# Does the "perfect" democracy exist?

*A comparison between standard and fuzzy social choice theory.*

Bachelor thesis TWIN

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# Abstract

In this thesis the question does the ”perfect” democracy exist, is central. To answer this question a broad view of democracy will be presented. It will be followed by some theorems from social choice theory after which Arrow’s theorem will be put in a fuzzy logic perspective. In the end it will be concluded that in fuzzy social choice theory Arrow’s theorem will not always hold. By doing so, is shown how interesting the extension of social choice in a fuzzy logic setting can be.

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# 1 Introduction

This thesis will show different aspects of a democracy. First a historic background on the development of, and the views on democracies will be provided. This will be followed by examining some social choice theorems and look at how these theorems change in a fuzzy setting. Then interpretations for these theorems will be given and discussed. In this thesis chapter 3,4 and 5 are all focussed on (fuzzy) social choice theory and these chapters are all strongly based on the book "Fuzzy Social Choice Theory"[7] most of the proofs have been adjusted until I found them complete, some are unadjusted since I found them complete already. At the start of each of these sections will be announced on which chapters of the book they are based.

## 1.1 Purpose and relevance of this thesis

In the Netherlands a lot of people claim to be dissatisfied by the Dutch democracy. There are several reasons for people to be dissatisfied by the Dutch democracy and everyone that has heard of social choice theory could be thinking of that right now. Because in social choice theory there are a some theorems that are often referred to as a mathematical proof that the perfect democracy can not exist. This is because according to these theorems there does not exist a "good" election system and therefore a democracy which relies on an "bad" election system can not be perfect.
In these theorems is defined when an election system is "good" and then proven that such an election system can not exist. That brings up the question: when is an election system good, and do the definitions these theorems use translate to reality? An example of a condition these "good" election systems have is that it is impossible to vote strategically. Strategically voting is not voting for your most preferred candidate on purpose to get an outcome that is more preferred than you expect to get otherwise. This condition for a election system would already be enough to show that the Dutch election system is not "good" since it is possible to vote strategically.
Another question may arise, how bad is it that it is possible to vote strategically? Though strategic voting may not seem bad, it does have some unwanted consequences. An example of these consequences is that even if an alternative that people voted for wins, they are still not truly satisfied (because it is not their actual choice). Another example is that votes becomes less valuable since the people that are chosen can not know whether this was a strategic choice or that all voters really wanted to vote for them and their ideas. Lastly, a third example is that it also leads to an election favouring the biggest parties and therefore making smaller parties even smaller and often unheard while a lot of people may actually agree with these smaller parties.
The purpose of this thesis is to learn how and why these theorems, follow up research and follow up theorems function, to give a background about democracies and to investigate if there is a better form for a election system that is not used.

# 2 Philosophical and historical background

This section will provide a general background on democracies and some of the criticism on democracy. Doing so will give purpose to the rest of this thesis which focusses on (fuzzy) social choice theory. This section is mainly based on Wikipedia. Since the main purpose for this section is being a motivation for the rest of the thesis, I found this source to be reliable enough. Nevertheless the reader should take this into account while reading.

## 2.1 Different ways of leading

As we will see later Plato separated all governance forms in five different types, in this subsection all forms will be introduced and there will be a closer look into democracy.

The word aristocracy is derived from the Greek word aristokratia which translates to "rule of the best-born". In an aristocracy the power is in the hands of of a small, privileged group. This privilege is inheritable.

A timocracy is a governance form in which only the people that own property are allowed to participate in government. This kind of governance form feels really natural since it would be strange to let people that do not own property decide what people that do own property can do with it. Though there is the obvious downside that the people that own property can be a small minority and could be unsuited to be leaders.

In an oligarchy the power is, just as in an aristocracy, only with a few, though the power does not necessarily pass down via a bloodline. The people in power are often either nobility, very wealthy, high up religiously, well educated or in control of some military force. Notable is that in the 20th century Robert Michels developed a theory that said that democracies have the tendency to turn into oligarchies, because in a large scale democracy the necessary division of labor will lead to a ruling class that will mainly try to protect their own power.

In a tyranny there is one absolute ruler that has all power and authority, this ruler is called a tyrant. Often a tyrant wields its power cruel, unjust and oppressive and in this way tries to guarantee its position

A democracy is a form of governance by the whole population or the eligible members of the population. In a democracy people can either directly vote on political decisions, in this case we call it a direct democracy, or people can vote on representatives that will then lead, in this case we call it a indirect democracy.

The first democracy in history was in Athens where a direct democracy was used to rule the city. This type of democracy seems ideal since every decision is made together and when ruling over a city with around 8.000 eligible voters this is still manageable. However today's democracies are often used to rule over entire nations with millions of eligible voters that has made a direct democracy impossible until now. The counting of the votes of an election takes hours of work throughout the entire country, though technological advancements such as the computers might make a direct democracy possible again in the future.

Indirect democracies take many forms the two most notable are a presidential democracy and a parliamentary democracy.
In a presidential democracy the president of a state is elected either directly by the population or indirectly by representatives of the population and has a significant amount of power in the government. In this type of government the president is both the head of state and the head of government, and will as such represent the country. In this type of government it is often possible, in extreme cases, for the president to dismiss the Parliament, and for the Parliament to remove the president from his/her office which shows how much power both parties have in this type of democracy.
In a parliamentary democracy most of the power is with the Parliament and the head of state is often different from the head of government. The head of government and the head of state are still important, either ceremonial or by leading the agenda and debates in the parliament but they do not have significantly more power then rest of the Parliament.[1]

## 2.2 The evolution of democracy

Democracy knows many forms and the form currently used in the Netherlands has not existed until recently, because in 1970 it changed a little. Since then Dutch voters are not obliged to vote for an election. This change might be a minor one but it shows that a democracy is not one static concept and that there can be multiple different types of democracies.[2]

The first time that is recorded that a democracy was used as a form of government was in the city of Athens in ancient Greece. In the year 510 B.C. Cleisthenes, who would later be remembered as the father of democracy, was important in expelling a tyrant named Hippias out of Athens, after which an oligarchy was put in place. Cleisthenes later on reformed the government and ended the Athens oligarchy to found the first direct democracy.
This first democracy differs a lot from the democracy now used in the Netherlands. In the direct democracy that was used in Athens all free men, that were from Athens and older than 18 years were allowed to vote. Slaves, women, foreigners and children could not vote. These voters would be called upon about 40 times per year to vote or pose their view on a subject that was chosen by leaders of the city. At such a gathering were approximately 6000-8000 men, and they voted by either raising their hand or handing in a bronze plate. For the voters it was also possible to have some control over the leaders of the city, even when the leaders were already in their position. To do this a majority had to vote against that leader and the leader would be exiled for some time. Thus this form of democracy is different in almost every way possible from the form of democracy that is currently used in the Netherlands, except that people could vote to have some kind of influence on the government.

When the Romans took over Europe this first democracy stopped, in Athens and got overtaken by the sole ruler that was the emperor. Democracy would not be used until the Roman empire had shattered and new nations had formed. In 1215 the English King John Lackland signed the Magna Carta which was a small step towards democracies as they are today. One of the things the Magna Carta said was that before the King decide things he had to consult a council with in it two knights of every province. This council can be seen as a predecessor of Parliament today. In following centuries this would gradually give extra rights and freedom to the people in England. In the rest of Europe slowly developed similarly.

In the 18th century some nations, including the Netherlands, had already some kind of council that made decisions for the entire nation. Though these councils were still only available for the rich and mighty people in the nation. In France, however, was an absolute monarchy with Louis XIV as monarch. Since the age of enlightenment had started and people like Jean-Jacques Rousseau and Charles de Montesquieu wrote about how politics should not be decided by the person with most weapons but by the people it effects and how there should be a separation of power in the leading of a nation. These ideas were not accepted by the current rulers but would eventually lead to the French revolution in 1789. The French revolution brought a lot of chaos throughout Europe which led to the rising of some new monarchs. However the ideals of the revolution had spread as well and at the end of the 19th century many countries would introduce some form of democracy.

In 1814 the Netherlands had just become independent from France and a new constitution was written. In this constitution the provincial governments could vote for the house of representatives. This construction was drastically changed in 1848 when the King of the Netherlands agreed to let Johan Rudolf Thorbecke construct a new constitution to prevent a revolution. He was afraid a revolution would take place because that had happened in the neighbouring countries, Belgium en Germany. This new constitution introduced a direct election for the members of the house of representatives and forms the basis of the current Dutch democracy.[3]

At this point only Dutch men that could pay a minimum amount of taxes could vote, this would first change in 1917 such that all Dutch men from a certain age were allowed to vote and two years later it would change again such that from then on all Dutch citizens were allowed to vote.[4]

In the last century many more countries would become democratic and would allow everyone to vote. In 2015 the majority of countries (115 out of 195 countries) around the world were democracies, however in some of these countries the governments developed in such a way that it is hardly possible to still call it a democracy. For example by arresting anyone from different political parties without a clear cause, this way the choice is practically restricted to just one political party.

## 2.3 Plato's and Habermas' view

In this subsection the different views on democracy by the famous Greek philosopher Plato and comparing it to the view Habermas . We will start by looking at Plato's sceptical view on democracy and will then look at the more positive view of Habermas.

When Plato was born in ancient Greece Athens had already known a democracy for about 80 years. Though Plato had another vision on what was the best way of leading a city, in his book Politiea he analyses different governance forms and figures that there are five different types, aristocracy, timocracy, oligarchy, democracy and tyranny. From these five types he concludes that democracy is the single to worst form, just after tyranny. This conclusion of Plato is derived from his statement that the best governance form is the one where everyone does where he is fitted for. And his deduction that in a democracy often the people that are not fit to lead will become the leaders.
This last deduction starts with the way Plato views society, he said that society is an image of the soul. And that the soul is build up from three parts, a desiring, an aspiring and a knowing part. Plato then says that the fitted place for someone in society depends on the dominating part of his soul. In this way the greatest group of people have their soul dominated by the desiring part and should perform an economical function. The second to greatest group of people have their soul dominated by the aspiring part of their soul, and should become the protectors of the city. The last group of people have their soul dominated by the knowing part of their soul, these people should be the leaders and philosophers for a city.
Plato then says that in a democracy the wrong people lead the city. This critic is of course especially applicable on the direct democracy that Athens used, since the people could vote on far more than we can do today in the Dutch indirect democracy where representatives are chosen. Plato further argues that the ordinary people often lack the knowledge needed to govern and will therefore easily be manipulated and convinced by skilled speakers to make the wrong decisions. This can still be seen in many democracies today where populistic parties are often successful while often not having actual plans to fulfil the promises they make. Plato also says that the people will be prone to choosing short term solutions for problems, this often goes hand in hand with the previous argument Plato made. This last argument was also a bit personal for Plato since he had felt the consequences of this in the war against Sparta, where was decided to on the marketplace.
All this made Plato conclude that a democracy is the single to worst option for a governance form.[5]

Jürgen Habermas is a German philosopher that does, contrary to Plato, believe that a democracy is good governance form, though he does think that the form of democracy that existed in Plato's time was at least suboptimal, he would plead for a deliberative constitutional democracy.
A deliberative constitutional democracy is an extension for both the direct democracy as in Athens as for the indirect democracy such as currently in the Netherlands in which there is a space in between the society and the state where everyone could discuss topics freely as equals. He adds to this that people should then also speak truthfully.
In this space between society and the state would be the place for discourse in which everyone would then be able able to speak truthfully and without coercion discuss any topic. In these discussions the power of the argument should be central and not the way these

arguments were displayed. In this way everyone would have gathered enough knowledge about a topic to be able to vote wisely and Habermas concluded that this would make a well working governance form.[6]

Habermas and Plato clearly did not agree on the topic of democracy in general. However both of their conclusions imply that the democracy the Athens used as well as the democracy we currently use in the Netherlands are not the best governance forms.
Furthermore both Plato spoke about democracies that are used to rule over a city while today democracies rule over entire countries and even international collaborations such as the European Union. And on such a large scale other factors will play a roll, when looking at the European Union for example most of the people only know the general statements of a party they are voting for which is far from an ideal fully informed citizen.

# 3 Social Choice Theory

In this section the concept of social choice is going to be introduced and the informal proofs of two famous social choice theorems will be presented. First Arrows Theorem and later the Gibbard-Satterthwaite Theorem will be presented. After that the chapter will be concluded by taking a look at the consequences these theorems have had in mathematics and politics.
This section is based on chapter 2 of the book "Fuzzy Social Choice Theory"[7].

## 3.1 Social Choice

Social Choice is a mathematical model for elections. In those models there is a set of voters that can vote on a set of alternatives and an election system which links all the votes to a result. To make this more formal a few definitions are needed.

Let $N = \{1, 2, 3, ..., n\}$ be a set of actors (the word actor and voter will be used interchangeably) and let $X$ be a set of alternatives, in this thesis it will always be assumed that $|X| \geq 3$.

To be able to say anything about the election system it is essential to know what the voters want to get out of the election in the first place. Therefore there will be assigned a preference relation to each actor, which can be interpreted as what outcome they prefer most, second and so on. In this thesis it will always be assumed that an actor has a preference between each pair of alternatives (this preference does not have to be strict) and that if an actor prefers $x$ to $y$ and $y$ to $z$ that it than also prefers $x$ to $z$ (completeness and transitivity).

**Definition 3.1 (Preference relation).** A preference relation is a reflexive, complete and transitive binary relation, $R$, on the set of alternatives, this is also called a weak order on the set of alternatives. Writing $R_{\mathrm{i}}$ for the preference relation of actor $i$, and if $x, y \in X$ and $(x, y) \in R_{\mathrm{i}}$ write $xR_{\mathrm{i}}y$ and say that actor $i$ prefers $x$ to $y$.

It would be nice to make a distinction between an actor that is indifferent between two alternatives and an actor that has a strict preference for one of two alternatives. Therefore we will introduce symmetric and asymmetric derivation of a preference relation.

**Definition 3.2 ((A)Symmetric derivation).** Given a preference relation, $R$, call $P$ the asymmetric derivation of $R$ if $\forall x, y \in X$ $xPy \iff (xRy$ and not $yRx)$. Call $I$ the symmetric derivation of $R$ if $\forall x, y \in X$ $xIy \iff (xRy$ and $yRx)$.

Often in lemmas and theorems it is assumed that all voters to have a strict preference between all alternatives. Therefore it is practical to define a strict preference relation.

**Definition 3.3 (Strict preference relation).** A strict preference relation is a preference relation that is equal to its asymmetric derivation. (This is equivalent to the relation itself being asymmetric.)

In an election there are $n$ preference relations, one from every actor. All these preference relations together form a preference profile.

**Definition 3.4 (Preference profile).** A preference profile, $\bar{R}$, is an $n$-tuple of preference relations such that $\bar{R}$=($R_1$,$R_2$,...,$R_n$)

To be able to say something about an election system it would be nice to be able to take an arbitrary preference profile and see what can be said about the outcome of the election. Therefore $\mathcal{R}$ will now designate the set of all preference relations (or weak orders) on $X$ and let $\mathcal{R}^n$ designate the set of all preference profiles on $X$.
In some cases it is interesting to only look at the preference of an actor or all actors of a certain subset of the alternatives, $S$. In this case we will restrict $R$ or $\bar{R}$ to $S$ as follows.

**Definition 3.5 (Restriction).** Let $S{\neq}\emptyset$, $S{\subseteq}X$ and $R{\in}\ \mathcal{R}$. There will be written $R\rceil_S$ when looking at $R$ $\forall x, y{\in}S$. And There will be written $\bar{R}\rceil_S$ when looking at the restriction of all $R_i$ to $S$. Thus $\bar{R}\rceil_S{=}(R_1\rceil_S,R_2\rceil_S,...,R_n\rceil_S)$.

Being able to relate every preference profile to an outcome of the election is needed to complete this model. For now it is not desirable to use any specific election type to keep it as general as possible. Therefore only two facts will be used. First that an election has as input $n$ lists of preferences between all alternatives (in the election you don't have to use all information from these lists, but it will be all the information that is given). Note that that these lists of preferences don't have to be exactly the same as the actual preferences of the actors, since actors are allowed to vote different either by mistake or to vote strategically. And second the output of the election will give a list of preferences between all alternatives (also here it is not necessary that every output matters, for example when there is only one winner. But still the entire output is given). We will call our election a Preference Aggregation Rule (PAR) and define it as follows:

**Definition 3.6 (Preference Aggregation Rule (PAR)).** A Preference aggregation rule is a function $f : \mathcal{R}^n{\to}\mathcal{R}$ this will often be abbreviated as PAR. Also sometimes there will be referred to $f(\bar{R})$ as $R$.

The model for an election is now complete. The model will now find its use since it can be restricted to our wishes to create the perfect election system.

## 3.2 Arrows Theorem

Now there is a model for social choice it is interesting to look at the limitations of a PAR. In this subsection there will be looked at some properties a PAR can have, and explain why it would be favourable for an election to fulfil these properties. Then I will present a proof of Arrows Theorem and with that it is shown that there is a certain combination of desired properties for PAR's that cannot be combined in one PAR.

**Definition 3.7 (Unanimity).** Unanimity (or weakly Paretian) of a PAR means that if every actor agrees on $x$ over $y$, then it must be so for the PAR as well. Or more mathematically $f$ is called weakly Paretian if it holds that $\forall\ i{\in}\{1,2,...,n\}\ xR_iy \Rightarrow xRy$ with $R{=}f(\bar{R})$.

Unanimity is clearly a desired quality of an election, since if everyone values an alternative $x$ over $y$ and $y$ is still higher up in the election result, then the places of $x$ and $y$ could be switched in the result and everyone would get a more preferred outcome because of that change.

**Definition 3.8 (Independence of Irrelevant Alternatives (IIA)).** If a PAR is IIA then the ranking of $x$ and $y$ by the PAR depends only on the rankings between $x$ and $y$ by the actors. Or in other words, the position of a third alternative $z$ in the rankings of the actors should not influence the ranking between $x$ and $y$ by the PAR. More mathematically $f$ is called IIA if for any two preference profiles, $\bar{R}$ and $\bar{R}$', $\bar{R}\rceil_{\{x,y\}}{=}\bar{R}'\rceil_{\{x,y\}} \Rightarrow f(\bar{R})\rceil_{\{x,y\}}{=}f(\bar{R}')\rceil_{\{x,y\}}$.

It might not be directly clear why IIA should be desired for an election. But an example can make this clear.
*Example 1:* Suppose there are three alternatives in an election $x, y$ and $z$. Now everyone votes and the outcome is $xRyRz$. Then there is some new information which makes $z$ look like a better alternative for some voters, and there is decided to vote again. In this situation it would make sense that $z$ might become first or second in the second vote, but since no one changed their preference between $x$ and $y$ you would expect that $x$ would still rank above $y$. And this is exactly what IIA demands of the PAR.

**Definition 3.9 (Non-dictatorship).** Non-dictatorship means that there is not one actor that controls the entire election (is the dictator). This translates mathematically to: There is no $k{\in}\{1,2,...,n\}$ such that $\forall x, y{\in}X xR_ky \Rightarrow xRy$ (with $R{=}f(\bar{R})$), this is equivalent to saying $R_k{=}R$ $({=}f(\bar{R}))$.

Non-dictatorship is clearly desired for an election since an election that does not fulfil non-dictatorship has a so called dictator within the group of actors. It is important to note that this actor does not have to be the same in successive votes, therefore it is not a dictator in the same way as there is usually thought about a dictator (it is simply someone that dictates that one election). This still makes an election system extremely fragile since everyone might hate some result except for one actor who loves it and if that actor is the dictator, it will still happen.

Arrows Theorem will show that if a PAR is unanimous and IIA, then it can not also be a non-dictatorship. To prove this it will be shown that if a PAR is unanimous and IIA, then there is always an actor that dictates some part of the election, after which it shown that this same actor then dictates the entire election.

**Theorem 3.10 (Arrows Theorem).** A PAR that is unanimous and IIA is dictatorial.

*Proof:* Let $a, b, c \in X$ be distinct elements of $X$, and let $f$ be a PAR as above. In this proof will first be shown that there is an actor that dictates the choice of $b$ above $c$. After that it will be shown that this same person also dictates the the choice between $b$ and $a$ and the choice between $c$ and $a$. Then will be concluded that this actor has to dictate the entire PAR.

*Step 1:* Let $\bar{R}$ be a preference profile where every actor has $a$ as first preference and $b$ as second preference, thus $\forall i \in \{1, 2, ..., n\}$ $\forall x \in X$ $aR_ix$ and $\forall i \in \{1, 2, ..., n\}$ $\forall y \in X \backslash \{a\}$ $bR_iy$.
By unanimity $f(\bar{R})$ also has $a$ as first preference and $b$ as second preference.
Now construct the sequence of preference profiles $\bar{R} = \bar{R}_0$, $\bar{R}_1$,$\bar{R}_2$, ..., $\bar{R}_n = \bar{R}$' such that $\forall i, j \in \{0, 1, ..., n\}$ $\bar{R}_i \rceil_{X \backslash \{b,a\}} = \bar{R}_j \rceil_{X \backslash \{b,a\}}$ and in the $i$'th step in the sequence actor $i$ changes its preference between $b$ and $a$, such that from then on $b\bar{R}_ia$.
In $\bar{R}$' every actor now prefers $b$ over $a$, and by unanimity it has to hold that $f(\bar{R}$') prefers $b$ first and $a$ second. Because of this, there has to be a $k$ such that $\bar{R}_k$ was the first preference profile in the sequence to switch the preference of the outcome of the PAR from $a$ to $b$. In preference profile $\bar{R}_k$ only actor $k$ changed his preference between $b$ and $a$ with respect to the preference profile $\bar{R}_{k-1}$, therefore actor $k$ will be called the pivot of $b$ over $a$.
From now on there will be referred to the actors $1, ..., k-1$ as Block I and referred to the actors $k+1, ..., n$ as Block II.
Note that by IIA, as long as all actors in Block I and the pivot have $b$ over $a$ and all actors in Block II have $a$ over $b$ then the PAR will have $b$ over $a$. And as long as all actors in Block I have $b$ over $a$ and all actors in Block II and the pivot have $a$ over $b$ then the PAR will have $a$ over $b$. Now choose $c$ arbitrary in $X \backslash \{a, b\}$.

Construct $\bar{Q}$ as follows:
(i) Block I ranks $b$ above $c$ and $c$ above $a$.
(ii) The pivot ranks $a$ above $b$ and $b$ above $c$.
(iii) Block II ranks $a$ above $b$ and $b$ above $c$.
And let $Q$ be $f(\bar{Q})$. Since all actors have kept the relative position of $a$ and $b$ the same as in $\bar{R}_{k-1}$ of the sequence of preference profiles from before, by IIA we know that $aQb$. Furthermore we know by unanimity that $bQc$ and therefore by transitivity that $aQc$.

Now construct $\bar{Q}*$ as follows:
(i) $\bar{Q}*$ is the same as $\bar{Q}$ for all actors accept the pivot.
(ii) $\bar{Q}*$ The pivot ranks $b$ above $a$ and $a$ above $c$.
And let $Q*$ be $f(\bar{Q}*)$ Since all actors have kept the relative positions of $a$ and $b$ the same as in profile $\bar{R}_k$ of step (1) we find that by IIA that $bQ*a$.
in the same way since all actors have kept the relative positions of $a$ and $c$ the same as in profile $\bar{Q}$ we find by IIA that $aQ*c$. And by transitivity that $bQ*c$.
Now Note that that we are allowed to switch the positions of $b$ and $c$ for all actors in block I and block II to an profile $\bar{Q}$' without changing the relative positions of $a, b$ and $c$ of the 'election'. Since the relative position of $b$ to $a$ and $c$ to $a$ stay the same and therefore it

holds by IIA that $b\bar{Q}'a$ and $a\bar{Q}'c$. And thus also by transitivity that $b\bar{Q}'c$.
Now choose $\bar{Q}$' to be:
(i) $c\bar{Q}$'$b$ and $b\bar{Q}'a$ for the actors in block I.
(ii) $b\bar{Q}'a$ and $a\bar{Q}'c$ for the pivot.
(iii) $a\bar{Q}$'$c$ and $c\bar{Q}'b$ for the actors in block II.
The pivot is the only one to have $b\bar{Q}'c$ and still the outcome of the 'election' has $b\bar{Q}'c$. Thus in the same way it is possible to change the preference between $b$ and $c$ for any number of actors in block I and block II and still get $b$ over $c$ as outcome. Therefore by IIA the pivot of $b$ over $a$ dictates the position of $b$ over $c$.

*Step 2:* Now will be shown that the dictator for the position of $b$ over $c$ is the same person that dictates the relative positions of $a, b$ and $c$. This is done by starting over again but now looking for the pivot of $b$ over $c$ and $c$ over $b$, this has to be the dictator between $b$ over $c$. Since, when everyone is preferring $c$ over $b$ and all preferences are switched one by one, just as in step 1. The switch will be $y$ the dictator ot by any actor before the dictator, since he dictates $b$over $c$. When looking at the situation where everyone is preferring $b$ over $c$ and the preference profiles are again switched one by one. Then it has to change by the dictator or by any actor after the dictator, Since he dictates $b$ over $c$. So the pivot is both before or equal to the dictator and after or equal to the dictator. This leaves as only option that the pivot of $b$ over $c$ and $c$ over $b$ is the dictator of $b$ over $c$.
by following the proof from step 1 as before we find that actor $k$ is the dictator for$b$ over $c$ and $c$ and $a$. Now repeating the proof of step 2 will show that actor $k$ is also the pivot between $c$and $a$. Repeating step 1 of the proof again will prove that actor $k$ is the dictator between $a$ and $b$. Therefore by the proof of step 2 actor $k$ has to be the pivot for $a$ over $b$ as well and finally by repeating the proof of step 1 again it is shown that actor $k$ is the dictator between all pairs of alternatives $a, b$ and $c$.

Now since $c$ is chosen arbitrary $k$is the dictator for every pair of alternatives. This makes the PAR a dictatorship and therefore it is proven that a PAR that is unanimous and IIA has to be a dictatorship. □

By proving that Arrows theorem holds, it has been shown that it is at least complicated to form a good election system, and that it is not possible for any election system to fulfil every wanted quality.

## 3.3 Gibbard-Satterthwaite theorem

In the previous section was shown that Arrows theorem holds and that this restricts a PAR from having three desired qualities at the same time. This section will present a proof of the Gibbard-Satterthwaite theorem which makes a similar statement but now for three properties of a voting rule. First the needed definitions will be given and then a proof of the Gibbard-Satterthwaite theorem will be presented.

In this paragraph it is assumed that the preference relation for each actor is strict.
**Definition 3.11 (Utility function).** A utility function is a function, $u$: $X \to \mathbb{R}$, that maps each alternative to a real number.

A utility function always corresponds with a preference relation in the following way. Given a utility function $u$ construct the preference relation $R$ as $xRy \iff u(x)>u(y)$. The set of all possible utility functions will be denoted by $U$ and $U^{\mathrm{n}}$ by $\mathcal{U}$. Thus every preference profile is now equivalent to an element of $\mathcal{U}$, such that $\bar{u}=(u_1, u_2, ..., u_{\mathrm{n}})$. When only looking at the first $i$ utility functions of a preference profile it is often written as $u_{\mathrm{i}}$, and when looking at the other utility functions (the last $n-i$) write $u_{\text{-i}}$.

**Definition 3.12 (Voting rule).** A voting rule is a mapping $f$: $\mathcal{U} \to X$. Where $X$ is the set of alternatives.

Note: a voting rule is different from a PAR because a voting rule only selects one element whereas a PAR can form an entire preference relation. When thinking of elections this is comparable to a presidential election where only one candidate wins.

**Definition 3.13 (Strategy proof).** A voting rule is called strategy proof if for every actor the most preferred alternative is selected when using there true utility function.

Another way of defining a strategy proof voting rule is by first defining a manipulatable voting rule. A voting rule is manipulatable if there is some actor $i$ who can alter his utility function $u_{\mathrm{i}}$ to $u_{\mathrm{i}}'$ and getting a preferred outcome, thus $u_{\mathrm{i}}(f((u_{\mathrm{i}}', u_{\text{-i}})))>u_{\mathrm{i}}(f(\bar{u}))$. When thinking of elections again this means it is impossible to vote strategically, this is preferred for an election since an election tries to transfer the will of the people into a result. In this case we could thus define strategy proof as not manipulatable.

**Definition 3.14 (Monotonicity).** A PAR is called monotonous when the ranking of an alternative $x$ can not decrease if an actor increases the position of $x$ in their ranking.

Note: monotonicity is a quality of a PAR but since a voting rule can be seen as a PAR in which only the first place in the outcome matters it is also a quality of a voting rule.

**Theorem 3.15 (Gibbard-Satterthwaite).** A voting rule that is onto and strategy proof is dictatorial.

*Proof:* Let $f$ be a voting rule that is onto, and strategy proof. In three steps will be proven that $f$ is dictatorial. It will be shown that $f$ is monotonous, unanimous, and IIA. When this is all proven it follows from Arrow's theorem (Theorem 1.10) that the voting rule has to be dictatorial.

*Step 1 (Strategy proof ⇒ Monotonicity):* For $\bar{u}\in\mathcal{U}$ suppose that $f(\bar{u})=a$ and suppose that $\bar{v}\in\mathcal{U}$ such that $\forall x\in X$ and $i\in\{1,2,...,n\}$ it holds that $u_i(a)\geq u_i(x) \Rightarrow v_i(a)\geq v_i(x)$.
Now construct the sequence of utility profiles $\bar{w}_0$, $\bar{w}_1$, ..., $\bar{w}_n$ such that $\bar{w}_0=\bar{u}$ and $\bar{w}_n=\bar{v}$ by changing the utility profile of actor $i$ from $u_i$ to $v_i$ in the $i$'th step of the sequence.
assume $f(\bar{w}_i)=a$and $f(\bar{w}_{i+1})=b$. Then it must hold that $v_i(a)\leq v_i(b)$ and $u_{\rm i}(b)\leq u_{\rm i}(a)$, or else actor $i$ could strategically promote the favoured $a$, or $b$ by substituting preference $u_1$ for $v_1$ in $\bar{w}_i$ or $v_1$ for $u_1$ in $\bar{w}_{i+1}$ to get a for actor $i$ more preferred result which would contradict that $f$ is strategy proof of. Because $u_{\rm i}(a)\geq u_{\rm i}(x) \Rightarrow v_{\rm i}(a)\geq v_{\rm i}(x)$ $u_{\rm i}(b)\leq u_{\rm i}(a) \Rightarrow v_{\rm i}(b)\leq v_{\rm i}(a)$. Thus $v_{\rm i}(b)\leq v_{\rm i}(a)$ and $v_{\rm i}(b)\geq v_{\rm i}(a)$ therefore it has to be that $v_{\rm i}(b) = v_{\rm i}(a)$ because there is a strict preference it has to be that $a = b$.
This is true for every $i$ and $f(\bar{w}_0)=f(\bar{u})=a$ we know by induction that $f(\bar{v})=f(\bar{w}_n)=a$. This proves that $f$ is monotonous.

*Step 2 (Strategy proof ⇒ Unanimity):* For $\bar{u}\in\mathcal{U}$ suppose that there are two distinct elements $a, b\in X$ such that $\forall i\in\{1,2,...,n\}$ $u_{\rm i}(a)>u_{\rm i}(b)$.
Suppose $f(\bar{u})=b$. Since $f$ is onto we know that there is a profile, $\bar{v}$, such that $f(\bar{v})=a$.
Now create the preference profile $\bar{w}_0$ such that $\forall i\in\{1,2,...,n\}$ and $\forall x\in X\backslash\{a,b\}$ $w_{\rm i}(x) = u_{\rm i}(x)$ and assign large utility to $a$and $b$such that $\forall x\in X\backslash\{a,b\}$ $w_{\rm i}(a)>w_{\rm i}(b)>w_{\rm x}$.
By the monotonicity of $f$ (step 1) it follows that $f(\bar{w}_0)=f(\bar{u})=b$.
Now create the sequence of profiles starting from $\bar{w}_0$ to $\bar{w}_n$ by switching $u_{\rm i}(x)$ to $v_{\rm i}(x)$ $\forall x\in X\backslash\{a,b\}$ in step $i$. By the monotonicity of $f$ (step 1) it follows that $f(\bar{w}_n)=f(\bar{v})=a$. Thus somewhere in the sequence there has to be a pivotal actor who changes the outcome from $b$ to $a$. But now it is possible for this actor to vote strategically, which is in contradiction with $f$ being strategy proof. Therefore we conclude that $f(\bar{u})\neq b$, and thus $f$ is unanimous.

*Step 3 (Strategy proof ⇒ IIA):* Suppose that $f(\bar{u})=a$. Let some actor $i$, decrease its utility for choice $z$. By monotonicity (step 1), $a$ must remain the result of the voting rule. Now suppose actor $i$ increases its utility for $z$. Suppose the winning candidate is $b$ when actor $i$ uses preference $v$. Let $w_{\rm i}(b) = min(u_{\rm i}(b), v_{\rm i}(b))$. Under the preference $\bar{u}$ then $a$ is chosen, so by monotonicity $a$ is still selected with the profile $(w_{\rm i},\bar{u}_{-i})$. However under the preference $(v_{\rm i},\bar{u}_{-i})$ candidate $b$ is elected. And thus, by monotonicity, $b$ is elected in $(w_{\rm i},\bar{u}_{-i})$. Therefore it has to be that $a = b$, and thus $f$ is IIA.

Thus the voting rule is unanimous and IIA if it would also be non-dictatorial we would have a contradiction with Arrow's Theorem (theorem 1.10). Thus is concluded that the voting rule is dictatorial. □

By showing this it is even more clear then it already was by Arrows theorem that it can be really difficult to find a good election system. And when comparing these theorems to elections in real life it is often possible to, at least partially, understand why there is discontent after any election.

## 3.4 Online ratings

Today online ratings are frequently used, they can be found easily when searching for a good restaurant, club or anything else. In an online rating people can express there opinion on something by giving it a rating. The rating is often in the form of stars, with one star being terrible and five stars being (nearly) perfect. The final rating becomes the average of all ratings. The main reason people use online ratings is to find the best option out of several alternatives. When looking at a set of persons as alternatives an online rating has suddenly become some kind of election system. But would it be a good one?

An online rating is unanimous. Since, if everyone thinks one alternative is better than another alternative everyone will give more stars to that alternative and the average of all ratings, and thus the final rating, will become higher than the less liked alternative.

Online ratings are also IIA since if there has been voted and alternative A scores higher than an alternative B then it does not matter if the votes for a third alternative C change for the outcome of alternative A over alternative B. Since the averages of the votes for alternative A and alternative B stay the same as during the first vote.

If Arrow's theorem would hold for online ratings this would imply that online ratings have to be a dictatorship since it is unanimous and IIA. However it is easily checked that this is not the case. Suppose one person gives alternative A five stars and alternative B one star, and a second person gives alternative A one star and alternative B 3 stars. Then the outcome would be that alternative A has an average of three stars and alternative B has an average of two stars, thus only the first person can be the dictator since the second person already has a different vote than the outcome. But if the first person now changes his vote such that alternative A gets five stars and alternative B gets four stars then the outcome would be three stars for alternative A and 3,5 stars for alternative B. Note that this is not the same as the first person voted, thus there is no dictator.
This might seem like a contradiction with Arrow's theorem but it definitely is not. This is because the model on which Arrow's theorem is based does not include an election system like an online rating. The main difference is that in an online rating there is more possible input. First you can let two alternatives be equally good and secondly every voter can choose when an alternative becomes worse and it deserves less stars. Because of this it would also be necessary to adjust the definitions of unanimity, IIA and dictatorship before it is possible to claim that the result of Arrow's theorem can be completely avoided.

The extension of the model for election systems and adjusted definitions are given by extending this model with fuzzy logic to form fuzzy social choice theory. These will be presented in the next two sections.

# 4 Fuzzy Logic

In this section the basis of fuzzy logic will be presented. Then a model will be made to interpret the preferences of actors in a fuzzy manner. While making this model there are several options which would all be in line with the model that was made in section 1. In section 3 will be shown that these and some other choices, will be essential when examining Arrows theorem.
This section is based on chapter 1 and 3 of the book "Fuzzy Social Choice Theory"[7].

## 4.1 Fuzzy Sets

This subsection will focus on the basis of fuzzy logic and will provide the fuzzy interpretation of basic mathematical objects like sets, set-operations, relations and will introduce T-(co)norms.

In fuzzy logic every element of a set corresponds, through a function, to a real number in the interval $[0, 1]$. This number is interpreted as the degree of membership of an element of the set, to the set. In which 0 is not a member, 1 is a member and everything in between is partially a member. This function is also called a fuzzy subset of the initial set since the elements of the set are generally not all members of the set according to the function.

**Definition 4.1 (Fuzzy subset).** Let $S$ be a set and let $\mu$: $S \rightarrow [0, 1]$ be a function. Then $\mu$ is a fuzzy subset.

To be able to compare different fuzzy subsets the following definition is introduced:

**Definition 4.2 ($\subseteq$, $\subset$ and $=$).** Let $\mu$ and $\nu$ be fuzzy subsets of $S$. Then write:
(i) $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ $\forall$ $x \in S$;
(ii) $\mu \subset \nu$ if $\mu \subseteq \nu$ and $\exists x \in S$ such that $\mu(x) < \nu(x)$;
(iii) $\mu = \nu$ if $\mu \subseteq \nu$ and $\nu \subseteq \mu$.

For fuzzy subsets the definition for unions and intersections are less straight forward compared to regular sets though they are still important to be able to compare operations on fuzzy subsets to operations on regular sets.

**Definition 4.3 (Union and intersection ).** Let $\mu$ and $\nu$ be fuzzy subsets of $S$. Then:
(i) The union of $\mu$ and $\nu$ is the fuzzy subset $\mu \cup \nu$ of $S$ such that $\mu \cup \nu(x) = max(\mu(x), \nu(x))$ $\forall x \in S$;
(ii) the intersection of $\mu$ and $\nu$ is the fuzzy subset $\mu \cap \nu$ of $S$ such that $\mu \cap \nu(x) = min(\mu(x), \nu(x))$ $\forall x \in S$.

In fuzzy logic you could also define a union differently but this is the most used version of a union in standard fuzzy logic, in the next subsection we will also take a look at different unions that are often used in fuzzy social choice theory.

In fuzzy logic there is not only a degree of membership for sets but also for for relations. (elements of sets can relate very well, a bit or not at all). This gives rise to new definitions for fuzzy relations, compositions of these relations and some properties of these relations.

**Definition 4.4 (Fuzzy relation).** Let $S$ and $T$ be sets. A fuzzy relation $\rho$ of $S$ into $T$ is a function: $S{\times}T{\rightarrow}[0,1]$. Such a relation of $S$ into itself is called a fuzzy relation on $S$.

**Definition 4.5 (Composition).** Let $S,T$ and $U$ be sets and $\rho$ be a fuzzy relation of $S$ into $T$ and $\sigma$ a fuzzy relation of $T$ into $U$. Define the composition of $\rho$ with $\sigma$ as $\rho{\circ}\sigma$: $S{\times}U{\rightarrow}[0,1]$ with:
$\forall(x,z){\in}S{\times}U\ \rho{\circ}\sigma(x,z){=}max_{y\in T}\ (min(\rho(x,y),\sigma(y,z)))$.

**Definition 4.6 (Reflexive, max-min transitive, symmetric, asymmetric and complete).** Let $\rho$ be a fuzzy relation on $S$. Then $\rho$ is called:
(i) reflexive if $\rho(x,x){=}1\ \forall x{\in}S$;
(ii) max-min transitive if $\rho(x,z){\geq}max_{y\in S}\ (min(\rho(x,y),\rho(y,z)))\ \forall x,z{\in}S$;
(iii) symmetric if $\rho(x,y){=}\rho(y,x)\ \forall x,y{\in}S$;
(iv) asymmetric if $\rho(x,y){>}0{\Rightarrow}\rho(y,x){=}0\ \forall x,y{\in}S$;
(v) complete if $max(\rho(x,y),\rho(y,x)){>}0\ \forall x,y{\in}S$.

In the following subsection a relation is going to be split up in two components. To do this we will need some special operation that can combine the two components to the original relation. Therefore the following definitions are both needed.

**Definition 4.7 (T-norm).** A T-norm $i$ is a binary operation on the interval $[0,1]$ such that $\forall a,b,c{\in}[0,1]$ the following conditions are satisfied:
(i)$i(a,1)=a$
(ii) $b{\leq}c{\Rightarrow}i(a,b){\leq}i(a,c)$
(iii) $i(a,b)=i(b,a)$
(iv) $i(a,i(b,c))=i(i(a,b),c)$

An example of a T-norm is the operation $min(x,y)$.

**Definition 4.8 (T-conorm).** A T-conorm $u$ is a binary operation on the interval $[0,1]$ such that $\forall a,b,c{\in}[0,1]$ the following conditions are satisfied:
(i) $u(a,0)=a$;
(ii) $b{\leq}c{\Rightarrow}u(a,b){\leq}u(a,c)$;
(iii) $u(a,b)=u(b,a)$;
(iv) $u(a,u(b,c))=u(u(a,b),c)$.

An example for such a T-conorm is the operation $max(x,y)$.

The principles of fuzzy logic are now defined and ready to be used to construct a fuzzy model of an election.

## 4.2 Fuzzy Preference Relations

In this subsection a fuzzy model will generalize the model made in section 1. Starting by defining fuzzy weak preference relations, then splitting that up into its indifference and strict preference components after which we will examine the properties of these components using different, frequently used, unions.

In this subsection is assumed, just as in section 1, that $X$ is a finite set of alternatives with $|X|\geq 3$.

**Definition 4.9 (Fuzzy Weak Preference Relation (FWPR)).** A Fuzzy Weak Preference Relation, or often FWPR, is a function $\rho$: $X\times X\rightarrow[0,1]$. Or in other words, a fuzzy relation on $X$.

A FWPR $\rho(x, y)$ can be interpreted as the degree in which $x$ is at least as good as $y$, in which $\rho(x, y) = 0$ is interpreted as $x$ is not at least as good as $y$ and $\rho(x, y) = 1$ as $x$ is at least as good as $y$. In the case where $\rho$ can only be 0 or 1 this is a weak preference relation just as in section 1.

**Definition 4.10 (Components).** It is said that $\iota$ and $\pi$, both fuzzy weak preference relations, are components of a FWPR $\rho$ if there exists some t-conorm $\cup$: $[0,1]\times[0,1]\rightarrow[0,1]$ such that $\rho{=}\iota\cup\pi$.

In theory, $\iota$ and $\pi$ represent indifference and strict preference respectively. Therefore, from now on, will be assumed that $\iota$ is symmetric and $\pi$ is asymmetric. To be able to say more about $\iota$ and $\pi$ there will be some assumptions on the operation $\cup$, these are called the union axioms and will from now on be assumed.

**Definition 4.11 (Union axioms).** Let $A, B, C$ and $D$ be fuzzy subsets of $S$ and let $\cup$: $[0,1]\times[0,1]\rightarrow[0,1]$. Then $\cup$ is said to satisfy the union axioms if the following holds:
(i) $B(x) = 0 \Rightarrow (A\cup B)(x) = A(x)$;(boundary condition)
(ii) $A(x)\geq B(x)$ and $C(x)\geq D(x) \Rightarrow (A\cup C)(x)\geq(B\cup D)(x)$. (monotonicity)

With these extra assumptions it is possible to show what function describes $\iota$ as will now be shown.

**Proposition 4.12 ($\iota(x, y) = min(\rho(x, y),\rho(y, x))$).** Let $\rho$ be a FWPR and suppose that $\rho{=}\iota\cup\pi$ then $\iota(x, y) = min(\rho(x, y),\rho(y, x))$.

*proof:* Let $x, y\in X$. By the monotonicity of $\cup$ we know that $\iota(x, y)\cup\pi(x, y)\geq\iota(x, y)\cup 0$, when choosing $A(x, y) = \iota(x, y)$, $B(x, y) = \iota(x, y)$, $C(x, y) = \pi(x, y)$ and $D(x, y) = 0$. By the boundary condition of $\cup$ we know that $\iota(x, y)\cup 0{=}\iota(x, y)$ And by the assumption we know that $\rho(x, y){=}\iota(x, y)\cup\pi(x, y)$. Thus we know that $\rho(x, y)\geq\iota(x, y)$. This argument is symmetric thus we also know that $\rho(y, x)\geq\iota(y, x)$. And therefore, since $\iota$ is symmetric $\iota(x, y){=}\iota(y, x)\leq min(\rho(x, y),\rho(y, x))$.

Since $\pi$ is asymmetric we know that either $\pi(x, y) = 0$ or $\pi(y, x) = 0$. Suppose $\pi(x, y) = 0$ then, since we assumed that $\rho{=}\iota\cup\pi$, $\rho(x, y) = \iota(x, y)\cup 0 = \iota(x, y)$ by the boundary conditions of $\cup$. Hence $\iota(x, y) = min(\rho(x, y),\rho(y, x))$. Now suppose $\pi(x, y)\neq 0$ then $\pi(y, x) = 0$

and we can repeat the argument to find again that $\iota(x, y) = min(\rho(x, y),\rho(y, x))$.
□

At this point there are a lot of different operations that can be used as $\cup$. To be able to have one function that is $\pi$ more information is needed. Therefore two frequently used operations for $\cup$ will now be given and then will be proven what the corresponding functions, to these union operations with some extra assumptions, are that describe $\pi$.

**Definition 4.13 (Lukasiewicz union $\cup_1$).** Let $A, B$ be fuzzy subsets, the Lukasiewicz union, $\cup_1$, is defined as:
$(A\cup_1 B)(x){=}min(1, A(x) + B(x))$.

**Definition 4.14 (Strict union $\cup_2$).** Let $A, B$ be fuzzy subsets, the Strict union $\cup_2$, is defined as:

$$(A \cup_2 B)(x) = \begin{cases} B(x) & \text{if } A(x){=}0 \\ A(x) & \text{if } B(x){=}0 \\ 1 & \text{otherwise} \end{cases}$$

**Proposition 4.15 ($\cup_1 \Rightarrow \pi_{(1)}$).** Let $x, y \in X$and let $\rho$ a FWPR such that
(i) $\rho{=}\iota\cup_1\pi$, $\iota$ is symmetric and $\pi$ is asymmetric,
(ii) $\iota(x, y){+}\pi(x, y){\le}1$,
(iii) $\rho(x, y){<}1$ implies $\pi(y, x){>}0$.
Then $\pi(x, y)$ = 1-$\rho(y, x)$.

*proof:* by (iii) is known that if $\rho(x, y){<}1{\Rightarrow}\pi(y, x){>}0$ by (i) we then know that $\pi(x, y) = 0$ and therefore by (iii) it has to be that $\rho(y, x) = 1$.
Suppose without loss of generality that $\rho(x, y){\le}\rho(y, x)$, then $\rho(y, x) = 1$.
Now consider two cases, 1) $\rho(x, y) {=}\rho(y, x) = 1$ and 2) $\rho(x, y){<}\rho(y, x) = 1$

Case 1 $\rho(x, y) {=}\rho(y, x) = 1$
then, by proposition 4.12, $\iota{=}min(\rho(x, y),\rho(y, x)) = 1$.
So by (ii) it holds that $\pi(x, y) {=}\pi(y, x) = 0 =$ 1-$\rho(y, x) = 1{-}\rho(x, y)$

case 2 $\rho(x, y){<}\rho(y, x) = 1$
by (i) $\rho(x, y) = min(1{,}\iota(x, y){+}\pi(x, y))$ and then by (ii) $\rho(x, y){=}\iota(x, y){+}\pi(x, y)$.
Now as assumed $1\ {=}\rho(y, x)$ and thus $1\ {=}\iota(y, x){+}\pi(y, x)$ now , by proposition 4.12, $1\ {=}\iota(y, x){+}\pi(y, x)\ {=}\rho(x, y){+}\pi(y, x)$ and therefore $\pi(y, x) = 1{-}\rho(x, y)$
and because $\rho(y, x) = 1$ (iii) shows that $\pi(x, y) = 0$ and therefore $\pi(x, y) = 1{-}\rho(y, x)$.

Therefore it holds that $\pi(y, x) = 1{-}\rho(x, y)$. □

From now on this specific function for $\pi$ will be denoted as $\pi_{(1)}$.
To get to this $\pi_{(1)}$ some extra assumptions were needed but even without those you can say a lot about the function of $\pi$ when using $\cup_1$ as union, as will be shown in the following proposition.

**Proposition 4.16** **($\cup_1 \Rightarrow \pi_{(2)}$).** Let $\rho$ a FWPR such that
(i) $\rho = \iota \cup_1 \pi$, $\iota$ is symmetric and $\pi$ is asymmetric,
Then $\iota(x,y) + \pi(x,y) \leq 1 \iff \pi(x,y) = max(0, \rho(x,y) - \rho(y,x))$.

*proof:* By (i) $\rho(x,y) = (\iota \cup_1 \pi)(x,y) = min(1, \iota(x,y) + \pi(x,y))$
Now suppose $\iota(x,y) + \pi(x,y) \leq 1$ $min(1, \iota(x,y) + \pi(x,y)) = \iota(x,y) + \pi(x,y)$.
It now follows that $\pi(x,y) = \rho(x,y) - \iota(x,y)$ and by proposition 4.12
$\rho(x,y) - \iota(x,y) = \rho(x,y) - min(\rho(x,y), \rho(y,x)) = max((\rho(x,y) - \rho(x,y)), (\rho(x,y) - \rho(y,x))) = max(0, (\rho(x,y) - \rho(y,x)))$.
And thus one implication holds.

Suppose now that $\pi(x,y) = max(0, (\rho(x,y) - \rho(y,x)))$. Then, by proposition 4.12, it holds that
$\pi(x,y) + \iota(x,y) = max(0, (\rho(x,y) - \rho(y,x))) + min(\rho(x,y), \rho(y,x)) = \rho(x,y) \leq 1$hence, $\iota(x,y) + \pi(x,y) \leq 1$.
Therefore the implication holds both ways and the proposition is proven. □

From now on this specific function for $\pi$ will be denoted as $\pi_{(2)}$.

**Proposition 4.17** **($\cup_2 \Rightarrow \pi_{(3)}$).** Let $\rho$ be a FWPR such that:
(i) $\rho = \iota \cup_2 \pi$, $\iota$ is symmetric and $\pi$ is asymmetric.
(ii) $\pi(x,y) > 0 \Rightarrow \iota(y,x) = 0$.
Then $\forall x, y \in X$ it holds that:

$$\pi(x,y)) = \begin{cases} \rho(x,y) & \text{if } \rho(y,x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

*proof:* Suppose that $\rho(y,x) = 0$. Then by proposition 4.12 $\iota(x,y) = min(\rho(x,y), \rho(y,x)) = 0$. Thus $\rho(x,y) = \pi(x,y)$ by (i).
Suppose that $\rho(y,x) \neq 0$. Then by the (i) and (ii) we know that $\pi(x,y) > 0 \Rightarrow (\pi(y,x) = 0 = \iota(y,x)) \Rightarrow \rho(y,x) = 0$. The contraposition of this states that $\rho(y,x) \neq 0 \Rightarrow \pi(x,y) = 0$.
Therefore we conclude that $\pi$ fulfils the above equation. □

From now on this specific function for $\pi$ will be denoted as $\pi_{(3)}$.

Sometimes it is not necessary to know the specific function for $\pi$ in that case there are often other properties of the function $\pi$ that are needed. Because of that, regularity will be defined.

**Definition 4.18 (Regular).** Let $\rho$ be a FWPR. Then its asymmetric part $\pi$ is called regular if $\pi(x,y) > 0 \iff \rho(x,y) > \rho(y,x)$.

Examples of a regular preference relations $\pi$ are $\pi_{(1)}$ and $\pi_{(2)}$.

In the classical setting every preference relation is transitive, therefore transitivity should be translated to the fuzzy setting. In the fuzzy setting there are however multiple ways to define transitivity that reduce to the classical form of transitivity when looking at them for fuzzy preference relations that correspond to classical preference relations.

**Definition 4.19 (Max-star transitive/ weakly transitive/ partially quasi/ partially transitive).** Let $\rho$ a FWPR. Then $\rho$ is called:
(i) max-star transitive if $\forall x, y, z \in X \ \rho(x,z) \geq \rho(x,y) * \rho(y,z)$;
(ii) weakly transitive if $\forall x, y, z \in X \ (\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)) \Rightarrow \rho(x,z) \geq \rho(z,x)$;
(iii) partially transitive if $\forall x, y, z \in X \ (\rho(x,y) > 0$ and $\rho(y,z) > 0) \Rightarrow \rho(x,z) > 0$;
(iv) partially quasi transitive if $\forall x, y, z \in X (\pi(x,y) > 0$ and $\pi(y,z) > 0) \Rightarrow \pi(x,z) > 0$

In the definition above the star in max-star transitivity can be any relation, thus this needs to be specified when using that kind of transitivity.
Now that there is a clear view on fuzzy preference relations the next section will combine these in fuzzy preference profiles and show the implications these fuzzyfications have on Arrows theorem.

# 5 Fuzzy Social Choice Theory

This Section will conclude the thesis It will start by defining several fuzzy interpretations of the Arrowian conditions. Then multiple proofs will be presented to show that certain combinations of fuzzy Arrowian conditions will make Arrows theorem hold and others will. Finally there will be looked at some political election systems that are used today and what these theorems show about those political election systems.
This section is based on chapter 4 of the book "Fuzzy Social Choice Theory"[7].

## 5.1 Fuzzy Arrowian Conditions

In this subsection several fuzzy interpretations of the Arrowian conditions will be defined. These interpretations will later on in this section be used to formulate the concluding proofs of this thesis.

In this section will be assumed that $N = \{1, 2, ..., n\}$ is a finite set of actors where $n \geq 2$, $X$ is a finite set of alternatives such that $|X| \geq 3$ and that every actor possesses a FWPR, $\rho_i$, such that $\rho_i$ is reflexive and complete. Now $\rho_i$ is called a fuzzy weak order and the set of fuzzy weak orders on $X$ is denoted as $\mathcal{FR}$ . which gives rise to the following definitions.

**Definition 5.1 (Fuzzy preference profile).** A fuzzy preference profile is a set of $n$ fuzzy weak orders, and thus an element of $\mathcal{FR}^n$.

**Definition 5.2 (Fuzzy Preference Aggregation Rule (FPAR)).** A fuzzy preference aggregation rule is a function $\bar{f}$: $\mathcal{FR}^n \rightarrow \mathcal{FR}$. This is often abbreviated as FPAR

In the non fuzzy case a PAR is always transitive, in the fuzzy case transitivity can have several meanings which will now be defined.

**Definition 5.3 (Weakly transitive, partially transitive and partially quasi transitive).** An FPAR $\bar{f}$ is called
(i) weakly transitive if $\forall \bar{\rho} \in \mathcal{FR}^n$, $\bar{f}(\bar{\rho})$ is weakly transitive;
(ii) partially transitive if $\forall \bar{\rho} \in \mathcal{FR}^n$, $\bar{f}(\bar{\rho})$ is partially transitive;
(iii) partially quasi transitive if $\forall \bar{\rho} \in \mathcal{FR}^n$, $\bar{f}(\bar{\rho})$ is partially quasi transitive.

Not every FWPR is transitive, this is why often the domain of an FPAR is restricted. A domain that will often be used is $D_w$ which is the domain of all weakly transitive fuzzy weak orders. Such an FPAR will then be denoted as $\bar{f}$: ${D_w}^n \rightarrow \mathcal{FR}$.
Now the different fuzzy definitions of the Arrowian conditions will be given, starting with a fuzzy version of unanimity.

**Definition 5.4 (Weakly Paretian).** An FPAR, $\bar{f}$, is said to be weakly Paretian if $\forall \bar{\rho} \in \mathcal{FR}^n$ and $\forall x, y \in X$, $min_{i \in N}(\pi_i(x, y)) > 0 \Rightarrow \pi(x, y) > 0$.

**Definition 5.5 (Pareto condition).** An FPAR, $\bar{f}$, is said to satisfy the Pareto condition if $\forall \bar{\rho} \in \mathcal{FR}^n$ and $\forall x, y \in X$, $\pi(x, y) \geq min_{i \in N}(\pi_i(x, y))$

It is important to notice that the Pareto condition directly implies that an FPAR is weakly

Paretian. This will be a stronger assumption when forming a fuzzy version of Arrows theorem and a stronger result when showing that a fuzzy version of Arrows theorem does not hold.
Now some different definitions of fuzzy IIA will be presented.

**Definition 5.6 (IIA-1).** An FPAR, $\bar{f}$ is called independent of irrelevant alternatives, type 1 (IIA-1), if $\forall \bar{\rho}, \bar{\rho}' \in \mathcal{FR}^n$, $\forall\, x, y \in X$ and $\forall i \in N$ $\rho_i(x,y) = \rho'_i(x,y) \Rightarrow \bar{f}(\bar{\rho})(x,y) = \bar{f}(\bar{\rho}')(x,y)$.

In the second definition of the IIA condition the term support is used. The support of a function, $f$, is a restriction of the domain of the function $f$ to the points where $f$ is not zero, this is denoted as $Supp(f)$.

**Definition 5.7 (IIA-2).** An FPAR, $\bar{f}$ is called independent of irrelevant alternatives, type 2 (IIA-2), if $\forall \bar{\rho}, \bar{\rho}' \in \mathcal{FR}^n$, $\forall\; x, y \in X$ and $\forall i \in N$ $Supp(\rho_i\rceil_{\{x,y\}}) = Supp(\rho'_i\rceil_{\{x,y\}}) \Rightarrow Supp(\bar{f}(\rho_i)\rceil_{\{x,y\}}) = Supp(\bar{f}(\rho'_i)\rceil_{\{x,y\}})$

To define a third version of the IIA condition first another property will be introduced.

**Definition 5.8 (Equivalent).** Let $\rho$,$\rho'$ be FWPRs and let $Im(\rho) = \{s_1, s_2, ..., s_n\}$ and $Im(\rho') = \{t_1, t_2, ..., t_{\mathrm{m}}\}$ now order the elements of $Im(\rho)$ and $Im(\rho')$ such that $s_1 < s_2 < ... < s_{\mathrm{n}}$ and $t_1 < t_2 < ... < t_{\mathrm{n}}$. Then $\rho$ and $\rho'$ are equivalent, written $\rho \sim \rho'$, if and only if
(i) $s_1 = 0 \iff t_1 = 0$;
(ii) $n = m$;
(iii) $\forall i \in N$ $\{x \in X \mid \rho(x) \geq s_{\mathrm{i}}\} = \{x \in X \mid \rho'(x) \geq t_{\mathrm{i}}\}$.

**Definition 5.9 (IIA-3).** An FPAR, $\bar{f}$ is called independent of irrelevant alternatives, type 3 (IIA-3). If $\forall \bar{\rho}, \bar{\rho}' \in \mathcal{FR}^n$, $\forall\, x, y \in X$ and $\forall i \in N$ $\rho_i\rceil_{\{x,y\}} \sim \rho'_i\rceil_{\{x,y\}} \Rightarrow \bar{f}(\bar{\rho})\rceil_{\{x,y\}} \sim \bar{f}(\bar{\rho}')\rceil_{\{x,y\}}$.

Now will be defined what a dictatorial FPAR is.

**Definition 5.10 (Dictatorial).** An FPAR, $\bar{f}$ is called dictatorial if there exist a $i \in N$ such that $\forall \bar{\rho} \in \mathcal{FR}^n$ and $x, y \in X$ $\pi_i(x,y) > 0 \Rightarrow \pi(x,y) > 0$

In the following subsections these conditions will be combined into fuzzy versions of Arrow's theorem.

## 5.2 Fuzzy Arrow Holds

In this subsection will be proven that with certain combinations of the fuzzy Arrowian conditions, described in the previous subsection, Arrows theorem still holds. To do this some lemmas will first be proven.

To prove the theorems later on in this subsection will be shown that there is a subset of all actors that together are, in some sense, dictatorial and then will be shown that this subset contains only 1 actor. Therefore two definitions will be given.

**Definition 5.11 (Semidecisive).** Let $\bar{f}$ be an FPAR, $x, y{\in}X$ and $\lambda$ be a fuzzy subset of $N$. Then $\lambda$ is called semidecisive for $x$ against $y$, written $x\bar{D}_\lambda y$, if $\forall\bar{\rho}{\in}\mathcal{FR}^n$ $(\forall i{\in}Supp(\lambda)\ \forall j{\notin}Supp(\lambda)\ \pi_i(x,y){>}0$ and $\pi_j(y,x){>}0) \Rightarrow \pi(x,y){>}0$.

**Definition 5.12 (Decisive).** Let $\bar{f}$ be an FPAR, $x, y{\in}X$ and $\lambda$ be a fuzzy subset of $N$. Then $\lambda$ is called decisive for $x$ against $y$, written $xD_\lambda y$, if $\forall\bar{\rho}{\in}\mathcal{FR}^n$ $(\forall i{\in}Supp(\lambda)\ \pi_i(x,y){>}0) \Rightarrow \pi(x,y){>}0$.

The definitions show clearly that decisiveness and semidecisiveness are strongly related and that decisiveness⇒semidecisiveness. Not immediately clear is how much can be said about the two consents if there are some extra assumptions on the FPAR. To show this the following lemma will be proven.

**Lemma 5.13 (Semidecisive⇒decisive).** Let $\lambda$ be a fuzzy subset of $N$, $\bar{f}$ be a partially quasi-transitive FPAR that is weakly Paretian and either IIA-3 with $\pi$ regular or IIA-2 with $\pi{=}\pi_{(3)}$. If $\lambda$ is semidecisive for $x$ against $y$, then $\forall v, w{\in}X$ $\lambda$ is decisive for $v$ against $w$.

*proof:* Let everything be defined as above. This will be proven in two steps. First will be shown that $x\bar{D}_\lambda y{\Rightarrow}xD_\lambda z\ \forall z{\in}X$ and then will be proven that $x\bar{D}_\lambda v{\Rightarrow}vD_\lambda w\ \forall v, w{\in}X$.

*step 1:* Suppose $\lambda$ is semidecisive for $x$against $y$. Now let $\bar{\rho}$ be a preference profile such that $\forall i{\in}Supp(\lambda)$ and $\forall z{\in}X\backslash\{x,y\}$ $\pi_i(x,z){>}0$. Now construct a fuzzy preference profile, $\bar{\rho}$', such that

$\forall i{\in}N$, $\rho_i\rceil_{\{x,z\}}{=}\rho\text{'}_i\rceil_{\{x,z\}}$;
$\forall i{\in}Supp(\lambda)$, $\pi\text{'}_i(x,y){>}0$;
$\forall i{\in}N\backslash Supp(\lambda)$, $\pi\text{'}_i{>}0$;
$\forall i{\in}N$, $\pi\text{'}_i(y,z){>}0$.

Since $x\bar{D}_\lambda y$, $\forall i{\in}Supp(\lambda)$, $\pi\text{'}_i(x,y){>}0$, $\forall i{\in}N\backslash Supp(\lambda)$ and $\pi\text{'}_i{>}0$ it is known that $\pi\text{'}(x,y){>}0$. And since $\bar{f}$ is weakly Paretian it follows that $\pi\text{'}(y,z){>}0$ because$\forall i{\in}N$, $\pi\text{'}_i(y,z){>}0$. Because $\bar{f}$ is partially quasi transitive it follows that $\pi\text{'}(x,z){>}0$, and thus by the regularity of $\pi$ that $\rho\text{'}(x,z){>}\rho\text{'}(z,x)$.

When $\bar{f}$ is IIA-3 and $\pi$ is regular the following holds:
Since $\forall i{\in}N$ $\rho_i\rceil_{\{x,z\}}{=}\rho\text{'}_i\rceil_{\{x,z\}}$ it is clear that $\rho_i\rceil_{\{x,z\}}{\sim}\rho\text{'}_i\rceil_{\{x,z\}}$, and thus by IIA-3 $\rho(x,z){>}\rho(z,x)$. Hence by regularity $\pi(x,z){>}0$.

When $\bar{f}$ is IIA-2 and $\pi{=}\pi_{(3)}$ the following holds:
Since $(\forall i{\in}N\ \rho_i\rceil_{\{x,z\}}{=}\rho\text{'}_i\rceil_{\{x,z\}}){\Rightarrow}\ (\forall i{\in}N\ Supp(\rho_i\rceil_{\{x,z\}}){=}Supp(\rho\text{'}_i\rceil_{\{x,z\}}))$ and $\bar{f}$ is IIA-2 it is found that $Supp(\rho\rceil_{\{x,z\}}){=}Supp(\rho\text{'}\rceil_{\{x,z\}})$. Therefore $\rho(x,z){>}0$ and $\rho(z,x)=0$, and thus by the definition of $\pi_{(3)}$ it holds that $\pi_{(3)}(x,z){>}0$.

Because $\bar{\rho}$' does not give any extra information about $\bar{\rho}$ $\bar{\rho}$ is still arbitrary and thus it is concluded that $xD_\lambda z$ and since $z$ was a arbitrary element of $X\backslash\{x,y\}$ it holds that $x\bar{D}_\lambda y{\Rightarrow}\ xD_\lambda z\ \forall z{\in}X\backslash\{x,y\}$.
Since $\lambda$ is decisive for $x$ against $z$ implies $\lambda$ is semidecisive for $x$ against $z$, it is found that by now starting with the proven fact that $x\bar{D}_\lambda z$ implies that $xD_\lambda y$ and thus it is concluded that $x\bar{D}_\lambda y\Rightarrow xD_\lambda z\ \forall z{\in}X$.

*step 2:* Now let $\bar{\rho}^*$ be a preference profile such that $\forall i{\in}Supp(\lambda)\ \pi^*{}_i(y,z){>}0$. and let $\bar{\rho}^+$ be defined as follows,

$\forall i{\in}N,\ \rho^*{}_i\rceil_{\{y,z\}}{=}\rho^+{}_i\rceil_{\{y,z\}}$;
$\forall i{\in}N,\ \pi^+{}_i(y,x){>}0$;
$\forall i{\in}Supp(\lambda),\ \pi^+{}_i(x,z){>}0$;
$\forall i{\in}N\backslash Supp(\lambda),\ \pi^+{}_i(z,x){>}0$.

Then since $\forall i{\in}Supp(\lambda),\ \pi^+{}_i(x,z){>}0$ and $xD_\lambda z$, $\pi^+(x,z){>}0$.
Now because $\bar{f}$ is weakly Paretian $\forall i{\in}N,\ \pi^+{}_i(y,x){>}0\Rightarrow\pi^+(y,x){>}0$.
Since $\bar{f}$ is partially quasi transitive $\pi^+(y,x){>}0$ and $\pi^+(x,z){>}0$ implies that $\pi^+(y,z){>}0$.

When $\bar{f}$ is IIA 3 and $\pi$ is regular the following holds:
Since $(\forall i{\in}N\ \rho^*{}_i\rceil_{\{y,z\}}{=}\rho^+{}_i\rceil_{\{y,z\}}){\Rightarrow}(\rho^*{}_i\rceil_{\{y,z\}}{\sim}\rho^+{}_i\rceil_{\{y,z\}})$ and because $\bar{f}$ is IIA-3, $\rho^*\rceil_{\{y,z\}}{\sim}\rho^+\rceil_{\{y,z\}}$.
Thus follows that $\rho^*(y,z){>}\rho^*(z,y)$ since $\pi^+(y,z){>}0{\Rightarrow}\rho^+(y,z){>}\rho^+(z,y)$ and $\rho^*\rceil_{\{y,z\}}{\sim}\rho^+\rceil_{\{y,z\}}$.
Therefore by the regularity of $\pi$ follows that $\pi^*(y,z){>}0$.

When $\bar{f}$ is IIA-2 and $\pi{=}\pi_{(3)}$ the following holds:
Since $(\forall i{\in}N\ \rho^*{}_i\rceil_{\{y,z\}}{=}\rho^+{}_i\rceil_{\{y,z\}}){\Rightarrow}(\forall i{\in}N\ Supp(\rho^*{}_i\rceil_{\{y,z\}}){=}Supp(\rho^+{}_i\rceil_{\{y,z\}}))$ and $\bar{f}$ is IIA-2 it is found that $Supp(\rho^*\rceil_{\{y,z\}}){=}\ Supp(\rho^+\rceil_{\{y,z\}})$. Therefore $\rho^*(y,z){>}0$ and $\rho^*(z,y){=}0$ Since $\pi_{(3)}$ is regular and $\pi^+(y,z){>}0{\Rightarrow}(\rho^+(y,z){>}0$ and $\rho^+(z,y)=0$. Thus $\rho^*(y,z){>}0$ and $\rho^*(z,y)=0$, and therefore by the definition of $\pi_{(3)}$ it holds that $\pi_{(3)}(y,z){>}0$

Because $\bar{\rho}^+$ does not give any extra information about $\bar{\rho}^*$ $\bar{\rho}^*$ is still arbitrary and thus it is concluded that $yD_\lambda z$ and since $z$ was a arbitrary element of $X\backslash\{x,y\}$ it holds that $x\bar{D}_\lambda y\Rightarrow yD_\lambda z\ \forall z{\in}X\backslash\{x,y\}$.
Since $\lambda$ is decisive for $x$ against $z$ implies $\lambda$ is semidecisive for $x$ against $z$, we find that by now starting with the proven fact that $z\bar{D}_\lambda$y implies that $yD_\lambda x$ and thus is concluded that $x\bar{D}_\lambda y\Rightarrow yD_\lambda z\ \forall z{\in}X$.

To complete the proof the result of step 1 and step 2 will be combined. By step 1 $x\bar{D}_\lambda y{\Rightarrow}xD_\lambda v{\Rightarrow}x\bar{D}_\lambda v\ \forall v{\in}X$. And now by step 2 $x\bar{D}_\lambda v{\Rightarrow}vD_\lambda w\ \forall v,w{\in}X$.
Thus is concluded that $x\bar{D}_\lambda y{\Rightarrow}vD_\lambda w\ \forall v,w{\in}X$. □

Now before the first fuzzy version of Arrows theorem will be presented a last lemma that will be useful in the proof will be shown.

**Lemma 5.14.** Let $\rho$ be a FWPR on $X$. Then the following are equivalent:
(i) $\rho$ is weakly transitive
(ii) $\forall x, y, z \in X$ $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)$ with at least once a strict inequality holding, then $\rho(x,z) > \rho(z,x)$.

*proof:* Suppose (i) and now suppose without loss of generality that $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) > \rho(z,y)$. Then by weak transitivity $\rho(x,z) \geq \rho(z,x)$. Now suppose that $\rho(x,z) = \rho(z,x)$ that implies that $\rho(z,x) \geq \rho(x,z)$. By weak transitivity $\rho(z,x) \geq \rho(x,z)$ and $\rho(x,y) > \rho(y,x)$ imply $\rho(z,y) \geq \rho(y,z)$, this is in contradiction with the assumption that $\rho(y,z) > \rho(z,y)$. Thus $\rho(x,z) > \rho(z,x)$.

Suppose (ii) and now assume that $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)$. Now suppose that $\rho(x,z) < \rho(z,x)$, then by (ii) it follows that $\rho(x,z) < \rho(z,x)$ and $\rho(x,y) \geq \rho(y,x)$ imply $\rho(z,y) > \rho(y,z)$. This contradicts the assumption that $\rho(y,z) \geq \rho(z,y)$ hence is concluded that $\rho(x,z) \geq \rho(z,x)$. □

**Corollary 5.15 (Weakly transitive⇒partially quasi-transitive).** Let $\rho$ be a FWPR on $X$. Then if $\rho$ is weakly transitive, then $\rho$ is partially quasi-transitive.

Now that this last lemma has been proven it is time to show the first fuzzy version of Arrows theorem. Remark that $D_{\mathrm{w}}$ is the set of all weakly transitive fuzzy weak orders.

**Theorem 5.16 (Fuzzy Arrow 1).** Let $\bar{f} : {D_w}^n \rightarrow \mathcal{FR}$ be an FPAR. Suppose $\pi$ is regular and $\bar{f}$ is weakly Paretian, weakly transitive and IIA-3. Then $\bar{f}$ is dictatorial.

*proof:* Let $\bar{f}$ and $\pi$ be as above. Because $\bar{f}$ is weakly Paretian there has to exist a fuzzy subset of the set of actors $N$ such that it is decisive for any pair of alternatives, namely the group of all actors. Let $m(u,v)$ denote the minimal cardinality of a fuzzy subset of $N$ which is semidecisive for $u$ against $v$. Now let $m = min(\{m(u,v) | (u,v) \in X \times X\})$.
Suppose $m = 1$.
Then there is a $m(u,v) = 1$ choose the corresponding fuzzy subset, $\lambda$, with cardinality 1 that is semidecisive for $u$ against $v$. By corollary 5.15 $\bar{f}$ is partially quasi-transitive. Thus $\lambda$, $\pi$ and $\rho$ fulfil all assumptions of lemma 5.13 and therefore $\lambda$ is decisive for all pairs $(u,v) \in X \times X$ and thus $\bar{f}$ is dictatorial.
Suppose $m > 1$.
Then there is a $m(u,v) = m > 1$ choose the corresponding fuzzy subset, $\lambda$, with cardinality $m$ that is semidecisive for $u$ against $v$. By the same argument as when assuming $m = 1$ it is known that $\lambda$ is decisive for all pairs $(u,v) \in X \times X$. Now consider any fuzzy preference profile $\bar{\rho}$ such that:

$i \in Supp(\lambda)$, $\pi_i(u,v) > 0$, $\pi_i(v,w) > 0$ and $\pi_i(u,w) > 0$ ;
$\forall j \in Supp(\lambda) \backslash \{i\}$, $\pi_j(w,u) > 0$, $\pi_j(u,v) > 0$ and $\pi_j(w,v) > 0$;
$\forall k \in N \backslash Supp(\lambda)$, $\pi_k(v,w) > 0$, $\pi_j(w,u) > 0$ and $\pi_j(v,u) > 0$.

Since $\lambda$ is decisive for $u$ against $v$ and $\pi_j > 0$ $\forall j \in Supp(\lambda)$ it follows that $\pi(u,v) > 0$.
Suppose that $\pi(w,v) > 0$ then $\lambda$' is semidecisive for $w$ against $v$ where $Supp(\lambda\text{'}) = Supp(\lambda) \backslash \{i\}$, since only those actors have $\pi_j > 0$. However this contradicts with the minimality of $\lambda$.

Thus $\pi(w,v)$=0. Since $\pi$ is regular $\pi(w,v)$=0$\Rightarrow\rho(v,w)\geq\rho(w,v)$ and $\pi(u,v)>0\Rightarrow\rho(u,v)>\rho(v,u)$. Since $\rho$ is weakly transitive lemma 5.14 implies that $\rho(u,w)>\rho(w,u)$. From the regularity of $\pi$ it now follows that $\pi(u,w)>0$. Since actor $i$ was the only actor with $\pi_i(u,w)>0$ this and IIA-3 imply that $\{i\}$ is semidecisive for $u$ against $w$. But this is in contradiction with $m>1$. Thus $m>1$ is not possible which concludes the proof. □

To prove a second version of Arrows theorem one extra proposition is needed.

**Proposition 5.17 (Partially transitive⇒partially quasi-transitive).** Let $\rho$ be a FWPR that is partially transitive and $\pi=\pi_{(3)}$. Then $\rho$ is partially quasi-transitive.

*proof:* Let $\rho$ and $\pi$ be as above and let $x,y,z\in X$. Suppose that $\pi(x,y)>0$ and $\pi(y,z)>0$. Then by definition of $\pi_{(3)}$ $\rho(x,y)>0$, $\rho(y,x)=0$, $\rho(y,z)>0$ and $\rho(z,y)=0$. Because $\rho$ is partially transitive this implies that $\rho(x,z)>0$.
Now suppose that $\pi(x,z)=0$ then by the definition of $\pi_{(3)}$ it must be that $\rho(z,x)>0$. However it was already known that $\rho(x,y)>0$ which by partial transitivity would imply that $\rho(z,y)>0$ but it was already shown that $\rho(z,y)=0$. Thus there is a contradiction and is concluded that $\pi(x,z)>0$ and therefore that $\rho$ is partially quasi transitive. □

The proof of the second fuzzy version of Arrows theorem is very similar to the proof of the first fuzzy version of Arrows theorem because of that this proof might not be very insightful, the theorem still helps to get a better understanding of FPARs. Note that, $D_p$ is the set of all partially transitive fuzzy weak orders.

**Theorem 5.18 (Fuzzy Arrow 2).** Let $\bar{f} : {D_p}^n\rightarrow\mathcal{FR}$ be an FPAR. Suppose $\rho$ is complete, $\pi=\pi_{(3)}$ and $\bar{f}$ is weakly Paretian, partially transitive and IIA-2. Then $\bar{f}$ is dictatorial.

*proof:* Let $\rho$ and $\pi$ and $\bar{f}$ be as above. Because $\bar{f}$ is weakly Paretian there has to exist a fuzzy subset of the set of actors $N$ such that it is decisive for any pair of alternatives, namely the group of all actors. Let $m(u,v)$ denote the minimal cardinality of a fuzzy subset of $N$ which is semidecisive for $u$ against $v$. Now let $m = min(\{m(u,v)|(u,v)\in X\times X\})$.
Suppose $m = 1$.
Then there is a $m(u,v) = 1$ choose the corresponding fuzzy subset, $\lambda$, with cardinality 1 that is semidecisive for $u$ against $v$. By proposition 5.17 $\bar{f}$ is partially quasi-transitive. Thus $\lambda$, $\pi$ and $\rho$ fulfil all assumptions of lemma 5.13 and therefore $\lambda$ is decisive for all pairs $(u,v)\in X\times X$ and thus $\bar{f}$ is dictatorial.
Suppose $m>1$.
Then there is a $m(u,v) = m>1$ choose the corresponding fuzzy subset, $\lambda$, with cardinality $m$ that is semidecisive for $u$ against $v$. By the same argument as when assuming $m = 1$ is known that $\lambda$ is decisive for all pairs $(u,v)\in X\times X$. Now consider any fuzzy preference profile $\bar{\rho}$ such that:

$i\in$Supp$(\lambda)$, $\pi_i(u,v)>0$, $\pi_i(v,w)>0$ and $\pi_i(u,w)>0$ ;
$\forall j\in Supp(\lambda)\backslash\{i\}$, $\pi_j(w,u)>0$, $\pi_j(u,v)>0$ and $\pi_j(w,v)>0$;
$\forall k\in N\backslash Supp(\lambda)$, $\pi_k(v,w)>0$, $\pi_j(w,u)>0$ and $\pi_j(v,u)>0$.

Since $\lambda$ is decisive for $u$ against $v$and $\pi_j>0$ $\forall j\in Supp(\lambda)$ it follows that $\pi(u,v)>0$.

Suppose that $\pi(w,v)>0$ then $\lambda$' is semidecisive for $w$ against $v$ where $Supp(\lambda')=Supp(\lambda)\setminus\{i\}$, since only those actors have $\pi_j>0$. However this contradicts with the minimality of $\lambda$. Thus $\pi(w,v)=0$, and since $\pi=\pi_{(3)}$ and $\rho$ is complete this implies that $\rho(v,w)>0$. Since $\bar{f}$ is partially transitive, $\rho(u,v)>0$ and $\rho(v,w)>0$ imply that $\rho(u,w)>0$.
Suppose $\pi(u,w)=0$, then by the definition of $\pi_{(3)}$ and because $\rho(u,w)>0$, $\rho(w,u)>0$ as well. However by partial transitivity it follows that $\rho(v,w)>0$ and $\rho(w,u)>0$ imply that $\rho(v,u)X>0$. But $\rho(v,u)>0$ would imply that $\pi(u,v)=0$ but it is already known that $\pi(u,v)>0$, thus this is a contradiction and hence $\pi(u,w)>0$. Since actor $i$ was the only actor with $\pi_i(u,w)>0$ this and IIA-2 imply that $\{i\}$ is semidecisive for $u$ against $w$. But this is in contradiction with $m>1$. Thus $m>1$ is not possible which concludes the proof. $\square$

## 5.3 Fuzzy Arrow Does Not Hold

In this subsection will be proven that with certain combinations of the fuzzy Arrowian conditions, from subsection 4.1, Arrows theorem does not hold any more. This subsection will start with the definition of an FPAR. And then will be shown that this FPAR fulfils some combinations of the Arrowian conditions.

**Definition 5.19 (Weighted mean rule).** Let $\bar{f}$ be an FPAR, $\bar{f}$ is called the weighted mean rule if $\forall\bar{\rho}\in\mathcal{FR}^n$ and all $x,y\in X$

$\bar{f}(\bar{\rho})(x,y)=\sum_{i=1}^{n} w_i\rho_i(x,y)$,

where $\sum_{i=1}^{n} w_{\mathrm{i}}=1$ and $w_{\mathrm{i}}>0$ for all $i\in N$.

This FPAR can be interpreted as a group of voters that all vote on how much better an alternative is than another alternative, in which votes of certain people (for example experts on the topic about which is voted) can count more, and than taking the average over all these weighted votes.
It is claimed that this FPAR can fulfil a certain combination of the Arrowian conditions. This will be proven in several lemmas. First will be proven if $\pi$ is regular then $\bar{f}$ is weakly Paretian.

**Proposition 5.20 (Weakly Paretian).** Let $\bar{f}$ be the weighted mean rule and let $\pi$ be regular, then $\bar{f}$ is weakly Paretian.

*Proof:* Let $x,y\in X$ and $min_{i\in\mathrm{N}}(\pi_i(x,y))>0$ then $\forall i\in N$ $\pi_i(x,y)>0$ and thus by regularity $\rho_i(x,y)>\rho_i(y,x)$. Hence for all $i\in N$ $w_{\mathrm{i}}>0$ also $w_{\mathrm{i}}\rho_i(x,y)>w_{\mathrm{i}}\rho_i(y,x)$. Therefore it has to hold that:

$\bar{f}(\bar{\rho})(x,y)=\sum_{i=1}^{n} w_{\mathrm{i}}\rho_i(x,y)>\sum_{i=1}^{n} w_{\mathrm{i}}\rho_i(y,x)=\bar{f}(\bar{\rho})(y,x)$.

Now by regularity of $\pi$ follows that $\pi(x,y)>0$. Thus is concluded that $\bar{f}$ is weakly Paretian. □

For now this proof is just a proof that a weighted mean rule can be weakly Paretian. This is not that impressive though since Arrow's theorem only forbids an FPAR to fulfil all Arrowian conditions at the same time.
Now an extra quality of an FPAR will be given, and then follows a proof that $\bar{f}$ fulfils this quality.

**Definition 5.21 (Positive responsiveness ).** Let $\bar{f}$ be an FPAR, then $\bar{f}$ is said to satisfy positive responsiveness with respect to $\pi$ if, $\forall\bar{\rho},\bar{\rho}'\in\mathcal{FR}^n$ and all $x,y\in X$ it holds that $\bar{f}(\bar{\rho})(x,y)=\bar{f}(\bar{\rho})(y,x)$ and $\exists j\in N$ such that for all $i\neq j$ $\rho_i=\rho'_i$ and either,
i) $\pi_j(x,y)=0$ and $\pi'_j(x,y)>0$ or
ii) $\pi_j(y,x)>0$ and $\pi'_j(y,x)=0$
imply that $\pi'(x,y)>0$.

Positive responsiveness says, that if there is a preference profile with no preference between two alternatives and an actor that first was indifferent acquires a preference for one of the two alternatives, or an actor that had a preference for the other alternative loses that preference, then the FPAR should respond by now preferring that alternative.

**Proposition 5.22 (Positive responsiveness to $\pi_{(2)}$).** Let $\bar{f}$ be the weighted mean rule, then $\bar{f}$ satisfies positive responsiveness with respect to $\pi_{(2)}$.

*Proof:* Let $\bar{\rho}$, $\bar{\rho}$'$\in\mathcal{FR}^n$ and all $x, y\in X$, $\bar{f}(\bar{\rho})(x,y)=\bar{f}(\bar{\rho})(y,x)$ and let there be a $j\in N$ such that for all $i\neq j$ $\rho_i=\rho'_i$. Then $\pi'(x,y)=max(0,\rho'(x,y)\text{-}\rho'(y,x))$.

$\rho'(x,y)\text{-}\rho'(y,x)=\sum_{i=1}^{n}(w_i\rho'_i(x,y)\text{-}w_i\rho'_i(y,x))$
$=\sum_{i=1}^{j-1}(w_i\rho'_i(x,y)\text{-}w_i\rho'_i(y,x)) + (w_j\rho'_j(x,y)\text{-}w_j\rho'_j(y,x)) + \sum_{i=j+1}^{n}(w_i\rho'_i(x,y)\text{-}w_i\rho'_i(y,x))$
$=\sum_{i=1}^{n}(w_i\rho_i(x,y)\text{-}w_i\rho_i(y,x))$ $\text{-}(w_j\rho_j(x,y)\text{-}w_j\rho_j(y,x))$ + $(w_j\rho'_j(x,y)\text{-}w_j\rho'_j(y,x))$

Since $\bar{f}(\bar{\rho})(x,y)=\bar{f}(\bar{\rho})(y,x)$ the sum we just got is zero. Hence,

$\rho'(x,y)\text{-}\rho'(y,x)=(w_j\rho'_j(x,y)\text{-}w_j\rho'_j(y,x))\text{-}(w_j\rho_j(x,y)\text{-}w_j\rho_j(y,x))$ Now assume that either,
i) $\pi_j(x,y)=0$ and $\pi'_j(x,y)>0$ or
ii) $\pi_j(y,x)>0$ and $\pi'_j(y,x)=0$
If i) holds then $\rho(x,y)=\rho(y,x)$ and $\rho'(x,y)>\rho'(y,x)$. If ii) holds then $\rho'(x,y)=\rho'(y,x)$ and $\rho(y,x)>\rho(x,y)$. Therefore in both cases holds that:
$\rho'(x,y)\text{-}\rho'(y,x)=(w_j\rho'_j(x,y)\text{-}w_j\rho'_j(y,x))\text{-}(w_j\rho_j(x,y)\text{-}w_j\rho_j(y,x))>0$.
Therefore $\pi'(x,y)>0$ and thus $\bar{f}$ satisfies positive responsiveness with respect to $\pi_{(2)}$. □

This shows that the mean weighted rule can be positive responsive, though it would be better if it would not restrict $\pi$ to one option. Therefore now will be proven that any regular $\pi$ will do.

**Proposition 5.23 (Positive responsiveness to a regular $\pi$).** Let $\bar{f}$ be the weighted mean rule, then $\bar{f}$ satisfies positive responsiveness with respect to $\pi$ if $\pi$ is regular.

*Proof:* This will be proven by showing that for any two strict preferences with respect to $\rho$, $\pi$ and $\pi^*$, for which holds that $\forall x,y\in X$ $\pi(x,y)>0 \iff \pi^*(x,y)>0$. An FPAR $\bar{f}$ satisfies positive responsiveness with respect to $\pi$ if and only if $\bar{f}$ satisfies positive responsiveness with respect to $\pi^*$. When this is shown the proof is complete since the previous proposition already shows that the weighted mean rule satisfies positive responsiveness with respect to $\pi_{(2)}$ and every regular $\pi$ satisfies $\forall x,y\in X$ $\pi(x,y)>0 \iff \pi_{(2)}(x,y)>0$.

Suppose that $\bar{f}$ satisfies positive responsiveness with respect to $\pi$ and $\pi^*(x,y)>0 \iff \pi(x,y)>0$.
Now suppose that $\forall\bar{\rho},\bar{\rho}'\in\mathcal{FR}^n$ and all $x,y\in X$ holds that $\bar{f}(\bar{\rho})(x,y)=\bar{f}(\bar{\rho})(y,x)$ and $\exists j\in N$ such that for all $i\neq j$ $\rho_i=\rho'_i$ and either,
i) $\pi^*_j(x,y)=0$ and $\pi^{*\prime}_j(x,y)>0$ or
ii) $\pi^*_j(y,x)>0$ and $\pi^{*\prime}_j(y,x)=0$

If i) holds because $\pi^*(x,y)>0 \iff \pi(x,y)>0$ then follows that $\pi_j(x,y)=0$ and $\pi'_j(x,y)>0$.
If ii) holds because $\pi^*(x,y)>0 \iff \pi(x,y)>0$ then follows that $\pi_j(y,x)>0$ and $\pi'_j(y,x)=0$

Now by the positive responsiveness of $\bar{f}$ to $\pi$ it has to be that $\pi'(x,y)>0$. And thus by

$\pi^*(x,y){>}0 \iff \pi(x,y){>}0$ follows that $\pi^{*}$'$(x,y){>}0$. Hence $\bar{f}$ satisfies positive responsiveness with respect to $\pi^*$. This concludes the proof. □

**Theorem 5.25 (Contradiction Arrow 1).** Let $\pi$ be regular. Then a non dictatorial $\bar{f}$: $\mathcal{FR}^n \rightarrow \mathcal{FR}$ that is weakly Paretian, satisfies positive responsiveness and satisfies IIA-1 exists.

*Proof:* It will be shown that the weighted mean rule satisfies all above conditions. From proposition 5.20 and proposition 5.23 follows that the mean weight rule is weakly Paretian and satisfies positive responsiveness. Trivially the mean weight rule is not dictatorial. And the mean weight rule is IIA-1 since $\sum_{i=1}^{n} w_\mathrm{i}\rho_i(x,y){=}\sum_{i=1}^{n} w_\mathrm{i}\rho\text{'}_i(x,y)$ if for all $i{\in}N$ $\rho_i(x,y){=}\rho\text{'}_i(x,y)$. Hence the weighted mean rule satisfies all above conditions. □

**Proposion 5.26 ($\pi_{(2)}{\Rightarrow}$Pareto condition).** Let $\bar{f}$ be the weighted mean rule and let $\pi{=}\pi_{(2)}$. Then $\bar{f}$ satisfies the Pareto condition.

*Proof:* Let $x,y{\in}X$ and choose $m_{x,y}{=}min_{i\in N}(\pi_i(x,y))$ and choose $\pi_1{=}m_{x,y}$. If $m_{x,y}{=}0$ then $\bar{f}$ would always satisfy the Pareto condition thus the proof would be complete. Therefore suppose that $m_{x,y}{>}0$ then for all $i$ $\frac{\pi_\mathrm{i}(x,y)}{m_\mathrm{x,y}}{\geq}1$ and $\sum_{i=1}^{n}(w_\mathrm{i}){=}1$. Hence,

$1{\leq}\sum_{i=1}^{n}(w_\mathrm{i}\frac{\pi_\mathrm{i}(x,y)}{m_\mathrm{x,y}})$.

Now multiplying with $m_{x,y}$ gives:

$\pi_1(x,y){=}m_{x,y}{\leq}\sum_{i=1}^{n}(w_i\pi_i(x,y))$.

Since $\pi{=}\pi_{(2)}$ the previous equation is equivalent to:

$m_{x,y}{\leq}\sum_{i=1}^{n}(w_\mathrm{i}(\rho_i(x,y)\text{-}\rho_i(y,x)){=}\sum_{i=1}^{n}(w_\mathrm{i}\rho_i(x,y))\text{-}\sum_{i=1}^{n}(w_\mathrm{i}\rho_i(y,x)){=}\bar{f}(\bar{\rho})(x,y)\text{-}\bar{f}(\bar{\rho}){=}\pi(x,y)$.

Therefore is concluded that the mean weight rule satisfies the Pareto condition with respect to $\pi_{(2)}$. □

**Theorem 5.27.(Contradiction Arrow 2)** Let $\pi{=}\pi_{(3)}$. Then there exist a non dictatorial $\bar{f}$: $\mathcal{FR}^n \rightarrow \mathcal{FR}$ that is weakly Paretian, satisfies positive responsiveness, the Pareto condition and satisfies IIA-1.

*Proof:* Since $\pi_{(2)}$ is regular follows by Theorem 5.25 that all conditions except for the Pareto condition are satisfied. And by proposition 5.26 then follows that also the Pareto condition is satisfied, and thus all conditions are satisfied. □

# 6 Conclusion

## 6.1 Consequences for political elections

At the end of section 3 the motivation to look at fuzzy social choice was given by looking at online ratings as a possible election system. In this section we developed the weighted mean rule. Are these election systems the solution to get a better working democracy, or do they bring new problems? And if there is a good election system, would that change the criticism on democracy?

Online ratings are a special case of a weighted mean rule, thus to answer the question if these election systems could make a democracy better we only need to look at the weighted mean rule. In the third section after looking at Arrow's theorem we looked at the Gibbard-Satterthwaite theorem and we argued that being strategy proof is also desired for a election system. This is already enough not to choose for a weighted mean rule, since this is not strategy proof. To show this, suppose that a voter wants to have an alternative A to end up in the middle of all alternatives but the other voters all want to have that alternative A to end up on the first place. then that voter could vote strategically by placing the alternative lower to average closer to the middle of all alternatives. Thus the weighted mean rule is not perfect, but it does, in some sense, fulfil all Arrowian conditions what would make it theoretically a better election system.
Suppose now that everyone got convinced that a form of a weighted mean rule should replace our current election system, would this bring new problems? In current election systems it is almost always the case that the voters vote only on one alternative. If we switched to a form with a weighted mean rule this would no longer be the case since everyone needs to vote on every alternative. This would cause some direct and some indirect practical problems.
Direct problems are problems that arise when trying to execute this election. If we want this election to be in the same manner as our elections now, it would cost a lot more time to count the votes since one vote would take way longer to process. It would also be more difficult to count the votes since the ratings should be readable while currently everyone only has to colour one circle. In the future it might be possible to make online ratings so safe that we could trust them for a political election. In that case many of the direct problems would be resolved which would make the weighted mean rule a better option than it is at this point.
Indirect problems are problems that influence the outcome of the election which could be a reason to prefer the election system that we currently use over the weighted mean rule. When voting via the weighted mean rule every voter needs to vote on every alternative and therefore needs to be informed about every alternative to be able to make its vote represent his/her opinion accurately. When looking at all of these problems it is not evident if switching to the weighted mean rule would be an improvement from our current situation.

Now what would happen if in some way we managed to make an election system such that the chosen representatives represents the voters perfectly, would this then lead to a good way of leading a country? It is possible to compare this hypothetical situation with every voter representing itself in Parliament, though less chaotic. This situation is very similar to a direct democracy and this takes us back to section one, where we discussed the different views of Plato and Habermas. And we would have to conclude that even in this ideal situation it is not clear if a democracy would be the best way to lead a country.

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