Utrecht University

Mathematics

Bachelor Thesis

# Variational Obstacle Problems

## Existence, Regularity and Applications

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# Contents

# 1 Introduction

Obstacle problems are a special class of variational problems in the field of the calculus of variations. This field is concerned with minimizing functionals

$$\mathcal{F} : X \to \mathbb{R}$$

on some infinite-dimensional function space $X$. In other words, we need to find an element $u_0 \in X$, called a minimizer, such that $\mathcal{F}(u_0) = \min_{u \in X} \mathcal{F}(u)$. Such problems often arise in physics, where the functional $\mathcal{F}$ represents some physical quantity, like time or energy, which needs to be minimized. One of the most famous variational problems is the Brachistochrone problem, posed by Bernoulli in 1696, which aims to find a curve connecting two points $(0, 0)$ and $(1, -a)$ for $a > 0$ such that the time it takes for an object to slide along this curve by gravity is minimal among all curves connecting the two points. This curve is called the Brachistochrone and can be found by minimizing the functional

$$\mathcal{F}(u) = \int_0^1 \sqrt{\frac{1 + u'(x)^2}{-u(x)}}\, dx,$$

among the suitable $u : [0, 1] \to \mathbb{R}$ that satisfy $u(0) = 0$ and $u(1) = -a$, see [6, p.6]. Another example is the minimization of the Dirichlet functional

$$\mathcal{D}(u) = \int_\Omega \frac{|\nabla u(x)|^2}{2}\, dx$$

for a bounded domain $\Omega \subset \mathbb{R}^2$. We can interpret this functional as the linearized elastic energy of $u$, where we consider $u$ to be an elastic membrane. Indeed, the elastic energy is proportional to the stretching of the membrane and hence proportional to the area of the membrane. By approximating the integrand of the area integral via a Taylor approximation we obtain the Dirichlet functional. The principle of least potential energy now tells us that the minimizer of the Dirichlet functional among those suitable $u : \Omega \to \mathbb{R}$ with $u = g$ on $\partial\Omega$ can be used to model the equilibrium position of an elastic membrane fixed at the boundary at height $g$, see Section 6. The Dirichlet functional will play a central role in the classical obstacle problem, see (1.1).

Similarly to the above two examples most of the functionals that the calculus of variations aims to minimize can be given by the following form

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx,$$

for a domain $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. The set of functions over which we minimize is usually given by suitable $u : \Omega \to \mathbb{R}$

which satisfy a boundary condition $u = g$ on $\partial\Omega$. The first systematic way for solving these problems was developed by Euler and Lagrange around the 1750s. Lagrange showed that any minimizer must satisfy a certain (partial) differential equation, now named the Euler-Lagrange equation, which can be understood as setting the 'derivative' of $\mathcal{F}$ to be zero. When Euler learned about this idea from Lagrange he immediatly adopted it and later came up with the name 'Calculus of variations', see [22].

Obstacle problems are similar to the above minimization problems with the addition of an obstacle function $\psi : \Omega \to \mathbb{R}$. The idea is that we only want to minimize the functional over those functions that lie above the given obstacle $\psi$. The main example is the classical obstacle problem which reads

$$\text{Minimize } \mathcal{D}(u) = \int_\Omega \frac{|\nabla u(x)|^2}{2}\, dx \text{ over all } u \in K_\psi, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $K_\psi$ is the set of admissible functions

$$K_\psi = \{u : \Omega \to \mathbb{R} \text{ 'suitable' } \mid u \geq \psi \text{ and } u = g \text{ on } \partial\Omega\}.$$

For $n = 2$ we again have the interpretation of the Dirichlet functional as a linearized elastic energy. Hence the solution of this problem can be interpreted as an elastic membrane fixed at the boundary at height $g$ and constrained to lie above the obstacle. In this thesis we also consider more general obstacle problems with different functionals than the Dirichlet functional.

A first question regarding obstacle problems is to ask whether there exists a unique minimizer. This can be positively answered by using the direct method in the calculus of variations, which is based on the simple idea that lower semicontinuous functions attain their minimum on compact sets. This method was developed after Hilbert had posed his 23 problems around 1900 of which the 20th asks whether the minimization problems in the calculus of variations always have solutions. This seems like a natural question, but Weierstrass was the first in 1870 to show an example of a variational problem which has no minimizer [25]. Before this example by Weierstrass it was assumed that variational problems always have minimizers. A simple example of a variational problem without a minimizer different from the one by Weierstrass is given in [9, p.172]. The direct method in its current form was mainly developed by Tonelli around 1920, where he proved important semicontinuity results about integral functionals [19]. To apply the direct method we need to introduce Sobolev spaces, which generalize the well-known $C^k$ spaces. The reason for this is that the $C^k$ spaces do not have the compactness properties needed for the direct method unlike the Sobolev spaces. Therefore we can show under certain conditions on the functional that there exists a minimizer in a Sobolev space. Uniqueness of the minimizer can simply be proven by requiring strict convexity of the functional.

Because these Sobolev functions can be ill-behaved ([2, §5.2 Example 4]) a natural next step is to determine how regular the solution is, i.e. how smooth it is. This is quite a delicate problem so we only carry this out for the classical obstacle problem. By using the variational inequality for the Dirichlet integral, which can be seen as an analogue of the Euler-Lagrange equation for the classical obstacle problem, we show that the solution of the classical obstacle problem is superharmonic on $\Omega$ and harmonic on the set where it lies strictly above the obstacle. This enables us to establish the optimal regularity result which was first proven by Jens Frehse in 1972 [20]. It states that the solution of the classical obstacle problem lies in $C^{1,1}_{loc}(\Omega)$ if the obstacle lies in $C^{1,1}(\Omega)$. This is the most important theoretical result in this thesis. The proof splits the domain up in the coincidence set, where the solution and obstacle coincide, and the noncoincidence set, where the solution lies above the obstacle. In both these sets individually the solution is $C^{1,1}$ and it remains to show that the same holds around the free boundary (see next paragraph), which is the interface between the coincidence and noncoincidence set. This is done by using estimates of harmonic, subharmonic and superharmonic functions.

Obstacle problems are also free boundary problems, which can roughly be described as partial differential equations on a domain for which part of the boundary is unknown. Knowledge about the free boundary can be very useful for solving these problems and Caffarelli spurred the recent interest in regularity of free boundaries by his 1998 paper 'The Obstacle Problem Revisited' [3], in which he describes the structure of the free boundary for the classical obstacle problem. We mention some of the results about the free boundary of the classical obstacle problem.

The classical obstacle problem arises in many applications. We discuss three of them in this thesis. The first is that of modelling an elastic membrane above an obstacle, which we already discussed. This is a very natural interpretation of the obstacle problem. The second application is the dam problem which aims to describe the fluid flow through a porous dam. In this case there is no obstacle, but by using the Baiocchi transformation this problem can be turned into the classical obstacle problem such that the free boundary corresponds to the water level inside the dam. We implemented a numerical method to solve the obstacle problem and have used this to carry out simulations of the dam problem. Lastly, the classical obstacle problem can be used in financial mathematics to design an optimal stopping time.

An optimal stopping time maximizes the expected payoff of a stochastic process. It turns out that by solving the classical obstacle problem with the obstacle function chosen to be the payoff function, we can determine that the optimal stopping time for an $n$-dimensional Brownian motion is the first time at which the Brownian motion does not lie in the noncoincidence set.

The structure of the text is as follows. In Section 2 we discuss the necessary prerequisites to start our study of variational obstacle problems. These prerequisites consist of basic material on analysis, functional analysis and measure theory which is presented for the sake of self-containment. Additionally, we state some more advanced results and introduce the notion of Sobolev spaces which is essential in the direct method of the calculus of variations. In the next section we treat the problem of existence and uniqueness using the direct method, which can be applied to minimization problems with or without obstacles. In Section 4 we turn to regularity theory without obstacles. We derive the Euler-Lagrange equation which is necessarily satisfied by the minimizer. Then we go in depth into the regularity of harmonic functions, which are intimately related to the Dirichlet functional through the Euler-Lagrange equation. We also discuss properties of sub- and superharmonic functions which arise when considering the classical obstacle problem. With the general theory from Section 3 and 4 we can in Section 5 effectively go into the study of obstacle problems. From Section 2 existence and uniqueness immediately follows and we derive the variational inequalities as an analogue to the Euler-Lagrange equation. In the specific case of the classical obstacle problem the variational inequality and the results about (sub/super)harmonic functions enable us to prove the remarkable $C^{1,1}$ optimal regularity result. After this we briefly touch upon the regularity for more general obstacle problems and considerations about the free boundary. In the last section we discuss the three applications of the classical obstacle problem elaborated in the above paragraph.

## 2 Preliminaries

In this section we discuss the prerequisite results needed throughout this thesis. None of the proofs are necessary in order to understand the main part of the thesis and they can safely be skipped. The section about Sobolev spaces in particular contains more advanced results. Extra background information on any of the topics can be found in the referred sources.

### 2.1 Analysis

We are working on $\mathbb{R}^n$ and denote the Euclidean norm for $x \in \mathbb{R}^n$ by

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

We do not make a distinction between the (total) derivative of a function and its gradient or Jacobian matrix. Thus let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ differentiable then the gradient of $u$ is defined as the vector

$$\nabla u(x) = (D_{x_1} u(x), \dots, D_{x_n} u(x))^T,$$

where $D_{x_i}$ denotes the partial derivative with respect to $x_i$. Similarly if $u$ is twice differentiable then we define its second derivative as the matrix

$$D^2 u(x) = (D_{x_i x_j} u(x))_{1 \le i,j \le n}.$$

For more about differentiation and submanifolds of $\mathbb{R}^n$ see the book [14]. Next is the Gauss-Green theorem which we use frequently. We assume that $\Omega$ is now also bounded and has a $C^1$-boundary, i.e. its boundary is a $C^1$-manifold.

**Theorem 2.1** (Gauss-Green, [15, Theorem 7.6.1])**.** *Let $u \in C^1(\overline{\Omega})$ then we have for $i = 1, \dots n$*

$$\int_\Omega D_{x_i} u \, dx = \int_{\partial\Omega} u \nu_i \, dS,$$

*where $\nu = (\nu_1, \dots, \nu_n)$ is an outward unit normal to $\partial\Omega$.*

From this theorem we can obtain a generalized integration by parts formula.

**Theorem 2.2** (Integration by parts, [15, Corollary 7.6.2])**.** *Let $u, v \in C^1(\overline{\Omega})$ then we have for $i = 1, \dots n$ that*

$$\int_\Omega D_{x_i} u v \, dx = -\int_\Omega u D_{x_i} v \, dx + \int_{\partial\Omega} u v \nu_i \, dS.$$

For more about integration over manifolds see [15].

## 2.2 Functional Analysis

Most of the results in this section can be found in [17]. We use the notation $X$ for a normed vector space. Recall that a Banach space is a complete normed vector space and a Hilbert space is a complete inner product space. An important notion in functional analysis is that of a bounded linear functional, which are elements of the dual space.

**Definition 2.3.** The dual space $X'$ of a normed vector space $X$ is the set of continuous linear functionals on $X$. Explicitly, these are linear maps $f$ from $X$ to $\mathbb{R}$ which are continuous or equivalently satisfy

$$\sup_{x \in X, \|x\| \leq 1} |f(x)| < \infty.$$

The space $X'$ becomes a normed vector space when equipped with the norm

$$\|f\|' = \sup_{x \in X, \|x\| \leq 1} |f(x)|.$$

Since $X'$ is again a normed vector space we can subsequently take its dual as well. We then get the bidual $X''$. As the name dual suggests we expect the bidual to be related to $X$ itself. This relation is exhibited by the canonical map.

**Definition 2.4.** The canonical map $I_X : X \to X''$ is defined for $x \in X$ as

$$(I_X x) : X' \to \mathbb{R}, \qquad f \mapsto f(x).$$

It turns out that $I_X$ is a linear isometry, [17, Lemma 5.37]. However, it need not be surjective, which makes $X$ and $X''$ canonically isometrically isomorphic. Therefore we have a special name for spaces that do have this property.

**Definition 2.5.** A normed space $X$ is called reflexive if the canonical map $I_X : X \to X''$ is surjective.

Apart from functionals we also have linear maps between vector spaces. We can define the following which is used to prove that $L^p$ is reflexive.

**Definition 2.6** (Dual of a map)**.** Let $T : X \to Y$ be a (bounded) linear map between normed vector spaces. Then its dual is given by

$$T' : Y' \to X', \qquad T'(g)(x) = g(Tx), \quad \text{for } g \in Y' \text{ and } x \in X.$$

Furthermore, if $T$ is an isomorphism then $(T^{-1})' = (T')^{-1}$.

In our proof of existence of minimizers we need some kind of compactness on an infinite-dimensional vector space $X$. However, for every infinite-dimensional vector space the unit ball $B = \{x \in X \mid \|x\| \leq 1\}$ is not compact. Thus we cannot say that bounded sequences have convergent subsequences. We therefore need to introduce a weaker notion of convergence.

**Definition 2.7.** A sequence $\{x_i\}$ in $X$ is said to converge weakly to $x \in X$, denoted as $x_i \rightharpoonup x$, if for every $f \in X'$ we have

$$\lim_{i \to \infty} f(x_i) = f(x),$$

as a limit in $\mathbb{R}$.

Note that this definition is weaker than converging in the norm as any $f \in X'$ is continuous by definition. Now we have a nice analogue to compactness in infinite-dimensional spaces as well.

**Theorem 2.8** ([17, Theorem 5.73])**.** *If $X$ is a reflexive Banach space then any bounded sequence $\{x_i\}$ in $X$ has a weakly convergent subsequence converging to some $x \in X$.*

We also have the following which states that closed and convex sets are weakly sequentially closed.

**Theorem 2.9** ([17, Lemma 5.70 (d)])**.** *Let $X$ be a Banach space and $M \subset X$ a closed and convex subset. If a sequence $\{x_i\}$ in $M$ converges weakly to some $x \in X$ then $x$ must lie in $M$.*

A simple consequence of this is the following statement which allows us to go from weak to strong convergence.

**Theorem 2.10** (Mazur's Lemma)**.** *Let $\{x_i\}$ be a sequence in a Banach space $X$ converging weakly to $x \in X$. Then there exists a function $K : \mathbb{N} \to \mathbb{N}$ and scalars $\lambda(i)_k \in [0, 1]$ for $k = i, \ldots, K(i)$ such that the convex combinations*

$$y_i = \sum_{k=i}^{K(i)} \lambda(i)_k x_k \quad \textit{with} \quad \sum_{k=i}^{K(i)} \lambda(i)_k = 1$$

*converge to $x$ strongly, i.e. in the norm.*

*Proof.* We denote by co$A$ the convex hull of a set $A$, which is the smallest convex set containing $A$. By continuity of scalar multiplication and addition in a normed vector space we find that the closure of a convex set is still convex. In particular the sets

$$C_i = \overline{\mathrm{co}\{x_j \mid j \geq i\}}$$

are closed and convex. We conclude by Theorem 2.9 above that the sets $C_i$ are weakly sequentially closed and hence $x \in C_i$ for all $i \in \mathbb{N}$. From this we see that

$$0 = d(x, \mathrm{co}\{x_j \mid i \leq j \leq \infty\}) = \lim_{K\to\infty} d(x, \mathrm{co}\{x_j \mid i \leq j \leq K\}).$$

For any $i$ we can now choose $K = K(i)$ large enough and a $y_i \in \mathrm{co}\{x_j \mid i \leq j \leq K(i)\}$ with $\|x - y_i\| \leq 1/i$. This proves that $y_i$ is a convex combination as stated and that the sequence $\{y_i\}$ converges to $x$ in the norm. $\square$

### 2.3 Measure Theory

For an introduction on measure theory and Lebesgue integration see [16]. First we discuss two theorems about interchanging limits with the integral which are exclusive to Lebesgue integration. We restrict ourselves to integration over subsets of $\mathbb{R}^n$ for simplicity. Thus we set $\Omega$ to be an open subset of $\mathbb{R}^n$. By almost everywhere or a.e. we mean that something holds up to a set of measure zero.

**Theorem 2.11** (Fatou's Lemma, [16, Theorem 2.4.4])**.** *Let $\{f_i\}$ a be a sequence of nonnegative valued Borel measurable functions on $\Omega$. Then*

$$\int_\Omega \liminf_{i\to\infty} f_i \, dx \leq \liminf_{i\to\infty} \int_\Omega f_i \, dx.$$

More important is the following theorem which is extremely useful and knows no analogue in Riemann integration.

**Theorem 2.12** (Lebesgue's Dominated Convergence, [16, Theorem 2.4.5])**.** *Let $\{f_i\}$ be a sequence of Borel measurable functions on $\Omega$ converging pointwise (a.e.) to $f$. Assume furthermore that there exists an integrable function $g$ on $\Omega$ such that $|f_i| \leq g$ (a.e.) for all $i \in \mathbb{N}$. Then*

$$\lim_{i\to\infty} \int_\Omega f_i \, dx = \int_\Omega f \, dx.$$

Our next aim is to describe $L^p$ spaces which are the basis of the function spaces that we use. To formally define $L^p$ spaces we first introduce for an open set $\Omega \subset \mathbb{R}^n$ the following.

**Definition 2.13.** For $p \in [0, 1]$ the vector space of $p$-integrable functions is defined as

$$\mathcal{L}^p(\Omega) = \{f : \Omega \to \mathbb{R},\ \text{Borel measurable} \mid |f|_p < \infty\},$$

where $|.|_p$ is the seminorm given by

$$|f|_p = \begin{cases} \int_\Omega |f|^p \, dx & \text{if } p < \infty, \\ \inf\{C \in \mathbb{R} \mid |f| \leq C \text{ a.e. in } \Omega\} & \text{if } p = \infty. \end{cases}$$

To get a normed space from a seminormed space we can take the quotient by those elements whose seminorm is zero. These are exactly the functions which are zero almost everywhere and hence we obtain that elements of the quotient space are equivalence classes of functions which are equal almost everywhere. From this construction we obtain the well-known $L^p$ spaces.

**Definition 2.14.** For $p \in [0, 1]$ we define $L^p(\Omega)$ as

$$L^p(\Omega) = \mathcal{L}^p(\Omega)/\{f = 0 \text{ a.e. in } \Omega\}.$$

It comes equipped with the norm $\|\bar{f}\|_p = |f|_p$, where $f$ is any representative of $\bar{f}$.

Although elements of $L^p$ are not functions we will regard them as such by implicitly taking a representative. From now on we therefore just write $f$ for an element of $L^p(\Omega)$. We have the following slight adaptation to $L^p$ spaces.

**Definition 2.15.** For an open set $\Omega$ we define $L^p_{loc}(\Omega)$ as the space of measurable function on $\Omega$, which lie in $L^p(U)$ for every open and bounded set $U$ such that $\overline{U} \subset \Omega$.

A useful fact about $L^p$ is that it is complete.

**Theorem 2.16** ($L^p$ Banach space, [16, Theorem 3.4.1])**.** *For any $p \in [0, 1]$ the space $L^p(\Omega)$ is a Banach space.*

A ubiquitous inequality for $L^p$ spaces is Hölder's inequality which involves the conjugate exponent $p'$ of $p$. This is given by $p' = p/(p-1)$ with the convention that $1/0 = \infty$ and $\infty/\infty = 1$.

**Theorem 2.17** (Hölder's inequality, [16, Proposition 3.3.2])**.** *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ then we have that their product $fg$ is an element of $L^1(\Omega)$ and we have the inequality*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

The above theorem already displays the duality between $L^p$ and $L^{p'}$, but this goes further for $1 < p < \infty$.

**Theorem 2.18** (Dual of $L^p$ for $1 < p < \infty$, [16, Theorem 4.5.1])**.** *The map $T_{p'} : L^{p'}(\Omega) \to (L^p(\Omega))'$ defined by*

$$T_{p'}(g)(f) = \int_{\Omega} fg\, dx, \quad \textit{for } f \in L^p(\Omega) \textit{ and } g \in L^{p'}(\Omega),$$

*is an isometric isomorphism.*

Note that the continuity of $T_{p'}$ relies on Hölder's inequality. We can now state the result which turns out to be crucial for our existence result.

**Theorem 2.19** ($L^p$ reflexive for $1 < p < \infty$)**.** *The Banach space $L^p(\Omega)$ is reflexive for $1 < p < \infty$.*

*Proof.* This follows quickly from Theorem 2.18. We have that the map $T_{p'}$ goes from $L^{p'}(\Omega)$ to $(L^p(\Omega))'$ and hence its dual goes from $(L^p(\Omega))''$ to $(L^{p'}(\Omega))'$. Therefore the inverse of this map, which is given by $(T_{p'}^{-1})'$ goes from $(L^{p'}(\Omega))'$ to $(L^p(\Omega))''$. It is now a convoluted but straightforward check to show that the canonical map $I_{L^p(\Omega)}$ is given by the composition $(T_{p'}^{-1})' \circ T_p$. This is a composition of bijective maps by Theorem 2.18 and hence $I_{L^p(\Omega)}$ is surjective as desired. □

Since we use the convergence in $L^p$ it is useful to see how it relates to pointwise convergence. This is the following result.

**Theorem 2.20** ([16, Proposition 3.1.3 and 3.1.5])**.** *Let $\{f_i\}$ be a sequence of functions converging to $f$ in $L^p(\Omega)$. Then there is a subsequence $\{f_{i_k}\}$ which converges to $f$ pointwise almost everywhere.*

We often need a way to approximate functions by smooth functions. This can be done with mollifiers, which are smooth functions close to the Dirac delta distribution. As convolution with the Dirac delta does not change the function we expect that convolution with a mollifier yields a smooth approximation of the function. Firstly, we need a bump function.

**Definition 2.21.** The bump function $\delta \in C_c^\infty(\mathbb{R}^n)$ is defined as

$$\delta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $c$ is a suitably chosen constant to obtain $\int_{\mathbb{R}^n} \delta\, dx = 1$.

We have that this bump function is radially symmetric, nonnegative and has support inside $\overline{B_1(0)}$. Now we can use scaling to let $\delta$ approximate the Dirac delta. We do this by defining for $\epsilon > 0$ the standard mollifier as $\delta_\epsilon(x) := \frac{1}{\epsilon^n}\delta(x/\epsilon)$. We see that $\delta_\epsilon$ still is nonnegative, radially symmetric, has support inside $\overline{B_\epsilon(0)}$ and satisfies $\int_{\mathbb{R}^n} \delta_\epsilon\, dx = 1$. We can now define the following.

**Definition 2.22.** Let $u \in L^1_{loc}(\Omega)$ then we define for $\epsilon > 0$ the mollification

$$f_\epsilon(x) = \int_\Omega \delta_\epsilon(x-y)u(y)\,dy \quad \text{or} \quad u_\epsilon = \delta_\epsilon * u$$

which is well-defined for $x \in \Omega_\epsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \epsilon\}$.

We can now state the following extremely practical properties about mollification.

**Theorem 2.23** ([5, Theorem 4.1])**.**

1. *We have $u_\epsilon \in C^\infty(\Omega_\epsilon)$ for all $\epsilon > 0$.*
2. *If $u$ is continuous then $u_\epsilon$ converges uniformly to $u$ as $\epsilon$ goes to zero on compact subsets of $\Omega$.*
3. *If $u \in L^p_{loc}(\Omega)$ for $1 \leq p < \infty$ then $u_\epsilon \to u$ in $L^p_{loc}(\Omega)$ as $\epsilon$ goes to zero.*
4. *If $u \in C^1(\Omega)$ then $D_{x_i}u_\epsilon = \delta_\epsilon * D_{x_i}u$ on $\Omega_\epsilon$.*

Lastly, there is a slightly more nontrivial theorem that we require. In some sense it is a generalization of the fundamental theorem of calculus for Lebesgue integration in $\mathbb{R}^n$. However, we only need a certain part of it which we can actually prove ourselves. To state this result we first define

$$\fint_{B_r(x)} u(y)\,dy := \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,dy,$$

which represents an average integral. If we let $r$ go to zero then we are taking the average over smaller and smaller balls which we expect to converge to $u(x)$. Indeed, it is easy to see that this is the case for continuous functions. However, the result also holds almost everywhere for just integrable functions. Those points $x \in \Omega$ for which the average converges to $u(x)$ are called the Lebesgue points of $u$. Hence the theorem states that almost every point in the domain is a Lebesgue point for a locally integrable function.

**Theorem 2.24.** *Let $u \in L^1_{loc}(\Omega)$ and suppose that*

$$\lim_{r\to 0} \fint_{B_r(x)} u(y)\,dy$$

*exists for all $x \in \Omega$ (possibly $\pm\infty$). Then it holds for almost every $x \in \Omega$ that*

$$u(x) = \lim_{r\to 0} \int_{B_r(x)} u(y)\,dy.$$

*Proof.* Since this is local we assume for simplicity that $u \in L^1(\Omega)$ and by extending it to zero outside $\Omega$ we find $u \in L^1(\mathbb{R}^n)$. Define the function

$$v_r(x) = \fint_{B_r(x)} u(y)\,dy.$$

We show that $v_r$ converges to $u$ in $L^1$ as $r$ goes to zero. Hence a subsequence of $v_r$ converges pointwise a.e. to $u$, but as the pointwise limit of $v_r$ already exists we find that $v_r$ converges to $u$ a.e. as desired. Because $v_r$ can be seen as a convolution of $u$ with the characteristic function $\frac{1}{|B_r|}\chi_{B_r(0)}$ the $L^1$-convergence follows from the general case [23, Theorem 3.22]. In our case we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x) - v_r(x)|\,dx &= \int_{\mathbb{R}^n} |u(x) - \fint_{B_r(x)} u(y)\,dy|\,dx \\
&\leq \frac{1}{|B_r|}\int_{\mathbb{R}^n}\int_{B_r(x)} |u(x) - u(y)|\,dy\,dx \\
&= \frac{1}{|B_r|}\int_{B_r(0)}\int_{\mathbb{R}^n} |u(x) - u(x-z)|\,dx\,dz \\
&= \frac{1}{|B_r|}\int_{B_r(0)} \|u - \tau_z u\|_1\,dz \\
&\leq \sup_{|z|\leq r} \|u - \tau_z u\|_1,
\end{aligned}$$

where $\tau_z u(x) := u(x-z)$ is the function $u$ translated by the vector $z$. The third line follows by interchanging the order of integration (Fubini's theorem) and setting $z = x - y$. As translation is a continuous operator on $L^1$, [23, Theorem 3.6], we find that

$$\lim_{z\to 0} \|u - \tau_z u\|_1 = 0.$$

This shows that $v_r$ converges to $u$ in $L^1$ thereby proving the result. $\square$

The above theorem is a variant of Lebesgue's differentiation theorem, which can be found in [5, Theorem 1.32]. The reason why Lebesgue's differentiation theorem is stronger is the fact that it does not assume a priori that the pointwise limit of the averages exists.

## 2.4 Sobolev Spaces

In this section we introduce Sobolev spaces, which generalize $C^k$ spaces by utilizing the notion of $L^p$. The reason we do this is because $C^k$ is not reflexive and therefore we do not have the nice compactness properties from Theorem 2.8 needed to establish existence of minimizers. In the end it is of course our goal to show that the minimizer lies in some $C^k$ space, but this first requires the knowledge of existence in the more general Sobolev space. To make sure that the Sobolev space is reflexive we use that this is true for $L^p$ for $1 < p < \infty$ and try to cleverly combine this space with differentiability. Most of the theory can be found in [2, Chapter 5].

Let $\Omega$ be an open set in $\mathbb{R}^n$ then we first want to describe the Sobolev space which generalizes $C^1(\Omega)$. A natural norm inspired by $L^p$ for a $u \in C^1(\Omega)$ is the sum of the $L^p$ norms of $u$ and $\nabla u$. Of course this norm might not be finite for all $u \in C^1(\Omega)$ so we only consider those for which this norm is finite. To make it into a Banach space we need to take its completion. Hence, all limits with respect to this norm must be included. However, it turns out that under the limit the differentiability might not be preserved. This can be resolved by introducing a generalized or weak sense of differentiability which is preserved under these limits. We first note that for any $u \in C^1(\Omega)$ we have by integration by parts (Theorem 2.2) that

$$\int_\Omega u D_{x_i} \varphi \, dx = -\int_\Omega D_{x_i} u \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

There is no boundary term because of the compact support of $\varphi$. This property is nice to have and combines well with integration so we take this property as the definition of the weak derivative.

**Definition 2.25.** Let $u \in L^1_{loc}(\Omega)$. We say that $u$ is weakly differentiable if there are locally integrable functions $v_1, \dots, v_n$ such that

$$\int_\Omega u D_{x_i} \varphi \, dx = -\int_\Omega v_i \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

If this holds we denote $D_{x_i} u := v_i$ and $\nabla u := (D_{x_1} u, \dots D_{x_n} u)^T$.

In the above we already use the normal notation of the derivative for the weak derivative. This is justified because the weak derivative is unique (a.e.) if it exists and coincides with the classical derivative for continuously differentiable functions. These facts rely on the following result, which lies at the foundation of the theory of Sobolev spaces and distributions.

**Theorem 2.26** (Fundamental lemma of the calculus of variations)**.** *Let $v \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega v \varphi \, dx = 0 \quad \textit{for all } \varphi \in C_c^\infty(\Omega).$$

*Then $v = 0$ almost everywhere.*

*Proof.* We provide a sketch of the proof, which can be found in [6, Lemma 3.10]. Since this is a local result we may assume that $\Omega$ is bounded and $v \in L^1(\Omega)$. Informally, if we take $\varphi$ to be the sign of $v$, i.e. $+1$ when $v$ is positive, -1 when $v$ is negative and 0 else, then we find

$$\int_\Omega v\varphi\, dx = \int_\Omega |v|\, dx = \|v\|_1.$$

Hence if the above is zero then $v$ must be zero almost everywhere. This of course does not work as $\varphi$ would not lie in $C_c^\infty(\Omega)$. Therefore we use mollification to first approximate $v$ by a smooth function $f$ and then approximate the sign of $f$ by another smooth function $g$. Then we can use estimates to get

$$\|v\|_1 \leq \epsilon$$

for any $\epsilon > 0$ from which the result follows. □

It is now clear to see that the weak derivative is unique almost everywhere since if there are two $v_i, \tilde{v}_i$ then we have

$$\int_\Omega (v_i - \tilde{v}_i)\varphi\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

from which we conclude $v_i = \tilde{v}_i$ almost everywhere. Similarly, we can see that the classical derivative for continuously differentiable functions satisfies the integration by parts and hence must be equal to the weak derivative almost everywhere.

Having settled this we can easily generalize to higher order weak derivatives. Let $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$ be a multi-index and $D^\alpha$ denote taking the partial derivative with respect to the $i$th variable $\alpha_i$ times for $i = 1, \dots n$. Furthermore, denote the order of $\alpha$ by $|\alpha| = \sum_{i=1}^n \alpha_i$. The weak partial derivative $D^\alpha u$ (if it exists) is defined as the function that is locally integrable and satisfies

$$\int_\Omega u D^\alpha \varphi\, dx = -1^{|\alpha|} \int_\Omega D^\alpha u \varphi\, dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Note that the left hand side does not depend on the order in which we carry out the partial derivatives since $\varphi$ is smooth. Hence the same applies to the right hand side. From this we see that the order of taking partial derivatives does not matter for weak derivatives, which justifies the multi-index notation. We also use the convention $D^0 u := u$. We are now ready to define the Sobolev spaces, which provide alternatives for $C^k$ spaces.

**Definition 2.27** (Sobolev space)**.** The Sobolev space $W^{k,p}(\Omega)$ for $p \in [1, \infty]$ consists of the functions $u \in L^p(\Omega)$ for which all weak partial derivatives up to order $k$ exists and lie in $L^p(\Omega)$. Explicitly, for $|\alpha| \leq k$ the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$. The space is equipped with the norm

$$\|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

when $p < \infty$ and else

$$\|u\|_{k,\infty} = \|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

The summation $\sum_{|\alpha| \leq k}$ means taking the sum over all multi-indices of order less than or equal to $k$ including $\alpha = 0$. Checking that the Sobolev norm indeed satisfies the triangle-inequality follows by using the same fact for the $L^p$-norm and the norm on $\mathbb{R}^n$ given by

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

When $p = 2$ it is convention to write $H^k(\Omega) = W^{1,2}(\Omega)$ since $H^k(\Omega)$ is actually an inner product space with inner product

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v \, dx.$$

Like usual we define the following local Sobolev spaces.

**Definition 2.28.** The space $W^{k,p}_{loc}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ such that $u \in W^{k,p}(U)$ for any bounded open set $U$ with $\overline{U} \subset \Omega$.

By utilizing the weak derivatives we obtain that the Sobolev spaces inherit the nice functional analytic properties from $L^p$.

**Theorem 2.29** (Banach space, [2, §5.2 Theorem 2])**.** *For $p \in [1, \infty]$ the Sobolev space $W^{k,p}(\Omega)$ is a Banach space. In particular $H^k(\Omega)$ is a Hilbert space.*

*Proof.* We need to show that $W^{k,p}(\Omega)$ is complete. Hence we consider a Cauchy sequence $\{u_i\}$ in $W^{k,p}(\Omega)$ and need to show that it has a limit in $W^{k,p}(\Omega)$. From the definition of the Sobolev norm we see that each sequence $\{D^\alpha u_i\}$ is a Cauchy sequence in $L^p(\Omega)$ for $|\alpha| \leq k$. By completeness of $L^p(\Omega)$ we conclude that each of these sequences $\{D^\alpha u_i\}$ converges to some function

$v_\alpha \in L^p(\Omega)$. We denote the limit of the sequence $\{u_i\}$ in $L^p$ by $u$. Now, if we show that $v_\alpha$ is equal to the weak $\alpha$th partial derivative of $u$ then it follows that $u \in W^{k,p}(\Omega)$ and $u_i \to u$ in $W^{k,p}(\Omega)$ thus establishing completeness. To show this we note that for any $\varphi \in C_c^\infty(\Omega)$ the map

$$T_\varphi : L^p(\Omega) \to \mathbb{R}, \qquad T_\varphi(f) = \int_\Omega f\varphi\, dx,$$

is continuous by Hölder's inequality. Thus we find that

$$\begin{aligned}\int_\Omega uD^\alpha\varphi\, dx &= T_{D^\alpha\varphi}(u)\\ &= \lim_{i\to\infty} T_{D^\alpha\varphi}(u_i)\\ &= -1^{|\alpha|} \lim_{i\to\infty} T_\varphi(D^\alpha u_i)\\ &= -1^{|\alpha|} T_\varphi(v_\alpha)\\ &= -1^{|\alpha|} \int_\Omega v_\alpha\varphi\, dx,\end{aligned}$$

where the third equality follows by definition of the weak derivative $D^\alpha u_i$. We conclude that indeed $D^\alpha u = v_\alpha$ thereby proving the theorem. $\square$

Not only completeness but also reflexivity is inherited from $L^p$.

**Theorem 2.30** (Reflexivity $W^{k,p}(\Omega)$ for $1 < p < \infty$)**.** *The space $W^{k,p}(\Omega)$ is reflexive for $1 < p < \infty$.*

*Proof.* We can embed $W^{k,p}(\Omega)$ into the Cartesian product of $L^p$ spaces. Indeed, let $N$ be the number of multi-indices of order less than or equal to $k$ then we find that

$$E : W^{k,p}(\Omega) \to (L^P(\Omega))^N, \qquad u \mapsto (D^\alpha u)_{|\alpha|\le k}$$

is a linear isometry (if we choose the $p$-product norm on $(L^p(\Omega))^N$, which is one of the choices). By the previous theorem we find that $E(W^{k,p}(\Omega))$ is closed inside $(L^P(\Omega))^N$. Since $L^p(\Omega)$ is reflexive for $1 < p < \infty$ (Theorem 2.19) the same holds for $(L^P(\Omega))^N$ and therefore also for any closed subspace of $(L^P(\Omega))^N$. From this we conclude that $E(W^{k,p}(\Omega))$ is reflexive and hence the same applies to $W^{k,p}(\Omega)$. $\square$

Now that we have discussed the functional analytic properties of Sobolev spaces we can consider approximations by smooth functions. This can be very useful as this allows us to work with the functions we are used to. It is all based on the properties of mollifiers and we have an extra property for Sobolev functions.

**Proposition 2.31** ([5, Theorem 4.1]). *If $u \in W^{k,p}_{loc}(\Omega)$ for $p \in [1,\infty)$ and we set $u_\epsilon := \delta_\epsilon * u$ then $u_\epsilon$ converges to $u$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon$ goes to zero. Furthermore, we have in terms of weak derivatives that $D_{x_i}u_\epsilon = \delta_\epsilon * (D_{x_i}u)$ holds on $\Omega_\epsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \epsilon\}$.*

In particular this result can be used for the following.

**Theorem 2.32** (Density of smooth functions, [2, §5.3 Theorem 3]). *Let $\Omega$ be open, bounded and with $C^1$ boundary. Then the space $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $p < \infty$.*

When considering minimization problems we are dealing with boundary conditions. This is why we want to be able to assign a value to Sobolev functions at the boundary. However, some Sobolev functions are not even continuous ([2, §5.2 Example 4]) so there is no obvious way to do this. For functions in $C^\infty(\overline{\Omega})$ we can assign values at the boundary by just restricting the function to the boundary. Therefore, the idea is to use the density from the above theorem to extend this to Sobolev functions. We have the following remarkable result.

**Theorem 2.33** (Trace operator, [2, §5.5 Theorem 1]). *Let $\Omega$ be open, bounded and with $C^1$ boundary. For $p < \infty$ there exists a continuous linear map*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

*such that for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ the map $T$ coincides with restriction to the boundary, i.e. $Tu = u|_{\partial\Omega}$.*

The above map is called the trace operator and we usually just write $Tu = u|_{\partial\Omega}$ even if $u$ does not lie in $C(\overline{\Omega})$. A special class of Sobolev functions is the one consisting of trace zero functions.

**Definition 2.34.** We define $W^{1,p}_0(\Omega)$ for $p < \infty$ as the space of functions $u \in W^{1,p}(\Omega)$ with $u|_{\partial\Omega} = Tu = 0$.

In some sense these Sobolev functions go to zero near the boundary, but it is not really clear in what way exactly. This is answered in the following theorem.

**Theorem 2.35** (Density of $C^\infty_c(\Omega)$ in $W^{1,p}_0(\Omega)$, [2, §5.5 Theorem 2]). *Let $\Omega$ be open, bounded and with $C^1$ boundary. Then $C^\infty_c(\Omega)$ is dense in $W^{1,p}_0(\Omega)$, i.e. we can approximate trace zero function by smooth functions with compact support.*

Clearly any $u \in C^\infty_c(\Omega)$ has trace zero and by continuity of the trace operator the same holds for any function in the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(\Omega)$.

Hence we conclude by Theorem 2.35 that $W_0^{1,p}(\Omega)$ is equal to the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Sometimes this is taken as the definition of $W_0^{1,p}(\Omega)$. A very useful inequality for trace zero functions states that we can bound the $L^p$ norm of this function by the $L^p$ norm of its weak derivative.

**Lemma 2.36** (Poincaré inequality, [24, Theorem 13.19])**.** *Let $\Omega$ be open, bounded and with $C^1$ boundary. Then there exists a constant $C_\Omega$, which only depends on $\Omega$ and $p$, that satisfies for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq C_\Omega \left( \sum_{i=1}^n \|D_{x_i} u\|_{L^p(\Omega)}^p \right)^{1/p} = C_\Omega \|\nabla u\|_{L^p(\Omega)}.$$

*Proof.* By approximation it is enough to prove this for $u \in C_c^\infty(\Omega)$. Extend $u$ as zero outside $\Omega$ such that it is defined on $\mathbb{R}^n$. By boundedness of $\Omega$ we can choose $a$ and $b$ such that $\Omega \subset \mathbb{R}^{n-1} \times [a,b]$. For $x = (x_1, \dots, x_n)$ we denote $x' = (x_1, \dots, x_{n-1})$ and this gives us by compact support that

$$u(x) = u(x', x_n) - u(x', a) = \int_a^{x_n} D_{x_n} u(x', t)\, dt.$$

Hence we obtain

$$\begin{aligned}
\int_\Omega |u(x)|^p\, dx &\leq \int_\Omega \left( \int_a^{x_n} |D_{x_n} u(x', t)|\, dt \right)^p dx \\
&\leq \int_\Omega \left( \int_a^{x_n} 1\, dt \right)^{p/p'} \cdot \left( \int_a^{x_n} |D_{x_n} u(x', t)|^p\, dt \right) dx \\
&\leq (b-a)^{p/p'} \int_\Omega \int_a^b |D_{x_n} u(x', t)|^p\, dt\, dx,
\end{aligned}$$

where the second line follows from Hölder's inequality. We note that the last integrand is not dependent on $x_n$. Therefore we split up the integral over $\Omega$ as an integral over $\mathbb{R}^{n-1}$ and one over $[a,b]$ by Fubini to obtain

$$\begin{aligned}
\int_\Omega |u(x)|^p\, dx &\leq (b-a)^{p/p'} \int_a^b \int_{\mathbb{R}^{n-1}} \int_a^b |D_{x_n} u(x', t)|^p\, dt\, dx'\, dx_n \\
&= (b-a)^{p/p'+1} \int_\Omega |D_{x_n} u(x)|^p\, dx \\
&= (b-a)^{p/p'+1} \|D_{x_n} u\|_{L^p(\Omega)}^p,
\end{aligned}$$

from which the result follows. $\square$

In the proof we actually only used boundedness in the $x_n$-direction and only needed the partial derivative with respect to $x_n$ to bound the $L^p$-norm of $u$. We do not need this stronger result however, so we maintain the more simple formulation.

With this we have discussed most of the basic theory about Sobolev spaces. Lastly, we mention a few results that come to play when studying the regularity of obstacle problems. The first of which is a characterization of $W^{1,\infty}$ in terms of Lipschitz functions. We denote $C^{0,1}(\Omega)$ as the space of Lipschitz functions on $\Omega$.

**Theorem 2.37** ($W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$, [2, §5.8 Theorem 4])**.** *Let $\Omega$ be open, bounded and with $C^1$ boundary. Then $u$ lies in $W^{1,\infty}(\Omega)$ if and only if $u$ is Lipschitz continuous.*

Another question one can ask is whether a Sobolev function is $C^1$ under certain conditions. A sufficient condition turns out to be that the Sobolev function and its weak derivative are continuous.

**Lemma 2.38.** *Let $u \in W^{1,p}(\Omega)$ be continuous with continuous weak derivative $\nabla u$. Then $u$ lies in $C^1(\Omega)$.*

*Proof.* We consider the mollification of $u$ given by $u_\epsilon = \delta_\epsilon * u$. Since $u$ is continuous we find that $u_\epsilon$ converges uniformly to $u$ on compact subsets of $\Omega$ (Theorem 2.23). Furthermore, we have that $D_{x_i} u_\epsilon = \delta_\epsilon * (D_{x_i} u)$ so by continuity of the weak derivative $D_{x_i} u$ we also see that $D_{x_i} u_\epsilon$ converges uniformly to $D_{x_i} u$ on compact subsets of $\Omega$. Hence we see that $u_\epsilon$ converges to $u$ in the uniform $C^1$-norm. Continuous differentiability is preserved under this norm so we conclude that $u$ is continuously differentiable on any compact subset of $\Omega$. Since continuous differentiability is a local property we obtain $u \in C^1(\Omega)$ as desired. $\square$

We wish to extend Theorem 2.37 to the case of $W^{2,\infty}$ and $C^{1,1}$, where $C^{1,1}$ is the space of continuously differentiable functions with Lipschitz continuous derivative. To this end assume that $\Omega$ is open, bounded and with $C^1$ boundary and let $u \in W^{2,\infty}(\Omega)$. Then we find that $u$ and $\nabla u$ are Lipschitz by Theorem 2.37. Hence by Lemma 2.38 we conclude $u \in C^{1,1}(\Omega)$. Conversely, if $u \in C^{1,1}(\Omega)$ then we find by Theorem 2.37 that the weak second partial derivatives lie in $L^\infty(\Omega)$. Since furthermore $u$ and $\nabla u$ are bounded because they are Lipschitz on a bounded domain we conclude that $u \in W^{2,\infty}(\Omega)$. The fact that $u$ is Lipschitz follows since $\nabla u$ is bounded. Hence we have proven the following.

**Theorem 2.39.** *Let $\Omega$ be open, bounded and with $C^1$ boundary then we have $W^{2,\infty}(\Omega) = C^{1,1}(\Omega)$.*

Lastly, we want to state and prove a approximation of $C^{1,1}$ functions by their first order Taylor polynomial.

**Lemma 2.40.** *Let $u \in C^{1,1}(\Omega)$ then we have for all $x, y \in \Omega$ such that the line between $x$ and $y$ lies in $\Omega$ that*

$$|u(y) - u(x) - \nabla u(x) \cdot (y - x)| \leq L|y - x|^2,$$

*with $L$ the Lipschitz constant of $\nabla u$.*

*Proof.* As $u$ is continuously differentiable and the line between $x$ and $y$ lies in $\Omega$ we find by the mean value theorem that

$$u(y) - u(x) = \nabla u(\eta) \cdot (y - x),$$

with $\eta = tx + (1 - t)y$ for some $t \in [0, 1]$. This yields

$$\begin{aligned} |u(y) - u(x) - \nabla u(x) \cdot (y - x)| =& |(\nabla u(\eta) - \nabla u(x)) \cdot (y - x)| \\ &\leq |\nabla u(\eta) - \nabla u(x)| \cdot |y - x| \\ &\leq L|\eta - x| \cdot |y - x| \\ &\leq L|y - x|^2. \end{aligned}$$

□

# 3 Uniqueness and Existence

In this section we discuss the existence and uniqueness of minimizers of integral functionals $\mathcal{F}$ of the following form

$$\mathcal{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^1$ domain (i.e. open, connected and with $C^1$ boundary). We are minimizing $\mathcal{F}$ over functions $u$ in the set

$$W_g^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid u|_{\partial\Omega} = g\},$$

for $p \in (1, \infty)$ and $g \in C^1(\partial\Omega)$. Since $\partial\Omega$ is a closed $C^1$-manifold we can extend $g$ to a $C^1$ function $G$ on $\mathbb{R}^n$. The function $G$ and its derivative are bounded on $\Omega$ as $\Omega$ is bounded and thus we obtain that $G \in W_g^{1,p}(\Omega)$. Therefore the set $W_g^{1,p}(\Omega)$ is non-empty and it makes sense to minimize over this set. Furthermore, $W_g^{1,p}(\Omega)$ is a closed affine subset of $W^{1,p}(\Omega)$ as the trace is a linear and continuous operator. Because the space $W^{1,p}(\Omega)$ is reflexive for the choice $p \in (1, \infty)$ the space $W_g^{1,p}(\Omega)$ is perfectly suited for addressing the problem of existence of minimizers. Note that (3.1) might not be well-defined depending on the function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ but we come to this later. In this section we are following the treatment of [6, Chapter 2].

## 3.1 Direct Method

We first work towards the existence of minimizers and this is done by utilizing the so called direct method in the calculus of variations. It was developed after Hilbert had posed his 20th problem around 1900, which asked whether these variational problems had solutions. The method is based on a very simple idea, which reduces the problem of finding a minimizer to verifying certain properties on $\mathcal{F}$. Let us first illustrate this method in the finite-dimensional case. Suppose we wish to prove existence of a minimum of a function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies

(i) For every sequence $\{x_i\}$ with $\|x_i\| \to \infty$ we have $f(x_i) \to \infty$;

(ii) For every sequence $\{x_i\} \to x$ we have $f(x) \leq \liminf_{i\to\infty} f(x_i)$.

(Note that condition (ii) is automatically satisfied when $f$ is continuous). To find a minimum we need to find a $x_0 \in \mathbb{R}^n$ such that $f(x_0) = \inf_{\mathbb{R}^n} f$. The most direct thing we can do is choose a sequence $\{x_i\}$ such that $\lim_{i\to\infty} f(x_i) = \inf_{\mathbb{R}^n} f$. Our naive hope is that this sequence (or a subsequence thereof) converges to a point at which $f$ attains its minimum. To show that this is the case we note that by property (i) the minimizing sequence $\{x_i\}$ is bounded. Now using that closed and bounded sets are compact yields a convergent subsequence $\{x_{i_k}\} \to x_0$ for some $x_0 \in \mathbb{R}^n$. To show that $f$ attains its minimum at $x_0$ we apply property (ii) to get

$$f(x_0) \leq \liminf_{k\to\infty} f(x_{i_k}) = \lim_{k\to\infty} f(x_{i_k}) = \inf_{\mathbb{R}^n} f.$$

This proves the existence of a minimum.

The key in this simple proof was using the compactness of closed and bounded sets to extract a convergent subsequence. However, in infinite-dimensional spaces the closed unit ball is never compact. Hence we cannot guarantee that bounded sequences have convergent subsequences. A partial replacement for this can be achieved by instead considering weak convergence. Namely, for reflexive Banach spaces bounded sequences admit weakly convergent subsequences, see Theorem 2.8. Having this functional analytic property is the main reason why we consider Sobolev spaces as opposed to more standard spaces like $C^k$, which are not reflexive. In order to generalize the direct method to arbitrary reflexive Banach spaces $(X, \|.\|)$ we need to slightly adapt the second property imposed to cater to weak convergence. Thus, let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be a functional then we need the following criteria on $\mathcal{F}$:

(P1) **Coercivity:** For every sequence $\{u_i\}$ with $\lim_{i\to\infty}\|u_i\| = \infty$ we have that $\lim_{i\to\infty} \mathcal{F}(u_i) = \infty$;

(P2) **Weak lower semicontinuity:** For every sequence $\{u_i\} \rightharpoonup u$ we have $\mathcal{F}(u) \leq \liminf_{i\to\infty} \mathcal{F}(u_i)$,

where $\rightharpoonup$ indicates weak convergence. The property P1 is called coercivity and the property P2 is called weak lower semicontinuity. We carry out the proof of existence of minimizers when $\mathcal{F}$ satisfies P1 and P2.

**Theorem 3.1** (Direct method, [6, Theorem 2.1]). *Let $X \neq \varnothing$ be a reflexive Banach space or a closed convex subset of a reflexive Banach space. If the functional $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ satisfies P1 and P2 then there exists a minimizer $u_0 \in X$, i.e. $\mathcal{F}(u_0) = \inf_{u\in X} \mathcal{F}(u)$.*

*Proof.* We treat the case of a reflexive Banach space first and argue like in the finite-dimensional case. Assume without loss of generality that $\inf_{u\in X} \mathcal{F}(u) \neq \infty$, otherwise any $u$ trivially minimizes $\mathcal{F}$. Now we take a sequence $\{u_i\}$ in $X$ such that

$$\lim_{i\to\infty} \mathcal{F}(u_i) = \inf_{u\in X} \mathcal{F}(u) < \infty.$$

We claim that $\{u_i\}$ is bounded. Indeed, if this is not the case then a subsequence $\{u_{i_k}\}$ of $\{u_i\}$ diverges to infinity. However, this implies by coercivity that $\lim_{k\to\infty} \mathcal{F}(u_{i_k}) = \infty$ contradicting the above. Thus the sequence $\{u_i\}$ is bounded in the reflexive Banach space $X$. By virtue of Theorem 2.8 we can extract a subsequence $\{u_i\}$ (not relabelled) that converges weakly to some $u_0 \in X$. Now we apply the weak lower semicontinuity to obtain

$$\mathcal{F}(u_0) \leq \liminf_{i\to\infty} \mathcal{F}(u_i) = \lim_{i\to\infty} \mathcal{F}(u_i) = \inf_{u\in X} \mathcal{F}(u),$$

which proves the result in the first case.

If $X$ is a closed and convex subset of a reflexive Banach space then all the above steps are valid except possibly that $u_0 \in X$. Namely, this element could lie in the ambient Banach space. However, as closed and convex sets in Banach spaces are weakly sequentially closed as well (Theorem 2.9) it follows that in fact $u_0 \in X$ and therefore the proof is still valid. $\square$

Having established this theorem we can now focus on giving simple sufficient conditions for a functional as in (3.1) to be coercive and weakly lower semicontinuous.

## 3.2 Coercivity

In this section we discuss the well-definedness and coercivity property of the functional in (3.1)

$$\mathcal{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx.$$

We achieve these properties by imposing certain conditions on the function

$$f : \Omega \times \mathbb{R}^n \to \mathbb{R},$$

which results in well-definedness and coercivity of $\mathcal{F}$.

Well-definedness plays a role as the Lebesgue integral in (3.1) might not exist even if we allow the value infinity. First of all the map $x \mapsto f(x, \nabla u(x))$ needs to be measurable for the integral to be defined. We guarantee this by assuming $f$ to be continuous in both its arguments. Secondly, the integral of the negative part of $f$ cannot be infinite as this would either imply $\mathcal{F} = -\infty$ or that the integral is not well-defined. To avoid this we can either assume that $f$ is bounded in some sense such that the integral is always finite or that $f \geq C$ for $C \in \mathbb{R}$ such that the integral is well-defined albeit possibly infinite (this uses that $\Omega$ is bounded). As we will see the property $f \geq C$ is required for coercivity regardless so assuming this property does not lose generality. Therefore we assume from now on that $f$ is continuous and that $f \geq C$ for some $C \in \mathbb{R}$. This yields that (3.1) is well-defined.

Now we consider coercivity. Intuitively, we need that $\mathcal{F}$ grows just as quickly as the the norm such that any sequence diverging in terms of the norm does the same in terms of $\mathcal{F}$. In the case of the Sobolev norm $\|.\|_{1,p}$ it is therefore natural to assume the following $p$-coercivity bound

$$\mu|\xi|^p - C \leq f(x,\xi), \qquad (x,\xi) \in \Omega \times \mathbb{R}^n, \tag{3.2}$$

with $\mu, C > 0$ constants. This bound makes the functional $\mathcal{F}(u)$ comparable to a constant times the $p$-norm of $\nabla u$. This almost gives us want we want but we also need that it is comparable with the $p$-norm of $u$. However, by using the Poincaré inequality we actually obtain a way to bound the $p$-norm of $u$ in terms of the $p$-norm of $\nabla u$. Therefore the condition (3.2) is sufficient for coercivity as we will formalize in the following proposition.

**Proposition 3.2.** *Let $\mathcal{F}$ be as in (3.1). If $f$ is continuous and satisfies the $p$-coercivity bound (3.2) for $p \in (1,\infty)$ then $\mathcal{F}$ is coercive on $W_g^{1,p}(\Omega)$.*

*Proof.* Let $\{u_i\}$ be a sequence in $W_g^{1,p}(\Omega)$ such that $\|u_i\|_{1,p} \to \infty$. We fix any $v \in W_g^{1,p}(\Omega)$ then it follows that $u_i - v \in W_0^{1,p}(\Omega)$ and $\|u_i - v\|_{1,p} \to \infty$. By the Poincaré inequality (Lemma 2.36) we find that

$$\|u_i - v\|_{1,p} \leq (1 + (C_\Omega)^p)^{1/p} \cdot \|\nabla(u_i - v)\|_p,$$

which implies that $\|\nabla(u_i - v)\|_p \to \infty$ and subsequently that $\|\nabla u_i\|_p \to \infty$. Finally, by the $p$-coercivity bound (3.2) we find that

$$\mathcal{F}(u_i) \geq \mu\|\nabla u_i\|_p^p - C\,\mathrm{meas}(\Omega),$$

which shows that $\mathcal{F}(u_i) \to \infty$. This proves the result. □

### 3.3 Weak Lower Semicontinuity

In this section we describe a sufficient condition for $\mathcal{F}$ to be weakly lower semicontinuous. It is slightly more difficult to see what kind of condition on $f$ we need for weak lower semicontinuity. First of all, we still assume that $f \geq -C$ for some $C > 0$, which is needed for well-definedness and coercivity. It turns out that if furthermore

$$\xi \mapsto f(x,\xi) \text{ is convex for all } x \in \Omega \tag{3.3}$$

then $\mathcal{F}$ is weakly lower semicontinuous. This is the content of the following theorem. This result dates back to 1920 where Tonelli proved this weak lower semicontinuity result for slightly more general integral functionals. He published this in his paper 'La semicontinuità nel calcolo delle variazioni', see [19].

**Theorem 3.3** ([6, Theorem 2.6]). *Let $\mathcal{F}$ be as in (3.1) and assume that $f \geq -C$ and $f$ satisfies (3.3). Then $\mathcal{F}$ is weakly lower semicontinuous on $W^{1,p}_g(\Omega)$.*

*Proof.* We divide the proof into two steps.

*Step 1.* We first prove that $\mathcal{F}$ is strongly lower semicontinuous. We may assume that $f \geq 0$ as we can replace $f$ with $f + C$ without loss of generality ($\Omega$ is bounded). Take a sequence $\{u_i\}$ in $W^{1,p}_g(\Omega)$ converging to $u$ strongly then we show that

$$\mathcal{F}(u) \leq a := \liminf_{i\to\infty} \mathcal{F}(u_i).$$

To do this we can take a subsequence $\{u_i\}$ (not relabelled) such that $\mathcal{F}(u_i) \to a$. This limit remains unchanged if we take any further subsequence. In particular, since $\nabla u_i \to \nabla u$ in $L^p(\Omega)$ we can extract a subsequence by Theorem 2.20 such that $\nabla u_i \to \nabla u$ pointwise almost everywhere and $\lim_{i\to\infty} \mathcal{F}(u_i) = a$. As $f \geq 0$ we can now apply Fatou's lemma (Theorem 2.11) to obtain

$$\mathcal{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx \leq \liminf_{i\to\infty} \int_\Omega f(x, \nabla u_i(x))\, dx = \lim_{i\to\infty} \mathcal{F}(u_i) = a,$$

as desired.

*Step 2.* For the weak lower semicontinuity, let $\{u_i\}$ be a sequence converging weakly to $u$ in $W^{1,p}_g(\Omega)$. We have to show that

$$\mathcal{F}(u) \leq a := \liminf_{i\to\infty} \mathcal{F}(u_i).$$

Again we take a subsequence $\{u_i\}$ (not relabelled) such that $\lim_{i\to\infty} \mathcal{F}(u_i) = a$. In order to go from weak to strong convergence we use Mazur's lemma (Theorem 2.10). The lemma states that we can find a function $K : \mathbb{N} \to \mathbb{N}$ and scalars $\lambda(i)_k \in [0, 1]$ for $k = i, \ldots, K(i)$ such that the convex combinations

$$v_i = \sum_{k=i}^{K(i)} \lambda(i)_k u_k \quad \text{with} \quad \sum_{k=i}^{K(i)} \lambda(i)_k = 1$$

converge strongly to $u$ in $W^{1,p}_g(\Omega)$. Now we use that $f$ is convex in its second argument to obtain

$$\begin{aligned}\mathcal{F}(v_i) = \int_\Omega f\left(x, \sum_{k=i}^{K(i)} \lambda(i)_k u_k\right) dx \\ \leq \sum_{k=i}^{K(i)} \lambda(i)_k \mathcal{F}(u_k).\end{aligned}$$

As $\sum_{k=i}^{K(i)} \lambda(i)_k = 1$ and $\mathcal{F}(u_k) \to a$ we conclude that the limit of the right hand side is $a$. Thus we find that $\liminf_{i\to\infty} \mathcal{F}(v_i) \leq a$. Because $v_i$ converges strongly to $u$ we can use step 1 to conclude

$$\mathcal{F}(u) \leq \liminf_{i\to\infty} \mathcal{F}(v_i) \leq a,$$

which yields the result. $\square$

**Remark 3.4.** It can actually be shown that convexity is a neccesary condition for weak lower semicontinuity when $f$ does not depend on $x$. Although we do not prove this here, it is good to know that demanding convexity is not too strong of a criteria. See for example Proposition 2.9 in [6] for a proof.

Having established the criteria for coercivity and weak lower semicontinuity we can now state the following existence and uniqueness result.

**Theorem 3.5** (Existence and uniqueness)**.** *Let $\mathcal{F}$ be as in (3.1) and $K \neq \varnothing$ a closed and convex subset of $W_g^{1,p}(\Omega)$. If $f$ satisfies the $p$-coercivity bound (3.2) and the convexity in (3.3) then $\mathcal{F}$ has a minimizer in $K$. If furthermore $f$ is strictly convex in its second argument then this minimizer is unique.*

*Proof.* By Proposition 3.2 and Theorem 3.3 $\mathcal{F}$ is both coercive and weakly lower semicontinuous on $K$. Furthermore, $K$ is a non-empty closed and convex subset of the reflexive Banach space $W^{1,p}(\Omega)$. Now applying the direct method (Theorem 3.1) we find that there exists a minimizer $u_0 \in K$. For uniqueness, assume that to the contrary there exist two different minimizers $u_0$ and $v_0$ in $K$. Now define $u := u_0/2 + v_0/2$ which lies in $K$ as $K$ is convex. Since $u_0 - v_0 \in W_0^{1,p}(\Omega)$ and $\|u_0 - v_0\|_{1,p} \neq 0$ we have by Poincaré's inequality that $\|\nabla u_0 - \nabla v_0\|_p \neq 0$. This means that $\nabla u_0$ and $\nabla v_0$ differ on a set of positive measure. Using strict convexity of $f$ we now obtain that

$$\int_\Omega f(x, \nabla u(x))\, dx < \int_\Omega \frac{1}{2}\left(f(x, \nabla u_0(x)) + f(x, \nabla v_0(x))\right)\, dx.$$

Using linearity of the integral we find that

$$\mathcal{F}(u) < \frac{1}{2}(\mathcal{F}(u_0) + \mathcal{F}(v_0)) = \inf_K \mathcal{F},$$

which is a contradiction thus proving uniqueness. $\square$

In the above we can in particular take $K = W_g^{1,p}(\Omega)$ but we need this more general statement when we are dealing with obstacle problems in Section 5.

**Example 3.6.** Consider the Dirichlet functional

$$\mathcal{D}(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx.$$

In the above setting we have $f(x, \xi) = \frac{|\xi|^2}{2}$. It is readily seen that this $f$ is strictly convex in its second argument and satisfies the 2-coercivity bound. We therefore find by Theorem 3.5 that the problem

$$\text{Minimize } \ \mathcal{D}(u) \ \text{ over all } u \in H^1_g(\Omega) = W^{1,2}_g(\Omega)$$

has a unique solution $u_0 \in H^1_g(\Omega)$.

As a last remark on existence we want to note that the results can be extended to functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx.$$

In this case we need that $\xi \mapsto f(x, y, \xi)$ is convex for every $(x, y) \in \Omega \times \mathbb{R}$ to ensure weak lower semicontinuity and certain bounds for coercivity, see [6, Theorem 5.2] or [7, Theorem 3.23].

# 4 Regularity

In this section we discuss the regularity of minimizers of variational problems. It was necessary for our proof of existence to consider the more general Sobolev spaces in contrary to $C^k$ spaces. Now that we can show existence of minimizers the next step consists of showing that the minimizers satisfy certain smoothness properties. This is very important to know as a general Sobolev function can even be discontinuous, see [2, §5.2 Example 4]. Establishing these smoothness properties of the minimizer is called regularity theory and in this section we show useful methods from this area and work out the regularity for the specific case of the Dirichlet functional.

Our first aim is to derive an equation that is necessarily satisfied by the minimizer. It is inspired by the finite-dimensional analogue, where a necessary condition for a function to have a minimum is that its derivative vanishes. In the infinite-dimensional case we actually get a partial differential equation which is called the Euler-Lagrange equation and we discuss its weak form as well, see Lemma 4.2. The Euler-Lagrange equation does not imply certain regularity in general but it is the basis of proving the regularity in specific cases. For the Dirichlet functional the Euler-Lagrange equation turns out to be the Laplace equation. Thus we can obtain regularity of the minimizer of the Dirichlet functional by studying the regularity of weakly harmonic functions. This is what we carry out. Lastly, we also discuss properties of sub- and superharmonic functions, which turn up naturally when introducing an obstacle in the minimization problem of the Dirichlet functional.

## 4.1 Euler-Lagrange Equation

We first recall that for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ attaining a minimum at $x_0 \in \mathbb{R}^n$ implies for the directional derivative that

$$Df(x_0)v = \lim_{h\to 0} \frac{f(x_0 + hv) - f(x_0)}{h} = 0$$

for all $v \in \mathbb{R}^n$. This is because $f(x_0 + hv) - f(x_0) \geq 0$ so we find that

$$\lim_{h\downarrow 0} \frac{f(x_0 + hv) - f(x_0)}{h} \geq 0, \quad \text{and} \quad \lim_{h\uparrow 0} \frac{f(x_0 + hv) - f(x_0)}{h} \leq 0.$$

Since the limit exists it must therefore be equal to zero. Now define

$$\mathcal{F} : W^{1,p}_g(\Omega) \to \mathbb{R}, \quad \mathcal{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \tag{4.1}$$

with $\Omega \subset \mathbb{R}^n$ a $C^1$-domain, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous and $g \in C^1(\partial\Omega)$. We can define a similar directional derivative for $\mathcal{F}$ but now $W^{1,p}_g(\Omega)$ is not a vector space so the directional derivative vanishes only in certain directions. This is because $\mathcal{F}(u_0 + h\varphi) - \mathcal{F}(u_0) \geq 0$ can only be guaranteed if $u_0 + h\varphi \in W^{1,p}_g(\Omega)$. We can achieve this by picking $\varphi \in W^{1,p}_0(\Omega)$ but for simplicity we first restrict ourselves to the case $\varphi \in C^\infty_c(\Omega)$.

**Definition 4.1.** *The first variation of* $\mathcal{F}$ at $u \in W^{1,p}_g(\Omega)$ in the direction $\varphi \in C^\infty_c(\Omega)$ is defined as (if it exists)

$$\delta\mathcal{F}(u)(\varphi) := \lim_{h\to 0} \frac{\mathcal{F}(u + h\varphi) - \mathcal{F}(u)}{h}.$$

Suppose for now that the variation of $\mathcal{F}$ exists. By the argument in the finite-dimensional case we find that for a minimizer $u_0 \in W^{1,p}_g(\Omega)$ we have $\delta\mathcal{F}(u)(\varphi) = 0$ for all $\varphi \in C^\infty_c(\Omega)$. This gives us an equation which the minimizer has to satisfy, i.e. a necessary condition. Hence it is interesting to calculate this first variation explicitly (if it exists). For this we need certain bounds on $f$ and its derivatives, which allow us to interchange the limit with the integral. We write $(x, y, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and set $D_y f(x, y, \xi)$ to be the derivative with respect to the second argument and $D_\xi f(x, y, \xi)$ the gradient with respect to the third argument. Note that $D_y f(x, y, \xi)$ is a scalar while $D_\xi f(x, y, \xi)$ is a vector in $\mathbb{R}^n$. We will assume the bounds:

$$|f(x, y, \xi)|,\ |D_y f(x, y, \xi)|,\ |D_\xi f(x, y, \xi)| \leq C(1 + |y|^p + |\xi|^p), \tag{4.2}$$

for $(x, y, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and a constant $C > 0$. We can now state the following result.

**Lemma 4.2** ([6, Theorem 3.1])**.** *Let $\mathcal{F}$ be as in (4.1) and suppose $f$ is continuously differentiable in its second and third argument and satisfies (4.2). Then we have for all $u \in W^{1,p}_g(\Omega)$ that*

$$\delta\mathcal{F}(u)(\varphi) = \int_\Omega D_y f(x, u, \nabla u)\varphi + D_\xi f(x, u, \nabla u) \cdot \nabla\varphi \, dx \tag{4.3}$$

*for all $\varphi \in C^\infty_c(\Omega)$.*

*Proof.* For simplicity of notation we do not explicitly write the $x$ dependence everywhere. Note that by the bound on $f$ itself we find that $\mathcal{F}(u)$ is well-defined and finite. Intuitively, to calculate the first variation we need to differentiate under the integral sign. To formalize this we use the mean value theorem to write

$$\begin{aligned}\delta\mathcal{F}(u)(\varphi) &= \lim_{h\to 0} \int_\Omega \frac{f(x, u + h\varphi, \nabla u + h\nabla\varphi) - f(x, u, \nabla u)}{h} \, dx \\ &= \lim_{h\to 0} \int_\Omega D_y f(x, u + \eta\varphi, \nabla u + \eta\nabla\varphi)\varphi \\ &\qquad + D_\xi f(x, u + \eta\varphi, \nabla u + \eta\nabla\varphi) \cdot \nabla\varphi \, dx,\end{aligned}$$

for $\eta = \eta(h) \in [0, h]$. This is obtained by differentiating $f(x, u + h\varphi, \nabla u + h\nabla\varphi)$ with respect to $h$ and evaluating at $\eta$. By using the growth bounds on the derivatives and the fact that $\varphi$ and $\nabla\varphi$ are bounded we find that the absolute value of the integrand inside the last integral is bounded uniformly in $h$ by $C'(1 + (|u| + |\varphi|)^p + (|\nabla u| + |\nabla\varphi|)^p)$ if we assume $|h| \leq 1$. This bound is integrable since $\Omega$ is bounded so we can use Lebesgue's dominated convergence theorem (Theorem 2.12) to take the limit of $h$ inside the integral. Now that we have interchanged the limit we can undo the effects of the mean value theorem to obtain

$$\delta\mathcal{F}(u)(\varphi) = \int_\Omega \lim_{h\to 0} \frac{f(x, u + h\varphi, \nabla u + h\nabla\varphi) - f(x, u, \nabla u)}{h}\, dx,$$

which is equal to (4.3). □

For a minimizer $u_0 \in W^{1,p}_g(\Omega)$ of $\mathcal{F}$ we therefore obtain the equation

$$\delta\mathcal{F}(u_0)(\varphi) = \int_\Omega D_y f(x, u_0, \nabla u_0)\varphi + D_\xi f(x, u_0, \nabla u_0) \cdot \nabla\varphi\, dx = 0,$$

for all $\varphi \in C^\infty_c(\Omega)$. This is called the weak form of the Euler-Lagrange equation. If we recall the definition of weak derivative we see that this equation implies in a weak sense that

$$D_y f(x, u_0, \nabla u_0) - \operatorname{div}[D_\xi f(x, u_0, \nabla u_0)] = 0, \tag{4.4}$$

where the divergence must be taken of the map $x \mapsto D_\xi f(x, u_0(x), \nabla u_0(x))$. This partial differential equation is the Euler-Lagrange equation and it establishes a deep connection between the theory of variational problems and the theory of partial differential equations. Euler and Lagrange developed this equation during the 1750s when they were working on variational problems. This analytical method for solving such minimization problems inspired Euler to come up with the name 'Calculus of variations', which is still used today [22]. It is easy to check by integration by parts and the fundamental lemma of the calculus of variations (Lemma 2.26) that if $u_0$ and $f$ are $C^2$ then $u_0$ satisfies the weak form if and only if it satisfies the (strong) Euler-Lagrange equation. In general we only know that $u_0 \in W^{1,p}_g(\Omega)$, which limits us to working with the weak form of the Euler-Lagrange equation. However, this can be enough in some cases to show that $u_0$ is $C^2$ or even $C^\infty$.

**Example 4.3.** Consider again the Dirichlet functional

$$\mathcal{D}(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx,$$

which has a unique minimizer $u_0 \in H^1_g(\Omega)$ by Example 3.6. In this case $D_\xi f(\xi) = \xi$ so we find by Lemma 4.2 that

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$

This is a weak form of the Laplace equation $-\Delta u_0 = 0$. In the next subsection we will study the Laplace equation and its weak form and show that actually $u_0 \in C^\infty(\Omega)$.

Our next aim is to extend the first variation to directions in $W^{1,p}_0(\Omega)$ as well. We cannot repeat the proof of Lemma 4.2 as that required boundedness of the test function $\varphi$. We therefore impose the stronger bounds on the derivatives

$$|D_y f(x,y,\xi)|, \; |D_\xi f(x,y,\xi)| \leq C(1 + |y|^{p-1} + |\xi|^{p-1}). \tag{4.5}$$

We have the following:

**Corollary 4.4** ([7, Theorem 3.37 (III)])**.** *Suppose $\mathcal{F}$ and $f$ are as in Lemma 4.2 and $f$ satisfies furthermore the bound (4.5). Then (4.3) holds for all $\varphi \in W^{1,p}_0(\Omega)$.*

*Proof.* The only adaptation we need to make is to show that

$$|D_y f(x, u + \eta\varphi, \nabla u + \eta \nabla \varphi)\varphi|$$

and

$$|D_\xi f(x, u + \eta\varphi, \nabla u + \eta \nabla \varphi) \cdot \nabla\varphi|$$

are uniformly bounded in $\eta$ by an integrable function. We will show that this is the case for the first one as the second can be proven analogously. By (4.5) we have

$$\begin{aligned} |D_y f(x,&u + \eta\varphi, \nabla u + \eta\nabla\varphi)\varphi| \\ &\leq C(1 + |u + \eta\varphi|^{p-1} + |\nabla u + \eta \nabla\varphi|^{p-1})|\varphi| \\ &\leq C(1 + (|u| + |\varphi|)^{p-1} + (|\nabla u| + |\nabla \varphi|)^{p-1})|\varphi| \end{aligned}$$

for $|\eta| \leq 1$. We show that this last function is integrable. The term $C|\varphi|$ is fine since by boundedness of $\Omega$ and Hölder's inequality (Theorem 2.17) we have $L^p(\Omega) \subset L^1(\Omega)$ because the constant function 1 lies in $L^{p'}(\Omega)$. For the next term we have by Hölder's inequality that

$$\begin{aligned} \int_\Omega (|u| + |\varphi|)^{p-1}|\varphi| \, dx &\leq \|(|u| + |\varphi|)^{p-1}\|_{p'} \|\varphi\|_p \\ &= \||u| + |\varphi|\|_p^{p-1} \|\varphi\|_p < \infty, \end{aligned}$$

where $p' = p/(p-1)$. This bound follows similarly for all other terms when we keep in mind that $\nabla u$ and $\nabla\varphi$ also have finite $L^p$-norm and this yields integrability. $\square$

In general, if $u$ satisfies the weak Euler-Lagrange equation then $u$ need not be a minimum. Indeed, just like in the finite-dimensional case, we can just have a local minimum or even a maximum or saddle-point. We call functions $u$ that satisfy $\delta\mathcal{F}(u)(\varphi) = 0$ for all $\varphi \in W_0^{1,p}(\Omega)$ critical points of $\mathcal{F}$. We can now actually show that any critical point is a minimizer under the assumption of convexity.

**Proposition 4.5.** *Let $\mathcal{F}$ be as in (4.1) and $f$ jointly convex in its second and third argument. Then any $u_0 \in W_g^{1,p}(\Omega)$ for which*

$$\delta\mathcal{F}(u_0)(\varphi) = 0 \quad \text{for all } \varphi \in W_0^{1,p}(\Omega)$$

*is a minimizer of $\mathcal{F}$.*

*Proof.* Since $f$ is convex in its second and third argument we find that $\mathcal{F}$ is also convex. Now let $v \in W_g^{1,p}(\Omega)$ be arbitrary then we have $v - u_0 \in W_0^{1,p}(\Omega)$. Therefore

$$0 = \delta\mathcal{F}(u_0)(v - u_0) = \lim_{h\to 0} \frac{\mathcal{F}((1-h)u_0 + hv) - \mathcal{F}(u_0)}{h}.$$

For $h \in (0, 1]$ we find by convexity that

$$\begin{aligned} \frac{\mathcal{F}((1-h)u_0 + hv) - \mathcal{F}(u_0)}{h} &\leq \frac{(1-h)\mathcal{F}(u_0) + h\mathcal{F}(v) - \mathcal{F}(u_0)}{h} \\ &= -\mathcal{F}(u_0) + \mathcal{F}(v). \end{aligned}$$

Combining the two we obtain that $0 \leq -\mathcal{F}(u_0) + \mathcal{F}(v)$ and hence $u_0$ is a minimizer. □

With this we have a complete overview of the Euler-Lagrange equation and its weak form. We now focus on the weak Euler-Lagrange equation of the Dirichlet functional to obtain $C^\infty$ regularity of the minimizer.

## 4.2 Regularity of Harmonic Functions

In this section we establish certain properties of harmonic functions and prove in an elementary way that all harmonic and even weakly harmonic functions are infinitely differentiable. This implies in particular that the minimizer of the Dirichlet functional is smooth.

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u \in C^2(\Omega)$. The function $u$ is called harmonic if $\Delta u := \sum_{i=1}^n D_{x_i x_i} u = 0$. As seen from Example 4.3 any minimizer of the Dirichlet integral which is twice continuously differentiable is harmonic. We now prove an important property of harmonic functions, which will later allow us to prove smoothness of weakly harmonic functions. We recall the notation of an average integral

$$\fint_{B_r(x_0)} u(x)\,dx = \frac{1}{|B_r|}\int_{B_r(x_0)} u(x)\,dx$$

and

$$\fint_{\partial B_r(x_0)} u(x)\,dS(x) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x_0)} u(x)\,dS(x)$$

with $|B_r|$ the volume of the ball with radius $r$ and $|\partial B_r|$ the hyper area of the sphere with radius $r$.

**Lemma 4.6** (Mean value property, [2, §2.2 Theorem 2])**.** *Let $u \in C^2(\Omega)$ be harmonic. Then for all $x_0 \in \Omega$ the functions*

$$p_{x_0}(r) = \fint_{\partial B_r(x_0)} u(x)\,dS(x) \quad \textit{and} \quad q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\,dx \tag{4.6}$$

*are constant and equal for $0 < r < d(x_0, \partial\Omega)$.*

*Proof.* We use the substitution $x = x_0 + ry$ to obtain

$$p_{x_0}(r) = \fint_{\partial B_1(0)} u(x_0 + ry)\,dS(y),$$

where the Jacobian compensated for the change of area. As $u$ is $C^2$ on the entire ball we can differentiate under the integral sign to obtain

$$p'_{x_0}(r) = \fint_{\partial B_1(0)} \nabla u(x_0 + ry)\cdot y\,dS(y).$$

Because $y$ is an unit normal vector field of the sphere we can use the Gauss-Green theorem (Theorem 2.1) to obtain

$$p'_{x_0}(r) = \frac{1}{n}\fint_{B_1(0)} \Delta u(x_0 + ry)\,dy = 0,$$

which proves that $p_{x_0}$ is constant. For $q_{x_0}$ we use polar coordinates to get

$$\begin{aligned} q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\,dx &= \frac{1}{|B_r|}\int_0^r \int_{\partial B_s(x_0)} u(x)\,dS(x)\,ds \\ &= \frac{p_{x_0}(r)}{|B_r|}\int_0^r |\partial B_s|\,ds = p_{x_0}(r), \end{aligned}$$

which yields the result. $\square$

Since $u$ is $C^2$ we have in particular that $u(x_0) = \lim_{r\to 0} p_{x_0}(r) = \lim_{r\to 0} q_{x_0}(r)$ which yields as $p_{x_0}$ and $q_{x_0}$ are constant that

$$u(x_0) = \fint_{\partial B_r(x_0)} u(x)\, dS(x) = \fint_{B_r(x_0)} u(x)\, dx.$$

More generally, any $u \in L^1_{loc}(\Omega)$ for which the map

$$q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\, dx$$

is constant, satisfies $u(x) = q_x(r)$ almost everywhere by Lebesgue's differentiation theorem (Theorem 2.24). Thus we can change $u$ on a set of measure zero to find

$$u(x_0) = \fint_{B_r(x_0)} u(x)\, dx \tag{4.7}$$

for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$. We show that this implies that $u$ is continuous and that

$$p_{x_0}(r) = \fint_{\partial B_r(x_0)} u(x)\, dS(x)$$

is also constant for $0 < r < d(x_0, \partial\Omega)$. Note that continuity allows to integrate $u$ over $\partial B_r$.

**Lemma 4.7.** *Let $u \in L^1_{loc}(\Omega)$. If $q_{x_0}(r)$ is constant for all $0 < r < d(x_0, \partial\Omega)$ then there is a continuous representative of $u$ which satisfies $u(x_0) = q_{x_0}(r) = p_{x_0}(r)$ for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$.*

*Proof.* We can take the representative of $u$ such that (4.7) holds. We show that $u$ is continuous. Take a sequence $\{x_i\}$ in $\Omega$ such that $x_i \to x$ for some $x \in \Omega$. We have that $u \cdot \chi_{B_r(x_i)}$ converges to $u \cdot \chi_{B_r(x)}$ pointwise with $\chi_A$ denoting the characteristic function of the set $A$. We find by Lebesgue's dominated convergence theorem (Theorem 2.12) for small enough $r$ that

$$\lim_{i\to\infty} u(x_i) = \lim_{i\to\infty} \fint_{B_r(x_i)} u(y)\, dy = \fint_{B_r(x)} u(y)\, dy = u(x),$$

which proves that $u$ is continuous. Now it is easy to see that

$$p_{x_0}(r) = \fint_{B_1(0)} u(x_0 + ry)\, dS(y)$$

is continuous in $r$. Utilizing polar coordinates and the fundamental theorem of calculus we find

$$\begin{aligned}0 = \frac{d}{dr}q_{x_0}(r) &= \frac{d}{dr}\left(\frac{1}{|B_r|}\int_0^r |\partial B_s| p_{x_0}(s)\,ds\right)\\ &= \frac{-n}{r|B_r|}\int_0^r |\partial B_s| p_{x_0}(s)\,ds + \frac{|\partial B_r|}{|B_r|}p_{x_0}(r)\\ &= \frac{n}{r}(p_{x_0}(r) - q_{x_0}(r)).\end{aligned}$$

This shows that $p_{x_0}(r) = q_{x_0}(r)$ which proves the lemma. $\square$

The above lemma implies that any $u \in L^1_{loc}(\Omega)$, for which $q_{x_0}$ is constant, satisfies the mean value property. This mean value property is very powerful, as we can prove that it actually implies infinite differentiability. This result is surprisingly simple to prove.

**Proposition 4.8** ([2, §2.2 Theorem 6])**.** *Let $u \in L^1_{loc}(\Omega)$ satisfy the mean value property, then we have $u \in C^\infty(\Omega)$.*

*Proof.* By Lemma 4.7 we can take the continuous representative of $u$ such that

$$u(x_0) = ⨍_{\partial B_r(x_0)} u(x)\,dS(x) = ⨍_{B_r(x_0)} u(x)\,dx$$

for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$. Let $x_0 \in \Omega$ and take $0 < r < \frac{1}{2}d(x_0, \partial\Omega)$ then we show that $u \in C^\infty(B_r(x_0))$. We take the standard mollifier $\delta_\epsilon$ (Definition 2.22) and by radial symmetry we have a function $\tilde{\delta}_\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\tilde{\delta}_\epsilon(|x|) = \delta_\epsilon(x)$. Set $u_\epsilon = \delta_\epsilon * u$ which is well-defined on $B_r(x_0)$ for $\epsilon < r$ and satisfies $u_\epsilon \in C^\infty(B_r(x_0))$. We verify that $u = u_\epsilon$ on $B_r(x_0)$ thereby proving the result. Let $y \in B_r(x_0)$ then we have

$$\begin{aligned}u_\epsilon(y) &= \int_{B_\epsilon(y)} \delta_\epsilon(y - x)u(x)\,dx\\ &= \int_{B_\epsilon(y)} \tilde{\delta}_\epsilon\left(|y - x|\right)u(x)\,dx\\ &= \int_0^\epsilon \int_{\partial B_s(y)} \tilde{\delta}_\epsilon(s)u(z)\,dS(z)\,ds\\ &= u(y)\int_0^\epsilon |\partial B_s|\tilde{\delta}_\epsilon(s)\,ds = u(y).\end{aligned}$$

This last part follows as

$$\int_0^\epsilon |\partial B_s|\tilde{\delta}_\epsilon(s)\,ds = \int_{B_\epsilon(0)} \delta_\epsilon(x)\,dx = 1.$$

$\square$

With this lemma in place we are ready to prove regularity of weakly harmonic functions. A function $u \in L^1_{loc}(\Omega)$ is weakly harmonic if

$$\int_\Omega u\Delta\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Clearly, any harmonic function is weakly harmonic by using integration by parts. Similarly if $u \in H^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega),$$

then we obtain by integration by parts that $u$ is in fact weakly harmonic. In particular, we can use the regularity of weakly harmonic functions for the minimizer of the Dirichlet functional. The following classical regularity result was proven by Weyl in 1940 in his paper [21]. He proved this by utilizing the fundamental solution of the Laplacian, while we are using a more modern approach through mollifiers.

**Theorem 4.9** (Weyl's Lemma, [12, Lemma 1.19])**.** *Let $u \in L^1_{loc}(\Omega)$ be a weakly harmonic function. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

*Proof.* The idea of the proof is to show that $u$ satisfies the mean value property and then appeal to Proposition 4.8. Take $x_0 \in \Omega$ and some $r > 0$ with $0 < r < \frac{1}{2}d(x_0, \partial\Omega)$ implying that $u \in L^1(B_{2r}(x_0))$. Let $\delta_\epsilon$ be the standard mollifier and define $u_\epsilon = \delta_\epsilon * u$ which is well-defined on $B_r(x_0)$ for $\epsilon < r$ and lies in $C^\infty(B_r(x_0))$. We now wish to show that $u_\epsilon$ is harmonic. To this aim let $\varphi \in C_c^\infty(B_r(x_0))$ then we find that

$$\begin{aligned}
\int_{B_r(x_0)} u_\epsilon(x)\Delta\varphi(x)\, dx &= \int_{B_r(x_0)} \left( \int_{B_\epsilon(x)} \delta_\epsilon(x-y)u(y)\, dy \right) \Delta\varphi(x)\, dx \\
&= \int_{B_{r+\epsilon}(x_0)} \left( \int_{B_\epsilon(y)} \delta_\epsilon(y-x)\Delta\varphi(x)\, dx \right) u(y)\, dy \\
&= \int_{B_{r+\epsilon}(x_0)} (\delta_\epsilon * \Delta\varphi)(y)u(y)\, dy.
\end{aligned}$$

The second line follows by Fubini. Indeed, instead of integrating for every $x \in B_r(x_0)$ the variable $y$ over $B_\epsilon(x)$ we integrate for every $y \in B_{r+\epsilon}(x_0)$ the $x$-variable over $B_\epsilon(y) \cap B_r(x_0)$. As $\Delta\varphi$ has support inside $B_r(x_0)$ we can omit the intersection with $B_r(x_0)$. By properties of the convolution we have $\delta_\epsilon * \Delta\varphi = \Delta(\delta_\epsilon * \varphi)$ and thus we obtain

$$\begin{aligned}
\int_{B_r(x_0)} u_\epsilon(x)\Delta\varphi(x)\, dx &= \int_{B_{r+\epsilon}(x_0)} \Delta(\delta_\epsilon * \varphi)(y)u(y)\, dy \\
&= \int_\Omega u(y)\Delta(\delta_\epsilon * \varphi)(y)\, dy = 0,
\end{aligned}$$

as $\Delta(\delta_\epsilon * \varphi) \in C_c^\infty(B_{r+\epsilon}(x_0))$. We conclude that

$$\int_{B_r(x_0)} u_\epsilon(x)\Delta\varphi(x)\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B_r(x_0))$$

and as $u_\epsilon$ is smooth we obtain by integration by parts twice that

$$\int_{B_r(x_0)} \Delta u_\epsilon(x)\varphi(x)\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B_r(x_0)).$$

By the fundamental lemma of the calculus of variations we find that $u_\epsilon$ is harmonic. This implies by Lemma 4.6 that $u_\epsilon$ satisfies the mean value property and therefore the function

$$q_{x_0}^\epsilon(r) = \fint_{B_r(x_0)} u_\epsilon(x)\,dx$$

is constant. Since $u_\epsilon$ converges to $u$ in $L^1(B_r(x_0))$ as $\epsilon$ goes to zero (Theorem 2.23) we find that

$$q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\,dx = \lim_{\epsilon\to 0} \fint_{B_r(x_0)} u_\epsilon(x)\,dx = \lim_{\epsilon\to 0} q_{x_0}^\epsilon(r).$$

This implies that $q_{x_0}$ is constant as well and by Lemma 4.7 we obtain that $u$ satisfies the mean value property. By application of Proposition 4.8 we can now take a representative such that $u \in C^\infty(B_r(x_0))$ and as this holds for any $x_0 \in \Omega$ we have $u \in C^\infty(\Omega)$. Harmonicity of $u$ now follows from

$$\int_{B_r(x_0)} u(x)\Delta\varphi(x)\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B_r(x_0))$$

by using integration by parts and the Fundamental lemma of the calculus of variations. $\square$

This important result lies at the foundation of our regularity theory for the obstacle problem. Actually, we can even prove that harmonic functions are analytic but we do not need this. However, a bound on the partial derivatives of harmonic functions turns out to be very convenient.

**Lemma 4.10** ([2, §2.2 Theorem 7])**.** *Let $u$ be harmonic and bounded inside $B_{r_0}(x)$. Then we have a constant $K > 0$ such that*

$$r_0|\nabla u(x)| + r_0^2|D^2u(x)| \leq K\|u\|_{L^\infty(B_{r_0}(x))}.$$

*Proof.* We prove the bound for any $r < r_0$ from which the result follows by taking the limit. Since the Laplace operator commutes with partial derivatives for smooth functions we find that any partial derivative of $u$ is again harmonic. Thus we obtain by the mean value property and the Gauss-Green theorem that

$$\begin{aligned}|D_{x_i}u(x)| &= |\fint_{B_r(x)} D_{x_i}u(y)\,dy| \\ &= |\frac{1}{|B_r|}\int_{\partial B_r(x)} u(y)\nu_i(y)\,dS(y)| \\ &\leq \frac{|\partial B_r|}{|B_r|}\|u\|_{L^\infty(B_r(x))} = \frac{K_1}{r}\|u\|_{L^\infty(B_r(x))},\end{aligned}$$

where $\nu_i$ is the $i$th component of a unit normal vector field on the sphere. Next, for the second partial derivatives we find similarly that

$$|D_{x_ix_j}u(x)| \leq \frac{2K_1}{r}\|D_{x_i}u\|_{L^\infty(B_{r/2}(x))}.$$

Since for any $y \in B_{r/2}(x)$ we have that $D_{x_i}u$ is harmonic on $B_{r/2}(y)$ we can use the bound on the first partial derivatives to get

$$|D_{x_ix_j}u(x)| \leq \frac{4K_1^2}{r^2}\|u\|_{L^\infty(B_r(x))}.$$

This proves the lemma for an appropriately chosen $K$. □

For the dam problem in Section 6 we need the min-max principle for harmonic functions. This is a profound result about harmonic functions which is interesting on itself.

**Proposition 4.11** ([2, §2.2 Theorem 4])**.** *Let $u \in C(\overline{\Omega})$ be a non-constant and harmonic function on $\Omega$. If $\Omega$ is connected and bounded then for all $x_0 \in \Omega$ we have*

$$\min_{\overline{\Omega}} u < u(x_0) < \max_{\overline{\Omega}}.$$

*Proof.* We concern ourselves with the minimum case which is analogous to the maximum case. Suppose there exists an $x_0 \in \Omega$ such that $u(x_0) = m := \min_{\overline{\Omega}} u$, which is well-defined by compactness of $\overline{\Omega}$. Then the set $U = \{x \in \Omega \mid u(x) = m\}$ is non-empty and it is closed in $\Omega$ by continuity of $u$. Furthermore, for any $x \in U$ we have by the mean value property

$$m = u(x) = \fint_{B_r(x)} u(y)\,dy$$

for $0 < r < d(x, \partial\Omega)$. This can only hold if $u \equiv m$ on $B_r(x)$ which means that $B_r(x) \subset U$. We conclude that $U$ is a non-empty open and closed subset of $\Omega$ and hence equal to $\Omega$ by connectedness. This contradicts the fact that $u$ is not constant on $\Omega$ and thereby proves the result. □

## 4.3 Sub- and Superharmonic Functions

In this section we discuss certain properties of sub- and superharmonic functions which are very similar to those of harmonic functions. The reason we treat this is because the Euler-Lagrange equation becomes an inequality when we introduce an obstacle. Thus we will naturally be dealing with (weakly) sub- and superharmonic functions. The results proven in this section are mostly trivial adaptations of those concerning harmonic functions.

**Definition 4.12.** A function $u \in C^2(\Omega)$ is called subharmonic if $\Delta u \geq 0$ and superharmonic if $\Delta u \leq 0$.

In one dimension subharmonic functions correspond to convex functions and superharmonic correspond to concave functions. Additionally, harmonic functions correspond to functions of the form $u(x) = ax + b$ for $a, b \in \mathbb{R}$. Since the graph of any convex function between two points always lies below the straight line connecting these points we find that subharmonic functions lie below harmonic functions in some sense. This is also where the name subharmonic comes from and could be used as an alternate definition. Regardless, sub- and superharmonic functions satisfy a similar mean value property to harmonic functions which we now prove.

**Lemma 4.13** (Mean value property)**.** *Let $u \in C^2(\Omega)$ be subharmonic then for all $x_0 \in \Omega$ the functions*

$$p_{x_0}(r) = \fint_{\partial B_r(x_0)} u(x)\, dS(x) \quad \text{and} \quad q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\, dx$$

*are increasing for $0 < r < d(x_0, \partial\Omega)$. If $u$ is superharmonic then they are decreasing.*

*Proof.* We only treat the case of subharmonic functions since they are analogous. Exactly as in Lemma 4.6 we find that

$$p'_{x_0}(r) = \frac{1}{n} \fint_{B_1(0)} \Delta u(x_0 + ry)\, dy \geq 0.$$

This shows that $p_{x_0}$ is increasing. Now employing polar coordinates we find that

$$q_{x_0}(r) = \frac{|\partial B_r|}{|B_r|} \int_0^r \fint_{\partial B_s(x_0)} u(x)\, dS(x)\, ds = \frac{n}{r} \int_0^r p_{x_0}(s)\, ds.$$

Differentiating yields

$$q'_{x_0}(r) = \frac{n}{r} \left( p_{x_0}(r) - \frac{1}{r} \int_0^r p_{x_0}(s)\, ds \right) \geq 0$$

as $r p_{x_0}(r) \geq \int_0^r p_{x_0}(s)\, ds$. This proves the result. □

In particular the above lemma implies that for subharmonic functions

$$u(x_0) = \lim_{s\to 0} p_{x_0}(s) \le p_{x_0}(r) \quad \text{and} \quad u(x_0) = \lim_{s\to 0} q_{x_0}(s) \le q_{x_0}(r)$$

for $0 < r < d(x_0, \partial\Omega)$. For superharmonic functions we have

$$u(x_0) \ge p_{x_0}(r) \quad \text{and} \quad u(x_0) \ge q_{x_0}(r).$$

Contrary to harmonic functions, the mean value property for sub- and superharmonic functions does not imply smoothness. However, we can still show that weakly sub- and superharmonic functions satisfy the mean value property, which we will use later. A weakly subharmonic function is a function $u \in L^1_{loc}(\Omega)$ such that for all nonnegative $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u(x)\Delta\varphi(x)\, dx \ge 0.$$

Note that this implies in some weak sense that $\Delta u \ge 0$. The function is called weakly superharmonic if the above integral is always smaller or equal to zero. We have the following result showing that weakly sub- and superharmonic functions satisfy one part of the mean value property.

**Proposition 4.14.** *Suppose $u \in L^1_{loc}(\Omega)$ is weakly sub- or superharmonic. Then for all $x_0 \in \Omega$ we have that the function*

$$q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\, dx$$

*is increasing or decreasing respectively for $0 < r < d(x_0, \partial\Omega)$.*

*Proof.* This is completely analogous to the first part of Theorem 4.9. We mollify $u$ to get a smooth function $u_\epsilon$ which turns out to be sub- or superharmonic. Then using that $u_\epsilon$ satisfies the mean value property and taking the limit of $\epsilon$ to zero yields the mean value property for $u$. □

Lastly, we prove a modest regularity result for weakly sub- and superharmonic functions, which becomes essential when investigating the regularity for obstacle problems.

**Proposition 4.15** ([1, Corollary 3.2])**.** *Let $u \in L^1_{loc}(\Omega)$ be weakly subharmonic. Then $u$ has an upper semicontinuous representative which satisfies*

$$u(x_0) \le \fint_{B_r(x_0)} u(x)\, dx$$

*for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$. If $u$ is weakly superharmonic then it has a lower semicontinuous representative which satisfies*

$$u(x_0) \ge \fint_{B_r(x_0)} u(x)\, dx$$

*for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$.*

*Proof.* We treat the case of $u$ being subharmonic. We have by Proposition 4.14 that for any $x_0 \in \Omega$ the function

$$q_{x_0}(r) = \fint_{B_r(x_0)} u(x)\, dx$$

is increasing on $0 < r < d(x_0, \partial\Omega)$. Therefore we find that $\lim_{r\to 0} q_{x_0}(r)$ exists, albeit possibly equal to $-\infty$. By Lebesgue's differentiation theorem (Theorem 2.24) we can now choose a representative such that

$$u(x_0) = \lim_{s\to 0} \fint_{B_s(x_0)} u(x)\, dx \le \fint_{B_r(x_0)} u(x)\, dx$$

for all $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$. Regarding the upper semicontinuity, take a sequence $\{x_i\}$ converging to $x$ in $\Omega$. Since for small enough $r$ we have that $u\chi_{B_r(x_i)}$ converges pointwise to $u\chi_{B_r(x)}$ we find by Lebesgue's dominated convergence theorem that

$$\fint_{B_r(x)} u(y)\, dy = \lim_{i\to\infty} \fint_{B_r(x_i)} u(y)\, dy.$$

Since each integral on the right hand side is greater than or equal to $u(x_i)$ we find that

$$\fint_{B_r(x)} u(y)\, dy = \lim_{i\to\infty} \fint_{B_r(x_i)} u(y)\, dy \ge \limsup_{i\to\infty} u(x_i).$$

Letting $r$ go to zero now yields $u(x) \ge \limsup_{i\to\infty} u(x_i)$, which proves the upper semicontinuity. $\square$

# 5 Obstacle Problems

We are finally ready to formally define obstacle problems and to discuss the regularity of solutions for the classical obstacle problem. We first look at the uniqueness and existence of solutions, which follows almost immediatly from the results in Section 3. Then we derive a so called variational inequality, which is a replacement of the weak Euler-Lagrange equation for obstacle problems. Afterwards we dive in depth into the classical obstacle problem and prove the optimal $C^{1,1}$-regularity of solutions. Lastly, we touch upon some generalizations of the optimal regularity and results about the free boundary. For a good introduction on obstacle problems in one dimension see [8, Chapter 5.4], while we will be discussing the general case mostly following [1].

Obstacle problems are similar to the minimization problems that we have seen in Section 3 with the added constraint that we are minimizing over functions that lie above a given obstacle function. Therefore we define the set of admissible functions as

$$K_\psi = \{u \in W^{1,p}_g(\Omega) \mid u \geq \psi \text{ a.e. in } \Omega\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^1$-domain, $g \in C^1(\partial\Omega)$ and $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is the obstacle function. We also impose that $\psi|_{\partial\Omega} \leq g$ for $K_\psi$ to be non-empty. The obstacle problem now reads:

$$\text{Minimize } \mathcal{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx \text{ over all } u \in K_\psi, \tag{5.1}$$

where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Our first goal is to prove uniqueness and existence of the minimizer by applying Theorem 3.5.

**Theorem 5.1.** *Suppose $f$ satisfies the $p$-coercivity bound (3.2) and is strictly convex in its second argument. Then there exists a unique minimizer $u_0 \in K_\psi$ of $\mathcal{F}$.*

*Proof.* This follows immediately from Theorem 3.5 if we can prove that $K_\psi$ is a non-empty closed and convex subset of $W^{1,p}_g(\Omega)$. Let $u, v \in K_\psi$ then we have for $t \in [0, 1]$ that

$$tu + (1-t)v \geq \min\{u, v\} \geq \psi \quad \text{(a.e)}.$$

Thus $tu+(1-t)v$ lies in $K_\psi$, which proves that $K_\psi$ is convex. For closedness, let $\{u_i\}$ be a sequence in $K_\psi$ converging to $u \in W^{1,p}_g(\Omega)$. In particular this means that $u_i \to u$ in $L^p(\Omega)$ and thus a subsequence converges pointwise almost everywhere. We conclude that $u \geq \psi$ almost everywhere and hence

$u$ lies in $K_\psi$. This proves that $K_\psi$ is closed in $W_g^{1,p}(\Omega)$. Lastly, we need to show that $K_\psi$ is non-empty. To this aim let $G : \overline{\Omega} \to \mathbb{R}$ be a $C^1$-extension of $g$. Then we have that both $G$ and $\psi$ lie in $W^{1,p}(\Omega)$ and thus $w := \max\{G, \psi\}$ lies in $W^{1,p}(\Omega)$ ([5, Theorem 4.4 (iii)]). Moreover, $w$ also lies in $C(\overline{\Omega})$ so we find that the trace of $w$ is just its restriction to the boundary. This is equal to $g$ as we have $\psi|_{\partial\Omega} \leq g$. It is also clear that $w \geq \psi$ so we conclude that $w \in K_\psi$. This proves that $K_\psi$ is non-empty and thereby the theorem. □

From now on we denote the unique solution of (5.1) by $u_0$. Having this solution we can now try to determine what kind of properties it satisfies. Because of the obstacle one cannot hope that the weak Euler-Lagrange equation applies to the minimizer. This is because $u_0 + h\varphi$ with $\varphi \in W_0^{1,p}(\Omega)$ does not necessarily lie in $K_\psi$ even for small $h$. Indeed, if $\varphi$ takes negative values then $u_0 + h\varphi$ could lie below the obstacle. Therefore we cannot conclude that $\mathcal{F}(u_0) \leq \mathcal{F}(u_0 + h\varphi)$, which was necessary to show that $\delta\mathcal{F}(u)(\varphi) = 0$. However, by convexity of $K_\psi$ we have for any $v \in K_\psi$ that $u_0 + h(v - u_0) \in K_\psi$ for $h \in [0, 1]$. This gives us a type of Euler-Lagrange inequality. We assume that $f$ satisfies the hypotheses of Corollary 4.4 then we have the following result.

**Proposition 5.2.** *Let $u_0$ be the minimizer of (5.1). Then for all $v \in K_\psi$ we have $\delta\mathcal{F}(u_0)(v - u_0) \geq 0$. Explicitly, this means that*

$$\int_\Omega D_\xi f(x, \nabla u_0) \cdot \nabla(v - u_0)\, dx \geq 0. \tag{5.2}$$

*Proof.* We know that $\delta\mathcal{F}(u_0)(v - u_0)$ exists by Corollary 4.4. Set $\varphi = v - u_0$ then we find

$$\delta\mathcal{F}(u_0)(\varphi) = \lim_{h \to 0} \frac{\mathcal{F}(u_0 + h\varphi) - \mathcal{F}(u_0)}{h} = \lim_{h \downarrow 0} \frac{\mathcal{F}(u_0 + h\varphi) - \mathcal{F}(u_0)}{h}.$$

Since for $h \in [0, 1]$ we have $u_0 + h\varphi \in K_\psi$ and thus $\mathcal{F}(u_0) \leq \mathcal{F}(u_0 + h\varphi)$ we conclude that the last limit is nonnegative as desired. If we recall the explicit formula for $\delta\mathcal{F}(u_0)$ as in Corollary 4.4 then we obtain the result. □

The inequality in (5.2) is called a variational inequality. This inequality encodes the entire obstacle problem in the case when $\mathcal{F}$ is convex (which is guaranteed when $\xi \mapsto f(x, \xi)$ is convex). Namely, any $u_0 \in K_\psi$ which satisfies the variational inequality is then actually a minimizer.

**Proposition 5.3.** *Suppose $\mathcal{F}$ is convex and there is some $u_0 \in K_\psi$ for which $\delta\mathcal{F}(u_0)(v - u_0) \geq 0$ for all $v \in K_\psi$. Then $u_0$ is a minimizer of $\mathcal{F}$ in $K_\psi$.*

*Proof.* For any $v \in K_\psi$ we have $\delta\mathcal{F}(u_0)(v - u_0) \geq 0$. Now we can reason analogously to Proposition 4.5 with convexity of $\mathcal{F}$ to find $0 \leq -\mathcal{F}(u_0) + \mathcal{F}(v)$ which yields the result. □

We conclude that solving the variational inequality is equivalent with solving the obstacle problem when $\mathcal{F}$ is convex. Although we have shown the existence of minimizers through the direct method, in certain cases establishing solutions of the variational inequality is also a possibility. See for example [4, Chapter 1], where existence for obstacle problems is shown through the variational inequality.

Our next step is to show regularity of the solution $u_0$ by using the variational inequality. This delicate issue cannot be carried out for all general integrands $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ at once. Hence we restrict ourselves to the classical obstacle problem that is related to harmonic functions whose regularity we have studied.

## 5.1 Classical Obstacle Problem

The classical obstacle problem is defined as

$$\text{Minimize } \mathcal{D}(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx \text{ over all } u \in K_\psi,$$

where

$$K_\psi = \{u \in H^1_g(\Omega) \mid u \geq \psi \text{ a.e. in } \Omega\}.$$

Again we assume $\Omega$ is a $C^1$-domain, $g \in C^1(\partial\Omega)$ and $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\psi|_{\partial\Omega} \leq g$. This problem can be seen as a general obstacle problem with the choice $f(x, \xi) = \frac{|\xi|^2}{2}$. In one dimension the problem models an elastic string which is fixed at the boundary and suspended by the obstacle $\psi$, see Figure 1. For more examples and applications see Section 6. As we have seen in Example 3.6 the Dirichlet functional is coercive and weakly lower semicontinuous on $H^1_g(\Omega)$. Furthermore, it has a strictly convex integrand and thus Theorem 5.1 shows that we have a unique solution $u_0 \in K_\psi$ of the classical obstacle problem.

In Example 4.3 we have shown that the first variation of the Dirichlet functional is given by

$$\delta\mathcal{F}(u)(v) = \int_\Omega \nabla u \cdot \nabla v\, dx.$$

Hence the variational inequality for the Dirichlet functional becomes

$$\int_\Omega \nabla u_0 \cdot \nabla(v - u_0)\, dx \geq 0$$

for all $v \in K_\psi$. We may take $v = u_0 + \varphi$ for any nonnegative $\varphi \in C_c^\infty(\Omega)$ from which we see that $u_0$ is weakly superharmonic. By applying Proposition 4.15 we have the following.

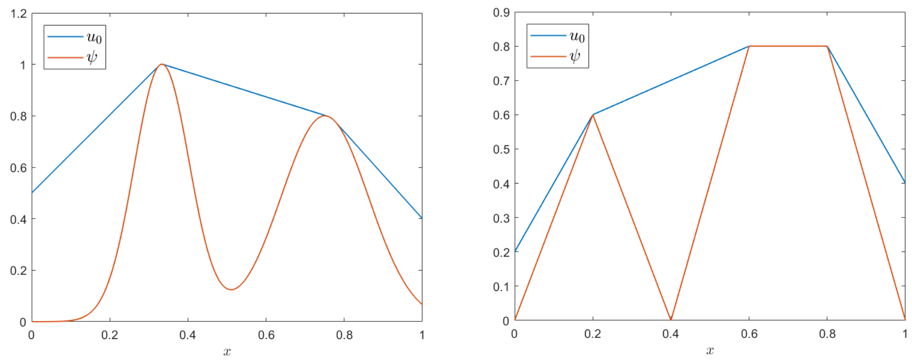


Figure 1: Two examples of the solution of the classical obstacle problem in one dimension.

**Corollary 5.4.** *The solution $u_0$ has a pointwise defined lower semicontinuous representative which satisfies*

$$u_0(x) = \lim_{s\to 0} \fint_{B_s(x)} u_0(y)\, dy \geq \fint_{B_r(x)} u_0(y)\, dy$$

*for all $0 < r < d(x, \partial\Omega)$.*

Note that for now $u_0 = \infty$ can occur on a set of measure zero, but this turns out not to be the case. From now on we identify $u_0$ with this lower semicontinuous representative. For this representative we have for all $x \in \Omega$ that

$$\begin{aligned} u_0(x) &= \lim_{r\to 0} \fint_{B_r(x)} u_0(y)\, dy \\ &\geq \lim_{r\to 0} \fint_{B_r(x)} \psi(y)\, dy = \psi(x), \end{aligned}$$

where the last part follows from continuity of $\psi$. We find that $u_0 \geq \psi$ holds everywhere in $\Omega$. Now we can define the following sets

$$\Lambda = \{x \in \Omega \mid u_0(x) = \psi(x)\} \quad \text{and} \quad N = \{x \in \Omega \mid u_0(x) > \psi(x)\}.$$

$\Lambda$ is called the contact or coincidence set and $N$ is the noncoincidence set. Furthermore, the set $\Gamma := \partial\Lambda \cap \Omega$ is called the free boundary. We have the following result.

**Proposition 5.5** ([1, Corollary 3.3 and 3.4])**.** *The set $\Lambda$ is closed in $\Omega$ and $N$ is open. In addition, $u_0 \in C^\infty(N)$ and $\Delta u_0 = 0$ on $N$.*

*Proof.* The function $u_0 - \psi$ is also lower semicontinuous and this implies that $\Lambda$ is closed in $\Omega$. Indeed, let $\{x_i\}$ be a sequence in $\Lambda$ converging to $x \in \Omega$ then we have

$$(u_0 - \psi)(x) \leq \liminf_{i \to \infty} (u_0 - \psi)(x_i) \leq 0,$$

which shows that $x \in \Lambda$. As $N$ is the complement of $\Lambda$ we find that $N$ is open. To prove the last claim let $\varphi \in C_c^\infty(N)$ be any function and let $M \subset N$ be its compact support. Since $u_0 - \psi$ is lower semicontinuous and positive on $M$ we find that there is a $x_0 \in M$ such that

$$\min_{x \in M} (u_0 - \psi)(x) = (u_0 - \psi)(x_0) > 0.$$

We have used the fact that a lower semicontinuous function attains its minimum on a compact set, which follows from the (finite-dimensional) direct method. We conclude that if we choose $\epsilon > 0$ small enough that $v = u_0 + \epsilon\varphi \geq \psi$ and hence $v \in K_\psi$. Now appealing to the variational inequality with $v = u_0 + \epsilon\varphi$ and dividing by $\epsilon$ yields

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx \geq 0.$$

Since this holds for any test function (not necessarily nonnegative) it must also hold for $-\varphi$. From this we conclude that

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(N),$$

which shows that $u_0$ is weakly harmonic on $N$. Using Weyl's lemma (Theorem 4.9) we conclude that $u_0$ is almost everywhere equal to a smooth harmonic function on $N$. However, since that representative is continuous it is exactly the representative $u_0$ that we have chosen. $\square$

With this proposition we already know a lot about $u_0$. Indeed, it is harmonic away from the contact set and equal to $\psi$ on the contact set. In particular, $u_0 < \infty$ holds everywhere in $\Omega$. Furthermore, $u_0$ will be as regular as the obstacle $\psi$ as long as there are no problems across the free boundary $\Gamma$. However, such problems can occur. If we look at the solution $u_0$ of the one-dimensional obstacle problem on the left side of Figure 1, then by the previous proposition we have that $u_0'' = 0$ when $u_0$ does not touch the obstacle. At the same time we can see that at all places where $u_0$ does touch the obstacle we have $u_0'' = \psi'' < 0$. This implies that $u_0''$ cannot be continuous even though $\psi$ is smooth. In general dimensions, it holds that $\Delta u_0$ can vary discontinuously across the free boundary as it is zero on $N$ and equal to $\Delta\psi$ on $\Lambda$ (if it exists). Therefore, the optimal regularity which we can hope to prove is that $u_0$ has bounded second derivative in the interior of $\Omega$, explicitly $u_0 \in W^{2,\infty}_{loc}(\Omega)$. It turns out that we can indeed prove this optimal regularity by assuming that $\psi \in W^{2,\infty}(\Omega)$. We note by Theorem 2.39 that this is the same as stating that $u_0 \in C^{1,1}_{loc}(\Omega)$ if $\psi \in C^{1,1}(\Omega)$.

We now turn to the proof of this optimal regularity result. Therefore we assume that $\psi \in C^{1,1}(\Omega)$ and for simplicity of notation we introduce $v_0 = u_0 - \psi$, which is zero on $\Lambda$ and positive on $N$. We prove the $C^{1,1}_{loc}$ regularity for the function $v_0$, which then automatically also holds for $u_0$. Since the regularity is concerned with what happens at the free boundary we first prove the following result which shows that the rate at which $v_0$ grows away from the free boundary is at most quadratic.

**Lemma 5.6** ([1, Lemma 4.2])**.** *Assume $\psi \in C^{1,1}(\Omega) = W^{2,\infty}(\Omega)$. There is a constant $C > 0$ which satisfies*

$$0 \leq v_0(x) \leq Cd(x,\Gamma)^2$$

*for $x \in N$ such that $d(x,\Gamma) \leq \frac{1}{3}d(x,\partial\Omega)$.*

*Proof.* The idea of the proof is to use the fact that $u_0$ is harmonic on $N$ and weakly superharmonic on $\Omega$. Then using a smart comparison we obtain the result. First we define for any $x \in \Omega$ the function

$$f_x : \Omega \to \mathbb{R}, \quad f_x(y) = \|\Delta\psi\|_\infty \frac{|x-y|^2}{2n}.$$

The crucial properties of this function are that $\Delta f_x \equiv \|\Delta\psi\|_\infty$ and $f_x(x) = 0$. Next, take $x_0 \in N$ such that $d(x_0,\Gamma) \leq \frac{1}{3}d(x_0,\partial\Omega)$ holds. We set $r_0 := d(x_0,\Gamma)$ from which we find that $u_0$ is harmonic on $B_{r_0}(x_0)$. Therefore the map $h_1(z) = v_0(z) + f_{x_0}(z)$ satisfies

$$\Delta h_1 = -\Delta\psi + \|\Delta\psi\|_\infty \geq 0 \qquad \text{a.e. in } B_{r_0}(x_0),$$

which shows that it is weakly subharmonic. By the mean value property (Proposition 4.14 with $r \to r_0$) we find

$$\begin{aligned} v_0(x_0) = h_1(x_0) &\leq \fint_{B_{r_0}(x_0)} v_0(z) + f_{x_0}(z)\,dz \\ &\leq \fint_{B_{r_0}(x_0)} v_0(z)\,dz + \|\Delta\psi\|_\infty \frac{r_0^2}{2n}. \end{aligned}$$

On the other hand we can take $y_0 \in \overline{B_{r_0}(x_0)} \cap \Lambda$, see Figure 2, such that $B_{2r_0}(y_0) \subset \Omega$ by choice of $x_0$. Now we define the function

$$h_2(z) := v_0(z) - f_{y_0}(z) = u_0(z) + (-\psi(z) - f_{y_0}(z)),$$

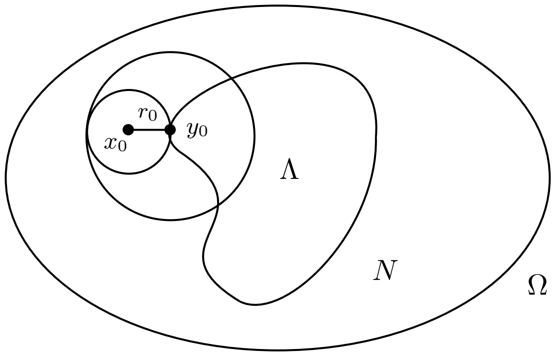


Figure 2: Illustration for Lemma 5.6.

which is the sum of two weakly superharmonic functions and thereby weakly superharmonic. In a similar way we obtain by the mean value property $(r \to 2r_0)$

$$0 = v_0(y_0) = h_2(y_0) \geq \fint_{B_{2r_0}(y_0)} v_0(z)\,dz - \|\Delta\psi\|_\infty \frac{2r_0^2}{n}.$$

Combining the two inequalities and using that $v_0 \geq 0$ and $B_{r_0}(x_0) \subset B_{2r_0}(y_0)$ we have

$$\begin{aligned} v_0(x_0) - \|\Delta\psi\|_\infty \frac{r_0^2}{2n} &\leq \fint_{B_{r_0}(x_0)} v_0(z)\,dz \leq \frac{|B_{2r_0}|}{|B_{r_0}|} \fint_{B_{2r_0}(y_0)} v_0(z)\,dz \\ &= 2^n \fint_{B_{2r_0}(y_0)} v_0(z)\,dz \leq 2^n \|\Delta\psi\|_\infty \frac{2r_0^2}{n}, \end{aligned}$$

which proves the result for $C = \|\Delta\psi\|_\infty(\frac{2^{n+1}}{n} + \frac{1}{2n})$. □

Quadratic growth is what we expect from a function with bounded second derivatives. Similarly, we expect the derivative to have linear growth and naturally the second derivative to be bounded. This is what we prove in the following slightly technical lemma, which enables us to intuitively obtain the optimal regularity.

**Lemma 5.7** ([1, Theorem 4.1])**.** *Assume $\psi \in C^{1,1}(\Omega) = W^{2,\infty}(\Omega)$. Then there is a constant $C'$ which satisfies*

$$|\nabla v_0(x)| \leq C' d(x, \Gamma) \quad \textit{and} \quad |D^2 v_0(x)| \leq C' \tag{5.3}$$

*for all $x \in N$ such that $d(x, \Gamma) \leq \frac{1}{6} d(x, \partial\Omega)$.*

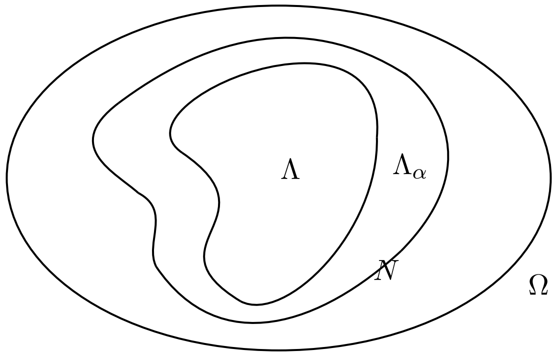


Figure 3: Schematic representation of the set $\Lambda_\alpha$.

*Proof.* For any $x_0 \in N$ and $r > 0$ such that $B_r(x_0) \subset N$ we can use a Taylor approximation of $\psi$ at $x_0$ to define the following approximation of $v_0$:

$$h(y) := u_0(y) - (\psi(x_0) + \nabla\psi(x_0) \cdot (y - x_0)).$$

Indeed, if we calculate $|h(y) - v_0(y)|$ then this is bounded by the error term of the Taylor approximation of the $C^{1,1}$ function $\psi$. Thus we have by Lemma 2.40 that

$$\begin{aligned} |h(y) - v_0(y)| &= |\psi(y) - \psi(x_0) - \nabla\psi(x_0) \cdot (y - x_0)| \\ &\leq Lr^2, \qquad \text{for } y \in B_r(x_0), \end{aligned}$$

with $L$ the Lipschitz constant of $\nabla\psi$. Next, define for $\alpha > 0$ the set

$$\Lambda_\alpha = \{x \in N \mid d(x, \Gamma) \leq \alpha \cdot d(x, \partial\Omega)\},$$

which is schematically represented in Figure 3. Our goal is to prove (5.3) for all $x \in \Lambda_{1/6}$. Take $x_0 \in \Lambda_{1/6}$ and set $r := d(x_0, \Gamma)/2$. It is an easy check that $B_r(x_0) \subset \Lambda_{1/3}$. In addition, for any $y \in B_r(x_0)$ we have $d(y, \Gamma) \leq r + d(x_0, \Gamma) = \frac{3}{2} d(x_0, \Gamma)$. Thus by using Lemma 5.6 we find that

$$\|v_0\|_{L^\infty(B_r(x_0))} \leq \frac{9}{4} C d(x_0, \Gamma)^2 = 9Cr^2.$$

Hence we obtain by this bound and the bound on the Taylor approximation that

$$\begin{aligned} \|h\|_{L^\infty(B_r(x_0))} &\leq \|v_0\|_{L^\infty(B_r(x_0))} + \|h - v_0\|_{L^\infty(B_r(x_0))} \\ &\leq (9C + L)r^2. \end{aligned}$$

Furthermore, as the Taylor polynomial is linear we find $\Delta h = \Delta u_0 = 0$ on $B_r(x_0) \subset N$. Therefore we can apply the bound for harmonic functions (Lemma 4.10) to $h$ on $B_r(x_0)$ to yield (divide by $r^2$)

$$\begin{aligned} \frac{|\nabla h(x_0)|}{r} + |D^2 h(x_0)| &\leq \frac{K}{r^2}\|h\|_{L^\infty(B_r(x_0))} \\ &\leq K(9C + L). \end{aligned}$$

Since $\nabla v_0(x_0) = \nabla h(x_0)$ and $D^2 v_0(x_0) = D^2 h(x_0) - D^2\psi(x_0)$ ($D^2 u_0(x_0)$ is well-defined since $u_0$ is smooth in $N$) we can rewrite the above to

$$\frac{|\nabla v_0(x_0)|}{r} + |D^2 v_0(x_0)| \leq K(9C + L) + \|D^2\psi\|_\infty =: C'.$$

We have chosen a representative of $D^2\psi$, which is a weak derivative, that satisfies $D^2\psi(x) \leq \|D^2\psi\|_\infty$ everywhere. Since $r \leq d(x_0, \Gamma)$ this proves the lemma. □

In some sense the previous two lemmas show that $v_0$ behaves like a $C^{1,1}$ function in $\Lambda_{1/6}$, which is exactly the crucial region. Therefore the optimal regularity result is now fairly simple to prove. This result was first proven by Jens Frehse in 1972, see [20].

**Theorem 5.8** (Optimal regularity, [1, Theorem 4.1])**.** *Let $\psi \in C^{1,1}(\Omega) = W^{2,\infty}(\Omega)$. Then the function $v_0$ lies in $C^{1,1}_{loc}(\Omega)$ and hence the solution of the classical obstacle problem $u_0$ lies in $C^{1,1}_{loc}(\Omega)$.*

*Proof.* We first prove that $v_0 \in C^1(\Omega)$. As $v_0 = 0$ on $\Lambda$ we have that $\nabla v_0 = 0$ a.e. on $\Lambda$ ([5, Theorem 4.4 (iv)]). Thus we take the representative of $\nabla v_0$ such that $\nabla v_0 = 0$ everywhere on $\Lambda$. We conclude that now $v_0$ and $\nabla v_0$ are continuous on $\Lambda$ and they are continuous on $N$ by harmonicity. In addition, Lemma 5.6 and 5.7 imply that for any sequence $\{x_i\}$ in $N$ with $d(x_i, \Gamma) \to 0$ we have $v_0(x_i) \to 0$ and $\nabla v_0(x_0) \to 0$. Therefore $v_0$ and $\nabla v_0$ are continuous on $\Omega$ (recall that discontinuities can only occur across the free boundary). This implies by Theorem 2.38 that $v_0 \in C^1(\Omega)$.

Now we prove that $\nabla v_0$ is locally Lipschitz. Since $u_0$ is smooth on $N$ we find at once that $\nabla v_0$ is locally Lipschitz on $N$. We prove the same on the set $\Lambda \cup \Lambda_{1/12}$, where $\Lambda_{1/12}$ is defined as in Lemma 5.7. For $x, y \in \Lambda_{1/12}$ we may assume that $x$ and $y$ do not lie both in $\Lambda$ otherwise $|\nabla u_0(x) - \nabla u_0(y)| = 0$. Assume without loss of generality that also $d(x, \Lambda) \geq d(y, \Lambda)$ which means that $x \in \Lambda_{1/12}$, while $y$ can lie in either $\Lambda$ or $\Lambda_{1/12}$. We consider two cases. The first case is that $|x - y| < d(x, \Gamma)/2 =: r$. An easy computation yields that $B_r(x) \subset \Lambda_{1/6}$ and therefore we have by Lemma 5.7

that $\|D^2 v_0\|_{L^\infty(B_r(x))} \leq C'$. Hence by Theorem 2.38 $\nabla v_0$ is Lipschitz on $B_r(x)$ (with constant $C'$) so

$$|\nabla v_0(x) - \nabla v_0(y)| \leq C'|x - y|.$$

The second case is when $|x - y| \geq d(x, \Gamma)/2$. Since $x, y \in \Lambda \cup \Lambda_{1/12}$ we have by Lemma 5.7 that

$$|\nabla v_0(x)| \leq C' d(x, \Gamma) \text{ and } |\nabla v_0(y)| \leq C' d(y, \Lambda),$$

since $\nabla v_0(y) = 0$ if $y \in \Lambda$. We now use a rough estimate and the assumption $d(x, \Gamma) \geq d(y, \Lambda)$ to get

$$|\nabla v_0(x) - \nabla v_0(y)| \leq |\nabla v_0(x)| + |\nabla v_0(y)| \leq 2C' d(x, \Gamma) \leq 4C'|x - y|.$$

This proves that $\nabla v_0$ is Lipschitz continuous on $\Lambda \cup \Lambda_{1/12}$ with Lipschitz constant $4C'$. As the interiors of $\Lambda \cup \Lambda_{1/12}$ and $N$ comprise $\Omega$ we conclude that $v_0 \in C^{1,1}_{loc}(\Omega)$. $\square$

**Remark 5.9.** Under weaker conditions we can also state regularity results. For example, we can show that $u_0$ is continuous if $\psi$ is continuous or $u_0$ is $C^1$ if $\psi$ is $C^1$. See for example [3]. We have restricted ourselves to the optimal case for simplicity.

## 5.2 Generalizations and the Free Boundary

In this section we discuss extensions of the optimal regularity result to elliptic obstacle problems and the regularity of the free boundary of the classical obstacle problem.

We first wish to generalize the $C^{1,1}_{loc}$ regularity result to a specific class of elliptic obstacle problems. These are obstacle problems with the functional

$$\mathcal{E}(u) = \int_\Omega \frac{\nabla u^T A \nabla u}{2} \, dx,$$

where $A$ is a symmetric positive definite matrix (i.e. it has strictly positive eigenvalues). Note that this functional corresponds to the Dirichlet functional when $A$ is the identity. The obstacle problem for this functional becomes

$$\text{Minimize } \mathcal{E}(u) \text{ over all } u \in K_\psi = \{v \in H^1_g(\Omega) \mid v \geq \psi \text{ a.e. in } \Omega\},$$

where $\Omega$, $\psi$ and $g$ are as in Section 5.1. I have come up with my own argument to show that this problem is equivalent to the classical obstacle problem by using a linear transformation. After showing this we then obtain that the optimal regularity also holds for the functional $\mathcal{E}$. Consider the decomposition $A = O^T DO$ with $O$ orthogonal and $D$ a diagonal matrix with positive entries. This decomposition exists as $A$ is symmetric and positive definite. Now we can define the principal square root of $A$ as $\sqrt{A} := O^T\sqrt{D}O$, where $\sqrt{D}$ is the diagonal matrix with as entries the square roots of those of $D$. We see that $\sqrt{A}$ is also symmetric and positive definite and $\sqrt{A}\sqrt{A} = A$. To make the distinction between a matrix and a linear map we define the smooth invertible map

$$L : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto (\sqrt{A})^{-1}x.$$

Since $L$ is smooth and invertible (with smooth inverse) we have that $\Omega' := L(\Omega)$ is a $C^1$-domain. Hence we can define the classical obstacle problem on $\Omega'$ as

$$\text{Minimize } \mathcal{D}(u) = \int_{\Omega'} \frac{|\nabla u|^2}{2}\,dx \ \text{ over all } u \in K'_\psi,$$

with $K'_\psi = \{v \in H^1(\Omega') \mid v|_{\partial\Omega'} = g \circ L^{-1},\ v \geq \psi \circ L^{-1} \text{ a.e.}\}$. We show equivalence of the two minimization problems. First, by our chosen definitions the map $L$ induces a bijection between $K_\psi$ and $K'_\psi$, which is explicitly given by

$$L^* : K'_\psi \to K_\psi, \qquad v \mapsto v \circ L.$$

We can also calculate for any $u \in K'_\psi$ that

$$\begin{aligned}
\mathcal{E}(L^*u) &= \int_\Omega \frac{(\nabla(L^*u(x)))^T A\nabla(L^*u(x))}{2}\,dx \\
&= \int_\Omega \frac{\nabla u(Lx)^T L^T AL\nabla u(Lx)}{2}\,dx \\
&= \int_\Omega \frac{\nabla u(Lx)^T \nabla u(Lx)}{2}\,dx,
\end{aligned}$$

where the third line follows as $L^TAL = (\sqrt{A})^{-1}A(\sqrt{A})^{-1} = \mathrm{Id}$. Now we make the substitution $y = Lx$ to obtain

$$\mathcal{E}(L^*u) = \det(L^{-1}) \cdot \int_{\Omega'} \frac{|\nabla u(y)|^2}{2}\,dy = C \cdot \mathcal{D}(u),$$

with $C = \det(L^{-1}) > 0$. We conclude that $L^*$ is a bijection between $K'_\psi$ and $K_\psi$ and relates the two functionals $\mathcal{E}$ and $\mathcal{D}$ by a positive constant. This indeed implies that the minimization problems are equivalent. In particular if $v_0$ is the minimizer of the Dirichlet integral over $K'_\psi$, then $u_0 := L^*v_0$ is the minimizer of $\mathcal{E}$ over $K_\psi$. This proves that the optimal regularity result also holds for obstacle problems with the functional $\mathcal{E}$. For the optimal regularity for more general second order elliptic variational inequalities see [4, Chapter 1.3 and 1.4].

Apart from generalizing the variety of obstacle problems one can try to investigate the regularity of the free boundary. Namely, in a lot of applications it is not only the minimizer that is of interest but also the free boundary. This is for example the case for the dam problem and the optimal stopping times explained in Section 6. Hence it would be interesting to know certain properties of the free boundary in both general and specific cases. Additionally, when the free boundary is known the classical obstacle problem reduces to solving the equation $\Delta u_0 = 0$ on the set $N$. The general regularity theory for the free boundary of the classical obstacle problem begins with the assumption that $\Delta\psi < 0$ on the set $\Lambda$ to avoid pathological cases. Then the free boundary can be divided up into regular and singular points. For regular points we find that there is a neighborhood in which the free boundary is an analytic $n-1$-dimensional manifold. For the singular points it is more subtle and difficult, and there are still some unanswered questions about the singular points. The first groundbreaking paper on this topic was [3] by Caffarelli, where he described the structure of the regular and singular points of the free boundary. For a great approachable text on this topic and some of the recent developments see the lecture notes by Figalli [1].

# 6 Applications

In this section we study motivating examples and applications of the classical obstacle problem studied in Section 5. The first example can be seen as the standard interpretation of the problem, while the second and third are a lot less obvious. The way they can be rewritten to a classical obstacle problem is very unexpected and shows the versatility of obstacle problems.

## 6.1 Modelling of Elastic Membrane

The prime motivating example for the obstacle problem, as discussed in [18, Chapter 1.2], is that of finding the equilibrium position of an elastic membrane which is suspended by the obstacle $\psi$. We explain this interpretation and show two examples.

Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^1$-domain and let $u_0 : \Omega \to \mathbb{R}$ describe the position of a membrane such that the membrane corresponds with the graph of $u_0$. The membrane is being held fixed at the boundary at height $g \in C^1(\partial\Omega)$. If there is no influence by gravity or other external forces we can assume that the potential energy of such a membrane is proportional to the area of this membrane. For a general $u \in H^1(\Omega)$ this area is given by the functional

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

By the principle of minimal potential energy we find that the equilibrium position of the membrane $u_0$ can be modelled by minimizer of the functional $\mathcal{A}$ over all functions $u \in H^1(\Omega)$ that satisfy $u \geq \psi$ and $u|_{\partial\Omega} = g$. However, the area functional $\mathcal{A}$ is difficult to analyse and hence we utilize the Taylor approximation $\sqrt{1 + |x|^2} \approx 1 + \frac{1}{2}|x|^2 + \mathcal{O}(|x|^4)$. Therefore for small $|\nabla u|$ the approximation

$$\mathcal{A}(u) \approx \int_\Omega 1 + \frac{|\nabla u|^2}{2}\, dx$$

is valid. As the constant 1 does not change the minimization problem we see that minimizing the right hand side is the same as minimizing the Dirichlet integral

$$\mathcal{D}(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx$$

over the set $K_\psi = \{u \in H^1_g(\Omega) \mid u \geq \psi\}$. Hence the membrane $u_0$ can be modelled by the solution of the classical obstacle problem, studied in the previous section. We will simulate the solution of the classical obstacle problem in two dimensions by using our numerical method given in the appendix. Two examples of obstacles and solutions are given in Figure 4.

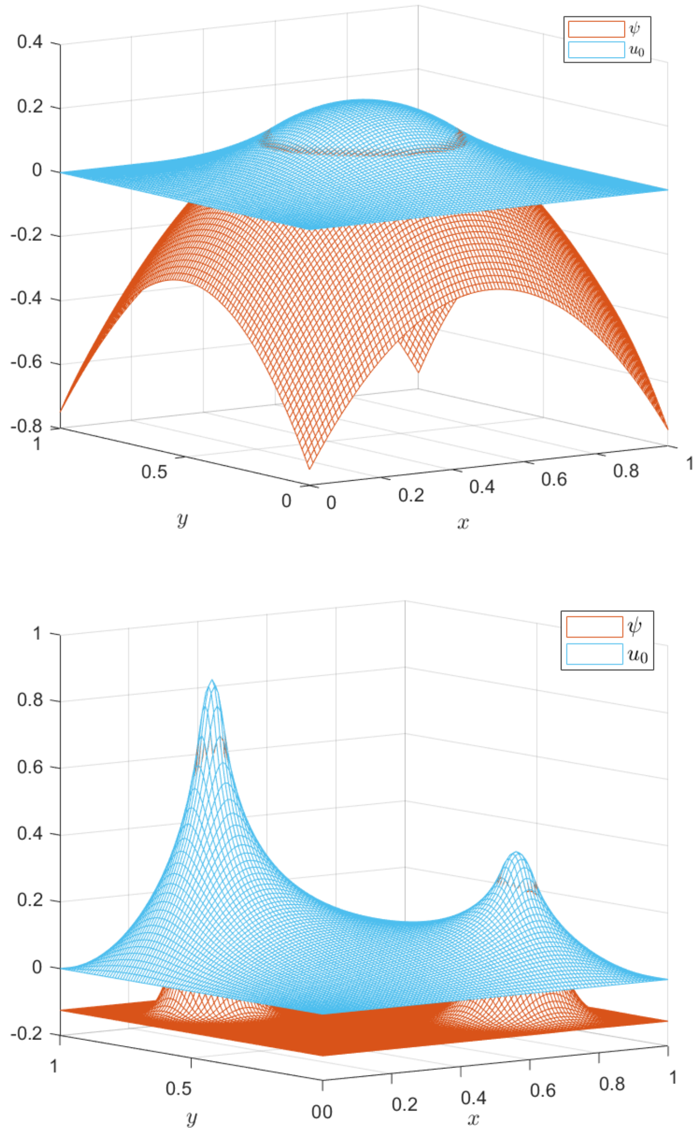


Figure 4: Two examples of the solution of the classical obstacle problem in two dimensions representing an elastic membrane. In the top figure we have $\psi(x, y) = 1/4 - 2((x-1/2)^2 + (y-1/2)^2)$ and in the bottom figure we have $\psi(x, y) = -1/8 + e^{-200((x-1/4)^2+(y-3/4)^2)} + 1/2\, e^{-100((x-3/4)^2+(y-1/4)^2)}$. Both examples have $g = 0$.

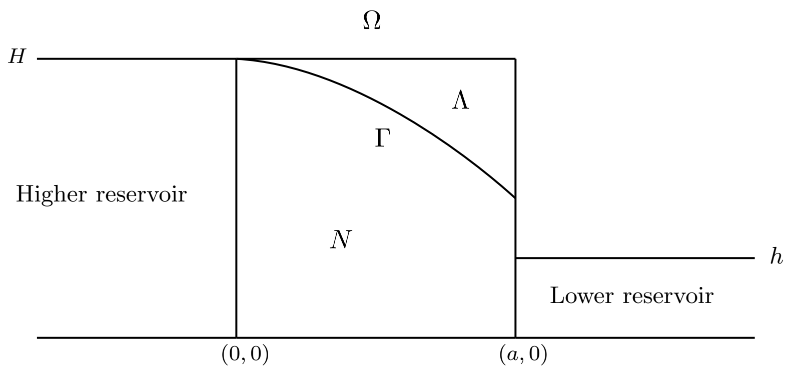


Figure 5: Cross-section of a dam.

## 6.2 The Dam Problem

This example is found in [4, Chapter 1.5] and [18, Chapter 2.3 and 2.4]. Consider the cross-section of a dam made out of a porous material like earth, which separates two reservoirs of water having different heights $H$ and $h$, see Figure 5. We are interested in how the water flows through the dam, which can have effects on the amount of seepage and the erosion inside the dam. Using $(x, y)$-coordinates we can represent the cross-section of the dam as $\Omega = (0, a) \times (0, H)$. We expect the water level to slowly decrease as we go from the high reservoir to the low reservoir. Therefore, if $N$ represents the wet part of the dam and $\Lambda$ the dry part then we can write their interface $\Gamma$, which describes the water level in the dam, as the graph of a function $\eta : (0, a) \to (0, H)$ which is decreasing. Because of this we also find

$$\Lambda = \{(x, y) \in \Omega \mid y \geq \eta(x)\} \quad \text{and} \quad N = \{(x, y) \in \Omega \mid y < \eta(x)\}.$$

Our goal is to determine this function $\eta$ or equivalently $\Gamma$ by considering an obstacle problem with free boundary exactly equal to $\Gamma$. First we describe the movement of the water by using the laws of fluid dynamics. By Darcy's law we have that

$$v(x, y) = k(x, y)\nabla w(x, y),$$

where $w : \overline{N} \to \mathbb{R}$ is the hydraulic head, $k$ is the permeability coefficient and $v$ is the velocity field of the fluid. The hydraulic head is a kind of pressure measurement above a certain vertical level. We take the vertical level $y = 0$ from which we obtain that the hydraulic head is given by $w(x, y) = y + p(x, y)$, where $y$ represents the elevation head, i.e. the gravitational force of the water and $p(x, y)$ is the pressure head, i.e. the inner pressure of the water. Assume for simplicity that $k$ is constant then we have by conservation of mass $\operatorname{div}(v) = 0$ that $\Delta w = 0$ on $N$. Since the flow of the water at $\Gamma$ is tangent to $\Gamma$ we have that $w$ satisfies the system

$$\Delta w = 0 \quad \text{on } N, \quad D_\nu w = 0 \quad \text{on } \Gamma, \tag{6.1}$$

where $\nu$ is an outward normal to $\Gamma$. Note that the size of $k$ has an influence on the velocity of the fluid, but it does not affect the above equation of the hydraulic head as long as it is constant. For the boundary conditions on $w$ we find that in the reservoirs we can assume there is negligible flow and hence the pressure at a point is given by the height of the liquid above it. Furthermore, the pressure is zero on $\Gamma$ and for $(a, y)$ with $a < y < \eta(a)$. Lastly, we assume that the bottom of the dam is impenetrable which translates to $D_y w(x, 0) = 0$. We can summarize this to the following boundary conditions.

$$\begin{cases} w(0,y) = H & \text{if } 0 \le y \le H, \\ w(a,y) = h & \text{if } 0 \le y \le h, \\ w(a,y) = y & \text{if } h \le y \le \eta(a), \\ w(x,y) = y & \text{on } \Gamma \\ D_y w(x,0) = 0 & \text{if } 0 \le x \le a. \end{cases} \tag{6.2}$$

We see that (6.1) and (6.2) define a partial differential equation on an unknown domain $N$. This is therefore called a free boundary problem and in this case the free boundary is $\Gamma$. We want to transform the problem into an obstacle problem now. To this aim suppose that we have a solution $w$ of (6.1), (6.2), which means that $\eta$ is $C^1$ and $w \in C^1(N \cup \Gamma) \cap C(\overline{N})$. Also $w$ is smooth on $N$ as $w$ is harmonic. First we show that $p(x, y) = w(x, y) - y > 0$ in $N$. As $p$ is harmonic on $N$ and continuous on $\overline{N}$ we find by the minimum principle (Proposition 4.11) that $p$ attains its minimum only at the boundary. We know that $p \ge 0$ on the boundary of $N$ except at the bottom. In addition, at the bottom we have $D_y(x, 0) = D_y w(x, 0) - 1 = -1$ which means that $p$ decreases when going up from the bottom. Hence the minimum of $p$ cannot be attained at the bottom of the boundary either. Thus we conclude that $p \ge 0$ on $\partial N$ and therefore $p > 0$ on $N$. Next, we will transform $w$ by using the Baiocchi transform, which is named after Claudio Baiocchi who first discovered it, see [11]. The transformed function is given by

$$u(x,y) = \begin{cases} \displaystyle\int_y^{\eta(x)} w(x,t) - t\,dt & \text{if } 0 < y \le \eta(x), \\ 0 & \text{if } \eta(x) < y < H, \end{cases}$$

which is a function defined on $\Omega = (0, a) \times (0, H)$. This function turns out to be a solution of an obstacle problem to which we can apply our developed theory. To this aim we calculate $\Delta u$ on $N$. First we have $D_y u(x, y) =$

$-w(x,y)+y$ and $D_{yy}u = -D_y w(x,y)+1$. Next, by using differentiation under the integral we obtain

$$\begin{aligned} D_x u(x,y) &= \int_y^{\eta(x)} D_x w(x,t)\,dt + \eta'(x)(w(x,\eta(x)) - \eta(x)) \\ &= \int_y^{\eta(x)} D_x w(x,t)\,dt, \end{aligned}$$

by the fourth condition of (6.2). Thus we have by differentiating again and using $D_{xx}w = -D_{yy}w$ that

$$\begin{aligned} D_{xx}u(x,y) &= \int_y^{\eta(x)} D_{xx}w(x,t)\,dt + \eta'(x)D_x w(x,\eta(x)) \\ &= -\int_y^{\eta(x)} D_{yy}w(x,t)\,dt + \eta'(x)D_x w(x,\eta(x)) \\ &= D_y w(x,y) - D_y w(x,\eta(x)) + \eta'(x)D_x w(x,\eta(x)) = D_y w(x,y), \end{aligned}$$

where the last equality follows by the second part of (6.1). We conclude that $\Delta u = 1$ on $N$. Additionally $u > 0$ on $N$ since $p > 0$ on $N$. Also we have $u = 0$ on $\Lambda$. To conclude that $u$ solves an obstacle problem we still need that $\Delta u \leq 1$ in distributional sense, explicitly this means that

$$\int_\Omega \nabla u \cdot \nabla \varphi\,dx \geq -\int_\Omega \varphi\,dx,$$

for all nonnegative $\varphi \in C_c^\infty(\Omega)$. By using the Gauss-Green theorem and $\Delta u = 1$ on $N$ we find

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla \varphi\,dx &= \int_N \nabla u \cdot \nabla \varphi\,dx \\ &= -\int_N \Delta u \varphi\,dx + \int_\Gamma D_\nu u \varphi\,dx \\ &= -\int_N \varphi\,dx + \int_\Gamma D_\nu u \varphi\,dx. \end{aligned}$$

We now show that the last integral is nonnegative. As $D_x u = 0$ on $\Gamma$ we can calculate

$$\begin{aligned} D_\nu u(x,y) &= c(x,y)(-\eta'(x)D_x u(x,y) + D_y u(x,y)) \\ &= c(x,y)(-w(x,y)+y) \\ &= c(x,y)p(x,y) > 0, \end{aligned}$$

where $c > 0$ scales to get a unit normal. The second line follows since $D_x u = 0$ on $\Gamma$. Returning to the original computation we now find by using $\varphi \geq 0$ that

$$\int_\Omega \nabla u \cdot \nabla \varphi\,dx \geq -\int_\Omega \varphi\,dx$$

as desired. Integrating the boundary conditions (6.2) we find that $u|_{\partial\Omega} = g$ where $g$ is the continuous and piecewise smooth function

$$\begin{cases} g(0,y) = \frac{1}{2}(H-y)^2 & \\ g(a,y) = \frac{1}{2}(h-y)^2 & \text{if } 0 < y \leq h, \\ g(x,0) = \frac{H^2}{2}(1-\frac{x}{a}) + \frac{h^2}{2}\frac{x}{a}, & \\ g(x,y) = 0 & \text{else.} \end{cases}$$

Combining everything that we have shown it is now an easy check by using that $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ that $u$ satisfies the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq -\int_\Omega (v-u)\,dx$$

for all $v \in K_0 := \{v \in H_g^1(\Omega) \mid v \geq 0\}$. This means that $u$ solves the problem

$$\text{Minimize } \mathcal{J}(v) = \int_\Omega \frac{|\nabla v|^2}{2} + v\,dx, \qquad \text{over all } v \in K_0. \tag{6.3}$$

Also $u = 0$ on $\Lambda$ and $u > 0$ on $N$ which shows that indeed $\Gamma$ is the free boundary for $u$. What is not mentioned in [4] is that we can reduce this to the classical obstacle problem by considering the function

$$\psi(x,y) = \frac{-(x^2+y^2)}{4},$$

which is smooth and satisfies $\Delta\psi = -1$. Any other smooth function with Laplacian -1 can be chosen as well. Then $u_0 = u + \psi$ satisfies

$$\begin{cases} \Delta u_0 = 0 & \text{on } N = \{u_0 > \psi\} \\ \displaystyle\int_\Omega \nabla u_0 \nabla\varphi\,dx \geq 0 & \text{for all } \varphi \in C_c^\infty(\Omega) \\ u_0 \geq \psi & \text{on } \Omega, \end{cases}$$

where the second line follows from the fact that $\Delta u \leq 1$ in distributional sense. Hence it is easy to check that $u_0$ satisfies the variational inequality for the classical obstacle problem with obstacle $\psi$ and boundary condition $g + \psi|_{\partial\Omega}$. Therefore we conclude by Proposition 5.3 that $u_0$ minimizes

$$\mathcal{D}(v) = \int_\Omega \frac{|\nabla v|^2}{2}\,dx$$

over all $v \in K_\psi = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = g + \psi|_{\partial\Omega},\ v \geq \psi\}$, which is the classical obstacle problem. (In this case $\partial\Omega$ is only piecewise smooth and so is $g$ but this is a straightforward generalization of what we have done).

We conclude by Section 5 that there exists a unique solution $u \in C^{1,1}_{loc}(\Omega)$ of (6.3) without assuming that (6.1), (6.2) has a solution. Therefore by approximating the solution and the corresponding free boundary we can determine in which part of the dam the water is located. Appealing to our numerical solver for the two-dimensional obstacle problem, see appendix, we can approximate the solution $u$ of (6.3) and from this approximation we can determine an approximation of the free boundary. We have carried this out for the values $H = 1$ and $h = 1/4$ and we vary the width of the dam $a$ to see how it affects the water level, see Figure 6.

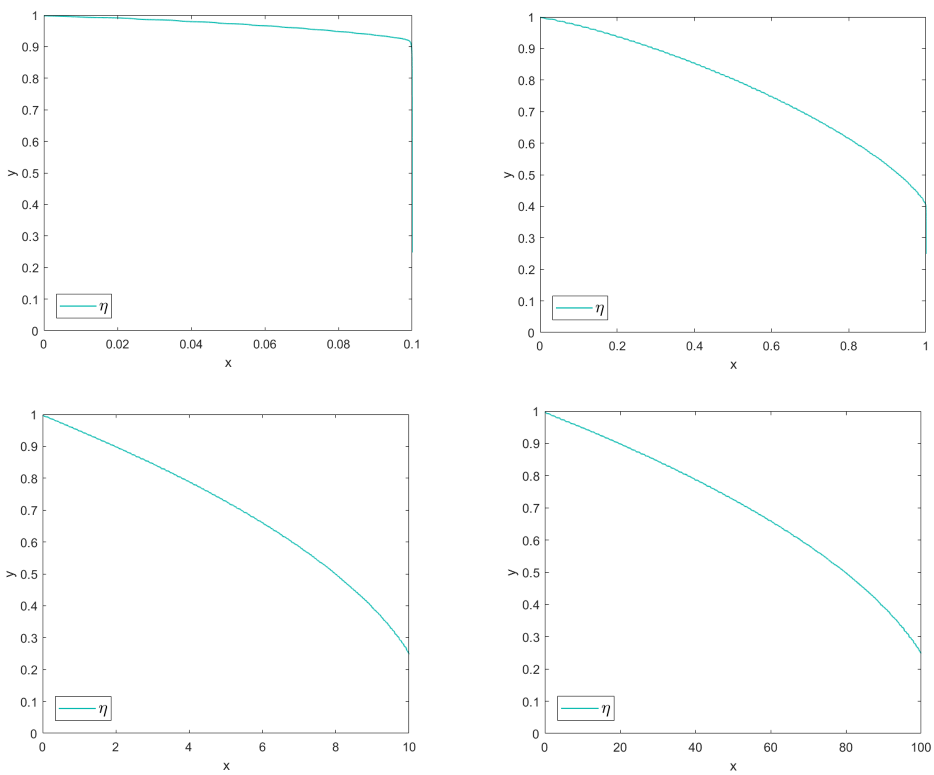


Figure 6: Simulation of the water level for the dam problem with $H = 1$, $h = 1/2$ and the width $a$ varying from 0.1 to 100.

When $a = 0.1$ the water level almost remains constant and ends at height 0.91. Hence it is natural to suspect, keeping in mind physical considerations, that as $a$ goes to zero the water level $\eta$ will become more and more flat. We expect that the function $\eta(x/a)$ converges to the constant function $H$ (on $(0, 1)$) as $a$ goes to zero. For $a = 1$ we can see the water level decreasing considerably, as expected, and end up at height 0.4. Noteworthy is that the end level is still higher than the lower reservoir at height 0.25. When $a = 10$ we have that the water level at the end is practically equal to that of the lower reservoir. As the water cannot drop below the level of the lower reservoir we imagine that this means that the solution has stabilized in some sense. Indeed, if we look at the graph for $a = 100$ we can see that it has practically the same shape as for $a = 10$. So we see that once the water level at the end approaches $h$, the solution does not change that much when we increase $a$ (of course we need to scale space). Therefore we expect that the solution $\eta(x/a)$ converges to a curve similar to the last graph in figure 6 as $a$ goes to infinity. A different hypothesis is that there is some critical value $a_m$ such that $\eta(a_m) = h$ and after which the shape of the water does not change anymore. However, this hypothesis is incorrect as it is proven in Chapter 2.6 of [4] that $\eta(a)$ is always strictly larger than $h$.

### 6.3 Optimal Stopping Times

The following is an application to financial mathematics, from [10, p.110-114], where we want to design a strategy to stop a random process at an optimal time, i.e. maximizing the expected payoff. In [10] this is done for general stochastic processes but we restrict ourselves to a specific case in which the classical obstacle problem comes up.

Consider a bounded $C^1$-domain $\Omega \subset \mathbb{R}^n$ and a payoff function $\psi \in C^{1,1}(\Omega)$ (in particular this means $\psi \in C(\overline{\Omega})$ by extension of Lipschitz functions). This payoff function turns out to be the obstacle function in our example. For any $x \in \Omega$ we consider the stochastic process $\{X(t) = x + W(t) \mid t \geq 0\}$, where $W(\cdot)$ is an $n$-dimensional Brownian motion. A Brownian motion is a family of random variables $W(t)$ for $t \geq 0$ such that $t \mapsto W(t) \in \mathbb{R}^n$ is a continuous path starting at the origin. Also the value of $W(t) - W(s)$ for $t > s$ is independent of $W(s)$, i.e. it is memoryless in some sense. Therefore the Brownian motion can be interpreted as a random walk starting at the origin that has no jumps. There are some more subtle properties of Brownian motion but we will not go into detail about these. See for example [26, Chapter 3] for an introduction on Brownian motion. From the definition of $X(\cdot)$ it follows that $X(\cdot)$ is a random walk starting at the point $x$ in $\Omega$, see Figure 7.

The game is that we can stop the process $X(\cdot)$ at any time $t$ and receive the payoff $\psi(X(t))$. If the process $X(\cdot)$ exits the domain $\Omega$ at time $t$, i.e. $X(t) \in \partial\Omega$, then the process stops immediately and we receive $\psi(X(t))$. Our problem is to find a strategy to determine at which time we need to stop the process to maximize the expected payoff. Of course, at which time we need to stop is dependent on the process $X(\cdot)$, which is random, and hence

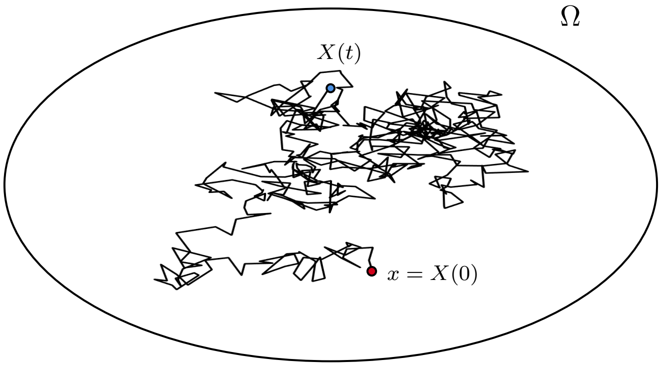


Figure 7: Representation of a 2-dimensional Brownian motion starting at $x$, which is simulated up to time $t$.

the stopping time should be a random variable as well. This is because it depends on the outcome of the experiment $X(\cdot)$ whether we want to stop at time $t_0$. Note that if we stop the process $X(\cdot)$ at time less than $t_0$ then this decision can only be based on the outcome of $X(t)$ for $t \leq t_0$. Indeed, if we knew beforehand what $X(t)$ is for all $t$ then it is easy and uninteresting to find the optimal stopping time. That the stopping time is independent of the future can be captured mathematically, but this will wander too far away from our goal. For our purposes it is enough to see a stopping time $\tau$ as a nonnegative valued random variable which determines when to stop and does not use information about the future. Now our goal is to find an optimal stopping time $\tau^*$ such that

$$\mathbb{E}[\psi(X(\tau^*))] = \max_{\tau \text{ stopping time}} \mathbb{E}[\psi(X(\tau))],$$

where $\mathbb{E}$ denotes the expected value. Informally this formula states that the expected outcome when applying the strategy $\tau^*$ is the maximal outcome over all strategies $\tau$. Instead of trying to solve this directly we shift the attention to the starting point of the process $X(\cdot)$. We emphasize the starting point by writing $X_x(\cdot) = X(\cdot)$ and this allows us to define the function

$$v_0(x) = \sup_{\tau \text{ stopping time}} \mathbb{E}[\psi(X_x(\tau))].$$

Now if we want to find an optimal stopping time $\tau^*$ (for any starting point $x$) then this is equivalent to showing that

$$\mathbb{E}[\psi(X_x(\tau^*))] = v_0(x). \tag{6.4}$$

About $v_0$, because $\tau \equiv 0$ is a valid stopping time we observe that $v_0(x) \geq \mathbb{E}[\psi(X_x(0))] = \psi(X_x(0)) = \psi(x)$. We therefore have $v_0 \geq \psi$ on $\Omega$. Note also that on the boundary $\partial\Omega$ any process immediately stops and hence $v_0 = \psi$ on $\partial\Omega$. This already hints at the fact that $v_0$ has something to do with an obstacle problem, but we show this later. For now, let $u_0 \in C^{1,1}_{loc}(\Omega)$ be the unique solution of the obstacle problem

$$\text{Minimize } \mathcal{D}(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx \qquad \text{over all } u \in K_\psi,$$

where $K_\psi = \{u \in H^1(\Omega) \mid u \geq \psi,\ u|_{\partial\Omega} = \psi|_{\partial\Omega}\}$. Our strategy is to find a stopping time $\tau^*$ such that $\mathbb{E}[\psi(X_x(\tau^*))] = u_0(x)$ for any $x$ and establish that $u_0 = v_0$ (as anticipated) from which we obtain (6.4) thus proving the optimality of $\tau^*$.

As usual we define the sets

$$\Lambda = \{x \in \Omega \mid u_0(x) = \psi(x)\} \quad \text{and} \quad N = \{x \in \Omega \mid u_0(x) > \psi(x)\}.$$

Using these sets we can define the following stopping time $\tau^*$. If $X_x(0) = x \in N$ then we continue the process until the first time $t$ such that $X_x(t) \in \Gamma = \partial N \cap \Omega$. If $X_x(0) = x \in \Lambda$ then we stop the process immediately. Also if at any time $X_x(t) \in \partial\Omega$ then the process is forced to stop. Intuitively, this means that $N$ is the continuation set and $\Lambda$ is the stopping set. Let us now show that $\mathbb{E}[\psi(X_x(\tau^*))] = u_0(x)$. If $x \in \Lambda$ then $\tau^* \equiv 0$ which implies $\mathbb{E}[\psi(X_x(\tau^*))] = \psi(X_x(0)) = \psi(x) = u_0(x)$. If on the other hand $x \in N$ then $\tau^*$ is exactly the time at which $X_x(\cdot)$ leaves the region $N$. We now need a formula from stochastic calculus which connects the Brownian motion with the Laplacian. This deep connection is called Itô's formula and for stopping times it takes the form

$$\mathbb{E}[u(X_x(\tau))] = u(x) + \mathbb{E}\left[\int_0^\tau \frac{1}{2}\Delta u(X(s))\, ds\right]. \tag{6.5}$$

We do not prove this formula but it can be found in [10, p.105]. This formula captures the way the expected value of a function applied to the Brownian motion varies over time. Because for $s < \tau^*$ we have $X(s) \in N$ we find that $\Delta u_0(X(s)) = 0$. Therefore applying (6.5) to $u_0$ and $\tau^*$ yields

$$\mathbb{E}[u_0(X_x(\tau^*))] = u_0(x) + \mathbb{E}\left[\int_0^{\tau^*} \frac{1}{2}\Delta u_0(X(s))\, ds\right] = u_0(x).$$

Furthermore, $X_x(\tau^*)$ will always lie in $\partial N$ (by design of $\tau^*$) and on this set we have $u_0 = \psi$. Therefore we can conclude that

$$\mathbb{E}[\psi(X_x(\tau^*))] = \mathbb{E}[u_0(X_x(\tau^*))] = u_0(x),$$

which shows $\mathbb{E}[\psi(X_x(\tau^*))] = u_0(x)$ for all $x \in \Omega$. Because this trivially implies that $u_0 \leq v_0$ we only need to prove $u_0 \geq v_0$ to find (6.4) which proves that $\tau^*$ is optimal. To this end let $\tau$ be any other stopping time. Then we have by rewriting (6.5) that

$$u_0(x) = \mathbb{E}[u_0(X_x(\tau))] + \mathbb{E}\left[\int_0^\tau -\frac{1}{2}\Delta u_0(X_x(s))\, ds\right].$$

Now $\Delta u_0 \leq 0$ by superharmonicity of $u_0$ and $u_0 \geq \psi$ thus we find

$$u_0(x) \geq \mathbb{E}[u_0(X_x(\tau))] \geq \mathbb{E}[\psi(X_x(\tau))].$$

Since this holds for any stopping time $\tau$ we find $u_0(x) \geq v_0(x)$ as desired.

In short, we have established that finding an optimal stopping time for the Brownian motion starting at $x$ with payoff function $\psi$ can be done by solving the obstacle problem with obstacle and boundary condition given by $\psi$. The optimal stopping time $\tau^*$ is then given by the first time at which $X_x(\cdot)$ does not lie in the continuation set $N$.

# Appendix

Here we want to include the code which was used to simulate the obstacle problem in one and two dimensions, which is used in Figures 1,4 and 6. For the one-dimensional case it can be readily seen from the theory discussed that the solution $u_0$ corresponds to the concave envelope of the obstacle $\psi$ with certain boundary conditions, see [8, Chapter 5.4]. There is an exact formula for this namely

$$u_0(x) = \sup_{t \leq x, s \geq x} \frac{(s-x)\psi(t) + (x-t)\psi(s)}{s-t},$$

where we redefine $\psi$ at the boundary to equal the boundary condition. We can simply implement this in Matlab R2017b as follows:

```
%obstacle function, boundary values and interval [0,L]
psi= @(x) exp(-(x-1/2)^2);
a=0.2;
b=0.4;
L=1;

n=501;
dx=L/(n-1);
v=[a, zeros(1,499), b];
for i=2:500
    v(i)=psi((i-1)*dx);
end
u0=[a,zeros(1,n-2),b];
m=-inf;
for i=2:n-1
    for j=1:i-1
        for k=i+1:n
            m=max(m,((k-i)*v(j)+(i-j)*v(k))/(k-j));
        end
    end
    u0(i)=m;
    m=-inf;
end
v(1)=psi(0);
v(n)=psi(1);
plot((0:dx:L),[u0;v])
leg1=legend('$u_0$', '$\psi$','location', 'southeast');
set(leg1,'Interpreter','latex');
x1=xlabel('$x$');
```

```
set(x1,'Interpreter','latex');
```

For the two-dimensional case we have implemented the algorithm from [13, p.26]. This algorithm solves the equation $\max(\Delta u, \psi - u) = 0$ by using finite differences. Note that this equation implies $u \geq \psi$, $\Delta u \leq 0$ and $\Delta u = 0$ when $u = \psi$, which is the case for the obstacle problem. The code implementing this in Matlab R2017b is given below.

```
%domain=[0,a]x[0,b]
a=1;
b=1;

%obstacle psi
psi=@(x) 1/4-2*((x(1)-1/2)^2+(x(2)-1/2)^2);

%boundary condition g
g=@(x) 0;

%stepsize
h=1/100;
im=a/h+1;
jm=b/h+1;

%solution table
u=zeros(im,jm);

%boundary condition
for i=1:im
    u(i,1)=g([(i-1)*h,0]);
    u(i,jm)=g([(i-1)*h,b]);
end
for j=2:jm-1
    u(1,j)=g([0,(j-1)*h]);
    u(im,j)=g([a,(j-1)*h]);
end


%initial condition
for i=2:im-1
    for j=2:jm-1
        u(i,j)=max(0,psi([(i-1)*h,(j-1)*h]));
    end
end
```

```
%algorithm
v=u;
for i=2:im-1
    for j=2:jm-1
        v(i,j)=max((1/4)*(u(i+1,j)+u(i,j+1)+u(i-1,j)+u(i,j-1)),
        psi([(i-1)*h,(j-1)*h]));
    end
end
l=0;
s=abs(u-v);
while max(s(:))>0.00001
    l=l+1;
    u=v;
    for i=2:im-1
        for j=2:jm-1
            v(i,j)=max(1/4*(u(i+1,j)+u(i,j+1)+u(i-1,j)+u(i,j-1)),
            psi([(i-1)*h,(j-1)*h]));
        end
    end
    s=abs(u-v);
end
ob=zeros(im,jm);
for i=1:im
    for j=1:jm
        ob(i,j)=psi([(i-1)*h,(j-1)*h]);
    end
end
x=0:h:a;
y=0:h:b;
mesh(x,y,transpose(ob),'EdgeColor',[0.8500 0.3250 0.0980])
hold on
mesh(x,y,transpose(v),'EdgeColor',[0.3010 0.7450 0.9330])
leg1=legend('$\psi$','$u_0$','location', 'best');
set(leg1,'Interpreter','latex');
x1=xlabel('$x$');
set(x1,'Interpreter','latex','FontSize',12);
y1=ylabel('$y$');
set(y1,'Interpreter','latex','FontSize',12);
```

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