So set theory is able to unify all kinds of math into one language. The set theoretic language itself is not useful to make proofs with. It is the idea that as a foundation all kinds of math now have underlying connections as they should have.

In set theory there are finite and infinite sets. As the names suggest, finite sets contain a finite amount of elements and infinite sets contain an infinite amount of elements. Difficulties arise when addressing the *size* of an infinite set. This problem stems back from at least the time of Galileo Galilei. Galileo described a paradox that led him to conclude that infinite sets did not have the relations "greater", "smaller" and "equal" (Galilei, 1638/1954). Many years later Georg Cantor appeared and proved that there are multiple cardinalities of infinite sets (Hosch, 2016). The cardinality of a set could be seen as the size of a set. If there are infinite sets of different cardinalities then there could also be infinite sets of different sizes. This means that relations such as "greater", "smaller" and "equal" do exist. The works of Galileo and Cantor clashed but neither work apart seemed problematic. Galileo's work was more philosophical while Cantor's work was mathematical. We will see that Whittle's paper is also more philosophical in nature. We will be analysing Whittle's paper and offer some criticism on his arguments. Our conclusion will include a review on Whittle's paper and a few remarks about future papers on the subject. This thesis is in no way a hostile attack on Whittle's position or academic status. The author of this thesis was interested in the subject and coincidentally chose Whittle's paper.

The relevance of this thesis regarding Artificial Intelligence revolves around computability theory. In simple words the theory is about whether a mathematical problem can be solved or not. Computers are used to try to compute some of these mathematical problems. Some infinite sets, such as the natural numbers, are important to talk about in computability theory but also in general. The natural numbers occur throughout the daily life of a simple citizen but for scholars they means much more. Scholars use the natural numbers to count, to calculate and it is with these numbers that we can partially convey mathematical ideas with each other.

For example, if we asked a mathematical question to a computer such as, "What are the even numbers?", that computer will run an algorithm that will try and solve this question. This algorithm is capable of classifying numbers. A number could be assigned to have the value of "true" or "false". The number 1 could be mapped to false and the number 2 could be mapped to true. We can call this mapping of numbers to truth-values an answering schema. Now let us take another set that contains all the possible algorithms that we, as humans, can build and that computers can run. The set that contains all the possible answering schemas has a greater cardinality than the set of all possible algorithms. This means that there are mathematical questions that cannot be computed by algorithms. In other words, there are incomputable mathematical problems.

The relevance of this thesis regarding Artificial Intelligence also revolves around infinity itself. In order to understand infinity we have to understand how infinite sets behave. The set of natural numbers or the set of real numbers are often used in math but these sets are also infinite sets. In order to understand infinite sets, we need to at least understand these two individual sets mentioned before.

Key concepts

In this section a few concepts revolving around infinite set theory will be explained. They are needed in order to gain a better understanding of the analysis of Whittle's paper and how they are at play in Whittle's paper and in general. The concepts are explained in a form for those who are not familiar with them, or need a refresher.

Part-whole principle

There is a common and intuitive notion called the part-whole principle. If we were to have a regular banana then this would be *the whole*. Suppose we now cut the banana in half leaving us with two *parts*. It seems obvious that one part banana is not as much as a whole banana in terms of mass. We, as humans, do not expect to cut a banana in two pieces and to end up with each part to be equal to a whole banana. This would result in one banana turning into two. If this was the case we could end world starvation! In other words, a part of a banana surely cannot be greater than or equal to the whole banana.

This principle dates back to at least around 300BCE (van der Warden & Tasibak, 2019) in Euclid's *Elements*. In this textbook there are five *Euclid's common notions.* The fifth one is, "The whole is greater than a part". Even if the textbook only concerned itself with geometry, this principle does not only hold in geometry. We don't even think about this principle consciously as it is so integrated into our daily lives. We want to drink some milk and understand that the carton will be empty after a few glasses. We understand that biting off a piece of a cookie will not result in a whole cookie in our mouth.

In set theory this principle is also present and is most clearly observed in the finite space. Suppose a set A with the numerical elements 1, 2 and 3, and let us suppose this is our whole. Now let us take a part, set B with the numerical elements 1 and 2. It is true that the whole (set A) is greater than its part (set B), because A has one more element in it than B. So if there is a proper subset X of the set Y, then Y will be greater than X.

Hume's principle, cardinality and Cantor's theorem

To clarify for further use: a one-to-one correspondence is between two sets and a one-to-one correlation is specifically between the elements of two sets.

As Crispin Wright (1999) wrote:

> It was George Boolos who, following Frege's somewhat charitable lead at *Grundlagen* §63, first gave the name, "Hume's Principle," to the constitutive principle for identity of cardinal number: that the number of *F*s is the same as the number of *G*s just in case there exists a one-to-one correlation between the *F*s and the *G*s. (p. 6)

The *F*s can be seen as elements from a set and the *G*s can be seen as elements from another set. The one-to-one correlation between the *F*s and *G*s means that every element *F* corresponds to one and only one element *G* and vice versa. So if the *F*s are elements of a set *A*, and the *G*s would be elements of the set *B*, then there would be a one-to-one correspondence between the sets *A* and *B*. Since the set *A* and the set *B* have a one-to-on correspondence, it would mean that *A* and *B* share the same cardinal number in Cantor's terminology. The cardinal number indicates the number of elements in a set, so the orthodox view agrees that the cardinal number of a set is the same as the size of a set. Thus the size of *A* and the size of *B* would be the same. A one-to-one correspondence, also called a bijection, is formally defined in terms of the notions *injective* and *surjective*:

> **Definition 4.4.2**. Let *A* and *B* be sets, and let $f : A \rightarrow B$ be a function.
>
> (1) The function f is **injective** (also called **one-to-one** or **monotonic**) if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in A$; equivalently, if $f(x) = f(y)$ implies $x = y$ for all $x, y \in$ A.
> (2) The function f is **surjective** (also known as **onto** or **epic**) if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$; equivalently if $f_\star(A) = B$.
> (3) The function *f* is **bijective** if it is both injective and surjective. (c.f. Bloch 2011, 155)

Cantor defined that sets would have the same cardinality (also the cardinal number) if and only if there is a bijection (also a one-to-one correspondence) between them. So the set *A* containing the numerical elements 1, 2 and 3 has the same cardinality as the set *B* containing alphabetical elements a, b and c, since we can let the elements in one set uniquely correlate with elements in the other set. For example a function from A to B, 1 maps to a, 2 maps to b, and 3 maps to c. So we can construct a bijection between *A* and *B* and this results in the two sets sharing the same cardinality.

So if there is no bijection between two sets, their cardinality would not be the same. Cantor showed that the cardinality of a set is strictly smaller than the cardinality of its powerset, the set of all subsets (Hosch, 2016). This is also known as Cantor's theorem. He used this on infinite sets to show that it is possible to have infinite sets that do not share the same cardinality. Suppose were to iterate this powerset over an initial infinite set, we would get *the power set of the power set of the power set*… and so on. Our first infinite set cannot share the same cardinality of its powerset and this powerset in turn cannot share the same cardinality of its powerset *and on and on*. So there are multiple infinite sets that do not share the same cardinality.

An infinite set is called countably infinite when this set has a bijection with the set of natural numbers. An infinite set is called uncountably infinite when this set does not, or cannot have a bijection with the set of natural numbers.

Galileo's paradox

Galileo Galilei describes an interesting paradox in which the terms part-whole principle and one-to-one correspondence are important. What follows is not what Galileo said in his exact own words but a reconstruction in modern words. The paradox consists of a few premises that together result in a contradictory conclusion. We will briefly explain some (mathematical) terms before presenting the premises and conclusion.

Squares are numbers that are the result by multiplying a number with itself. The latter number is also called the root. So the square 9 would have 3 as its root because multiplying 3 with itself results in 9. Non-squares are thus the numbers that are not squares, for example 2 and 5.

Premise 1: The number of positive integers, consisting of squares and non-squares together, is greater than just the number of squares.

Premise 2: The number of squares is the same as the number of roots, because every square has its own root and vice versa.

Premise 3: The number of positive integers is the same as the number of roots because every positive number is a root.

Notice that premise 1 has the part-whole principle at play, and that premise 3 is actually a one-to-one correspondence between the positive integers and the roots. Combining the second and third premise results in Statement 1.

Statement 1: The number of squares is the same as the number of positive integers.

Combining Premise 1 with Statement 1, it results in a contradiction. It cannot be true that the number of positive integers is simultaneously equal to or greater than the number of squares. Galileo believed that the relations "greater", "smaller" and "equal" between infinite sets did not exist in combination with Premises 1 to 3 because if they did exist then it would lead to this paradox.

Cantor's notion of size, the orthodox stance

The followers of the orthodox stance follow Cantor's notion of size[1]. This notion consists of a few basic claims, which can be combined together to form implications. The claims are what Cantor believed in or has mathematically proven. We are not directly quoting Cantor in this, as "Cantor's notion of size" is a simplified term of what we think Cantor believed was the meaning of size correlated with his mathematics.

Fact 1: Sets A and B have the same cardinality iff there is a bijection between them.

Fact 2: The cardinality of a set A is strictly lower than the cardinality of its powerset $P(N)$. (See Cantor's theorem)

[1] This is a simplification of the actual worldly representation. We are constructing "Cantor's notion of size" to simplify the way of addressing what we mean with the orthodox view but without being too vague about some key elements such as size and cardinality. Followers of an unorthodox view are those who do not agree on all the listed Claims and Implications of "Cantor's notion of size" and thus those who do not agree with the orthodox view. So in the actual worldly representation of the orthodox view there are more concepts than just the ones we have listed.

Claim 1: The word 'size' means the same as cardinality.

Applying Fact 2 with an infinite sets results in Implication 1.

Implication 1: There are infinite sets with smaller cardinalities than their powersets.

Combining Fact 1 with Implication 1, or just following Implication 1, it is clear that there is a second implication.

Implication 2: There are infinite sets that do not share the same cardinality.

Now lastly if we were to combine Claim 1 with Implication 2, we end up with a third Implication.

Implication 3: There are infinite sets that do not share the same size.

Fact 1 and Fact 2 will be called the *basics.*

Due to the fact that the basics are mathematically sound it follows that when Cantor's notion of size is attacked, it will be on the other fronts. To attack a sound theorem would not seem to be very fruitful. To attack Implication 1 would also be the same as attacking the basics. This is simply a re-iteration of Fact 2 with infinite sets. The same follows for Implication 2. What is left to attack is then Claim 1 and Implication 3. To attack this Claim and Implication means to argue with the definition of size if it is not cardinality for infinite sets.

Analysis of Whittle's paper

Whittle (2015) starts by agreeing that he is not challenging Cantor's mathematics but the significance of it. He says, "I should underscore that I am not, in any way, going to challenge Cantor's mathematics: my arguments are aimed solely at the standard account of the significance of this mathematics." (p.3). He continues with "Cantor *cannot* be said to have established that there are different sizes of infinity." (p. 3). So Whittle argues against the idea that Cantor sufficiently proved that there are different sizes of infinity. Note that this is the same as Implication 3. He presents two claims that follow from combining the Cantorian notion of cardinality and the Cantorian notion of size and calls them C1 and C2. Whittle's "Cantorian notion of cardinality" is equal to Fact 1 and Whittle's "Cantorian notion of size" is equal to Claim 1. Whittle's goal in the paper is to show that two claims, C1 and C2, are not justified. They are listed below.

(C1) For any infinite sets A and B, A is the same size as B iff there is a one-to-one correspondence from A to B.
(C2) For any infinite sets A and B, A is at least large as B iff there is a one-to-one function from B to A.

There are a few topics on which Whittle gives some arguments in order to refute these two Cantorian claims. In the first topic Whittle indirectly attacks the implication of Cantor's theorem. He claims that Cantor's theorem cannot have proven that there are multiple sizes of cardinality. His argument revolves around the similarity between proofs for Russell's Paradox and Cantor's theorem. The second topic is a direct attack on the logical structure of the claims; specifically the size-to-function direction.

Whittle's first topic

Whittle's goal in this first topic is to show that the implication of Cantor's theorem cannot have anything to do with size. He claims that his version of the proof for Russell's Paradox is very similar to the proof for Cantor's theorem. Because of this similarity, Whittle claims that the implications of Cantor's theorem and Russell's Paradox fall or stand together. He shows that the implication of Russell's Paradox cannot be about size through an indirect matter and thus concludes that the implication of Cantor's theorem cannot be about size either.

Russell's paradox arises when we consider a set that contains all sets that are not members of themselves. If that set contains itself, then it should not contain itself. If that set does not contain itself, it should contain itself. Whittle gives a derivation of a proof of this paradox. The structure of Whittle's version of the proof for Russell's paradox is written as a proof by contradiction and is structured to be very similar to the proof for Cantor's theorem. First an assumption is presented that there is a one-to-one function[2] from pluralities to objects. Whittle calls this (V) and explains that uppercase variables range over pluralities, lowercase letters range over objects, and that the law 'ext' denotes a function from pluralities to objects. So 'ext(X)' denotes the extension of X

(V) $\forall X \forall Y(\text{ext}(X) = \text{ext}(Y) \leftrightarrow \forall z(Xz \leftrightarrow Yz))$

Second it is supposed a plurality R that consists of objects x such that for some plurality Y, ext(Y) = x and x is not in x. Now there are two cases. The first case is that if ext(R) is in R, then it follows that ext(R) is not in R. The second case is that if ext(R) is not in R, then it follows that ext(R) is in R. Either way a contradiction follows, thus the first assumption has to be false.

To make the similarity between the two proofs clearer we will quote Whittle on the way he presented the proof for Cantor's theorem:

> *Proof.* Suppose that f is a one-to-one function from P(A) into A, and consider $C = \{x \in A: \exists y \in P(A)$ such that $f(y) = x$ and $x \notin y\}$. But now consider f(C). And suppose first that $f(C) \in C$. Then (by the definition of C, and the fact that f is one-to-one) it follows that $f(C) \notin C$. So $f(C) \notin C$. But then (by the definition of C again) $f(C) \in C$: which is a contradiction. (p.5)

The assumptions made in the proof for Russell's Paradox share the same structure as the ones in the proof for Cantor's theorem. The proof for Cantor's theorem is also written as a proof by contradiction and also starts with the assumption of a one-to-one function, but between a set and its powerset. Secondly, in the proof of Cantor's theorem there is an assumption C that serves the same role and is defined in the same way as plurality R according to Whittle. Then the two cases of whether f(C) is in C or not in C are similar to the cases of whether ext(R) is in R or not in R. In both proofs we arrive at a contradiction and conclude that the first assumption made has to be false. This means that it is shown that a one-to-one function cannot exist in both proofs. So we see that Whittle has purposefully structured a proof for Russell's paradox to be very similar to the proof of Cantor's theorem.

[2] A one-to-one function must not be confused with a one-to-one correspondence. A one-to-one function can also be referred to as an injective function.

Whittle makes this analogy between the two proofs so he can claim that the proof for Cantor's theorem and his version of the proof for Russell's paradox are "really just the same argument in slightly different settings." (p. 5). He also claims that there is a close analogy between the implications that come forth from these proofs. The implication from Cantor's theorem is that a powerset *P(A)* is always greater than *A*. The implication from Russell's paradox is that there are more pluralities than objects. Whittle believes this analogy is so tight "that these two implications must stand or fall together." (p. 6).

Whittle is not yet directly arguing directly against the definition of size in this case. However to say that Cantor's theorem cannot have anything to do with size while it does uses cardinality seems to be an implicit attack on Claim 1.

Counter argumentation about the similarity of the two proofs

Whittle does not give an argumentation as to why exactly these proofs are similar. We have given a shallow explanation why these proofs are similar. We have only touched the surface of this similarity of proofs but not given any in depth mathematical proof as to why exactly they are so similar and that we thus can claim that their implications fall or stand together. As Whittle has not provided any proof at all we can say that this claim of similarity is mathematically weak in this case.

There is however an paper written by Yanofsky (2003) that does link Cantor's theorem with Russell's Paradox using mathematical concepts. So it is possible that there is a strong mathematical resemblance between these two concepts. However to say that their implications fall or stand together is a whole other claim to make. To demonstrate we will use an example of plants. Suppose that there are two plants named Pimmel and Frummel, and that they are growing in similar ways. The plants need around the same amount of water, sunlight and love. This similar process of growth is like the similarity between the proof of Cantor's theorem and the proof of Whittle's version of Russell's Paradox. However, because the plants grow in similar ways, it does not mean that Pimmel and Frummel will blossom with the same colour of flowers. We can indeed say that the flowers look alike but are not identical just like the implications from the proofs. However, we cannot claim that "because Pimmel did not grow a pink flower, Frummel cannot grow a pink flower either.". This is similar to Whittle's idea that because the implication of Russell's Paradox cannot be about size, then the implication of Cantor's theorem cannot be about size either. We conclude that regardless of the proofs being similar or not, the claim that the implications of these proofs fall or stand together is mathematically weak in Whittle's case.

Whittle's first topic continued

Whittle believing he has established that the implications of the two proofs fall or stand together, he continues by trying to let the implication of the proof for Russell's Paradox fall. This would mean that the implication of Cantor's theorem also falls and thus an attack on the initial two claims.

Whittle gives another version of (V) that he calls (V*). Whittle says the $\varphi(z)$ stands for a formula of our language. So (V*) is a schema.

(V*) $\forall X(\forall z(Xz \leftrightarrow \varphi(z)) \rightarrow \forall Y(ext(X) = ext(Y) \leftrightarrow \forall z(Xz \leftrightarrow Yz)))$

The second part of (V*) that starts with ∀Y is actual the same as (V). The part that is different in (V*) is that Whittle added the part in the beginning with $(\forall z(Xz \leftrightarrow \varphi(z))$. According to Whittle, this part ensures that there are not more definable pluralities than objects. Recall the implication of Russell's Paradox that there are more pluralities than objects. So (V*) cannot be problematic like (V). However, Whittle shows that (V*) is problematic and in the same way as the proof for Russell's Paradox with the exact same steps. So even if the attention is restricted to definable pluralities we still arrive at a contradiction. This idea now is that (V*) "cannot have anything to do with size because there are not too many definable pluralities to allow each to get its own object." (p.8) and since (V) and (V*) have the same diagnoses, (V) cannot have anything to do with size either. This is also supported by the fact that (V*) is a weaker version of (V). What cannot be true in (V*) is a fortiori a reason that (V) cannot be true either. According to Whittle the implications of the proof for Cantor's theorem and Russell's Paradox would stand or fall together. So if the implication of (V) and its diagnosis, which is essentially a proof for Russell's Paradox, cannot be about size, then the implication of Cantor's theorem cannot be about size either.

Counter argumentation using Skolem's paradox

The way Whittle argues that (V*) cannot be about size, so (V) cannot be about size due to their similar diagnoses sounds awfully familiar like the similarity between the proof of Cantor's theorem and the proof of Whittle's version of Russell's Paradox and that their implication fall or stand together. As explained before with the plants Pimmel and Frummel we will say that even if there is a mathematical similarity between (V*) and (V), it does not necessarily mean that their implications fall or stand together.

Our next argument will revolve around (V*); specifically the claim that formulas are just objects. We notice that Whittle might be mixing up different mathematical levels when he claimed that there are no more definable pluralities than objects. He supports this claim by saying "there are no more definable pluralities than there are formulas to do the defining; since formulas *are* objects" (p.7). By saying that a formula is an object does not seem right. The formula is part of the language and thus cannot be in the domain or model itself. It does not seem plausible in this case to quantify over formulas with a language that is made up from these same formulas. We think it is similar to an interesting concept named Skolem's Paradox that also revolves around different levels of math. Vann McGee (2015) wrote a response to Whittle that also revolves around this concept of different levels of math. We will first quote and elaborate on McGee before explaining Skolem's Paradox.

McGee wrote as a response to Whittle:

> For a given countable language L, we can enumerate the real numbers definable in L, and we can define a real number *r*, different from every number on the list. The definition of *r* isn't given in L, however, *r* is defined semantically, and the semantic theory of L is developed, not in L itself, but in the metalanguage richer than L in expressive power. (p. 26)

McGee says that the definition of an object r isn't given in a language L. Whittle claims that formulas are just objects. This means that the definition of Whittle's formulas are not given in a language. However, Whittle's formulas are what the language is composed of. If the formulas cannot be given in the language that is composed of these same formulas then it seems like a clear contradiction. McGee also mentions that there is a metalanguage richer than L in expressive power that is related to our mention of different levels of math. In semantics and philosophy there is the metalanguage and the object language. The metalanguage is used to talk about the object language and the analysis thereof. The object language is the language of the system itself. The metalanguage would be of a higher order logic. (Hodges 2018). We provide an example with a sentence such as "This sentence is false" written in the object language that is English. In order to analyse this sentence we need a metalanguage. In this case our metalanguage is also partially in English. This sentence could be classified as true or false and this is done in the metalanguage. In the object language a word such as "false" as seen in the sentence does not hold such higher order logic. We would arrive at a problematic situation if we do not separate the object language from the metalanguage. "This sentence is false" could be true in the metalanguage but if we take "This sentence is false" literally and with the same higher order logic, then the sentence is simultaneously true and false. In Whittle's case he could be mixing up the metalanguage and the object language in claim that formulas are just objects.[3]

We will now explain Skolem's Paradox that uses two theorems, the Löwenheim-Skolem theorem and Cantor's theorem. Whittle does not mention this in his paper but these concepts express the situation of different levels of math better than the object language and metalanguage example and could cause complications for his position.

Starting off as written by Bays (2014):

> In 1915, Leopold Löwenheim proved that if a first-order sentence has a model, then it has a model whose domain is countable. In 1922, Thoralf Skolem generalized this result to whole sets of sentences. He proved that if a countable collection of first-order sentences has an infinite model, then it has a model whose domain is only countable. This is the result which typically goes under the name the Löwenheim-Skolem theorem.

Skolem's paradox arises when we take the axioms of set theory that are a collection of first-order sentences. If there is a model that corresponds to this collection then there is also a model that is only countable as explained by the Löwenheim-Skolem theorem. Remember that with the implication of Cantor's theorem that is proved by these axioms that there are uncountable sets. So in this only countable model whose domain is only countable there are uncountable sets. It is this strange idea that makes Skolem's Paradox seem problematic at first. As further written by Bays (2014):

> Skolem's paradox shows that the line between countable and uncountable sets is, in a fairly deep sense, the first place where our model theory loses the ability to capture cardinality notions. This fact helps to explain why Skolem's Paradox may continue to look paradoxical.

[3] One could say that the formulas Whittle mentions are in the metalanguage but treats them as objects in the object language by saying it is an object. To fully analyse Whittle's claim of formulas being objects in terms of object language and metalanguage is beyond the scope of this thesis. We are simply trying to convey that there are different levels of math and languages that can be considered important in an argument against Whittle's claim.

In the case of Skolem's Paradox in which a countable model contains uncountable objects, one must ask how the model is defined as countable and also how an object in this model is defined as uncountable. To say that a model is countable means that this model has a one-to-one correspondence with the set of natural numbers. This one-to-one correspondence does not take place in the model itself. To say that an object is uncountable (for example the set of real numbers), it would mean that there cannot be a bijection between this object and the set of natural numbers. This attempt to create a bijection but not being able to takes place within the model itself. This is why Skolem's Paradox seems problematic at first but after closer inspection does not really seem problematic anymore. Skolem's Paradox does demonstrate that notions such as cardinality (and size) cannot always be captured clearly in set theory.

For explanatory purposes we will sketch a scene in which a box contains orbs. The box is equal to the model and the orbs are equal to objects within that model. The box is countable so we can successfully count the box[4]. In this case we can point to the elements that coincidentally are orbs and count them. So each orb would count as one element. This counting is done outside of the box. Now if we want to count an individual orb in the same way we counted the box, we need to step inside the box otherwise the orb will just be counted as one element like before. So now what happens when we are counting the elements of the orb, we are not outside of the box anymore. The place of counting the box and the place of counting the orb is not the same. It is this idea of different levels that matters.

This mix up of the difference of the places of where functions are defined is similar to Whittle saying that objects are just formulas. The two functions that do the counting in the previous explanation are not on the same level because if we do treat them as being on the same level we arrive at a problematic situation. Whittle says that objects are formulas but by doing so he pulls the formulas from the language level and the object level of the model into one level, which is essentially the same mistake that happens in Skolem's Paradox but with functions of different levels.

Whittle's second topic

Whittle's goal in his second topic is to attack C1 and thus C2. He does this in two parts. In the first part Whittle's briefly argues that C1 "does not state what it is for two sets to be of the same size" (p.9). The second part is about splitting C1's bidirectionality into two parts; the function-to-size direction and the size-to-function direction. Whittle then argues that the size-to-function direction has not been sufficiently proved. He does this by arguing against an orthodox argument for the size-to-function direction and by presenting another way to establish that there cannot be an one-to-one function between a set and its powerset that does not revolve around size. The latter is essentially a different way of presenting Cantor's theorem but without using the concept of size.

Whittle says:

[4] The one-to-one correspondence can be seen as counting. So a countable set can be successfully counted. To count a box seems trivial because there is only one box. But in this case we are interested in the amount of elements that are contained in this box. So we want to know the cardinality of the box with regard to what is in the box. For the sake of the explanation you do not need to worry about infinite concepts because counting would take forever and the box would never be counted and thus no one-to-one correspondence could be established.

> "For the size of a set (infinite or otherwise) is an intrinsic property of that set: that is, it is a property that a set had purely in virtue of what *it* is like; it is not a property that it has in virtue of its relations to distinct sets, or to functions between it and such sets." (p. 9)

Recall C1 - two sets being the same size if and only if there is a one-to-one-correspondence between them. According to Whittle this one-to-one correspondence does not define what size is between two sets. This means that C1 cannot say anything about the size of two sets. Whittle does say that C1 and C2 can still be true but that they do not capture the nature of the same-size relation and in turn the at-least-as-large relation. Unfortunately Whittle gives no alternative on what he thinks size should be for infinite sets, if cardinality and a one-to-one correspondence are not sufficient enough.

Counter argumentation on Whittle not giving an alternative

Even if it is true that size does not equal cardinality and that the nature of how size behaves is not entirely captured in C1 and C2, there is little in Whittle's argument to justify this. Whittle also says that it is still possible for C1 and C2 to be true. However by doing so it would mean that he agrees with Cantor's notion of size and so the discussion would end here. Another possibility could be that Whittle thinks that size does not equal cardinality and this leads to the claim that C1 and C2 do not fully capture the nature of size but can still be true. So perhaps Whittle means to say that cardinality partially describes size.

To give a stronger argument to argue that size does not equals cardinality would be to redefine the word size that does not cause complications like in Galileo's Paradox. This is exactly what Benci and Di Nasso (2003) have done by developing numerosity. Unfortunately Whittle does not define what size is and only that it is not to be defined in terms of relations with other sets.

Whittle's second topic continued

Whittle does think that the function-to-size direction of C1 is true - this meaning that if there is a one-to-one correspondence between two sets, then these two sets are the same size. In his argument he says it does not matter what elements a set has for its size but it matters how many the set has. Taking a set and removing one element and replacing it with another does not result in the size having changed. He continues by saying that a one-to-one correspondence is like replacing the elements of each set with one another. This does not result in change of size so the two sets have to be the same size.

However, Whittle does not believe that the size-to-function direction of C1 is true - this meaning that if two sets are the same size, then there is a one-to-one correspondence between these two sets. For the followers of the orthodox stance, it is sufficient to show the size-to-function direction by constructing a one-to-one correspondence between two sets that share the same size. The idea behind this is that if we suppose that two infinite sets are the same size, then one can just construct a one-to-one correspondence between these sets. Due to their same size, one set cannot run out of members before the other when mapping an element from one set to the other. So this leads to conclude that if two sets share the same size, a one-to-one correspondence can just be constructed.

Whittle claims this is hopeless in the infinite case and tries to show this by using an example of the set of natural numbers. He says that the set of natural numbers with an obvious order like 0, 1, 2, etc will not end up with a one-to-one correspondence with the set of natural numbers that has the order like 0, 2, 4, etc . Whittle does not continue to explain why he thinks this is the case.

We think that Whittle means to say some numbers, such as 1, 3 and 5, from the second set of natural numbers with order 0, 2, 4, etc will be missed out. It will not be obvious where these elements will correlate to with elements of the other set with the order 1, 2, 3 etc. The two ways of ordering the elements in the set of natural numbers does not change the size of the set of natural numbers. It is logical that these two sets share the same size because they are the same set. So even if two sets share the same size, it is not always possible to *just* construct a one-to-one correspondence.

We will demonstrate Whittle's argument with something of our own that we think Whittle means. Let us take the set of natural numbers and the set of integers and construct a bijective function like this: 0 maps to 0, 1 maps to -1, 2 maps to 1, 3 maps to -2, 4 maps to 2 and so on. The order of the elements in the set of integers is now like {0, -1, 1, -2, 2, …}. If one changed this ordering into {0, 1, 2, …, -1, -2, …} in which the negative numbers are "glued onto" the end of the positive integers, then it feels like the set of natural numbers is equal to the first (positive) part of the set of integers. The negative integers then seems to be excess and cannot be linked with a natural number in an obvious way. So in this case the construction of a one-to-one correspondence is not clear.

Counter argumentation on construction

Whittle does not explain at all why he think this orthodox argument of simply constructing a one-to-one correspondence if two infinite sets are the same size holds. He only presents the two ways of ordering the elements of the set of natural numbers. It is true that these two sets share the same size, because they are the same set. There is no obvious way to construct a one-to-one correspondence between these two sets. This means that even if two sets share the same size, it does not always mean that there will be a one-to-one correspondence between them. However, to say that it is not obvious how to construct a one-to-one construction does not mean that it is impossible. We think that these kinds of arguments – that not knowing something implies that it is perhaps impossible - are not helpful for scientific advancements.

There is a story that we don't know the origin of that revolves around the idea that non-obvious matters are magic or not possible. Before a cavemen ever witnesses a fire, the fire is an unknown subject. The caveman has no idea what fire is or how it can be used. So for a caveman the idea of a fire is impossible and non-existing. We could say that the fire is like magic. One day the caveman stumbles across a fire for the first time. He notices certain things like the heat and its colour. After some studying, the caveman can now make his own fire and cook meals. However we assume that cavemen do not formally know what fires are. Perhaps they know how to start and maintain a fire, but not that there are multiple factors that contribute to these processes. Over time the caveman might be deceased but science has uncovered the truth about what fire actually is. Fire is not magic anymore.

Whittle's argument reminds us of this story because the non-obvious way of constructing a one-to-one correspondence between infinite sets seems to imply that it is not possible. The construction is like magic. It seems that Whittle also thinks that the way elements are ordered in a set matters. However, in set theory the order of elements in a set does not matter and might not even exist. For example, the finite sets {1, 2, 3} or {3, 1, 2} are the same. This way of thinking might be very intuitive and finite-minded like the caveman. Just because we do not know how the process of constructing a one-to-one correspondence with infinite sets exactly does not mean it is magic or not possible.

Whittle's second topic continued

The second argument Whittle gives against the size-to-function direction revolves around the proof of Cantor's theorem. Cantor's theorem proves that there cannot be a one-to-one correspondence between a set and its powerset. We think that Cantor's theorem could be a proof by contraposition for the size-to-function direction. The contrapositive version of size-to-function direction would be that no one-to-one function between to sets implies that the size of these two sets differ. By assuming there is no one-to-one function in the proof for Cantor's theorem it follows that the sets are of different sizes. Whittle argues against this by giving a proof of his own that is very similar to the proof of Cantor's without using the concept of size.

Whittle show that there cannot be an onto-function[5] between the set of natural numbers and its powerset. This means that there cannot be a one-to-one correspondence either. However in Whittle's proof he does not use the concept of size and thus concludes that there cannot be a onto function between a set and its powerset by reasons other than that of size. By simply not mentioning size, Whittle's argues that he has shown that the set of natural numbers and its powerset cannot have a one-to-one correspondence not because they do not share the same size but because of intrinsic properties.

Counter argumentation on onto function

Whittle's argument is about showing another way to proof there cannot be a one-to-one correspondence between a set and its powerset. We do not think is a strong argument. We think it is possible to have a process but not mention the term explicitly. For example if we are talking about making a banana milkshake, we would need a blender, banana, ice, milk, sugar and ice cream. However if we are saying we are putting these ingredients in the blender and turn the machine on, we do not use the word banana milkshake. We have simply blended these particular ingredients and end up with a beverage. To say that this process has nothing to do with a banana milkshake seems very strange because this process is in fact the creation of a milkshake banana. It is thus possible to explicitly leave certain words or terms out but this does not mean that they are not in some way relevant.

Discussion

When we first read Whittle's paper, there were many question marks. Not all of these question marks have been answered by reading the paper multiple times. We are unsure if this is because of how Whittle constructed his paper or because of our inexperience on the

[5] Recall that an onto function is also a surjective function. If a function is either not one-to-one or onto, then the function cannot be a bijection (a one-to-one correspondence) between two sets.

subject. Whittle is asking questions and that is not wrong by itself. However he rejects or disagrees with certain subjects but does not give an alternative.

Whittle has not been very clear about his arguments as we think he could have. This made it harder to understand the paper and thus some arguments we have provided might not be on point. Many times have we had to think about what Whittle meant to convey with his arguments and thus our interpretation could be wrong in some cases.

Conclusion

There are many terms that come forth in Whittle's paper that are not explicitly mentioned and are to be expected of the reader to know. We do not think this is a clear way of constructing papers. As most papers are meant to educate and transmit knowledge we do not think this is very helpful. It would be well-appreciated if papers that are not extremely in-depth could be written in a more clear manner that is understandable for those are new to the subject but have an interest in it.

The only way to attack Cantor's notion of size is on a philosophical front because his work is mathematically sound. For followers of an unorthodox stance they have to hit hard with philosophical arguments. However we think this is hard because we humans don't really know much about the infinite. It is also easier to agree with sound math than something philosophical that has not been proven or cannot be proven. We do not say that philosophical arguments cannot be strong, because we see that Galileo's Paradox still holds power today. We believe Whittle's paper is philosophical in nature due to him indirectly attacking Cantor's work and not giving an alternative like Benci and Di Nasso (2003) have done.

Whittle has some arguments that he does not fully explain. It seems that Whittle lays the *burden of the proof* on the followers of the orthodox view. Whittle is not satisfied by the answers provided by the followers of the orthodox view, so he seems to lay the matter in their court. We understand why Whittle feels this way, but we do not think this is how scientific advancements work. The ultimate goal is to educate and progress on the subject. If one disagrees with something, then that is fine. However, we do not think that by simply claiming that certain statements are not sufficiently supported while providing no alternative helps to progress on the subject.

We found Whittle's paper lacking in arguments in general. This made it harder to provide arguments against his claims. Although we do not see a direct link between Whittle's paper and Galileo's Paradox, it seems that the intuitive part-whole principle is somehow related to Whittle's arguments. These arguments that Whittle provide seem intuitive in the same way as the part-whole principle. Due to Whittle's silence on alternatives it seems to echo this intuitiveness at the reader as something being not right but not having the proof to explain why. This is similar to our gut feelings that we classify as intuitive.

Perhaps it is good to know that finite concepts are always at play inside the minds of humans. So if we are to take a step into a world filled with infinite concepts, it is possible we are limiting ourselves with concepts of the finite world. Scholars could take this into account when researching infinity in every way. The author too has struggled to understand Cantor's work and felt that is was unnatural at first. However after more reading the author has decided to agree with Cantor's work but keep an open mind for other possibilities and that Cantor's work is not absolute.

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