Whittaker functions and residues of a Fourier-Whittaker inverse transform for $\mathbf{SL_2}(\mathbb{R})$

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Whittaker vectors for the discrete series representations of $\mathbf{SL_2}(\mathbb{R})$

Abstract

In the Harmonic analysis of real semisimple Lie groups, *Whittaker vectors* play an important role in the *Whittaker-Plancherel* decomposition. These elements consist of generalised vectors that transform by a character under a certain nilpotent subgroup of G. In the group $\mathrm{SL}_2(\mathbb{R})$, we will see that they are the main building blocks of certain type of matrix coefficients of the principal series representations, that can be identified with special functions satisfying the classical Whittaker differential equation. In recent work of E. van den Ban, a new inversion formula for the so-called *Whittaker-Fourier transform* on a semisimple group has been derived. In $\mathrm{SL}_2(\mathbb{R})$, the residues of this inversion transform appear in terms of the previously mentioned principal series representations matrix coefficients; and they turn out to be Whittaker functions associated with representations of the discrete series. In this thesis, we will introduce the notion of Whittaker vector and Whittaker matrix coefficient for both the discrete and the principal series represetations of $\mathrm{SL}_2(\mathbb{R})$. We will make explicit the connection with the Whittaker differential equation and analyse the nature of the aforementioned residues for the Lie group $\mathrm{SL}_2(\mathbb{R})$.

Contents

Introduction

In 1982, Harish-Chandra announced he had a proof of the Whittaker-Plancherel formula for the case of connected real semisimple Lie groups. Unfortunately, because of his passing this work remained unpublished for a long time only having been communicated by private correspondence. Independent tretaments of the Whittaker-Plancherel formula have appeared throughout these years, see [15]. In 2018, this work is made published in [2]. However there is a step in Harish-Chandra's proof that appears to be missing. This is addressed in recently announced work of E. van den Ban, relying on a new inversion formula for the Fourier-Whittaker transform for a real connected semisimple Lie group. The motivation behind this thesis is to understand the nature of the residues that this formula produces in the case of the group $\mathrm{SL}_2(\mathbb{R})$. We will profit from the rich structure of this group as it will make the theory simpler. We will see that the theory of Whittaker vectors for this group is intimately related to the classical theory of Whittaker functions.

Structure of this monograph

Let us outline the structure of this thesis. This monograph is divided into three chapters and an appendix. In the first half of this chapter, we will introduce some prior knowledge and notation that will be used throughout the text. The other half of the chapter will be dedicated to the study of the principal series representations and the discrete series representations. The main reference for this chapter is [13].

The second chapter is the core of this thesis. Whittaker vectors were firstly introduced by Jacquet in [3]. We will study the concept of Whittaker vector for both the principal and discrete series representations of $\mathrm{SL}_2(\mathbb{R})$. For the latter, we specifically outline the construction of an example of a Whittaker vector provided by E. van den Ban. Further on in the chapter we introduce the standard intertwining operator. It will serve us to subsequently study the Harish-Chandra c_n-functions. This will be of relevance in the study of the residues previously mentioned. Most of this theory has been studied from [9]; except for the c_n-functions, for which [15] has been followed.

In the final chapter, we will see that Whittaker matrix coefficients are associated to special functions that solve the Classical Whittaker equation for $\mathrm{SL}_2(\mathbb{R})$. In the end we will introduce the aforementioned Fourier-Whittaker transform in the case of $\mathrm{SL}_2(\mathbb{R})$ and compute the residues for the inversion formula, establlishing the connection with the discrete series representations.

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Chapter 1

Preliminaries

This chapter will be devoted to the development of a few tools regarding the representation theory on real connected semisimple Lie groups that will be used through the coming chapters. The reader should be acquainted with some Lie theory basics to make the most of the subsequent sections. Should it not be the case, the reader is always invited to check [11]. The chapter is clearly differentiated in two parts. In the first one we will learn about two important tools in representation theory; namely, the Iwasawa decomposition and induced representations. Most of the proofs will be omitted, referring to [13, Ch. 15-22] for a detailed account of the first part of this chapter. Secondly, we treat the principal and discrete series representations due to its relevance for the theory that will thereafter be presented; along with its construction for $\mathrm{SL}_2(\mathbb{R})$. Lastly, we discuss how discrete series representations may be seen as subrepresentations of the principal series representations.

1.1 The Iwasawa decomposition

In this section, we will be concerned with finding an appropriate decomposition of a general connected semisimple Lie group and describing it specifically for $\mathrm{SL}_2(\mathbb{R})$. We will derive such decomposition starting at the Lie algebra level.

Let $\mathfrak{g}$ be a real semisimple[1] Lie algebra. An involution of $\mathfrak{g}$ is defined to be an automorphism of $\mathfrak{g}$ that squared equals the identity. If $\mathfrak{g}$ has such an involution, observe that ± 1 are the only possible eigenvalues of the involution and therefore we may write

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-,$$

where $\mathfrak{g}_\pm$ denote the plus and minus one eigenspaces, respectively. These eigenspaces are orthogonal with respect to the Killing form associated to $\mathfrak{g}$ because this is invariant under automorphisms. Furthermore, since automorphisms are in particular Lie algebra homomorphisms, $\mathfrak{g}_+$ is a Lie subalgebra.

Definition 1.1.1 (Cartan involution)**.** An involution for which the Killing form is negative definite on $\mathfrak{g}_+$ and positive definite on $\mathfrak{g}_-$ is called *Cartan involution.*

Remark. We shall denote $\mathfrak{g}_\pm$ by $\mathfrak{k}$ and $\mathfrak{p}$, respectively and Cartan involutions by θ. We see that $\mathfrak{k}$ is compact[2] in $\mathfrak{g}$ as the Killing form is negative definite on $\mathfrak{k}$.

[1] A *semisimple* algebra is a direct sum of simple algebras. A Lie algebra is said to be *simple* if it is non-abelian and it has no proper ideals

[2] Meaning, it is isomorphic to the Lie algebra of a compact Lie group

Lemma 1.1.1 (Existence of Cartan involutions)**.** *[13, Proposition 15.4] Any real semisimple Lie algebra has a Cartan involution. Furthermore, Cartan involutions are unique up to conjugation by interior automorphisms of the Lie algebra.*

Having the Killing form B and a fixed Cartan involution θ at hand, the following expression defines an inner product on $\mathfrak{g}$:

$$\langle X, Y\rangle := -B(X, \theta Y) \qquad \text{for } X, Y \in \mathfrak{g}. \tag{1.1}$$

This map is readily seen to be bilinear. Both symmetry and invariance under automorphisms of the Killing form make $\langle\cdot,\cdot\rangle$ symmetric. Let $X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$. Then

$$\langle X, X\rangle = \langle X_{\mathfrak{k}} + X_{\mathfrak{p}}, X_{\mathfrak{k}} + X_{\mathfrak{p}}\rangle = B(X_{\mathfrak{p}}, X_{\mathfrak{p}}) - B(X_{\mathfrak{k}}, X_{\mathfrak{k}})$$

The expression on the right side is positive definite as a consequence of the definition of Cartan involution. This means that expression (1.1) defines indeed, an inner product on $\mathfrak{g}$. With such inner product, the previous direct sum decomposition stays orthogonal because so it is with respect to the Killing form.

Lemma 1.1.2. $\mathrm{ad}(\mathfrak{k})$ *and* $\mathrm{ad}(\mathfrak{p})$ *are respectively contained in the spaces of antisymmetric and symmetric endomorphisms of* $\mathfrak{g}$*.*

Proof. For $X, Y, Z \in \mathfrak{g}$

$$\langle \mathrm{ad}(X)Y, Z\rangle = -B(\mathrm{ad}(X)Y, \theta Z) = B(Y, \mathrm{ad}(X)\theta Z) = -\langle Y, \mathrm{ad}(\theta X)Z\rangle.$$

Thus $\mathrm{ad}(X)^t = -\mathrm{ad}(\theta X)$ and the result follows. □

By Lemma 1.1.2, we observe that $\mathrm{ad}(\mathfrak{p})$ consists of real symmetric maps, all these automatically diagonalisable with real eigenvalues. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Since $\mathfrak{p}$ is finite-dimensional, such subalgebras always exist. For $\lambda \in \mathfrak{a}^*$, we define the λ-*weight space*

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X,\ \forall H \in \mathfrak{a}\}.$$

Definition 1.1.2 (Root system)**.** We say that $\alpha \in \mathfrak{a}^*\backslash\{0\}$ is a *root* if $\mathfrak{g}_\alpha \neq 0$. The set of roots is called *root system* and it is denoted by $\Sigma(\mathfrak{a}, \mathfrak{g})$.

We shall write Σ instead $\Sigma(\mathfrak{a}, \mathfrak{g})$ when the dependence on $\mathfrak{a}$ and $\mathfrak{g}$ is clear. With the previous definition, we have the following *root space decomposition*.

Lemma 1.1.3 (Root space decomposition)**.** *[13, Lemma 16.4; Corollary 16.10]. The set Σ is finite and*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha\right).$$

Moreover, this root decomposition is orthogonal with respect to the Cartan inner product 1.1.

Remark. We have that $\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p}$. For any $X \in \mathfrak{g}_0$, $\mathfrak{a} + \mathbb{R}X$ is abelian. In consequence, $X \in \mathfrak{a}$ by maximality. This implies that $\mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$.

In the same spirit, denote by $\mathfrak{m}$ the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$; that is to say, $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{g}_0$. It is easy to verify that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ orthogonally, since the Cartan involution leaves $\mathfrak{a}$ invariant. Let $\Sigma^+ \subset \Sigma$ denote a choice of positive roots. We may consider the following subalgebras

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \qquad \overline{\mathfrak{n}} := \bigoplus_{\alpha \in -\Sigma^+} \mathfrak{g}_\alpha$$

These two subalgebras are related by means of the Cartan involution; via $\theta(\mathfrak{n}) = \overline{\mathfrak{n}}$. We may rewrite the root space decomposition as follows:

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

in which the summands are mutually orthogonal by Lemma 1.1.3. We have all the ingredients to introduce the *infinitesimal Iwasawa decomposition*, that is at the Lie algebra level.

Theorem 1.1.1 (Infinitesimal Iwasawa decomposition). *[13, Lemma 17.3]. As linear spaces,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Remark. Observe that the infinitesimal Iwasawa decomposition highly depends on the choice of positive roots. As $\theta(\mathfrak{n}) = \overline{\mathfrak{n}}$, we have

$$\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \overset{\theta}{\simeq} \mathfrak{k} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}.$$

We see that the right-hand side corresponds to taking $-\Sigma^+$ as a preferred choice of positive roots.

A decomposition at the group level shall become handy for our purpose. Assume that G is a connected semisimple real Lie group with associated Lie algebra $\mathfrak{g}$. Recall that we can endow $\mathfrak{g}$ with a Cartan involution θ. In this setup, we have the following decomposition of G.

Theorem 1.1.2 (Cartan decomposition). *[13, Theorem 15.12]. Let $K \subset G$ be the analytic subgroup with Lie algebra $\mathfrak{k}$. Then K is closed in G and the map*

$$\varphi : K \times \mathfrak{p} \to G, (k, X) \mapsto k \exp X$$

is a diffeomorphism.

The Cartan decomposition of G allows us to define a unique involution at the group level that is compatible with the Cartan involution given in $\mathfrak{g}$. Such an involution will be called the Cartan involution of G. This is expressed in the following lemmas, which correspond to [13, Lemmas 15.13 & 15.14].

Lemma 1.1.4 (Cartan involution of G). *There exists a unique involution Θ on G such that $d\Theta(e) = \theta$. The involution Θ is given by the following expression:*

$$\Theta(k \exp X) = k \exp(-X)$$

Lemma 1.1.5. *K is the subgroup of fixed points under Θ. Moreover, K is compact if and only if G has finite centre.*

For $\mathfrak{a}$ as in Theorem 1.1.1, let A to be its associated analytic subgroup. Then we can write $A = \exp \mathfrak{a}$ as $\mathfrak{a}$ is abelian. We can also regard $\mathfrak{a}$ as the closed submanifold $\{e\} \times \mathfrak{a}$ inside $K \times \mathfrak{p}$. It follows by Theorem 1.1.2 that $\exp : \mathfrak{a} \to A$ is a diffeomorphism with A closed. In fact, because $\mathfrak{a}$ is abelian, the exponential map is an isomorphism of Lie groups between $(\mathfrak{a}, +, 0)$ and A. Denote the inverse of this isomorphism by $\log := \exp^{-1}$. For fixed $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$[3], we can define the following character on A,

$$(\cdot)^{\lambda} : A \to (0, \infty), \quad a^{\lambda} = e^{\lambda(\log a)}. \tag{1.2}$$

Lemma 1.1.6. *[13, Lemma 17.5] $\mathrm{Ad}(a)|_{\mathfrak{g}_\alpha} = a^{\alpha} I_{\mathfrak{g}_\alpha}$ for $a \in A$ and $\alpha \in \Sigma \cup \{0\}$. In particular $\mathrm{Ad}(a)$ preserves the root space decomposition and the subalgebra $\mathfrak{n}$.*

Proof. $\mathrm{Ad}(a)|_{\mathfrak{g}_\alpha} = \mathrm{Ad}(\exp H)|_{\mathfrak{g}_\alpha} = e^{\mathrm{ad}(H)|_{\mathfrak{g}_\alpha}} = e^{\alpha(H) I_{\mathfrak{g}_\alpha}} = e^{\alpha(\log a)} I_{\mathfrak{g}_\alpha} = a^{\alpha} I_{\mathfrak{g}_\alpha}$. □

It will be useful to consider both the associated analytic subgroups N and $\overline{N}$ with Lie algebras $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ respectively. With this framework, we are in the position to define the *global Iwasawa decomposition.*

Theorem 1.1.3 (Global Iwasawa decomposition)**.** *[13, Theorem 17.6] The map*

$$\begin{array}{rccc} \varphi : & K \times A \times N & \longrightarrow & G \\ & (k, a, n) & \longmapsto & kan \end{array}$$

is a diffeomorphism.

Analogously to the group A, we may prove that N is closed in G except that in this case we make use of the global Iwasawa decomposition instead of the Cartan decomposition. A more involved argument proves that $N = \exp \mathfrak{n}$ and that the exponential on $\mathfrak{n}$ is a diffeomorphism. For details, we refer to [13, Lemma 17.13].

Remark. Similarly to the infinitesimal Iwasawa decomposition, if we prefer to work with the opposite choice of positive roots, the previous discussion yields $G \simeq K \times A \times \overline{N}$, with $\overline{N} = \exp \overline{\mathfrak{n}}$. Since

$$\Theta(N) = \Theta(\exp \mathfrak{n}) = \exp d\Theta(e)(\mathfrak{n}) = \exp \theta(\mathfrak{n}) = \exp \overline{\mathfrak{n}} = \overline{N},$$

it follows that

$$K \times A \times N \overset{\Theta}{\simeq} K \times A \times \overline{N}.$$

Define M as the centraliser of A of K. With such definition, it is readily seen that MA is a subgroup of G. It also shows that M is a closed subgroup of G contained in the compact K, hence compact itself. We introduce the *minimal parabolic subgroup* P of G to be

$$P = MAN.$$

By Theorem 1.1.3, we see that $G = KP$. We can apply the Iwasawa decomposition to the minimal parabolic subgroup itself, yielding the following results. Proof of these facts are to be found in [13, Section 19]

Lemma 1.1.7. *P is a closed subgroup of G. The Iwasawa decomposition restricts to a diffeomorphism between $M \times A \times N$ and $P = MAN$.*

[3] This notation stands for the complexification of the Lie algebra $\mathfrak{a}$. For a real linear space V, we define its complexification $V_{\mathbb{C}}$ to be the linear space $V \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 1.1.8. *The group MA normalises N and N is a normal subgroup of P.*

Lemma 1.1.9. *The inclusion $i_K : K \to G$ induces a diffeomorphism*

$$\overline{i_K} : K/M \to G/P.$$

Remark. One may also talk about the *opposite minimal parbolic* that corresponds to $\overline{P} = MA\overline{N}$. In fact, we shall mostly consider this one in accordance with the notation used by Harish-Chandra.

It will be useful to take into account the projections onto the different components of the Iwasawa decomposition. Let φ as in Theorem 1.1.3 in the KAN-decomposition. For $g \in G$, we put

$$k(g) = (\mathrm{pr}_K \circ \varphi^{-1})(g), \quad a(g) = (\mathrm{pr}_A \circ \varphi^{-1})(g) \quad n(g) = (\mathrm{pr}_N \circ \varphi^{-1})(g);$$

where pr_K, pr_A and pr_N are the smooth projections of $K \times A \times N$ onto the respective components K, A and N. One ought to be aware that these projection maps depend on the form of the Iwasawa decomposition considered. The reader should keep in mind that throughout the text other forms of the Iwasawa decomposition may be more convenient to work with, for instance the NAK- or $KA\overline{N}$-decompositions.

1.1.1 Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$

This subsection will be dedicated to work out the theory that has bee previously laid out, for the group $\mathrm{SL}_2(\mathbb{R})$. In the rest of this subsection we set $G = \mathrm{SL}_2(\mathbb{R})$.

Recall that G is the (connected) semisimple Lie group of 2-dimensional square matrices with determinant 1. Its associated Lie algebra $\mathfrak{g}$ consists of the traceless 2-dimensional square matrices. The following elements form the so-called $\mathfrak{g}$-standard triple:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These satisfy the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{1.3}$$

The standard Cartan involution on $\mathfrak{g}$ is defined by $\theta(X) = -X^t$. We choose $\mathfrak{a} = \mathbb{R}H$. By the previous relations, one gets $\Sigma = \{\alpha, -\alpha\} \subset \mathfrak{a}^*$ determined by $\alpha(H) = 2$. Folowing the definitions we have that

$$\begin{aligned}
&\mathfrak{k} = \{2 \times 2\,\text{antisymmetric matrices}\} = \mathbb{R}(Y - X) \\
&\mathfrak{p} = \{2 \times 2\,\text{traceless symmetric matrices}\} = \mathbb{R}H \oplus \mathbb{R}(Y + X) \\
&\mathfrak{a} = \mathfrak{g}_0 = \mathbb{R}H \\
&\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k} = 0 \\
&\mathfrak{n} = \mathfrak{g}_\alpha = \mathbb{R}X \\
&\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha} = \mathbb{R}Y
\end{aligned}$$

At the level of G we get the following list

$$\begin{aligned}
&K = \exp \mathbb{R}(Y-X) = \left\{ k_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi) \right\} = \mathrm{SO}(2) \simeq S^1 \\
&M = \{\pm I\} \\
&A = \exp \mathbb{R}H = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R}_+ \right\} \simeq \mathbb{R}_+ \\
&N = \exp \mathbb{R}X = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \simeq \mathbb{R} \\
&\overline{N} = \exp \mathbb{R}Y = \left\{ \overline{n}_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\} \simeq \mathbb{R} \\
&P = MAN = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ t \in \mathbb{R}\backslash\{0\} \right\} \\
&\overline{P} = MA\overline{N} = \left\{ \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ t \in \mathbb{R}\backslash\{0\} \right\}
\end{aligned}$$

Lemma 1.1.10 (Iwasawa projections for $\mathrm{SL}_2(\mathbb{R})$)**.** *For $g \in G$ it holds that*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & (\sqrt{a^2+c^2})^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix}.$$

Furthermore, in the $KA\overline{N}$ decomposition

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{b^2+d^2}} \begin{pmatrix} d & b \\ -b & d \end{pmatrix} \begin{pmatrix} (\sqrt{b^2+d^2})^{-1} & 0 \\ 0 & \sqrt{b^2+d^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{ab+cd}{b^2+d^2} & 1 \end{pmatrix}.$$

Proof. The proof is by computation and it is left to the reader. □

The Iwasawa decomposition of G is intimately related to the geometry of the upper half plane. In Section 1.4.1 we shall exploit that connection in order to construct a model for the discrete series representations of G.

We recall that the upper-half plane $\mathcal{H}^+$ consists of all complex numbers with strictly positive imaginary part. This subset carries a complex structure of an open subset of the Riemann sphere $\widehat{\mathbb{C}}$. The group G acts smoothly and transitively on $\widehat{\mathbb{C}}$ by fractional linear transformations; that is

$$g \cdot z = T_g(z) = \frac{az+b}{cz+d} \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ and } z \in \mathcal{H}^+$$

The reader may prove that there are three orbits, namely, $G \cdot i = \mathcal{H}^+$, $\widehat{\mathbb{R}} = G \cdot 0$ and $G \cdot (-i) = \mathcal{H}^-$. The stabiliser[4] of i in G coincides with K and the stabiliser of 0 in G is $\overline{P}$. By the Orbit-Stabiliser theorem, the map j determined by $g \mapsto g \cdot i$ induces a diffeomorphism from $G/K \simeq NA$ onto $\mathcal{H}^+$. Furthermore, the complex structure on $\mathcal{H}^+$ may be transferred so that this diffeomorphism becomes a biholomorphic map. More specifically, this map is given by

$$\begin{array}{rccc} j: & NA & \longrightarrow & \mathcal{H}^+ \\ & n_x a_t & \longmapsto & x + ie^{2t} \end{array} \qquad \begin{array}{rccc} j^{-1}: & \mathcal{H}^+ & \longrightarrow & NA \\ & x+iy & \longmapsto & n_x a_{\log\sqrt{y}} \end{array}$$

[4] We use the notation G_a to indicate the stabiliser of the element a in the group G.

In the remaining part, set $G_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$. The group $G_{\mathbb{C}}$ consists of all 2-dimensional complex matrices with determinant one. This a complexification[5] of the group G. In the following we compile a list with all complexifications that we shall use throughout the text.

$$
\begin{aligned}
&K_{\mathbb{C}} := \mathrm{SO}(2)_{\mathbb{C}} = \exp \mathbb{C}(Y-X) = \left\{ \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \,\middle|\, c^2+s^2=1,\ c,s \in \mathbb{C} \right\} \\
&A_{\mathbb{C}} = \exp \mathbb{C}H = \left\{ a_z = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \,\middle|\, z \in \mathbb{C}\backslash\{0\} \right\} \\
&N_{\mathbb{C}} = \exp \mathbb{C}X = \left\{ n_w = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \,\middle|\, w \in \mathbb{C} \right\} \\
&\overline{N}_{\mathbb{C}} = \exp \mathbb{C}Y = \left\{ \overline{n}_w = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \,\middle|\, w \in \mathbb{C} \right\} \\
&\overline{P}_{\mathbb{C}} = A_C \overline{N}_{\mathbb{C}} = \left\{ \begin{pmatrix} z & 0 \\ w & z^{-1} \end{pmatrix} \,\middle|\, w \in \mathbb{C}, z \in \mathbb{C}\backslash\{0\} \right\} = (G_{\mathbb{C}})_0 \\
&\overline{B}_{\mathbb{C}} = (G_{\mathbb{C}})_i
\end{aligned}
$$

In particular, we see that $\overline{P}_{\mathbb{C}}$ is connected, as $M \subset A_{\mathbb{C}}$. Mimicking the previous discussion, the assignation $g \mapsto g \cdot i$ for $g \in G_{\mathbb{C}}$ induces again a biholomorphic map between $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ and $\widehat{\mathbb{C}}$. We also note that G/K is open in $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ by the following commutative diagram

$$
\begin{tikzcd}
G/K \arrow[r, "\simeq"] \arrow[d, hook] & \mathcal{H} \arrow[d, hook, "i"] \\
G_{\mathbb{C}}/\overline{B} \arrow[r, "\simeq"] & \widehat{\mathbb{C}}
\end{tikzcd}
$$

Next lemma shall be convenient in in Section 1.4.1.

Lemma 1.1.11. *There exists an element $g_0 \in G_{\mathbb{C}}$ such that*

1. $g_0 \cdot 0 = i$.
2. $K_{\mathbb{C}} = g_0\, A_{\mathbb{C}}\, g_0^{-1}$.
3. $\overline{B}_{\mathbb{C}}$ *is diffeomorphic to* $K_{\mathbb{C}} \times g_0\, \overline{N}_{\mathbb{C}}\, g_0^{-1}$.

Proof. Note that for any element $c \in G_{\mathbb{C}}$ such that $c \cdot 0 = i$, we have that $\overline{B}_{\mathbb{C}} = (G_{\mathbb{C}})_i = c\,(G_{\mathbb{C}})_0\, c^{-1} = c\,\overline{P}_{\mathbb{C}}\, c^{-1} = c\, A_{\mathbb{C}} \overline{N}_{\mathbb{C}}\, c^{-1} \simeq c\, A_{\mathbb{C}}\, c^{-1} \times c\, \overline{N}_{\mathbb{C}}\, c^{-1}$. From this we also observe that $\overline{B}_{\mathbb{C}}$ is connected as $\overline{P}_{\mathbb{C}}$ is connected. That is to say, $\overline{B}_{\mathbb{C}}$ is generated by its associated Lie algebra $\overline{\mathfrak{b}}_{\mathbb{C}}$. Furthermore, $\overline{\mathfrak{b}}_{\mathbb{C}} = \mathrm{Ad}(c)\mathrm{Lie}(\overline{P}_{\mathbb{C}})$. We shall impose the condition that $\mathrm{Ad}(c)H \in \mathbb{C}(Y-X) = \mathfrak{k}_{\mathbb{C}}$. If we diagonalise the matrix $Y - X$, we observe that there exists g_0 orthogonal matrix with determinant 1, such that $Y - X = g_0\,(iH)\, g_0^{-1}$. This implies that $K_{\mathbb{C}} = g_0\, A_{\mathbb{C}}\, g_0^{-1}$. Computing, we see that

$$
g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}
$$

and that $g_0 \cdot 0 = i$. $\square$

[5] For a definition, see [5, Page 437]

1.2 Induced representations

Given a general continuous representation of a Lie group, any closed subgroup induces a continuous representation of itself, just by restriction of the representation. The other way around, there is a method to construct a representation of a given Lie group G by means of a representation of a Lie subgroup. This resultant representation is called *induced representation*. In this text we will give an explicit construction. We will study different realisations of the induced representation in terms of function spaces that shall be useful throughout the theory. The reader is referred to [13, Section 19]

Assume that we are given a continuous finite dimensional representation (ξ, V_ξ) of a closed subgroup H of a Lie group G(The reader should note that this is the only generality needed for the development of the subsequent theory). Then H acts naturally from the right on the space $G \times V_\xi$ by means t

$$(g, v) \cdot h = (gh, \xi(h^{-1})v) \qquad (g \in G, h \in H, v \in V_\xi) \tag{1.4}$$

This action is free and proper[6].This allows us to endow $G \times_H V_\xi := (G \times V_\xi)/H$ with a unique C^∞-structure for which the quotient map $\pi_{G\times_H V} : G \times V_\xi \to G \times_H V_\xi$ is a smooth submersion. Analogously, we observe that H acts on G by right multiplication and that this action is again free and proper. Hence G/H may also be endowed with a C^∞ structure making its associated quotient map $\pi_{G/H} : G \to G/H$ become a smooth submersion as well. These quotient maps, along with the projection onto the first component of $G \times V_\xi$, $\mathrm{pr}_G : G \times V_\xi \to G$ induce a new smooth map $p : G \times_H V_\xi \to G/H$ such that the following diagram commutes,

$$\begin{array}{ccc} G \times V_\xi & \xrightarrow{\pi_{G\times_H V_\xi}} & G \times_H V_\xi \\ {\scriptstyle \mathrm{pr}_G}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow[\pi_{G/H}]{} & G/H \end{array}$$

We observe that each fibre $p^{-1}(gH) \subset G \times_H V_\xi$ can be endowed with a linear structure such that the map $\phi_g(v) = \pi_{G\times_H V_\xi}(g, v)$ becomes a linear isomorphim. In fact, this shows that $p : G \times_H V_\xi \to G/H$ has a unique vector bundle structure making the map $\pi_{G\times_H V_\xi}$ a vector bundle morphism. Furthermore, the natural action of G on $G \times V_\xi$ given by left multiplication in the first component induces a smooth action of G on $G \times_H V$ turning it into a G-equivariant vector bundle over G/H. With such structure we can consider the space of continuous sections $\Gamma(G \times_H V_\xi)$ of $G \times_H V_\xi$, endowed with the usual Fréchet topology[7], on which we can define a Fréchet representation Ξ of G given by

$$\Xi(g)(s)(xH) = g \cdot s(g^{-1}xH)$$

[6] We say that a smooth right action of a Lie group G on a manifold M is *proper* if the map $\psi : M \times G \to M \times M$ defined by $\psi(h, m) = (m, m \cdot g)$ is proper in the topological sense; that is, the preimage of a compact set is compact.

[7]A total trivialisation of $G \times_H V$; that is a local trivialisation (U_i, τ_i) of $G \times_H V_\xi$ which is also a chart for G/H, induces a linear a linear isomorphism $\phi_i : \Gamma(E\,|_{U_i}) \to C^\infty(\tau_i(U_i))^r$ for some $r \in \mathbb{N}_{>0}$. Given an index (i, K) where i corresponds to the $i - th$ total trivialisation of $G \times_H V_\xi$ and $K \subset \tau_i(U_i)$ compact, we define the seminorm on $\Gamma(G \times_h V_\xi)$ to be $\|s\|_{(i,K)} = \|\phi_i(s|_{U_i})\|_{\infty,K}$.

This representation is called the *induced representation* of G from the representation ξ of H. This particular realisation is called the *vector bundle picture* and it is customary to denote Ξ by $\mathrm{ind}_H^G(\xi)$. Define

$$C(G:H:\xi) = \{f \in C(G, V_\xi) \mid R_h f = \xi(h)^{-1} f,\ \forall\, h \in H\}$$

This space is a closed subspace of the space $C(G, V_\xi)$ endowed with the usual Fréchet topology; hence a Fréchet space itself. Moreover, since the functions in this space only involve behaviour from the right, $C(G:H:\xi)$ can be regarded as a Fréchet G-module with the left regular representation.

Lemma 1.2.1 (Induced picture)**.** *[13, Lemma 19.3]. $(\mathrm{ind}_H^G(\xi), \Gamma(G \times_H V_\xi)) \simeq (C(G:H:\xi), L)$ as representations. This equivalence is given by the map*

$$\begin{array}{rccc} \Phi: & \Gamma(G \times_H V_\xi) & \longrightarrow & C(G:H:\xi) \\ & s & \longmapsto & \Phi(s)(x) = x^{-1} \cdot s(\pi_{G/H}(x)) \end{array}$$

Let (π, V) be a locally convex complex G-module.

Definition 1.2.1 (Conjugate adjoint of π)**.** We define the *conjugate adjoint representation of* G to be the pair (π^*, V^*)[8] defined by

$$\pi^*(x) = \pi(x^{-1})^* \quad \text{for}\, x \in G.$$

It is straightforward that $\pi^* = \pi$ if and only if π is unitary.

In general it would be convenient that if we start with some Hilbert structure on V_ξ, this is preserved by the induction process. Unfortunately, the unitarity of the representation is not generally preserved when inducing. Then we have to performed what is called *normalised induction*. Consider a finite dimensional Hilbert H-module (ξ, V_ξ). We define $C_c(G:H:\xi)$ as the subset of functions φ of $C(G:H:\xi)$ such that $\pi_{G/H}(\mathrm{supp}\ \varphi)$ is compact. May the reader observe that $C_c(G:H:\xi)$ is a G-invariant subspace with the left regular representation. Therefore, it is a G-module on its own.

The square root of the modular function $\Delta^{1/2}$ [9] defines a one-dimensional representation of H that we may tensor with the H-module V_ξ. The tensor product representation is isometrically realised in V_ξ endowed with the representation

$$(\xi \otimes \Delta^{1/2})(h)v = \Delta^{1/2}(h)\xi(h)v \quad \text{for}\ v \in V_\xi, h \in H.$$

The inner product in V_ξ induces the following sesquilinear map

$$\langle \cdot, \cdot \rangle_\xi : C_c(G:H:\xi \otimes \Delta^{1/2}) \times C_c(G:H:\xi^* \otimes \Delta^{1/2}) \longrightarrow C_c(G:H:\Delta).$$

given by $\langle \varphi, \psi \rangle_\xi(x) = \langle \varphi(x), \psi(x) \rangle_\xi$. Then for $\omega \in \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})^*$; that is, a top order differential form on $\mathfrak{g}/\mathfrak{h}$ we can define the following compactly supported H-invariant map

$$\langle \varphi, \psi \rangle_{\xi,\,\omega}(x) = \langle \varphi, \psi \rangle_\xi(x)((l_{x^{-1}})^*|\omega|)(x), \tag{1.5}$$

[8] For V a complex topological vector space, we shall henceforth denote by V^* the topological antilinear dual and by V' the topological linear dual. Hence $V^* = \overline{V'}$

[9] See appendix for a concrete treatment of the modular function.

where l_x means left multiplication by the element $x \in G$. This descends to a compactly supported density on G/H with which we define the following G-equivariant sesquilinear map

$$\langle \varphi, \psi \rangle = \int_{G/H} \langle \varphi, \psi \rangle_{\xi,\, \omega} \tag{1.6}$$

Lemma 1.2.2. *[13, lemma 19.12]. If ξ is a unitary representation, the pairing (1.6) defines a pre-Hilbert structure on $C_c(G : H : \xi \otimes \Delta^{1/2})$ for which the representation $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\xi \otimes \Delta^{1/2})$ extends to a unitary map in the Hilbert completion $L^2(G : H : \xi \otimes \Delta^{1/2})$.*

Remark. In the rest of the text, we write $\mathrm{Ind}_H^G(\xi) = \mathrm{ind}_H^G(\xi \otimes \Delta^{1/2})$. Note the change to capital letters.

1.2.1 K-finite vectors and C^∞-vectors of a representation

In the rest of the theory we will work with two type of vectors: K-finite and C^∞-vectors. In the present subsection, we give a brief description of what they consist of. Define $\widehat{K}$ to be the set of equivalence classes of irreducible finite-dimensional continuous representations of K.

Definition 1.2.2 (K -finite vector)**.** Let K be a compact group and (π, V) a locally convex K-module. We say that a vector $v \in V$ is K-finite if Span $\pi(K)v$ is a finite dimensional vector space. The space of all such vectors shall be denoted by V_K.

Let $V[\delta]$ denote the *isotypic component of type* $\delta \in \widehat{K}$; that is, the space of vectors $v \in V$ for which Span $\pi(K)v$ is equivalent as a representation to $V^{\oplus\, m(\delta)}$ for some natural number $m(\delta)$. Clearly $V[\delta] \subset V_K$.

Lemma 1.2.3 (Decomposition in K-types)**.** *[13, Proposition 3.5]. Let (π, V) be a continuous locally convex representation of K on V. The following statements are true:*

1. *For each $\delta \in \widehat{K}$, $(V_\delta \otimes \mathrm{Hom}_K(V_\delta, V), \delta \otimes 1) \simeq \left(V[\delta], \pi|_{V[\delta]}\right)$ as representations.*

2. *We have the K-type decomposition*

$$V_K = \bigoplus_{\delta \in \widehat{K}} V[\delta].$$

It is well-known in Lie theory that any continuous finite dimensional representation is smooth. This is basically because any continuous homomorphism between Lie groups is automatically a Lie group homomorphism. However, we are interested in considering infinite dimensional G-modules. This is where the notion of *smooth vector of a representation* comes in.

Definition 1.2.3 (C^∞-vector of a representation)**.** Let (π, V) be a continuous Fréchet G-module. We say that $v \in V$ is a C^∞ vector if the function $\psi_v(g) := \pi(g)v$ belongs to $C^\infty(G, V)$. We shall denote the space of smooth vectors of (π, V) by V^∞.

The first that we observe is that the space V^∞ is a G-invariant subspace of V. Let r_x denote the right multiplication map by the element $x \in G$. Then

$$\psi_{\pi(x)v}(g) = \pi(gx)v = (r_x^*\psi_v)(g),$$

meaning that $\pi(x)v$ is a smooth vector for every $x \in G$ if and only if v is also a smooth vector. Then V^∞ becomes a G-module by restriction of π to V^∞. Denote this representation by π^∞. The following shows that V^∞ may also be endowed with a natural $\mathfrak{g}$-module structure.

Lemma 1.2.4. *[13, lemma 21.4 and lemma 21.5]. Let (π, V) be a G-module. The map*

$$\pi_* : \mathfrak{g} \longrightarrow \mathrm{End}(V^\infty), \quad \pi_*(X)v = \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)v$$

is a Lie algebra representation of $\mathfrak{g}$. Moreover, the following holds

$$\pi(x) \circ \pi_*(X) = \pi_*(\mathrm{Ad}(x)X) \circ \pi(x) \qquad \forall x \in G, \forall X \in \mathfrak{g}.$$

We observe that we have endowed the space V^∞ with two module structures, namely, the $\mathfrak{g}$-module structure from Lemma 1.2.4 and the G-module structure π^∞ earlier defined. If we restrict the latter to the associated maximal compact subgroup K, we get what is commonly known in Lie theory as *the underlying $(\mathfrak{g}, K)$-module of* V. This notion was introduced by Harish-Chandra in a more general way as follows

Definition 1.2.4 ($(\mathfrak{g}, K)$-module)**.** A $\mathbb{C}$-linear space V is said to be a $(\mathfrak{g}, K)$-module if it has both structures of $\mathfrak{g}$-module and K-module such that

1. $V = V_K$, endowed with coarsest topology that makes the K-intertwining embeddings $V_\delta \hookrightarrow V_K$ continuous for all $\delta \in \widehat{K}$.
2. The identity $\pi(k) \circ X = \mathrm{Ad}(k)X \circ \pi(k)$ must be satisfied for all $k \in K$ and $X \in \mathfrak{g}$.
3. The action of $X \in \mathfrak{k}$ on $v \in V$ follows the rule:

$$Xv = \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)v.$$

Lemma 1.2.5. *[4, Lemmas 8.1, 8.5] K-finite vectors of a unitary irreducible representation of a connected semisimple Lie group are smooth vectors.*

It is convenient to work with the space of smooth vectors of a G-module because it sits densely inside the G-module. An account of this fact can be found in [13, Lemma 21.8]. We can apply the previous discussion to the induced representations. Let (ξ, V_ξ) be a continuous finite dimensional Hilbert representation of a closed subgroup H of a Lie group G. The G-equivariant map Φ in Lemma 1.2.1 restricts to a G-equivariant map $\tilde{\Phi}$ in the respective spaces of smooth vectors such that the following diagram commutes

$$\begin{array}{ccc}
(\mathrm{ind}_H^G(\xi), \Gamma(G \times_H V_\xi)) & \xrightarrow{\ \Phi\ } & (L, C(G : H : \xi)) \\
\downarrow & & \downarrow \\
(\mathrm{ind}_H^G(\xi)^\infty, \Gamma^\infty(G \times_H V_\xi)) & \xrightarrow[\tilde{\Phi}]{} & (L, C^\infty(G : H : \xi))
\end{array}$$

and where Γ^∞ is the space of smooth sections of the bundle $G \times_H V$ and $C^\infty(G : H : \xi)$ is the subspace of smooth functions in $C(G : H : \xi)$.

1.3 The principal series

With the Iwasawa decomposition and the induction process we are ready to define one of the main objects in this thesis: *the principal series representations*. Roughly speaking, these are normalised parabolic induced representations with respect to certain characters that we will hereunder describe. At a later stage, we shall be interested in certain matrix coefficients of the principal series (see Definition 2.3.1) that will satisfy certain ODE, yet to be specified. In this section we shall be working with the following representations:

- A unitary representation ξ of M on a finite dimensional Hilbert space V_ξ.
- The representation induced by constant unitary character 1 on N.
- The representation induced by the character $(\cdot)^\lambda$ on A, for $\lambda \in \mathfrak{a}_\mathbb{C}$, which is given by the expression (1.2).

In view of lemma 1.1.8, we can consider the tensor product representation of the three of them as a representation of P on $V_\xi \otimes \mathbb{C}_\lambda \otimes \mathbb{C}_1$, which can be naturally realised on V_ξ as

$$(\xi \otimes \lambda \otimes 1)(man)v = a^\lambda \xi(m)v \quad \text{for } v \in V_\xi.$$

The next definition is in order.

Definition 1.3.1 (Principal series representation)**.** The G-module $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ is called the normalised principal series representation of G with parameters ξ and λ, where $\xi \in \widehat{M}$ is irreducible and unitary and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. In particular, if $\xi = 1$ we speak about the *spherical induced representations*.

Remark. We point out to the reader the use of capital letters in $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ since we are considering normalised induction.

We consider

$$\rho_P(H) = \frac{1}{2}\mathrm{tr}\left(\mathrm{ad}(H)|_\mathfrak{n}\right) = \frac{1}{2}\sum_{\alpha \in \Sigma^+} \dim \mathfrak{g}_\alpha \, \alpha(H) \quad (\text{for } H \in \mathfrak{a}_\mathbb{C})$$

Lemma 1.3.1. *[13, Lemma 20.3]. The modular function of P, as defined in the previous section, is given by*

$$\Delta(p) = \Delta(m(p)a(p)n(p)) = a^{2\rho_P}$$

Accordingly, $(\xi \otimes \lambda \otimes 1) \otimes \Delta^{1/2} = \xi \otimes (\lambda + \rho_P) \otimes 1$. Consequently,

$$\pi_{P,\xi,\lambda} := \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1) = \mathrm{ind}_P^G(\xi \otimes (\lambda + \rho_P) \otimes 1).$$

By Lemma 1.2.1, we can consider the induced realisation of the principal series that looks like

$$C(G : P : \xi \otimes \lambda \otimes 1) = \{f \in C(G, V_\xi) \mid R_p f = \xi(m(p))^{-1} a(p)^{-\lambda - \rho_P} f, \, \forall \, p \in P\}$$

equipped with the left regular representation. In order that the text does not become notationally very heavy, we shall as of now denote the previous space by $C(P : \xi : \lambda)$ and the left regular representation by $\pi_{P,\xi,\lambda}$.

We observe by a simple computation that $(\xi\otimes\lambda\otimes 1)^* = \xi^*\otimes -\overline{\lambda}\otimes 1 = \xi\otimes -\overline{\lambda}\otimes 1$, as we have assumed that ξ is unitary. In particular, the one dimensional representation defined by $\xi\otimes\lambda\otimes 1$ is unitary if and only if $\lambda \in i\mathfrak{a}^*$. Since G/P is compact, $C_c(P:\xi:\lambda) = C(P:\xi:\lambda)$ and the bilinear form given in equation (1.6) provides the principal series representation with a non-degenerate sesquilinear G-equivariant pairing with respect to the representations $\pi_{P,\xi,\lambda}$ and $\pi_{P,\xi,-\overline{\lambda}}$. According to Lemma 1.2.2, the pairing defines a pre-Hilbert structure on $C(P:\xi:\lambda)$ extending $\pi_{P,\xi,\lambda}$ to a unitary representation on $L^2(P:\xi:\lambda)$. However, the structure presented by the Iwasawa decomposition, simplifies the pairing to integration over K, in the following sense.

Theorem 1.3.1. *[13, Theorem 20.5]. Let dk be the normalised Haar measure on K. Then the form $\omega \in \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$ defined in formula (1.6) can be normalised so that for $\varphi \in C(P:\xi:\lambda)$ and $\psi \in C(P:\xi:-\overline{\lambda})$*

$$\langle \varphi,\psi\rangle = \int_{G/P} \langle \varphi,\psi,\rangle_{\xi,\omega} = \int_K \langle \varphi(k),\psi(k)\rangle_\xi \, dk$$

Another realisation of the principal series plays an important role. This is the *compact picture*. Observe that M is a closed subgroup of K, hence we can make sense of $\mathrm{ind}_M^K(\xi)$ for $\xi \in \widehat{M}$. Since K is compact, $\Delta_K \equiv 1$ and therefore $\mathrm{ind}_M^K(\xi) = \mathrm{Ind}_M^K(\xi)$. By Lemma 1.2.1, we can realise it as

$$C(K:M:\xi) = \{f \in C(K,V_\xi) \mid R_m f = \xi^{-1}(m) f, \, \forall\, m \in M\}$$

with the restriction to K of the left regular representation.

Lemma 1.3.2. *[13, Lemma 20.6]. The map $r_K(\lambda) : C(P:\xi:\lambda) \to C(K:M:\xi)$, defined as the restriction to K, is a K-equivariant topological linear isomorphism.*

Via the linear isomorphism, $r_K(\lambda)$ we endow $C(K:M:\xi)$ with the structure of G-module making this linear isomorphism G-equivariant. This realisation is called the *compact picture* and we shall denote it by $(\pi_{P,\xi}, C(K:M:\xi))$. By means of the Iwasawa decomposition as in Theorem 1.1.3, an easy computation shows that if $f \in C(K:M:\xi)$, it follows that

$$(\pi_{P,\xi}(x)f)(\tilde{k}) = (r_K(\lambda)^{-1} f)(x^{-1}\tilde{k}) = a(x^{-1}\tilde{k})^{-\lambda-\rho_P} f(k(x^{-1}\tilde{k}))$$

for $f \in C(K:M:\xi)$ and all $x \in G$ and $\tilde{k} \in K$.

Remark. We will see further on that the advantage in considering the compact picture realisation is that the dependence on λ is removed from the representation space. This will allow to perform analytic continuation in certain functions to be regarded in the next chapter.

1.3.1 Generalised section

In this subsection, we describe what generalised sectionsof the principal series are. This will be central to define Whittaker vectors in the next chapter. For a more general definition, we refer to [12].

Definition 1.3.2 (Generalised vectors for the principal series)**.** The space of generalised vectors for the principal series representation is defined to be the antilinear topological dual space of $C^\infty(P:\xi:-\overline{\lambda})$ equipped with the strong topology, we denote it by

$$C^{-\infty}(P:\xi:\lambda) := C^\infty(P:\xi:-\overline{\lambda})^*$$

Remark. In general, generalised sections are defined as some topological dual of compactly supported sections. However, we have seen before that $C_c^\infty(P:\xi:\lambda) = C^\infty(P:\xi:\lambda)$.

The sesquilinear pairing given in (1.6) provides a linear continuous embedding $C(P:\xi:\lambda) \hookrightarrow C^{-\infty}(P:\xi:\lambda)$. The space of generalised vectors has a G-module structure that can be described in the following manner: $\pi_{P,\xi,\lambda}(g)\eta = \eta\circ\pi_{P,\xi,-\overline{\lambda}}(g^{-1})$ for $g\in G$ and $\eta \in C^{-\infty}(P:\xi:\lambda)$. This representation extends the one in $C^\infty(P,\xi,\lambda)$ justifying the slight abuse of notation.

The G-equivariant topological linear isomorphism in Lemma 1.3.2 induces a G-equivariant topological linear isomorphism by dualising between $C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(K:M:\xi)$ where $C^{-\infty}(K:M:\xi)$ is defined in the same manner as for the principal series and it is endowed with the strong topology, as well. In the next lemma, we see that the behaviour that the functions of the induced picture of the principal series representation have can be extended to the space of the generalised sections of the principal series representation.

Lemma 1.3.3. *[9, Lemma 1.42]. The embedding $C^\infty(P:\xi:\lambda) \hookrightarrow C^\infty(G,V_\xi)$ extends uniquely to an embedding $C^{-\infty}(P:\xi:\lambda) \hookrightarrow \Gamma^{-\infty}(G\times V_\xi)$, which is G-equivariant when $\Gamma^{-\infty}(G,V_\xi)$ in equipped with the left regular representation. Furthermore, the image of this embedding sits in*

$$\left\{\eta\in C^{-\infty}(G,V_\xi) \mid R_p\eta = \xi(m(p))^{-1}a(p)^{-\lambda-\rho_P}\eta,\ \forall p\in P\right\}$$

1.4 The discrete series

The *discrete series representation* is the second main character of our story. In the last chapter we will see that residues in the Fourier-Whittaker inversion formula for $\mathrm{SL}_2(\mathbb{R})$ (see Section 3.3) appear as matrix coefficients of this sort of representations. This means that there is a contribution of the discrete series in the Whittaker-Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$. In this section we start by defining the concept of matrix coefficient and discrete series representation. We shall see that in the case of $\mathrm{SL}_2(\mathbb{R})$, we will be inducing from certain unitary characters of $\mathrm{SO}(2)$ in order to construct the discrete series representations. We will study a particular model in which to realise the discrete series representations of $\mathrm{SL}_2(\mathbb{R})$. We shall begin by introducing the concept of *matrix coefficient map* of a representation.

Definition 1.4.1 (Matrix coefficient map)**.** Let (π, V) be a G-module. The *matrix coefficient map* is the G- equivariant map $m:(V\otimes V^*, \pi\otimes\pi^*) \to (C(G), L\times R)$ given by $m(v\otimes\eta) = \langle\pi(g^{-1})v,\eta\rangle$. Whenever, $v\otimes\eta\in V\otimes V^*$ is fixed we speak of a matrix coefficient.

We are now in place to define what a discrete series representation is.

Definition 1.4.2 (Discrete series representation of G)**.** We say that an irreducible unitary representation π of G is of the discrete series if some non-zero K-finite matrix coefficient is in $L^2(G)$.

The following is criteria to check whether a representation is of the discrete series can be found in [4, Proposition 9.6]. This is

Lemma 1.4.1. *The following statements are equivalent for an irreducible unitary representation π of G.*

1. *π is of the discrete series.*
2. *All matrix coefficients of π are square integrable.*
3. *If π is a irreducible subrepresentation of $L^2(G)$ endowed with the left regular representation.*

1.4.1 Holomorphic discrete series representation for $\mathrm{SL}_2(\mathbb{R})$

Once familiarised with the concept of discrete series representations, we shall study a model for the discrete series representations for $\mathrm{SL}_2(\mathbb{R})$. The result corresponding [4, Theorem 12.21] guarantees that the model we are about to construct is unique up to equivalence. This model is based on the upper-half plane, so we retain the same framework as in subsection 1.1.1.

Firstly, we recall which form the K-types have. Since K is compact and abelian, all K-types are continuous one-dimensional representations, hence they are given by continuous characters on K. Again, compactness of K implies that each one of these characters is unitary, meaning, $\tau(k_\varphi) = e^{i\varphi\, d\tau(0)(Y-X)}$. Since $\tau(k_0)$ has to be equal to $\tau(k_{2\pi})$, $d\tau(0)(Y-X)$ is an integer. Thence

$$\widehat{K} = \{\tau_n : K \to \mathbb{C} \mid \tau_n(k_\varphi) = e^{in\varphi}, n \in \mathbb{Z}\}.$$

Let g_0 be as in Lemma 1.1.11. The K-types can be extended to characters on $\overline{B}_{\mathbb{C}}$ given by

$$\tau_n\left(\exp zi(Y-X)(\exp w\mathrm{Ad}(g_0)Y)\right) = e^{nz}$$

for $z, w \in \mathbb{C}$. We henceforward denote by $\mathbb{C}_n$ the one-dimensional $\overline{B}_{\mathbb{C}}$-module induced by the character τ_n. The discrete series representation for G will arise from a holomorphic induction procedure. Specifically, the holomorphic structure will be derived from the holomorphic line bundle $\mathcal{L}_n = G_{\mathbb{C}} \times_{\overline{B}_{\mathbb{C}}} \mathbb{C}_n$.

Unfortunately, if we desire to proceed and induce like in the previous section, it might be that the space of global holomorphic sections is trivial. Hence, we must induce locally. Let U_0 be an open set in $G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$ and U the preimage of U_0 under the quotient map $\pi_{G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}} : G_{\mathbb{C}} \to G_{\mathbb{C}}/\overline{B}_{\mathbb{C}}$. The space of local holomorphic sections over U_0 of the bundle $\mathcal{L}_n$, say $\Gamma^{\mathbb{C}}(U_0, \mathcal{L}_n)$, can be identified with the space

$$\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n) := \left\{f \in \mathcal{O}(U) \mid R_{\overline{b}} f = \tau_n(\overline{b})^{-1} f,\ \forall\, \overline{b} \in \overline{B}_{\mathbb{C}}\right\}$$

where $\mathcal{O}(U)$ means the space of holomorphic complex valued functions on U. Clearly, with the left regular representation, this becomes a G-module if U_0 is G-invariant. Now we apply the discussion of the previous paragraph to $U_0 = G/K \simeq NA$. Then

U in our case corresponds to $NA\overline{B}_{\mathbb{C}}$, which equals $G\overline{B}_{\mathbb{C}}$ because $K \subset G \cap \overline{B}_{\mathbb{C}}$. By right invariance, functions in $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ are determined if we know their value at NA. That is to say, if $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$, then

$$f(g\bar{b}) = f(nak\bar{b}) = \tau_n(\bar{b})^{-1}\tau_n(k)^{-n}f(na) \quad (\text{for } nak \in NAK, \bar{b} \in \overline{B}_{\mathbb{C}})$$

Lemma 1.4.2. *The function $\sigma_n : NA \to \mathbb{C}$ given by*

$$\sigma_n(na) = a^{-n\rho}$$

defines a nowhere vanishing function in $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ for $n \in \mathbb{Z}$. Furthermore, $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n) = \mathcal{O}(NA)\sigma_n$

Proof. We treat the case $n = 1$ in the first place. Consider the natural representation of $G_{\mathbb{C}}$ on $\mathbb{C}^2$ given by matrix multiplication. Define the standard pairing $\beta : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by

$$\beta(z, w) = z^t w = \begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The bilinear map β induces a linear isomorphism between $\mathbb{C}^2$ and $(\mathbb{C}^2)'$. Our aim is to find vectors $u, v \in \mathbb{C}^2$ such that $k \cdot u = \tau_1(k)^{-1}u$ for all $k \in K$ and $\beta(n^t v, \cdot) = \beta(v, \cdot)$ for all $n \in N$. One can easily prove that $u = e_1 + ie_2$ and $v = -ie_2$ do the job. Then we consider the matrix coefficient of the $G_{\mathbb{C}}$- module $\mathbb{C}^2$ given by

$$m_{-ie_2, e_1+ie_2}(na) = \beta(-ie_2, na \cdot (e_1 + ie_2)) = \beta(-ie_2, a \cdot (e_1 + ie_2)) = a^{-\rho}.$$

The representation of $G_{\mathbb{C}}$ on $\mathbb{C}^2$ is holomorphic, since $G_{\mathbb{C}}$ has a structure of complex Lie group and any finite dimensional representation of $G_{\mathbb{C}}$ is holomorphic. Therefore, the previous matrix coefficient is holomorphic. By holomorphic continuation, we can extend to $k \in K_{\mathbb{C}}$

$$k \cdot (e_1 + ie_2) = \tau_1(k)^{-1}(e_1 + ie_2)$$

We also see that $g_0 \overline{n} g_0^{-1} \cdot (e_1 + ie_2) = (e_1 + ie_2)$ for $\overline{n} \in \overline{N}_{\mathbb{C}}$, as $g_0^{-1} \cdot e_2 = e_1 + ie_2$ Then it follows that

$$\bar{b} \cdot (e_1 + ie_2) = \tau_1(\bar{b})^{-1}(ie_1 + e_2) \quad (\forall\, \bar{b} \in \overline{B})$$

Hence $m_{-ie_2, e_1+ie_2} \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_1)$. For general σ_n, we observe that $\sigma_n = (\sigma_1)^n$, thus nowhere vanishing and $\sigma_n \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$. It is also rather clear that $\mathcal{O}(NA)\,\sigma_n \subset \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$. To prove the other inclusion, we observe that if $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ then $f\sigma_n^{-1} \in \mathcal{O}(G : K : \tau_0) = \mathcal{O}(NA)$. □

Recall that the exponential map defines diffeomorphisms $\mathfrak{a} \to A$ and $\mathfrak{n} \to N$. As finite-dimensional vector spaces, we can consider dt and dx the respective Lebesgue measures on $\mathfrak{a}$ and $\mathfrak{n}$. We now may define da, dn to be the left Haar measures on A and N, respectively; such that $\exp^* da = dt$ and $\exp^* dn = dx$. Let X_n be the subspace of functions f belonging to $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ such that $f\sigma_n^{-1}$ is $L^2(NA, a^{-2\rho}dadn)$. One can show that the inner product in $L^2(NA, a^{-2\rho_{\overline{P}}}dadn)$ defines a pre-Hilbert structure on X_n. By means of the upper-half plane realisation, we will see that it is actually a Hilbert space and that the left regular representation is unitary, for $n \leqslant 2$.

Definition 1.4.3 (Holomorphic discrete series representation of $\mathrm{SL}_2(\mathbb{R})$)**.** We call the G-module (X_{-n}, L) for $n \geqslant 2$ the *holomorphic discrete series representation.*

Remark. The reader should be aware that we have not proven yet that the previously defined representation is of the discrete series. This comes later on.

The holomorphic discrete series representation can be realised on the upper-half plane, as well. Consider the following G-module: for $n \geqslant 2$ and $f \in H_n^+ = \mathcal{O}(\mathcal{H}^+) \cap L^2(\mathcal{H}^+, y^{n-2}dxdy)$, define

$$D_n^+(g)f(z) = (a - cz)^{-n} f(g^{-1}z) \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

with the following pairing

$$\langle \cdot, \cdot \rangle : H_n^+ \times H_n^+ \longrightarrow \mathbb{C}, \quad \langle \varphi, \psi \rangle = \int_{\mathcal{H}} \varphi(x + iy)\overline{\psi(x + iy)}\, y^{n-2}dxdy$$

In addition, one can show by the Cauchy integral formula, that the space H_n^+ is closed. This means that H_n^+ is already complete with the L^2-norm, hence a Hilbert space itself. Moreover, it is readily seen that the D_n^+ is unitary with respect to this inner product. The following facts will be useful in the theory to come. They can be proven by computation so the proofs are left to the reader.

Lemma 1.4.3.

$$\mathrm{Im}(g \cdot z) = \frac{\mathrm{Im}(z) \det g}{|cz + d|^2} \quad \textit{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

Lemma 1.4.4. *Let $j : NA \to \mathcal{H}$ be the map as in Section 1.1.1. For $f \in X_{-n}$ with $n \geqslant 2$, we have that*

$$f(g^{-1}j^{-1}(z)) = \left(\frac{a - cz}{|a - cz|}\right)^{-n} f(j^{-1}(g^{-1} \cdot z)) \quad \textit{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \textit{ and } z \in \mathbb{C}$$

We are now ready to establish the equivalence between the two realisations of the holomorphic discrete series previously defined.

Lemma 1.4.5. $(X_{-n}, L) \simeq (H_n^+, D_n^+)$ *as representations of G.*

Proof. In order to prove the statement, we define the following linear map

$$T : (X_{-n}, L) \to (H_n^+, D_n^+), \quad f \mapsto T(f) = \frac{1}{\sqrt{2}}\, f\sigma_n|_{NA} \circ j^{-1}.$$

We prove first that T is well-defined. Since $f \in \mathcal{O}(\Omega : \overline{B}_{\mathbb{C}} : \tau_{-n}) = \mathcal{O}(NA)\sigma_{-n}$. Then $f\sigma_n \in \mathcal{O}(NA)$, hence $T(f) \in \mathcal{O}(\mathcal{H})$. We can rewrite $T(f)$ for $f \in X_{-n}$ in the following form

$$\begin{aligned} \sqrt{2}T(f)(z) &= f\sigma_n(j^{-1}(z)) = f(j^{-1}(x + iy))\sigma_n(j^{-1}(x + iy)) \\ &= f(j^{-1}(x + iy))\sigma_n(n_x a_{\log\sqrt{y}}) = f(j^{-1}(x + iy))y^{-n/2} \\ &= f(j^{-1}(z))\mathrm{Im}(z)^{-n/2} \end{aligned}$$

Computing the L^2-norm

$$2\|T(f)\|^2_{H_n^+} = \int_{-\infty}^{\infty}\int_0^{\infty}((j^{-1})^*|f|^2)(x+iy)y^{-n}y^{n-2}\,dydx = \int_{NA}|f|^2 j^*\left(\frac{dxdy}{y^2}\right)$$

The last equality holds by change of variables under for the diffeomorphism j^{-1}. By computing, we get that

$$j^*\left(\frac{dxdy}{y^2}\right) = 2a^{-4\rho}a^{2\rho}dnda = 2a^{-2\rho}dnda$$

Hence $\|T(f)\|_{H_n^+} = \|f\|_{X^{-n}} < \infty$. This shows that T is well-defined and that it is a linear isometry. One can readily show that T is bijective with inverse $T^{-1}(f) = (f \circ j)\sigma_{-n}$. Only G-equivariance of T remains to be shown. One may observe that this is equivalent to showing that, for $f \in X_{-n}$, $g \in G$ and $z \in \mathcal{H}$,

$$f(g^{-1}j^{-1}(z))\mathrm{Im}(z)^{-n/2} = (a-cz)^{-n}\mathrm{Im}(g^1 \cdot z)f(j^{-1}(g^{-1}\cdot z))$$

The previous expression clearly holds as a consequence of Lemmas 1.4.3 and 1.4.4 combined. □

The conjugate dual with respect the H_n^+-pairing of the holomorphic discrete series representation plays a crucial role in this theory . These will be denoted by (H_n^-, D_n^-), which under the previous pairing, it can be described as

$$D_n^-(g)f(z) = (c\overline{z}+a)^{-n}f(g^{-1}z) \quad \text{for } f \in H_n^- = \overline{\mathcal{O}(H) \cap L^2(\mathcal{H}, y^{n-2}dxdy)}$$

This is usually called the *antiholomorphic discrete series representation.*

Lemma 1.4.6. *[4, Proposition 2.7]. The discrete series representations D_n^+ are irreducible.*

Lemma 1.4.7. *$D_n^\pm$ for $n \geqslant 2$ are of the discrete series.*

Proof. According to Definition 1.4.2, we only need to find a K-finite square integrable matrix coefficient. Define the function $f(z) = (z+i)^{-n}$ on $\mathcal{H}$. It is clearly holomorphic and its L^2-norm is finite, therefore $f \in H_n^+$. Consider $m_{f,f}$ the matrix coefficient of the discrete series D_n^+. In [4, Proposition 5.28] can be found the computation showing that its L^2 norm is finite. This is based on another realisation of the discrete series. Analogously, D_n^- is also a discrete series representation of G. □

It will be useful to consider the underlying $(\mathfrak{g}, K)$-module of the discrete series representation of G. Consider the functions on H_n^+ given by

$$\psi_{n,k}(z) = \frac{(z-i)^k}{(z+i)^{n+k}}$$

It is easily shown by computation that $D_n^+(k_\varphi)\psi_{n,k} = e^{i(n+2k)\varphi}\psi_{n,k}$, meaning that $\psi_{n,k}$ lies in the isotypic component of type τ_{n+2k}. Performing the K-type decomposition in Lemma 1.2.3, we find that

$$(H_n^+)_K = \bigoplus_{k\in\mathbb{N}}\mathbb{C}\psi_{n,k}.$$

By Lemma 1.2.5, $\mathrm{Span}\{\psi_{n,k} \mid k \in \mathbb{N}\} \subset (H_n^+)^\infty$. We may also compute the associated $\mathfrak{g}$-module. Assume $f \in (H_n^+)^\infty$, it follows that

$$\begin{aligned}
((D_n^+)_*(H)f)(z) &= \left.\frac{d}{dt}\right|_{t=0} e^{-tn} f(e^{-2t}z) = -nf(z) - 2zf'(z) \\
((D_n^+)_*(X)f)(z) &= \left.\frac{d}{dt}\right|_{t=0} f(z-t) = -f'(z) \\
((D_n^+)_*(Y)f)(z) &= \left.\frac{d}{dt}\right|_{t=0} (1-tz)^{-n} f\left(\frac{1}{1-tz}\right) = nzf(z) + f'(z)z^2
\end{aligned}$$

Remark. From this, we observe that if $f \in (H_n^+)^\infty$, then f', zf' are also smooth vectors for the representation. Hence, by induction, $z^k f^{(m)}$ is also a smooth vector for $1 \leqslant k \leqslant m$. This will be important at a latter stage.

1.4.2 Discrete series vs Principal series in $\mathrm{SL}_2(\mathbb{R})$

In this section, we will compare the discrete series representation with the principal series representation of $\mathrm{SL}_2(\mathbb{R})$. The former can be embedded in the latter for the appropriate choice of characters. This discussion will be helpful to understand theory to come further on. In order to establish this embedding we will use Casselman's subrepresentation theorem.

Theorem 1.4.1 (Casselman's subrepresentation Theorem)**.** *[4, Theorem 8.37] Let (V,π) be an irreducible admissible $(\mathfrak{g}, K)$-module. Then there exists $\xi \in \widehat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which there exists $(\mathfrak{g}, K)$-equivariant linear embedding*

$$V_K \hookrightarrow \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)_K$$

Definition 1.4.4. A G-module is said *admissible* if all its isotypic components are finite dimensional.

Remark. It can be shown that all irreducible unitary representations and the principal series representations are admissible. This corresponds to [4, Theorem 8.1 & 8.4], respectively.

In the rest of this section we let $G = \mathrm{SL}_2(\mathbb{R})$ and recover all notation in Section 1.1.1. The following result is a consequence of Lemma 1.4.6, the previous remark and the unitarity of the discrete series representations.

Corollary 1.4.1.1. *Let $n \geqslant 2$. Then $(H_n^+)_K \hookrightarrow \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)_K$ for some $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

The aim for the rest of the section is to compute for which $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\xi \in \widehat{M}$ we have such embedding. First we observe that the holomorphic discrete series has the same behaviour on M as the principal series. Indeed, in the construction of the holomorphic discrete series we require the functions to have τ_n behaviour in the same manner as in the principal series. Recall that the M-types of G are just the K-types of G restricted to M. Since $M = \{\pm I\}$, we have that $\tau_n|_M = 1$ if n is even and $\tau_n|_M = \varepsilon$ if n is odd, where ε denotes the sign function. This means that $\widehat{M} = \{1, \varepsilon\}$.

In order to determine which $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ corresponds to H_n^+, we shall make use of the Casimir operator[10]. Let the $\mathfrak{g}$-standard triple be as in Section 1.1.1. This basis is not normalised with respect to the Killing form. In this basis, the Killing form B is given by the following matrix

$$B = \begin{array}{cc} \begin{array}{ccc} H & X & Y \end{array} & \\ \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} & \begin{array}{c} H \\ X \\ Y. \end{array} \end{array}$$

Then the Casimir operator has the following form

$$\Omega = \frac{1}{8}H^2 + \frac{1}{4}(XY + YX).$$

Nevertheless, after appropriately rescaling the Killing form B, we shall make all computations with the following rescaled Casimir operator

$$\Omega = H^2 + 2(XY + YX).$$

Lemma 1.4.8. *The Casimir operator Ω acts on $\mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ by the scalar $n^2 - 2n$. Furthermore, it acts on $\mathrm{Ind}_{\overline{P}}^G(\xi \otimes \lambda \otimes 1)$ by the scalar $(\lambda + \rho_{\overline{P}})(H)^2 + 2(\lambda + \rho_{\overline{P}})(H)$.*

Proof. We start with the first statement. Let Ω denote the Casimir operator in G and let $g_0 \in G_{\mathbb{C}}$ as in Lemma 1.1.11. By Lemma A.2.5, we know that

$$\Omega = \mathrm{Ad}(g_0)\Omega = \mathrm{Ad}(g_0)H^2 + 2\mathrm{Ad}(g_0)(XY + YX) = \tilde{H}^2 + 2\tilde{X}\tilde{Y} + \tilde{Y}\tilde{X}$$

where $\tilde{H}, \tilde{X}, \tilde{Y}$ mean the respective images of H, X, Y under the map $\mathrm{Ad}(g_0)$. We firstly observe that if $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$ then

$$R_{\tilde{Y}} f(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp t\mathrm{Ad}(g_0)Y) = \left.\frac{d}{dt}\right|_{t=0} f(g) = 0.$$

Secondly, note that the Casimir operator can be rewritten as

$$\Omega = \tilde{H}^2 + 2(\tilde{X}\tilde{Y} + \tilde{Y}\tilde{X}) = \tilde{H}^2 - 2\tilde{H} + 4\tilde{X}\tilde{Y}.$$

These observations together with Lemma A.2.6 yield

$$L_\Omega f = R_\Omega f = R_\Omega f = R_{\tilde{H}}^2 f - 2R_{\tilde{H}} f + 4R_{\tilde{X}} R_{\tilde{Y}} f = R_{\tilde{H}}^2 f - 2R_{\tilde{H}} f.$$

Now, we only need to compute the $R_{\tilde{H}} f$, for $f \in \mathcal{O}(U : \overline{B}_{\mathbb{C}} : \tau_n)$. One can see that $\tilde{H} = \mathrm{Ad}(g_0)H = -i(Y - X)$ and therefore

$$R_{\tilde{H}} f(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp -it(Y - X)) = \left.\frac{d}{dt}\right|_{t=0} e^{nt} f(g) = n.$$

Consequently,

$$L_\Omega f = R_{\tilde{H}}^2 f - 2R_{\tilde{H}} f = (n^2 - 2n)f.$$

[10]For a general overview of the Casimir operator we refer to appendix A.2.2

We prove the second assertion in the same spirit. For $f \in C^\infty(\overline{P} : \xi : \lambda)$ we have

$$R_Y f(g) = \frac{d}{dt}\bigg|_{t=0} f(g\overline{n}_t) = \frac{d}{dt}\bigg|_{t=0} f(g) = 0.$$

$$R_H f(g) = \frac{d}{dt}\bigg|_{t=0} f(ga_{e^t}) = \frac{d}{dt}\bigg|_{t=0} e^{-t(\lambda+\rho_{\overline{P}})(H)} f(g) = -(\lambda + \rho_{\overline{P}})(H) f(g).$$

We have again then

$$L_\Omega f = R_\Omega f = (R_H)^2 f - 2R_H f = ((\lambda + \rho_{\overline{P}})(H)^2 + 2(\lambda + \rho_{\overline{P}})(H)) f.$$

□

Since two irreducible equivalent $\mathfrak{g}$-modules need to have the same infinitesimal character (See [13, Corollary 13.18]), we can equate both actions of the Casimir elements of the previous lemma, yielding

$$n^2 - 2n = ((\lambda + \rho_{\overline{P}})(H)^2 + 2(\lambda + \rho_{\overline{P}})(H)) \iff (n-1)^2 = (\lambda + \rho_{\overline{P}} + 1)(H)^2$$

Using that $\rho_{\overline{P}}(H) = -1$ in G, we have that $\lambda(H) = \pm(n-1)$. The Langland's classification will allow us to exclude one of the previous values. This states that if $\langle \mathrm{Re}\,\lambda, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$ then $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ has a unique irreducible quotient. By dualising we get that if $\langle \mathrm{Re}\,\lambda, \alpha \rangle < 0$ for all $\alpha \in \Sigma^+$ then $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ has a unique irreducible subrepresentation. For a more detailed exposition of the Langlands decomposition and its proof, we refer to [10, Theorem 5.1]. This means that $\mathrm{Ind}_{\overline{P}}^G(\xi \otimes \lambda \otimes 1)$ has a unique irreducible subrepresentation if $\langle \mathrm{Re}\,\lambda, \alpha \rangle > 0$. By the general theory, one can show that the finite dimensional representation of highest weight $-(n-2)\rho_{\overline{P}}$ is embedded in the induced representation $\mathrm{Ind}_{\overline{P}}(\xi \otimes -(n-1)\rho_{\overline{P}} \otimes 1)$. Since $-(n-1)\rho_{\overline{P}}(H) = n-1 > 0$, the principal series $\mathrm{Ind}_{\overline{P}}^G(\xi \otimes -(n-1)\rho_{\overline{P}} \otimes 1)$ cannot have any more subrepresentations, meaning that $(H_n^+)_k$ cannot be embedded in $\mathrm{Ind}_{\overline{P}}^G(\xi \otimes -(n-1)\rho_{\overline{P}} \otimes 1)_K$. Hence

$$(H_n^+)_K \hookrightarrow \mathrm{Ind}_{\overline{P}}^G(\xi \otimes (n-1)\rho_{\overline{P}} \otimes 1)_K.$$

Chapter 2

Whittaker matrix coefficients

Whittaker matrix coefficients wiil be the core of this thesis. In short, these are continuous linear functionals with certain character behaviour on the group N resulting from the Iwasawa decomposition. We motivate them by studying the Whittaker vectors for the principal series representations. Next, we introduce the standard intertwining operator for $\mathrm{SL}_2(\mathbb{R})$, which will allow us to study the asymptototicts of the so-called Whittaker matrix coefficients. This will be a particular matrix coefficient of great interest for the following chapter. When performing the aforementioned asymptotics we shall be left with a particular family of functions: the c-functions. In the sequel of this chapter we shall study their poles, which will be of considerable importance. We wil finish this chapter with a construction of Whittaker vectors for the discrete series representations. Most of this chapter concentrates on the group $\mathrm{SL}_2(\mathbb{R})$.

2.1 Whittaker vectors for the principal series

In the section we motivate the concept of Whittaker vector in generality by constructing the space of Whittaker vectors of the principal series representations. These are generalised sections of the principal series representations with certain character behaviour with respect to N.

Let G be a connected semisimple real Lie group. Consider $\chi : N \to S^1$ be a continuous character on the $N \subset G$ from the Iwasawa decomposition. As we have seen before, a unitary character induces a continuous unitary one-dimensional representation. For such a unitary character it always holds that $d\chi(e)(\mathfrak{n}) \in i\mathbb{R}$.

Definition 2.1.1 (Regular character)**.** We say that the continuous character $\chi : N \to S^1$ is regular if $d\chi(e)\mathfrak{g}_\alpha \neq 0$ for all α simple root[1].

As for now, retain the notation as in Section 1.1.1. According to Definition 2.1.1, regularity in this case is equivalent to requiring that $r_\chi := -i\, d\chi(0) \neq 0$ as $\mathfrak{g}$ has only one simple root (namely $\alpha(H) = 2$). Take the following G-module into consideration,

$$C^\infty(G/N;\chi) = \{f \in C^\infty(G) \mid R_n f = \chi(n)f,\ \forall\, n \in N\}$$

[1]We say that $\alpha \in \Sigma^+$ is simple if it cannot be written as the sum of two other positive roots.

endowed with the left-regular representation. By Lemma 1.2.3, we may decompose the space $C^\infty(G/N;\chi)$ as a direct sum of K-types, that is

$$C^\infty(G/N;\chi)_K = \bigoplus_{n\in\mathbb{Z}} C^\infty(G/N;\chi)[\tau_n]$$

By application of Lemma 1.2.3 again, observe that the τ_n-isotypic component is isomorphic as a K-representation to the space

$$C^\infty(\tau_{-n};\ G/N;\ \chi) := \{f \in C^\infty(G/N;\chi) \mid L_k f = \tau_n(k)f,\ \forall\ k \in K\} \quad (\text{for } n \in \mathbb{Z})$$

Hence, we can write

$$C^\infty(G/N;\chi)_K = \bigoplus_{n\in\mathbb{Z}} C^\infty(\tau_n;\ G/N;\ \chi).$$

Remark. By the Iwasawa decomposition as in Theorem 1.1.3, the restriction to KA induces a topological linear isomorphism r_A between $C^\infty(G/N;\chi)$ and $C^\infty(KA)$, by the Iwasawa decomposition. Analogously, the restriction to A provides a topological linear isomorphism between $C^\infty(\tau_n;G/N;\chi)$ and $C^\infty(A)$.

According to Definition 1.4.1, the matrix coefficient map for the principal series $\pi_{\overline{P},\xi,\lambda}$ corresponds to the map

$$m : C^\infty(\overline{P}:\xi:\lambda) \otimes C^{-\infty}(\overline{P}:\xi:-\overline{\lambda}) \to C^\infty(G)$$

given by $m(\varphi\otimes\eta)(g) = \langle \pi_{\overline{P},\xi,\lambda}(g^{-1})\varphi, \eta\rangle$. Our goal is to find conditions on $\varphi\otimes\eta$ so that it belongs to $m^{-1}(C^\infty(\tau_n;\ G/N;\ \chi))$. Let $\varphi\otimes\eta \in m^{-1}(C^\infty(\tau_n;G/N;\chi))$ and let $k \in K$ and $g \in G$. On the one hand it holds that

$$L_k m(\varphi\otimes\eta)(g) = \langle \pi_{\overline{P},\xi,\lambda}(g^{-1}k)\varphi, \eta\rangle = m(\pi_{\overline{P},\xi,\lambda}(k)\varphi\otimes\eta)(g)$$

Whereas on the other hand, we see that

$$\tau_n(k)^{-1} m(\varphi\otimes\eta)(g) = m(\tau_n(k)^{-1}\varphi\otimes\eta)(g)$$

Equating both sides we get that

$$m((\pi_{\overline{P},\xi,\lambda}(k)\varphi - \tau_n(k^{-1})\varphi)\otimes\eta) = 0 \quad (\forall\, k \in K).$$

Therefore, imposing $\pi_{\overline{P},\xi,\lambda}(k)\varphi = \tau_n(k^{-1})\varphi$ for all $k \in K$, we find that φ is of the form

$$\varphi(g) = \varphi(ka\overline{n}) = \tau_n(k)\varphi(a\overline{n}) = \tau_n(k)a^{-\lambda-\rho_{\overline{P}}}$$

since $\varphi \in C^\infty(\overline{P}:\xi:\lambda)$. In consequence, we observe that φ must be the extension, up to a scalar, of τ_n to $C^\infty(\overline{P}:\xi:\lambda)$. Analogously, we also must have

$$R_n m(\varphi\otimes\eta) = m(\varphi\otimes\pi_{\overline{P},\xi,-\overline{\lambda}}(n)\eta) \quad (\forall\, n \in N)$$

In order that $m(\varphi\otimes\eta)$ belongs to $C^\infty(\tau_n;G/N;\chi)$, it must happen that

$$m(\varphi\otimes\pi_{\overline{P},\xi,-\overline{\lambda}}(n)\eta) = \chi(n)m(\varphi\otimes\eta) = m(\varphi\otimes\chi(n)^{-1}\eta).$$

If in this case we require the condition $\pi_{\overline{P},\xi,-\overline{\lambda}}(n)\eta = \chi(n)^{-1}\eta$, we say that such an η is a *Whittaker vector for the principal series representation* $C^\infty(\overline{P}:\xi:\lambda)$. Then we define

$$\mathrm{Wh}_\chi(\overline{P}:\xi:\lambda) = \left\{\eta \in C^{-\infty}(\overline{P}:\xi:\lambda) \mid \pi_{\overline{P},\xi,\lambda}(n)\eta = \chi(n)^{-1}\eta,\ \forall\ n \in N\right\}$$

Remark. The reader should note that the space that it has been defined above is a subspace of the generalised sections of the principal series, whereas the discussion outlined prior to the definition concerns functions in the induced picture.

The previous definition motivates the following generalisation

Definition 2.1.2. Let (V, π) a G-module and let χ be a character on N. We define the space of Whittaker vectors as

$$\mathrm{Wh}_\chi(V) = \left\{\lambda \in V' : \lambda \circ \pi(n)^{-1} = \chi(n)\lambda,\ \forall\, n \in N\right\}.$$

Remark. In the previous definition, we have used the topological linear dual instead of the toplogical antilinear dual. This is done in accordance with the notation that Wallach uses. Nevertheless, we stick to the notation previously defined for the principal series representation. The relation between both notations is as follows

$$\mathrm{Wh}_\chi(\overline{P} : \xi : \lambda) = \overline{\mathrm{Wh}_\chi\left(\mathrm{Ind}_{\overline{P}}^{G}(\xi \otimes -\overline{\lambda} \otimes 1)\right)^{\infty}}$$

In [3], Hervé Jacquet proved the following result in 1967.

Theorem 2.1.1. *The space of Whittaker vectors of the principal series is one dimensional.*

2.2 Standard intertwining operator

The *standard intertwinning opearator* is an important tool in the study of principal series representations. Among its many applications, we shall be concerned with its relation with Whittaker matrix coefficients. More concretely, we will see in the next section that standard intertwining operators are closely related to the asymptotics of the aforementioned Whittaker vectors. In this section, we shall give a construction of such intertwining operators and study some of its properties in the context of $\mathrm{SL}_2(\mathbb{R})$.

We recall from the first chapter that $\overline{N}$ coming from the Iwasawa decomposition is a closed subgroup, hence a Lie subgroup of G. This means that there exists a choice of left Haar measure $d\overline{n}$. According to lemma A.1.4, $\overline{N}$ is unimodular as it is nilpotent (more concretely abelian). This means that $d\overline{n}$ is also right invariant. Recall the setup of Sections 1.3 and 1.1.1. Consider the following function: for $f \in C^\infty(P : \xi : \lambda)$, define

$$A_{P,\xi,\lambda} f : G \to \mathbb{C}, \quad (A_{P,\xi,\lambda} f)(x) = \int_{\overline{N}} (L^*_{x^{-1}} f)(\overline{n})\, d\overline{n},$$

whenever it makes sense.

Lemma 2.2.1 (Absolute convergence of $A_{P,\xi,\lambda}$ for $\mathrm{SL}_2(\mathbb{R})$)**.** *The previous integral converges if* $\langle \mathrm{Re}\,\lambda, \alpha\rangle > 0$.

Proof. Without loss of generality, we only need to prove that the integral

$$\int_{\overline{N}} |f(\overline{n})|\, d\overline{n} < \infty$$

for every $f \in C^\infty(P : \xi : \lambda)$. This is because the general statement will also be valid for the functions $L^*_{x^{-1}} f \in C^\infty(P : \xi : \lambda)$ and $x \in G$, once proven the previous. The estimate goes as follows

$$\begin{aligned}\int_{\overline{N}} |f(\overline{n})|\, d\overline{n} &= \int_{\overline{N}} |f(k(\overline{n})a(\overline{n})n(\overline{n}))|\, d\overline{n} = \int_{\overline{N}} |f(k(\overline{n}))| a(\overline{n})^{\operatorname{Re}\lambda - \rho_P}\, d\overline{n} \\ &\leqslant \|f\|_{\infty,K} \int_{\overline{N}} a(\overline{n})^{-\operatorname{Re}\lambda - \rho_P}\, d\overline{n} = \|f\|_{\infty,K} \int_0^\infty (1+x^2)^{-\frac{1}{2}(\operatorname{Re}\lambda + \rho_P)(H)}\, dx.\end{aligned}$$

The last integral is convergent if and only if $\frac{1}{2}(\operatorname{Re}\lambda + \rho_P)(H) > \frac{1}{2}$. This is, if and only if $\langle \operatorname{Re}\lambda, \alpha \rangle > 0$ □

Accordingly, we have that if $f \in C^\infty(P : \xi : \lambda)$ and $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$, then $A_{P,\xi,\lambda} f$ is a complex valued function on G. In fact, $A_{P,\xi,\lambda} f \in C^\infty(G)$ because f is smooth and absolutely convergent. The following lemma shows that $A_{P,\xi,\lambda}$ defines an intertwining operator that is called the *standard intertwining operator*.

Lemma 2.2.2 (Standard intertwining operator)**.** *If $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$, the map*

$$\mathcal{A}(P : \overline{P} : \xi : \lambda) := A_{P,\xi,\lambda} : C^\infty(P : \xi : \lambda) \to C^\infty(\overline{P} : \xi : \lambda)$$

defines a linear G-intertwining operator called standard intertwining operator. Furthermore, this map is continuous with respect to the supremum norm $\|\cdot\|_{\infty,K}$.

Proof. By Lemma 2.2.1 it is clear that $A_{P,\xi,\lambda}$ is well-defined. We have to check that its image lies in $C^\infty(\overline{P} : \xi : \lambda)$. We will show this, by showing how it behaves separately in MA, and in $\overline{N}$. Denote by $C_x : G \to G$ the 'conjugation by x' map for $x \in G$. We notice that for $ma \in MA$,

$$(C_{ma})^* d\overline{n} = |\det \operatorname{Ad}(ma)|_{\mathfrak{n}}|\, d\overline{n} = \Delta_P(ma)^{-1} d\overline{n} = a^{2\rho_P} d\overline{n}.$$

Therefore, we have that

$$\begin{aligned}(A_{P,\xi,\lambda} f)(xma) &= \xi(m)^{-1} a^{-\lambda-\rho_P} \int_{\overline{N}} (C_{ma})^* f(x\overline{n}) d\overline{n} \\ &= \xi(m)^{-1} a^{-\lambda-\rho_P} \int_{\overline{N}} f(x\overline{n}) (C_{a^{-1}m^{-1}})^*\, d\overline{n} \\ &= \xi(m)^{-1} a^{-\lambda-\rho_{\overline{P}}} (A^{\overline{P}}_{P,\xi,\lambda} f)(x).\end{aligned}$$

This proves the correct behaviour in MA. The behaviour in $\overline{N}$ follows by left invariance of $d\overline{n}$. Altogether,

$$(A_{P,\xi,\lambda} f)(xma\overline{n}) = \xi(m)^{-1} a^{-\lambda-\rho_{\overline{P}}} (A^{\overline{P}}_{P,\xi,\lambda} f)(x)$$

Hence $A_{P,\xi,\lambda} f \in C^\infty(\overline{P} : \xi : \lambda)$ if $f \in C^\infty(P : \xi : \lambda)$. It is clear that this operator intertwines the left regular representations as

$$\begin{aligned}\mathcal{A}(P : \overline{P} : \xi : \lambda) L_g f(x) &= \int_{\overline{N}} (L_{x^{-1}g} f)(\overline{n})\, d\overline{n} \\ &= \mathcal{A}(P : \overline{P} : \xi : \lambda) f(g^{-1}x) \\ &= (L_g \circ \mathcal{A}(P : \overline{P} : \xi : \lambda) f)(x)\end{aligned}$$

for every $g, x \in G$ and for every $f \in C^\infty(P : \xi : \lambda)$. By the equivariance of the map and the estimate in proof of Lemma 2.2.1

$$\left|\mathcal{A}(P : \overline{P} : \xi : \lambda)f(k)\right| = \int_{\overline{N}} |L_{k^{-1}}f(\overline{n})|\, d\overline{n} \leqslant C\|L_{k^{-1}}\|_{\infty,K} \leqslant C\|f\|_{\infty,K};$$

from where the continuity of $\mathcal{A}(P : \overline{P} : \xi : \lambda)$ follows. □

The standard intertwining operator can be realised in the space $C^\infty(K : M : \xi)$ satisfying the following commutative diagram

$$\begin{array}{ccc} C^\infty(P : \xi : \lambda) & \xrightarrow{\mathcal{A}(P:\overline{P}:\xi:\lambda)} & C^\infty(\overline{P} : \xi : \lambda) \\ \big\downarrow r_K(\lambda)^P & & \big\downarrow r_K(\lambda)^{\overline{P}} \\ C^\infty(K : M : \xi) & \xrightarrow{A_\lambda(P:\overline{P})} & C^\infty(K : M : \xi) \end{array}$$

where $r_K(\lambda)^P$ and $r_K(\lambda)^{\overline{P}}$ are the G-equivariant linear isomorphisms given in 1.3.2 for P and $\overline{P}$ respectively. It will be of importance to us to give a formula for $A_\lambda(P : \overline{P})$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Let $f \in C^\infty(K : M : \xi)$.

$$\begin{aligned} A_\lambda(P : \overline{P})f(k) =& r^{\overline{P}}_\lambda \circ \mathcal{A}(P : \overline{P} : \lambda : \xi) \circ (r^P_\lambda)^{-1} f(k) = \int_{\overline{N}} (r^P_\lambda)^{-1} f(k\overline{n})\, d\overline{n} \\ =& \int_{\overline{N}} f(kk(\overline{n}))a(\overline{n})^{-\lambda-\rho_P}\, d\overline{n} \end{aligned}$$

It is of considerate importance that A_λ acts on a space that does not depend on the parameter λ. This was not the case of $\mathcal{A}(P : \overline{P} : \xi : \lambda)$. The two following lemmas can be found in [14] and they are related to the holomorphicity of the assignation $\lambda \to A_\lambda$.

Lemma 2.2.3. *The map* $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \langle \operatorname{Re}\lambda, \alpha\rangle > 0, \forall \alpha \in \Sigma^+\} \to \operatorname{End}_G(C^\infty(K : M : \xi))$ *given by* $\lambda \mapsto A_\lambda$ *is holomorphic.*

Theorem 2.2.1 (Vogan-Wallach)**.** *The map in Lemma 2.2.3 can be meromorphically extended to* $\mathfrak{a}^*_{\mathbb{C}}$*.*

Remark. In this section, we have studied the standard intertwining operator for the very specific case of $\mathrm{SL}_2(\mathbb{R})$. Moreover, we have developed the theory for the standard intertwining $\mathcal{A}(\overline{P} : P : \xi : \lambda)$ from P to $\overline{P}$. The reader should notice that the analogous is also possible, namely, the standard intertwining operator from $\overline{P}$ to P. In fact, the latter is the one that we will be henceforward considering. In a more describing note, the standard intertwining operator on a general semisimple Lie group G can also be treated and the different choices of parabolic subgroups that G might have provides a wide range of standard intertwining operators. A clear exposition of the standard intertwining operator can be found in [4, Chapter 7].

2.3 Whittaker matrix coefficient

In the discussion of Section 2.1, we have seen that $\eta \in \mathrm{Wh}_\chi(\overline{P} : \xi : \lambda)$ must behave according to the expression $\pi_{\overline{P},\xi,-\overline{\lambda}}(n)\eta = \chi(n)^{-1}\eta$ for every $n \in N$. However, we could aim for better and look for a function instead of a generalised section. Let $f \in \mathrm{Wh}_\chi(\overline{P} : \xi : -\overline{\lambda}) \cap C^\infty(\overline{P} : \xi : -\overline{\lambda})$. Then clearly, $f(nma\overline{n}) = \chi(n)^{-1}\xi(m)^{-1}a^{\overline{\lambda}-\rho_{\overline{P}}}$, for $nma\overline{n} \in N\overline{P}$. We define the following function

$$\eta_{\overline{P},\xi,-\overline{\lambda}}(x) = \begin{cases} \chi(n)^{-1}\xi(m)^{-1}a^{\overline{\lambda}-\rho_{\overline{P}}} & \text{if} \quad x = nma\overline{n} \in N\overline{P} \\ 0 & \quad\text{otherwise} \end{cases} \tag{2.1}$$

It is clear that this function has the appropriate character behaviour to lie in the space $\mathrm{Wh}_\chi(\overline{P} : \xi : -\overline{\lambda}) \cap C^\infty(\overline{P} : \xi : -\overline{\lambda})$. However, this function is not continuous in general, for all choice of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Lemma 2.3.1. *[9, Proposition 2.11]. The function $\eta_{\overline{P},\xi,\lambda}$ is continuous on G if and only if $\lambda \in U = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \mathrm{Re}\,\lambda + \rho_{\overline{P}}, \alpha\rangle > 0, \forall\alpha \in \Sigma^+\}$.*

Lemma 2.3.1 implies that the map $U \ni \lambda \longmapsto \eta_{\overline{P},\xi,-\overline{\lambda}}$ has no antiholomorphic extension from $\mathfrak{a}_{\mathbb{C}}^*$ to $C(P : \xi : -\overline{\lambda})$ in the usual way. Nonetheless, this map induces another one, denoted by Ψ, in accordance with the following commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[rrr, bend right, "\Psi"'] & C(P : \xi : -\overline{\lambda}) \arrow[r, hook] & C^{-\infty}(P : \xi : -\overline{\lambda}) \arrow[r, "\simeq"] & C^{-\infty}(K : M : \xi) \end{tikzcd}$$

that can be holomorphically extended to $\mathfrak{a}_{\mathbb{C}}^*$. Here we should clarify in what sense we mean "holomorphically". We observe that $C^{-\infty}(K : M : \xi)$ carries a natural filtration given by $C_k^{-\infty}(K : M : \xi) := C^k(K : M : \xi)^*$ for $k \in \mathbb{N}$ and for which the inductive limit topology coincides with the strong dual topology. The space $C^k(K : M : \xi)$ is a complex Banach space, since K is compact. Hence $C_k^{-\infty}(K : M : \xi)$ as well. Let V be a complex linear space and $O \in V$ and open subset. We say that $f : O \to C^\infty(K : M : \xi)$ is *holomorphic* at $z \in O$, if there exists an open neighbourhood O_z of z and $k \in \mathbb{N}$ such that $f : U \to C_k^{-\infty}(K : M : \xi)$ is a holomorphic map of complex Banach spaces. The following theorem corresponds to [9, Proposition 2.18]

Lemma 2.3.2 (Holomorphic extension of $\eta_{\overline{P},\xi,-\overline{\lambda}}$)**.** *Let χ be a regular character on N. Then Ψ extends holomorphically to $\mathfrak{a}_{\mathbb{C}}^*$ in the sense of the previous paragraph.*

Definition 2.3.1 (Whittaker matrix coefficient)**.** We call the following matrix coefficient of the principal series $\mathrm{Ind}_{\overline{P}}^G(\xi \otimes (-\lambda) \otimes 1)$ the $n-th$ *Whittaker matrix coefficient*:

$$\mathrm{Wh}_n(\overline{P}, \lambda)(g) := m(\tau_{n,\overline{P},-\lambda} \otimes \eta_{\overline{P},\xi,\overline{\lambda}})(g) = \langle \pi_{\overline{P},\xi,-\lambda}(g)^{-1}\tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}}\rangle$$

where $\tau_{n,\overline{P},-\lambda}(kan) = \tau_n(k)a^{\lambda-\rho_{\overline{P}}}$ and $\eta_{\overline{P},\xi,\overline{\lambda}}$ follows the expression in (2.1). In the particular case of $\xi = 1$, we say that $\mathrm{Wh}_0(\overline{P}, \lambda)$ is the *Spherical Whittaker matrix coefficient.*

As we have seen in Section 2.1, $\mathrm{Wh}_n(\overline{P}, \lambda) \in C^\infty(\tau_n, G/N, \chi)$. As we have remarked in Section 2.1, the restriction to A according to the Iwasawa decomposition given by $G = KAN$, induces a topological linear isomorphism between

$C^\infty(\tau_n, G/N, \chi)$ and $C^\infty(A)$. That is to say, we only need to study $\mathrm{Wh}_n(\overline{P}, \lambda)\big|_A$. In the following, we shall be concerned with the asymptotic behaviour of the Whittaker matrix coefficients. We will observe that the standard intertwining operator arises in a natural manner leading to the so-called Harish-Chandra c-function. But first a lemma on integration that corresponds to [9, Proposition 1.36]

Lemma 2.3.3. *Assume that $dg, da, dn, d\overline{n}$ are normalised so that*

$$dg = a^{2\rho_{\overline{P}}} dn dm da d\overline{n}.$$

Then for every $f \in C(K)$ it follows that

$$\int_K f(k)\, dk = \int_N \int_M f(k(n)m) a(n)^{-2\rho_{\overline{P}}}\, dm dn,$$

for $k(n)$ and $a(n)$ Iwasawa projections in $KA\overline{N}$.

Using Lemma 2.3.3 and the pairing given in 1.3.1, we may unravel the definition of Whittaker matrix coefficient (restricted to A) in terms of an integral, yielding

$$\begin{aligned}
\mathrm{Wh}_n(\overline{P}, \lambda)(a) &= \int_K \langle \pi_{\overline{P},\xi,-\lambda}(a)^{-1} \tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}} \rangle_\xi (k)\, dk \\
&= \int_K \langle \tau_{n,\overline{P},-\lambda}(k), \eta_{\overline{P},\xi,\overline{\lambda}}(a^{-1}k) \rangle_\xi\, dk \\
&= \int_N \int_M \langle \tau_n(k(n)m), \eta_{\overline{P},\xi,\overline{\lambda}}(a^{-1}k(n)m) \rangle_\xi\, a(n)^{-2\rho_{\overline{P}}}\, dm dn \\
&= \int_N \tau_n(k(n))\, \overline{\eta_{\overline{P},\xi,\overline{\lambda}}}(a^{-1}k(n))\, a(n)^{-2\rho_{\overline{P}}}\, dn;
\end{aligned}$$

where the last equality follows from the normalisation of dm. By means of the Iwasawa decomposition, we may write $k(n) = n\overline{n}(n)^{-1}a(n)^{-1}$ and therefore

$$\begin{aligned}
\mathrm{Wh}_n(\overline{P}, \lambda)(a) &= \int_N \tau_n(k(n))\, \overline{\eta_{\overline{P},\xi,\overline{\lambda}}}(a^{-1}k(n))\, a(n)^{-2\rho_{\overline{P}}}\, dn \\
&= \int_N \tau_{n,\overline{P},-\lambda}(n\overline{n}(n)^{-1}a(n)^{-1})\, \overline{\eta_{\overline{P},\xi,\overline{\lambda}}}(a^{-1}n\overline{n}(n)^{-1}a(n)^{-1}) a(n)^{-2\rho_{\overline{P}}}\, dn \\
&= \int_N \tau_{n,\overline{P},-\lambda}(n) a(n)^{-\lambda+\rho_{\overline{P}}}\, \overline{\eta_{\overline{P},\xi,\overline{\lambda}}}(a^{-1}n) a(n)^{\lambda+\rho_{\overline{P}}} a(n)^{-2\rho_{\overline{P}}}\, dn \\
&= \int_N \tau_{n,\overline{P},-\lambda}(n)\, \overline{\eta_{\overline{P},\xi,\overline{\lambda}}}(a^{-1}n)\, dn \\
&= a^{\lambda+\rho_{\overline{P}}} \int_N \tau_{n,\overline{P},-\lambda}(n)\, \chi(a^{-1}na)\, dn
\end{aligned}$$

Lemma 2.3.4. $\mathrm{Wh}_n(\overline{P}, \lambda) \sim a^{\lambda+\rho_{\overline{P}}} \mathcal{A}(\overline{P} : P : \xi : \lambda) \tau_{n,\overline{P},-\lambda}(e)$ *when* $a \xrightarrow{A^+} \infty$[2].

Proof. According to the previous, it suffices to show that

$$\int_N \tau_{n,\overline{P},-\lambda}(n)\, \chi(a^{-1}na)\, dn \longrightarrow \mathcal{A}(\overline{P} : P : \xi : \lambda) \tau_{n,\overline{P},-\lambda}(e) \quad \text{as} \quad a \xrightarrow{A^+} \infty.$$

[2]This notation means that $a_t \to \infty$ if $t \to \infty$.

Computing the norm of the difference and taking the limit $a \xrightarrow{A^+} \infty$, we have that

$$\left|\int_N \tau_{n,\overline{P},-\lambda}(n)\,\chi(ana)\,dn - \int_N \tau_{n,\overline{P},-\lambda}(n)\,dn\right| \leqslant \int_N \left|\tau_{n,\overline{P},-\lambda}(n)\right| \left|1-\chi(a^{-1}na)\right|\,dn$$
$$= \int_{-\infty}^{\infty} \left|\tau_{n,\overline{P},-\lambda}(n_x)\right| \left|1-\chi(a^{-1}n_x a)\right|\,dx = \int_{-\infty}^{\infty} \left|\tau_{n,\overline{P},-\lambda}(n_x)\right| \left|1-e^{ira^{-\alpha}x}\right|\,dx \longrightarrow 0.$$

□

2.3.1 c-functions

In the previous, we have defined what Whittaker matrix coefficients are and studied their asymptotic behaviour. We have seen that $\mathrm{Wh}_n(\overline{P},\lambda)$ behaves asymptotically as $a^{\lambda+\rho_{\overline{P}}}\mathcal{A}(\overline{P}:P:\xi:\lambda)\tau_{n,\overline{P},-\lambda}(e)$ when a tends to infinity in the positive Weyl chamber A^+. We define the c_n-*function* to be the function

$$c_n(\lambda) = \mathcal{A}(\overline{P}:P:\xi:\lambda)\tau_{n,\overline{P},-\lambda}(e) = \int_N \tau_{n,\overline{P},-\lambda}(k(n)a(n)\overline{n}(n))\,dn$$

In [15], an explicit formula for the c_n-function is given in the case of $\mathrm{SL}_2(\mathbb{R})$:

$$\frac{i^{-n}\pi^{1/2}\Gamma\left(\frac{\lambda(H)}{2}\right)\Gamma\left(\frac{\lambda(H)+1}{2}\right)}{\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)\Gamma\left(\frac{\lambda(H)-n+1}{2}\right)}. \tag{2.2}$$

However, we have not been able to understand how the factor i^{-n} comes out in this formula. Therefore, in the first part of this section we find some recurrence relations between the c_n-functions and check that the following formula satisfies them:

$$c_n(\lambda) = \frac{\pi^{1/2}\Gamma\left(\frac{\lambda(H)}{2}\right)\Gamma\left(\frac{\lambda(H)+1}{2}\right)}{\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)\Gamma\left(\frac{\lambda(H)-n+1}{2}\right)}. \tag{2.3}$$

At the end of this section, we will study the poles of the c_n-functions with the formulas that we have derived as it will be useful in the next chapter. As it is clear from the formula from above, we shall need several properties of the Γ-function. Let us compile a list of facts that will be used throughout this section before we start our exposition of the c_n-function. The following can be found in [1].

Lemma 2.3.5. *For* $\operatorname{Re} z > 0$, $\Gamma(z+1) = z\Gamma(z)$ *and* $\Gamma(1) = 1$. *Particularly,* $\Gamma(n+1) = n!$. $\Gamma(1/2) = \pi^{1/2}$

Lemma 2.3.6. Γ *is holomorphic on* $\operatorname{Re} z > 0$ *and it can be analytically extended to* $\operatorname{Re} z < 0$, *having simple poles in the non-positive integers and no zeroes.*

Proof. Let us show the computation for the residue, as it will be of importance later on.

$$\lim_{z\to -n}(z+n)\Gamma(z) = \lim_{z\to -n}\frac{\Gamma(z+n+1)}{\prod_{k=0}^{n-1}(z+k)} = \frac{(-1)^n}{n!}.$$

□

We also introduce the β-function and its relation with the Γ-function.

Lemma 2.3.7. *For $x > 0$ and $x > 0$ define*

$$\mathrm{B}(x, y) := 2\int_0^{\frac{\pi}{2}} (\cos\varphi)^{2x-1}(\sin\varphi)^{2y-1}\, d\varphi.$$

With the previous definition we have the following

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

By definition of c_n, it holds that

$$c_n(\lambda) = \int_N \tau_{n,\overline{P},-\lambda}(k(n)a(n)\overline{n}(n))\, dn = \int_{-\infty}^{\infty} \tau_n(k(n_x))a(n_x)^{\lambda-\rho_{\overline{P}}}\, dx.$$

Using Lemma 1.1.10, we continue integrating

$$\begin{aligned} c_n(\lambda) &= \int_{-\infty}^{\infty} e^{in\arctan(-x)}(1+x^2)^{-\frac{1}{2}(\lambda-\rho_{\overline{P}})}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{in\varphi}(1+\tan^2\varphi)^{1-\frac{1}{2}(\lambda-\rho_{\overline{P}})(H)}\, d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{in\varphi}(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\, d\varphi. \end{aligned}$$

One can easily check that $c_n = c_{-n}$ as the cosine is an even function (in particular, one may notice that Wallach's formula for the c_n-function cannot satisfy this symmetry). Making use of the previous fact,

$$\begin{aligned} c_n(\lambda) =& \frac{c_n(\lambda) + c_{-n}(\lambda)}{2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{in\varphi} + e^{-in\varphi}}{2}(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\, d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos n\varphi\,(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\, d\varphi = 2\int_0^{\frac{\pi}{2}} \cos n\varphi\,(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\, d\varphi \end{aligned}$$

Note that $\rho_{\overline{P}}(H) = -1$ Using the n-th Chebyshev polynomial, we may write

$$\cos n\varphi = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} (\sin\varphi)^{2k}(\cos\varphi)^{n-2k}$$

Substituting this in the previous integral, we get

$$\begin{aligned} c_n(\lambda) =& 2\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} \int_0^{\frac{\pi}{2}} (\sin\varphi)^{2k}(\cos\varphi)^{n-2k+(\lambda-\rho_{\overline{P}})(H)-2}\, d\varphi \\ &= \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} \mathrm{B}\left(k+\frac{1}{2}, \frac{(\lambda-\rho_{\overline{P}})(H)+n-2k-1}{2}\right) \end{aligned}$$

Pluging $\rho_{\overline{P}}(H) = -1$ in the equation, we have that

$$c_n(\lambda) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} \mathrm{B}\left(k+\frac{1}{2}, \frac{\lambda(H)+n-2k}{2}\right) \tag{2.4}$$

Nevertheless, there is a much simpler expression in order to study the poles of the c_n-function. The c_n-functions can also be expressed in terms of a recurrence relation. Using the fact that $\cos(n+1)\varphi = \cos n\varphi \cos\varphi - \sin n\varphi \sin\varphi$, we have that

$$\begin{aligned} c_{n+1}(\lambda) &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(n+1)\varphi(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\,d\varphi \\ &= \underbrace{2\int_0^{\frac{\pi}{2}} \cos n\varphi(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-1}\,d\varphi}_{I_1} - \underbrace{2\int_0^{\frac{\pi}{2}} \sin n\varphi \sin\varphi(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-2}\,d\varphi}_{I_2} \end{aligned}$$

From the definition we see that $2I_1 = c_n(\lambda - \rho_{\overline{P}})$ and integrating by parts in I_2 yields, if $n > 0$

$$\begin{aligned} I_2 &= -\sin n\varphi \frac{(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-1}}{(\lambda-\rho_{\overline{P}})(H)-1}\Bigg|_0^{\frac{\pi}{2}} + \frac{n}{(\lambda-\rho_{\overline{P}})(H)-1}\int_0^{\frac{\pi}{2}} \cos n\varphi(\cos\varphi)^{(\lambda-\rho_{\overline{P}})(H)-1}\,d\varphi \\ &= \frac{n}{(\lambda-\rho_{\overline{P}})(H)-1}\frac{c_n(\lambda-\rho_{\overline{P}})}{2}. \end{aligned}$$

Adding all up together with $\rho_{\overline{P}}(H) = -1$, we get that

$$c_{n+1}(\lambda) = \left(1 - \frac{n}{\lambda(H)}\right) c_n(\lambda - \rho_{\overline{P}}) \qquad (n > 0). \tag{2.5}$$

For $n = 0$, according to formula 2.4, we have that

$$\begin{cases} c_0(\lambda) = \mathrm{B}\left(\dfrac{1}{2}, \dfrac{(\lambda-\rho_{\overline{P}})(H)-1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{(\lambda-\rho_{\overline{P}})(H)-1}{2}\right)}{\Gamma\left(\frac{(\lambda-\rho_{\overline{P}})(H)}{2}\right)} \\ \\ c_1(\lambda) = \mathrm{B}\left(\dfrac{1}{2}, \dfrac{(\lambda-\rho_{\overline{P}})(H)}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{(\lambda-\rho_{\overline{P}})(H)}{2}\right)}{\Gamma\left(\frac{(\lambda-\rho_{\overline{P}})(H)+1}{2}\right)} \end{cases} \tag{2.6}$$

using the expression of the Euler Beta function in terms of the Euler Gamma function. We may observe that $c_1(\lambda) = c_0(\lambda - \rho_{\overline{P}})$, satisfying the recurrence as well. Hence it yields the following for $n \geqslant 0$

$$c_n(\lambda) = \left(1 - \frac{n-1}{\lambda(H)}\right) c_{n-1}(\lambda - \rho_{\overline{P}}) \quad \text{with} \quad c_0(\lambda) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\lambda(H)}{2}\right)}{\Gamma\left(\frac{\lambda(H)+1}{2}\right)} \tag{2.7}$$

Lemma 2.3.8. *The function defined by* (2.3) *is the unique solution satisfying the recurrence relation with initial value* c_0 *given in* (2.7).

Proof. By repeatedly use of Lemma 2.3.5,

$$\begin{aligned} \left(1 - \frac{n-1}{\lambda(H)}\right) c_{n-1}(\lambda - \rho_{\overline{P}}) &= \frac{(\lambda(H)-n+1)/2}{\lambda(H)/2} \frac{\pi^{1/2}\Gamma\left(\frac{\lambda(H)+1}{2}\right)\Gamma\left(\frac{\lambda(H)+2}{2}\right)}{\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)\Gamma\left(\frac{\lambda(H)-n+3}{2}\right)} \\ &= \frac{\pi^{1/2}\Gamma\left(\frac{\lambda(H)+1}{2}\right)\Gamma\left(\frac{\lambda(H)}{2}\right)}{\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)\Gamma\left(\frac{\lambda(H)-n+1}{2}\right)} = c_n(\lambda) \end{aligned}$$

Checking that the initial conditions are also satisfied in 2.3 is routine, so we leave it to the reader. Uniqueness is trivial. □

By induction, the reader may prove that the following is also valid as an expression for the c_n-function.

$$c_n(\lambda) = c_0(\lambda - n\rho_{\overline{P}}) \prod_{j=1}^{n} \left(1 - \frac{n-j}{\lambda(H) + (j-1)}\right) \tag{2.8}$$

Using the initial condition of c_0, we may write

$$c_n(\lambda) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\lambda(H)+n}{2}\right)}{\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)} \prod_{j=1}^{n} \frac{\lambda(H) + 2j - n - 1}{\lambda(H) + (j-1)}. \tag{2.9}$$

It will be of our interest to study the zeroes and poles of the c_n-function and their order. We recall that the Gamma function has no zeroes and has simple poles in the non-positive integers making $1/\Gamma$ contributing with zeroes in the non-positive integers and no poles to the c_n-function. Thus, we may observe that in 2.9 the factors

$$\Gamma\left(\frac{\lambda(H)+n}{2}\right) \quad \text{and} \quad \lambda(H) + (j-1) \quad \text{for } j \in \{1, \ldots, n$$

contribute with simple poles to c_n, that is to say, they contribute with order -1 to the expression; whereas the factors

$$\Gamma\left(\frac{\lambda(H)+n+1}{2}\right) \quad \text{and} \quad \lambda(H) - 1 + n + 2j \quad \text{for } j \in \{1, \ldots, n$$

contribute with simple zeroes to the expression, meaning, with order $+1$. Nonetheless, it might occur that some of this poles cancel with some of the zeroes. Assume firstly that n is odd positive, then

- $\Gamma\left(\frac{\lambda(H)+n}{2}\right)$ contributes with simple poles at each odd integer $\leqslant -n$.
- $\Gamma\left(\frac{\lambda(H)+n+1}{2}\right)$ contributes with zeroes of order 1 at each even integer $< -n$.
- $-(j-1)$ for $1 \leqslant j \leqslant n$ contributes with simple poles at every negative integer between $-n+1$ and 0.
- $1 - n - 2j$ for $1 \leqslant j \leqslant n$ contributes with simple zeroes at each even integer between $-n+1$ and $n-1$ have order $+1$.

In summary, the c_n-function with n odd has first order poles in the negative odd integers and simple zeroes in the even integers that are smaller than n. Analogously, we carry out the same argument for n even, yielding the opposite; namely, simple poles in the even non-positive integers and simple zeroes in the odd integers strictly smalller than n. In the following we attach two charts, when n is either odd positive or even positive respectively, with the contributions of each of the terms in the formula of the c_n-function given by (2.3). In these charts, the reader has a more visual account which is in accordance with the previous discussion.

degree contributions when n **odd**												
Factors $\setminus \mathbb{Z}$	-n-1	-n	-n+1	$\cdots$	-2	-1	0	1	2	$\cdots$	n-1	n
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}}{\mathbf{2}}\right)$	-1		-1		-1		-1					
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}+\mathbf{1}}{\mathbf{2}}\right)$		-1				-1						
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}+\mathbf{n}+\mathbf{1}}{\mathbf{2}}\right)$	+1											
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}-\mathbf{n}+\mathbf{1}}{\mathbf{2}}\right)$	+1		+1		+1		+1		+1		+1	
Total	+1	-1	0	$\cdots$	0	-1	0	0	+1	$\cdots$	+1	0

degree contributions when n **even**												
Factors $\setminus \mathbb{Z}$	-n-1	-n	-n+1	$\cdots$	-2	-1	0	1	2	$\cdots$	n-1	n
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}}{\mathbf{2}}\right)$	-1		-1		-1		-1					
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}+\mathbf{1}}{\mathbf{2}}\right)$		-1				-1						
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}+\mathbf{n}+\mathbf{1}}{\mathbf{2}}\right)$	+1											
$\mathbf{\Gamma}\left(\frac{\boldsymbol{\lambda_0}-\mathbf{n}+\mathbf{1}}{\mathbf{2}}\right)$	+1		+1		0	+1	0	+1	0		+1	
Total	+1	-1	0	$\cdots$	-1	0	-1	+1	0	$\cdots$	+1	0

Remark. The reader may observe that all degrees at every point oscillate between -1, 0 and 1. This means that all poles and zeroes are simple. This fact shall be crucial in the theory to come.

2.4 Whittaker vectors for the discrete series

Using Definition 2.1.2, we can consider the Whitaker vectors for both the holomorphic and antiholomorphic discrete series representations. Firstly, we present a construction of a Whittaker vectors for the holomorphic discrete series representation based on the complex Fourier transform. We should mention that this construction has been provided by E. van den Ban to us. Afterwards, we shall relate this construction to a result that can be found in [16].

To begin with, we need to construct certain seminorm. This is guaranteed by the following lemma, that we state without proof.

Lemma 2.4.1. *There exists $C > 0$ such that for all $f \in H_n^+$, one has $|f(z)| \leqslant C\,\mathrm{Im}(z)^{1-n}\|f\|_{H_n^+}$.*

Proof. Let $z = x + iy \in \mathcal{H}^+$ and define $R_1(z) = \frac{1}{4}y$ and $R_2(z) = \frac{3}{4}y$. For $r \in$

$[R_1(z), R_2(z)]$, we may write

$$\begin{aligned}f(z) =& \frac{1}{2\pi i}\int_{\{\|w-z\|=r\}} \frac{f(w)}{z-w}\, dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\varphi})}{re^{i\varphi}} ire^{i\varphi}\, d\varphi \\ =& \frac{1}{2\pi i(R_2-R_1)}\int_{R_1}^{R_2}\int_0^{2\pi} \frac{f(z+re^{i\varphi})}{re^{i\varphi}} ire^{i\varphi}\, d\varphi dr\end{aligned}$$

Defining $A(z) = \{z + re^{-\varphi} \mid (r,\varphi) \in [R_1(z), R_2(z)] \times (0, 2\pi)\}$, we get the following estimate

$$\begin{aligned}|f(z)| \leqslant& \frac{1}{2\pi(R_2-R_1)} \iint\limits_{A(z)} \frac{|f(u+iv)|}{|u+iv-z|}\, dudv \\ \leqslant& \frac{1}{2\pi(R_2-R_1)} \iint\limits_{A(z)} \frac{|f(u+iv)|}{R_1(z)} v^{2-n}v^{n-2}\, dudv \\ \leqslant& \frac{1}{2\pi(R_2-R_1)} \left(\iint\limits_{A(z)} |f(u+iv)|^2 v^{n-2}\, dudv\right)^{1/2} \left(\iint\limits_{A(z)} \frac{1}{R_1^2(z)} v^{4-2n}v^{n-2}\, dudv\right)^{1/2},\end{aligned}$$

where the last inequality is obtained by the Cauchy-Schwartz inequality. If we continue, we see that

$$\begin{aligned}|f(z)| \leqslant& \frac{1}{2\pi}\frac{2}{y}\|f\|_{H_n^+} \operatorname{Area}(A(z))^{1/2} \frac{1}{|R_1(z)|}|R(z)|^{2-n} \leqslant 4^{1-n}\sqrt{2}\, y^{1-n}\|f\|_{H_n^+} \\ =& C\operatorname{Im}(z)^{1-n}\|f\|_{H_n^+}.\end{aligned}$$

□

Lemma 2.4.2. *There exists a continuous seminorm ν on $(H_n^+)^\infty$ such that for all $f \in (H_n^+)^\infty$,*

$$|f''(z)| \leqslant (1+|z|)^{-2}\operatorname{Im}(z)^{1-n}\nu(f)$$

Proof. By Lemma 2.4.1 and the last remark in Section 1.4.1, it follows that

$$|f''(z)| \leqslant C\operatorname{Im}(z)^{1-n}\|f''\|_{H_n^+} \quad \& \quad |z^2f''(z)| \leqslant C\operatorname{Im}(z)^{1-n}\|z^2f''\|_{H_n^+}$$

Hence, by adding up, the result follows with $\nu(f) = \|f''\|_{H_n^+} + \|z^2f''\|_{H_n^+}$ □

For $r \in \mathbb{R}$, define the following functional on $(H_n^+)^\infty$ by

$$\eta_y(f) = \int_{-\infty}^{\infty} f''(x+iy)e^{-irx}\, dx \qquad (\text{for } y > 0). \tag{2.10}$$

Lemma 2.4.3 (Whittaker vector in the discrete series representation)**.** *For $y > 0$ and χ a regular unitary character on N, the functional η_y as in (2.10) is in* $\mathrm{Wh}_\chi(H_n^+)$.

Proof. The map η_y is clearly linear. To show continuity, we have the following estimate:

$$|\eta_y(f)| \leqslant C\nu(f)\int_{-\infty}^{\infty} \frac{y^{1-n}}{\left(1+\sqrt{x^2+y^2}\right)^2}\, dx \leqslant 2Cy^{1-n}\nu(f)\int_0^{\infty}\frac{1}{1+x^2}\, dx = C_y\nu(f).$$

This shows that η_y is a continuous linear functional. Regarding its behaviour with respect to χ, let $n_\xi \in N$. It yields then

$$\eta_y(D_n^+(n_\xi^{-1})f) = \int_{-\infty}^{\infty} f(z+\xi)'' e^{-irx}\, dx = \int_{-\infty}^{\infty} f''(z+\xi) e^{-irx}\, dx = e^{+ir\xi}\eta_y(f).$$

$\square$

Observe that $\eta_y = e^{r(1-y)}\eta_1$. Indeed, for $y > 1$,

$$e^{ry}\eta_y(f) = \int_{-\infty}^{\infty} f''(x+iy)e^{ir(x+iy)}\, dx = \int_{\mathbb{R}+iy} f''(z)e^{-irz}\, dz.$$

We note that the integrand is holomorphic in the upper-half plane, thus by Cauchy's theorem, we know that the line integral over the border of the complex rectangle $[-R, R] \times [1, y]$ is zero. However the integral over the vertical edges of the rectangle tend to 0 as R goes to infinity. Indeed,

$$\left|\int_1^y f''(R+it)e^{-ir(R+it)}\, idt\right| \leqslant \int_1^y \frac{t^{1-n}e^{tr}}{1+|R+it|}\nu(f)\, dt \leqslant \frac{e^{ry}(y-1)\nu(f)}{1+R} \underset{R\to\infty}{\longrightarrow} 0.$$

The analogous result holds in in the other vertical edge of the rectangle. Hence $e^{ry}\eta_y = e^r\eta_1$ for $y > 1$. Perform the same strategy for $0 < y < 1$, and then the equality holds for $y > 0$. In particular, this means that if $r < 0$ and by using the estimate in the proof of Lemma 2.10, it yields

$$|\eta_1(f)| \leqslant 2C\,\nu(f)\, e^{r(y-1)}y^{1-n} \to 0 \quad \text{for}\, y \to \infty.$$

Therefore we have that for every $f \in (H_n^+)^\infty$,

$$0 = \eta_1(f) = e^{r(1-y)}\eta_y(f) \implies \eta_y(f) = 0 \quad \text{for}\, f \in (H_n^+)^\infty$$

On the other hand, $\eta_y \neq 0$ if $r > 0$. To see this, we need to find a function in $(H_n^+)^\infty$ for which the functional is not 0. For a function $\varphi \in C_c^\infty(\mathbb{R})$, define the complex Fourier transform

$$\mathcal{F}\varphi(z) := \int_{\mathbb{R}} \varphi(t)e^{-izt}\, dt = \int_{\mathbb{R}} \varphi(t)e^{yt}e^{-ixt}\, dt = \mathcal{F}(\varphi e^{yt})(x) \quad \text{if}\, z = x+iy$$

The following lemma will be necessary.

Lemma 2.4.4. *Let $\varepsilon > 0$ and $\varphi \in C_c^\infty(-\infty, -\varepsilon)$, then $\mathcal{F}\varphi|_{\mathcal{H}^+} \in (H_n^+)^\infty$.*

Proof. Recall that

$$\mathcal{F}\varphi(x+iy) = \int_{\mathbb{R}} \varphi(t)e^{yt}e^{-ixt}\, dt$$

This function is holomorphic in the variable z because the integrand is of compact support and it has holomorphic dependence on z given by the term e^{-izt}. Now we estimate its L^2 norm on the space H_n^+. We must first make the observation that $|\mathcal{F}\varphi(x+iy)| \leqslant e^{-\varepsilon y}\|\varphi\|_{L^1(-\infty,-\varepsilon)}$ and thus, for any $N \in \mathbb{N}$,

$$|(x+iy)^N\mathcal{F}\varphi(x+iy)| \leqslant |\mathcal{F}\varphi^{(N)}(x+iy)| \leqslant e^{-\varepsilon y}\|\varphi^{(N)}\|_{L^1(-\infty,-\varepsilon)}.$$

Then it is readily seen that for $N \in \mathbb{N}$, $|\mathcal{F}\varphi(x+iy)| \leqslant C_N(1+|x+iy|^N)^{-1}e^{-\varepsilon y}$. Applying the last estimate, the following holds for N sufficiently large,

$$\begin{aligned}\|\mathcal{F}\varphi\|^2_{H_n^+} =& \int_{-\infty}^{\infty}\int_0^{\infty} |\mathcal{F}\varphi(x+iy)|^2 y^{n-2}\,dydx \\ \leqslant& C_N^2 \left(\int_{-\infty}^{\infty} \frac{1}{(1+|x|^N)^2}\,dx\right)\left(\int_0^{\infty} e^{-2\varepsilon y}y^{n-2}\,dy\right) \\ =& C_N^2 \frac{(n-2)!}{(2\varepsilon)^{n-1}}\left(\int_{-\infty}^{\infty} \frac{1}{(1+|x|^N)^2}\,dx\right) \leqslant \infty.\end{aligned}$$

This means that $\mathcal{F}\varphi \in H_n^+$. It remains to show that $(D_n^+)_*(u)\mathcal{F}\varphi \in H_n^+$, for every $u \in U(\mathfrak{g})$. In Section 1.4.1, we computed the associated $\mathfrak{g}$-module of the holomorphic discrete series representation for the $\mathfrak{g}$-standard triple. This was:

$$((D_n^+)_*(H)f)(z) = -nf(z) - 2zf'(z) \quad ((D_n^+)_*(X)f)(z) = -f'(z)$$

$$((D_n^+)_*(Y)f)(z) = nzf(z) + f'(z)z^2.$$

By Theorem A.2.1, we may see that $(D_n^+)_*(u)$ is a differential operator that takes the following form.

$$(D_n^+)_*(u) = \sum_{j=0}^{l} p_j(z)\frac{d^j}{dz^j} = P\left(z, \frac{d}{dz}\right).$$

By the properties of the Fourier trasform, we see that

$$(D_n^+)_*(u)\mathcal{F}\varphi = P\left(z, \frac{d}{dz}\right)\mathcal{F}\varphi = \mathcal{F}\left(P\left(t, -i\frac{d}{dt}\right)\varphi\right).$$

Since $\varphi \in C_c^\infty(-\infty,-\varepsilon)$, then $P\left(t, -i\frac{d}{dt}\right)\varphi$ is also in $C_c^\infty(-\infty,-\varepsilon)$. This implies, by the discussion in the first part of the proof that $(D_n^+)_*(u)\mathcal{F}$ for every $u \in U(\mathfrak{g})$. Hence $\mathcal{F}\varphi \in (H_n^+)^\infty$. □

Let now $0 < \varepsilon < r$ and pick any $\varphi \in C_c^\infty(-\infty,-\varepsilon)$ with $\varphi(-r) \neq 0$, then by the previous lemma,

$$\eta_y(\mathcal{F}\varphi) = \int_{\mathbb{R}} (\mathcal{F}\varphi)''(x+iy)e^{-irx}\,dx = \int_{\mathbb{R}} \mathcal{F}(-t^2e^{yt}\phi)(x)e^{-irx}\,dx = -2\pi r^2 e^{ry}\varphi(-r)$$

In conclusion, we have found a Whittaker vector η_y (for $y > 0$) in the holomorphic discrete series representation, such that if $r < 0$ it is zero and if $r > 0$ is non-zero. The following result corresponding to [16, Theorem 2] confirms that the distinction between $r > 0$ or $r < 0$ must happen. The original proof of the theorem is attributed to C. Moore, although Wallach provides another proof.

Theorem 2.4.1. *For χ regular character on N. If $r_\chi > 0$ then* $\dim \overline{\mathrm{Wh}_\chi(H_n^+)} = 1$ *and* $\dim \overline{\mathrm{Wh}_\chi(H_n^-)} = 0$. *If $r_\chi < 0$, we have* $\dim \overline{\mathrm{Wh}_\chi(H_n^+)} = 0$ *and* $\dim \overline{\mathrm{Wh}_\chi(H_n^-)} = 1$.

Chapter 3

Whittaker ODE for $\mathrm{SL}_2(\mathbb{R})$

The classical theory of Whittaker functions is very well-known. It can be found in any standard book concerning the confluent hypergeometric function. We recommend [18, Chapter 16] and [8] if the reader is curious about the topic. However, let us briefly describe it for a motivational purpose. Whittaker functions are simply solutions to the Whittaker ODE. In 1904, E. Whittaker introduced a new variation of the hypergeometric ODE. In short, the hypergeometric ODE is a second order differential equation with three regular singularities at $0, 1$ and ∞ on the Riemann sphere. The Whittaker ODE is a modified version of the latter in which 1 and ∞ have been forced to meet at an irregular singularity, whereas the other singularity at 0 remains regular. The Whittaker differential equation takes the following form

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) W = 0, \tag{3.1}$$

where k, m are parameters in the complex plane. In the following we devote ourselves to develop what we call the *Whittaker ODE for* $\mathrm{SL}_2(\mathbb{R})$. In few words, this is a differential equation derived from the action of the radial component of the Casimir operator in $\mathrm{SL}_2(\mathbb{R})$. In particular, we will see that it results in a Whittaker ODE in the classical sense for parameters yet to be specified. Afterwards, we will proceed to define the Fourier-Whittaker transform and to state its inverse for the group $\mathrm{SL}_2(\mathbb{R})$. AS a conclusion, we shall deal with the residues of this formula and relate them to the discrete series representations of $\mathrm{SL}_2(\mathbb{R})$.

3.1 Derivation of the Whittaker ODE for $\mathrm{SL}_2(\mathbb{R})$

Let us return to the notation of Sections 1.1.1, 1.4.1 and 2.1. We recall that the Casimir has the following form (up to rescaling) for the standard $\mathfrak{g}$-triple:

$$\Omega = H^2 + 2(XY + YX) = H^2 + 2H + 4YX.$$

One may observe that $R_\Omega \in \mathrm{End}(C^\infty(\tau_n, G/N, \chi))$. Indeed, by left invariance of R_Ω it yields that for $f \in C^\infty(\tau_n, G/N, \chi)$

$$L_k R_\Omega f = R_\Omega L_k f = R_\Omega(\tau_n(k)^{-1} f) = \tau_n(k)^{-1} R_\Omega f \quad \text{for } k \in K.$$

The behaviour with respect to χ follows from the fact that Ω is Ad-invariant. Letting $f \in C^\infty(\tau_n, G/N, \chi)$ and $n \in \mathbb{N}$, it holds that

$$R_n R_\Omega f = R_{\mathrm{Ad}(n)\Omega} R_n f = R_\Omega R_n f = R_\Omega \chi(n) f = \chi(n) R_\Omega.$$

It is known from the previous chapter that the restriction to A (earlier denoted by r_A) induces a topological linear isomorphism from $C^\infty(\tau_n, G/N, \chi)$ onto $C^\infty(A)$, in accordance with the Iwasawa decomposition described in Section 1.1.3. Thus it is worth asking how the Casimir operator acts on $C^\infty(A)$. This gives rise to the *radial component* of Ω.

Definition 3.1.1 (Radial component of Ω)**.** We define the *radial component of* Ω to be the linear operator $\operatorname{rad}\Omega \in \operatorname{End}(C^\infty(A))$ satisfying the following commutative diagram

$$\begin{array}{ccc} C^\infty(\tau_n, G/N, \chi) & \xrightarrow{R_\Omega} & C^\infty(\tau_n, G/N, \chi) \\ {\scriptstyle r_A}\big\downarrow & & \big\downarrow{\scriptstyle r_A} \\ C^\infty(A) & \xrightarrow{\quad \operatorname{rad}\Omega \quad} & C^\infty(A) \end{array}$$

where r_A denotes the restriction to A in the Iwasawa decomposition $G = KAN$.

In order to explicitly compute $\operatorname{rad}\Omega$ we first need to know how the Casimir acts on $C^\infty(\tau_n, G/N, \chi)$. As usual, it is only necessary to know the action of the elements of the standard $\mathfrak{g}$-triple:

$$R_H f(a_t) = \left.\frac{d}{ds}\right|_{s=0} f(a_t \exp sH) = \left.\frac{\partial}{\partial s}\right|_{s=0} f(\exp(t+s)H) = \frac{d}{dt} f(a_t).$$
$$R_X f(a_t) = \left.\frac{d}{ds}\right|_{s=0} f(a_t \exp sX) = \left.\frac{d}{ds}\right|_{s=0} f(a_t)\chi(\exp sX) = irf(a_t).$$

The right action of the Casimir of the element Y is a bit more involved. We see that

$$R_Y f(a_t) = L_{-\operatorname{Ad}(a_t)Y} f(a_t) = L_{\operatorname{Ad}(a_t)^{-1}X - \operatorname{Ad}(a_t)Y} f(a_t) - L_{\operatorname{Ad}(a_t)^{-1}X} f(a_t) = L_1 + L_2.$$

By Lemma 1.1.6, we may write $\operatorname{Ad}(a_t)^{-1}X - \operatorname{Ad}(a_t)Y = a^{-\alpha}X - a^{-\alpha}Y = a^{-\alpha}(X - Y)$. Consequently, it follows that

$$\begin{aligned} L_1 =& a^{-\alpha} L_{Y-X} f(a_t) = a^{-\alpha} \left.\frac{d}{ds}\right|_{s=0} f(\exp s(Y-X)a_t) \\ =& a^{-\alpha} \left.\frac{d}{ds}\right|_{s=0} \tau_n(\exp(s(Y-X))) f(a_t) = ina^{-\alpha} f(a_t). \end{aligned}$$

In the case of L_2,

$$L_2 = L_{\operatorname{Ad}(a_t)^{-1}X} f(a_t) = R_{-\operatorname{Ad}(a_t)^{-2}X} f(a_t) = a_t^{-2\alpha} R_x f(a_t) = a_t^{-2\alpha} irf(a_t)$$

Adding up L_1 and L_2 we obtain the right action of Y on $C^\infty(\tau_n, G/N, \chi)$,

$$R_Y f(a_t) = (ina_t^{-\alpha} + ira_t^{-2\alpha}) f(a_t).$$

The radial component has finally the following expression for $f \in C^\infty(\tau_n, G/N, \chi)$:

$$\begin{aligned} \operatorname{rad}\Omega\ (f|_A)(a_t) =& (R_\Omega f)(a_t) = ((R_H)^2 f + 2R_H f + 4R_X R_Y f))(a_t) \\ =& \left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 4ir(ina_t^{-\alpha} + ira_t^{-2\alpha})\right) f(a_t) \\ =& \left(\frac{d^2}{dt^2} + 2\frac{d}{dt} - 4(rna_t^{-\alpha} + r^2 a_t^{-2\alpha})\right) f(a_t). \end{aligned}$$

To relate the last formula to the classical theory, define the change of variables $f(a_t) = \psi(x(t))$ where $x(t) = 2ra_t^{-\alpha} = 2re^{-2t}$ and $t > 0$. A simple computation shows that

$$\frac{d}{dt}(\psi \circ x) = -2x\frac{d\psi}{dx} \quad \text{and} \quad \frac{d^2}{dt^2}(\psi \circ x) = 4x\frac{d\psi}{dx} + 4x^2\frac{d^2\psi}{dx^2}.$$

Applying this change to the radial operator we get the following expression

$$\operatorname{rad}\Omega(f|_A)(a_t) = \left(4x^2\frac{d^2}{dx^2} - x^2 - 2nx\right)\psi.$$

Recall from Lemma 1.4.8 that the Casimir operator acts on the principal series of G by a scalar. By definition of the radial component, it follows that the radial component of the Casimir element will also act by the same scalar on the functions of the principal series. Therefore, it will be useful to consider, as for now, the eigenvalue problem for the radial component of Ω. It will be convenient, to reparametrise the eigenvalue problem so that $\operatorname{rad}\Omega = (\lambda^2 - 1)f$ with $\lambda \in \mathbb{C}$. The reader might observe that this consideration traces back to the use of $\lambda(H)^2 - 1$, which is the scalar by which the Casimir acts on the principal series of $\mathrm{SL}_2(\mathbb{R})$. Therefore the expression in coordinates for the eigenvalue problem is

$$\frac{d^2}{dx^2}\psi + \left(\frac{(-n)/2}{x} + \frac{1/4 - (\lambda/2)^2}{x^2} - \frac{1}{4}\right)\psi = 0; \tag{3.2}$$

which is a Whittaker ODE in the classical sense for $k = -\frac{n}{2}$ and $m = \frac{\lambda(H)}{2}$. Equation (3.2) will be called the *Whittaker ODE for* $\mathrm{SL}_2(\mathbb{R})$. One should note that the classical Whittaker ODE has been defined for the complex variable, while in our case we restrict to the positive real line. Nevertheless, considering the equation in the complex plane will be advantageous to us. In the following lemma, we finally learn why every element in Chapter 2 was named after 'Whittaker'.

Lemma 3.1.1. *The Whittaker matrix coefficient* $\mathrm{Wh}_n(\overline{P}, \lambda)$ *as in Definition 2.3.1 satisfies the Whittaker ODE for* $\mathrm{SL}_2(\mathbb{R})$ *(after the aforementioned changes of variables).*

Proof. We only need to see how the Casimir operator acts on $\mathrm{Wh}_n(\overline{P}, \lambda)$. The result inmediately follows form the discussion above. Let $Z \in \mathfrak{g}$ and $g \in G$, we see then that

$$\begin{aligned}
L_Z\mathrm{Wh}_n(\overline{P}, \lambda)(g) &= \left.\frac{d}{dt}\right|_{t=0} \langle \pi_{\overline{P},\xi,-\lambda}(g^{-1})\pi_{\overline{P},\xi,-\lambda}(\exp tZ)\tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}}\rangle \\
&= \left\langle \pi_{\overline{P},\xi,-\lambda}(g^{-1}) \left.\frac{d}{dt}\right|_{t=0} \pi_{\overline{P},\xi,-\lambda}(\exp tZ)\tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}} \right\rangle \\
&= \left\langle \pi_{\overline{P},\xi,-\lambda}(g^{-1})(\pi_{\overline{P},\xi,-\lambda})_*(Z)\tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}} \right\rangle.
\end{aligned}$$

Since the identity holds for any $Z \in \mathfrak{g}$, apply it successively followed by Lemmas 1.4.8 and A.2.6 to find

$$\begin{aligned}
L_\Omega\mathrm{Wh}_n(\overline{P}, \lambda)(g) =& \left\langle \pi_{\overline{P},\xi,-\lambda}(g^{-1})(\pi_{\overline{P},\xi,-\lambda})_*(\Omega)\tau_{n,\overline{P},-\lambda}, \eta_{\overline{P},\xi,\overline{\lambda}} \right\rangle \\
=& (\lambda(H)^2 - 1)\mathrm{Wh}(\overline{P}, \lambda)(g).
\end{aligned}$$

∎

Set $\lambda_\alpha = \lambda(H)/2$ so that $\lambda = \lambda_\alpha \alpha$. Consider the complex version of the Whittaker ODE for $\mathrm{SL}_2(\mathbb{R})$, derived from the standard one, just by multiplying by z^2

$$z^2 \frac{d^2}{dz^2}\psi + \left(-\frac{n}{2}z - \frac{1}{4}z^2 + \frac{1}{4} - \lambda_\alpha^2\right)\psi = 0. \tag{3.3}$$

We shall proceed with examining whether there are solutions defined on a neighbourhood around $z = 0$, that is to say around the regular singularity of our equation. The reader should know that the following is standard theory that can be found in any comprehensive book on complex ODEs, for example in [17]. Nevertheless, we have decided to include it for subsequent references. As an educated guess we try out functions of the form $\psi(z) = z^s \varphi$ where φ is a holomorphic function on an open neighbourhood around 0; and $s \in \mathbb{C}$. We may assume without loss of generality that $\varphi(0) = 1$. If we substitute our candidate in the equation, the following expression holds:

$$z^2\varphi'' + 2sz\varphi' + \left(-\frac{n}{2}z - \frac{1}{4}z^2 + s(s-1) + \frac{1}{4} - \lambda_\alpha^2\right)\varphi = 0.$$

Since the equation holds for every value of $z \in \mathbb{C}$, in particular it must for $z = 0$. Evaluating at $z = 0$ (recall that $\varphi(0) \neq 0$), we observe that

$$s(s-1) + \frac{1}{4} - \lambda_\alpha^2 = 0 \quad \text{meaning} \quad s_\pm = \mp\lambda_\alpha + \frac{1}{2} = \frac{1}{2}(\mp\lambda - \rho_{\overline{P}})(H).$$

From the general theory of 2nd order complex differential equations with a regular singularity, it is known that if the difference between the exponents $s_+ - s_-$ is not an integer, the solutions form a fundamental system of the ODE. The reader may find more details in [17, Chapter 5, Section 25]. In regards with our case, $\{z^{s_+}\varphi_{s_+}, z^{s_-}\varphi_{s_-}\}$ form a fundamental system if and only if $\lambda(H) \notin \mathbb{Z}$. Assume for the moment that $\lambda(H) = 2\lambda_\alpha$ is not an integer. In the case of $\psi_{s_+} = z^{s_+}\varphi_{s_+}$ we have the following differential equation

$$z^2\varphi_{s_+}'' + 2s_+z\varphi_{s_+}' + \left(-\frac{n}{2}z - \frac{1}{4}z^2\right)\varphi_{s_+} = 0.$$

Since we have assumed that φ_{s_+} is holomorphic around a neighbourhood of 0, it must be given by a power series of the form

$$\varphi_{s_+}(z) = \sum_{j=0}^{\infty} \Gamma_j^+(\lambda) z^j \quad \text{with} \quad \Gamma_0^+ = 1$$

Substitute the series above in the previous equation to obtain recurrence relations that determine the coefficients Γ_j^+,

$$\begin{aligned}
0 =& z^2 \sum_{j=0}^{\infty} j(j-1)\Gamma_j^+ z^{j-2} + 2s_+ z \sum_{j=0}^{\infty} j\Gamma_j^+ z^{j-1} - \frac{n}{2} z \sum_{j=0}^{\infty} \Gamma_j z^j - \frac{1}{4} z^2 \sum_{j=0}^{\infty} \Gamma_j^+ z^j \\
=& \sum_{j=0}^{\infty} j(j-1)\Gamma_j^+ z^j + \sum_{j=0}^{\infty} 2s_+ j\Gamma_j^+ z^j - \sum_{j=0}^{\infty} \frac{n}{2}\Gamma_j z^{j+1} - \sum_{j=0}^{\infty} \frac{1}{4}\Gamma_j^+ z^{j+2} \\
=& \sum_{j=0}^{\infty} j(j-1)\Gamma_j^+ z^j + \sum_{j=0}^{\infty} 2s_+ j\Gamma_j^+ z^j - \sum_{j=1}^{\infty} \frac{n}{2}\Gamma_{j-1} z^j - \sum_{j=2}^{\infty} \frac{1}{4}\Gamma_{j-2}^+ z^j \\
=& 0z^0 + \left(2s_+\Gamma_1^+ - \frac{n}{2}\Gamma_0^+\right) z + \sum_{j=2}^{\infty} \left(j(j-1)\Gamma_j^+ + 2s_+ j\Gamma_j^+ - \frac{n}{2}\Gamma_{j-1}^+ - \frac{1}{4}\Gamma_{j-2}^+\right) z^j
\end{aligned}$$

We get the following recurrences

$$\begin{cases} \Gamma_0^+(\lambda) = 1 \\ \Gamma_1^+(\lambda) = \dfrac{n}{2(1-\lambda(H))} \\ \Gamma_j^+(\lambda) = \dfrac{1}{j(j-\lambda(H))}\left(\dfrac{n}{2}\Gamma_{j-1}^+(\lambda) + \dfrac{1}{4}\Gamma_{j-2}^+(\lambda)\right) & \text{if } j \geqslant 2 \end{cases}$$

In an analogous way, we may find the same recurrences for the solution φ_-, just substituting in the previous s_+ by s_-. By the theory of complex differential equations, we observe that the power series that we have just computed can be holomorphically extended to the whole complex plane because the coefficients of the differential equation are holomorphic everywhere except at 0 (for which there is a regular singularity). This last statement corresponds to [17, Theorem 3, Section 24].

Remark. It is worth mentioning that the coefficients $\Gamma_j^\pm$ for $j \in \mathbb{N}$ are actually functions of λ. It can be shown that $\Gamma_j^-(\lambda) = \Gamma_j^+(-\lambda)$ since $s_-(\lambda) = s_+(\lambda)$. Furthermore, it is the case that they are meromorphic in the variable λ. This will be of relevance to us.

Put $W_\lambda(x) = \mathrm{Wh}(\overline{P}, \lambda)(a)$. As we have seen in Lemma 3.1.1, W_λ is a solution of the Whittaker ODE for $\mathrm{SL}_2(\mathbb{R})$. We remind the reader that, we are in the case of $\lambda(H) \notin \mathbb{Z}$ fixed. Then there exist complex coefficients $C_n^+(\lambda)$ and $C_n^-(\lambda)$ such that

$$W_\lambda(x) = x^{-\frac{1}{2}(\lambda+\rho_{\overline{P}})(H)} C_n^+(\lambda)\varphi_{s_+}(x) + x^{\frac{1}{2}(\lambda-\rho_{\overline{P}})(H)} C_n^-(\lambda)\varphi_{s_-}(x),$$

as an identity of holomorphic functions on x. If we undo the change of variables $x = 2ra^{-\alpha}$, we have that

$$\mathrm{Wh}_n(\overline{P}, \lambda)(a) = a^{\lambda+\rho_{\overline{P}}} \widetilde{C_n^+}(\lambda)\Phi_{\lambda+\rho_{\overline{P}}}(a) + a^{-\lambda+\rho_{\overline{P}}} \widetilde{C_n^-}(\lambda)\Phi_{-\lambda+\rho_{\overline{P}}}(a)$$

where $\widetilde{C_n^+}(\lambda) = (2r)^{\frac{1}{2}(\lambda+\rho_{\overline{P}})(H)} C_n^+(\lambda)$ and $\widetilde{C_n^-}(\lambda) = (2r)^{\frac{1}{2}(\lambda-\rho_{\overline{P}})(H)} C_n^-(\lambda)$. Moreover, we see that

$$\varphi_{s_+}(x) = \Phi_{\lambda+\rho_{\overline{P}}}(a) = \sum_{j=0}^{\infty} (2r)^j \Gamma_j^+(\lambda) a^{-j\alpha}, \quad \varphi_{s_-}(x) = \Phi_{-\lambda+\rho_{\overline{P}}}(a) = \sum_{j=0}^{\infty} (2r)^j \Gamma_j^-(\lambda) a^{-j\alpha}.$$

Lemma 3.1.2. *If $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$, then $\widetilde{C_n^+} = c_n$, for c_n the c_n-function as in formula (2.3).*

Proof. Firstly, we observe that the functions $\Phi_{\lambda+\rho_{\overline{P}}}(a), \Phi_{-\lambda+\rho_{\overline{P}}}(a) \to 1$ and $a^{-\lambda} \to 0$ as $a \xrightarrow{A^+} \infty$. Since $\varphi_{s_\pm}$ are holomorphic about a neighbourhood of 0, $\varphi_{s_\pm}(x) \to \varphi_{s_\pm}(0) = 1$, when $x \to 0$. By Lemma 2.3.4 we may write

$$a^{\lambda+\rho_{\overline{P}}} c_n(\lambda) \sim a^{\lambda+\rho_{\overline{P}}} \widetilde{C_n^+}(\lambda)\Phi_{\lambda+\rho_{\overline{P}}} + a^{-\lambda+\rho_{\overline{P}}} \widetilde{C_n^-}(\lambda)\Phi_{-\lambda+\rho_{\overline{P}}}$$

This is equivalent to saying

$$\frac{1}{c_n}\left(\widetilde{C_n^+}(\lambda)\Phi_{\lambda+\rho_{\overline{P}}} + a^{-2\lambda}\widetilde{C_n^-}(\lambda)\Phi_{-\lambda-\rho_{\overline{P}}}\right) \longrightarrow 1 \quad \text{as } a \xrightarrow{A^+} \infty$$

By the observations made at the beginning of the proof, the result follows. □

Remark. The coefficient $\widetilde{C_n^-}$ can be determined using the classical theory of Whittaker functions. This coefficient will not be very important for the purpose of this theory, therefore we omit its treatment in this text. The expression for the Whittaker matrix coefficient is as follows

$$\mathrm{Wh}_n(\overline{P},\lambda)(a) = a^{\lambda+\rho_{\overline{P}}} c_n^+(\lambda)\Phi_{\lambda+\rho_{\overline{P}}}(a) + a^{-\lambda+\rho_{\overline{P}}} c_n^-(\lambda)\Phi_{-\lambda+\rho_{\overline{P}}}(a) \tag{3.4}$$

Remark. In the case when $\lambda(H) \in \mathbb{Z}$, the theory of complex ODEs allows us to state the same equality as in 3.4, but in this case as an equality of meromorphic functions.

For the subsequent sections we need some properties of the power series coefficients of the function $\Phi_{\lambda+\rho_{\overline{P}}}$.

Lemma 3.1.3. *The following statements are true about the coefficcients* $\{\Gamma_j^+(\lambda)\}_{j\in\mathbb{N}}$.

1. *On the region* $\langle \mathrm{Re}\,\lambda, \alpha\rangle < 0$, *they have neither zeroes nor poles in the variable* λ.

2. *For* $j > k \in \mathbb{N}$, $\lambda(H) = j$ *cannot be a pole of the coefficient* $\Gamma_k^+(\lambda)$.

3. *All poles of* $\Gamma_j^+(\lambda)$ *are simple.*

Proof.

1. We observe that Γ_1^+ has a pole at $\lambda(H) = -\rho_{\overline{P}}(H) = 1$, thus it has no negative poles. By induction, assume that the statement is true for $j \leqslant n-1$. Then Γ_n^+ has no negative poles because the term $\frac{n}{2}\Gamma_{j-1}^+ + \frac{1}{4}\Gamma_{j-2}^+$ is holomorphic on $\langle \mathrm{Re}\,\lambda, \alpha\rangle$ by hypothesis; and the only candidate could be a pole at $\lambda(H) + \rho_{\overline{P}}(H) = (j-1)$. The latter is impossible since $\langle \mathrm{Re}\,\lambda, \alpha\rangle < 0$. The statement corresponding to the zeroes of Γ_j^+ follows automatically by induction as well and the fact that Γ_j^+ are positive on the region $\langle \mathrm{Re}\,\lambda, \alpha\rangle < 0$.

2. By induction, Γ_1^+ satisfies the hypothesis. Assume it true for every $j < k$. If we let $j_0 > k$ be a pole of the coeffiecient $\Gamma_k^+(\lambda)$, then it is immediate that the factor $\frac{n}{2}\Gamma_{k-1}^+ + \frac{1}{4}\Gamma_{k-2}^+$ would have a pole at j_0. Hence, either Γ_{k-1}^+ or Γ_{k-2}^+ would have a pole at j_0. This is a contradiction with the induction hypothesis.

3. we see that Γ_1^+ has a simple pole at $\lambda(H) = 1$. Assume that the hypothesis is true for all $j < k$. Assume that Γ_k^+ has a pole of order m at $\lambda(H) = \lambda_0 \neq j$. This means that the factor $\frac{n}{2}\Gamma_{k-1}^+ + \frac{1}{4}\Gamma_{k-2}^+$ has a pole of order m at $\lambda(H) = \lambda_0$. But this is impossible because the sum of two functions that have all its poles simple cannot increase the order of the pole. In the case that the pole is at $\lambda(H) = k$, one can see that it has to be simple by (2) in Lemma 3.1.3.

∎

3.2 The Fourier-Whittaker transform

Let us retain the notation as in the previous section and let χ be a unitary regular character on N. We shall consider the space $L^2(G/N,\chi)$ of measurable complex valued functions f such that $|f| \in L^2(G/N)$ and $R_n f = \chi(n)f$ for all $n \in N$. The reader may readily check that it is G-module with the left regular representation. The decomposition in K-types as in Lemma 1.2.3 yields

$$L^2(G/N,\chi) = \bigoplus_{n\in\mathbb{Z}} L^2(\tau_n, G/N, \chi)$$

where $L^2(\tau_n, G/N,\chi) = \{f \in L^2(G/N,\chi) \mid L_k f = \tau_n(k)^{-1} f, \, \forall k \in K\}$.

Lemma 3.2.1. *The space $C_c^\infty(\tau_n, G/N,\chi)$ is dense in $C^\infty(\tau_n, G/N,\chi)$. Furthermore, $C_c^\infty(\tau_n, G/N,\chi) \hookrightarrow L^2(\tau_n, G/N,\chi)$ continuously.*

Proof. These follow from [9, Propositions 3.4 & 3.5]. □

We have earlier remarked in Section 2.1 that the restriction to A map (denoted by r_A) in the Iwasawa decomposition KAN is a topological linear isomorphism between $C^\infty(\tau_n, G/N,\chi)$ and $C^\infty(A)$. This implies the following commutative diagram

$$\begin{tikzcd} C_c^\infty(\tau_n, G/N,\chi) \arrow[r, hook] \arrow[d, "r_A"'] & L^2(\tau_n, G/N,\chi) \arrow[d, "r_A"] \\ C_c^\infty(A) \arrow[r, hook] & L^2(A, a^{-2\rho_{\overline{P}}} da) \end{tikzcd}$$

In these terms, we may define the Fourier-Whittaker transform as follows

Definition 3.2.1 (Fourier-Whittaker transform)**.** Let $f \in C_c^\infty(A)$. We define the following function in the variable $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$$\mathcal{F}_n^{\mathrm{Wh}}(f)(\lambda) := \int_A f(a)\overline{\mathrm{Wh}_n(\overline{P}, -\overline{\lambda})}(a) a^{-2\rho_{\overline{P}}} da$$

Remark. The reader may observe that the choice of conjugating λ in $\mathrm{Wh}(\overline{P}, -\overline{\lambda})$ is not by chance. This is made so that the Fourier type transform defined above becomes holomorphic in the parameter λ. This is seen in the coming lemma.

Lemma 3.2.2. *We have that $\mathcal{F}_n^{\mathrm{Wh}} : C_c^\infty(A) \to \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)$.*

Proof. That f is a compactly supported function on A implies that the integral is absolutely convergent for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. This means that we may differentiate under the integral sign with respect to λ. Since the integrand of $\mathcal{F}_n^{\mathrm{Wh}}(f)$ depends on λ in a holomorphic fashion, the result follows. □

3.3 Residues of the Fourier-Whittaker inversion formula

As it has been commented in the introduction to this thesis a new inversion formula for the Fourier-Whittaker transform has been derived by E. van den Ban for a connected semisimple real Lie groups. For this announcement can be found in [refer to slides]. The result for the case of $\mathrm{SL}_2(\mathbb{R})$ may be stated as follows.

Theorem 3.3.1 (Fourier-Whittaker inversion formula)**.** *There exists $\eta \in \mathfrak{a}^*$ with $\langle \eta, \alpha\rangle << 0$ and for which for all $f \in C_c^\infty(A)$*

$$f(a) = 2\int_{i\mathfrak{a}^*+\eta} a^{\lambda+\rho_{\overline{P}}} \Phi_{\lambda+\rho_{\overline{P}}}(a) \frac{\mathcal{F}_n^{\mathrm{Wh}}(f)(\lambda)}{\overline{c_n^+(-\overline{\lambda})}} \, d\lambda.$$

As we may observe in the theorem the shift is made towards $-\infty$. In the process of shifting the domain, some poles in the negative real line will be collected. Our task for the rest of the section is to investigate the following expression when $\mu \in \mathfrak{a}^*$ is such that $\mu(H) <$:

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}(f)(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right). \tag{3.5}$$

Remark. The reader ought to be wary that this expression should be understood as taking the residue at $\lambda(H) = \mu(H)$. In other words, the map $\mathrm{ev}_H : \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ given by $\mathrm{ev}_H(\lambda) = \lambda(H)$ establishes a topological linear isomorphism, which implies that we may transfer the notion of residue of functions defined on $\mathbb{C}$ to complex valued functions on $\mathfrak{a}_{\mathbb{C}}^*$.

In the remainder of this section, we need to deal with the sets of zeroes of the functions $\overline{c_n(-\overline{\lambda})}$. In particular , we will be concern with

$$Z_n = \left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^+ \middle| \, \langle \operatorname{Re}\lambda, \alpha\rangle < 0, \, \overline{c_n(-\overline{\lambda})} = 0\right\}.$$

Then the following properties are readily seen by the discussion in Section 2.3.1.

- Since $c_n = c_{-n}$, it follows that $Z_n = Z_{-n} = Z_{|n|}$.
- If n is odd, $Z_n = \{-2, -4, \ldots, -|n| + 1\} \cdot \alpha/2$.
- If n is even, $Z_n = \{-1, -3, \ldots, -|n| + 1\} \cdot \alpha/2$.

By means of these properties together with the discussion in Section 2.3.1, we may conclude that if n and $\mu(H) < 0$ have the same parity, then

$$\operatorname*{Res}_{\lambda=\mu}\left(\overline{c_n^+(-\overline{\lambda})}\right)^{-1} = 0$$

Before proceeding with the next lemma, we shall be in need of two basic results of complex analysis. They are as follows,

Lemma 3.3.1. *[7, Section 18.7]. Let $a \in \mathbb{C}$, and denote by $D_r(a)$, the open disk in the complex plane of radius r around a. If $g = h/k$ with $h, k \in \mathcal{O}(D_r(a))$ with $h(a) \neq 0$, $k(a) = 0$ and $k'(a) \neq 0$; then $\operatorname*{Res}_{z=a} g = h(a)/k'(a)$.*

Lemma 3.3.2. *In the framework of Lemma 3.3.1, if $f \in \mathcal{O}(D_r(a))$ and g a complex valued function with a simple pole at a, then $\operatorname*{Res}_{z=a} f \cdot g = f(a)\operatorname*{Res}_{z=a} g$.*

Proof. By definition of residue,

$$\operatorname*{Res}_{z=a} f \cdot g = \lim_{z\to a}(z - a)f(z)g(z) = f(a)\operatorname*{Res}_{z=a} g$$

$\square$

Lemma 3.3.3. *Let $\mu \in Z_n$. Then*

$$\operatorname*{Res}_{\lambda=\mu}\left(\overline{c_n^+(-\overline{\lambda})}\right)^{-1} = \frac{\Gamma\left(\frac{-\mu(H)+n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{-\mu(H)+n}{2}\right)}\frac{n-1-\mu(H)}{2}\prod_{\substack{j=1\\ \mu(H)\neq 2j-n-1}}^{n}\frac{-\mu(H)+j-1}{-\mu(H)+2j-n-1}$$

Proof. According to equation (2.9),

$$\operatorname*{Res}_{\lambda=\mu}\left(\overline{c_n^+(-\overline{\lambda})}\right)^{-1} = \operatorname*{Res}_{\lambda=\mu}\left(\overline{\frac{\Gamma\left(\frac{-\overline{\lambda(H)}+n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{-\overline{\lambda(H)}+n}{2}\right)}}\prod_{j=1}^{n}\frac{-\lambda(H)+(j-1)}{-\lambda(H)+2j-n-1}\right).$$

First of all, since $\mu \in Z_n$ is neither a pole of $\Gamma\left(\frac{-\overline{\lambda(H)}+n+1}{2}\right)$ nor $\Gamma\left(\frac{-\overline{\lambda(H)}+n}{2}\right)$, we have that

$$\overline{\frac{\Gamma\left(\frac{-\overline{\lambda(H)}+n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{-\overline{\lambda(H)}+n}{2}\right)}} = \frac{\Gamma\left(\frac{-\lambda(H)+n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{-\lambda(H)+n}{2}\right)}.$$

Furthermore, we observe that this last expression is holomorphic in $\lambda(H)$ around a neighbourhood of $\mu(H)$; hence it can be pulled out in the residue computation by means of Lemma 3.3.2 yielding

$$\operatorname*{Res}_{\lambda=\mu}\left(\overline{c_n^+(-\overline{\lambda})}\right)^{-1} = \frac{\Gamma\left(\frac{-\mu(H)+n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{-\mu(H)+n}{2}\right)}\operatorname*{Res}_{\lambda=\mu}\left(\prod_{j=1}^{n}\frac{-\lambda(H)+(j-1)}{-\lambda(H)+2j-n-1}\right).$$

Using Lemma 3.3.2, it is not difficult to see that we obtain the desired result. ☐

Remark. As for now, we shall denote $\operatorname*{Res}_{\lambda=\mu}\left(\overline{c_n^+(-\overline{\lambda})}\right)^{-1}$ by $\Lambda_{n,\mu}$.

Lemma 3.3.4. *The function $a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}$ is holomorphic on $\langle \operatorname{Re}\lambda, \alpha\rangle < 0$.*

Proof. It is readily seen that the function $a^{\lambda+\rho_{\overline{P}}}$ is holomorphic in λ. It remains to be shown that $\Phi_{\lambda+\rho_{\overline{P}}}$ is holomorphic for $\langle \operatorname{Re}\lambda, \alpha\rangle < 0$. To do so, it is sufficient to study the poles of the coefficients $\{\Gamma_j^+(\lambda)\}_{j=1}^{\infty}$. By Lemma 3.1.3, we see that they have no poles in the variable λ in the aforementioned region, hence $a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}$ is holomorphic on $\langle \operatorname{Re}\lambda, \alpha\rangle < 0$. ☐

By means of Lemmas 3.3.1, 3.2.2 and 3.3.4, we can now take a step forward in the calculation of the residue defined in (3.5): if $\mu(H) < 0$,

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}f(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right) = \Lambda_{n,\mu}a^{\mu+\rho_{\overline{P}}}\Phi_{\mu+\rho_{\overline{P}}}(a)\mathcal{F}_n^{\mathrm{Wh}}f(\mu).$$

We observe that if $\mu \notin Z_n$, the whole expression vanishes. This will have its consequences that we shall discuss further on. Assume that $\mu \in Z_n$. Using the definition of the Fourier-Whittaker transform, we get that

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}f(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right) = \Lambda_{n,\mu}\int_A a^{\mu+\rho_{\overline{P}}}\Phi_{\mu+\rho_{\overline{P}}}(a)\overline{\mathrm{Wh}_n(\overline{P},-\mu)(b)}b^{-2\rho_{\overline{P}}}f(b)db.$$

Lemma 3.3.5. *There exists a possibly zero constant $C_{n,\mu} \in \mathbb{C}$ for $\mu \in Z_n$ such that $\mathrm{Wh}_n(\overline{P},-\mu)(b) = C_{n,\mu}b^{\mu+\rho_{\overline{P}}}\Phi_\mu(b)$.*

Proof. We consider the map $\varphi(\lambda, a) := c_n^+(-\lambda)a^{-\lambda+\rho_{\overline{P}}}\Phi_{-\lambda}(a)$. As a function of λ, it is holomorphic around μ. This is because we know that μ is a zero of $c_n^+(-\lambda)$ and $\Phi_{-\lambda}$ has at most simple poles by Lemma 3.1.3. We have earlier seen that the function solves the eigenvalue problem if $\lambda(H) \notin \mathbb{Z}$. Since φ is holomorphic around μ, we can apply holomorphic continuation concluding that $\varphi(\mu, \cdot) \in \mathcal{E}_{n,\mu}$. The function $\varphi(\mu, \cdot)$ has the following description in power series.

$$\varphi(\mu, a) = a^{-\mu+\rho_{\overline{P}}} \sum_{j \geqslant 0} c_n^+(-\lambda)\Gamma_j^+(-\lambda)\big|_{\lambda=\mu} a^{-j\alpha}$$

We note for instance that the independent coefficient of the power series is 0 as $\Gamma_1^+(-\mu) = 1$. Moreover, this power series cannot be zero. The reason is that at least, the power series has a non-zero coefficient; namely, $c_n^+(-\lambda)\Gamma_{-\mu(H)}(-\lambda)$. This coefficient is non-zero because the term $\Gamma_{-\mu(H)}(-\lambda)$ has a simple pole at μ by Lemma 3.1.3 and $c_n(-\lambda)$ has a simple zero at μ. Thus, let us take the smallest $k \in \mathbb{N}\backslash\{0\}$ for which $c_n^+(-\lambda)\Gamma_k(-\lambda)|_{\lambda=\mu}$ is not zero. Since the smallest pole of $\Gamma_j(-\lambda)$ is at $\lambda(H) = -j$ by Lemma 3.1.3, k must be $-\mu(H)$. Then we may write,

$$\varphi(\mu, b) = Cb^{\mu+\rho_{\overline{P}}} \sum_{j \geqslant 0} d_j(\lambda)a^{-j\alpha}$$

with C a non-zero constant a $d_0 = 1$. In order to conclude, we observe that we have the expression

$$\mathrm{Wh}(\overline{P}, -\lambda)(a) = \varphi(\lambda, a) + a^{\lambda+\rho_{\overline{P}}}c_n^-(-\lambda)\Phi_\lambda(a)$$

and that the left hand side and the first term of the right hand side are holomorphic maps around μ. This means that the expression $c_n^-(-\lambda)a^{\lambda+\rho_{\overline{P}}}\Phi_\lambda(a)$ is also holomorphic at μ, hence $c_n^-(-\lambda)$ is holomorphic at μ by Lemma 3.3.4. Hence we may finish writing that

$$\mathrm{Wh}(\overline{P}, -\mu)(b) = \varphi(\mu, b) + a^{\mu+\rho_{\overline{P}}}c_n^-(-\mu)\Phi_{-\mu}(b) = (C + c_n^-(-\mu))a^{\mu+\rho_{\overline{P}}}\Phi_\mu.$$

□

Lemma 3.3.6. *Let $\mu \in Z_n$. Then there exists a unique discrete series representation π of G such that*

- $\pi^\infty(\Omega) = (\mu(H)^2 - 1)I$.
- τ_n *is a K-type of π.*

In fact, $\pi = D_{m(\mu)}^{\varepsilon(n)}$, where $m(\mu) = |\mu(H)| + 1$ and $\varepsilon(n) = \mathrm{sign}(n)$.

Proof. From the properties of Z_n, we deduce that when $\mu \in Z_n$, then $1 \leqslant |\mu(H)| \leqslant |n| - 1$ and $\mu(H)$ and $|n| - 1$ have the same parity. Set $m(\mu) := |\mu(H)| + 1$. This means that $2 \leqslant m(\mu) \leqslant |n|$, with $m(\mu)$ and n having the same parity. If we want the first condition to be fulfilled by a holomorphic discrete series representation of parameter m, there is no other chance that that $m = \pm\mu(H) + 1$ due to lemma 1.4.8. In the case of $n > 0$, this means that $m(\mu) \leqslant n$. We also have seen that the K-types for the holomorphic discrete series are $m(\mu) + 2k$ for $k \in \mathbb{N}$. Since $m(\mu)$ and n have the same parity, then $n = m + 2k_0$ for some $k_0 \in \mathbb{N}$. Then τ_n appears as K-type for $D_{m(\mu)}^+$. In the other case, the same argument can be carried out yielding the antiholomorphic discrete series. Hence the representation of the discrete series that we are looking for is $D_{m(\mu)}^{\varepsilon(n)}$. □

Lemma 3.3.7. *Let $\mu \in Z_n$. Let χ be a regular character on N. If r_χ and n have opposite signs, then $C_{\mu,n} = 0$.*

Proof. In Section 1.4.2, we have seen that $D^{\pm}_{m(\mu)}$ may be embedded into the principal series representation corresponding to the parameters $\xi_{m(\mu)}$ and μ. More specifically, there exist embeddings of $(\mathfrak{g}, K)$-modules

$$j^{\pm} : (H_n^{\pm})_K \hookrightarrow \mathrm{Ind}_{\overline{P}}^{G}(\xi_{m(\mu)} \otimes \mu \otimes 1)_K.$$

Since the holomorphic and antiholomorphic discrete series are irreducible by Lemma 1.4.6, we may define the embedding

$$j = j^+ \oplus j^- : H_n^+ \oplus H_n^- \to \mathrm{Ind}_{\overline{P}}^{G}(\xi_{m(\mu)} \otimes \mu \otimes 1)_K.$$

By the Casselman-Wallach globalization functor [15, Theorem 11.6.7 and Lemma 11.5.7], j has a unique extension to a continuous G-equivariant linear map

$$j : (H^+_{m(\mu)})^\infty \oplus (H^-_{m(\mu)})^\infty \longrightarrow C^\infty(\overline{P} : \xi_{m(\mu)} : \mu).$$

Let $\eta_{\overline{P}, \xi_{m(\mu)}, -\mu}$ according to formula (2.1). Then

$$\eta_\mu := \langle \cdot, \eta_{\overline{P}, \xi_{m(\mu)}, -\mu} \rangle \in \mathrm{Wh}_\chi(C^\infty(\overline{P} : \xi_{m(\mu)} : \mu)).$$

Set $\eta_\mu^{\pm} := (j^{\pm})^* \eta_\mu \in \mathrm{Wh}_\chi((H^{\pm}_{m(\mu)})^\infty)$ and put $\varepsilon = \mathrm{sign}(n)$. Let $\tau_{n,\mu} \in C^\infty(\overline{P} : \xi_{m(\mu)} : \mu)$ defined as in 2.3.1. Since K-types have multiplicity one in $C^\infty(\overline{P} : \xi_{m(\mu)} : \mu)$, there exists a unique v_n in the the τ_n-isotypical component of $H_n^{\varepsilon(n)}$ such that $\tau_{n,\mu} = j^{\varepsilon(n)}(v_n)$. It is immediate that

$$\mathrm{Wh}_\chi(\overline{P}, -\mu)(a) = \eta_\mu \left(\pi_{\overline{P}, \xi, \mu}(a)^{-1} j^{\varepsilon(n)}(v_n) \right) = \eta_\mu \left(j^{\varepsilon}(D^{\varepsilon(n)}_{m(\mu)} v_n) \right). \tag{3.6}$$

By Theorem 2.4.1, if $\varepsilon(n)$ and r_χ have opposite signs then $\mathrm{Wh}_\chi\left(\left(H^{\varepsilon(n)}_{m(\mu)} \right)^\infty \right) = 0$. This implies that if the n and r_χ have opposite signs $\eta_\mu^{\varepsilon(n)} = 0$. We see clearly in equation 3.6 that this implies

$$\mathrm{Wh}_\chi(\overline{P}, -\mu)(a) = 0$$

if n and r_χ have opposite signs. ☐

Lemma 3.3.8. *Let $\mu \in Z_n$. If $\mathrm{Wh}_n(\overline{P}, -\mu)$ is a non-zero function in the variable a then it is a Whittaker matrix coefficient of the discrete series representation $D^{\varepsilon(n)}_{m(\mu)}$.*

Proof. If $\mathrm{Wh}_n(\overline{P}, -\mu)$ is a non-zero function in the variable a, by Lemma 3.3.7, we know that then n and r_χ have the same sign. We immediately see that expression 3.6 has the form of a non-zero matrix coefficient of the discrete series representation $D^{\varepsilon(n)}_{m(\mu)}$. ☐

Now we can conclude with the following theorem.

Theorem 3.3.2. *Let $\mu \in Z_n$ and suppose that*

$$\operatorname*{Res}_{\lambda=\mu} \left(a^{\lambda+\rho_{\overline{P}}} \Phi_{\lambda+\rho_{\overline{P}}}(a) \frac{\mathcal{F}_n^{\mathrm{Wh}} f(\lambda)}{c_n^+(-\overline{\lambda})} \right) \tag{3.7}$$

is a non-zero function of the variable a. Then it is a Whittaker matrix coefficient of the discrete series representation $D^{\varepsilon(n)}_{m(\mu)}$.

Proof. We have seen in this section that

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}f(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right)=\Lambda_{n,\mu}\int_A a^{\mu+\rho_{\overline{P}}}\Phi_{\mu+\rho_{\overline{P}}}(a)\overline{\mathrm{Wh}_n(\overline{P},-\mu)(b)}b^{-2\rho_{\overline{P}}}f(b)db.$$

By Lemma 3.3.5, we may write

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}f(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right)=\Lambda_{n,\mu}\int_A a^{\mu+\rho_{\overline{P}}}\Phi_{\mu+\rho_{\overline{P}}}(a)\overline{C_{n,\mu}b^{\mu+\rho_{\overline{P}}}\Phi_{\mu}(b)}b^{-2\rho_{\overline{P}}}f(b)db.$$

Since the residue is a non-zero function, we see that $C_{n,\mu}$ cannot be zero. Then, we may rewrite the residue as follows:

$$\operatorname*{Res}_{\lambda=\mu}\left(a^{\lambda+\rho_{\overline{P}}}\Phi_{\lambda+\rho_{\overline{P}}}(a)\frac{\mathcal{F}_n^{\mathrm{Wh}}f(\lambda)}{\overline{c_n^+(-\overline{\lambda})}}\right)=\frac{\Lambda_{n,\mu}}{C_{n,\mu}}\mathrm{Wh}_n(\overline{P},-\mu)(a)\mathcal{F}_n^{\mathrm{Wh}}(f)(\mu).$$

In this last expression we see that the residue given by the expression 3.5 is a constant multiple of a Whittaker matrix coefficient of the discrete series representation $D_{m(\mu)}^{\varepsilon(n)}$. $\square$

A

Appendix

The purpose of this appendix is to collect basic prior knowledge that has been studied beforehand the making of this thesis or general constructions that are used throughout the thesis without mentioning. In this appendix we will give an overview about Haar measures and the Casimir operator.

A.1 Haar measures

Let G be a locally compact group and let μ be a Borel measure on G, denoting by $\mathfrak{B}$ the Borel σ-algebra of G. We say that the measure μ is left invariant if $l_g^*\mu(A) = \mu(l_g(A)) = \mu(A)$ for every $A \in \mathfrak{B}$ and for all $g \in G$. Analogously, we can define right invariant measures via multiplication on the right. When the measure is both right and left invariant, we speak of a bi-invariant measure.

Definition A.1.1 (Haar measure)**.** A *left Haar measure* on G is a regular Borel left invariant measure, that is finite on the compact subsets of G.

We are interested in the construction of a Haar measure on a Lie group G. We assume the reader familiar with the basic theory of densities and their integration. We shall denote by $\mathcal{D}(TG)$, the vector bundle of densities over G, with fibre $\mathcal{D}(T_pG)$ at the point $p \in G$. We denote the fibre at the identity by $\mathcal{D}\mathfrak{g}$. Recall, that the space of densities of a complex vector space is always one-dimensional. Whenever we want to refer to the positive densities, we write $\mathcal{D}_+(TG)$.

Lemma A.1.1 (Construction of a density on TG)**.** *Let $\omega_0 \in \mathcal{D}_+\mathfrak{g}$. Then the element ω given by*

$$\omega(x) = (dl_{x^{-1}}(x))^*\omega_0 \quad \textit{for } x \in G$$

is a positive left-invariant smooth density on TG. Furthermore, positive left-invariant densities are unique up to a positive scalar multiples.

Proof. We shall check in the first place that ω is indeed a density. So for fixed $x \in G$ and $A \in \mathrm{End}(T_xG)$, we have that

$$A^*\omega(x) = A^*(dl_{x^{-1}}(x))^*\omega_0 = |\det dl_{x^{-1}}(x)|\,|\det A|\,\omega_0 = |\det A|\,\omega(x).$$

Thus ω is a density. It is clearly left-invariant as

$$(l_g^*\omega)(x) = (dl_g(x))^*\omega(gx) = (dl_g(x))^*(dl_{x^{-1}g^{-1}}(x))^*\omega_0 = (dl_{x^{-1}}(x))^*\omega_0 = \omega(x).$$

Precisely as $\omega(x) = |\det dl_{x^{-1}}(x)|\ \omega_0$, the density is positive for every $x \in G$ since so is ω_0. To prove the last assertion, suppose we have $\omega_1, \omega_2 \in \Gamma(\mathcal{D}_+TG)$. Then $\omega_1(e), \omega_2(e) \in \mathcal{D}\mathfrak{g}$, which is one dimensional. Then there exists $C > 0$ such that $\omega_1(e) = C\omega_2(e)$. Hence, for $x \in G$

$$\omega_1(x) = (dl_{x^{-1}}(x))^*\omega_1(e) = C(dl_{x^{-1}}(x))^*\omega_2(e) = C\omega_2(x)$$

□

In a Lie group, we benefit from the C^∞- structure by taking the density $\omega_0 = |\widetilde{\omega_0}|$ where $\widetilde{\omega_0} \in \bigwedge^{\text{top}} \mathfrak{g}$ and apply A.1.1. The theory of integration over densities provides us the following integral

Lemma A.1.2. *The map $I : C_c(G) \to C$ given by $I(f) = \int_G f\omega$ is a positive continuous complex linear functional such that $I(l_g^* f) = I(f)$ for all $f \in C_c(G)$.*

The Riesz representation theorem for positive complex linear functionals, applied to the positive functional in Lemma A.1.2, yields the existence of a Haar measure on G. This Haar measure will be denoted by dg.

Remark. If G is compact, then there exists a unique left Haar measure dg for which $\int_G dg = 1$. Let $d\tilde{g}$ be a Haar measure for G. Then $0 < C = \int_G d\tilde{g} < \infty$. Define $dg = C^{-1}d\tilde{g}$. By left-invariance, if there where another left Haar measure ω such that $\int_G \omega = 1$, then $1 = \int_G dg = \int_G c_0\omega = c_0 \int_G \omega = c_0$ for some $c_0 > 0$. Thus $dg = \omega$

Now we explore the situation when our Haar measure is bi-invariant

Definition A.1.2 (Unimodular Lie group)**.** We say that a Lie group is unimodular if $|\det \mathrm{Ad}(g)| = 1$ for every $g \in G$.

Lemma A.1.3. *If G is unimodular, then every left invariant density is right invariant.*

Proof. Let ω be a left invariant density, then for $g \in G$

$$r_g^*\omega = r_g^* l_{g^{-1}}^*\omega = (C_{g^{-1}})^*\omega = \mathrm{Ad}(g^{-1})^*\omega = |\det \mathrm{Ad}(g^{-1})|\omega = \omega$$

□

Lemma A.1.4. *[5, Corollary 8.31] A Lie group is unimodular if it is either abelian, compact, semisimple or nilpotent.*

A.1.1 Positive densities on homogeneous spaces

The goal is now to investigate the existence of positive invariant densities on G/H, where G is a Lie group and H a closed subgroup. It is well-known, since left multiplication on G defines a free and proper action, that there exists a unique structure of C^∞ manifold making the projection map $\pi_{G/H} : G \to G/H$ a smooth surjective submersion. By differentiating, $d\pi_{G/H}(e)$ induces a linear isomorphism between $\mathfrak{g}/\mathfrak{h}$ and $T_e(G/H)$. We observe that the maps $C_h(x) = hxh^{-1}$ for $x \in G$ and $h \in H$, leave H invariant, meaning that $\mathrm{Ad}(h)|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$. Therefore the map $\pi_{\mathfrak{g}/\mathfrak{h}} \circ \mathrm{Ad}(h)$ induces a linear automorphism of $\mathfrak{g}/\mathfrak{h}$ that we shall denote by $A(h)$. Now we identify $\mathcal{D}(T_{eH}(G/H))$ with $\mathcal{D}(\mathfrak{g}/\mathfrak{h})$. Then for $\omega \in \mathcal{D}(\mathfrak{g}/\mathfrak{h})$, we see that

$$A(h)^*\omega = |\det A(h)|\omega = \frac{|\det \mathrm{Ad}(h)|_{\mathfrak{h}}|}{|\det \mathrm{Ad}(h)|}\,\omega$$

Definition A.1.3 (Modular function)**.** We call $\Delta : H \to \mathbb{R}_{>0}$ given by

$$\Delta(h) = \frac{|\det \mathrm{Ad}(h)|_{\mathfrak{h}}|}{|\det \mathrm{Ad}(h)|}$$

the modular function. The modular function is a character.

Lemma A.1.5 (Existence of an invariant positive density on G/H)**.** *[11, Corollary 19.19]. Let G be a Lie group and H be a compact subgroup. Then G/H has a H-invariant positive density that is unique up to positive scalar multiples.*

A.2 The Casimir element

The *Casimir element* plays an important role in the theory developed. That is why, it has been decided to include its construction in this appendix.We will begin by giving a brief description of the *universal enveloping algebra* in terms of the the tensor and symmetric algebras of a Lie algebra . To find a more detatiled account of the tensor and symmetric algebras, we recommend to consult [13, Chapter 9] and [6].

Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra. Consider the two-sided ideal $\mathcal{J} \subset T(\mathfrak{g})$ generated by all elements of the form $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$.

Definition A.2.1 (Universal enveloping algebra)**.** The universal enveloping algebra is the associative unital algebra given by the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{J}$.

Again, the canonical injective map $j : \mathfrak{g} \hookrightarrow T(\mathfrak{g})$ induces a linear map $\nu : \mathfrak{g} \to U(\mathfrak{g})$ when composed with the quotient map $T(\mathfrak{g}) \to U(\mathfrak{g})$. We have the following universal property for the universal enveloping algebra that determines it up to algebra isomorphism: let A be any associative unital algebra and $\varphi : \mathfrak{g} \to (A, [\cdot,\cdot]_{\mathrm{com}})$ a Lie algebra homomorphism. Then there exists a unique algebra homomorphism $\tilde{\varphi}$ making the following diagram commmute

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\nu"'] & (A, [\cdot,\cdot]_{\mathrm{com}}) \\ U(\mathfrak{g}) \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

The Poincaré-Birkhoff-Witt theorem gives an explicit basis for the universal enveloping algebra once given a basis for the Lie algebra.

Theorem A.2.1 (Poincaré-Birkhoff-Witt Theorem)**.** *[5, Theorem 3.18] Let $\{X_i\}_{i=1}^n$ be a basis of $\mathfrak{g}$. Then the products $X_1^{j^1} \cdots X_n^{j_n}$ for $j_i \geqslant 0$ form a basis of $U(\mathfrak{g})$.*

Remark. The PBW theorem implies that the map ν is injective since it sends linearly independent vectors to linearly independent vectors.

The motivation behind considering the universal enveloping algebra in representation theory is that we can extend any $\mathfrak{g}$-module to a $U(\mathfrak{g})$-module by the universal property. On the other hand, any $U(g)$-module can be restricted to a $\mathfrak{g}$-module. Moreover, this establishes an isomorphism between the categories of $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules.

One last tool that we need is the *symmetrisation map*. Its importance lies in the fact that it is an effective attempt to parametrise an non-abelian algebra by an abelian one. Details of the construction of this map can be found in [13, Chapter 10]

Lemma A.2.1 (Symmetrisation map)**.** *There exists a unique linear map $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ such that $s(X^m) = X^m$ for all $X \in \mathfrak{g}$.*

We are now prepared to make the construction of the Casimir element. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra[1] of $\mathfrak{g}$ and let R be a root system with R^+ a choice of positive roots. Recall that B the Killing form on $\mathfrak{g}$. is given by $B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$. The Killing form is symmetric and invariant under automorphisms and under the Lie bracket.

Lemma A.2.2. *The following holds*

1. *For all $\lambda, \mu \in \mathfrak{h}^*$, if $\lambda + \mu \neq 0$. Then $\mathfrak{g}_\lambda \perp^B \mathfrak{g}_\mu$.*

2. *There are elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $B(X_\alpha, Y_\alpha) = 1$ for all root $\alpha \in R$.*

3. *$B|_{\mathfrak{h}_\mathbb{R}}$ defines a positive definite inner product.*

Proof.

1. Let $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$. For $H \in \mathfrak{h}$. Since $B([H,X],Y) = -B(X,[H,Y])$, the result follows.

2. Assume by contradiction that $X_\alpha \mathfrak{g}_\alpha \backslash \{0\}$ there does not exist $Y \mathfrak{g}$ suc that $B(X_\alpha, Y) = 1$. This means that for $X_\alpha \in \mathfrak{g}_\alpha$ there does not exist $Y \in \mathfrak{g}_\alpha$ such that $B(X_\alpha, Y) \neq 0$. Thus $B(X_\alpha, Y) = 0$ for all $Y \in \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, B is non-degenerate. Hence $X_\alpha = 0$, which is a contradiction. By the root decomposition 1.1.3 and the previous result $Y = Y_\alpha \in \mathfrak{g}_{-\alpha}$.

3. Let $X \in \mathfrak{h}_\mathbb{R}$. Then $B(X,X) = \mathrm{Tr}(\mathrm{ad}(X)^2)$. We observe that $\mathrm{ad}(X)$ is anti-symmetric, then $\mathrm{ad}(X)^2$ is symmetric. Then it diagonalises in real positive eigenvalue. Then $B(X,X) \geqslant 0$. And if $B(X,X) = 0$, the sum of all positive eigenvalues are zero, hence all are zero. Then $\mathrm{ad}(X) = 0$. Thus $X = 0$ because B is non-degenerate.

□

We proceed to construct the Casimir element. Let $\{H_j\}_{j=1}^k$ be a B-orthonormal basis of $\mathfrak{h}_\mathbb{R}$. We complete the basis up to a basis for $\mathfrak{g}$ with elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ as in Lemma A.2.2. Since each $\mathfrak{g}_\alpha$ has dimension 1 for $\alpha \in R$. Then $\{H_j, X_\alpha, Y_\alpha\}_{j=1,\, \alpha\in R^+}^n$ is a basis for $\mathfrak{g}$. Denote $\{H^j, X^\alpha, Y^\alpha\}_{j=1,\, \alpha\in R^+}^n$ its associated basis in $\mathfrak{g}^*$.

Lemma A.2.3. *In the previous bases the Killing form can be written as*

$$B = \sum_{j=1}^{k} H^j \otimes H^j + \sum_{\alpha \in R^+} X^\alpha \otimes Y^\alpha + Y^\alpha \otimes X^\alpha.$$

[1] A Cartan subalgebra $\mathfrak{h}$ is an abelian subalgebra which is maximal and for which $\mathrm{ad}(\mathfrak{h})$ is contained in the diagonalisible endomorphisms of $\mathfrak{g}$

Proof. Write

$$X = \sum_{j=1}^{k} h_j H^j + \sum_{\alpha\in R^+} x_\alpha X_\alpha + y_\alpha Y_\alpha \quad \tilde{X} = \sum_{j=1}^{k} \widetilde{h_j} H^j + \sum_{\alpha\in R^+} \widetilde{x_\alpha} X_\alpha + \widetilde{y_\alpha} Y_\alpha$$

We compute

$$\begin{aligned} B(X,\widetilde{X}) =& B\left(\sum_{j=1}^{k} h_j H_j + \sum_{\alpha\in R^+} x_\alpha X_\alpha + y_\alpha Y_\alpha, \sum_{j=1}^{k} \widetilde{h_j} H_j + \sum_{\alpha\in R^+} \widetilde{x_\alpha} X_\alpha + \widetilde{y_\alpha} Y_\alpha\right) \\ =& \sum_{j=1}^{k} h_j \widetilde{h_j} B(H_j, H_j) + \sum_{\alpha\in R^+} x_\alpha \widetilde{y_\alpha} B(X_\alpha, Y_\alpha) + y_\alpha \widetilde{x_\alpha} B(Y_\alpha, X_\alpha) \\ =& \sum_{j=1}^{k} H^j(X) H^j(\widetilde{X}) + \sum_{\alpha\in R^+} X^\alpha(X) Y^\alpha(\widetilde{X}) + Y^\alpha(X) X^\alpha(\widetilde{X}). \end{aligned}$$

□

With the previous Lemma we can dualise the Killing form, which takes the form of

$$B^* = B = \sum_{j=1}^{k} H_j \otimes H_j + \sum_{\alpha\in R^+} X_\alpha \otimes Y_\alpha + Y_\alpha \otimes X_\alpha.$$

We consider now the the polynomial $p \in P(\mathfrak{g}^*)$, given by $p(\xi) = B^*(\xi,\xi)$. Thus, seen in $S(\mathfrak{g})$,

$$p = \sum_{i=1}^{k} H_j^2 + 2 \sum_{\alpha\in R^+} X_\alpha Y_\alpha. \tag{A.1}$$

Definition A.2.2 (Casimir element)**.** The Casimir element is the image under the symmetrisation map of the polynomial given in (A.1).

Then the Casimir operator takes the following form

$$\Omega = \sum_{i=1}^{k} H_j^2 + \sum_{\alpha\in R^+} s(2X_\alpha Y_\alpha) = \sum_{i=1}^{k} H_j^2 + 2 \sum_{\alpha\in R^+} X_\alpha Y_\alpha + Y_\alpha X_\alpha.$$

Lemma A.2.4. *Ω is an element in the centre of the universal enveloping algebra.*

Proof. We have to prove that Ω commutes with every element in the universal enveloping algebra. Observe that it is enough to show that it commutes with the elements of the basis $\{H_j, X_\alpha, Y_\alpha\}$. For all $j = 1,\dots,k$, it follows

$$\begin{aligned} \Omega H_j &= \sum_{i=1}^{k} H_j^3 + 2 \sum_{\alpha\in R^+} X_\alpha Y_\alpha H_j + Y_\alpha X_\alpha H_j \\ &= \sum_{i=1}^{k} H_j^3 + 2 \sum_{\alpha\in R^+} X_\alpha (H_j Y_\alpha - [H_j, Y_\alpha]) + Y_\alpha (H_j X_\alpha - [H_j, X_\alpha]) \\ &= \sum_{i=1}^{k} H_j^3 + 2 \sum_{\alpha\in R^+} H_j X_\alpha Y_\alpha + H_j Y_\alpha X_\alpha \\ &= H_j \Omega. \end{aligned}$$

With the same techniques, one can show that $\Omega X_\alpha = X_\alpha \Omega$ and $\Omega Y_\alpha = y_\alpha \Omega$ for all $\alpha \in R^+$ and the result follows. □

Usually, Ω is also refered as the Casimir operator. Consider $(C^\infty(G), R)$ the right regular representation of a Lie group G. Then its associated $\mathfrak{g}$-module can be computed by means of the differential given by the following

$$R : \mathfrak{g} \longrightarrow \mathrm{End}(C^\infty(G)), \quad X \longmapsto (R_X f)(g) = \left.\frac{d}{dt}\right|_{t=0} R_{\exp tX} f(g).$$

Then for every $X \in \mathfrak{g}$, the map R_X defines a first order linear differential operator. Moreover, R_X is G-equivariant with the left regular representation, which implies that R_X is left-invariant linear differential operator. Left invariant linear differential operators form a Lie algebra with the commutator and it will be denoted by $\mathbb{D}(G)$[2]. By the universal property of the universal enveloping algebra, we can extend $R : U(\mathfrak{g}) \to \mathbb{D}(G)$ (note the abuse of notation) satisfying the following commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "R"] \arrow[d, "\nu"'] & \mathbb{D}(G) \\
U(\mathfrak{g}) \arrow[ur, "R"'] &
\end{tikzcd}$$

This extension allows us to consider $R_\Omega \in \mathbb{D}(G)$, thence we can speak of the Casimir operator. Analogously, the discussion equally applies to the construction of L_Ω.

Lemma A.2.5. *Let G be a connected semisimple Lie group,* $\mathrm{Ad}(x)\Omega = \Omega$ *for all* $x \in G$.

Proof. Since G is a connected Lie group, it is enough to prove the statement for $x = \exp X$ for some $X \in \mathfrak{g}$. First of all, the universal property of the universal enveloping algebra yields an extensension of the adjoint action on G in the following way

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{Ad}(x)"] \arrow[d, "\nu"'] & \mathfrak{g} \arrow[r, "\nu"] & U(\mathfrak{g}) \\
U(\mathfrak{g}) \arrow[urr, "\tilde{\mathrm{Ad}}(x)"'] & &
\end{tikzcd}$$

Hence $\mathrm{Ad}(x)$ is well-defined. Then $\mathrm{Ad}(x)\Omega = \mathrm{Ad}(\exp X)\Omega = e^{\mathrm{ad}(X)}\Omega = \Omega$, since $\mathrm{ad}(X)|_{\mathfrak{z}(\mathfrak{g})} = 0$. $\square$

Lemma A.2.6. *In G, $L_\Omega = R_\Omega$.*

Proof. Let $g \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G)$

$$\begin{aligned}
L_X f(g) &= \left.\frac{d}{dt}\right|_{t=0} f(\exp -tX\, g) = \left.\frac{d}{dt}\right|_{t=0} f(gg^{-1}(\exp -tX)g) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(g \exp -t\mathrm{Ad}(g^{-1})X) = R_{-\mathrm{Ad}(g)^{-1}X} f(g).
\end{aligned}$$

With the Casimir operator of the form $\Omega = H^2 + 2(XY + YX)$,

$$\begin{aligned}
L_\Omega f(g) =& (L_H L_H + 2(L_X L_Y + L_Y L_X))f(g) \\
=& R_{(-\mathrm{Ad}(g^{-1})H)^2} f(g) + 2R_{\mathrm{Ad}(g^{-1})(XY+YX)} f(g) = R_{\mathrm{Ad}(g^{-1})\Omega} f(g) = R_\Omega f(g).
\end{aligned}$$

$\square$

[2]See [13, Chapters 11 & 12] for a more detailed account on $\mathbb{D}(G)$.

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