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Configuration Spaces and Operads

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Contents

Introduction

Configuration spaces have been objects of interest in topology for a long time, going back at least as far as the work of Faddell and Neuwirth ([FN62]). Despite their deceptively simple definition, these spaces carry substantial amounts of geometric information and can be used to distinguish different topological spaces. In recent years, configuration spaces have continued to be objects of interest via their connection with e.g. operads (e.g. [And10], [AT14], [DHK19], [BW18a]), factorisation homology ([AF15], [Lur17]), non-abelian Poincaré duality (e.g. [Lur17], [AF15]), etc.

This thesis is principally concerned with the connections between configuration spaces and operads. These connections have been acknowledged since the inception of operads – May demonstrated weak equivalences between the configuration spaces of Euclidean spaces and the little cubes operads in his paper [May72].

More recent research has built on this foundation, as witnessed in the separate approaches taken in the work of Dwyer, Hess and Knudsen [DHK19], [HK18], and Boavida and Weiss [BW18a], [BW18b]. In the course of this thesis, we have become interested in both approaches, leading us to attempt to expand upon some of their results, and to tackle some of their conjectures.

Overview

- We begin with some reminders on model categories and the left Bousfield localisation of a simplicial model category – a tool to which we will have repeated recourse. As a special example of this technique in action, we describe the category of simplicial spaces with the Segal model structure. The Segal property is intimately linked with operads: in a sense, we can view it as one of the narrative through-lines of this thesis – hence, even though much of this material is standard, we believe it is appropriate to treat it as a foundation on which we can build our later work.

- In chapters 2 and 3, we discuss first strict operads and then several models of ∞-operads. The purpose of the discussion on strict operads is two-fold: first, we wish to provide a sound intuition on this material before we proceed to the more challenging generalisation in the ∞-categorical setting; second, we need to have a solid understanding of strict operads to deal with the notion of *right modules over operads*, a concept of central importance in the work of Dwyer, Hess and Knudsen.

  In our discussion of ∞-operads, we try to sketch some of the ideas which have been presented in [Bar18] and [CHH18] to demonstrate that the various models of ∞-operads are equivalent. This is necessary since we later attempt to harness similar approaches in our own efforts to deduce an equivalence between two models of (reduced) ∞-operads. We also spend some time discussing the Dunn additivity theorem for ∞-categorical little cubes operads from [Lur17]: this theorem is another of the recurring characters in this thesis, making its presence felt when we deal with configuration modules à la Dwyer, Hess and Knudsen, and separately when we work with the configuration category introduced by Boavida and Weiss.

- In chapter 4, we record some of the basic facts about right modules over operads (in spaces) and we explain how this notion can be translated into the setting of *right fibrations over a (fixed) Segal space* (this is just one of the many opportunities we get to further our acquaintance with the Segal condition). Using this adapted perspective, we show that it is possible to give a new/alternative proof of the main result in [DHK19]. We also propose a modified construction of this translated result which appears to generalise the main result in that paper.

- In chapter 5, we introduce the *configuration category* as defined in [BW18a] and [BW18b] and describe some of its properties. We draw a connection between the configuration category of Euclidean spaces and the little cubes operads, which leads us to a pair of interesting questions: the first pertains to an equivalence of models for reduced ∞-operads; while the second is in regard to a possible alternative proof of the ∞-categorical Dunn additivity theorem using the configuration category of Euclidean spaces in conjunction with some results from [BW18b] relating the configuration category of a product of manifolds to the *box-product* of configuration categories.

- In chapters 6 and 7, we study the aforementioned questions/conjectures in greater depth. We first attempt to deduce a Quillen equivalence between two models for reduced ∞-operads. Based on this, we study whether it is feasible to use the configuration category to produce an alternative proof of the Dunn additivity theorem by relating the box-product with the ∞-categorical tensor product for reduced ∞-operads. We give an outline of how such a proof of this result would work in principle – however, the reader should be warned that this proof is based around the incomplete proofs of Lemma 7.1.7 and Lemma 7.1.8.

Additionally, we have included two short appendices. The first is an attempt to take the reader on a whistlestop (and intuition-heavy) tour of some relevant quasi-categorical notions, relying on the many parallels that exist between quasi-categories and Segal spaces. In particular, since the notion of a coCartesian morphism is so critical in the definition of an ∞-operad, we devote quite some space to building up this idea from the 1- and 2-categorical versions to furnish the reader with some intuition. Much of the material in this appendix is covered in more depth in [Lur09a] and [Har], and our treatment is enormously indebted to (and an utterly inadequate reflection of) those sources. The second appendix seeks to give a skeleton of the proof of the Dunn additivity theorem as stated in [Lur17] – the reason we have seen fit to add this is mainly due to the sheer frequency with which we refer to that result. We have tried to impart the main steps of this proof while hoping to avoid getting too bogged down in some of the more technical details which run to many pages of the voluminous [Lur17].

We have attempted as far as possible to give a thorough outline of the foundational material needed to understand the contents of the first three chapters, though we have stinted somewhat on proofs of this material, preferring to let the curious reader investigate these details from the sources.

In later chapters, as the balance of the contents shifts more towards our own work (and towards more recent developments in research), we have opted for a slightly more concise approach in the exposition of background material, especially since many of our proofs are formally similar to the ones used in our primary sources – [DHK19], [BW18a], [BW18b], [CHH18] and [Lur17]. We hope that the reader will forgive this somewhat uneven level of coverage, and will feel encouraged and inspired to examine these source texts for the wealth of beautiful and interesting ideas that they contain.

The relation between configuration spaces and operads continues to be a fertile ground for research, with many possible avenues of interest being opened all the time. Among the topics which we have been unable to treat here are

- The connections between configuration categories, operads and the Goodwillie-Weiss embedding calculus, following [BW18a] and [BW13].
- The study of configuration spaces for manifolds with boundary, as in [Cam+18].
- The link between configuration spaces, operads and spaces of long knots – many of the techniques which we describe in this thesis have also found application in the study of spaces of long knots, as in [BW18a], [AT14] and [DH12].
- The possible utility of pro-functors in studying these subjects, as already evinced in [BH19], where the authors use the properties of the configuration category and techniques around pro-functors to capture a number of nice results pertaining to the little disks operads and the Grothendieck-Teichmüller group.

Hopefully the reader may find this thesis to be a useful starting point from which to explore any and all of these exciting areas of research.

Prerequisites

Some familiarity with certain notions from the world of quasi-categories is assumed: specifically, the ideas in [Lur09a, Chapter 1]; the various notions of fibrations of quasi-categories (i.e. left/inner/right/Cartesian) and the Joyal model structure (all in [Lur09a, Chapter 2]); as well as some familiarity with the straightening and unstraightening constructions of [Lur09a, Section 3.2]. (Essentially, Higher Topos Theory is the topologist's version of the Hitchhiker's Guide to the Galaxy.)

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Chapter 1

Model Categories, Localisations and Segal Spaces

In the first section of this chapter, we give a very brief recap of model categories before going on to discuss localisations, focusing especially on the left Bousfield localisation of a proper simplicial model category. Most of this material is drawn from [Hir98], with some additional material coming from [Rie14], [Lur09a], [Bar07] and [Hov07]. We then direct our focus on discussing the category of simplicial spaces with the Reedy model structure. We conclude the chapter with a discussion of the categories of Segal and complete Segal spaces introduced by Rezk in [Rez01] and show how they arise as a left Bousfield localisation. These categories (or their fibrewise versions) will prove to have repeated utility for us and will also give us a hands-on application of the localising machinery we develop in the course of this chapter.

1.1 A Short Recap on Model Categories

Definition 1.1.1. A **model category** $\mathcal{C}$ is the data of a complete and cocomplete category with three distinguished classes of morphisms: the weak equivalences, the cofibrations and the fibrations. These are required to satisfy the following axioms:

[M1] Given two composable maps f, g in $\mathcal{C}$, if any two of f, g, gf are weak equivalences, then so is the third.

[M2] If g belongs to one of the distinguished classes of morphisms, then any retract of g also belongs to that class.

[M3] Given any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the dashed lift exists if either of the following holds:

(i) The morphism i is a cofibration and p is both a fibration and a weak equivalence (we say p *is a trivial fibration*).

(ii) The morphism p is a fibration and i is both a cofibration and a weak equivalence (we say i *is a trivial cofibration*).

[M4] Every morphism $f : A \to B$ in $\mathcal{C}$ admits two functorial factorizations:

(i) $A \xrightarrow{i} A' \xrightarrow{p} B$, where i is a trivial cofibration and p is a fibration.

(ii) $A \xrightarrow{j} B' \xrightarrow{q} B$, where j is a cofibration and q is a trivial fibration.

An object A in $\mathcal{C}$ is said to be **cofibrant** if its map from the initial object in $\mathcal{C}$ is a cofibration; while A is **fibrant** if its map to the terminal object in $\mathcal{C}$ is a fibration. A model category is said to be **cofibrantly-generated** if there exist two sets of maps I, J such that the collection of fibrations consists of those maps which have the right-lifting property with respect to all maps in I, and the collection of trivial fibrations consists of those maps which have the right-lifting property with respect to all maps in J. Examples of cofibrantly-generated model structures include the Kan-Quillen model structure on simplicial sets, the Quillen model structure on topological spaces, or the Hovey model structure on bounded chain complexes (where the weak equivalences are quasi-isomorphisms and the fibrations are the levelwise surjections).

1.1.1 Homotopy Categories and Quillen Functors

Given a category $\mathcal{C}$ with a model structure, it is possible to define a notion of a homotopy between morphisms – in particular, for morphisms between fibrant-cofibrant objects, this notion of homotopy determines an equivalence relation; given such a pair of objects X, Y we let $[X, Y]$ denote the set of homotopy classes of morphisms between them. With this equivalence relation available to us, we can define the **homotopy category** of $\mathcal{C}$ – this is the category $\mathrm{Ho}(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$ and given a pair of objects X, Y, with fibrant-cofibrant replacements $\tilde{X}, \tilde{Y}$ respectively, we take $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = [\tilde{X}, \tilde{Y}]$. There is a natural functor $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ which acts as the identity on objects and sends a morphism $f : X \to Y$ to the homotopy class of the corresponding morphism between the fibrant-cofibrant replacements of X and Y respectively.

In a suitable sense, we can view the category $\mathrm{Ho}(\mathcal{C})$ as being obtained from $\mathcal{C}$ by formally inverting all of the weak equivalences in the model structure on $\mathcal{C}$. This approach lies at the heart of localisation – in particular, the method of left Bousfield localisations will act as a kind of machinery by which we can add extra weak equivalences to a model structure which will in turn give rise to more equivalences in the corresponding homotopy category.

Given model categories $\mathcal{C}$ and $\mathcal{D}$, with natural functors $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$, and a functor $F : \mathcal{C} \to \mathcal{D}$, we can define the **total left derived functor** of F, $\mathbb{L}F$, to be the right Kan extension (if it exists) of $\gamma_{\mathcal{D}} \circ F$ along $\gamma_{\mathcal{C}}$. Likewise, $\mathbb{R}F$ is the **total right derived functor** of F and is defined as the left Kan extension (if it exists) of $\gamma_{\mathcal{D}} \circ F$ along $\gamma_{\mathcal{C}}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is said to be a **left Quillen functor** if F preserves cofibrations and trivial cofibrations; we say F is a **right Quillen functor** if F preserves fibrations and trivial fibrations. Given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between model categories $\mathcal{C}$ and $\mathcal{D}$, we say that (F, G) are a **Quillen pair** if F is left Quillen and G is right Quillen. Given a Quillen pair (F, G), there is an induced adjunction on the homotopy categories

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$$

We will say the pair (F, G) is a pair of Quillen equivalences (or more typically, we will say that either F or G is a **Quillen equivalence**) if the derived adjunction above is an equivalence of categories.

1.2 Proper Model Categories and Localisations

Let $\mathcal{C}$ be a model category and $\mathcal{W}$ be a set of morphisms in $\mathcal{C}$. A **left localisation** of $\mathcal{C}$ with respect to $\mathcal{W}$ is the data of a model category $L_{\mathcal{W}}\mathcal{C}$ with a left Quillen functor $j : \mathcal{C} \to L_{\mathcal{W}}\mathcal{C}$ such that

- For every element g in $\mathcal{W}$ with image $\gamma(g)$ in $\mathrm{Ho}(\mathcal{C})$, the total left-derived functor $\mathbb{L}j$ takes $\gamma(g)$ to an isomorphism in the homotopy category of $L_{\mathcal{W}}\mathcal{C}$.
- Given any other model category $\mathcal{D}$ and a functor $j' : \mathcal{C} \to \mathcal{D}$ such that the image of $\gamma(g)$ is an isomorphism in $\mathrm{Ho}(\mathcal{D})$ for every g in $\mathcal{W}$, there exists unique left Quillen functor $\delta : L_{\mathcal{W}}\mathcal{C} \to \mathcal{D}$ such that $\delta j = j'$.

In particular, we wish to discuss a method for systematically producing localisations with respect to a given set of maps on a cofibrantly-generated presentable left proper simplicially-enriched model category – this will be the subject of left Bousfield localisations. To arrive at that juncture, we first remind ourselves of the definitions of presentable categories, proper model categories and simplicial model categories.

Definition 1.2.1. Let λ be a regular cardinal. A category $\mathcal{C}$ is said to be **λ-presentable** if $\mathcal{C}$ is cocomplete, and there is a set of objects S in $\mathcal{C}$ such that

(i) every object in $\mathcal{C}$ can be written as a colimit of a diagram valued in the subcategory of $\mathcal{C}$ spanned by the objects of S;

(ii) for each object $s \in S$, the functor $\mathrm{Hom}_{\mathcal{C}}(s, -) : \mathcal{C} \to \mathrm{Set}$ preserves λ-filtered colimits.

The category $\mathcal{C}$ is **presentable** if it is λ-presentable for some λ.

Definition 1.2.2. Let $\mathcal{C}$ be a model category. We say that $\mathcal{C}$ is **left proper** if every pushout of a weak equivalence along a cofibration is a weak equivalence. Dually, we say $\mathcal{C}$ is **right proper** if every pullback of a weak equivalence along a fibration is a weak equivalence. A **proper model category** is a model category which is both left and right proper.

To aid us in producing some examples of proper model categories, we invoke this next result due to Reedy.

Theorem 1.2.3. *[Hir98, Theorem 13.1.2] Let $\mathcal{C}$ be a model category. Then, every pushout along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence; dually, every pullback along a fibration of a weak equivalence between fibrant objects is a weak equivalence.*

In the special case where every object is cofibrant (e.g. for the Kan-Quillen model structure on the category of simplicial sets), the above result tells us that the category is automatically left proper. Similarly, if every object is fibrant (e.g. with the Quillen model structure on the category of topological spaces), the category is automatically right proper. In fact, with a little bit more work it can be shown that the categories Top and sSet are both proper. One final useful fact which we note about proper model structures is the following:

Theorem 1.2.4. *[Hir98, Theorem 13.1.14] Let $\mathcal{D}$ be a small category, and let $\mathcal{C}$ be a cofibrantly-generated model category which is left proper (resp. right proper/proper). Then, with respect to the projective model structure $\mathcal{C}^{\mathcal{D}}$ is also left proper (resp. right proper/proper).*

The next ingredient we need to introduce before we define left Bousfield localisations is that of the simplicial model category. First, we recall that a category $\mathcal{C}$ is said to be a **simplicial category** (or is enriched in simplicial sets) if it satisfies the following properties:

- for any two objects X, Y in $\mathcal{C}$, there is a simplicial set $\mathrm{Map}(X, Y)$ (*the simplicial mapping space*);
- given any object X in $\mathcal{C}$, there is a map of simplicial sets $i_X : \Delta[0] \to \mathrm{Map}(X, X)$ (*the unit map*);
- given any objects X, Y, Z in $\mathcal{C}$, there is a *composition* rule:

$$c_{X,Y,Z} : \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

  and this composition rule must be associative and unital in the obvious way;
- for any two objects X, Y in $\mathcal{C}$ there is a bijection of sets $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\cong} \mathrm{Map}(X, Y)_0$.

An obvious example of a simplicial category is the category of simplicial sets itself: given a pair of simplicial sets X, Y, we can define

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta[n], Y)$$

The category of simplicial sets is (in a somewhat tautological sense) tensored and cotensored over itself, since it is a monoidal category with respect to the Cartesian product: specifically, we can define $X \otimes Y = X \times Y$ and $Y^X = \mathrm{Map}(X, Y)$.

Another example of a simplicial model category is the category of topological spaces: we can define the simplicial mapping spaces in this category by $\mathrm{Map}(X, Y) = \mathrm{Hom}_{\mathrm{Top}}(X \times |\Delta[n]|, Y)$.

Definition 1.2.5. A **simplicial model category** is a simplicial category $\mathcal{C}$, equipped with a model structure which is compatible with the simplicial structure in the sense that it satisfies two further axioms:

[M5] For any two objects X, Y in $\mathcal{C}$ and any simplicial set K, there exist objects $X \otimes K$ and Y^K in $\mathcal{C}$ such that we have the adjunction:

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(X, Y^K)$$

where in the middle term we use $\mathrm{Map}(,)$ interchangeably to denote both the simplicial mapping space in the category of simplicial sets and in the category $\mathcal{C}$ (we say that $\mathcal{C}$ is *tensored and cotensored over simplicial sets*);

[M6] For any cofibration $i : A \to B$ in $\mathcal{C}$ and any fibration $p : X \to Y$, the map of simplicial sets

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a Kan fibration that is trivial if either i is a trivial cofibration or p is a trivial fibration.

The category of topological spaces with the Quillen model structure and the aforementioned mapping spaces is a simplicial model category. Using this fact in conjunction with Theorem 1.2.4, it is evident that the category of simplicial spaces (with the projective model structure) is also a simplicial model category.

Definition 1.2.6. Given a simplicial model category $\mathcal{C}$, equipped with a cofibrant replacement functor $\mathscr{C} : \mathcal{C} \to \mathcal{C}$ and a fibrant replacement functor $\mathscr{F} : \mathcal{C} \to \mathcal{C}$, we define for any pair of objects X, Y in $\mathcal{C}$

$$\mathbb{R}\mathrm{Map}(X, Y) = \mathrm{Map}(\mathscr{C}X, \mathscr{F}Y)$$

The above mapping spaces are only well-defined up to weak equivalence of simplicial sets, since our choice of (co)fibrant replacement is only well-defined up to weak equivalence in $\mathcal{C}$. This may seem to be an undesirable position; however, we note that these mapping spaces *do* give the homotopy category $\mathrm{Ho}(\mathcal{C})$ the structure of an enriched category over the homotopy category of simplicial sets, which will be sufficient for the purposes of defining the left Bousfield localisation.

Definition 1.2.7. Consider a simplicial model category $\mathcal{C}$, with a set of maps $\mathcal{W}$.

- An object X of $\mathcal{C}$ is said to be **$\mathcal{W}$-local** if X is fibrant and, for every $f : A \to B$ in $\mathcal{W}$, the induced map $\mathbb{R}\mathrm{Map}(f, X) : \mathbb{R}\mathrm{Map}(B, X) \to \mathbb{R}\mathrm{Map}(A, X)$ is a weak equivalence of simplicial sets.
- A map $g : Y \to Z$ is said to be a **$\mathcal{W}$-local equivalence** if $\mathbb{R}\mathrm{Map}(g, X) : \mathbb{R}\mathrm{Map}(Z, X) \to \mathbb{R}\mathrm{Map}(Y, X)$ is a weak equivalence of simplicial sets for every $\mathcal{W}$-local object X.

We are now finally in a position to produce the main definition of interest for us in this section:

Definition 1.2.8. Let $\mathcal{C}$ be a simplicially-enriched left proper presentable cofibrantly-generated model category, $\mathcal{W}$ a set of maps in $\mathcal{C}$. The **left Bousfield localisation** of $\mathcal{C}$ with respect to $\mathcal{W}$ (if it exists) is a presentable simplicial model category structure $L_{\mathcal{W}}\mathcal{C}$ on the underlying category of $\mathcal{C}$ such that:

- the weak equivalences of $L_{\mathcal{W}}\mathcal{C}$ are the $\mathcal{W}$-local equivalences;
- the cofibrations of $L_{\mathcal{W}}\mathcal{C}$ are the cofibrations of $\mathcal{C}$;
- the fibrations of $L_{\mathcal{W}}\mathcal{C}$ are those maps which have the right lifting property with respect to all maps which are both cofibrations and $\mathcal{W}$-local equivalences;
- the fibrant objects in this model structure are the $\mathcal{W}$-local objects of $\mathcal{C}$.

It is immediate from the above that every weak equivalence in $\mathcal{C}$ is a weak equivalence in $L_{\mathcal{W}}\mathcal{C}$, while every fibration of $L_{\mathcal{W}}\mathcal{C}$ is a fibration of $\mathcal{C}$; since the classes of cofibrations are the same for both model structures, it also follows that the class of trivial fibrations of $L_{\mathcal{W}}\mathcal{C}$ coincides with the trivial fibrations of $\mathcal{C}$.

The existence of this localisation is not at all a trivial fact – the principal challenge lies in showing that a set of generating trivial cofibrations exists, the solution to which resides in the Bousfield-Smith cardinality argument (see [Hir98, Section 4.5] for details). Based on that argument, we have the following result:

Theorem 1.2.9. *[Bar07, Theorem 2.11] Let $\mathcal{C}$ be a simplicially-enriched, left proper, cofibrantly-generated presentable model category, $\mathcal{W}$ a set of maps in $\mathcal{C}$. Then, the left Bousfield localisation of $\mathcal{C}$ with respect to $\mathcal{W}$ exists.*

As remarked previously, the value of the Bousfield localisation is that it gives us a procedure to add weak equivalences to a model category. In particular, if we wish to isolate certain objects in our category with some distinguished properties, the method of Bousfield localisations can prove exceptionally useful. We will see this principal in action repeatedly throughout this thesis. A closely related scenario in which the question of localising with respect to some set of morphisms comes from the following theorem:

Theorem 1.2.10. *[GZ67, Proposition 1.3] Consider a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, with counit $\eta : FG \to 1_{\mathcal{C}}$, and let $\mathcal{W}$ be the set of all morphisms h in $\mathcal{C}$ such that Fh is an isomorphism in $\mathcal{D}$. If we denote by γ the localisation functor $\mathcal{C} \to L_{\mathcal{W}}\mathcal{C}$, then the following are equivalent:*

(a) G is fully faithful.

(b) η is invertible.

(c) The functor $H : L_{\mathcal{W}}\mathcal{C} \to \mathcal{D}$ defined by $F = H \circ \gamma$ is an equivalence.

A corollary of this is in the case where G is homotopically fully faithful (i.e. the right derived functor $\mathbb{R}G$ is a fully faithful between the homotopy categories): the theorem then tells us that the homotopy category $\mathrm{Ho}(\mathcal{C})$ can be further localised with respect to some class of morphisms such that the resulting localisation is equivalent to $\mathrm{Ho}(\mathcal{D})$.

An instructive first example of the idea of using Bousfield localisation to isolate some special class of objects will be in the setting of Segal spaces, which may be identified as the $\mathcal{W}$-local objects for a set of morphisms $\mathcal{W}$ in the category of simplicial spaces. Before we can discuss this example in detail, we need to make a quick review of the general theory of Reedy model structures.

1.3 Simplicial Spaces and the Reedy Model Structure

Definition 1.3.1. A **Reedy category** is the data of a category $\mathcal{B}$ and two subcategories, $\mathcal{B}_+$ and $\mathcal{B}_-$, together with a functor $d : \mathcal{B} \to \lambda$ (*the degree function*) for some ordinal λ, such that

- every non-identity morphism in $\mathcal{B}_+$ raises the degree;
- every non-identity morphism in $\mathcal{B}_-$ lowers the degree;
- every morphism f in $\mathcal{B}$ may be factored uniquely as $f = gh$ with h in $\mathcal{B}_-$ and g in $\mathcal{B}_+$.

We will sometimes refer to $\mathcal{B}_+$ as the **direct subcategory** of $\mathcal{B}$, and to $\mathcal{B}_-$ as the **inverse subcategory**.

For us, the most relevant example will be the simplicial index category Δ^{op}: we let Δ^{op}_+ be the subcategory of Δ spanned by all the objects of Δ, and whose morphisms are the opposites of surjective maps; and we let Δ^{op}_- be the subcategory spanned by all objects of Δ, and whose morphisms are the opposites of injective maps. Taking λ to be the first countable ordinal, the degree function $d : \Delta^{op} \to \lambda$ is given by $[n] \mapsto n$.

Given a Reedy category $\mathcal{B}$ and a functor $F : \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is a category with all small limits, we define for each object $n \in \mathcal{B}$, the n^{th} **latching object** of F:

$$L_nF = \mathrm{colim}_{\partial(\mathcal{B}_+ \downarrow n)} L$$

where the objects of the category $\partial\,(\mathcal{B}_+ \downarrow n)$ are the non-identity arrows in $\mathcal{B}_+$ with codomain n. Since L_nF is defined by a colimit it follows that there is a natural map $L_nF \to F(n)$ for each object n in $\mathcal{B}$. Dually, we define the n^{th} **matching object** to be the object M_nF of $\mathcal{C}$ given by

$$M_nF = \lim_{\partial(n \downarrow \mathcal{B}_-)} L$$

where the objects of the category $\partial\,(n \downarrow \mathcal{B}_-)$ are the non-identity arrows in $\mathcal{B}_-$ with domain n. By the dual properties of limits, there is a natural map $F(n) \to M_nF$ for every object n in $\mathcal{B}$. (In the case of simplicial sets, these are the familiar latching and matching spaces that we frequently encounter in topological applications.)

The construction of the n^{th} matching and latching objects is functorial in the sense that for any $X \to Y$ in the functor category $\mathcal{C}^{\mathcal{B}}$, there are induced maps $L_nX \to L_nY$ and $M_nX \to M_nY$ in $\mathcal{C}$ for every object n in $\mathcal{B}$. (This is immediate from the definition of these objects.) Putting this together with the information of the natural latching/mapping maps, we obtain the following diagrams:

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & X_n \arrow[d] \arrow[ddr, bend left] & \\
L_nY \arrow[r] \arrow[drr, bend right] & X_n \sqcup_{L_nX} L_nY \arrow[dr, dotted] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & Y_n
\end{tikzcd}
\qquad
\begin{tikzcd}
X_n \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& M_nX \times_{M_nY} Y_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M_nX \arrow[d] \\
& Y_n \arrow[r] & M_nY
\end{tikzcd}$$

We refer to the dotted morphism in the diagram on the left as the relative latching map, while the dashed morphism in the diagram on the right is the relative matching map.

Definition 1.3.2. Let $\mathcal{C}$ be a model category and let $\mathcal{B}$ be a Reedy category. The **Reedy model structure** on $\mathcal{C}^{\mathcal{B}}$ is the model structure such that a map $f : X \to Y$ in $\mathcal{C}^{\mathcal{B}}$ is

- a Reedy weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence for every object n in $\mathcal{B}$;
- a Reedy cofibration if the relative latching map $X_n \sqcup_{L_n X} L_n Y \to Y_n$ is a cofibration for each n in $\mathcal{B}$;
- a Reedy fibration if the relative matching map $X_n \to M_n X \times_{M_n Y} Y_n$ is a fibration for each n in $\mathcal{B}$.

The category of simplicial spaces may be endowed with a Reedy model structure instead of the projective one, and it can be shown that there is a Quillen equivalence between these two model structures. On the surface, it may seem like the new model structure only serves to substantially complicate our lives; however, one advantage to using the Reedy structure is that we have an explicit description of both the fibrations and cofibrations, whereas in the case of the projective structure, cofibrations are defined by the fact that they satisfy the left-lifting property with respect to all levelwise trivial fibrations.

1.4 Segal Spaces and Complete Segal Spaces

As it is sometimes instructive to work backwards, we produce first the definitions of Segal and complete Segal spaces, and then reverse-engineer to show how we arrive at such definitions in the context of a left Bousfield localisation. The first step in this process is to familiarise ourselves with the Segal condition.

Typically, when we first encounter the Segal condition for simplicial sets, it is motivated in terms of the notion of the nerve of a category. Specifically, we have the following elegant result which gives a characterisation of those simplicial sets which are equivalent to the nerves of categories:

Theorem 1.4.1. *A simplicial set X arises as the nerve of a category if and only if there are bijections*

$$X_n \xrightarrow{\simeq} \lim(X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \dots \xleftarrow{d_0} X_1)$$

for all $n \geq 2$.

The above is generally referred to as the **Segal condition**, having appeared in a paper [Seg68] by that author (even though Segal himself attributes the original assertion to Grothendieck). With the above definition in mind, we can consider the following special class of simplicial spaces, originally defined by Rezk in [Rez01]:

Definition 1.4.2. A simplicial space X is said to be a **Segal space** if it is fibrant with respect to the Reedy model structure and it satisfies the **Segal condition** – for all $n \geq 2$, there is a weak equivalence of spaces

$$X_n \xrightarrow{\simeq} \lim(X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \dots \xleftarrow{d_0} X_1) \tag{1.1}$$

Remark 1.4.3. In some papers authors forego the Reedy fibrancy condition and instead state that a simplicial space X is Segal if there is a weak equivalence

$$X_n \xrightarrow{\simeq} \operatorname{holim}\Big(X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \dots \xleftarrow{d_0} X_1\Big)$$

where holim denotes the homotopy limit of the diagram, i.e. in this case, the homotopy pullback. However, we will prefer to demand the Reedy fibrancy upfront, as per the original definition of Rezk.

Given a Segal space X, we can define so-called **mapping spaces** – namely, for a pair of vertices $x, y \in X_0$, we let $\operatorname{map}_X(x, y) = \{x\} \times_{X_0} X_1 \times_{X_0} \{y\} = \{f \in X_1 : d_1 f = x,\, d_0 f = y\}$. For $x \in X_0$, we let $\mathrm{id}_x \in \operatorname{map}_X(x, x)$ denote the element $s_0 x \in X_1$.

It will occasionally be convenient for us to utilise an alternative definition of the Segal condition in terms of mapping spaces. First, for each $i = 0, \dots, n$, let $\alpha_i^n : [0] \to [n]$ be the map $0 \mapsto i$. Then for any fixed collection of vertices $x_0, \dots, x_n \in X_0$, we write $\operatorname{map}_X(x_0, x_1, \dots, x_n)$ to mean the fibre of the map $(\alpha_0^{n*}, \dots, \alpha_n^{n*}) : X_n \to X_0^{n+1}$ over the point $(x_0, \dots, x_n)$. In this context, we can rephrase the condition that X is Segal as a requirement that we have weak equivalences

$$\varphi_n : \operatorname{map}_X(x_0, \dots, x_n) \xrightarrow{\sim} \prod_{1 \leq i \leq n} \operatorname{map}_X(x_{i-1}, x_i)$$

for every $(n+1)$-tuple of vertices $(n \geq 1)$, $x_0, \ldots, x_n$.

There is also a notion of "composition" for these mapping spaces: given any triple of vertices $x, y, z \in X_0$ and a pair of maps between them, $f \in \mathrm{map}_X(x,y)$ and $g \in \mathrm{map}_X(y,z)$, we let $g \circ f \in \mathrm{map}_X(x,z)$ denote $d_1\xi$ where $\xi \in X_2$ satisfies $d_2\xi = f$, $d_0\xi = g$. Intuitively, a picture of the situation is as follows:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\xi"] & \\ x \arrow[ur, "f"] \arrow[rr] & {} & z \end{tikzcd}$$

where ξ is supposed to represent the whole 2-simplex whose vertices are x, y, z. Obviously, this "composition" isn't strictly well-defined, since many such 2-simplices ξ may exist; however, a suitable notion of homotopy may be contrived such that $g \circ f$ is well-defined up to homotopy. More precisely, given a pair of edges $f, f' \in \mathrm{map}_X(x,y)$, we write $f \sim f'$ if $[f] = [f']$ in $\pi_0\mathrm{map}_X(x,y)$. Hence, if we are given a pair of 2-simplices ξ, ξ' which both satisfy $d_2\xi = d_2\xi' = f$ and $d_0\xi = d_0\xi' = g$, then we can use the Segal condition (1.1) for $n = 2$ to see that $d_1\xi$ and $d_1\xi'$ must lie in the same path-component, i.e. the composition of maps makes sense up to homotopy. In what follows, we thus write $g \circ f$ to mean a representative of the desired homotopy class.

We note that composition satisfies notions of associativity and unitality:

Proposition 1.4.4. *[Rez01, Proposition 5.4] Given $f \in \mathrm{map}_X(x,y)$, $g \in \mathrm{map}_X(y,z)$, $h \in \mathrm{map}_X(z,w)$, we have*

$$h \circ (g \circ f) \sim (h \circ g) \circ f \text{ and } \mathrm{id}_y \circ f \sim f \sim f \circ \mathrm{id}_x$$

The foregoing proposition tells us that there is a well-defined notion of composition on the Segal space X, at least up to homotopy. With this in mind, we are able to define the **homotopy category** of X, written $\mathrm{Ho}(X)$: the objects of this category are the vertices of X; given two vertices $x, y \in X_0$, we take $\mathrm{Hom}_{\mathrm{Ho}(X)}(x,y)$ to be $\pi_0\mathrm{map}_X(x,y)$.

We say further that a map $f \in \mathrm{map}_X(x,y)$ is a **homotopy equivalence** (or is homotopy invertible) if there exist maps $g, g' \in \mathrm{map}_X(y,x)$ such that $[f \circ g] = [\mathrm{id}_y]$ and $[g' \circ f] = [\mathrm{id}_x]$. The Segal condition tells us that such a triple $(g, f, g') \in \mathrm{map}_X(y,x) \times \mathrm{map}_X(x,y) \times \mathrm{map}_X(y,x)$ allows us to specify an element (unique up to homotopy) $\xi \in \mathrm{map}_X(y,x,y,x)$ such that $d_0 \circ d_0\xi = g$, $d_1 \circ d_0\xi = f$ and $d_1 \circ d_1\xi = g'$. In fact, we can turn this argument around to give an alternative characterisation of homotopy equivalence: an edge $f \in \mathrm{map}_X(x,y)$ is a homotopy equivalence if and only if the triple $(\mathrm{id}_x, f, \mathrm{id}_y) \in X_1 \times_{X_0} X_1 \times_{X_0} X_1$ admits a lift to an element of X_3.

We note that for any vertex $x \in X_0$, it is trivially the case that id_x is a homotopy equivalence. The next result is key in showing that homotopy equivalences satisfy a kind of transitivity relation:

Proposition 1.4.5. *[Rez01, Lemma 5.8] Let X be a Segal space. Let g and g' lie in the same path component of X_1 and suppose that g' is a homotopy equivalence (in the sense above). Then g is also a homotopy equivalence.*

Armed with this information, we can now define a subspace $X_1^{he} \subseteq X_1$ consisting of all those edges which are homotopy equivalences. Since id_x is a homotopy equivalence for each $x \in X_0$, we find that the map $s_0 : X_0 \to X_1$ must factor through X_1^{he}. This leads us to the next definition:

Definition 1.4.6. A Segal space X is said to be a **complete Segal space** if the map $s_0 : X_0 \to X_1^{he}$ is a weak equivalence.

An important variant of the notion of a complete Segal space is that of a fibrewise complete Segal space:

Definition 1.4.7. A map of Segal spaces $w : X \to B$ is said to be a **fibrewise complete Segal space** (or, leaving the reference map w implicit, we say that X is a fibrewise complete Segal space over B) if the below diagram is homotopy Cartesian:

$$\begin{tikzcd} X_1^{he} \arrow[r, "d_0"] \arrow[d, "w"'] & X_0 \arrow[d, "w"] \\ B_1^{he} \arrow[r, "d_0"] & B_0 \end{tikzcd}$$

Before we go on to give some examples of Segal spaces we turn our attention to the concept of a Dwyer-Kan equivalence, which is intimately linked with the theory of complete Segal spaces.

Definition 1.4.8. Let $f : X \to Y$ be a map of Segal spaces. We say f is a **Dwyer-Kan equivalence** if it satisfies the following properties:

- The induced map on homotopy categories $\mathrm{Ho}(f) : \mathrm{Ho}(X) \to \mathrm{Ho}(Y)$ is an equivalence of categories.
- For each pair of vertices $x, x' \in X_0$, the induced map $\mathrm{map}_X(x, x') \to \mathrm{map}_Y(f(x), f(x'))$ is a weak equivalence.

1.4.1 Examples of Segal Spaces, Complete Segal Spaces and fibrewise Complete Segal Spaces

Example 1.4.9. Given any small category $\mathcal{C}$, we can view the nerve of $\mathcal{C}$ as being a simplicial space, where for each i, $(N\mathcal{C})_i$ is given the discrete topology. In the context of Segal spaces, this construction is typically known as the **discrete nerve construction** to draw a distinction with other notions of "nerve" which also arise in this setting.

The reader is warned that, in general, it is not the case that the nerve of a small category is a complete Segal space. As a counterexample, consider the discrete category $\mathcal{C}$ depicted as follows:

$$\begin{tikzcd} x \arrow[rr, bend left, "g"] & & y \arrow[dl, bend left, "f"] \\ & z \arrow[ul, bend left, "h"] \arrow[ur, bend left, "f^{-1}"] & \end{tikzcd}$$

where we have suppressed the identity morphisms in our diagram, and where the only non-trivial compositions which are isomorphic to identity morphisms are $f \circ f^{-1}$ and $f^{-1} \circ f$. We see immediately that $(N\mathcal{C})_0 = \{x, y, z\}$, while $(N\mathcal{C})_1^{he} = \{\mathrm{id}_x, \mathrm{id}_y, \mathrm{id}_z, f, f^{-1}\}$, so clearly this discrete simplicial space is Segal, but not complete.

Example 1.4.10. Let $\mathcal{C}$ be a category with a class of weak equivalences, $\mathcal{W}$. Given any small category $\mathcal{D}$ and a pair of functors $F, G : \mathcal{D} \to \mathcal{C}$, we say a natural transformation α from F to G is a weak equivalence if α_d is an element of $\mathcal{W}$ for each $d \in \mathcal{D}$. We write $\mathrm{We}(\mathcal{C}^{\mathcal{D}})$ to denote the subcategory of $\mathcal{C}^{\mathcal{D}}$ whose collection of objects is all the functors $\mathcal{D} \to \mathcal{C}$ and whose morphisms are the weak equivalences between them. As a special case, we remark that $\mathrm{We}(\mathcal{C})$ is just $\mathcal{W}$ itself.

Viewing each element $[n]$ of the simplicial indexing category Δ as a poset, we define the **classification diagram** $N(\mathcal{C}, \mathcal{W})$ to be the simplicial space whose collection of n-simplices is the nerve of the category $\mathrm{We}(\mathcal{C}^{[n]})$. In this sense, the classification diagram is a bisimplicial set whose (m, n)-simplices (i.e. n-simplices of the nerve of $\mathrm{We}(\mathcal{C}^{[m]})$) are $m \times n$ grids of composable arrows, where each vertical string of n composable arrows consists of weak equivalences. We can in fact view the discrete nerve as a special case of this construction, where we let the class $\mathcal{W}$ consist only of the identity morphisms between objects of $\mathcal{C}$. Using the equivalence between the geometric realisation of a bisimplicial set and its diagonal, we observe that the discrete nerve corresponds exactly to the diagonal of this classication diagram.

At the other extreme, we can instead consider $\mathcal{W}$ to consist of all isomorphisms in $\mathcal{C}$ (alternatively, we can look at a general $\mathcal{W}$ as a category with the same objects as $\mathcal{C}$ but a smaller collection of morphisms between objects: from this perspective, the case where $\mathcal{W}$ consists of all isomorphisms corresponds to the maximal subgroupoid of $\mathcal{C}$). We write $\mathcal{NC}$ for the associated classification diagram and we refer to it as the **classifying diagram** of the category $\mathcal{C}$. The name is more than a little suggestive: usually we recover the classifying space of a group(oid) as the geometric realisation of its nerve – but, viewed as a constant simplicial space, the nerve of a groupoid $\mathcal{G}$ is weakly equivalent to the classifying diagram of $\mathcal{G}$.

Example 1.4.11. Let C be a locally small category enriched in spaces and consider any topological functor $F : \mathsf{C} \to \mathcal{S}$. Let $\mathcal{G}(F)$ denote the Grothendieck construction (which we will discuss in greater detail in Chapter 4) of this functor; in a natural way, this is a topological category, equipped with a forgetful functor to C. In particular, it follows that the nerves of both $\mathcal{G}(F)$ and C have the structure of simplicial spaces and that there is a reference map $N\mathcal{G}(F) \to N\mathsf{C}$, which exhibits $N\mathcal{G}(F)$ as a fibrewise complete Segal space over $N\mathsf{C}$.

1.4.2 Localisations for Segal Spaces and the Model Structures

We have already gone some way implicitly towards describing the Segal space localisation; at this juncture, we make the relevant set of localising maps explicit.

Recall that for each n and each $0 \le i \le n-1$, we specified maps $\alpha_i^n : [0] \to [n]$ in the ordinal category by $0 \mapsto i$. Similarly, for $2 \le n$, we can define $\tilde{\alpha}_i^n : [1] \to [n]$ by $j \mapsto j+i$ For each n, let $F(n)$ denote the simplicial set $\Delta[n]$ given the discrete topology (i.e. viewed as a simplicial space) and for $n \ge 2$, let $G(n) = \cup_{1 \le i \le n-1} (\tilde{\alpha}_i^n)_* F(1) \subseteq F(n)$. For each n, there is a natural inclusion map $G(n) \hookrightarrow F(n)$, which we denote by φ^n. Note that for any Reedy-fibrant simplicial space X, the simplicial mapping space $\mathrm{Map}(F(n), X)$ is equal to the space X_n by the Yoneda lemma; on the other hand $\mathrm{Map}(G(n), X)$ is the (homotopy) limit of the diagram $X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \dots \xleftarrow{d_0} X_1$.

Having recast these spaces in this light, we see that a Reedy-fibrant simplicial space X is a Segal space precisely if there is a weak equivalence of simplicial sets

$$\varphi^{n*} : \mathrm{Map}(F(n), X) \to \mathrm{Map}(G(n), X)$$

for all $n \ge 2$. In other words, taking a left Bousfield localisation with respect to the set of maps $\mathcal{W} = \{\varphi^n : G(n) \to F(n)\}$, we find that the Segal spaces are precisely the $\mathcal{W}$-local objects, i.e. the fibrant objects in the new model structure. This observation leads to the following theorem of Rezk:

Theorem 1.4.12 (Segal Space Model Structure)**.** *There exists a simplicial closed model category structure on the category of simplicial spaces, with the following properties:*

- *the cofibrations are the monomorphisms;*
- *the fibrant objects are the Segal spaces;*
- *the weak equivalences are those maps f such that* $\mathrm{Map}(f, W)$ *is a weak equivalence of simplicial sets for every Segal space W;*
- *a Reedy weak equivalence between any two objects is a weak equivalence in the Segal space model structure; and a weak equivalence of Segal spaces in the Segal space model structure is a weak equivalence with respect to the Reedy model structure.*

Let $I[1]$ denote the groupoid with two objects x, y and exactly one non-trivial isomorphism between the objects, and let E be its nerve (from the discussion above on the various types of nerves, we recall that the discrete nerve of a groupoid is weakly equivalent (as a simplicial space) to its classifying diagram).

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd} \qquad\qquad 0 \xrightarrow{0<1} 1 \tag{1.2}$$

By the pair of diagrams in (1.2), we see that $I[1]$ (the diagram on the left) is closely related to the space $F(1)$ (the diagram on the right), except that we have "allowed an inverse" to the unique map defined by $0 < 1$ which induces the face map $d_0 : X_1 \to X_0$. From this heuristic perspective, the inverse arrow $y \to x$ in E should be viewed as capturing the data of a homotopy inverse of the map $s_0 : X_0 \to X_1$. We note that there is a natural map of simplicial spaces $x : F(0) \to E$ given in degree 0 by picking out the object x. As before, the Yoneda lemma tells us that for any Segal space X, we have $\mathrm{Map}(F(0), X) \cong X_0$.

We note that there is an obvious map $F(1) \to E$ given by sending the edge $0 < 1$ to the arrow $x \to y$. This map is key in proving the following result:

Theorem 1.4.13. *[Rez01, Theorem 6.2] If X is a Segal space, then* $\mathrm{Map}(E, X) \to X_1$ *factors through $X_1^{he} \subseteq X_1$, and induces a weak equivalence* $\mathrm{Map}(E, X) \simeq X_1^{he}$.

Thus, a Segal space X is complete just when there is a weak equivalence of simplicial sets

$$x^* : \mathrm{Map}(E, X) \to \mathrm{Map}(F(0), X)$$

i.e. the complete Segal space model structure is obtained from the Segal space model structure by taking a further Bousfield localisation with respect to the single-element set of maps $\mathcal{W}' = \{x : F(0) \to E\}$; and complete Segal spaces are the fibrant objects in this new model structure. Again, we refer to the original paper of Rezk to state the result more precisely:

Theorem 1.4.14 (Complete Segal Space Model Structure)**.** *There exists a simplicial closed model category structure on the category of simplicial spaces, with the following properties:*

- *the cofibrations are the monomorphisms;*
- *the fibrant objects are the complete Segal spaces;*
- *the weak equivalences are precisely the maps f such that* $\mathrm{Map}(f, W)$ *is a weak equivalence of simplicial sets for every complete Segal space W;*
- *a Reedy weak equivalence between any two objects is a weak equivalence in the complete Segal space model structure; and a weak equivalence of complete Segal spaces in the complete Segal space model structure is a weak equivalence with respect to the Reedy model structure.*

As alluded to when we gave Definition 1.4.8, there is a strong connection between Dwyer-Kan equivalences and equivalences of complete Segal spaces. The following result of Rezk makes this relationship clear:

Theorem 1.4.15. *[Rez01, Theorem 7.7] Let $f : X \to Y$ be a map of Segal spaces. Then, f is a Dwyer-Kan equivalence if and only if it becomes a weak equivalence in the complete Segal space model structure.*

As a corollary of this, we may say that the homotopy category of complete Segal spaces is obtained from the category of Segal spaces by formally inverting all the Dwyer-Kan equivalences.

At this juncture, the reader should not be surprised to learn that only a minor modification is required to adapt these ideas to produce the model structure of fibrewise complete Segal spaces over some fixed Segal space B. In this case, the set of localising maps is $\mathcal{W} = \{G(n) \hookrightarrow F(n) \to B : n \geq 2\} \cup \{F(0) \hookrightarrow E \to B\}$, giving us a model structure in which the cofibrations are commutative triangles

$$\begin{tikzcd} X \arrow[rr, tail] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

such that the horizontal arrow is a monomorphism; the fibrant objects are the fibrewise complete Segal spaces over B; and a morphism $f \in \mathrm{Map}_B(X, Y)$ is a weak equivalence if and only if the induced morphisms $\mathrm{Map}_B(f, W) : \mathrm{Map}_B(Y, W) \to \mathrm{Map}_B(X, W)$ are weak equivalences for all fibrewise complete $W \to B$.

Chapter 2

Strict Operads

In this chapter, we introduce strict operads in a monoidal category. We will describe some explicit examples of the such strict operads, and then give various constructions relating to operads, such as the category of operators of an operad and the Boardman-Vogt tensor product of operads. We will conclude by drawing a connection between the category of trees and operads. The theory we examine here will form a conceptual basis for many of our subsequent investigations, when we proceed to discuss notions of ∞-operads in later chapters, or when we introduce the notion of a right module over an operad

2.1 Coloured Operads

In what follows, we let C denote a symmetric monoidal category with monoidal product $\times$, monoidal unit 1_{C} and initial object $\emptyset$. For our purposes, C will usually be the category of sets or spaces, but operads can also be defined in categories which are not Cartesian monoidal, e.g. the category of graded modules over some fixed ring R with the tensor product $\otimes_R$.

Definition 2.1.1. A coloured operad $\mathcal{O}$ in C is the data of

- A set of objects $c, d, \ldots$ called colours, which we denote by $\mathrm{Col}(\mathcal{O})$.
- For a finite set I and a collection of colours, $\{c_i\}_{i\in I}$, d, there is a C-object of morphisms from $\{c_i\}_{i\in I}$ to d, denoted $\mathcal{O}(\{c_i\}_{i\in I}; d)$.
- For a map of finite sets $I \to J$ with fibres $\{I_j\}_{j\in J}$ and a collection of colours $\{c_i\}_{i\in I}, \{d_j\}_{j\in J}, e$ of $\mathcal{O}$, we have a composition law:

$$\prod_{j\in J} \mathcal{O}(\{c_i\}_{i\in I_j}; d_j) \times \mathcal{O}(\{d_j\}_{j\in J}; e) \xrightarrow{\gamma} \mathcal{O}(\{c_i\}_{i\in I}; e)$$

- This composition law satisfies an associativity requirement: for $I \to J \to K$ and a collection of colours $\{b_i\}_{i\in I}, \{c_j\}_{j\in J}, \{d_k\}_{k\in K}, e$ of $\mathcal{O}$, the following commutes:

$$\begin{tikzcd}
\prod_{j\in J} \mathcal{O}(\{b_i\}_{i\in I_j}; c_j) \times \prod_{k\in K} \mathcal{O}(\{c_j\}_{j\in J_k}; d_k) \times \mathcal{O}(\{d_k\}_{k\in K}; e) \arrow[r, "\gamma\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\gamma"] & \prod_{k\in K} \mathcal{O}(\{b_i\}_{i\in I_k}; d_k) \times \mathcal{O}(\{d_k\}_{k\in K}; e) \arrow[d, "\gamma"] \\
\prod_{j\in J} \mathcal{O}(\{b_i\}_{i\in I_j}; c_j) \times \mathcal{O}(\{c_j\}_{j\in J}; e) \arrow[r, "\gamma"] & \prod_{j\in J} \mathcal{O}(\{b_i\}_{i\in I}; e)
\end{tikzcd}$$

- A collection of units $\{id_c \in \mathcal{O}(\{c\}; c)\}_{c\in \mathrm{Col}(\mathcal{O})}$ which are left and right units for composition on $\mathcal{O}$.
- Equivariance properties with respect to action of the symmetric groups $\mathfrak{S}_n$: given $p \in \mathcal{O}(\{c_i\}_{1\leq i\leq n}; c)$ and $q_i \in \mathcal{O}(\{d^i_j\}_{1\leq j\leq m_i}; c_i)$ for all $1 \leq i \leq n$, and given $\sigma \in \mathfrak{S}_n$ and $\tau_i \in \mathfrak{S}_{m_i}$ for all i, we have

$$\gamma(\sigma^* p; q_{\sigma(1)}, \ldots, q_{\sigma(n)}) = \gamma(p; q_1, \ldots, q_n)$$

We observe that operads come equipped with a notion of **partial composition**: given operations $p \in \mathcal{O}(d_1, \ldots, d_n; e)$ and $q \in \mathcal{O}(c_1, \ldots, c_m; d_i)$, we can define a new operation

$$p \circ_i q = \gamma\left(p; (1_{d_1}, \ldots, 1_{d_{i-1}}, q, 1_{d_{i+1}}, \ldots, 1_{d_n})\right) \in \mathcal{O}(d_1, \ldots, d_{i-1}, c_1, \ldots, c_m, d_{i+1}, \ldots, d_n; e)$$

Remark 2.1.2. In particular, we can think of coloured operads with one just one colour, $*$. In this case, we just write $\mathcal{O}(n)$ to mean the collection of morphisms $\mathcal{O}\left(\{*\}_{1 \leq i \leq n}; *\right)$. We refer to such operads as *plain* operads.

One of the original motivations behind the definition of the operad is that they "parametrise multiplications": certainly, this was the case in the work of Boardman and Vogt [BV68] and May [May72], where the little-cubes operads (or closely related objects in the case of the earlier paper) were introduced to parametrise how closely the multiplicative structure induced by the concatenation of loops in a space comes to defining an associative group operation. In what follows, we will discuss some examples which will recur throughout our studies. In particular, the suggestive names of the "commutative" and "associative" operads are indicative of their main purpose: they capture the data of commutative/associative algebra structures. We will later see that the little 1-cubes operad is weakly equivalent to the associative operad, while the little ∞-cubes operad (defined via a suitable colimiting procedure) is weakly equivalent to the commutative operad.

Example 2.1.3 (Trivial Operad)**.** The trivial operad $\mathcal{J}$ in C is the operad with unique colour such that $\mathcal{J}(n) = 1_{\mathsf{C}}$ if $n = 1$ and $\mathcal{J}(n) = \emptyset$ for all other values of n.

Example 2.1.4 (Commutative Operad)**.** The commutative operad, which we denote by Com is the operad with one colour for which $\mathrm{Com}(n) = \{1_{\mathsf{C}}\}$ for all n and the action of the symmetric group is the trivial one.

Example 2.1.5 (Associative Operad)**.** The associative operad, denoted Ass is the operad with one colour for which $\mathrm{Ass}(n) \cong \mathfrak{S}_n$. The composition $\gamma : \mathrm{Ass}(k) \times \mathrm{Ass}(j_1) \times \ldots \mathrm{Ass}(j_k) \to \mathrm{Ass}(j)$ (where $j = \sum_i j_i$) is determined by letting $\gamma(e_k; e_{j_1}, \ldots, e_{j_k}) = e_j$, where e_i is the identity element of $\mathfrak{S}_i$. The equivariance properties tell us that for each $\sigma \in \mathfrak{S}_k$ and for $\tau_{j_i} \in \mathfrak{S}_{j_i}$, we have

$$\begin{aligned} \gamma(e_k \cdot \sigma; e_{j_1}, \ldots, e_{j_k}) &= \gamma(e_k; e_{j_{\sigma^{-1}(1)}}, \ldots, e_{j_{\sigma^{-1}(k)}}) \cdot \sigma(j_1, \ldots, j_k) \\ \gamma(e_k; e_{j_1} \cdot \tau_{j_1} \ldots e_{j_k} \cdot \tau_{j_k}) &= \gamma(e_k; e_{j_1}, \ldots, e_{j_k}) \cdot \tau_1 \oplus \ldots \oplus \tau_k \end{aligned}$$

There is a close relationship between categories and operads – indeed, in some sources, operads appear under the name *multicategories*. This relationship is partially elucidated in the following series of examples. We will later see that the nerve construction which associates a simplicial set to a category can be generalised in a suitable way to a nerve construction which associates to an operad an object in a new category – the category of dendroidal sets.

Example 2.1.6. Let D be a small category internal in some monoidal category C. We can associate a particularly simple operad $\mathcal{D}$ to D as follows: the colours of $\mathcal{D}$ are the objects of D, while

$$\mathcal{D}\left(\{d_1, \ldots, d_n\}; d\right) = \begin{cases} \mathrm{Hom}_{\mathsf{D}}\left(d_1; d\right) & n = 1 \\ \emptyset & n \neq 1 \end{cases}$$

Remark 2.1.7. This construction admits a kind of converse: given an operad $\mathcal{O}$ in C, we can define its **underlying category**: this is the small category whose objects are the colours of $\mathcal{O}$, and whose hom-objects are precisely given by the 1-ary operations of $\mathcal{O}$, i.e. given colours c, c' of $\mathcal{O}$, the collection of morphisms between them is $\mathcal{O}(\{c\}; c')$. Since $\mathcal{O}$ is an operad in C, there is a well-defined composition rule, which ensures that the underlying category of $\mathcal{O}$ is internal in C.

Returning to the question of producing operads from categories, we can also consider the case where D is itself a symmetric monoidal category internal in C: in this case we obtain a slightly more elaborate construction.

Example 2.1.8 (Symmetric Monoidal Category)**.** Let $(\mathsf{D}, \otimes)$ be a symmetric monoidal category. We can view D as an operad in sets (denoted by $\mathcal{D}$) by letting the colours of $\mathcal{D}$ be the objects of D and given a collection of objects $\{c_i\}_{i \in I}, d$, (where I is a finite set) we define

$$\mathcal{D}\left(\{c_i\}_{i \in I}; d\right) \cong \mathrm{Hom}_{\mathsf{D}}\left(\bigotimes_{i \in I} c_i, d\right)$$

Remark 2.1.9. To bookend our discussion on the relationship between operads and categories, it can in fact be shown that there is a canonical isomorphism of categories

$$\mathrm{Op}_{/\mathcal{J}} \simeq \mathrm{Cat}$$

There is a natural Kan extension $j_! : \mathrm{Cat} \to \mathrm{Op}$, whose image is given by the construction described in Example 2.1.6, while the restriction $j^* : \mathrm{Op} \to \mathrm{Cat}$ is defined by the underlying category construction of Remark 2.1.7. When we discuss dendroidal sets, this functor will make a reappearance.

Example 2.1.10 (Endomorphism Operad)**.** In a similar vein to the previous example, we let $(\mathsf{D}, \otimes)$ be a symmetric monoidal category and fix an object $d \in \mathsf{D}$. The endomorphism operad $\mathcal{E}\mathrm{nd}_d$ is the single-colour operad with $\mathcal{E}\mathrm{nd}_d(n) = \mathrm{Hom}_{\mathsf{D}}(d^{\otimes n}, d)$. The composition law is determined by the composition of morphisms in D, i.e. given $f : d^{\otimes k} \to d$ and $\{g_j : d^{\otimes i_j} \to d\}_{j=1,\dots,k}$, there is an obvious composition $f \circ (g_1 \otimes \dots g_k) : d^{\otimes i_1 + \dots + i_k} \to d$. Since the category D is symmetric monoidal, this composition satisfies associativity properties and is equivariant with respect to the action of the symmetric group. The unit for the composition is determined by the unit map for d, $\eta_d : 1_{\mathsf{D}} \to d$, which is again part of the monoidal structure of the category D.

Among the most important examples of operads which we will encounter in this thesis is the little d-cubes operad:

Example 2.1.11 (Little Cubes Operad)**.** A **little d-cube** is an embedding $(-1,1)^d \to (-1,1)^d$ of the form

$$(x_1, \dots, x_d) \mapsto (a_1 x_1 + b_1, \dots, a_d x_d + b_d)$$

for $a_i > 0$ and $b_i \in \mathbb{R}$. Let $\mathrm{Rect}(\sqcup_n (-1,1)^d, (-1,1)^d) \subseteq \mathrm{Emb}(\sqcup_n (-1,1)^d, (-1,1)^d)$ be the subset of those embeddings such that the restriction to each $(-1,1)^d$ in the domain is a little d-cube. We then define the topological little d-cubes operad ${}^t\mathbb{E}_d$ as the operad which has one colour and for each $n \geq 1$,

$${}^t\mathbb{E}_d(n) = \mathrm{Rect}\left(\sqcup_n (-1,1)^d, (-1,1)^d\right)$$

We give each of these sets the relative topology as subsets of the continuous maps of spaces from $\sqcup_n (-1,1)^d$ to $(-1,1)^d$ (with the compact-open topology). Thus ${}^t\mathbb{E}_d$ is an operad in spaces.

The multiplication maps in ${}^t\mathbb{E}_d$ are defined by composition of embeddings, i.e. writing $f + g$ for the action of two little d-cubes on disjoint copies of $(-1,1)^d$, we set $\gamma(f_n; f_{j_1}, \dots, f_{j_n}) = f_n \circ (f_{j_1} + \dots + f_{j_n})$.

We finally note that for each d, there is a natural inclusion of spaces ${}^t\mathbb{E}_d(n) \hookrightarrow {}^t\mathbb{E}_{d+1}(n)$ defined by sending each little d-cube f to $f \times \mathrm{id} : (-1,1)^d \times (-1,1) \to (-1,1)^d \times (-1,1)$. Taking the colimit of the chain

$${}^t\mathbb{E}_1(n) \hookrightarrow {}^t\mathbb{E}_2(n) \hookrightarrow ... \hookrightarrow {}^t\mathbb{E}_d(n) \hookrightarrow \dots$$

gives a new space ${}^t\mathbb{E}_\infty(n)$, and the collection of these spaces inherits the structure of an operad, which we denote by ${}^t\mathbb{E}_\infty$.

Remark 2.1.12. We note that there is a weak homotopy equivalence between the little 1-cubes operad and the associative operad: specifically, if we consider an element $f = \langle f_1, \dots, f_n \rangle$ of ${}^t\mathbb{E}_1(n)$, this is a collection of n rectilinear embeddings $f_i : (-1,1) \to (-1,1)$ into itself, with disjoint images. By evaluating each f_i at the origin we obtain a collection of n disjoint points in $(-1,1)$. Since these points are naturally linearly ordered, it follows that this evaluation determines a map ${}^t\mathbb{E}_1(n) \to \mathfrak{S}_n$. A result of [May72] shows that these maps in fact determine weak equivalences ${}^t\mathbb{E}_1(n) \xrightarrow{\sim} \mathrm{Ass}(n)$ for all n.

The case of 1-cubes is special insofar as there is a natural linear ordering on the image when we evaluate elements of ${}^t\mathbb{E}_1$ at the origin. Evidently, if $\langle f_1, \dots, f_n \rangle \in {}^t\mathbb{E}_d(n)$ for $d > 1$, there is no obvious linear ordering on the set $\{f_1(0), \dots, f_n(0)\}$. However, this evaluation does determine a configuration of n points in $\mathbb{R}^d$, and in fact, it can be shown that by choosing a framed diffeomorphism $(-1,1)^d \cong \mathbb{R}^d$, we obtain a weak equivalence ${}^t\mathbb{E}_d(n) \xrightarrow{\sim} \mathrm{Conf}_n(\mathbb{R}^d)$, where $\mathrm{Conf}_n(\mathbb{R}^d)$ denotes the set of ordered configurations of n points in $\mathbb{R}^d$.

The case of the ${}^t\mathbb{E}_\infty$ operad also warrants some special comment – the paper [May72] also shows that there are weak equivalences ${}^t\mathbb{E}_\infty(n) \xrightarrow{\sim} \mathrm{Com}(n)$ for all n.

Example 2.1.13 (Skew Little Cubes Operad)**.** As a variation on the above, we can also consider the skew little-cubes introduced in [DHK19]. First, we let $\Lambda(d) \subseteq GL_d(\mathbb{R})$ be the subgroup of diagonal matrices with positive entries. A representation $\rho : G \to GL_d(\mathbb{R})$ is said to be a **dilation representation** if $\mathrm{im}(\rho) = (\mathrm{im}(\rho) \cap O(d)) \cdot \Lambda(d)$ – which ensures that the image of ρ contains the entirety of $\Lambda(d)$ and given any $g \in G$, the QR decomposition of $\rho(g)$ is of the form QR, where $R \in \Lambda(d)$.

Given a dilation representation $\rho : G \to GL_d(\mathbb{R})$, a **$G$-skew little cube** is an embedding $(-1,1)^d \to (-1,1)^d$ of the form

$$(x_1, \ldots, x_d) \mapsto (v_1 + \rho(g)x_1, \ldots, v_d + \rho(g)x_d)$$

for $-1 < v_i < 1$ for all i and $g \in G$. We write ${}^t\mathbb{E}_d^G$ for the topological operad of G-skew little cubes. In the

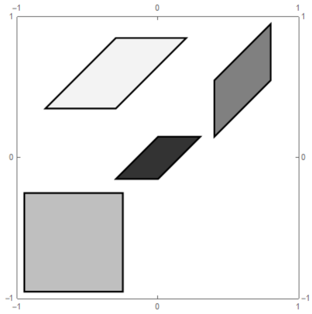

Figure 1: Example of four skew little G-cubes with disjoint images, where $G = \mathfrak{S}_3 \times \Lambda(2)$

graphic 1, we have depicted an element of ${}^t\mathbb{E}_2^G(4)$, where the group G is $\mathfrak{S}_3 \times \Lambda(2)$ (given the product of the standard representation and the identity representation).

Remark 2.1.14. The condition that ρ be a dilation representation is perhaps unduly strict, and appears to stem from a desire to ensure a weak equivalence between the *G-framed configuration spaces* of k points in $(-1,1)^d$ (which we will discuss in detail at a later stage) and the spaces ${}^t\mathbb{E}_d^G(k)$. If we relax this condition, we can still obtain operads, but with the knowledge that they are no longer so conveniently related with G-framed configuration spaces.

As alluded to previously, one of the reasons why we discuss operads is that they can be used to parametrise certain kinds of multiplication. To make this idea more precise, we need to define the notion of an algebra over an operad:

Definition 2.1.15. Let $\mathcal{O}$ be an operad in C. An **algebra over $\mathcal{O}$** is a set of C-objects $\{\mathcal{A}_d\}$, indexed by colours of $\mathcal{O}$, plus maps

$$\begin{aligned} \mathcal{O}(d_1, \ldots, d_n; d) \times \mathcal{A}_{d_1} \times \ldots \times \mathcal{A}_{d_n} &\to \mathcal{A}_d \\ (o, (a_1, \ldots, a_n)) &\longmapsto o \cdot (a_1, \ldots, a_n) \end{aligned}$$

for all tuples of colours $d_1, \ldots, d_n, d$. These are required to satisfy the following unitality, equivariance and associativity properties:

- For all colours d of $\mathcal{O}$ and all $a \in \mathcal{A}_d$, we have $1_d \cdot (a) = a$.
- For all $\sigma \in \mathfrak{S}_n$, all $p \in \mathcal{O}(d_1, \ldots, d_n)$ and $a_i \in \mathcal{A}_{d_i}$, we have
$$(\sigma^* p) \cdot (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = p \cdot (a_1, \ldots, a_n)$$
- For $p \in \mathcal{O}(e_1, \ldots, e_n; f)$, $q_i \in \mathcal{O}(d_1^i, \ldots, d_{m_i}^i; e_i)$ and $a_j^i \in \mathcal{A}_{d_j^i}$, we have
$$\gamma(p; q_1, \ldots, q_n) \cdot \left(a_1^1, \ldots, a_{m_1}^1, \ldots, a_1^n \ldots, a_{m_n}^n\right) = p \cdot \left(q_1 \cdot (a_1^1, \ldots, a_{m_1}^1), \ldots, q_n \cdot (a_1^n \ldots, a_{m_n}^n)\right)$$

Equivalently, using Example 2.1.8 to view C as an operad (which we write as $\mathcal{C}$), we can use the tensor-hom adjunction in C to see that this notion is equivalent to a morphism of operads $\mathcal{O} \to \mathcal{C}$. There is also a notion of an $\mathcal{O}$-algebra morphism: given $\mathcal{O}$-algebras $\{\mathcal{A}_d\}_d$ and $\{\mathcal{B}_d\}_d$, a morphism between them is a collection of maps $f_d : \mathcal{A}_d \to \mathcal{B}_d$ such that

$$p \cdot (f_{d_1} a_1, \ldots, f_{d_n} a_n) = f_d\left(p \cdot (a_1, \ldots, a_n)\right)$$

for all $p \in \mathcal{O}(d_1, \ldots, d_n; d)$ and all $a_i \in \mathcal{A}_{d_i}$. In this way, we obtain a category of $\mathcal{O}$-algebras, which we denote by $\mathsf{Alg}_{\mathcal{O}}$.

Example 2.1.16. Let C be the category of sets. An algebra over the associative operad is the data of a set $\mathcal{A}$, equipped with a collection of maps

$$\varphi_n : \mathfrak{S}_n \times \mathcal{A}^n \to \mathcal{A}$$

or by adjunction, a collection of maps $\widetilde{\varphi}_n : \mathfrak{S}_n \to \mathrm{Hom}_{\mathrm{Set}}(\mathcal{A}^n, \mathcal{A})$. In particular, if e_2 is the identity element of $\mathfrak{S}_2$, then $m = \widetilde{\varphi}_2(e_2) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ specifies a multiplication map. The associativity of the operadic composition maps ensures that m defines an associative multiplication law, and the unitality of the operadic composition ensures that m is unital. Thus, we see that an algebra over the associative operad in Set coincides with our usual notion of an associative algebra.

We can give a similar recasting of a commutative algebra as an algebra over the commutative operad. However, for our purposes, the principal motivation for studying algebras over operads is as a motivation for discussing the Boardman-Vogt tensor product, so we will not spend too much time on this notion (admittedly, since the tensor product was originally introduced to study algebras over operads, this may seem like a case of the tail wagging the dog).

2.2 Plain Operads as Symmetric Sequences

In the study of plain operads, an alternative description of operads exists, which will prove particularly useful in some of our applications. In this section, we will examine this characterisation in some depth and recast the ideas of Section 2.1 in this new light.

As in Section 2.1, we let $(\mathsf{C}, \times)$ be a symmetric monoidal category with coproduct $\coprod$, initial object $\emptyset$ and unit object 1_{C}. In what follows, we will let N be the discrete category of finite sets, and Σ be the category whose objects are finite sets and whose morphisms are the bijections between finite sets.

A **sequence** in C is a functor $\mathcal{X} : \mathsf{N} \to \mathsf{C}$; a **symmetric sequence** in C is a functor $\mathcal{X} : \Sigma^{op} \to \mathsf{C}$. Unpicking this definition, we see that a sequence $\mathcal{X}$ is a collection $\{\mathcal{X}(n)\}_{n \geq 0}$, where each $\mathcal{X}(n) =: \mathcal{X}(\langle n \rangle)$ is an object of C. Similarly, a symmetric sequence $\mathcal{Y}$ is a collection $\{\mathcal{Y}(n)\}_{n \geq 0}$ of objects of C such that each $\mathcal{Y}(n)$ is equipped with a right action by the group $\mathfrak{S}_n$. By the **arity-n component** of a (symmetric) sequence $\mathcal{X}$, we mean the element $\mathcal{X}(n) \in \mathsf{C}$. A morphism of (symmetric) sequences $\mathcal{X} \to \mathcal{Y}$ is a natural transformation of functors $\mathsf{N} \to \mathsf{C}$ (resp, a natural transformation of functors $\Sigma \to \mathsf{C}$, which is compatible with the right actions in each arity). We write $\mathsf{Seq}(\mathsf{C})$ for the category of sequences in C and $\mathsf{SymSeq}(\mathsf{C})$ for the category of symmetric sequences.

The category of symmetric sequences may be equipped with a monoidal structure, defined as follows: given two symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$, we define their **composition product** $\mathcal{X} \circ \mathcal{Y}$ by

$$\mathcal{X} \circ \mathcal{Y}(n) = \coprod_{k \geq 0} \coprod_{i_1 + \ldots + i_k = n} \mathcal{X}(k) \times_{\mathfrak{S}_k} \left(\prod_{j=1}^{k} \mathcal{Y}(i_j) \times_{\prod_{j=1}^{k} \mathfrak{S}_{i_j}} \mathfrak{S}_n \right)$$

Evidently, the operation $\circ$ is not symmetric.

Let $\mathcal{J}$ be the (symmetric) sequence with $\mathcal{J}(1) = 1_{\mathsf{C}}$ and $\mathcal{J}(k) = \emptyset$ for all other k. It is immediate from the definition that $\mathcal{X} \circ \mathcal{J}(n) = \mathcal{X}(n) = \mathcal{J} \circ \mathcal{X}(n)$. Hence, we see that $\mathcal{J}$ is a unit for the composition product. It can also be seen (after some computations) that the composition product is an associative operation, which ensures that $(\mathsf{SymSeq}(\mathsf{C}), \circ, \mathcal{J})$ is a monoidal category. This perspective leads us to the following definition of (plain) operads in C:

Definition 2.2.1. An **operad** $\mathcal{O}$ is a monoid in the monoidal category $(\mathsf{SymSeq}(\mathsf{C}), \circ, \mathcal{J})$, i.e. there are maps $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ and $\eta : \mathcal{J} \to \mathcal{O}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} \arrow[r, "\mathrm{Id}_{\mathcal{O}} \circ \gamma"] \arrow[d, "\gamma \circ \mathrm{Id}_{\mathcal{O}}"'] & \mathcal{O} \circ \mathcal{O} \arrow[d, "\gamma"] \\ \mathcal{O} \circ \mathcal{O} \arrow[r, "\gamma"] & \mathcal{O} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J} \circ \mathcal{O} \arrow[r, "\eta \circ \mathrm{Id}_{\mathcal{O}}"] \arrow[dr, "="'] & \mathcal{O} \circ \mathcal{O} \arrow[d, "\gamma"] & \mathcal{O} \circ \mathcal{J} \arrow[l, "\mathrm{Id}_{\mathcal{O}} \circ \eta"'] \arrow[dl, "="] \\ & \mathcal{O} & \end{tikzcd}$$

To see that this definition coincides with the "classical" notion of operad (recalling the specialisation of Definition 2.1.1 to the case of one colour), we note that, since γ is a morphism of symmetric sequences, for each $n \geq 0$, γ encodes the data of a $\mathfrak{S}_n$-equivariant map $\gamma(n) : \mathcal{O} \circ \mathcal{O}(n) \to \mathcal{O}(n)$. Spelling this out explicitly using the definition of the composition product, we see that for each partition of n into k pieces, $n_1 + \ldots n_k = n$, we get a map

$$\gamma_{(k; n_1, \ldots, n_k)} : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \ldots \times \mathcal{O}(n_k) \to \mathcal{O}(n)$$

The associativity and unitality constraints of Definition 2.1.1 are encoded in the pair of diagrams above.

It is perhaps enlightening to examine a handful of our earlier examples of operads in this light to reassure ourselves of the fact that these concepts are in fact the same.

- For a symmetric monoidal category D (taking C to be the category of sets) and a fixed object $d \in \mathsf{D}$, the **endomorphism operad** $\mathcal{E}\mathrm{nd}_d$ is the symmetric sequence whose arity-n component is $\mathrm{Hom}_{\mathsf{D}}(d^{\otimes n}, d)$. Given $f : d^{\otimes k} \to d$ and $\{g_j : d^{\otimes n_j} \to d\}_{j=1,\dots,k}$ with $n_1+\dots+n_k = n$, we have a composition $f \circ (g_1, \dots, g_k) : d^{\otimes n} \to d$. Assembling all such compositions together gives the data of a map $\gamma(n) : \mathcal{E}\mathrm{nd}_d \circ \mathcal{E}\mathrm{nd}_d(n) \to \mathcal{E}\mathrm{nd}_d(n)$ which is associative, unital and equivariant with respect to the action of $\mathfrak{S}_n$, and we can produce such maps for each n, giving a morphism of symmetric sequences $\gamma : \mathcal{E}\mathrm{nd}_d \circ \mathcal{E}\mathrm{nd}_d \to \mathcal{E}\mathrm{nd}_d$.
- The **commutative operad** is the symmetric sequence Com which in arity-n is given by $\mathrm{Com}(n) = \{1_{\mathsf{C}}\}$. For any $n_1 + \dots n_k = n$, we have a canonical isomorphism $\mathrm{Com}(k) \times \mathrm{Com}(n_1) \times \dots \times \mathrm{Com}(n_k) \xrightarrow{\cong} \mathrm{Com}(n)$, and the collection of all such isomorphisms in arity n gives an associative unital map $\gamma(n) : \mathrm{Com} \circ \mathrm{Com}(n) \to \mathrm{Com}(n)$.
- The **associative operad** Ass is the symmetric sequence with $\mathrm{Ass}(n) = \mathfrak{S}_n$. The multiplication $\gamma : \mathrm{Ass} \circ \mathrm{Ass} \to \mathrm{Ass}$ is defined by assembling the maps $\mathrm{Ass}(k) \times \mathrm{Ass}(n_1) \times \dots \times \mathrm{Ass}(n_k) \to \mathrm{Ass}(n)$ specified in Example 2.1.5 for all partitions $n_1 + \dots + n_k = n$.

2.3 The Category of Operators of an Operad

We have seen already that we can associate a very simple category to an operad, namely the underlying category of Remark 2.1.7 – however, there is also a much more elaborate construction called the *category of operators* associated to an operad. It is clear that in moving from an operad to its underlying category, we lose lots of information – all we retain is information about the 1-ary operations of the operad. The category of operators, while a more complicated object, has the advantage that we can completely reconstruct the operad from it. In a sense, we can view the category of operators as reversing the ideas of Example 2.1.8.

This construction and certain variations on it will have applications for us when we come to discuss right modules over operads. Moreover, we will also see the construction appear when we come to discuss operads in the ∞-categorical setting, where the defining properties of this category of operators will be utilised to give a suitable generalisation of operads to the quasi-categorical world.

Before introducing this notion, we first need to define an useful auxiliary category, which will appear at several stages of our exposition.

Definition 2.3.1. Let Fin_* be the category whose objects are finite sets with an added distinguished point, e.g. we will write the set $\langle n \rangle$ to mean the set $\{1, \dots, n\} \cup \{*\}$, where $*$ is our distinguished point. (We will refer to such an object as a set with a point.) Given two objects I, J in Fin_*, a morphism between them is a map of finite sets which sends the distinguished point $*_I$ of I to the distinguished point $*_J$ of J.

Given such a map $f : I \to J$, let f_o denote the map of finite sets given by the restriction of f to $I \setminus f^{-1}\{*_J\}$. We say f is **inert** if f_o is a bijection of finite sets, and we say f is **active** if $f^{-1}\{*_J\} = \{*_I\}$. If Fin is the subcategory of Fin_* with the same objects, but whose morphisms are the active maps, then we can and do identify Fin with the usual category of sets (in this case, we will ignore the point $*$, and view $\langle k \rangle$ as the set $\{1, \dots, k\}$). We remark that the collection of inert and active maps forms a factorization system on Fin_*.

Remark 2.3.2. The category Fin_* can be described in a different way: from this perspective, the objects of Fin_* are just sets in the usual sense; however, the morphisms are the so-called **partial maps**: a partial map $f : I \to J$ is the data of a map of sets $f_o : I_o \to J$ for some subset $I_o \subseteq I$. In this context, the partial map f is inert if f_o is injective, and f is active if $I = I_o$.

An example of an inert map which appears frequently in the ∞-categorical version of operads is the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$, defined by $\rho^i(i) = 1$ and $\rho^i(j) = *$ for all $j \neq i$.

Definition 2.3.3. To any coloured operad $\mathcal{O}$ in a monoidal category C (with coproduct $\coprod$ and product $\prod$), we can associate the so-called **category of operators** $\mathcal{O}^{\otimes}$. Objects of this category are sequences of colours $\{c_i\}_{i=1}^n$. Given two such sequences $\{c_i\}_{i=1}^n$, $\{d_j\}_{j=1}^m$, the hom-object of morphisms between them is

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i=1}^n, \{d_j\}_{j=1}^m) = \coprod_{f:\langle n \rangle \to \langle m \rangle} \prod_{j=1}^m \mathcal{O}(\{c_i\}_{i \in f^{-1}\{j\}}; d_j)$$

where the coproduct is taken over all maps $f \in \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle m\rangle)$. In other words, a morphism of two such sequences $\{c_i\}_{i=1}^n$, $\{d_j\}_{j=1}^m$ is the data of

- a map $f : \langle n\rangle \to \langle m\rangle$ in Fin_*;
- a tuple of operations $(\phi_j)_{j=1}^m$ where $\phi_j \in \mathcal{O}(\{c_i\}_{i\in f^{-1}\{j\}}; d_j)$

The composition properties of this category are determined by the composition properties of the operad $\mathcal{O}$ and the category Fin_*.

In the case where $\mathcal{O}$ is a single-coloured operad, the objects of $\mathcal{O}^{\otimes}$ can be identified with objects $\langle n\rangle$ of Fin_*, and a morphism from $\langle n\rangle$ to $\langle m\rangle$ in the category $\mathcal{O}^{\otimes}$ is the data of a morphism $f \in \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle m\rangle)$ and a tuple of operations $(o_1, \dots, o_m) \in \prod_{j=1}^m \in \mathcal{O}(f^{-1}(j))$.

Remark 2.3.4 (Variants of the category of operators)**.** In addition, this construction admits some variants of interest obtained by considering subcategories of Fin_* which have the same objects, but fewer morphisms: let B be such a subcategory, then we can define a category $\mathsf{B}(\mathcal{O})$ whose objects are the same as those of $\mathcal{O}^{\otimes}$, namely sequences of colours of $\mathcal{O}$, $\{c_i\}_{i\in I}$, while the hom-object of morphisms in $\mathsf{B}(\mathcal{O})$ from $\{c_i\}_i \in I$ to $\{d_j\}_{j\in J}$ is

$$\coprod_{f:\langle n\rangle\to\langle m\rangle} \prod_{j=1}^{m} \mathcal{O}(\{c_i\}_{i\in f^{-1}\{j\}}; d_j)$$

where the coproduct is now taken over maps $f \in \mathrm{Hom}_{\mathsf{B}}(\langle n\rangle, \langle m\rangle)$. It is evident that this construction is functorial in B. Some examples of subcategories of interest for us will be:

- Fin – in this case, we will write $\mathsf{F}(\mathcal{O})$ to denote $\mathsf{Fin}(\mathcal{O})$: the distinction between $\mathsf{F}(\mathcal{O})$ and $\mathcal{O}^{\otimes}$ is that the tuples of operations $(\phi_j)_{j\in J}$ from $\{c_i\}_{i\in I}$ to $\{d_j\}_{j\in J}$ which appear in Fin_* can include operations which "forget colours", e.g. a morphism $\{c_i\}_{i\in\langle 2\rangle} \to \{d\}$ in $\mathcal{O}^{\otimes}$ lying over the inert map $\rho^1 : \langle 2\rangle \to \langle 1\rangle$ is determined by an operation in $\mathcal{O}(\{c_1\}; d)$, while any morphism $\{c_i\}_{i\in\langle 2\rangle} \to \{d\}$ in $\mathsf{F}(\mathcal{O})$ will be determined by an operation in $\mathcal{O}\left(\{c_i\}_{i\in\langle 2\rangle}; d\right)$. We refer to $\mathsf{F}(\mathcal{O})$ as the **active category of operators** of $\mathcal{O}$.
- let N be the discrete subcategory of Fin – thus a morphism in $\mathsf{N}(\mathcal{O})$ from $\{c_i\}_{i\in I}$ to $\{d_j\}_{j\in J}$ exists if and only if $I = J$; and in such a case, a morphism is a tuple of operations $(\phi_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}(c_i; d_i)$.
- let Σ be the subcategory of Fin with the same objects and whose only morphisms are the bijections of sets – as in the case of $\mathsf{N}(\mathcal{O})$, we can only have morphisms between sequences of the same length, but a morphism from $\{c_i\}_{i\in I}$ to $\{d_j\}_{j\in I}$ is now determined by a permutation $\sigma \in \mathfrak{S}_I$ and a collection of unary operations $(\phi_j)_{j\in I} \in \prod_{j\in I} \mathcal{O}(c_{\sigma^{-1}(j)}; d_j)$.

2.4 The Boardman-Vogt Tensor Product of Operads

In Definition 2.1.15, we met with the notion of an algebra over an operad. One question we might wish to study is what happens when an object in C has the structure of an algebra over two operads? Is there a way in which these separate algebraic structures can be made compatible? The Boardman-Vogt tensor product can be used as a tool to study such questions. In fact, the Boardman-Vogt tensor product of two operads $\mathcal{O}$ and $\mathcal{P}$ is the operad $\mathcal{O} \star \mathcal{P}$ defined by the property that there are equivalences of categories:

$$\mathsf{Alg}_{\mathcal{O}}(\mathsf{Alg}_{\mathcal{P}}) \simeq \mathsf{Alg}_{\mathcal{O}\star\mathcal{P}} \simeq \mathsf{Alg}_{\mathcal{P}}(\mathsf{Alg}_{\mathcal{O}})$$

This operad admits a more explicit description as follows:

Definition 2.4.1. The Boardman-Vogt tensor product of $\mathcal{O}$ and $\mathcal{P}$ is the operad $\mathcal{O} \star \mathcal{P}$, whose collection of colours consists of all $c \star d$, where c is a colour of $\mathcal{O}$ and d is a colour of $\mathcal{P}$. The operations of $\mathcal{O} \star \mathcal{P}$ are generated by operations of the form

- $p \star d \in \mathcal{O} \star \mathcal{P}(c_1 \star d, \dots, c_n \star d; c \star d)$ where $p \in \mathcal{O}(c_1, \dots, c_n; c)$ and d is a colour of $\mathcal{P}$;

  or
- $c \star q \in \mathcal{O} \star \mathcal{P}(c \star d_1, \dots, c \star d_m; c \star d)$ where $q \in \mathcal{P}(d_1, \dots, d_m; d)$ and c is a colour of $\mathcal{O}$.

These operations must be compatible in the sense of satisfying the following relations (with $p \in \mathcal{O}(c_1, \ldots, c_n; c)$ and $q \in \mathcal{P}(d_1, \ldots, d_m; d)$):

- $(p \star d) \circ (p_1 \star d, \ldots, p_n \star d) = (p \circ (p_1, \ldots, p_n)) \star d$.
- $(c \star q) \circ (c \star q_1, \ldots, c \star q_m) = c \star (q \circ (q_1, \ldots, q_m))$.
- $\sigma^*(p \star d) = (\sigma^* p) \star q$, where $\sigma \in \mathfrak{S}_n$.
- $\delta^*(c \star q) = c \star (\delta^* q)$, where $\delta \in \mathfrak{S}_m$.
- $(p \star d) \circ (c_1 \star q, \ldots, c_n \star q) = \sigma^*_{m,n} ((c \star q) \circ (p \star d_1, \ldots, p \star d_m))$

where $\sigma_{m,n} \in \mathfrak{S}_{mn}$ is defined by $\sigma_{m,n} ((i-1)n + j) = (j-1)m + i$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We will occasionally refer to this last relation as the *Boardman-Vogt interchange relation.*

In the case where both $\mathcal{O}$ and $\mathcal{P}$ are plain operads, this somewhat unwieldy description simplifies, and the only one of the above relations that we need to consider is the final relation.

Example 2.4.2. The Eckmann-Hilton argument tells us that if we have two associative multiplications m, m' on a set $\mathcal{A}$ which are compatible in the sense that they have the same unit and $m(m'(a, b), m'(c, d)) = m'(m(a, b), m(c, d))$, then $m = m'$ and m defines a commutative product map. In other words, an object in Set which has a pair of compatible associative algebra structures is the same as a commutative algebra object, i.e. $\mathrm{Ass} \star \mathrm{Ass} = \mathrm{Com}$.

Example 2.4.3. The Additivity Theorem for the little d-cubes operads ([Dun88], [Bri00]) asserts that there are levelwise weak equivalences of operads

$$^t\mathbb{E}_d \star ^t\mathbb{E}_{d'} \simeq {^t\mathbb{E}_{d+d'}}$$

We recall from Remark 2.1.12 that there are weak equivalences $^t\mathbb{E}_1(n) \xrightarrow{\sim} \mathrm{Ass}(n)$ and $^t\mathbb{E}_\infty(n) \xrightarrow{\sim} \mathrm{Com}(n)$. We note however that the Additivity Theorem tells us $^t\mathbb{E}_1 \star ^t\mathbb{E}_1(n)$ is weakly equivalent to $^t\mathbb{E}_2(n)$, which is evidently not weakly equivalent to $^t\mathbb{E}_\infty(n) \simeq \mathrm{Com}(n) \simeq \mathrm{Ass}(n) \star \mathrm{Ass}(n)$.

This last example tells us that the Boardman-Vogt tensor product of operads is poorly-behaved from a homotopy-theoretic point of view. Indeed, one of the advantages of working with ∞-operads and the associated ∞-categorical version of the tensor product is a more satisfactory behaviour with respect to such questions.

Remark 2.4.4. It is an open conjecture from [DHK19] that there is a kind of additivity theorem for the skew little-cubes operads of Example 2.1.13. Specifically, given dilation representations $\rho : G \to GL_d(\mathbb{R})$ and $\rho' : G' \to GL_{d'}(\mathbb{R})$, the authors propose that there are levelwise weak equivalences of of operads

$$^t\mathbb{E}^G_d \star ^t\mathbb{E}^{G'}_{d'} \simeq {^t\mathbb{E}^{G \times G'}_{d+d'}}$$

where the group $G \times G'$ is given the product representation $G \times G' \xrightarrow{\rho \times \rho'} GL_d(\mathbb{R}) \times GL_{d'}(\mathbb{R}) \to GL_{d+d'}(\mathbb{R})$ (which is also a dilation representation). A suitable ∞-categorical version of this conjecture has been devised and proven in [Lur17] – we will discuss this in the next chapter.

2.5 Operads and Trees

There is a close relationship between operads and trees which will play a substantial role in much of the theory we go on to develop. To discuss this in detail, we first need to provide some background on the category of trees, Ω. Much of our exposition on this material stems from [HM18].

A graph with half-edges is a pair (V, E), comprising a set of vertices, V, and a set of edges, E, which are one- or two-element subsets of V. An edge e corresponding to a one-element subset is an **external edge**, while an edge corresponding to a two-element subset is called an **internal edge**. For us, a **tree** will be a graph with half-edges, $T = (V(T), E(T))$, in which there is a distinguished external edge and in which there is an unique path between any two edges (here and in what follows, we will only be considering finite trees). The distinguished external edge will be referred to as the **root** of the tree; all the other external edges of the tree are known as **leaves**. At this point, a visual example will help to make sense of some of this terminology: consider

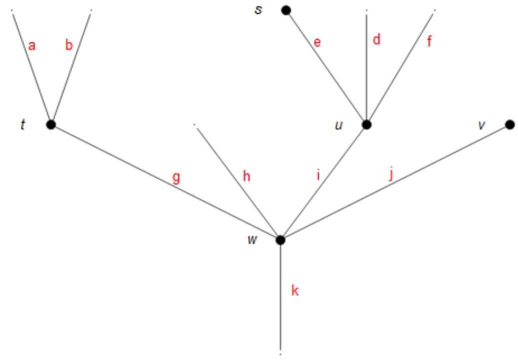

Figure 2: A typical example of a rooted tree – vertex labels are in black, edge labels are in red

the rooted tree depicted in Figure 2. The set of edges in this tree is $E = \{a, b, d, e, f, g, h, i, j, k\}$, with k as the root element. The set of vertices V is $\{s, t, u, v, w\}$. The set of leaves on the tree is $\{a, b, d, f, h\}$.

The existence of a root element $\mathfrak{R}$ in T imparts a natural partial order structure to the set of edges: we can view $E(T)$ as a poset with an unique maximal element $\mathfrak{R}$ and for any other edges e, f, we say that $e \leq f$ if the unique path from e to the root passes through f (thus with respect to this partial order, the leaves of the tree become the minimal elements). We say the edges e and f are **incomparable** or **independent** if e and f are not related by this partial order, and in that case, we write $e \perp f$. In turn, this partial order structure on the edges allows us to talk about the incoming and outgoing edges of a vertex, e.g. given a vertex v which connects the edges $e_1, \dots, e_n$ and e', we say that e' is an outgoing edge from v if $e_i \leq e'$ for all i, and we say that the edges $e_1, \dots, e_n$ are the incoming edges; we write $out(v)$ (resp. $in(v)$) to denote the sets of outgoing (incoming) edges. By definition, any vertex will have a single outgoing edge. A vertex whose set of incoming edges is empty is said to be a **nullary** vertex (in Figure 2, the only nullary vertices are s and v). A tree which has no nullary vertices is said to be **open**, while a tree which has no leaves is said to be **closed**.

Note that when giving such a visualisation as in Figure 2 we are compelled to put some kind of linear order on the set of incoming edges of each vertex; however, this linear ordering is not part of the data of the tree.

It can be shown that the foregoing poset description of trees is almost enough to completely characterise a tree:

Lemma 2.5.1. *[HM18, Lemma 3.2] Let E be a poset with a unique maximal element $\mathfrak{R}$ and assume that E satisfies the following property:*

- *For every $e \in E$, the poset $E_{e\leq} = \{f \in E : e \leq f\}$ is linearly ordered.*

Let $L \subseteq E$ be the subset of all minimal elements of E. Then there exists a tree T with $E(T) = E$, whose set of leaves is L.

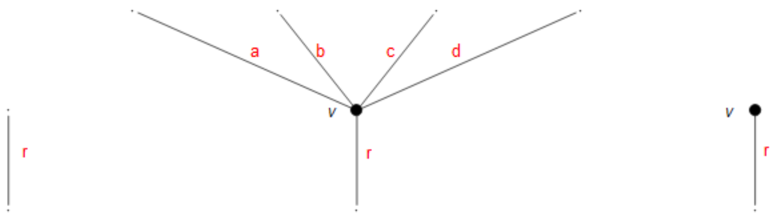

Figure 3: A graphic of the root η, the 4-corolla, C_4, and the 0-corolla, C_0.

Example 2.5.2. The root tree η is the tree with a single edge and no vertices, which can be visualised as the tree on the left in Figure 3. In terms of a poset description, we can write η as the poset $\{r\}$.

For $n \geq 0$, the n-corolla C_n is the tree with 1 vertex and n incoming edges. We have a visual example of the 4-corolla and the 0-corolla in Figure 3. We observe in particular that C_0 is obtained from the root tree simply by adjoining a nullary vertex.

If we denote the leaves of C_n by $\ell_1, \dots, \ell_n$, then as a poset, we can describe C_n as $\{r, \ell_1, \dots, \ell_n\}$ with $\ell_i \leq r$ for all i and $\ell_i \perp \ell_j$ for all $i \neq j$.

Given a tree T with leaves $t_1, \dots, t_n$ and trees $T_1, T_2, \dots, T_n$, we can define a new tree $T \star_{t_1,\dots,t_n} (T_1, \dots, T_n)$ by gluing the root of T_i to the leaf t_i. This procedure is known as **grafting**. Inductively, we see that any tree can be built by grafting together corollas and roots. This is convenient since we will frequently rely on making inductive arguments by proving results for n-corollas, and then deducing for arbitrary trees by grafting.

One of the first places where we see this principle in action is when discussing the automorphism group of a tree, $\mathrm{Aut}(T)$: in the case where $T = C_n$, then $\mathrm{Aut}(T) = \mathfrak{S}_n$. If $T = C_n \star_{\ell_1,\dots,\ell_n} (T_1, \dots, T_n)$ such that all T_i are isomorphic, then we define

$$\mathrm{Aut}(T) = \mathrm{Aut}(C_n) \ltimes \prod_{i=1}^{n} \mathrm{Aut}(T_i) = \mathfrak{S}_n \ltimes \prod_{i=1}^{n} \mathrm{Aut}(T_i)$$

If $T = C_n \star_{\ell_1,\dots,\ell_n} (T_1, \dots, T_n)$, and we partition the set of trees $\{T_1, \dots, T_n\}$ into k equivalence classes, where trees T_i and T_j are in the same equivalence class if and only if they are isomorphic, and we let n_i be the size of the equivalence class represented by the object T_i (so $n_1 + \dots + n_k = n$), then we define

$$\mathrm{Aut}(T) = (\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}) \ltimes (\mathrm{Aut}(T_1)^{n_1} \times \dots \times \mathrm{Aut}(T_k)^{n_k})$$

Example 2.5.3. Another important example of a tree is the linear tree $[n]$ ($n \geq 0$): this is the tree which has $(n+1)$ edges and n vertices, each of which has unique incoming and outgoing edges. We note that $[0] = \eta$. This construction defines a fully faithful functor $u : \Delta \hookrightarrow \Omega$.

Construction 2.5.4 (Operads from Trees)**.** Given a tree T, we can construct a coloured operad in sets from T, which we denote by $\Omega(T)$: the colours of $\Omega(T)$ are the edges of T; given a collection of edges $c_1, \dots, c_n, d$ an operation $q \in \mathcal{O}(c_1, \dots, c_n; d)$ is a subtree with leaves $c_1, \dots, c_n$ and root d. By construction, at most one such subtree exists for any given collection $c_1, \dots, c_n, d$. In particular, we will identify a vertex v with the unique operation in $\Omega(T)(in(v); out(v))$. Referring to Figure 2 again, we see for example that there is a subtree whose leaves are d, f and whose root is i – thus $\Omega(T)(d, f; i) = \{u\}$; we also see that we have an unique operation $\Omega(-; e)$, given by the nullary vertex $\{s\}$. The composition is determined by considering the subtree containing both vertices which are to be composed: in the case of our example, $\Omega(-; e) \times \Omega(d, f; i) \ni (s, u) \mapsto u \circ s \in \Omega(-, d, f; i)$.

Note also that the action of the symmetric group $\mathfrak{S}_n$ on the set of operations $\Omega(T)(c_1, \dots, c_n; d)$ is given by permuting the order of the leaves – however, as asserted before, the linear ordering on the incoming edges of a vertex doesn't play any role in determining the structure of a (sub)tree, which ensures that the operad $\Omega(T)$ is equivariant with respect to the action of $\mathfrak{S}_n$ for all n.

The above construction is functorial in T and and in fact gives us a way of defining morphisms between trees: a morphism of trees $T_1 \to T_2$ is a morphism of operads $\Omega(T_1) \to \Omega(T_2)$. Unpicking this definition, we see that to give a morphism of trees from T_1 to T_2, we must specify a map of sets φ from $E(T_1)$ to $E(T_2)$, and for any subset of edges $c_1, c_2, \dots, c_n, d \in E(T_1)$, we must ensure that if $\Omega(T_1)(c_1, \dots, c_n; d) \neq \emptyset$, then also $\Omega(T_2)(\varphi(c_1), \dots, \varphi(c_n); d) \neq \emptyset$ (as we see from even this foray into unravelling the definition, giving a completely general explanation of a morphism between two trees in explicit terms can be a somewhat haphazard affair).

The category of closed trees Ω_c is a subcategory of Ω in which we will be especially interested. Conveniently, since the leaf sets of closed trees are empty, this category admits a succinct alternative description using Lemma 2.5.1: a closed tree is a poset E with an unique maximal element and such that for every $e \in E$, the subposet $E_{e\leq}$ is linearly ordered; a morphism of closed trees is a map of posets which respects the incomparability relation (i.e. given two such posets E and E', a poset morphism $\varphi : E \to E'$ is a morphism of closed trees if and only if $\varphi(e) \perp \varphi(f)$ whenever $e \perp f$ in E).

The inclusion functor $\Omega_c \hookrightarrow \Omega$ admits a left adjoint, called the closure, which we denote by cl: given a tree $T \in \Omega$, the closure of T is the tree obtained by placing a nullary vertex on each of its leaves. For example, we have already seen that the closure of the root η is just the 0-corolla. Given a closed tree T, each colour of the associated operad $\Omega(T)$ has is a singleton set of nullary operations. This will play a role for us when we come to discuss reduced ∞-operads.

Chapter 3

∞-Operads

In this chapter, we discuss a weakened or homotopy-coherent notion of operads, the so-called ∞-operads. Many models exist to describe such ∞-operads and in the course of this thesis, it has become not only useful but of crucial importance to become at least conversant with a number of these models.

In this chapter, we will give an exposition of three of these models: the ∞-operads of Lurie's quasi-categorical formulation [Lur17] (which we will hereafter refer to as quasi-operads), the complete Segal dendroidal spaces of Moerdijk and Cisinski [CM13] and the complete Segal operads of Barwick [Bar18]. In the course of our discussion on quasi-operads, we will also devote some time to discussing a quasi-categorical version of the little cubes operad, and the quasi-categorical version of the Dunn additivity theorem to which we alluded in our discussion of the topological little cubes.

Proving that these models of ∞-operad are in fact equivalent is a highly non-trivial exercise (indeed, one of the chapters in this thesis is given over to such a proof of equivalence in a restricted setting) and a veritable cottage industry has grown up around proving such assertions – see for example [Bar18], [CM11], [CM13], [CHH18] and [HHM16]. We will highlight some of the constructions employed to deduce these equivalences, but for the most part we direct the reader to the original texts to find more thorough and enlightening arguments of the equivalences.

3.1 Quasi-Operads

3.1.1 From Coloured Operads to Quasi-Operads

We recall from our discussion of coloured operads in spaces that to a coloured operad $\mathcal{O}$ we could associate its category of operators $\mathcal{O}^{\otimes}$. This was the category enriched in spaces whose objects were finite sequences of colours of $\mathcal{O}$; and the space of all morphisms between two such sequences, $\{c_i\}_{i\in I}$ and $\{d_j\}_{j\in J}$, was given by

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J}) = \coprod_{f:I\to J}\prod_{j\in J}\mathcal{O}(\{c_i\}_{i\in f^{-1}\{j\}}; d_j)$$

where the coproduct is taken over all maps $f : I \to J$ in Fin_*. This category comes naturally equipped with an obvious forgetul functor $\pi : \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, and in fact the operad $\mathcal{O}$ can be reconstructed from this category and the forgetful functor π.

We note that while the map π is not a coCartesian fibration, coCartesian lifts may be found for a special class of morphisms in Fin_*: namely, the inert maps. For example, given an inert morphism $f : I \to J$, a map $\alpha : \{c_i\}_{i\in I} \to \{d_j\}_{j\in J}$ lying over f is specified by choosing for each $j \in J$ an operation $o_j \in \mathcal{O}(\{c_i\}; d_j)$, where i is the unique element of I in the preimage of f. Then the map $\alpha \in \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J})$ is π-coCartesian if and only if each o_j is an isomorphism in the underlying category of $\mathcal{O}$. As a result, each inert $f : I \to J$ admits a π-coCartesian lift, starting at an arbitrary object over I.

Related to the above observations, we note that the underlying category of $\mathcal{O}$ is precisely the fibre of π over the pointed set $\langle 1\rangle$, which we denote by $\mathcal{O}^{\otimes}_{\langle 1\rangle}$. Furthermore, if $\rho^i : \langle n\rangle \to \langle 1\rangle$ is the unique inert morphism defined by $i \mapsto 1$, then by the coCartesian lift property above, there is an induced map

$$\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$$

In particular, the collection of all such maps determines an equivalence

$$\prod_{i\in\langle n\rangle}(\rho^i_!) : \mathcal{O}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \prod_{i\in\langle n\rangle}\mathcal{O}^{\otimes}_{\langle 1\rangle}$$

Also related to the two observations above is the next property of the map $\pi : \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$: given two sequences of colours $\{c_i\}_{i\in I}, \{d_j\}_{j\in J}$ in $\mathcal{O}$ and a map $f : I \to J$ in Fin_*, we let $\mathrm{Hom}^f_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J})$ denote the fibre of the projection map

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J}) \to \mathrm{Hom}_{\mathsf{Fin}_*}(I, J)$$

over f. If for each $j \in J$, we choose a π-coCartesian morphism $\{c_i\}_{i\in I} \to d_j$ lying over the inert morphism $\rho^j : I \to \langle 1\rangle$, then the following square is homotopy Cartesian:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J}) \arrow[r] \arrow[d] & \prod_{j\in J}\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}) \arrow[d] \\
\mathrm{Hom}_{\mathsf{Fin}_*}(I, J) \arrow[r] & \prod_{j\in J}\mathrm{Hom}_{\mathsf{Fin}_*}(I, \langle 1\rangle)
\end{tikzcd}$$

i.e. we have weak equivalences $\mathrm{Hom}^f_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J}) \xrightarrow{\simeq} \prod_{j\in J}\mathrm{Hom}^{\rho^j\circ f}_{\mathcal{O}^{\otimes}}(\{c_i\}_{i\in I}, \{d_j\})$

In a sense, these properties are the crucial criteria which determine a coloured operad, so when attempting to define a quasi-operad, it makes sense to enforce analogous conditions (in what follows we work with quasi-categorical mapping spaces, denoted $\mathrm{map}_{\mathcal{O}^{\otimes}}$, rather than our previous 0-categorical Hom-objects).

Definition 3.1.1. A quasi-operad is the data of a functor between quasi-categories $\pi : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ satisfying the following properties:

1. For each inert morphism $f : \langle n\rangle \to \langle m\rangle$ in $N\mathsf{Fin}_*$ and every object $c \in \mathcal{O}^{\otimes}_{\langle n\rangle} = \pi^{-1}\{\langle n\rangle\}$, there exists a π-coCartesian morphism $c \to d$ in $\mathcal{O}^{\otimes}$ over f. (In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle m\rangle}$.)

2. For every morphism $f : \langle n\rangle \to \langle m\rangle$ and every pair of objects $c \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, $d \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, if we choose morphisms $\mathcal{O}^{\otimes}_{\langle n\rangle} \ni c \to d_j \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$ lying over the composite $\rho^j \circ f$, then we have weak homotopy equivalences

$$\mathrm{map}^f_{\mathcal{O}^{\otimes}}(c, d) \xrightarrow{\simeq} \prod_{1\le j\le m}\mathrm{map}^{\rho^j\circ f}_{\mathcal{O}^{\otimes}}(c, d_j)$$

3. For every $n \geq 1$, the induced functors $\rho^j_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ give rise to an equivalence of quasi-categories

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \simeq \prod_n \mathcal{O}^{\otimes}_{\langle 1\rangle}$$

The coloured operads in spaces we previously encountered naturally generalise to quasi-operads by means of the homotopy-coherent nerve, thus supplying us with a smorgasbord of examples of quasi-operads.

Example 3.1.2. Let Com denote the commutative operad in spaces, as in Example 2.1.4. Since Com has a unique colour and for every n there is an unique n-ary operation, we see that the category of operators of Com coincides with the category Fin_*. It follows that the natural projection map $\mathrm{Com}^{\otimes} \to \mathsf{Fin}_*$ is actually the identity map; taking homotopy-coherent nerves of both categories (which coincide with the usual discrete nerves since these categories are discrete), we see that $\mathrm{id} : N\mathrm{Com}^{\otimes} = N\mathsf{Fin}_* \to N\mathsf{Fin}_*$ is a quasi-operad.

Likewise, the other examples of coloured operads we encountered such as the associative operad and the trivial operad also yield quasi-operads.

We can also construct quasi-operads with a particularly simple collection of operations from quasi-categories. This method will have some utility when we come to discuss some of the variants of the ∞-categorical version of the little cubes operads. Before we can describe these, we need to define an auxiliary category, Γ^*: objects of Γ^* are pairs $(\langle n\rangle, i)$, where $i \in \langle n\rangle_o$. A morphism from $(\langle n\rangle, i)$ to $(\langle m\rangle, j)$ in Γ^* is given by a map $\alpha : \langle n\rangle \to \langle m\rangle$ in Fin_* such that $\alpha(i) = j$.

Example 3.1.3. Let $\mathcal{C}$ be a quasi-category. We can define a simplicial set $\mathcal{C}^{\sqcup}$ which satisfies the following adjunction

$$\mathrm{Hom}_{N\mathsf{Fin}_*}(K, \mathcal{C}^{\sqcup}) = \mathrm{Hom}_{s\mathrm{Set}}(K \times_{N\mathsf{Fin}_*} N\Gamma^*, \mathcal{C})$$

for each simplicial set K. Unraveling the above definition, we see that for each simplicial set K, we get a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}_{s\mathrm{Set}}(K \times_{N\mathsf{Fin}_*} N\Gamma^*, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{s\mathrm{Set}}(K, \mathcal{C}^{\sqcup}) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{s\mathrm{Set}}(K, N\mathsf{Fin}_*)
\end{tikzcd}$$

In particular, this tells us that an n-simplex of $\mathcal{C}^{\sqcup}$ is determined by the data of a pair (ρ, δ), where $\rho : \Delta[n] \to N\mathsf{Fin}_*$ and $\delta : \Delta[n] \times_{N\mathsf{Fin}_*} N\Gamma^* \to \mathcal{C}$ such that the fibre product is defined along the reference map $\rho : \Delta[n] \to N\mathsf{Fin}_*$.

We can zero in on this observation for the case $n = 0$: such a pair (ρ, δ) will then correspond to an object $\langle m \rangle$ in Fin_* and a map $\{\langle m \rangle\} \times_{N\mathsf{Fin}_*} N\Gamma^* \cong \langle m \rangle_o \to \mathcal{C}$. In other words, we can identify the vertices of $\mathcal{C}^{\sqcup}$ with tuples $(c_1, \ldots, c_m)$ where each c_i is a vertex of $\mathcal{C}$. Similarly, we can examine the case where $n = 1$: in this case, a pair (ρ, δ) will correspond to an object $f : \langle m \rangle \to \langle k \rangle$ and a map $\delta : \Delta[1] \times_{N\mathsf{Fin}_*} N\Gamma^* \to \mathcal{C}$. We can identify such a pair (ρ, δ) with a triple $\Big((c_i)_{1 \le i \le m}, (d_j)_{1 \le j \le k}, \{\delta_i : c_i \to d_{f(i)}\}_{i \in f^{-1}\{\langle m \rangle_o\}}\Big)$. It can be shown that the obvious reference map $\mathcal{C}^{\sqcup} \to N\mathsf{Fin}_*$ gives the structure of a quasi-operad ([Lur17, Proposition 2.4.3.3]).

Morally speaking, our observations above tell us that we can view $\mathcal{C}^{\sqcup}$ as the ∞-operad whose colours are the vertices of $\mathcal{C}$ and whose spaces of operations are defined by

$$\mathcal{C}^{\sqcup}(\{c_i\}_{1 \le i \le m}; d) = \prod_{1 \le i \le m} \mathrm{map}_{\mathcal{C}}(c_i, d)$$

i.e. the n-ary operations are just products of 1-ary operations. We refer to this construction as the **coCartesia quasi-operad** associated to the quasi-category $\mathcal{C}$.

3.1.2 Morphisms of Quasi-Operads and a Model Structure

Definition 3.1.4. Given a quasi-operad $\pi : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$, we say a morphism $g : c \to d$ in $\mathcal{O}^{\otimes}$ is **inert** precisely if it is π-coCartesian and lies over an inert morphism in Fin_*.

A **map of quasi-operads** $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of quasi-categories $\alpha : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ over $N\mathsf{Fin}_*$ such that α sends inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{P}^{\otimes}$. We let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ be the full subcategory of $\mathrm{Fun}_{N\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ spanned by maps of quasi-operads.

We remark that a map over $N\mathsf{Fin}_*$ of quasi-categories $f : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ between two quasi-operads preserves all inert morphisms if and only if it preserves those inert morphisms lying over the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for all i and n.

Example 3.1.5. We saw previously that a quasi-category $\mathcal{C}$ gives rise to a particularly simple kind of quasi-operad, which we denoted $\mathcal{C}^{\sqcup}$. In the special case where $\mathcal{C} = \mathcal{O}^{\otimes}$ is already a quasi-operad, there is a natural map of quasi-operads $\mathcal{O}^{\otimes} \to \mathcal{O}^{\sqcup}$ (where we write $\mathcal{O}^{\sqcup} = (\mathcal{O}^{\otimes})^{\sqcup}$), which we will try to describe intuitively. First, by the properties of a quasi-operad, we know that there is a weak homotopy equivalence $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n$, so loosely, we can say that the 0-simplices of $\mathcal{O}^{\otimes}$ are given by tuples $\{c_i\}_{1 \le i \le n}$, as is the case for $\mathcal{O}^{\sqcup}$. An edge between two such tuples, $\{c_i\}_{1 \le i \le m}$ and $\{d_j\}_{1 \le j \le n}$ in $\mathcal{O}^{\otimes}$ is the data of a map $f : \langle m \rangle \to \langle n \rangle$ in Fin_* and a tuple of "operations" $\big(o_j : \{c_i\}_{i \in f^{-1}\{j\}} \to d_j\big)_{1 \le j \le n}$.

By the third property in the definition of quasi-operads, we know that for each $1 \le j \le n$, there is a weak equivalence

$$\prod_{j_i \in f^{-1}\{j\}} \rho_!^{j_i} : \mathcal{O}^{\otimes}_{f^{-1}\{j\}} \xrightarrow{\sim} \Big(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\Big)^{f^{-1}\{j\}}$$

Hence, for each $o_j \in \mathrm{map}_{\mathcal{O}^{\otimes}}\big(\{c_i\}_{i \in f^{-1}\{j\}}, \{d_j\}\big)$, there is a corresponding tuple

$$(o_{j_i})_{j_i \in f^{-1}\{j\}} \in \prod_{j_i \in f^{-1}\{j\}} \mathrm{map}_{\mathcal{O}^{\otimes}}(\{c_{j_i}\}, \{d_j\})$$

and the map

$$\left(\langle n\rangle \xrightarrow{f} \langle m\rangle, (o_j)_{1\leq j\leq m}\right) \mapsto \left(\langle n\rangle \xrightarrow{f} \langle m\rangle, \left((o_{j_i})_{j_i\in f^{-1}\{j\}}\right)_{1\leq j\leq m}\right)$$

defines the morphism on 1-simplices. The Segal condition ensures that this is sufficient to determine the morphism of simplicial sets in all degrees. By our remarks after Definition 3.1.4, to show that this is a map of quasi-operads, it suffices to show the preservation of inert morphisms over the maps $\rho^i : \langle n\rangle \to \langle 1\rangle$, but this follows from the second property of the definition of a quasi-operad, and the fact that a morphism $\{c_i\}_{1\leq i\leq n} \to c$ in $\mathcal{O}^{\sqcup}$ is inert if and only if each of the edges $c_i \to c$ is an equivalence in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$.

Based on Definition 3.1.4, we obtain a simplicial category $\widetilde{\mathrm{Op}}$, whose objects are quasi-operads, and for any two such objects $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$, the mapping spaces are given by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. Taking the homotopy-coherent nerve of this category gives us a quasi-category Op_∞. Our next goal will be to explain how Op_∞ arises as the underlying ∞-category of a model structure on a certain simplicial category. We will later make use of this model structure when we come to describe the ∞-categorical version of the Boardman-Vogt tensor product of operads.

Before we arrive at that point, we first need to introduce the preliminary notion of a **marked simplicial set**: this is a pair (X, M), where X is a simplicial set and M is a collection of 1-simplices of X which contains all the degenerate 1-simplices. We refer to $M \subseteq X_1$ as the marking on X. If X is a simplicial set, we write $X^\flat$ to denote the marked simplicial set whose markings consist only of the degenerate 1-simplices; and we write $X^\sharp$ to denote the marked simplicial set whose markings consist of all 1-simplices. If $\mathcal{O}^{\otimes}$ is a quasi-operad, we write $\mathcal{O}^{\otimes,\natural}$ to indicate the marked simplicial set whose markings consist of all the inert morphisms in $\mathcal{O}^{\otimes}$ (evidently this subset of $\mathcal{O}^{\otimes}_1$ contains all the degenerate 1-simplices). The notion of a map of marked simplicial sets is the obvious one: namely, a map of simplicial sets which sends marked edges to marked edges. Having prepared the ground with this notion of marked simplicial sets, we are now in a position to describe the desired simplicial category...

Definition 3.1.6. A **quasi-preoperad** is a marked simplicial set (X, M) equipped with a map of simplicial sets $f : X \to N\mathsf{Fin}_*$ which satisfies the property that for each marked edge $e \in M$, the image $f(e)$ is an inert morphism in Fin_*. A morphism of quasi-preoperads is a map of marked simplicial sets $g : (X, M) \to (X', M')$ which commutes with the respective reference maps to $N\mathsf{Fin}_*$.

This determines a category $\mathcal{P}\mathrm{Op}_\infty$, which is tensored over simplicial sets as follows: given a quasi-preoperad (X, M) and a simplicial set K, we let $(X, M) \otimes K = (X \times K, M \times K_1)$. Thus $\mathcal{P}\mathrm{Op}_\infty$ is in fact a simplicial category.

Obviously, if $\mathcal{O}^{\otimes}$ is a quasi-operad, then $\mathcal{O}^{\otimes,\natural}$ is a quasi-preoperad. In fact, the following proposition tells us that quasi-operads form a distinguished class in the simplicial category $\mathcal{P}\mathrm{Op}_\infty$:

Proposition 3.1.7. *[Lur17, Proposition 2.1.4.6] There exists a left proper combinatorial simplicial model structure (**the ∞-operadic model structure**) on the category $\mathcal{P}\mathrm{Op}_\infty$ characterised by the following properties:*

- *a morphism $(X, M) \to (X', M')$ is a cofibration if and only if $X \to X'$ is a monomorphism of simplicial sets;*
- *a map $f : (X, M) \to (X', M')$ is a weak equivalence if and only if, for each quasi-operad $\mathcal{O}^{\otimes}$, the induced map*

$$\mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}((X', M'), \mathcal{O}^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}((X, M), \mathcal{O}^{\otimes,\natural})$$

  is a weak homotopy equivalence of simplicial sets.

The fibrant objects in the ∞-operadic model structure on $\mathcal{P}\mathrm{Op}_\infty$ are precisely the objects of the form $\mathcal{O}^{\otimes,\natural}$, where $\mathcal{O}^{\otimes}$ is a quasi-operad.

3.1.3 Tensor Products of Quasi-Operads

We recall from our discussion of algebras over operads in spaces that the Boardman-Vogt tensor product of two strict operads $\mathcal{O}$ and $\mathcal{P}$ is the operad $\mathcal{O} \star \mathcal{P}$ defined in terms of the universal property that there is an equivalence of categories

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{P}}) \simeq \mathrm{Alg}_{\mathcal{O}\star\mathcal{P}} \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Alg}_{\mathcal{O}}) \tag{3.1}$$

When discussing tensor products of operads in that setting it was possible to give a fairly explicit description of this tensor product; however, in the context of quasi-operads, the crucial thing we want to generalise is this universal property. Key to the definition we describe in this setting will be the ∞-operadic model structure we introduced in Proposition 3.1.7.

First, we define the **smash product functor** for pointed finite sets – this is the functor $\wedge : \mathsf{Fin}_* \times \mathsf{Fin}_* \to \mathsf{Fin}_*$ which acts

- on objects by: $(\langle m \rangle, \langle n \rangle) \longmapsto \langle mn \rangle$
- on morphisms by: $((f : \langle m \rangle \to \langle m' \rangle), (g : \langle n \rangle \to \langle n' \rangle)) \mapsto (f \wedge g : \langle mn \rangle \to \langle m'n' \rangle)$, where

$$(f \wedge g)(an + b - n) = \begin{cases} * & f(a) = * \text{ or } g(b) = * \\ f(a)n' + g(b) - n' & \text{otherwise} \end{cases}$$

Definition 3.1.8. Given quasi-operads, $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ and $\mathcal{Q}^{\otimes}$ a map $f : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ is said to be a **bifunctor of quasi-operads** if it satisfies the following properties:

- The diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "f"] \arrow[d] & \mathcal{Q}^{\otimes} \arrow[d] \\ N\mathsf{Fin}_* \times N\mathsf{Fin}_* \arrow[r, "\wedge"] & N\mathsf{Fin}_* \end{tikzcd}$$

  commutes.

- If α is an inert morphism in $\mathcal{O}^{\otimes}$ and β is an inert morphism in $\mathcal{P}^{\otimes}$, then $f(\alpha, \beta)$ is inert in $\mathcal{Q}^{\otimes}$.

We let $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes}) \subseteq \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes})$ denote the full subcategory spanned by bifunctors of quasi-operads. We say $\mathcal{Q}^{\otimes}$ exhibits as the tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ if there is an equivalence

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O}') \simeq \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{O}'^{\otimes})$$

for every quasi-operad $\mathcal{O}'$. In such a case, we write $\mathcal{Q}^{\otimes} = \mathcal{O}^{\otimes} \odot \mathcal{P}^{\otimes}$.

We see immediately that this definition captures the essential universal property of the tensor product described in (3.1). However, the definition is also decidedly gnomic and, in general, it is unclear whether such a tensor product even exists.

To resolve the question of existence, we must rely on the ∞-operadic model structure. First, given quasi-preoperads (X, M) and (X', M'), we set $(X, M) \odot (X', M') = (X \times X', M \times M')$. It should be noted that this marked simplicial set has the structure of a quasi-preoperad via the composition

$$X \times X' \to N\mathsf{Fin}_* \times N\mathsf{Fin}_* \xrightarrow{\wedge} N\mathsf{Fin}_*$$

It follows from our definition of a bifunctor of operads that $\mathcal{Q}^{\otimes}$ exhibits as the tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ if and only if for each ∞-operad $\mathcal{O}'^{\otimes}$, we have an equivalence:

$$\mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\mathcal{Q}^{\otimes,\natural}, \mathcal{O}'^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\mathcal{O}^{\otimes,\natural} \odot \mathcal{P}^{\otimes,\natural}, \mathcal{O}'^{\otimes,\natural})$$

In other words, $\mathcal{Q}^{\otimes} = \mathcal{O}^{\otimes} \odot \mathcal{P}^{\otimes}$ if and only if the quasi-preoperad $\mathcal{Q}^{\otimes,\natural}$ is a fibrant replacement for $\mathcal{O}^{\otimes,\natural} \odot \mathcal{P}^{\otimes,\natural}$ in the ∞-operadic model structure.

This assertion at least makes it clear that the tensor product exists. However, it is not in general obvious how one might go about actually producing such a fibrant replacement. One fairly explicit approach involves the **wreath product** construction ([Lur17, Section 2.4.4]) which we will not discuss here, but even this is somewhat tricky to handle.

3.2 The Little Cubes Operads

In this section we discuss a quasi-categorical version of the little cubes operads we encountered in the topological setting, and we will discuss some of the variants of these operads. We will also review some of the properties exhibited tensor products of these operads, where we meet a quasi-categorical version of the additivity theorem of Dunn and Brinkmeier.

3.2.1 Quasi-Categorical Little Cubes and Dunn Additivity

We recall that a little d-cube is a rectilinear embedding of the d-cube $(-1,1)^d$ in itself and that the topological operad ${}^t\mathbb{E}_d$ is the operad with one colour whose collection of n-ary operations consists of all n-tuples of such rectilinear embeddings with disjoint image. Taking the homotopy-coherent nerve of this operad gives a quasi-operad, which we denote by $\mathbb{E}_d^{\otimes}$. In Example 2.4.3 we saw that these operads satisfy a kind of additivity relation with respect to the Boardman-Vogt tensor product of strict operads. The question we now wish to study is whether a similar relation holds in the quasi-categorical setting when we work with the tensor product $\odot$ instead.

We recall that an operad $\mathcal{Q}^{\otimes}$ exhibits as the tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ if there is a bifunctor $f : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ over the smash product $\wedge : N\mathsf{Fin}_* \times N\mathsf{Fin}_* \to N\mathsf{Fin}_*$ which induces an equivalence

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O}') \simeq \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{O}'^{\otimes})$$

Let us first consider the bifunctor of topological categories

$$\rho :\ {}^t\mathbb{E}_d^{\otimes} \times {}^t\mathbb{E}_{d'}^{\otimes} \to {}^t\mathbb{E}_{d+d'}^{\otimes} \tag{3.2}$$

defined on objects by $(\langle n\rangle, \langle m\rangle) \mapsto \langle nm\rangle$, and on hom-objects by

$$((\alpha, (\varepsilon_i)_{1\le i\le n'}), (\beta, (\delta_j)_{1\le j\le m'})) \longmapsto (\alpha \wedge \beta, (\varepsilon_i \times \delta_j)_{1\le i\le n', 1\le j\le m'})$$

where $\alpha \in \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle n'\rangle)$, $\varepsilon_i \in {}^t\mathbb{E}_d(\alpha^{-1}\{i\})$ for each $1 \le i \le n'$, $\beta \in \mathrm{Hom}_{\mathsf{Fin}_*}(\langle m\rangle, \langle m'\rangle)$ and $\delta_j \in {}^t\mathbb{E}_{d'}(\beta^{-1}\{j\})$ for each $1 \le j \le m'$.

This induces a map (which we also denote by ρ) on homotopy-coherent nerves and we note that there is an obvious commutative diagram

$$\begin{tikzcd} \mathbb{E}_d^{\otimes} \times \mathbb{E}_{d'}^{\otimes} \arrow[r, "\rho"] \arrow[d] & \mathbb{E}_{d+d'}^{\otimes} \arrow[d] \\ N\mathsf{Fin}_* \times N\mathsf{Fin}_* \arrow[r, "\wedge"] & N\mathsf{Fin}_* \end{tikzcd}$$

Theorem 3.2.1. *[Lur17, Theorem 5.1.2.2] Let d, d' be non-negative integers. Then the bifunctor $\rho : \mathbb{E}_d^{\otimes} \times \mathbb{E}_{d'}^{\otimes} \to \mathbb{E}_{d+d'}^{\otimes}$ exhibits $\mathbb{E}_{d+d'}^{\otimes}$ as a tensor product of $\mathbb{E}_d^{\otimes}$ and $\mathbb{E}_{d'}^{\otimes}$.*

The proof of this theorem is highly non-trivial and will not be discussed in detail here. At its heart is the notion of weak approximations of operads and the wreath product construction to which we alluded above (with more than a hint of trepidation) – specifically, Lurie demonstrates that the proof of the above can be reduced to proving that there is a weak approximation $\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes} \to \mathbb{E}_{1+d}^{\otimes}$. The notion of weak approximation in particular is quite technical, and we refer the reader to [Lur17, Section 2.3.3] or [Har, Section 4.2] for details. We can however say a little about the wreath-product $\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}$, as it can be realised as the homotopy-coherent nerve of a certain topological category, W:

- objects of W are finite sequences of objects in Fin_*, e.g. $(\langle s_0\rangle, \langle s_1\rangle, \dots, \langle s_n\rangle)$
- for a pair of such sequences $(\langle s_0\rangle, \langle s_1\rangle, \dots, \langle s_n\rangle)$, $(\langle s'_0\rangle, \langle s'_1\rangle, \dots, \langle s'_m\rangle)$, the space of hom-objects between them is given by
$$\coprod_{\alpha:\langle n\rangle\to\langle m\rangle} \mathrm{Hom}^{\alpha}_{{}^t\mathbb{E}_1^{\otimes}}(\langle n\rangle, \langle m\rangle) \times \prod_{\alpha(i)=j} \mathrm{Hom}_{{}^t\mathbb{E}_d^{\otimes}}(\langle s_i\rangle, \langle s'_j\rangle)$$
where the coproduct is taken over all maps $\alpha \in \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle m\rangle)$, and by $\mathrm{Hom}^{\alpha}_{{}^t\mathbb{E}_1^{\otimes}}(\langle n\rangle, \langle m\rangle)$, we mean all those morphisms from $\langle n\rangle$ to $\langle m\rangle$ in ${}^t\mathbb{E}_1^{\otimes}$ which lie over the morphism α.

There are functors of topological categories ${}^t\mathbb{E}_1^{\otimes} \times^t \mathbb{E}_d^{\otimes} \to W$ and $W \to^t \mathbb{E}_{1+d}$ given respectively by

$$(\langle n\rangle, \langle m\rangle) \mapsto \left(\underbrace{\langle m\rangle, \dots, \langle m\rangle}_{\text{n times}}\right) \quad \text{and} \quad (\langle s_1\rangle, \dots, \langle s_n\rangle) \mapsto \langle s_1 + \dots + s_n\rangle$$

and these induce maps on the homotopy-coherent nerves. Showing that the latter functor defines a weak approximation forms the bulk of the work in Lurie's proof.

3.2.2 Variants on the Little Cubes Operads

As a variant on the usual little cubes operads, we can instead consider the topological category ${}^t\mathbb{E}^{\otimes}_{B\mathrm{Top}(d)}$, defined as follows:

- The objects of ${}^t\mathbb{E}^{\otimes}_{B\mathrm{Top}(d)}$ are the objects $\langle n\rangle$ of Fin_*.
- For objects $\langle m\rangle, \langle n\rangle$, the space of all maps between them is

$$\mathrm{Hom}_{{}^t\mathbb{E}^{\otimes}_{B\mathrm{Top}(d)}}(\langle m\rangle, \langle n\rangle) = \coprod_{f:\langle m\rangle\to\langle n\rangle} \coprod_{1\le i\le n} \mathrm{Emb}\left(\sqcup_{f^{-1}\{i\}}\mathbb{R}^d, \mathbb{R}^d\right)$$

  where the first coproduct is taken over all maps $f : \langle m\rangle \to \langle n\rangle$ in Fin_*.
- The composition of morphisms in this category is determined by the composition law in Fin_* and the composition of embeddings of $\mathbb{R}^d$ in itself.

Taking the homotopy-coherent nerve of this category, we obtain a quasi-operad, which we denote by $B\mathrm{Top}(d)^{\otimes}$. We can view this as an unframed version of the little cubes operads – indeed, the choice of a homeomorphism $\mathbb{R}^d \cong (-1,1)^d$ leads to an inclusion of quasi-operads $\mathbb{E}^{\otimes}_d \hookrightarrow B\mathrm{Top}(d)^{\otimes}$. If we consider the fibre of $B\mathrm{Top}(d)^{\otimes} \to N\mathsf{Fin}_*$ over the element $\langle 1\rangle$, we see that the edges of this simplicial set can be identified with the space $\mathrm{Emb}(\mathbb{R}^d, \mathbb{R}^d)$. A theorem of Kister and Mazur tells us that this space is homotopy-equivalent to the topological group of all homeomorphisms from $\mathbb{R}^d$ into itself (which we denote by $\mathrm{Top}(d)$). From this perspective, we can view $B\mathrm{Top}(d)^{\otimes}_{\langle 1\rangle}$ as the nerve of the topological group $\mathrm{Top}(d)$ – or more loosely, we identify $B\mathrm{Top}(d)^{\otimes}_{\langle 1\rangle}$ with the classifying space of $\mathrm{Top}(d)$ (indeed, it is for this reason that the rather suggestive name for this operad was chosen).

This unframed version of the little cubes operad opens up a whole range of potential variations on the little cubes operad: for example, let B be a Kan complex (so B is automatically a quasi-groupoid) and suppose we have a Kan fibration $\alpha : B \to B\mathrm{Top}(d)$. Then we can define a quasi-operad $\mathbb{E}^{\otimes}_B \to N\mathsf{Fin}_*$ by

$$\mathbb{E}^{\otimes}_B = B\mathrm{Top}(d)^{\otimes} \times_{B\mathrm{Top}(d)^{\sqcup}} B^{\sqcup} \to B\mathrm{Top}(d)^{\otimes} \to N\mathsf{Fin}_*$$

The fact that this map satisfies the conditions of a quasi-operad relates to the stability of coCartesian morphisms under base-change; additionally, we know by Example 3.1.3 that since both B and $B\mathrm{Top}(d)$ are both quasi-categories, it is possible to define their associated coCartesia quasi-operads $B^{\sqcup}, B\mathrm{Top}(d)^{\sqcup}$.

To further develop this example, we could consider a group G, with a representation α in $\mathrm{Top}(d)$. The representation induces a map on classifying spaces, $B\alpha : BG \to B\mathrm{Top}(d)$ (which may, if necessary, be replaced by a Kan fibration). We will write $\mathbb{E}^{G,\otimes}_d$ for the corresponding quasi-operad $\mathbb{E}^{\otimes}_{BG}$.

Remark 3.2.2. Let us pause a beat to examine this quasi-operad from a more geometric perspective. The 0-simplices of the fibre $\mathbb{E}^{\otimes}_B$ over $\langle n\rangle$ are pairs of the form $(c, \{b_i\}_{1\le i\le n}) \in B\mathrm{Top}(d)^{\otimes}_{\langle n\rangle} \times B^n$ such that $\alpha(b_i) = \rho^i_!(c)$ for each i, i.e. identifying c with a tuple $\langle c_1, \dots, c_n\rangle$ of embeddings $\mathbb{R}^d \hookrightarrow \mathbb{R}^d$ with disjoint images, we have $\alpha(b_i) = c_i$. Similarly, a 1-simplex in $\mathbb{E}^{\otimes}_B$ lying over the map of finite pointed sets $f : \langle m\rangle \to \langle n\rangle$ is the data of

- an edge $\sigma : c \to c'$ in $B\mathrm{Top}(d)^{\otimes}$ lying over f – i.e. if we identify c with $\langle c_1, \dots, c_m\rangle$ and we identify c' with $\langle c'_1, \dots, c'_n\rangle$, then for each $1 \le j \le n$ we have edges $\sigma_j : \langle c_{i_j}\rangle_{i_j\in f^{-1}\{j\}} \to c'_j$ in $B\mathrm{Top}(d)^{\otimes}$.
- for each $f(i_j) = j$, a collection of edges $\delta_{i_j} : b_{i_j} \to b'_j$ in B
- these edges must be compatible in the sense that $\sigma_j = \left(\alpha\left(\delta_{i_j}\right)\right)_{i_j\in f^{-1}\{j\}}$ as maps

$$\left(c_{i_j}\right)_{i_j\in f^{-1}\{j\}} = \alpha\left(b_{i_j}\right)_{i_j\in f^{-1}\{j\}} \to \alpha\left(b'_j\right) = c'_j$$

  for each $1 \le j \le n$.

In particular, for the case where $B = BG$ for some group G, then a 0-simplex $(c, \{g_i\}_{1\le i\le n})$ of this quasi-operad can be regarded as a tuple of disjoint embeddings each of which arises from the action of the group G on the space $\mathbb{R}^d$ – in other words, these operads can be seen as playing the role of the skew little cubes which we discussed in Example 2.1.13.

To give an example of this principle in action, let us fix a homeomorphism from $\mathbb{R}^d$ to its open unit ball $B(1)$. A map $\lambda : B(1) \to B(1)$ is said to be a *projective isometry* if there exist $g \in O(d)$, $c > 0$

and $v_0 \in B(1)$ such that $\lambda(x) = cg(x) + v_0$. Let $\mathrm{Isom}^+(\sqcup_I B(1), B(1)) \subseteq \mathrm{Emb}(\sqcup_I B(1), B(1))$ denote those embeddings $\langle \varepsilon_i \rangle_{i\in I} : \sqcup_I B(1) \hookrightarrow B(1)$ such that each ε_i is a projective isometry. We can define the topological category ${}^t\mathbb{E}^{\otimes}_{SO(d)}$ which is the subcategory of ${}^t\mathbb{E}^{\otimes}_{B\mathrm{Top}(d)}$ with the same objects and whose morphism spaces are given by

$$\mathrm{Hom}_{{}^t\mathbb{E}^{\otimes}_{SO(d)}}(\langle n\rangle, \langle m\rangle) = \coprod_{f:\langle n\rangle\to\langle m\rangle} \coprod_{1\le j\le m} \mathrm{Isom}^+\left(\sqcup_{f^{-1}\{j\}} B(1), B(1)\right)$$

The homotopy-coherent nerve of this operad is a quasi-operad, and there is a natural inclusion $N({}^t\mathbb{E}^{\otimes}_{SO(d)}) \hookrightarrow B\mathrm{Top}(d)^{\otimes}$, which induces an equivalence with the operad $\mathbb{E}_d^{SO(d),\otimes}$.

Most pertinently for our concerns, in the case where we have a dilation representation of the group G in $GL_d(\mathbb{R})$, the natural inclusion $N({}^t\mathbb{E}_d^{G,\otimes}) \hookrightarrow B\mathrm{Top}(d)^{\otimes}$ will factor through $\mathbb{E}_d^{\otimes}$. Hence, this really *does* provide us with a homotopy-coherent version of the skew-little cubes which we described in Example 2.1.13.

Moreover, unlike in the case of the topological skew little cubes operads, [Lur17, Remark 5.4.2.14] gives a positive answer to the question of whether there is a quasi-categorical additivity theorem for the homotopy-coherent skew-little cubes. First, we note that the homeomorphism $\mathbb{R}^{d+d'} \cong \mathbb{R}^d \times \mathbb{R}^{d'}$ induces a map of Kan complexes

$$B\mathrm{Top}(d)^{\otimes}_{\langle 1\rangle} \times B\mathrm{Top}(d')^{\otimes}_{\langle 1\rangle} \to B\mathrm{Top}(d+d')^{\otimes}_{\langle 1\rangle}$$

and this in turn gives rise to bifunctors of quasi-operads

$$B\mathrm{Top}(d)^{\sqcup} \times B\mathrm{Top}(d')^{\sqcup} \to B\mathrm{Top}(d+d')^{\sqcup} \quad \text{and} \quad B\mathrm{Top}(d)^{\otimes} \times B\mathrm{Top}(d')^{\otimes} \to B\mathrm{Top}(d+d')^{\otimes}$$

In particular, given representations $G \to GL_d(\mathbb{R})$ and $G \to GL_{d'}(\mathbb{R})$ (and hence a representation $G \times G' \to GL_{d+d'}(\mathbb{R})$), we obtain maps of Kan complexes $BG \to B\mathrm{Top}(d)^{\otimes}_{\langle 1\rangle}$, $BG' \to B\mathrm{Top}(d')^{\otimes}_{\langle 1\rangle}$ and $BG \times BG' \to B\mathrm{Top}(d+d')^{\otimes}_{\langle 1\rangle}$, inducing a bifunctor of quasi-operads

$$\theta_{G,G'} : \mathbb{E}_d^{G,\otimes} \times \mathbb{E}_{d'}^{G',\otimes} \to \mathbb{E}_{d+d'}^{G\times G',\otimes}$$

(In the special case where G and G' are both contractible topological groups, the bifunctor $\theta_{G,G'}$ is weakly equivalent to the functor ρ which we described below (3.2)). To see that $\theta_{G,G'}$ exhibits $\mathbb{E}_{d+d'}^{G\times G',\otimes}$ as the tensor product of $\mathbb{E}_d^{G,\otimes}$ and $\mathbb{E}_{d'}^{G',\otimes}$, it suffices to observe that the construction $B \mapsto \mathbb{E}_B^{\otimes}$ (and consequently $BG \mapsto \mathbb{E}_d^{G,\otimes}$) is functorial and carries homotopy colimits of Kan complexes over $B\mathrm{Top}(d)$ to homotopy colimits of quasi-operads: thus using e.g. a hypercover argument, it is enough to deduce the claim in the case of contractible G and G'. In other words, it is only necessary show that the bifunctor $\rho : \mathbb{E}_d^{\otimes} \times \mathbb{E}_{d'}^{\otimes} \to \mathbb{E}_{d+d'}^{\otimes}$ exhibits $\mathbb{E}_{d+d'}^{\otimes}$ as the tensor product of $\mathbb{E}_d^{\otimes}$ and $\mathbb{E}_{d'}^{\otimes}$ – but this is precisely the statement of Theorem 3.2.1. For future reference we state this as a theorem:

Theorem 3.2.3. *Let d, d' be non-negative integers, and let G, G' be groups with dilation representations in $GL_d(\mathbb{R})$, $GL_{d'}(\mathbb{R})$ respectively. Then the bifunctor $\theta_{G,G'} : \mathbb{E}_d^{G,\otimes} \times \mathbb{E}_{d'}^{G',\otimes} \to \mathbb{E}_{d+d'}^{G\times G',\otimes}$ exhibits $\mathbb{E}_{d+d'}^{G\times G',\otimes}$ as a tensor product of $\mathbb{E}_d^{G,\otimes}$ and $\mathbb{E}_{d'}^{G',\otimes}$.*

3.3 Complete Segal Dendroidal Spaces

In Section 2.5 we saw a close connection between operads and the category of trees. This connection can be exploited to produce another model of ∞-operads, the so-called *complete Segal dendroidal spaces.*

3.3.1 Dendroidal Objects

We recall that there is a fully faithful inclusion functor u from the ordinal indexing category Δ to Ω defined by sending the object $[n]$ to the linear tree with n vertices and $(n+1)$ edges – in this light, we can regard Ω as a generalisation of Δ. We can likewise investigate a generalisation of the simplicial sets: the category, $d\mathrm{Set}$ of **dendroidal sets** is the category of presheaves on Ω. Via the inclusion functor u, we see that there are natural adjoint functors

$$sSet \underset{u_*}{\overset{u_!}{\rightrightarrows}} \!\!\!\xleftarrow{u^*}\! dSet$$

Identifying the root tree η with its Yoneda embedding in $d\mathrm{Set}$, we in fact find that there is a canonical isomorphism of categories

$$d\mathrm{Set}_{/\eta} \simeq s\mathrm{Set}$$

so that u^* is just the forgetful functor. This relationship between simplicial sets and dendroidal sets is more than a little reminiscent of the relationship between operads and categories discussed in Remark 2.1.9. Moreover, the natural extension of the functor $\Omega \ni T \mapsto \Omega(T) \in \mathrm{Op}$ to a functor $\tau : d\mathrm{Set} \to \mathrm{Op}$ admits a right adjoint – the **dendroidal nerve functor**, $N_d : Op \to d\mathrm{Set}$, defined by

$$N_d\mathcal{O}(T) = \mathrm{Hom}_{\mathrm{Op}}(\Omega(T), \mathcal{O})$$

for an operad $\mathcal{O}$ in sets, and a tree T. The dendroidal nerve shows that "dendroidal sets generalise simplicial sets" in the same way that "operads generalise categories" in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Op} \arrow[r, "N_d"] & d\mathrm{Set} \\ \mathrm{Cat} \arrow[u, "j_!"] \arrow[r, "N"] & s\mathrm{Set} \arrow[u, "u_!"'] \end{tikzcd}$$

where $j_!$ is the functor discussed in Remark 2.1.9.

We can also consider dendroidal objects in the category of spaces, or dendroidal spaces. These presheaves again generalise the category of simplicial spaces, and we can find a suitable generalisation of the dendroidal nerve functor simply by giving each of the operads $\Omega(T)$ the structure of an operad in spaces in the obvious way.

We observe also that the category of dendroidal sets can be equipped with a monoidal structure which is compatible with both the Boardman-Vogt tensor product of operads and the Cartesian product of simplicial sets. We first define this tensor product on elementary objects in the category of dendroidal sets (i.e. trees) and then extend it to all dendroidal sets. We define $- \otimes - : \Omega \times \Omega \to d\mathrm{Set}$ by

$$(T, S) \mapsto N_d\left(\Omega(T) \star \Omega(S)\right)$$

where $\star$ is the Boardman-Vogt tensor product of strict operads. By taking left Kan extensions with respect to both variables, this gives a tensor product $- \otimes - : d\mathrm{Set} \times d\mathrm{Set} \to d\mathrm{Set}$. We list here some of the properties of this operation:

- The tensor product is symmetric, i.e. $X \otimes Y \simeq Y \otimes X$
- $\mathrm{colim}_i(X \otimes Y_i) \simeq X \otimes (\mathrm{colim_i} Y_i)$
- $\mathrm{colim}_i(X_i \otimes Y) \simeq (\mathrm{colim}_i X_i) \otimes Y$
- For any simplicial sets X, Y, we have $u_!X \otimes u_!Y \simeq u_!(X \times Y)$
- For dendroidal sets X and Y, we have $\tau(X \otimes Y) \simeq \tau(X) \star \tau(Y)$

A proof of the above properties can be found in [HM18, Proposition 4.2] (in fact, the property that the tensor product commutes with colimits in both variables is actually one of the properties defining this extension.) As with the Boardman-Vogt tensor product of strict operads, this tensor product is not necessarily homotopically well-behaved.

3.3.2 Model Structures on Dendroidal Sets and Dendroidal Spaces

Before we can discuss ∞-operads through the prism of dendroidal spaces, we first need to produce suitable model structures for our various categories of dendroidal objects. To do this, it is necessary to say a few words about boundaries and horns of trees (which play a role similar to the boundaries and horns of the elementary simplicial sets $\Delta[n]$).

Given a tree T with a vertex v attached to a leaf of T, we define the leaf face $\partial_v T$ as the tree obtained by pruning away the vertex v and all leaves attached to v. If a tree T has an unique inner vertex connected to its root, then we can also define its root face $\partial_r T$, which is the tree obtained by pruning away that inner vertex and all edges lying above it. Together, we refer to the leaf faces and the root face (if it exists) of a tree T as the **external faces** of T. On the other hand, given an internal edge e in the tree T, we can define the internal

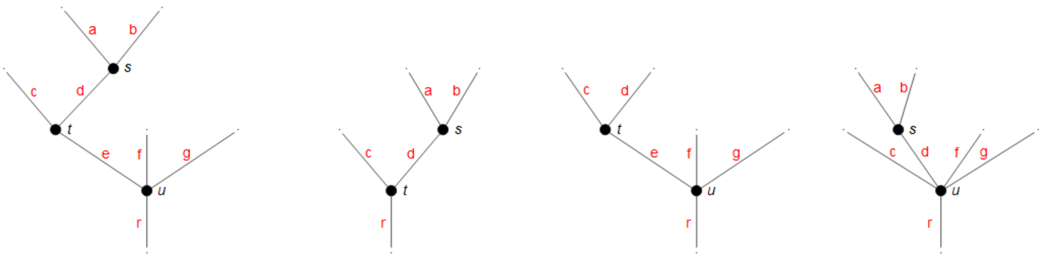

Figure 4: A tree T, together with its root face $\partial_r T$, a leaf face $\partial_s T$ and an internal face $\partial_e T$

face $\partial_e T$ as the tree obtained by contracting the edge e and identifying the vertices at either end of e. Each of these face maps $\partial_x T \hookrightarrow T$ gives rise to a map on the associated operads $\Omega(\partial_x T) \to \Omega(T)$ – we define $\partial\Omega(T)$ to be the union of all these faces, and we note that there is a boundary inclusion $\partial\Omega(T) \hookrightarrow \Omega(T)$. In Figure 4, we give an example of each of these faces.

There is a corresponding **boundary inclusion** for dendroidal sets associated to a tree T as well: if $\Omega[T]$ is the Yoneda embedding of T in the category of dendroidal sets, and $\partial\Omega[T]$ denotes the colimit over $\Omega[\partial_x T]$ for all faces $\partial_x T$ of T, then there is a natural boundary inclusion $\partial\Omega[T] \hookrightarrow \Omega[T]$ of dendroidal sets.

We can also define horns on a tree T as well. Let x denote either an external vertex or an inner edge of a tree T, then the horn $\Lambda^x[T]$ is the subobject of the dendroidal set $\Omega[T]$ which is the union of all faces except $\partial_x T$. In the case where x is an inner edge, we say that $\Lambda^x[T]$ is an inner horn.

We define the class of **normal monomorphisms** on dendroidal sets to be the closure of these boundary inclusions under transfinite composition, pushouts and retracts. It can be shown that a morphism $u : X \to Y$ of dendroidal sets is a normal monomorphism if and only if, for each tree T, the automorphism group $\mathrm{Aut}(T)$ acts freely on $Y(T) \setminus u(X(T))$.

We can also define the class of **inner anodyne extensions** on dendroidal sets to be the closure of the inner horn inclusions $\Lambda^e[T] \hookrightarrow \Omega[T]$ under transfinite composition, pushouts and retracts. We say a map of dendroidal sets $X \to Y$ is an inner fibration if for all trees T and all internal edges e in T, the dashed lift exists in the diagram

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Omega[T] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

A fibrant dendroidal set X is said to be a **dendroidal inner Kan complex** or **dendroidal ∞-operad**. Unsurprisingly, the reason we flag these classes of morphisms in $d\mathrm{Set}$ is that they give rise to a model structure:

Theorem 3.3.1. *[CM11] The category $d\mathrm{Set}$ carries a cofibrantly generated model structure – the **model category structure for dendroidal ∞-operads** – whose cofibrations are the normal monomorphisms and whose fibrant objects are the dendroidal ∞-operads. The induced model structure on $d\mathrm{Set}_{/\eta} = s\mathrm{Set}$ coincides with the Joyal model structure on $s\mathrm{Set}$.*

The set of generating cofibrations is the collection of all boundary inclusions $\partial\Omega[T] \hookrightarrow \Omega[T]$.

Definition 3.3.2. A face map $S \to T$ is said to be a **subtree** of T if $S \to T$ is a composition of external face maps, i.e. S is obtained from T by successively pruning leaves or by removing the root vertex if the root vertex only has one internal edge attached to it. We define the **Segal core** of a tree T with at least one vertex to be the dendroidal set

$$T_{\mathrm{Seg}} = \mathrm{colim}_{C_n \to T}\Omega[C_n]$$

where the colimit is taken over all subtrees of T which are corollas $C_n \to T$. By convention, we declare that $\eta_{\mathrm{Seg}} = \Omega[\eta]$. We also define the **external boundary of a tree** T as the dendroidal set

$$\partial^{\mathrm{ext}}T = \mathrm{colim}_{S \to T}\Omega[S]$$

where the colimit runs over all external faces $S \to T$ (i.e. all those faces obtained by pruning away external vertices).

It is evident that for any tree T with at least 2 vertices, and any internal edge e, there are inclusions of presheaves:

$$T_{\text{Seg}} \to \partial^{\text{ext}} T \to \Lambda^e[T] \to \Omega[T]$$

Moreover, it can be shown that the map $T_{\text{Seg}} \to \Omega[T]$ is inner anodyne. This last fact can be used to show the following important result:

Theorem 3.3.3. *A dendroidal set X is the dendroidal nerve of a coloured operad $\mathcal{P}$ in sets if and only if, for any tree T, the map*

$$X(T) = \text{Hom}_{d\text{Set}}(\Omega[T], X) \to \text{Hom}_{d\text{Set}}(T_{\text{Seg}}, X)$$

is an equivalence.

In other words, any coloured operad $\mathcal{P}$ satisfies the property that there are equivalences

$$N_d\mathcal{P}(T) \simeq \prod_{v \in \text{vert}(T)} N_d\mathcal{P}(C_{|v|})$$

for any tree T, where the product is taken over all the vertices of T and $|v|$ is the number of incoming edges of a vertex v. In particular, if $\mathcal{P}$ is a plain operad then $N_d\mathcal{P}(C_n) = \mathcal{P}(n)$, so that the above formula tells us that a plain operad satisfies:

$$N_d\mathcal{P}(T) \simeq \prod_{v \in \text{vert}(T)} \mathcal{P}(|v|)$$

Another special case of this theorem is when we view a simplicial set X as a dendroidal set via the functor $u_!$, and we consider the linear tree $[n]$ with n vertices: in that scenario, the statement of the theorem devolves to the original Segal condition for simplicial sets, i.e. X is the nerve of a category if and only if for each n, there is an equivalence

$$X_n = (u_!X)([n]) \simeq \text{Hom}_{d\text{Set}}([n]_{\text{Seg}}, u_!X) = X_1 \times_{X_0} \ldots \times_{X_0} X_1$$

This suggests that in the theory of dendroidal spaces, the Segal core T_{Seg} is a suitable generalisation of the simplicial spaces $G(n) = \cup_{1 \le i \le n} (\alpha_i^n)^* F(1) = [n]_{\text{Seg}}$ which we introduced when discussing the Segal model structure on simplicial spaces (see Theorem 1.4.12 and the preceding discussion for a reminder of the notation).

By identifying spaces and simplicial sets, we can consider the category of dendroidal spaces as the category of simplicial objects in dendroidal sets, written $\mathcal{P}(\Omega)$ (in [CM13] this category is denoted *sd*Set). It can be shown that Ω is an example of a *generalised Reedy category* in the sense of [BM11], and by analogy with simplicial spaces, we can endow the category of dendroidal spaces with the so-called **generalised Reedy model structure**:

Theorem 3.3.4. *[CM13, Proposition 5.2] The category $\mathcal{P}(\Omega)$ admits a cofibrantly generated proper model structure whose*

- *weak equivalences are the termwise simplicial weak homotopy equivalences (i.e. $X \to Y$ is a weak equivalence of dendroidal spaces if and only if $X(T) \to Y(T)$ is a weak equivalence of simplicial sets for each tree T);*
- *cofibrations are normal monomorphisms;*
- *fibrations are those maps $X \to Y$ such that for all trees T, we have Kan fibrations*

$$X^{\Omega[T]} \to X^{\partial\Omega[T]} \times_{Y^{\partial\Omega[T]}} Y^{\Omega[T]}$$

The generating cofibrations for this model structure are of the form

$$\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \to \Delta[n] \times \Omega[T]$$

while the generating trivial cofibrations are those maps

$$\Lambda[n,k] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \to \Delta[n] \times \Omega[T]$$

for all $n \ge 0$, $0 < k < n$ and all trees T.

Further extending the analogy between simplicial spaces and dendroidal spaces, we arrive at the following definition:

Definition 3.3.5. A dendroidal space X is **Segal** if it is fibrant with respect to the generalised Reedy model structure on dendroidal spaces and it satisfies the property that for all trees T, there is a weak equivalence of simplicial sets

$$\mathrm{Map}(\Omega[T], X) \xrightarrow{\sim} \mathrm{Map}(T_{\mathrm{Seg}}, X)$$

The Segal model structure on simplicial spaces (Theorem 1.4.12) was obtained by a Bousfield localisation of the Reedy model structure on simplicial spaces with respect to the maps $[n]_{\mathrm{Seg}} = G(n) \hookrightarrow F(n) = \Omega[n]$. The same approach on the category of dendroidal spaces yields the **Segal model structure on** $\mathcal{P}(\Omega)$: namely, by taking the Bousfield localisation with respect to all maps $T_{\mathrm{Seg}} \to \Omega[T]$, we obtain a model structure on $\mathcal{P}(\Omega)$ whose:

- cofibrations are levelwise monomorphisms of dendroidal spaces;
- fibrant objects are Segal dendroidal spaces;
- weak equivalences are those maps of dendroidal spaces $f : Y \to Z$ such that for each Segal dendroidal space X, the induced map
$$\mathrm{Map}(f, X) : \mathrm{Map}(Z, X) \to \mathrm{Map}(Y, X)$$
is a weak equivalence of simplicial sets;
- weak equivalences between Segal dendroidal spaces are weak equivalences in the Reedy generalised model structure on $\mathcal{P}(\Omega)$.

Combining the definition of the weak equivalences in the generalised Reedy model structure on $\mathcal{P}(\Omega)$ with the statement of Theorem 3.3.3, we see that a map between Segal dendroidal spaces $f : X \to Y$ is a weak equivalence if and only if f induces weak homotopy equivalences of simplicial sets

$$Y(C_n) \xrightarrow{\sim} X(C_n) \text{ and } X(\eta) \xrightarrow{\sim} Y(\eta)$$

for all $n \geq 0$.

Definition 3.3.6. A Segal dendroidal space X is said to be **complete** if for each tree T, there is a weak equivalence

$$\mathrm{Map}(\Omega[T], X) \xrightarrow{\sim} \mathrm{Map}(u_! E \otimes \Omega[T], X)$$

where we recall that E was the discrete simplicial space (i.e. simplicial set) defined as the nerve of the groupoid $I[1]$ with two objects and one non-trivial isomorphism.

We note that a Segal dendroidal space X is complete if and only if the underlying simplicial space $u^* X$ is a complete Segal space. Again, this leads to a new model structure on the category of dendroidal spaces: the **complete Segal model structure on** $\mathcal{P}(\Omega)$ arises from the Segal model structure on $\mathcal{P}(\Omega)$ by taking left Bousfield localisations with respect to the maps $\Omega[T] \otimes u_! E \to \Omega[T]$ for all trees T.

Remark 3.3.7. We will write $\mathcal{P}_{\mathrm{Seg}}(\Omega)$ to denote the category of presheaves on Ω given the Segal model structure, and $\mathcal{P}_{CS}(\Omega)$ for the category of presheaves on Ω given the complete Segal model structure. We observe that $\mathcal{P}_{\mathrm{Seg}}(\Omega) \to \mathcal{P}_{CS}(\Omega)$ is an accessible localisation.

3.4 Complete Segal Operads

The third model we consider is due to Barwick [Bar18] and goes by the name complete Segal operads. Barwick, in his paper, defines quite a broad framework of so-called *operator categories* (not to be confused with the notion of the category of operators which we introduced previously). For our purposes however, such generality will not be needed and it will suffice to restrict our attention to the complete Segal operads on Fin.

Definition 3.4.1. We can define a category of **Fin-sequences**, denoted $\Delta_{\mathbb{F}}$ (following notation used by [CHH18]) whose objects are pairs $([n], S)$, where $[n]$ is an object of the simplicial indexing category Δ and S is a functor $[n] \to \mathsf{Fin}$ – in other words, we can and frequently do identify a pair $([n], S)$ with a string of maps of finite sets of the form $(S_0 \to S_1 \to \ldots \to S_n)$. A morphism between two Fin-sequences, $([n], S)$ and $([m], S')$ is the data of a map $u : [n] \to [m]$ in Δ and a natural transformation $\eta : S \to S' \circ u$ which satisfies the following properties:

- for each $0 \leq i \leq n$, the map $\eta_i : S_i \to S'_{u(i)}$ is an injection;
- for any $0 \leq i \leq l \leq n$, the commutative diagram

$$\begin{tikzcd} S_i \arrow[r, "\eta_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S'_{u(i)} \arrow[d] \\ S_l \arrow[r, "\eta_l"] & S'_{u(l)} \end{tikzcd}$$

is Cartesian

We note that there is an obvious inclusion of categories $\nu : \Delta \hookrightarrow \Delta_{\mathbb{F}}$ given on objects by

$$\nu([n]) = ([n], \langle 1 \rangle) = \underbrace{(\langle 1 \rangle \to \ldots \to \langle 1 \rangle)}_{n+1 \text{ times}}$$

A morphism $(u, \eta) : ([n], S) \to ([m], S')$ is said to be **inert** if the map u is inert; we say (u, η) is **active** if u is active in Δ and, for all i, the map $\eta_i : S_i \to S'_{u(i)}$ is an isomorphism. The collection of inert and active morphisms defines a factorisation system on $\Delta_{\mathbb{F}}$.

In this category of Fin-sequences, we can also formulate analogous versions of the Segal condition and the complete Segal condition. Given a length-1 sequence $S_0 \to S_1$ and an element $i \in S_1$, let us write $S_{0,i}$ for the fibre of the map $S_0 \to S_1$ over i. Then for each $i \in S_1$ we obtain inclusion morphisms in $\Delta_{\mathbb{F}}$ of the form $(S_{0,i} \to \{i\}) \hookrightarrow (S_0 \to S_1)$. Gluing these inclusions together, we get a map (here $S_1 = \{i_1, \ldots, i_n\}$)

$$(S_{0,i_1} \to \{i_1\}) \underset{S_1}{\cup} \ldots \underset{S_1}{\cup} (S_{0,i_n} \to \{i_n\}) \longrightarrow (S_0 \to S_1) \tag{3.3}$$

A presheaf $\mathcal{X}$ on $\Delta_{\mathbb{F}}$ valued in spaces is said to satisfy the **product Segal condition** if it is local with respect to these maps, i.e., if the induced maps

$$\begin{aligned} \mathrm{Map}\left((S_0 \to S_1), \mathcal{X}\right) \quad &\to \mathrm{Map}\left((S_{0,i_1} \to \{i_1\}) \underset{S_1}{\cup} \ldots \underset{S_i}{\cup} (S_{0,i_n} \to \{i_n\}), \mathcal{X}\right) \\ \iff \mathcal{X}(S_0 \to S_1) \quad &\to \textstyle\prod_{i_j \in S_1} \mathcal{X}(S_{0,j} \to \{i_j\}) \end{aligned}$$

are weak homotopy equivalences.

Given any sequence of length at least 2, say $(S_0 \to S_1 \to \ldots \to S_n)$, the inclusions

$$(S_i \to S_{i+1}) \hookrightarrow (S_0 \to S_1 \to \ldots S_n)$$

also glue to give a map

$$(S_0 \to S_1) \underset{S_1}{\cup} (S_1 \to S_2) \underset{S_2}{\cup} \ldots \underset{S_{n-1}}{\cup} (S_{n-1} \to S_n) \longrightarrow (S_0 \to S_1 \to \ldots \to S_n) \tag{3.4}$$

We say a presheaf $\mathcal{X}$ on $\Delta_{\mathbb{F}}$ satisfies the **pullback Segal condition** if $\mathcal{X}$ satisfies the property that the induced maps

$$\begin{aligned} \mathrm{Map}\left((S_0 \to S_1 \to \ldots \to S_n), \mathcal{X}\right) \quad &\to \mathrm{Map}\left((S_0 \to S_1) \underset{S_1}{\cup} \ldots \underset{S_{n-1}}{\cup} (S_{n-1} \to S_n), \mathcal{X}\right) \\ \iff \mathcal{X}(S_0 \to S_1 \to \ldots \to S_n) \quad &\to \mathcal{X}(S_0 \to S_1) \times_{\mathcal{X}(S_1)} \ldots \times_{\mathcal{X}(S_{n-1})} \mathcal{X}(S_{n-1} \to S_n) \end{aligned}$$

are weak homotopy equivalences. Finally, we can formulate a completeness condition for those presheaves $\mathcal{X}$ which already satisfy both the Segal product and Segal pullback conditions. Since we have a map $\nu : \Delta \to \Delta_{\mathbb{F}}$, the presheaf $\mathcal{X}$ gives rise to a simplicial space $\nu^*\mathcal{X}$, and this is a Segal space (this can be seen by applying the Segal conditions for the finite sequences $\nu([n])$ for each $n \geq 1$). We say $\mathcal{X}$ satisfies the **completeness condition** if $\nu^*\mathcal{X}$ is a complete Segal space, i.e. we have weak homotopy equivalences

$$(\nu^*\mathcal{X})_0 = \mathrm{Map}(\nu^*\mathcal{X}, \Delta[0]) \xrightarrow{\simeq} \mathrm{Map}(\nu^*\mathcal{X}, E) \simeq (\nu^*\mathcal{X})_1^{he}$$

Definition 3.4.2. Let $\mathcal{X}$ be a presheaf on $\Delta_{\mathbb{F}}$ taking values in Kan complexes. We say $\mathcal{X}$ is a **Segal Fin-operad** (or simply a Segal operad) if it satisfies both the product and pullback Segal conditions. We say $\mathcal{X}$ is a **complete Segal Fin-operad** if $\mathcal{X}$ is a Segal operad and $\mathcal{X}$ satisfies the completeness condition.

Remark 3.4.3. By the straightening and unstraightening constructions of [Lur09a, Section 3.2], we can identify a presheaf $\mathcal{X}$ on $\Delta_{\mathbb{F}}$ valued in Kan complexes with a left fibration of simplicial sets $q : \widetilde{\mathcal{X}} \to N\Delta_{\mathbb{F}}$. From this perspective, we can (at least intuitively – these statements are given in a more precise sense in [Bar18, Proposition 10.6]) examine some of the similarities between quasi-operads in the sense of Definition 3.1.1 and the Segal operads we introduced above. For example, the product Segal condition tells us that we can identify a point in the fibre $\mathcal{X}(I) = q^{-1}(I)$ with a tuple of objects $(x_j)_{j\in I} \in \prod_{j\in I} q^{-1}(\{j\})$, which corresponds to the third condition of the definition of a quasi-operad. Likewise, given a partial map of finite sets $f : I \to J$ the pullback and product Segal conditions tell us that we have weak equivalences

$$\begin{aligned}\mathcal{X}(I) \times_{\mathcal{X}(I_o)} \mathcal{X}(I_o \xrightarrow{f_o} J) &\simeq \mathcal{X}(I) \times_{\prod_{j\in J} \mathcal{X}(I_j)} \textstyle\prod_{j\in J} \mathcal{X}(I_j \to \{j\}) \\ &\to \mathcal{X}(I) \times \textstyle\prod_{j\in J} \mathcal{X}(\{j\}) \\ &\simeq \mathcal{X}(I) \times \mathcal{X}(J)\end{aligned}$$

where we recall the notation that the partial map $f : I \to J$ is defined as a map of sets $f_o : I_o \to J$ for some subset $I_o \subseteq I$. Given $x \in \mathcal{X}(I)$, $y \in \mathcal{X}(J)$ and $y_j \in \mathcal{X}(\{j\})$ for each $j \in J$, the map constructed here corresponds to the map

$$\mathrm{map}^{f}_{\mathcal{O}^{\otimes}}(x, y) \xrightarrow{\simeq} \prod_{j\in J} \mathrm{map}^{\rho^j \circ f}_{\mathcal{O}^{\otimes}}(x, y_j)$$

which appears in second condition in the definition of a quasi-operad $\mathcal{O}^{\otimes}$.

If we equip the category of space-valued presheaves on $\Delta_{\mathbb{F}}$ with the injective model structure (which category we will denote by $\mathcal{P}(\Delta_{\mathbb{F}})$), then by means of left Bousfield localisation with respect to the maps (3.3) and (3.4), we obtain a model structure of presheaves on $\Delta_{\mathbb{F}}$ whose fibrant objects are the Segal operads. We can further localise with respect to the completeness condition to arrive at the following model structure:

Theorem 3.4.4. *[Bar18, Proposition 2.17] The category $\mathcal{P}(\Delta_{\mathbb{F}})$ admits a left proper tractable model structure – the **operadic model structure** – whose:*

- *cofibrations are the levelwise monomorphisms $\mathcal{X} \to \mathcal{Y}$;*
- *fibrant objects are the complete Segal operads;*
- *weak equivalences are those maps $\mathcal{Y} \to \mathcal{Z}$ of presheaves such that for each fibrant object $\mathcal{X}$, the induced map*

$$\mathrm{Map}(\mathcal{Z}, \mathcal{X}) \to \mathrm{Map}(\mathcal{Y}, \mathcal{X})$$

  is a weak homotopy equivalence.

Remark 3.4.5. Writing $\mathcal{P}_{\mathrm{Seg}}(\Delta_{\mathbb{F}})$ for the category of presheaves on Fin-sequences localised with respect to the Segal properties, and $\mathcal{P}_{CS}(\Delta_{\mathbb{F}})$ for the category of presheaves on Fin-sequences further localised with respect to the completeness condition, then as in the case of Remark 3.3.7 we see immediately that the passage $\mathcal{P}_{\mathrm{Seg}}(\Delta_{\mathbb{F}}) \to \mathcal{P}_{CS}(\Delta_{\mathbb{F}})$ is an accessible localisation.

3.5 Constructions relating the Models

3.5.1 Quasi-Operads and Complete Segal Operads

In this section, we briefly discuss some of the constructions given to relate the various models of ∞-operad which we have described above. First, we consider a construction given in [Bar18] to pass from Segal operads to quasi-operads. To make sense of this construction, we first need to introduce some notation:

- Given a map $f : I \to J$ in Fin_*, let us denote by $[f]$ the pullback of the diagram

$$\begin{tikzcd} & J \arrow[d, "\mathrm{id}"] \\ I_o \arrow[r, "f_o"] & J \end{tikzcd}$$

- For a string of maps in Fin_* of the form

$$S_0 \xrightarrow{\sigma^1} S_1 \xrightarrow{\sigma^2} \ldots \xrightarrow{\sigma^n} S_n \tag{3.5}$$

  and $0 \leq i \leq j \leq n$, we write $\sigma^{i,j}$ to mean the composite $\sigma^j \circ \ldots \circ \sigma^i : S_{i-1} \to S_j$. In particular, we will just write $\sigma^j = \sigma^{j,j}$.

Given a string of maps in Fin_* of the form in 3.5, we notice that for any fixed $0 \leq i \leq j \leq n$, there is an induced map of strings of finite sets

$$F_\sigma(i,j) := \left([\sigma^{1,j}] \to [\sigma^{2,j}] \to \ldots \to [\sigma^{i,j}]\right)$$

Since the construction here may not be entirely obvious, we include a hands-on example:

Example 3.5.1. Consider a string of partial maps

$$\sigma = \left(\langle 4\rangle \xrightarrow{\sigma^1} \langle 2\rangle \xrightarrow{\sigma^2} \langle 3\rangle \xrightarrow{\sigma^3} \langle 2\rangle\right)$$

with

$$\sigma^1(i) = \begin{cases} 1 & i = 1,4 \\ 2 & i = 3 \end{cases} \quad ; \quad \sigma^2(i) = \begin{cases} 1 & i = 2 \\ 2 & i = 1 \end{cases} \quad ; \quad \sigma^3(i) = \begin{cases} 2 & i = 2 \end{cases}$$

Suppose we wish to study both $F_\sigma(2,2)$ and $F_\sigma(2,3)$. Working through the compositions, we see that

$$\sigma^{1,2}(i) = \begin{cases} 2 & i = 1,4 \\ 1 & i = 3 \end{cases} \quad ; \quad \sigma^{1,3}(i) = \begin{cases} 2 & i = 1,4 \end{cases} \quad ; \quad \sigma^{2,3}(i) = \begin{cases} 2 & i = 1 \end{cases}$$

Hence, it follows that

$$\left[\sigma^{1,2}\right] = \{(1,2),(4,2),(3,1)\}, \quad \left[\sigma^{1,3}\right] = \{(1,2),(4,2)\}, \quad \left[\sigma^{2,3}\right] = \{(1,2)\}, \quad \left[\sigma^{2}\right] = \{(1,2),(2,1)\}$$

which in turn tells us that

$$F_\sigma(2,2) = (\{(1,2),(4,2),(3,1)\} \to \{(1,2),(2,1)\})$$

where a pair (x,y) gets sent to the element $(\sigma^1(x), y)$ in the codomain, i.e. $(1,2) \mapsto (1,2)$, $(4,2) \mapsto (1,2)$ and $(3,1) \mapsto (2,1)$. The map is evident in

$$F_\sigma(2,3) = (\{(1,2),(4,2)\} \to \{(1,2)\})$$

Given an element $[m]$ in the simplicial indexing category, let us denote by $\mathscr{O}(m)$ the associated twisted arrow category – that is, the poset whose objects are pairs (i,j) where $0 \leq i \leq j \leq m$, and for two such pairs, (i,j) and (i',j'), the partial order relation is defined by $(i,j) \leq (i',j')$ if and only if $i' \leq i \leq j \leq j'$. For any length-n sequence σ of maps in Fin_*, we define

$$\Delta_{\mathbb{F}}^\sigma = \operatorname{colim}\left(\mathscr{O}(n) \xrightarrow{F_\sigma(i,j)} \Delta_{\mathbb{F}} \to \mathcal{P}(\Delta_{\mathbb{F}})\right)$$

For each n, the construction $\sigma \mapsto \Delta_{\mathbb{F}}^\sigma$ defines a functor $\mathrm{Fun}([n], \mathsf{Fin}_*) \to \mathcal{P}(\Delta_{\mathbb{F}})$. If $\mathcal{X}$ is any presheaf on $\Delta_{\mathbb{F}}$, then by right Kan extension, we can define a simplicial set

$$\mathcal{X}(\Delta_{\mathbb{F}}^\sigma) = \lim_{(i,j)\in\mathscr{O}(n)} \mathcal{X}(F_\sigma(i,j))$$

It follows that the assignment $\mathrm{Fun}([n], \mathsf{Fin}_*) \ni \sigma \mapsto \mathcal{X}(\Delta_{\mathbb{F}}^\sigma) \in s\mathrm{Set}$ is functorial.

With these preliminaries in place, we can finally state our construction for passing from complete Segal operads to quasi-operads.

Definition 3.5.2. Given a presheaf $\mathcal{X}$ on $\Delta_{\mathbb{F}}$, we can define a simplicial set $P^{\otimes}(\mathcal{X})$ whose set of n-simplices consists of pairs (σ, x) where $\sigma : [n] \to \mathsf{Fin}_*$ and $x \in \mathcal{X}(\Delta_{\mathbb{F}}^\sigma)_0$. We note that this simplicial set is equipped with a natural map to $N\mathsf{Fin}_*$, given by $(\sigma, x) \mapsto \sigma$.

In [Bar18, Proposition 10.6], it is shown that if $\mathcal{X}$ is a complete Segal operad, then $P^{\otimes}(\mathcal{X})$ is a quasi-operad. The proof of this claim runs essentially along the lines of our earlier remarks (3.4.3) on the connection between complete Segal operads and quasi-operads via the straightening and unstraightening constructions.

Remark 3.5.3. Barwick's construction of the functor $P^{\otimes}(\mathcal{X})$ stems from a slightly more general object which is built using the notion of the *nerve of a category $\mathcal{C}$ relative to a functor $f : \mathcal{C} \to s\mathrm{Set}$* which appears in [Lur09a, Definition 3.2.5.2]. We recall the construction here.

Starting from the functor $\mathrm{Fun}([n], \mathsf{Fin}_*) \ni \sigma \mapsto \mathcal{X}(\Delta^{\sigma}_{\mathbb{F}}) \in s\mathrm{Set}$, we can define a simplicial set $\widetilde{P}^{\otimes}(\mathcal{X})_n$, a k-simplex of which is the data of

- an element $\Sigma \in N\mathrm{Fun}([m], \mathsf{Fin}_*)_k$;
- for every subposet $J \subseteq [k]$ with maximal element $\max J$, a map of simplicial sets

$$\tau(J) : \Delta^J \to \mathcal{X}\left(\Delta^{\Sigma(\max J)}_{\mathbb{F}}\right)$$

- compatibility between these maps in the sense that for every pair of subposets $J' \subseteq J \subseteq [k]$, the following diagram commutes

$$\begin{tikzcd} \Delta^{J'} \arrow[r, "\tau(J')"] \arrow[d] & \mathcal{X}\left(\Delta^{\Sigma(\max J')}_{\mathbb{F}}\right) \arrow[d] \\ \Delta^{J} \arrow[r, "\tau(J)"] & \mathcal{X}\left(\Delta^{\Sigma(\max J)}_{\mathbb{F}}\right) \end{tikzcd}$$

This construction is functorial in $[n]$, so it in fact yields a simplicial space, $\widetilde{P}^{\otimes}(\mathcal{X})$. Moreover, we note that the map $(\Sigma, \{\tau(J)\}_{J\subseteq[k]}) \mapsto \Sigma$ defines a map of simplicial spaces from $P^{\otimes}(\mathcal{X})$ to the classifying diagram of Fin_* (for a recollection on the concept of the classifying diagram, see Example 1.4.10).

It can be shown that if $\mathcal{X}$ is a complete Segal operad, then $\widetilde{P}^{\otimes}(\mathcal{X})$ is a complete Segal space. In fact, Barwick uses this observation, and the Quillen equivalence between complete Segal spaces and quasi-categories in [JT06], $X_{\bullet,*} \mapsto X_{\bullet,0}$, as an inspiration for the definition of the simplicial set $P^{\otimes}(\mathcal{X})$ from Definition 3.5.2, which, as we readily see, coincides with $\widetilde{P}^{\otimes}(\mathcal{X})_{\bullet,0}$.

3.5.2 Complete Segal Operads and Complete Segal Dendroidal Spaces

We discuss here some of the methods utilised in the paper [CHH18] to define a Quillen equivalence between complete Segal operads and complete Segal dendroidal spaces. In fact, in this paper, the authors demonstrate an equivalence between Segal operads and Segal dendroidal spaces and then use the fact that in both cases, passage to complete Segal objects in the respective presheaf categories is an accessible localisation. To exhibit the equivalence, [CHH18] introduce a subcategory $\Delta^1_{\mathbb{F}}$ of $\Delta_{\mathbb{F}}$ consisting of all those Fin-sequences which terminate in the set $\langle 1\rangle$. The intuition for this choice is obvious: we can identify a set $\langle n\rangle$ with a collection of n edges in a single stratum of a tree – in particular, the root of a tree should be a stratum with a single edge, corresponding to the set $\langle 1\rangle$. More precisely, we have the following construction...

Construction 3.5.4. Consider an object in $\Delta^1_{\mathbb{F}}$ of the form

$$([n], S) = \left(S_0 \xrightarrow{\sigma^1} S_1 \to \ldots \xrightarrow{\sigma^n} S_n = \langle 1\rangle\right)$$

To this string, we associate an open tree $\tau([n], S)$ as follows: the set of edges of $\tau([n], S)$ is $\sqcup_{0\leq i\leq n} S_i$ and the partial order relation $\leq$ defining the tree is determined as follows:

- if $0 \leq i \leq l \leq n$ and $y \in S_i$ and $z \in S_l$, then $y \leq z$ if and only if $\sigma^{i,l}(y) = z$ (where, as before, we write $\sigma^{i,l}$ to mean the composite $\sigma^l \circ \ldots \sigma^i : S_{i-1} \to S_l$).

The set of leaves of $\tau([n], S)$ is precisely the collection of minimal elements with respect to this partial order relation.

Describing the action of τ on general morphisms is somewhat challenging (for much the same reasons that it is challenging to describe morphisms in the category Ω in general) – however, in the case of inert morphisms in $\Delta^1_{\mathbb{F}}$, we can describe the action of the functor τ in a straightforward way: if $(u,\zeta) : ([n],S) \to \left([m],\widetilde{S}\right)$ is an inert morphism in $\Delta^1_{\mathbb{F}}$, then u must be of the form $i \mapsto u(0)+i$. Then, for each $0 \le i \le n$, the map $\zeta_i : S_i \to \widetilde{S}_{u(0)+i}$ is an injection, so there is an induced injective map between the sets of edges

$$\sqcup_{0\le i\le n} S_i \rightarrowtail \sqcup_{u(0)\le i\le m} \widetilde{S}_i \hookrightarrow \sqcup_{0\le i\le m}\widetilde{S}_i$$

and the condition that we have Cartesian commutative diagrams for all $0 \le i \le l \le n$

$$\begin{tikzcd} S_i \arrow[r, tail, "\zeta_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{S}_{u(0)+i} \arrow[d] \\ S_l \arrow[r, tail, "\zeta_l"] & \widetilde{S}_{u(0)+l} \end{tikzcd}$$

ensures that the partial order relations on the trees $\tau([n],S)$ and $\tau\left([m],\widetilde{S}\right)$ coincide on their "common edges" (i.e. those edges of $\tau\left([m],\widetilde{S}\right)$ which lie in the image of ζ).

To clarify the above discussion somewhat we provide some visual examples of how the functor τ acts.

Example 3.5.5. Consider the following pair of objects in $\Delta^1_{\mathbb{F}}$:

$$([2],S) = \left(\langle 4\rangle \xrightarrow{\sigma^1} \langle 2\rangle \xrightarrow{\sigma^2} \langle 1\rangle\right) \quad \text{and} \quad \left([3],\widetilde{S}\right) = \left(\langle 8\rangle \xrightarrow{\widetilde{\sigma}^1} \langle 6\rangle \xrightarrow{\widetilde{\sigma}^2} \langle 3\rangle \xrightarrow{\widetilde{\sigma}^3} \langle 1\rangle\right)$$

where

$$\sigma^1(i) = \begin{cases} 1 & i=1,2 \\ 2 & i=3,4 \end{cases} \quad ; \quad \widetilde{\sigma}^1(i) = \begin{cases} 1 & i=1,2 \\ 2 & i=3,4 \\ 5 & i=5,6 \\ 6 & i=7,8 \end{cases} \quad ; \quad \widetilde{\sigma}^2(i) = \begin{cases} 1 & i=1,2 \\ 2 & i=3,4 \\ 3 & i=5,6 \end{cases} \quad ;$$

and σ^2 and $\widetilde{\sigma}^3$ are the unique maps to the terminal object in Fin. We can define an obvious inert map

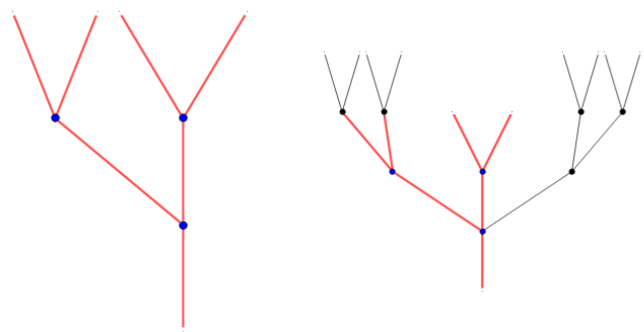

Figure 5: The open trees $\tau([2],S)$ (on the left) and $\tau\left([3],\widetilde{S}\right)$.

$(u,\zeta) : ([2],S) \to ([3],\widetilde{S})$ by $u(j) = 1+j$ and $\zeta_j(i) = i$ for each $0 \le j \le 2$. We have depicted the resulting trees in Figure 5, and the image of the map $\tau(u,\zeta)$ is depicted by the thick red edges in the subtree on the right.

Apart from the fact that there is a more natural way of associating trees to objects of $\Delta^1_{\mathbb{F}}$, there is a second advantage to working with this subcategory: since we are working with strings which terminate with the object $\langle 1\rangle$, the product Segal condition becomes vacuous.

In [CHH18], a zig-zag of Quillen equivalences is deduced

$$\mathcal{P}_{\mathrm{Seg}}(\Delta_{\mathbb{F}}) \underset{\sim}{\xrightarrow{\iota^*}} \mathcal{P}_{\mathrm{Seg}}(\Delta^1_{\mathbb{F}}) \underset{\sim}{\xleftarrow{\tau^*}} \mathcal{P}_{\mathrm{Seg}}(\Omega)$$

where $\iota : \Delta^1_{\mathbb{F}} \hookrightarrow \Delta_{\mathbb{F}}$ is the fully faithful subcategory inclusion. Proving that ι^* is a Quillen equivalence is *relatively* straightforward; proving that the map $\tau^* : \mathcal{P}(\Omega) \to \mathcal{P}(\Delta^1_{\mathbb{F}})$ restricts to a map of Segal objects is also fairly painless. The main work of the paper is based on deducing that τ^* is a Quillen equivalence. Since we will be drawing on many similar ideas for a proof of our own, it behoves us to outline some of the steps towards this deduction.

Definition 3.5.6. Let $\Delta^1_{\mathbb{F},\mathrm{int}}$ denote the subcategory of $\Delta^1_{\mathbb{F}}$ with the same objects, but whose morphisms are just the inert morphisms. We can also define $\Delta^{\mathrm{el}}_{\mathbb{F}}$ to be the subcategory of elementary objects in $\Delta^1_{\mathbb{F}}$, i.e. the full subcategory spanned by objects of the form $(\langle n \rangle \to \langle 1 \rangle)$.

Analogously, let Ω_{int} be the subcategory of Ω with the same objects, but whose morphisms are the inert morphisms of trees; let Ω^{el} be the subcategory spanned by the corollas C_n, $n \geq 1$ and the root η.

It is possible to rephrase the Segal conditions for presheaves on $\Delta^1_{\mathbb{F}}$ and Ω using the above subcategories (in what follows, all the presheaves have values in Kan complexes): we first do this for $\Delta^1_{\mathbb{F}}$.

Given an object $J \in \Delta^1_{\mathbb{F}}$, let $(\Delta^{\mathrm{el}}_{\mathbb{F}})_{/J}$ be the pullback category $(\Delta_{\mathbb{F},\mathrm{int}})_{/J} \times_{\Delta_{\mathbb{F},\mathrm{int}}} \Delta^{\mathrm{el}}_{\mathbb{F}}$. We can associate a presheaf

$$J_{\mathrm{Seg}} = \underset{E \in (\Delta^{\mathrm{el}}_{\mathbb{F}})_{/J}}{\mathrm{colim}} E$$

(we are being somewhat lax in our notation here by identifying E with its Yoneda embedding). We note that there is a natural map of presheaves $J_{\mathrm{Seg}} \to J$. A presheaf $\mathcal{X} \in \mathcal{P}(\Delta^1_{\mathbb{F}})$ satisfies the Segal condition if and only if the restriction $\mathcal{X}\mid_{\Delta_{\mathbb{F},\mathrm{int}}}$ is local with respect to each such map $J_{\mathrm{Seg}} \to J$.

We can take an analogous approach with presheaves on Ω – in fact, by constructing the pullback categories $\Omega^{\mathrm{el}}_{/T}$, and considering the colimits

$$S_{\mathrm{Seg}} = \underset{S \in \Omega^{\mathrm{el}}_{/T}}{\mathrm{colim}} S$$

we obtain the notion of Segal core which we encountered in Definition 3.3.2. As in the case of $\Delta^1_{\mathbb{F}}$, a presheaf $\mathcal{X} \in \mathcal{P}(\Omega)$ satisfies the Segal condition if and only if the restriction $\mathcal{X}\mid_{\Omega_{\mathrm{int}}}$ is local with respect to each map of presheaves $T_{\mathrm{Seg}} \to T$ (this is in fact just a restatement Definition 3.3.5).

Moreover, it is evident that the map τ restricts to an equivalence of categories $\Delta^{\mathrm{el}}_{\mathbb{F}} \to \Omega^{\mathrm{el}}$. Using a theorem of Lurie ([Lur09a, Lemma 4.3.2.15]), this implies that τ^* induces an equivalence:

$$\mathcal{P}_{\mathrm{Seg}}(\Omega_{\mathrm{int}}) \simeq \mathcal{P}_{\mathrm{Seg}}(\Delta^1_{\mathbb{F},\mathrm{int}})$$

If $e_{\Delta_{\mathbb{F}}} : \Delta^1_{\mathbb{F},\mathrm{int}} \hookrightarrow \Delta^1_{\mathbb{F}}$ and $e_\Omega : \Omega_{\mathrm{int}} \hookrightarrow \Omega$ denote the obvious subcategory inclusions, then we obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{Seg}}(\Omega) \arrow[r, "\tau^*"] \arrow[d, "e^*_\Omega"'] & \mathcal{P}_{\mathrm{Seg}}(\Delta_{\mathbb{F}}) \arrow[d, "e^*_{\Delta_{\mathbb{F}}}"] \\
\mathcal{P}_{\mathrm{Seg}}(\Omega_{\mathrm{int}}) \arrow[r, "\tau^*_{\mathrm{int}}", "\simeq"'] & \mathcal{P}_{\mathrm{Seg}}(\Delta_{\mathbb{F},\mathrm{int}})
\end{tikzcd}$$

It can be shown that both functors $e^*_\bullet$ admit left adjoints and that the respective adjunctions are monadic, so that we can use the following consequence of the Barr-Beck Theorem:

Lemma 3.5.7. *[Lur17, Corollary 4.7.3.16] Given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "U"] \arrow[dr, "G"'] & & \mathcal{C}' \arrow[dl, "G'"] \\
& \mathcal{D} &
\end{tikzcd}$$

such that

(a) the functors G and G' have left adjoints, F and F' respectively,

(b) the pairs $(F \dashv G)$ and $(F' \dashv G')$ are monadic,

(c) for all objects d in $\mathcal{D}$, the unit map $d \to GFd \simeq G'UFd$ induces an equivalence $\alpha_d : F'd \to UFd$ in $\mathcal{C}'$,

then U is an equivalence of ∞-categories.

This is applied to the diagram

$$\begin{tikzcd} \mathcal{P}_{\mathrm{Seg}}(\Omega) \arrow[rr, "\tau^*"] \arrow[dr, "e^*_\Omega"'] & & \mathcal{P}_{\mathrm{Seg}}(\Delta^1_{\mathbb{F}}) \arrow[dl, "(\tau^*_{\mathrm{int}})^{-1}\circ e^*_{\Delta_{\mathbb{F}}}"] \\ & \mathcal{P}_{\mathrm{Seg}}(\Omega_{\mathrm{int}}) & \end{tikzcd}$$

The key to being able to apply the lemma is in proving that the functor τ^* preserves Segal equivalences of the form $\partial^{\mathrm{ext}}T \xrightarrow{\sim} T$, for any tree T (where $\partial^{\mathrm{ext}}T$ is the exterior boundary of T, as defined in 3.3.2). Proving this relies on two somewhat technical concepts:

Definition 3.5.8. The **barycentric subdivision** of $\Delta[n]$ is the partially ordered set $sd(\Delta[n])$ of all ordered non-empty subsets $(i_1, i_2, \dots, i_k)$ of $\{0, \dots, n\}$. Given an object $([n], J) = (J_0 \to \dots J_n) \in \Delta^1_{\mathbb{F}}$ and a subposet $\mathcal{I} \subseteq sd(\Delta[n])$, we can define an object $J(\mathcal{I})$ in $\mathcal{P}(\Delta_{\mathbb{F}})$ as the colimit over all $\varphi^* J$ for all $\varphi \in \mathcal{I}$. In particular, we write $\Lambda[n, n-1]\ J$ to mean $J(sd(\Lambda[n, n-1]))$, where $sd(\Lambda[n, n-1])$ is the subposet of $sd(\Delta[n])$ containing all elements except $(0, 1, \dots, n-2, n)$ and $(0, 1, \dots, n-1, n)$.

In [CHH18, Lemma 5.10], the following result is proven

Lemma 3.5.9. *For any length n object $([n], J) := J_* \in \Delta^1_{\mathbb{F}}$, the map $\Lambda[n, n-1]J \to J_*$ is a Segal equivalence.*

Definition 3.5.10. Let $J_* = (J_0 \to \dots \to J_n) \in \Delta^1_{\mathbb{F}}$ and let $T \in \Omega$. We say a map $y : J_* \to \tau^* T$ is **admissible of length n** if

- J_* is non-degenerate, i.e. none of the arrows $J_i \to J_{i+1}$ are identity maps;
- the adjoint map $y^\natural : \tau(J_*) \to T$ sends the root to the root.

Using this notion of admissible simplices, it is possible to define a filtration of subpresheaves, F_r, of $\tau^* T$, as follows: let $J_* \in \Delta^1_{\mathbb{F}}$ and consider a map of presheaves $x : J_* \to \tau^* T$. We say $x \in F_r(J_*)$ if the map satisfies one of the following conditions

(a) $x \in \tau^*(\partial^{\mathrm{ext}}T)(J_*)$;

(b) $x = \varphi^* y$ for some $y \in (\tau^* T)(I_*)$, where $y : I_* \to \tau^* T$ is admissible of length at most r and there is a morphism $\varphi : I_* \to J_*$ in $\Delta^1_{\mathbb{F}}$.

In particular, we observe that $\tau^* \partial^{\mathrm{ext}} T = F_0$ and $\tau^* T = \mathrm{colim} F_r$, so proving the desired Segal equivalence $\partial^{\mathrm{ext}}T \xrightarrow{\sim} T$ reduces to proving that the natural inclusion maps $F_{r-1} \hookrightarrow F_r$ are Segal equivalences for all r. In [CHH18, Proposition 5.6], it is shown that if $([n], J) = J_*$ and Q_n is the set of isomorphism classes of admissible maps $J_* \to \tau^* T$, then taking a collection of representative elements $\{\varphi\}$, we obtain a pushout diagram of the form

$$\begin{tikzcd} \coprod_{[\varphi] \in Q_n} \varphi^* \Lambda[n, n-1] J_* \arrow[r] \arrow[d] & F_{n-1} \arrow[d] \\ \coprod_{[\varphi] \in Q_n} \varphi^* J_* \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & F_n \end{tikzcd}$$

But Lemma 3.5.9 tells us that the left hand vertical is a Segal equivalence, and since these are preserved under pushouts, it follows also that $F_{n-1} \to F_n$ is a Segal equivalence, thus giving the claim.

Chapter 4

Modules over Plain Operads and Right Fibrations

In this chapter, we discuss the theory of (right) modules over plain operads and the closely related notion of right fibrations from the category of operators of an operad. As an application of this theory, we translate the concept of configuration modules of manifolds to the setting of right fibrations, where it is possible to give a generalisation of a theorem of [DHK19].

4.1 Modules over Plain Operads

We recall from Section 2.2 that it was possible to cast plain operads over some category C as monoids in a certain monoidal category of *symmetric sequences in* C. One of the pleasing aspects of this formulation of operads is that it presents us with the following slick definition of the notion of an algebra over an operad (as compared to the rather more unwieldy version of Definition 2.1.15).

Definition 4.1.1. An **algebra** over an operad $(\mathcal{O}, \gamma, \eta)$ is a symmetric sequence $\mathcal{A}$ together with a morphism of symmetric sequences $\nu : \mathcal{O} \circ \mathcal{A} \to \mathcal{A}$ such that the following commute

$$\begin{tikzcd} \mathcal{O} \circ \mathcal{O} \circ \mathcal{A} \arrow[r, "\mathrm{Id}_{\mathcal{O}} \circ \nu"] \arrow[d, "\gamma \circ \mathrm{Id}_{\mathcal{A}}"'] & \mathcal{O} \circ \mathcal{A} \arrow[d, "\nu"] \\ \mathcal{O} \circ \mathcal{A} \arrow[r, "\nu"] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J} \circ \mathcal{A} \arrow[r, "\eta \circ \mathrm{Id}_{\mathcal{A}}"] \arrow[dr, "="'] & \mathcal{O} \circ \mathcal{A} \arrow[d, "\nu"] \\ & \mathcal{A} \end{tikzcd}$$

A morphism of $\mathcal{O}$-algebras, $f : (\mathcal{A}, \nu_{\mathcal{A}}) \to (\mathcal{B}, \nu_{\mathcal{B}})$ is a morphism of symmetric sequences $f : \mathcal{A} \to \mathcal{B}$ which is compatible with the respective algebra morphisms. In this way, we see that there is a category of $\mathcal{O}$-algebras, denoted $\mathsf{Alg}_{\mathcal{O}}$

In some texts, the phrase "algebra over an operad" is replaced by "left module over an operad", stemming from the notion of algebra defined above. This suggestive nomenclature also implies the existence of *right* modules...

Definition 4.1.2. A **right module** over an operad $\mathcal{O}$ is a symmetric sequence $\mathcal{M}$ together with a map $\zeta : \mathcal{M} \circ \mathcal{O} \to \mathcal{M}$ which is associative and unital in the sense that the following diagrams commute:

$$\begin{tikzcd} \mathcal{M} \circ \mathcal{O} \circ \mathcal{O} \arrow[r, "\zeta \circ \mathrm{Id}_{\mathcal{O}}"] \arrow[d, "\mathrm{Id}_{\mathcal{M}} \circ \gamma"'] & \mathcal{M} \circ \mathcal{O} \arrow[d, "\zeta"] \\ \mathcal{M} \circ \mathcal{O} \arrow[r, "\zeta"] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \circ \mathcal{J} \arrow[r, "\mathrm{Id}_{\mathcal{M}} \circ \eta"] \arrow[dr, "="'] & \mathcal{M} \circ \mathcal{O} \arrow[d, "\eta"] \\ & \mathcal{M} \end{tikzcd}$$

where $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ is the monoidal product map, and $\eta : \mathcal{J} \to \mathcal{O}$ is the monoidal unit.

A morphism of right $\mathcal{O}$-modules, $f : (\mathcal{M}, \zeta_{\mathcal{M}}) \to (\mathcal{N}, \zeta_{\mathcal{N}})$ is a map of symmetric sequences $f : \mathcal{M} \to \mathcal{N}$, which is compatible with the respective module morphisms in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathcal{M} \circ \mathcal{O} \arrow[r, "\zeta_{\mathcal{M}}"] \arrow[d, "f \circ \mathrm{id}_{\mathcal{O}}"'] & \mathcal{M} \arrow[d, "f"] \\ \mathcal{N} \circ \mathcal{O} \arrow[r, "\zeta_{\mathcal{N}}"] & \mathcal{N} \end{tikzcd}$$

We let $\mathsf{Mod}_{\mathcal{O}}$ denote the category of right $\mathcal{O}$-modules.

Example 4.1.3. Let $(\mathsf{D}, \otimes)$ be a symmetric monoidal category and consider the endomorphism operad $\mathcal{E}\mathrm{nd}_d$ for some fixed object $d \in \mathsf{D}$, as defined in Example 2.1.10. For each fixed object $d' \in \mathsf{D}$, we can define a right $\mathcal{E}\mathrm{nd}_d$-module $\mathcal{H}\mathrm{om}_{d,d'}$ by $\mathcal{H}\mathrm{om}_{d,d'}(n) = \mathrm{Hom}_{\mathsf{D}}(d^{\otimes n}, d')$. The module maps are given by the obvious composition of morphisms. We remark also that the construction

$$\mathsf{D} \ni d' \mapsto \mathcal{H}\mathrm{om}_{d,d'} \in \mathsf{Mod}_{\mathcal{E}\mathrm{nd}_d}$$

is functorial for each fixed d in D.

Suppose now that $(\mathcal{M}, \nu) \in \mathsf{Mod}_{\mathcal{O}}$, and let id denote the image of the unit map in $\mathcal{O}(0)$. By analogy with the notion of partial composition for operads, the map ν induces a partial composition map for each $1 \leq i \leq m$:

$$\begin{aligned} - \circ_i - : \mathcal{M}(m) \times \mathcal{P}(n) &\to \mathcal{M}(m+n-1) \\ (x, y) &\mapsto \nu(x; \mathrm{id}, \dots, y, \mathrm{id}, \dots, \mathrm{id}) \end{aligned}$$

where we compose with n terms such that y is the i^{th} input. The associativity of the map ν ensures that the partial composition maps satisfy the following rules:

- For $1 \leq i \leq m$ and $1 \leq k \leq n$, the following commutes:

$$\begin{tikzcd} \mathcal{M}(m) \times \mathcal{O}(n) \times \mathcal{O}(p) \arrow[r, "(-\circ_i -)\times \mathrm{Id}_{\mathcal{O}}"] \arrow[d, "\mathrm{Id}_{\mathcal{M}} \times (-\circ_k -)"'] & \mathcal{M}(m+n-1) \times \mathcal{O}(p) \arrow[d, "-\circ_{m+k-1} -"] \\ \mathcal{M}(m) \times \mathcal{O}(n+p-1) \arrow[r, "-\circ_i -"] & \mathcal{M}(m+n+p-2) \end{tikzcd}$$

- For $1 \leq i \neq j \leq m$, the following commutes:

$$\begin{tikzcd} \mathcal{M}(m) \times \mathcal{O}(n) \times \mathcal{O}(p) \arrow[r, "(-\circ_i -)\times \mathrm{Id}_{\mathcal{O}}"] \arrow[d, "(-\circ_j -)\times \mathrm{Id}_{\mathcal{O}}"'] & \mathcal{M}(m+n-1) \times \mathcal{O}(p) \arrow[d, "-\circ_j -"] \\ \mathcal{M}(m) \times \mathcal{O}(n+p-1) \arrow[r, "-\circ_i -"] & \mathcal{M}(m+n+p-2) \end{tikzcd}$$

Using these partial composition maps, we can give another characterisation of right modules over an operad as a kind of presheaf.

Proposition 4.1.4. *The data of a right $\mathcal{O}$-module in the category C is equivalent to a contravariant functor from the active category of operators $\mathsf{F}(\mathcal{O})$ to C.*

Proof. Let $\mathcal{M}$ be a right module over $\mathcal{O}$. Define a functor $\mathcal{M} : \mathsf{F}(\mathcal{O})^{op}$ to C as follows. For a finite set $\langle n \rangle$, $\mathcal{M}$ acts by $\langle n \rangle \mapsto \mathcal{M}(n)$. For a morphism of finite sets $\alpha : \langle n \rangle \to \langle m \rangle$, and an element $(d_{i_1}, \dots, d_{i_m}) \in \prod_{j=1}^{m} \mathcal{O}(\alpha^{-1}(j))$, the functor $\mathcal{M}$ associates a map $\mathcal{M}(m) \to \mathcal{M}(n)$ by taking successive partial compositions with the elements o_j, i.e. for an element $a \in \mathcal{M}(m)$, the map $\mathcal{M}(\alpha)$ acts by $a \mapsto (\dots(a \circ_{i_1} d_{i_1}) \circ_{i_2} d_{i_2}) \dots) \circ_{i_m} d_{i_m} \in \mathcal{M}(n)$. The partial composition $- \circ_1 -$ gives a map from arity m to arity $m + \alpha^{-1}\{1\} - 1$; then applying the composition $- \circ_2 -$ gives a map from arity $(m + \alpha^{-1}\{1\} - 1)$ to arity $(m + \alpha^{-1}\{1\} + \alpha^{-1}\{2\} - 2)$, and so on, so that by the end of the process (i.e. applying all m possible partial composition maps once), the final arity is $(m + \sum_{j=1}^{m} \alpha^{-1}\{j\} - m) = n$, as expected.

Conversely, given a functor $\mathcal{M} : \mathsf{F}(\mathcal{O})^{\mathrm{op}} \to \mathsf{C}$, we can define a symmetric sequence $\mathcal{M}$ whose arity-n component is $\mathcal{M}(\langle n \rangle)$. By the functoriality of $\mathcal{M}$, for each element $\left(\alpha : \langle n \rangle \to \langle k \rangle, \{d_j^{\alpha}\}_{1 \leq j \leq k}\right) \in \mathrm{Hom}_{\mathsf{F}(\mathcal{O})}(\langle n \rangle, \langle k \rangle)$, we have a map in C:

$$\mathcal{M}\left(\alpha, \{d_j^{\alpha}\}_{1 \leq j \leq k}\right) : \mathcal{M}(k) \to \mathcal{M}(n)$$

In turn, this induces a map in C:

$$\mathrm{Hom}_{\mathsf{F}(\mathcal{O})}(\langle n \rangle, \langle k \rangle) \to \mathrm{Hom}_{\mathsf{C}}(\mathcal{M}(k), \mathcal{M}(n))$$

or by tensor-hom adjunction, and taking coinvariants with respect to the action of the symmetric group:

$$\mathcal{M}(k) \times_{\mathfrak{S}_k} \mathrm{Hom}_{\mathsf{F}(\mathcal{O})}(\langle n \rangle, \langle k \rangle) \to \mathcal{M}(n)$$

Taking the coproduct over all $k \geq 0$, and recalling the definition of the hom-objects in the category of operators, we obtain a map in C:

$$\coprod_{k\geq 0} \mathcal{M}(k) \times_{\mathfrak{S}_k} \coprod_{\alpha:\langle n\rangle\to\langle k\rangle} \prod_{1\leq j\leq k} \mathcal{O}\left(\alpha^{-1}\{j\}\right) \times_{\prod_{1\leq j\leq k}\mathfrak{S}_{\alpha^{-1}\{j\}}} \mathfrak{S}_n \to \mathcal{M}\left(n\right)$$

In other words, we have a map $\mathcal{M}\circ\mathcal{O}(n) \to \mathcal{M}(n)$, and by the functoriality of $\mathcal{M}$, this is unital and associative, thus giving the symmetric sequence $\mathcal{M}$ the structure of a right module over $\mathcal{O}$. $\square$

As the construction $\mathcal{O} \mapsto \mathsf{F}(\mathcal{O})$ is functorial, it follows that a map of operads $\varphi : \mathcal{O} \to \mathcal{P}$ induces a functor, which we also denote by φ, between the respective categories of operators. The foregoing result tells us that φ gives rise to base-change adjunctions:

$$\begin{tikzcd} \mathsf{Mod}_{\mathcal{O}} \arrow[r, shift left=2, "\varphi_!"] \arrow[r, shift right=2, "\varphi_*"'] & \mathsf{Mod}_{\mathcal{P}} \arrow[l, "\varphi^*"'] \end{tikzcd}$$

4.1.1 Configuration Modules of Manifolds

Definition 4.1.5. Let G be a Lie group with a representation $\rho : G \to GL_d(\mathbb{R})$. A d-dimensional manifold M is said to admit a **G-framing** if its frame-bundle $\mathrm{Fr}(M)$ has a sub-G-bundle $\mathrm{Fr}^G(M)$ such that there is an isomorphism of principal $GL_d(\mathbb{R})$-bundles over the identity map on M:

$$\varphi_M : \rho_*\mathrm{Fr}^G(M) \xrightarrow{\cong} \mathrm{Fr}(M) \tag{4.1}$$

Remark 4.1.6. A G-framing on a d-dimensional manifold M carries potentially interesting geometric information. For example, an $O(d)$-framing on M corresponds to a Riemannian metric, while (if d is even) a $Sp(d/2)$-framing corresponds to the existence of an almost-symplectic form on M. We note that $\mathbb{R}^d$ is canonically equipped with a G-framing for any G.

In the case where G is the trivial group $\{e\}$, a G-framing on M corresponds to the assertion that the tangent bundle is trivial, in which case we say that the manifold M is *framed*. We observe that every Lie group is framed; relatively speaking however, the class of framed manifolds is quite restricted.

Given an embedding of manifolds $f : M \to N$, there is a canonical induced map on the frame bundles, which we denote by $\mathrm{Fr}(f)$. We will leverage this map to define a homotopically suitable notion of a framed embedding, following [And10].

Definition 4.1.7. Let M, N be G-framed d-dimensional manifolds. A **G-framed embedding** from M to N is the data of

- a map of manifolds $f : M \to N$;
- a G-bundle map $F : \mathrm{Fr}^G(M) \to \mathrm{Fr}^G(N)$ which covers f;
- a $GL_d(\mathbb{R})$-equivariant homotopy $H : \mathrm{Fr}(M)\times[0,1] \to \mathrm{Fr}(N)$, which covers f, from $\mathrm{Fr}(f)$ to $\varphi_N\circ\rho_*F\circ\varphi_M^{-1}$.

We denote by $\mathrm{Emb}^G(M,N)$ the set of all G-framed embeddings from M to N, and we give this set the compact-open topology. In particular, for the case where $G = \{e\}$, we will write $\mathrm{Emb}^{\{e\}}$ as $\mathrm{Emb}^{\mathrm{fr}}$ – this is the space of so-called framed embeddings.

Based on the foregoing, we can produce a topological category Mfld_d^G, whose objects are G-framed d-dimensional manifolds, and whose mapping objects are (for objects M, N in Mfld_d^G)

$$\mathrm{Hom}_{\mathsf{Mfld}_d^G}(M,N) = \mathrm{Emb}^G(M,N)$$

If we let P denote the homotopy-pullback of the diagram

$$\begin{tikzcd} & \mathrm{Hom}^G\left(\mathrm{Fr}^G(M),\mathrm{Fr}^G(N)\right) \arrow[d, "F\mapsto\varphi_N\circ\rho_*F\circ\varphi_M^{-1}"] \\ \mathrm{Emb}\left(M,N\right) \arrow[r, "f\mapsto\mathrm{Fr}(f)"] & \mathrm{Hom}^{GL_d(\mathbb{R})}\left(\mathrm{Fr}(M),\mathrm{Fr}(N)\right) \end{tikzcd}$$

where M, N are objects in Mfld_d^G and $\mathrm{Hom}^G\left(\mathrm{Fr}^G(M), \mathrm{Fr}^G(N)\right)$ is the G-equivariant mapping space, then the arguments of [And10, Section V.9.1-2] tell us that the natural inclusion $\mathrm{Emb}^G(M, N) \hookrightarrow P$ is a weak equivalence of spaces. More importantly for us, if we define the G-framed configuration space of a G-framed manifold M by

$$\mathrm{Conf}_k^G(M) := \mathrm{Fr}^{G^k}(\mathrm{Conf}_k(M))$$

then we obtain the following important result:

Proposition 4.1.8. *[And10, Proposition 14.4] Evaluation at the origin defines a weak equivalence*

$$\mathrm{Emb}^G(\sqcup_k \mathbb{R}^d, M) \xrightarrow{\sim} \mathrm{Conf}_k^G(M)$$

Recall that given a dilation representation of a locally compact Hausdorff Lie group, $\rho : G \to GL_d(\mathbb{R})$, we were able to discuss the notion of *G-skew little cubes* (see Example 2.1.13). Given such a representation, we can construct a map

$$E : {}^t\mathbb{E}_d^G(n) \to \mathrm{Hom}^G\left(\sqcup_n(-1,1)^d \times G, (-1,1)^d \times G\right)$$

as follows: if $\varepsilon = \langle \varepsilon_1, ..., \varepsilon_n\rangle \in {}^t\mathbb{E}_d^G(n)$ with $\varepsilon_i(x) = \rho(g_i)x + v_i$ for some fixed $g_i \in G$ and $v_i \in (-1,1)^d$, then we define the G-equivariant map $E(\varepsilon)$ by

$$E(\varepsilon)(z, h) = (\varepsilon_i(z), g_i h)$$

where $(z, h) \in \sqcup_n(-1,1)^d \times G$ such that z lies in the i^{th} copy of $(-1,1)^d$. There is also an inclusion of ${}^t\mathbb{E}_d^G(n)$ in $\mathrm{Emb}\left(\sqcup_n(-1,1)^d, (-1,1)^d\right)$. Finally, we note that the skew-embedding $\varepsilon = \langle \varepsilon_1, ..., \varepsilon_n\rangle$ satisfies

$$\mathrm{Fr}(\varepsilon) = (\rho(g_1), \ldots, \rho(g_n))$$

thus ensuring that the following solid arrow diagram commutes

$$\begin{tikzcd}
{}^t\mathbb{E}_d^G(n) \arrow[dr, dashed, "\varphi_d^G(n)"] \arrow[drr, bend left, "E"] \arrow[ddr, bend right] & & \\
& \mathrm{Emb}^G\left(\sqcup_n(-1,1)^d, (-1,1)^d\right) \arrow[r] \arrow[d] & \mathrm{Hom}^G\left(\sqcup_n(-1,1)^d \times G, (-1,1)^d \times G\right) \arrow[d] \\
& \mathrm{Emb}\left(\sqcup_n(-1,1)^d, (-1,1)^d\right) \arrow[r, "\mathrm{Fr}"] & \mathrm{Hom}^{GL_d(\mathbb{R})}\left(\sqcup_n(-1,1)^d \times GL_d(\mathbb{R}), (-1,1)^d \times GL_d(\mathbb{R})\right)
\end{tikzcd}$$

– and hence there is a map to the pullback which we can compose with the natural map to the homotopy-pullback, ensuring that the indicated dashed arrow exists. In fact, if we consider

$$\mathcal{E}\mathrm{nd}^G_{(-1,1)^d}(n) = \mathrm{Hom}_{\mathsf{Mfld}_d^G}\left(\sqcup_n(-1,1)^d, (-1,1)^d\right) = \mathrm{Emb}^G\left(\sqcup_n(-1,1)^d, (-1,1)^d\right)$$

(we have decorated our usual notation for the endomorphism operad to emphasise the choice of framing group) then we see that the maps $\varphi_d^G(n)$ determine a morphism of topological operads $\varphi_d^G : {}^t\mathbb{E}_d^G \to \mathcal{E}\mathrm{nd}^G_{(-1,1)^d}$. Indeed, we can even say more:

Proposition 4.1.9. *[DHK19, Theorem 4.14] For each n, there is a weak equivalence of spaces*

$$\varphi_d^G : {}^t\mathbb{E}_d^G(n) \xrightarrow{\simeq} \mathrm{Emb}^G\left(\sqcup_n(-1,1)^d, (-1,1)^d\right)$$

This proposition, combined with 4.1.8, gives a generalisation of Remark 2.1.12 to the case of G-skew little cubes. Armed with these results, we are finally in a position to define the main object of interest for us from these discussions:

Definition 4.1.10. Let M be a d-dimensional G-framed manifold. The **G-framed configuration module** of M, denoted $\mathcal{C}_M^G$ is the right ${}^t\mathbb{E}_d^G$-module defined by

$$\mathcal{C}_M^G(n) = \left(\varphi_d^G\right)^* \mathcal{H}\mathrm{om}_{(-1,1)^d, M}(n) = \left(\varphi_d^G\right)^* \mathrm{Emb}^G\left(\sqcup_n(-1,1)^d, M\right)$$

If M is a framed manifold, then we will just write $\mathcal{C}_M$ for its configuration module.

Remark 4.1.11. One might have assumed that the G-framed configuration module of a manifold M should be the symmetric sequence with $\mathcal{C}_M^G(n) = \mathrm{Conf}_n^G(M)$. However, taking this approach we wouldn't be able to avail of the structure of a module over the little cubes operads. Instead, by working with the spaces $\mathcal{C}_M^G(n)$ as defined above, we have sufficient "wriggle-room" to have the structure of a module over the G-skew little cubes operad, while simultaneously being weakly homotopy-equivalent to the more classically defined configuration spaces via the evaluation at the origin.

4.1.2 The Boardman-Vogt Tensor Product of Modules

One of the main ideas in the paper [DHK19] is the notion of the Boardman-Vogt tensor product of right modules. We sketch an outline of how this works. Let $\mathcal{O}$ and $\mathcal{P}$ be operads in spaces (we assume single-coloured for simplicity). There is a functor relating their respective active categories of operators to the active category of operators of their Boardman-Vogt tensor product,

$$\mu : \mathsf{F}(\mathcal{O}) \times \mathsf{F}(\mathcal{P}) \to \mathsf{F}(\mathcal{O} \star \mathcal{P}) \tag{4.2}$$

which acts on objects by $(\langle n \rangle, \langle m \rangle) \mapsto \langle nm \rangle$, while the map on hom-objects is specified by

$$\begin{aligned} \mathrm{Hom}_{\mathsf{F}(\mathcal{O})}\left(\langle n \rangle, \langle m \rangle\right) \times \mathrm{Hom}_{\mathsf{F}(\mathcal{P})}\left(\langle n' \rangle, \langle m' \rangle\right) &\longrightarrow \mathrm{Hom}_{\mathsf{F}(\mathcal{O}\star\mathcal{P})}\left(\langle nn' \rangle, \langle mm' \rangle\right) \\ \Big(\alpha, (o_i)_{1\le i\le m}\Big), \Big(\beta, (p_j)_{1\le j\le m'}\Big) &\longmapsto \Big(\alpha \times \beta, (o_i \star p_j)_{1\le i\le m, 1\le j\le m'}\Big) \end{aligned}$$

where by $o_i \star p_j$ we mean the equivalence class in the operad $\mathcal{O} \star \mathcal{P}$ represented by the operations o_i in $\mathcal{O}$ and p_j in $\mathcal{P}$. In the case where both $\mathcal{O}$ and $\mathcal{P}$ are the commutative operad, we see that this map μ is essentially an "unpointed" version of the smash product of pointed finite sets which we introduced in Section 3.1.3.

Definition 4.1.12. Viewing a right module over $\mathcal{O}$ as a space-valued presheaf on $\mathsf{F}(\mathcal{O})$, the **Boardman-Vogt tensor product** of an $\mathcal{O}$-module $\mathcal{M}$ and a $\mathcal{P}$-module $\mathcal{N}$ is given by the indicated dashed left Kan extension in the following diagram

$$\begin{tikzcd} \mathsf{F}(\mathcal{O})^{\mathrm{op}} \times \mathsf{F}(\mathcal{P})^{\mathrm{op}} \arrow[r, "\mathcal{M}\times\mathcal{N}"] \arrow[d, "\mu"'] & \mathcal{S} \times \mathcal{S} \arrow[r, "-\times-"] & \mathcal{S} \\ \mathsf{F}(\mathcal{O} \star \mathcal{P})^{\mathrm{op}} \arrow[urr, dashed, "\mu_!(\mathcal{M}\hat{\times}\mathcal{N}) := \mathcal{M}\star\mathcal{N}"'] & & \end{tikzcd}$$

where we write $\mathcal{M}\hat{\times}\mathcal{N}$ to indicate the *external tensor product* of $\mathcal{M}$ and $\mathcal{N}$, which is given by the horizontal composition in the diagram.

If we fix a $\mathcal{P}$-module $\mathcal{N}$, then the functor $- \star \mathcal{N} : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}\star\mathcal{P}}$ admits a right adjoint, which we denote $\widetilde{\mathrm{Hom}}_{\mathcal{P}}(\mathcal{N}, \mu^*(-))$. Specifically, for a $(\mathcal{O} \star \mathcal{P})$-module $\mathcal{L}$ and an object $\langle n \rangle$ in $\mathsf{F}(\mathcal{O} \star \mathcal{P})$, the value of this right adjoint is:

$$\widetilde{\mathrm{Hom}}_{\mathcal{P}}\left(\mathcal{N}, \mu^*(\mathcal{L})\right)(\langle n \rangle) = \int_{\langle m \rangle \in \mathsf{F}(\mathcal{P})} \mathrm{Hom}_{\mathcal{S}}\left(\mathcal{N}(m), \mathcal{L}(nm)\right) \tag{4.3}$$

where the integral denotes an end formula.

Endowing the category of right modules over a topological operad with the projective structure (where $\mathcal{S}$ has the usual Quillen model structure), it can be shown that these functors form a Quillen pair. In particular, this means that we can produce a left derived Boardman-Vogt tensor product of modules:

$$- \star^{\mathbb{L}} - : \mathrm{Ho}(\mathrm{Mod}_{\mathcal{O}}) \times \mathrm{Ho}(\mathrm{Mod}_{\mathcal{P}}) \to \mathrm{Ho}(\mathrm{Mod}_{\mathcal{O}\star\mathcal{P}})$$

The key element to showing that the above pair is Quillen is the fact that $\widetilde{\mathrm{Hom}}_{\mathcal{P}}(\mathcal{N}, -)$ is a right Quillen functor for any fixed cofibrant $\mathcal{P}$-module $\mathcal{N}$.

In particular, for the case of a framed d-manifold M and a framed d'-manifold N, the tensor product of their configuration modules is a ${}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}$-module. At this point, we recall the Dunn/Brinkmeier Additivity Theorem for little cubes, which tells us that

Theorem 4.1.13. *There is a weak equivalence* $\iota : {}^t\mathbb{E}_d \star {}^t\mathbb{E}_{d'} \to {}^t\mathbb{E}_{d+d'}$

By definition of the tensor product of modules, $\mathcal{C}_M \star \mathcal{C}_N$ is a ${}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}$-module. On the other hand, the product of M and N is a $d+d'$-manifold, so $\mathcal{C}_{M\times N}$ is a ${}^t\mathbb{E}_{d+d'}$-module and hence $\iota^*\mathcal{C}_{M\times N}$ is a ${}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}$-module. The question then arises as to whether there is any connection between these two objects – the main theorem in the paper of Dwyer, Hess and Knudsen asserts that there is:

Theorem 4.1.14. *[DHK19, Theorem 1.1] Let M be a framed d-manifold and N a framed d'-manifold. Then there is a natural isomorphism*

$$\mathrm{Ho}(\iota^*)(\mathcal{C}_{M\times N}) \cong \mathcal{C}_M \star^{\mathbb{L}} \mathcal{C}_N$$

in $\mathrm{Ho}(\mathrm{Mod}_{{}^t\mathbb{E}_d \star {}^t\mathbb{E}_{d'}})$.

The key argument in this proof is that the configuration modules are *multi-local* – that is, for a suitably fine cover $\mathcal{U}$ of the manifold M by open disks, there is a levelwise weak equivalence

$$\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{C}_U(n) \xrightarrow{\sim} \mathcal{C}_M(n)$$

(We will return to this notion of multi-locality in more detail at a later point in this chapter.) In particular, the covering of M by embeddings of disjoint copies of $\mathbb{R}^d$ is such a suitable cover, so deducing the result of Theorem 4.1.14 reduces to the matter of showing that there is a weak equivalence in the case where $M = \sqcup_i \mathbb{R}^d$ and $N = \sqcup_j \mathbb{R}^{d'}$. This in turn can be shown to be a direct consequence of the additivity theorem.

4.2 Right Fibrations and the Grothendieck Construction

In the paper of Boavida [Boa16], the concept of a right fibration of Segal spaces is introduced. This notion appears to have some utility for our setting.

Definition 4.2.1. Let B be a Segal space. A **right fibration** over B is a map of Segal spaces $X \to B$ such that the square

$$\begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{d_0} & B_0 \end{array}$$

is homotopy Cartesian.

By analogy with the world of quasi-categories, there is a suitable version of a Grothendieck construction which provides a way of communicating between right fibrations and space-valued presheaves on categories which are internal/enriched in spaces.

Construction 4.2.2. Let D be an internal/enriched category in spaces and let $F : \mathsf{D}^{\mathrm{op}} \to \mathcal{S}$ be a (possibly enriched) functor. We associate to the functor F a simplicial space, $\mathcal{G}(F)$, which we call the **Grothendieck construction** of F; this is the simplicial space whose space of n-simplices is given by

$$\mathcal{G}(F)_n = \coprod_{c_0,\dots,c_n\in\mathsf{D}} \prod_{0\le i\le n-1} \mathrm{Hom}_{\mathsf{D}}(c_{n-i+1}, c_{n-i}) \times F(c_0)$$

For simplicity, we will describe face maps in degrees 0 and 1, but the same idea obviously extends to higher degrees. The map $d_0 : \mathcal{G}(F)_1 \to \mathcal{G}(F)_0$ is defined by the composition

$$\mathrm{Hom}_{\mathsf{D}}(c_1, c_0) \times F(c_0) \ni (f,\zeta) \mapsto F(f)(\zeta) \in F(c_1)$$

using the fact that F is a contravariant functor. The map $d_1 : \mathcal{G}(F)_1 \to \mathcal{G}(F)_0$ is defined by the projection

$$\mathrm{Hom}_{\mathsf{D}}(c_1, c_0) \times F(c_0) \ni (f,\zeta) \mapsto \zeta \in F(c_0)$$

On the other hand, the degeneracy maps $s_i : \mathcal{G}(F)_j \to \mathcal{G}(F)_{j+1}$ are given by inserting an identity morphism of the object c_i in $\mathcal{C}$, e.g.

$$\begin{array}{rl} \mathrm{Hom}_{\mathsf{D}}(c_j, c_{j-1}) \times \ldots \times \mathrm{Hom}_{\mathsf{D}}(c_1, c_0) \times F(c_0) & \xrightarrow{s_i} \mathrm{Hom}_{\mathsf{D}}(c_j, c_{j-1}) \times \ldots \times \mathrm{Hom}_{\mathsf{D}}(c_1, c_0) \times F(c_0) \\ (f_j, \dots, f_{i+1}, f_i, \dots, f_0) & \mapsto (f_j, \dots, f_{i+1}, \mathrm{id}_{c_i}, f_i, \dots, f_0) \end{array}$$

It is immediate that the Grothendieck construction of F is a Segal space. There is also an evident reference map of Segal spaces $\mathcal{G}(F) \to N\mathsf{D}$, where we view $N\mathsf{D}$ as a discrete simplicial space: in degree 0, this is given by $F(c_0) \ni \zeta \mapsto c_0$ and in higher degree, it is just the evident projection. We note furthermore that for any fixed object c_1 in D, the homotopy fibre of the map $d_0 : N\mathsf{D}_1 \to N\mathsf{D}_0$ over c_1 is $\mathrm{Hom}_{\mathsf{D}}(c_1, c_0)$, which coincides with the homotopy fibre of $d_0 : \mathcal{G}(F)_1 \to \mathcal{G}(F)_0$ over $F(c_1)$ (i.e. the preimage of c_1 under the reference map). Hence, this reference map gives the Grothendieck construction of a contravariant functor $F : \mathsf{D}^{\mathrm{op}} \to \mathcal{S}$ the structure of a right fibration over $N\mathsf{D}$.

Our interest in this construction stems from the fact that we can view a right module over an operad in spaces $\mathcal{O}$ as a space-valued presheaf on $\mathsf{F}(\mathcal{O})$, so we can view a right $\mathcal{O}$-module as "being the same thing" as a right fibration over $N\mathsf{F}(\mathcal{O})$.

In other models of ∞-categories, we see an analogous notion of right/left fibration appear (a left fibration of Segal spaces $X \to B$ is just a right fibration $X^{\mathrm{op}} \to B^{\mathrm{op}}$) – see e.g. [HM15], [Lur17]. In all cases, an associated model structure is derived. The same procedure may be repeated for Segal spaces with right fibrations.

Theorem 4.2.3. *Let B be a Segal space. The **right fibration model structure** on $s\mathcal{S}_{/B}$ is determined by the following data:*

- *fibrant objects in $s\mathcal{S}_{/B}$ are those maps of Segal spaces $X \to B$ which are right fibrations;*
- *weak equivalences of fibrant objects are maps $X \to Y$ over B which are weak equivalent in degree 0, i.e. for every vertex $b \in B$, the map $X_b \to Y_b$ is a weak equivalence of spaces;*
- *cofibrations are monomorphisms of simplicial spaces over B.*

As with the iterations of left/right fibration model structures for other ∞-categorical schemata, it is worth pointing out a few facts regarding the homotopy properties of a base change. For example, if $\iota : B \to B'$ is a map of Segal spaces, then there is an adjoint base-change pair

$$s\mathcal{S}_{/B} \underset{\iota^*}{\overset{\Sigma_\iota}{\rightleftarrows}} s\mathcal{S}_{/B'}$$

It can be shown straightforwardly that right fibrations (i.e. fibrant objects) are preserved under ι^*, and likewise that trivial fibrations of simplicial spaces over B are stable under ι^*, so this is actually a Quillen pair with respect to the right fibration model structures on $s\mathcal{S}_{/B}$ and $s\mathcal{S}_{/B'}$ respectively.

We saw above that the Grothendieck construction gives us a way to rephrase space-valued presheaves on categories internal in spaces in terms of right fibrations of Segal spaces. We might ask ourselves whether this construction admits a left adjoint and whether these functors are homotopically well-behaved with respect to the projective/right fibration model structures.

Construction 4.2.4 (A left adjoint to the Grothendieck Construction)**.** The point of this construction will be to use the fact that left adjoints preserve colimits, so in particular it will suffice to construct our left adjoint $\mathcal{L}$ in terms of representables; in the case of simplicial spaces over the nerve of a small category D which is internal in spaces, these will be objects of the form $\Delta[n] \times K \xrightarrow{\beta} N\mathsf{D}$, where K is a space.

Let $\alpha_n^n : [0] \to [n]$ be the monotone map with image $n \in [n]$, which induces the so-called *target map* $(\alpha_n^n)^* : N\mathsf{D}_n \to N\mathsf{D}_0$: this acts on a string of n composable morphisms in D by $(c_n \to \ldots \to c_0) \mapsto c_0$. Writing $\beta^\# : K \to N\mathsf{D}_n$ for the adjoint of the reference map β, we set β' to be the composite $(\alpha_n^n)^* \circ \beta^\# : K \to N\mathsf{D}_0$. With this notation in place, we can give an expression for the left adjoint $\mathcal{L}$:

$$\mathcal{L}(\Delta[n] \times K \xrightarrow{\beta} N\mathsf{D})(d) = \coprod_{c_0 \in \mathrm{Im}(\beta')} K_{c_0} \times \mathrm{Hom}_{\mathsf{D}}(d, c_0)$$

for some object d in D. We remark that as a special case of this construction, we can restrict our attention to the case where K is just a point, in which case the map β' will just pick out the target object of one fixed string of n-composable morphisms, say $(c_n \to \ldots \to c_0)$, in D, so that $\mathcal{L}(\Delta[n] \xrightarrow{\beta} N\mathsf{D})(d) = \mathrm{Hom}_{\mathsf{D}}(d, c_0)$.

The next lemma affirms the value of this construction:

Lemma 4.2.5. *For any small category D which is internal in spaces, the functors*

$$\mathcal{L} : s\mathcal{S}_{/N\mathsf{D}} \rightleftarrows \mathcal{S}^{\mathsf{D}^{\mathrm{op}}} : \mathcal{G}$$

form an adjoint pair. Moreover, with respect to the right fibration model structure on $s\mathcal{S}_{/N\mathsf{D}}$ and the projective model structure on $\mathcal{S}^{\mathsf{D}^{\mathrm{op}}}$, they form a Quillen pair.

Proof. That the pair is adjoint follows from a series of applications of the Yoneda lemma. To see the homotopical properties, we observe that two presheaves F and H are weak equivalent in the projective structure if they are objectwise-weak equivalent, i.e. $F(c) \simeq H(c)$ for all objects $c \in \mathsf{D}$; on the other hand, $\mathcal{G}(F)$ and $\mathcal{G}(H)$ are fibrant objects by the properties of the Grothendieck construction, so they are weak equivalent if and only if they are weak equivalent in degree 0, i.e. if and only if

$$\coprod_{c\in \mathsf{D}} F(c) \simeq \coprod_{c\in \mathsf{D}} H(c)$$

which is obviously true. Hence, $\mathcal{G}$ preserves weak equivalences and thus all trivial fibrations.

Now let $F \to H$ be a fibration of presheaves on $\mathcal{C}$, that is, $F(c) \to H(c)$ is a fibration for each object c in D. The induced map $\mathcal{G}(F) \to \mathcal{G}(H)$ is a fibration in $s\mathcal{S}_{/N\mathsf{D}}$ precisely if it has the right lifting property with respect to all cofibrations, i.e. all morphisms of simplicial spaces over $N\mathsf{D}$ which are monomorphisms in degree 0. However, since $F \to H$ is an objectwise fibration, we see in degree 0 that we can solve the following lifting problem for any such cofibration $X \to Y$:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, tail] & \mathcal{G}(F)_0 = \coprod_c F(c) \arrow[d] \\ Y_0 \arrow[r] \arrow[ur, dotted] & \mathcal{G}(H)_0 = \coprod_c H(c) \end{tikzcd}$$

By definition of the Grothendieck construction, it is evident that we can also solve the lifting problem in all higher degrees, thus ensuring that $\mathcal{G}(F) \to \mathcal{G}(H)$ is a fibration. Hence, $\mathcal{G}$ is right Quillen. □

Putting the foregoing information together, we obtain the following square of Quillen adjunctions for any functor between small categories which are internal in spaces, $f : \mathsf{D} \to \mathsf{E}$:

$$\begin{tikzcd} \mathcal{S}^{\mathsf{D}^{\mathrm{op}}} \arrow[r, shift left, "f_!"] \arrow[d, shift right, "\mathcal{G}"'] & \mathcal{S}^{\mathsf{E}^{\mathrm{op}}} \arrow[l, shift left, "f^*"] \arrow[d, shift right, "\mathcal{G}"'] \\ s\mathcal{S}_{/N\mathsf{D}} \arrow[u, shift right, "\mathcal{L}"'] \arrow[r, shift left, "\Sigma_f"] & s\mathcal{S}_{/N\mathsf{E}} \arrow[u, shift right, "\mathcal{L}"'] \arrow[l, shift left, "f^*"] \end{tikzcd} \tag{4.4}$$

4.3 Configuration Modules in Terms of Right Fibrations

Before diving in to the work of stating a putative equivalent result to that of [DHK19] in the language of right fibrations, we take a moment to explicitly describe how the Grothendieck construction and the relevant base-change functors work in the case of our configuration modules. In particular, we initially focus our attention on the case of framed manifolds and only use the "standard" topological little cubes operads of Example 2.1.11 where we can fall back on the additivity result of Dunn and Brinkmeier. We will later broaden the scope of our attentions to the (∞-categorical) G-skew little cubes which we discussed in Section 3.2.2. We fix M to be a framed d-manifold. To spare ourselves some notational clutter, we write $\mathcal{E}_d$ to mean the nerve of the category $\mathsf{F}({}^t\mathbb{E}_d)$.

By our foregoing discussion, the right module $\mathcal{C}_M : \mathsf{F}({}^t\mathbb{E}_d)^{\mathrm{op}} \to \mathcal{S}$ gives rise to a right fibration of Segal spaces $\mathcal{G}(\mathcal{C}_M) \to \mathcal{E}_d$. The 0-simplices of this simplicial space take the form

$$\mathcal{G}(\mathcal{C}_M)_0 = \coprod_{\langle n\rangle \in \mathsf{F}({}^t\mathbb{E}_d)} \mathcal{C}_M(n) \simeq \coprod_{n\geq 0} \mathrm{Emb}^{\mathrm{fr}}(\sqcup_n \mathbb{R}^d, M)$$

Since M is a framed manifold, $\mathrm{Emb}^{\mathrm{fr}}(\sqcup_n \mathbb{R}^d, M)$ is weakly equivalent to $\mathrm{Emb}(\sqcup_n \mathbb{R}^d, M)$ for each n. The 1-simplices of this simplicial space satisfy the weak equivalence

$$\mathcal{G}(\mathcal{C}_M)_1 \simeq \coprod_{n_1,n_0\geq 0} \coprod_{\alpha:\langle n_1\rangle\to\langle n_0\rangle} \prod_{1\leq i\leq n_0} \mathrm{Emb}^{\mathrm{fr}}(\sqcup_{\alpha^{-1}\{i\}} \mathbb{R}^d, \mathbb{R}^d) \times \mathrm{Emb}^{\mathrm{fr}}(\sqcup_{n_0} \mathbb{R}^d, M)$$

We next turn our attention to the question of base-change for these configuration modules of manifolds. Recall that N is the discrete subcategory of Fin. For any operad $\mathcal{O}$, the inclusion of N in Fin gives rise to a functor $j : \mathsf{N}(\mathcal{O}) \to \mathsf{F}(\mathcal{O})$ which in turn induces a base-change adjunction:

$$\begin{tikzcd} s\mathcal{S}_{/N\mathsf{N}(\mathcal{O})} \arrow[r, shift left, "\Sigma_j"] & s\mathcal{S}_{/N\mathsf{F}(\mathcal{O})} \arrow[l, shift left, "j^*"] \end{tikzcd}$$

We recall that the trivial/unit operad in spaces, $\mathcal{J}$, is the operad whose arity-1 component is the one-point space $*$ and whose arity-n component is the empty set for all $n > 1$. Based on our interpretation of plain operads in spaces as monoids in the monoidal category $(\mathrm{Op}(\mathcal{S}), \circ, \mathcal{J})$, we know that, for any operad $\mathcal{O}$, we have an unit map $\hat{\nu}_{\mathcal{O}} : \mathcal{J} \to \mathcal{O}$, which leads to a base-change adjunction:

$$s\mathcal{S}_{/N\mathsf{F}(\mathcal{J})} \underset{\hat{\nu}_{\mathcal{O}}^*}{\overset{\Sigma_{\hat{\nu}_{\mathcal{O}}}}{\rightleftarrows}} s\mathcal{S}_{/N\mathsf{F}(\mathcal{O})}$$

We write $\nu_{\mathcal{O}}$ for the composite map $\mathsf{N}(\mathcal{J}) \to \mathsf{F}(\mathcal{J}) \to \mathsf{F}(\mathcal{O})$. In particular, for the little d-cubes operad we get a base-change pair:

$$s\mathcal{S}_{/N\mathsf{N}(\mathcal{J})} \underset{\nu_{t_{\mathbb{E}_d}}^*}{\overset{\Sigma_{\nu_{t_{\mathbb{E}_d}}}}{\rightleftarrows}} s\mathcal{S}_{/\mathcal{E}_d}$$

We remark that the collection of presheaves on $\mathsf{N}(\mathcal{J})$ corresponding to the simplicial spaces on the left side of our adjunction above are precisely the *sequences* in $\mathcal{S}$.

Example 4.3.1. Let $\mathfrak{S}_n$ denote the sequence whose arity-k term is the n^{th} symmetric group if $k = n$ (viewed as a discrete space) and is the empty set otherwise. As a toy example, we want to examine the action of $\Sigma_{\nu_{t_{\mathbb{E}_d}}}$ on the Grothendieck construction of this sequence.

First, note that the 0-simplices of $\mathcal{G}(\mathfrak{S}_n)$ is just the discrete space $\mathfrak{S}_n$. By definition of the trivial operad, we see furthermore, that the k-simplices are also the discrete space $\mathfrak{S}_n$ for all $k \geq 1$. In other words,

$$\mathcal{G}(\mathfrak{S}_n) = \Delta[0] \times \mathfrak{S}_n$$

Now let us examine those 0-simplices of the space $\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n)$ which lie over the vertex $\langle m \rangle \in (\mathcal{E}_d)_0$ – we will denote this fibre space by $\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n)_{/\langle m \rangle}$. We note that an element in this fibre arises as an element of $\mathcal{G}(\mathfrak{S}_n)$ lying over some vertex $\langle k \rangle \in (N\mathsf{N}(\mathcal{J}))_0$ such that $\langle k \rangle \in \nu_{t_{\mathbb{E}_d}}^{-1}(\langle m \rangle)$. Thus

$$\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n)_{/\langle m \rangle} = \coprod_{k \geq 0} \mathcal{G}(\mathfrak{S}_n)_{/\langle k \rangle} \times_{\mathfrak{S}_k} \coprod_{\alpha : \langle m \rangle \to \langle k \rangle} {}^t\mathbb{E}_{d/\alpha}$$

where by ${}^t\mathbb{E}_{d/\alpha}$ we mean those elements of $(\mathcal{E}_d)_1$ which lie over the map $\alpha : \langle m \rangle \to \langle k \rangle$; the second coproduct in our formula is taken over all such maps in Fin. We observe that $\mathcal{G}(\mathfrak{S}_n)_{/\langle k \rangle}$ is empty unless $k = n$, so our first coproduct collapses to one term; hence, we get

$$\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n)_{/\langle m \rangle} = \left(\coprod_{\alpha : \langle m \rangle \to \langle n \rangle} {}^t\mathbb{E}_{d/\alpha} \right)_{\mathfrak{S}_n}$$

For a fixed map of finite sets $\alpha : \langle m \rangle \to \langle n \rangle$, the space ${}^t\mathbb{E}_{d/\alpha}$ is given by

$${}^t\mathbb{E}_{d/\alpha} = \prod_{1 \leq i \leq n} {}^t\mathbb{E}_d \left(\alpha^{-1}\{i\} \right)$$

The coinvariants of the action of the group $\mathfrak{S}_n$ on this space are:

$$\prod_{1 \leq i \leq n} {}^t\mathbb{E}_d \left(\alpha^{-1}\{i\} \right) \times_{\prod_{1 \leq i \leq n} \mathfrak{S}_{\alpha^{-1}\{i\}}} \mathfrak{S}_m$$

Hence, taking the coproduct over all such α is weakly equivalent to considering the space $\mathrm{Emb}^{\mathrm{fr}}(\sqcup_m \mathbb{R}^d, \sqcup_n \mathbb{R}^d)$. In summary,

$$\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n)_{/\langle m \rangle} \simeq \mathcal{G}(\mathcal{C}_{\sqcup_n \mathbb{R}^d})_{/\langle m \rangle}$$

for all $m \geq 0$, so there is a weak equivalence $\Sigma_{\nu_{t_{\mathbb{E}_d}}} \mathcal{G}(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{G}(\mathcal{C}_{\sqcup_n \mathbb{R}^d})$ in the category of right fibrations over $\mathcal{E}_d$ for each n.

Remark 4.3.2. The above result should not be especially surprising in light of the almost identical property utilised by Dwyer, Hess and Knudsen in the proof of their main theorem. Writing $\nu_{{}^t\mathbb{E}_d!}$ for the left Kan extension from sequences to right ${}^t\mathbb{E}_d$ modules, they demonstrate that

$$\mathcal{C}_{\sqcup_n \mathbb{R}^d} \cong \mathfrak{S}_n \circ {}^t\mathbb{E}_d = \nu_{{}^t\mathbb{E}_d!}(\mathfrak{S}_n)$$

Hence, we can rephrase our above result as saying that $\mathcal{G} \circ \nu_{{}^t\mathbb{E}_d!} \simeq \Sigma_{\nu_{{}^t\mathbb{E}_d}} \circ \mathcal{G}$, which is very much to be expected given diagram (4.4) above, relating the various base-change adjunctions.

The next step is to examine how the base-change adjunction works for products of manifolds. Let $\mathcal{O}$ and $\mathcal{P}$ be operads in spaces. As we remarked in our discussion preceding Definition 4.1.12, there is a map relating the respective active categories of operators of $\mathcal{O}$ and $\mathcal{P}$ to the active category of operators of their Boardman-Vogt tensor product

$$\mu : \mathsf{F}(\mathcal{O}) \times \mathsf{F}(\mathcal{P}) \to \mathsf{F}(\mathcal{O} \star \mathcal{P})$$

This gives rise to a base-change adjunction

$$s\mathcal{S}_{/N\mathsf{F}(\mathcal{O})\times N\mathsf{F}(\mathcal{P})} \underset{\mu^*}{\overset{\Sigma_\mu}{\rightleftarrows}} s\mathcal{S}_{/N\mathsf{F}(\mathcal{O}\star\mathcal{P})}$$

We will again provide an example to demonstrate how this base-change works in practice:

Example 4.3.3. Let $\mathcal{X}$ be a right $\mathcal{O}$-module and $\mathcal{Y}$ be a right $\mathcal{P}$-module. (For our purposes, we will usually work in the situation where $\mathcal{O}$ and $\mathcal{P}$ are (skew) little-cubes operads and $\mathcal{X}$ and $\mathcal{Y}$ are configuration modules; however, at this juncture, a greater level of generality is perhaps more illuminating.)

We write $\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})$ to mean the simplicial space over $N\mathsf{F}(\mathcal{O}) \times N\mathsf{F}(\mathcal{P})$ whose fibre over the vertex $(\langle i\rangle, \langle j\rangle)$ is the space $\mathcal{G}(\mathcal{X})_{/\langle i\rangle} \times \mathcal{G}(\mathcal{Y})_{/\langle j\rangle}$ – we will refer to this as the **external product of simplicial spaces** over $N\mathsf{F}(\mathcal{O})$ and $N\mathsf{F}(\mathcal{P})$. This construction is in a sense analogous to the external tensor product of presheaves which we encountered when defining the Boardman-Vogt tensor product of modules.

Our goal in this example is to understand the fibre of $\Sigma_\mu(\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y}))$ over the vertex $\langle m\rangle$ in $N\mathsf{F}(\mathcal{O}\star\mathcal{P})$. As before, we note that such a vertex will arise as an element of $\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})$ lying over the fibre of some vertex $(\langle i\rangle, \langle j\rangle) \in (N\mathsf{F}(\mathcal{O}))_0 \times (N\mathsf{F}(\mathcal{P}))_0$ such that $(\langle i\rangle, \langle j\rangle) \in \mu^{-1}(\langle m\rangle)$, which is to say that there is a morphism $\langle m\rangle \to \langle ij\rangle$ in the active category of operators of $\mathcal{O}\star\mathcal{P}$. However, each such element $(\langle i\rangle, \langle j\rangle)$ in the fibre will be subject to quotient relations arising from the morphisms in the active categories of operators of $\mathcal{O}$ and $\mathcal{P}$ respectively: hence the fibre of $\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})$ over $\langle m\rangle$ takes the form

$$\Sigma_\mu\Big(\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})\Big)_{/\langle m\rangle} = \coprod_{i,j\geq 0}\left(\frac{\coprod_{\alpha:\langle k\rangle\to\langle ij\rangle}\big(\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})\big)_{/(\langle i\rangle,\langle j\rangle)} \times (N\mathsf{F}(\mathcal{O}\star\mathcal{P}))_{/\alpha}}{\coprod_{i'\geq 0}\coprod_{\beta:\langle i\rangle\to\langle i'\rangle}(N\mathsf{F}(\mathcal{O}))_{/\beta} \times \coprod_{j'\geq 0}\coprod_{\gamma:\langle j\rangle\to\langle j'\rangle}(N\mathsf{F}(\mathcal{P}))_{/\gamma}}\right)$$

We note again that there is an interplay here between the base-changes in the right fibration picture and the presheaf picture: specifically, recalling the the external tensor product of spaces $\mathcal{X}\hat{\times}\mathcal{Y} : \mathsf{F}(\mathcal{O})^{\mathrm{op}} \times \mathsf{F}(\mathcal{P})^{\mathrm{op}} \to \mathcal{S}$, we see that

$$\mathcal{G}(\mathcal{X}\hat{\times}\mathcal{Y}) = \mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})$$

and

$$\mathcal{G}(\mu_!(\mathcal{X}\hat{\times}\mathcal{Y}))_0 = \Sigma_\mu\big(\mathcal{G}(\mathcal{X}) \square \mathcal{G}(\mathcal{Y})\big)_0$$

In other words, the functor $\Sigma_\mu(-\square-)$ here is playing the same role as the Boardman-Vogt tensor product of modules. With this correspondence in mind, we will occasionally refer to $\Sigma_\mu(-\square-)$ as the **Boardman-Vogt tensor product of right fibrations**.

Remark 4.3.4. As a remark on the functorial behaviour of this Boardman-Vogt tensor product, we observe that for a pair of *sequences* $\mathcal{X}$, $\mathcal{Y}$, we have a weak equivalence of right fibrations over $N\mathsf{F}(\mathcal{O}\star\mathcal{P})$:

$$\Sigma_\mu\left(\Sigma_{\nu_\mathcal{O}}\mathcal{G}\left(\mathcal{X}\right) \square \Sigma_{\nu_\mathcal{P}}\mathcal{G}\left(\mathcal{Y}\right)\right) \simeq \Sigma_{\nu_{\mathcal{O}\star\mathcal{P}}}\left(\Sigma_\mu\left(\mathcal{G}\left(\mathcal{X}\right) \square \mathcal{G}\left(\mathcal{Y}\right)\right)\right)$$

where $\mu : \mathsf{F}(\mathcal{J}) \times \mathsf{F}(\mathcal{J}) \to \mathsf{F}(\mathcal{J})$ because the trivial operad $\mathcal{J}$ is the unit for the Boardman-Vogt tensor product. This relation is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{F}(\mathcal{J}) \times \mathsf{F}(\mathcal{J}) \arrow[r, "\nu_\mathcal{O}\times\nu_\mathcal{P}"] \arrow[d, "\mu"'] & \mathsf{F}(\mathcal{O}) \times \mathsf{F}(\mathcal{P}) \arrow[d, "\mu"] \\
\mathsf{F}(\mathcal{J}) \arrow[r, "\nu_{\mathcal{O}\star\mathcal{P}}"] & \mathsf{F}(\mathcal{O}\star\mathcal{P})
\end{tikzcd}$$

Remark 4.3.5. Since we will also take advantage of the right adjoint functor μ^* in our later proofs, we remind ourselves that for a simplicial space $\mathcal{X}$ over $N\mathsf{F}(\mathcal{O}\star\mathcal{P})$,

$$\mu^*(\mathcal{X}) = \mathcal{X} \times^h_{N\mathsf{F}(\mathcal{O}\star\mathcal{P})} N\mathsf{F}(\mathcal{O}) \times N\mathsf{F}(\mathcal{P})$$

where $\times^h$ indicates the homotopy pullback. We will use the model of the homotopy pullback given in [Lur09b, Definition 2.1.10].

4.3.1 A left-derived version of the right fibration tensor product

In [DHK19], it is shown that the Boardman-Vogt tensor product of modules is homotopically well-behaved with respect to the projective model structures on the respect categories of right modules. Critical to this assertion is the existence of a right adjoint to the external tensor product, as defined in Equation (4.3). Moreover, in [DHK19, Corollary 3.8], it is shown that for any fixed cofibrant $\mathcal{P}$-module, $\mathcal{N}$, the right adjoint functor $\widetilde{\mathrm{Hom}}_{\mathcal{P}}(\mathcal{N}, \mu^*(-))$ preserves weak equivalences and fibrations in the projective model structure, i.e., it is a right Quillen functor. From this, it can be deduced that the Boardman-Vogt tensor product descends to a derived version on the homotopy categories. Likewise, to be able to make suitable homotopical assertions about the Boardman-Vogt tensor product of right fibrations, we also require a suitable right adjoint to the functor $-\Box\,\mathcal{Y}$, given some fixed $\beta : \mathcal{Y} \to N\mathsf{D}$ which is cofibrant with respect to the right fibrations model structure on $s\mathcal{S}_{/N\mathsf{D}}$. We will work initially in the general setting of right fibrations over categories internal in spaces, before we restrict our attention to working with right fibrations over categories of operators. In what follows, we will explicitly write simplicial spaces as bisimplicial sets.

Construction 4.3.6. Let C and D be categories internal in spaces. We write $\Delta[n,m]$ to be the representable simplicial space (alias bisimplicial set) whose (k,l)-simplices are given by $\Delta[n]_k \times \Delta[m]_l$. Fix a cofibrant object $\beta : \mathcal{Y} \to N\mathsf{D}$ in $s\mathcal{S}_{/N\mathsf{D}}$ (with its right fibration model structure), and a map of simplicial spaces $\gamma : \mathcal{Z} \to N\mathsf{C} \times N\mathsf{D}$. Then we can define a simplicial space $\mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})$ as follows: an (n,m)-simplex of $\mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})$ is given by a pair (α, f), where α is a map of simplicial spaces $\alpha : \Delta[n,m] \to N\mathsf{C}$, and we have a commutative triangle

$$\begin{tikzcd}
\Delta[n,m] \times \mathcal{Y} \arrow[rr, "f"] \arrow[dr, "\alpha\times\beta"'] & & \mathcal{Z} \arrow[dl, "\gamma"] \\
& N\mathsf{C} \times N\mathsf{D} &
\end{tikzcd}$$

This simplicial space is equipped with an obvious reference map to $N\mathsf{C}$, sending an (n,m)-simplex, (α, f) to the element of $N\mathsf{C}_n$ associated to the map α. By construction, given a simplicial space $\Delta[n,m]^{(\alpha)}$ corresponding to a fixed map of simplicial spaces $\alpha : \Delta[n,m] \to N\mathsf{C}$, we have an isomorphism

$$\mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}}}\left(\Delta[n,m]^{(\alpha)}, \mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})\right) \cong \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}}\left(\Delta[n,m]^{(\alpha)} \,\Box\, \mathcal{Y}, \mathcal{Z}\right)$$

We can take advantage of this latter property of the simplicial space $\mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, -)$ to show that it is indeed a right adjoint for the map $-\Box\,\mathcal{Y}$. The key to this argument is the fact that we can write any presheaf as a colimit of representables; in particular, given an element $\delta : \mathcal{X} \to N\mathsf{C}$ of $s\mathcal{S}_{/N\mathsf{C}}$, we can write $\mathcal{X} = \mathrm{colim}_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}} \Delta[n,m]^{(\alpha)}$, where each $\Delta[n,m]^{(\alpha)}$ in the image of the diagram corresponds to a map of simplicial spaces $\alpha : \Delta[n,m] \to N\mathsf{C}$. Thus, we see that

$$\begin{aligned}
\mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}}}\left(\mathcal{X}, \mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})\right) &= \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}}}\left(\operatorname*{colim}_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}}\left(\Delta[n,m]^{(\alpha)}\right), \mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})\right) \\
&\cong \lim_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}} \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}}}\left(\Delta[n,m]^{(\alpha)}, \mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y}, \mathcal{Z})\right) \\
&\cong \lim_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}} \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}}\left(\Delta[n,m]^{(\alpha)} \,\Box\, \mathcal{Y}, \mathcal{Z}\right) \\
&\cong \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}}\left(\operatorname*{colim}_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}}\left(\Delta[n,m]^{(\alpha)} \,\Box\, \mathcal{Y}\right), \mathcal{Z}\right) \\
&\overset{\dagger}{\cong} \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}}\left(\left(\operatorname*{colim}_{\Delta[n,m]^{(\alpha)}\downarrow\mathcal{X}} \Delta[n,m]^{(\alpha)}\right) \Box\, \mathcal{Y}, \mathcal{Z}\right) \\
&\cong \mathrm{Hom}_{s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}}\left(\mathcal{X} \,\Box\, \mathcal{Y}, \mathcal{Z}\right)
\end{aligned}$$

where at †, we utilised the fact that the external product of simplicial spaces preserves colimits in both variables. Hence, we have the desired right adjoint. Our next aim is to show that this adjunction is compatible with the respective model structures.

Proposition 4.3.7. *Let $\beta : \mathcal{Y} \to N\mathsf{D}$ be a fixed cofibrant object. Then the functor $-\square\, \mathcal{Y} : s\mathcal{S}_{/N\mathsf{C}} \to s\mathcal{S}_{/N\mathsf{C}\times N\mathsf{D}}$ preserves cofibrations and weak equivalences.*

Proof. If $\mathcal{X} \to \mathcal{X}'$ is a levelwise monomorphism of simplicial spaces over $N\mathsf{C}$, then it is immediate that $\mathcal{X} \,\square\, \mathcal{Y} \to \mathcal{X}' \,\square\, \mathcal{Y}$ is a levelwise monomorphism of simplicial spaces over $N\mathsf{C} \times N\mathsf{D}$, since, as a simplicial space, $\mathcal{X} \,\square\, \mathcal{Y} = \mathcal{X} \times \mathcal{Y}$.

To show that $-\square\, \mathcal{Y}$ preserves trivial cofibrations, it will suffice to show that it preserves the generating trivial cofibrations for the right fibration model structure. These in turn come from the generating cofibrations of the fibrewise Reedy model structure on $s\mathcal{S}_{/N\mathsf{C}}$, namely those commutative triangles of the form

$$\begin{tikzcd}
\partial F(k) \times \Delta[l] \underset{\partial F(k)\times\Lambda[l,t]}{\sqcup} F(k) \times \Lambda[l,t] \arrow[rr, tail, "\sim"] \arrow[dr] & & F(k) \times \Delta[l] \arrow[dl] \\
& N\mathsf{C} &
\end{tikzcd} \tag{4.5}$$

for $0 \le k$ and $0 \le t \le l$; and the generating trivial cofibrations in the right fibration model structure, given by commutative triangles of the form

$$\begin{tikzcd}
F(k,t) \times \Delta[l] \underset{F(k,t)\times\partial\Delta[l]}{\sqcup} F(k) \times \partial\Delta[l] \arrow[rr, tail, "\sim"] \arrow[dr] & & F(k) \times \Delta[l] \arrow[dl] \\
& N\mathsf{C} &
\end{tikzcd} \tag{4.6}$$

for $0 \le t \le k$ and $0 \le l$. We draw attention to our notation here: as bisimplicial sets, we have $F(k)_{n,m} = \Delta[k]_n$ for all m, while $\Delta[l]_{n,m} = \Delta[l]_m$ for all n; we also write $F(k,t)_{n,m} = \Lambda[k,t]_n$ for all m, while we write $\Lambda[l,t]_{n,m} = \Lambda[l,t]_m$ for all n. It is sufficient to demonstrate that for each $0 \le k$ and $0 \le t \le l$, we have weak equivalences of simplicial spaces over $N\mathsf{C} \times N\mathsf{D}$,

$$\left(\partial F(k) \times \Delta[l] \underset{\partial F(k)\times\Lambda[l,t]}{\sqcup} F(k) \times \Lambda[l,t]\right) \square\, \mathcal{Y} \simeq (F(k) \times \Delta[l]) \,\square\, \mathcal{Y}$$

and likewise for $0 \le l$ and $0 \le t \le k$

$$\left(F(k,t) \times \Delta[l] \underset{F(k,t)\times\partial\Delta[l]}{\sqcup} F(k) \times \partial\Delta[l]\right) \square\, \mathcal{Y} \simeq (F(k) \times \Delta[l]) \,\square\, \mathcal{Y}$$

Spelling this out explicitly in terms of the right fibration model structure, this means that for every right fibration $\mathcal{Q} \to N\mathsf{C} \times N\mathsf{D}$, we have weak equivalences between the derived simplicial mapping spaces

$$\mathbb{R}\mathrm{Map}_{N\mathsf{C}\times N\mathsf{D}}\left(\left(\partial F(k) \times \Delta[l] \underset{\partial F(k)\times\Lambda[l,t]}{\sqcup} F(k) \times \Lambda[l,t]\right) \square\, \mathcal{Y}, \mathcal{Q}\right) \simeq \mathbb{R}\mathrm{Map}_{N\mathsf{C}\times N\mathsf{D}}\left((F(k) \times \Delta[l]) \,\square\, \mathcal{Y}, \mathcal{Q}\right)$$

and

$$\mathbb{R}\mathrm{Map}_{N\mathsf{C}\times N\mathsf{D}}\left(\left(F(k,t) \times \Delta[l] \underset{F(k,t)\times\partial\Delta[l]}{\sqcup} F(k) \times \partial\Delta[l]\right) \square\, \mathcal{Y}, \mathcal{Q}\right) \simeq \mathbb{R}\mathrm{Map}_{N\mathsf{C}\times N\mathsf{D}}\left((F(k) \times \Delta[l]) \,\square\, \mathcal{Y}, \mathcal{Q}\right)$$

where the derived fibrewise mapping space $\mathbb{R}\mathrm{Map}_{N\mathsf{C}\times N\mathsf{D}}(\mathcal{Z}, \mathcal{Q})$ is defined as the limit of the diagram

$$* \to \mathbb{R}\mathrm{Map}\left(\mathcal{Z}, N\mathsf{C} \times N\mathsf{D}\right) \leftarrow \mathbb{R}\mathrm{Map}\left(\mathcal{Q}, N\mathsf{C} \times N\mathsf{D}\right) \tag{4.7}$$

Using the Quillen equivalence between simplicial spaces with the complete Segal model structure and simplicial sets with the Joyal model structure (as per [JT06]), we can prevail upon [Lur09a, Corollary 2.1.2.7] to see that the maps

$$\left(F(k,t) \times \Delta[l] \underset{F(k,t)\times\partial\Delta[l]}{\sqcup} F(k) \times \partial\Delta[l]\right) \times \mathcal{Y} \simeq (F(k) \times \Delta[l]) \times \mathcal{Y}$$

are weak equivalences in the Segal model structure, thus giving the desired weak equivalence in the fibrewise model structure.

For the other equivalence, we utilise [Dug01, Lemma 4.4], which tells us that since the top horizontal map in (4.5) is already a Reedy weak equivalence, then by taking products with a fixed simplicial space $\mathcal{Y}$, we still obtain a Reedy weak equivalence:

$$\left(\partial F(k) \times \Delta[l] \underset{\partial F(k) \times \Lambda[l,t]}{\sqcup} F(k) \times \Lambda[l,t]\right) \times \mathcal{Y} \simeq (F(k) \times \Delta[l]) \times \mathcal{Y}$$

Hence we are left with weak equivalences of simplicial sets

$$\mathbb{R}\mathrm{Map}\left(\left(\partial F(k) \times \Delta[l] \underset{\partial F(k) \times \Lambda[l,t]}{\sqcup} F(k) \times \Lambda[l,t]\right) \times \mathcal{Y}, N\mathsf{C} \times N\mathsf{D}\right) \simeq \mathbb{R}\mathrm{Map}\left((F(k) \times \Delta[l]) \times \mathcal{Y}, N\mathsf{C} \times N\mathsf{D}\right)$$

and

$$\mathbb{R}\mathrm{Map}\left(\left(F(k,t) \times \Delta[l] \underset{F(k,t) \times \partial\Delta[l]}{\sqcup} F(k) \times \partial\Delta[l]\right) \times \mathcal{Y}, N\mathsf{C} \times N\mathsf{D}\right) \simeq \mathbb{R}\mathrm{Map}\left((F(k) \times \Delta[l]) \times \mathcal{Y}, N\mathsf{C} \times N\mathsf{D}\right)$$

Based on the definition of the derived fibrewise mapping spaces in (4.7), we deduce the claim. □

Now we can use the fact that base-change is a Quillen adjunction with respect to the right fibration model structure to see that for any functor $\phi : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ (whose induced map on the respective nerves we also denote by ϕ), we also have a Quillen adjunction

$$s\mathcal{S}_{/N\mathsf{C}} \underset{\mathfrak{Hom}^{/\mathsf{C}}(\mathcal{Y},\phi^*(-))}{\overset{\Sigma_\phi \circ (- \square\, \mathcal{Y})}{\rightleftarrows}} s\mathcal{S}_{/N\mathsf{E}}$$

In particular, we can consider the case where $\mathsf{C} = \mathsf{F}(\mathcal{O})$, $\mathsf{D} = \mathsf{F}(\mathcal{P})$ and $\mathsf{E} = \mathsf{F}(\mathcal{O} \star \mathcal{P})$, and $\phi = \mu$ as defined in (4.2), which leads to our being able to define a left-derived version of the Boardman-Vogt tensor product of right fibrations:

$$\mathbb{L}\Sigma_\mu(-\square-) : \mathrm{Ho}\left(s\mathcal{S}_{/N\mathsf{F}(\mathcal{O})}\right) \times \mathrm{Ho}\left(s\mathcal{S}_{/N\mathsf{F}(\mathcal{P})}\right) \to \mathrm{Ho}\left(s\mathcal{S}_{/N\mathsf{F}(\mathcal{O}\star\mathcal{P})}\right) \tag{4.8}$$

4.4 An Alternative Proof of the Product Theorem

In this section, we provide a version of Theorem 4.1.14 from the right fibrations perspective. As in the work of Dwyer, Hess and Knudsen, our proof will essentially rely on a local-to-global argument and the Dunn additivity theorem. This may seem like an unnecessary exercise in translating from one perspective to another: however, at this point, we alert the reader that by working with right fibrations rather than right modules, we can produce a slightly modified version of the Boardman-Vogt tensor product of right fibrations which makes use of the quasi-categorical tensor product of quasi-operads. We will investigate this object in greater depth in the final section of this chapter, and deduce a somewhat generalised version of the product theorem we state in this section using very similar arguments.

The fact that we have the choice of these two approaches to the definition of the tensor product of right fibrations is a direct result of the fact that we have moved to the setting of simplicial spaces over Segal spaces – such a choice is not so readily available when we work with the notion of modules as presheaves on some fixed category. This should retrospectively justify the labour of translating the existing work of Dwyer, Hess and Knudsen into this new language. (The advantage of the modified version of the tensor product of right fibrations is that the ∞-categorical equivalents of the skew little-cube operads are known to satisfy a suitable analogue of the Dunn additivity relation, as per Theorem 3.2.3, so that we can extend the results of Dwyer, Hess and Knudsen to the setting of skew little-cubes.)

With a view to fixing our notation, we recall that the additivity theorem for little-cubes operads in spaces told us that there is a levelwise weak equivalence of operads $\iota :\ {}^t\mathbb{E}_d \star^t \mathbb{E}_{d'} \to\ {}^t\mathbb{E}_{d+d'}$. We can now state the "right fibrations version" of the original product theorem of Dwyer, Hess and Knudsen (Theorem 4.1.14):

Theorem 4.4.1 (Product Theorem for Framed Manifolds)**.** *Let M be a framed d-manifold and let N be a framed d'-manifold. Then, there is an isomorphism*

$$\mathbb{L}\Sigma_\mu\Big(\mathcal{G}(\mathcal{C}_M) \,\square\, \mathcal{G}(\mathcal{C}_N)\Big) \cong \mathrm{Ho}(\iota^*)\mathcal{G}(\mathcal{C}_{M\times N})$$

in the homotopy category of right fibrations over $N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})$.

Before giving a proof of this, we need to explain our local-to-global methodology and prove some results using this.

Definition 4.4.2. A **Weiss cover** of a manifold M is a cover $\mathcal{U}$ of M such that any finite subset of M is contained in an element of $\mathcal{U}$. A **complete Weiss cover** of M is a Weiss cover $\mathcal{U}$ which satisfies the property that for any finite subcollection $\mathcal{U}_0$ in $\mathcal{U}$, there is a Weiss cover of the intersection $\cap_{U\in\mathcal{U}_0} U$ contained in $\mathcal{U}$. If V is a model category, we say that a functor $H : \mathsf{Mfld}_d^G \to \mathsf{V}$ is **multi-local** if, for each object M in Mfld_d^G and every complete Weiss cover $\mathcal{U}$ of M, there is an isomorphism

$$\operatorname*{hocolim}_{U\in\mathcal{U}} H(U) \cong H(M)$$

in the homotopy category.

For a d-manifold M, let $\mathrm{Disk}(M)$ be the collection of open subsets of M diffeomorphic to a disjoint union of copies of $\mathbb{R}^d$. Note that $\mathrm{Disk}(M)$ is a complete Weiss cover since, for any finite subcollection $\mathcal{U}_0$ in $\mathrm{Disk}(M)$, we have $\cap_{U\in\mathcal{U}_0} U \in \mathrm{Disk}(M)$. Moreover, given manifolds M and N, the cover $\mathrm{Disk}(M)\times\mathrm{Disk}(N)$ also meets the defining requirement for being a complete Weiss cover on $M\times N$. (For details of this argument, see [DHK19, Lemma 5.3]). Our next result draws from the methods of both [DHK19, Lemma 5.5] and [BW18a, Corollary 3.6]:

Lemma 4.4.3. *The functor* $\mathsf{Mfld}_d^{\mathrm{fr}} \to s\mathsf{S}_{/\mathcal{E}_d}$, $M \mapsto \mathcal{G}(\mathcal{C}_M)$ *is multi-local.*

Proof. If $\mathcal{U}$ is a complete Weiss cover of M, then the collection $\{\mathrm{Conf}_k(U) : U\in\mathcal{U}\}$ is a complete cover (in the sense of [DI01]) of $\mathrm{Conf}_k(M)$ for each $k\geq 0$, from which it follows that

$$\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{C}_U(k) \cong \mathcal{C}_M(k)$$

is an isomorphism in the homotopy category for all k.

Taking the disjoint union over all $k\geq 0$ (and noting that the homotopy colimit commutes with this coproduct), we see that we have an isomorphism

$$\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{G}(\mathcal{C}_U)_0 \cong \mathcal{G}(\mathcal{C}_M)_0$$

in the homotopy category of spaces the homotopy category of right fibrations over $\mathcal{E}_d$. Now we note that for any $U\in\mathcal{U}$ and any $r\geq 1$, the commutative diagram

$$\begin{tikzcd} \mathcal{G}(\mathcal{C}_U)_r \arrow[r] \arrow[d, "d_1\circ\ldots\circ d_r"'] & \mathcal{G}(\mathcal{C}_M)_r \arrow[d, "d_1\circ\ldots\circ d_r"] \\ \mathcal{G}(\mathcal{C}_U)_0 \arrow[r] & \mathcal{G}(\mathcal{C}_M)_0 \end{tikzcd}$$

is homotopy Cartesian (this is essentially immediate when we recall the definition of the Grothendieck construction). Because homotopy colimits are stable under homotopy base-change, we find that $\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{G}(\mathcal{C}_U)_r$ is weak equivalent to the homotopy pullback of the diagram

$$\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{G}(\mathcal{C}_U)_0 \to \mathcal{G}(\mathcal{C}_M)_0 \leftarrow \mathcal{G}(\mathcal{C}_M)_r$$

but we have already demonstrated that the left arrow in this diagram is a weak equivalence, so that for each $r\geq 1$, we also have levelwise weak equivalences of simplicial spaces over $N\mathsf{F}({}^t\mathbb{E}_d)$

$$\operatorname*{hocolim}_{U\in\mathcal{U}} \mathcal{G}(\mathcal{C}_U)_r \simeq \mathcal{G}(\mathcal{C}_M)_r$$

and hence this is also a weak equivalence in the right fibration model structure. $\square$

Remark 4.4.4. To prevent ourselves drowning in a sea of notation (already a risk), we will identify a morphism

$$\gamma = \left(\langle j\rangle \xrightarrow{\alpha} \langle i\rangle, (c_r)_{1\leq r\leq i} \in \prod_r {}^t\mathbb{E}_d(\alpha^{-1}\{r\})\right)$$

in $\mathrm{Hom}_{\mathsf{F}({}^t\mathbb{E}_d)}(\langle i\rangle,\langle j\rangle)$ with the framed embedding $\sqcup_i\mathbb{R}^d \hookrightarrow \sqcup_j\mathbb{R}^d$ it induces.

Proposition 4.4.5. *For $U \in \mathrm{Disk}(M)$ and $V \in \mathrm{Disk}(N)$, there is an isomorphism:*

$$\mathbb{L}\Sigma_\mu\Big(\mathcal{G}(\mathcal{C}_U) \,\square\, \mathcal{G}(\mathcal{C}_V)\Big) \cong \mathrm{Ho}(\iota^*)\mathcal{G}(\mathcal{C}_{U\times V})$$

in the homotopy category of right fibrations over $N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})$.

Proof. We begin by exhibiting a suitable map in the category of right fibrations. The key to this will actually be to deduce an adjoint map

$$f : \mathcal{G}(\mathcal{C}_U) \,\square\, \mathcal{G}(\mathcal{C}_V) \to \mu^*\iota^*(\mathcal{G}(\mathcal{C}_{U\times V}))$$

The existence of such a map can be deduced once we recall that $\mu^*\iota^*$ acts on the simplicial space $\mathcal{G}(\mathcal{C}_{U\times V})$ over $N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})$ by:

$$\mu^*\iota^*\left(\mathcal{G}(\mathcal{C}_{U\times V})\right) = \left(N\mathsf{F}({}^t\mathbb{E}_d) \times N\mathsf{F}({}^t\mathbb{E}_{d'})\right) \times^h_{N\mathsf{F}({}^t\mathbb{E}_d\star^t\mathbb{E}_{d'})} N\mathsf{F}\left({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}\right) \times^h_{N\mathsf{F}({}^t\mathbb{E}_{d+d'})} \mathcal{G}\left(\mathcal{C}_{U\times V}\right)$$

We will see that the map f is essentially a natural inclusion in this space. It will suffice to give the map in degrees 0 and 1; the Segal property will determine the map in all higher degrees.

A vertex of $\mathcal{G}(\mathcal{C}_U) \,\square\, \mathcal{G}(\mathcal{C}_V)$ will take the form of a pair $(\rho_U, \rho_V) \in \mathcal{C}_U(i_0) \times \mathcal{C}_V(j_0)$. If we let $\mathrm{cons}^{\langle n\rangle}_{d\star d'}$ denote the constant path $\langle n\rangle$ to $\langle n\rangle$ in $(N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}))_0$, and we let $\mathrm{cons}^{\langle n\rangle}_{d+d'}$ denote the constant path from $\langle n\rangle$ to $\langle n\rangle$ in $(N\mathsf{F}({}^t\mathbb{E}_{d+d'}))_0$, then f acts on 0-simplices by

$$(\rho_U, \rho_V) \mapsto (\langle i_0\rangle, \langle j_0\rangle, \mathrm{cons}^{i_0j_0}_{d\star d'}, \langle i_0j_0\rangle, \mathrm{cons}^{i_0j_0}_{d+d'}, \rho_U \times \rho_V)$$

A 1-simplex of $\mathcal{G}(\mathcal{C}_U) \,\square\, \mathcal{G}(\mathcal{C}_V)$ is of the form

$$(\tilde\varepsilon_d, \rho_U, \tilde\varepsilon_{d'}, \rho_V) \in N\mathsf{F}({}^t\mathbb{E}_d)_{/\langle i_1\rangle\to\langle i_0\rangle} \times \mathcal{C}_U(i_0) \times N\mathsf{F}({}^t\mathbb{E}_{d'})_{/\langle j_1\rangle\to\langle j_0\rangle} \times \mathcal{C}_V(j_0)$$

On the other hand, a 1-simplex of $\mu^*\iota^*\mathcal{G}(\mathcal{C}_{U\times V})$ is of the form $(\tilde\varepsilon_d, \tilde\varepsilon_{d'}, \gamma_1, \tilde\varepsilon_{d\star d'}, \gamma_2, \tilde\varepsilon_{d+d'}, \rho_{U\times V})$ where

- $\tilde\varepsilon_d \in N\mathsf{F}({}^t\mathbb{E}_d)_{/\langle i_1\rangle\to\langle i_0\rangle}$
- $\tilde\varepsilon_{d'} \in N\mathsf{F}({}^t\mathbb{E}_{d'})_{/\langle j_1\rangle\to\langle j_0\rangle}$
- $\tilde\varepsilon_{d\star d'} \in N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})_{/\langle k_1\rangle\to\langle k_0\rangle}$
- γ_1 is a path in $(N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}))_1$ from $\mu(\tilde\varepsilon_d, \tilde\varepsilon_{d'})$ to $\tilde\varepsilon_{d\star d'}$,
- $\tilde\varepsilon_{d+d'} \in N\mathsf{F}({}^t\mathbb{E}_{d+d'})_{/\langle l_1\rangle\to\langle l_0\rangle}$
- γ_2 is a path in $(N\mathsf{F}({}^t\mathbb{E}_{d+d'}))_1$ from $\iota(\tilde\varepsilon_{d\star d'})$ to $\tilde\varepsilon_{d+d'}$,
- $\rho_{U\times V} \in \mathcal{C}_{U\times V}(l_0)$.

Again, if δ is a fixed 1-simplex in $N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})$, we write $\mathrm{cons}^\delta_{d\star d'}$ to mean the constant path from δ to itself; and likewise, for a fixed 1-simplex δ in $N\mathsf{F}({}^t\mathbb{E}_{d+d'})$, we write $\mathrm{cons}^\delta_{d+d'}$ for the constant path from δ to itself. We see that the map f should act on 1-simplices by

$$(\tilde\varepsilon_d, \rho_U, \tilde\varepsilon_{d'}, \rho_V) \mapsto \Big(\tilde\varepsilon_d, \tilde\varepsilon_{d'}, \mathrm{cons}^{\mu(\tilde\varepsilon_d,\tilde\varepsilon_{d'})}_{d\star d'}, \mu(\tilde\varepsilon_d, \tilde\varepsilon_{d'}), \mathrm{cons}^{(\iota\circ\mu)(\tilde\varepsilon_d,\tilde\varepsilon_{d'})}_{d+d'}, (\iota\circ\mu)(\tilde\varepsilon_d, \tilde\varepsilon_{d'}), \rho_U \times \rho_V\Big)$$

By the adjunction, it follows that there is a corresponding map

$$\Sigma_\mu\Big(\mathcal{G}(\mathcal{C}_U) \,\square\, \mathcal{G}(\mathcal{C}_V)\Big) \to \iota^*\mathcal{G}(\mathcal{C}_{U\times V})$$

It only remains to show that this map is a weak equivalence of right fibrations over $N\mathsf{F}({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'})$. We fix $U = \sqcup_n \mathbb{R}^d$ and let $V = \sqcup_m \mathbb{R}^{d'}$. By Example 4.3.1 we have weak equivalences

$$\mathcal{G}(\mathcal{C}_{\sqcup_n\mathbb{R}^d}) \simeq \Sigma_{\nu_{{}^t\mathbb{E}_d}}\mathcal{G}(\mathfrak{S}_n) \quad \text{and} \quad \mathcal{G}(\mathcal{C}_{\sqcup_m\mathbb{R}^{d'}}) \simeq \Sigma_{\nu_{{}^t\mathbb{E}_{d'}}}\mathcal{G}(\mathfrak{S}_m)$$

Hence, to show the required weak equivalence, the two-out-of-three property tells us that it is sufficient to show that the top horizontal map in the commutative diagram of simplicial spaces over $N\mathsf{F}\left({}^t\mathbb{E}_d \star {}^t\mathbb{E}_{d'}\right)$

$$\begin{tikzcd}
\Sigma_\mu\left(\Sigma_{\nu_{{}^t\mathbb{E}_d}}\mathcal{G}\left(\mathfrak{S}_n\right) \,\square\, \Sigma_{\nu_{{}^t\mathbb{E}_{d'}}}\mathcal{G}\left(\mathfrak{S}_m\right)\right) \arrow[r] \arrow[d, "\simeq"'] & \iota^*\Sigma_{\nu_{{}^t\mathbb{E}_{d+d'}}}\mathcal{G}\left(\mathfrak{S}_{nm}\right) \arrow[d, "\simeq"] \\
\Sigma_\mu\left(\mathcal{G}\left(\mathcal{C}_{\sqcup_n\mathbb{R}^d}\right) \,\square\, \mathcal{G}\left(\mathcal{C}_{\sqcup_m\mathbb{R}^{d'}}\right)\right) \arrow[r] & \iota^*\mathcal{G}\left(\mathcal{C}_{\sqcup_{nm}\mathbb{R}^{d+d'}}\right)
\end{tikzcd}$$

is a weak equivalence. Recall however from Remark 4.3.4 that we also have the relation

$$\Sigma_\mu\left(\Sigma_{\nu_{t_{\mathbb{E}_d}}}\mathcal{G}(\mathfrak{S}_n)\ \Box\ \Sigma_{\nu_{t_{\mathbb{E}_{d'}}}}\mathcal{G}(\mathfrak{S}_m)\right) \simeq \Sigma_{\nu_{t_{\mathbb{E}_d}\star^t\mathbb{E}_{d'}}}\left(\Sigma_\mu\left(\mathcal{G}(\mathfrak{S}_n)\ \Box\ \mathcal{G}(\mathfrak{S}_m)\right)\right)$$

and by the Dunn additivity theorem, this latter simplicial space is weakly equivalent to

$$\iota^*\Sigma_{\nu_{t_{\mathbb{E}_{d+d'}}}} \circ \Sigma_\mu\left(\mathcal{G}\left(\mathfrak{S}_n\right)\ \Box\ \mathcal{G}\left(\mathfrak{S}_m\right)\right)$$

Hence, to conclude, we only need to show that there is a weak equivalence of simplicial spaces lying over $N\mathsf{N}(\mathcal{J})$:

$$\Sigma_\mu\Big(\mathcal{G}(\mathfrak{S}_n)\ \Box\ \mathcal{G}(\mathfrak{S}_m)\Big) \simeq \mathcal{G}(\mathfrak{S}_{nm})$$

The fibre of the simplicial space on the left over $\langle k\rangle$ is the space

$$\sqcup_{ij=k}\mathcal{G}(\mathfrak{S}_n)_{/\langle i\rangle} \times \mathcal{G}(\mathfrak{S}_m)_{/\langle j\rangle}$$

Hence, we see that this is empty except in the case where $k = nm$. Thus the two discrete simplicial spaces are the same in degree 0 (so by discreteness, in all degrees), giving the desired equivalence. □

Having established the local version of our product theorem and the local-to-global principle for our right fibrations, we are finally in a position to prove Theorem 4.4.1:

Proof of 4.4.1. There are isomorphisms in the homotopy category of right fibrations over $N\mathsf{F}\left({}^t\mathbb{E}_d \star^t \mathbb{E}_{d'}\right)$:

$$\begin{aligned}
\mathbb{L}\Sigma_\mu\Big(\mathcal{G}(\mathcal{C}_M)\ \Box\ \mathcal{G}(\mathcal{C}_N)\Big) &\cong \mathbb{L}\Sigma_\mu\Big(\operatorname*{hocolim}_{U\in \mathrm{Disk}(M)} \mathcal{G}(\mathcal{C}_U)\ \Box \operatorname*{hocolim}_{V\in\mathrm{Disk}(N)} \mathcal{G}(\mathcal{C}_V)\Big)\\
&\cong \operatorname*{hocolim}_{U\in \mathrm{Disk}(M)}\ \operatorname*{hocolim}_{V\in\mathrm{Disk}(N)} \mathbb{L}\Sigma_\mu\Big(\mathcal{G}(\mathcal{C}_U)\ \Box\ \mathcal{G}(\mathcal{C}_V)\Big)\\
&\cong \operatorname*{hocolim}_{(U,V)\in\mathrm{Disk}(M)\times\mathrm{Disk}(N)} \mathrm{Ho}(\iota^*)\ \mathcal{G}(\mathcal{C}_{U\times V})\\
&\cong \mathrm{Ho}(\iota^*)\Big(\operatorname*{hocolim}_{(U,V)\in\mathrm{Disk}(M)\times\mathrm{Disk}(N)} \mathcal{G}(\mathcal{C}_{U\times V})\Big)\\
&\cong \mathrm{Ho}(\iota^*)\ \mathcal{G}(\mathcal{C}_{M\times N})
\end{aligned}$$

where in the first and fifth lines we use the multi-locality result of Lemma 4.4.3; in the second, we use the fact that $\mathbb{L}\Sigma_\mu(-\Box-)$ distributes over homotopy colimits in both variables; in the third line, we use Proposition 4.4.5; and in the fourth line we note that the restriction functor commutes with homotopy colimits. □

4.5 A Product Theorem for G-Framed Manifolds

As alluded to previously, one of the advantages of using the approach of describing configuration modules from the right fibrations perspective is that we can more easily dip into the world of ∞-categories. One of the restrictions to the utility of Theorem 4.1.14 is that it is only applicable in the setting where both manifolds are framed, but as we noted when discussing framings, the class of framed manifolds is relatively small. The reason for this limited applicability stems from the fact that the proof of that result (and our own analogue of it in the setting of right fibrations) requires the additivity theorem for the topological little d-cubes operads. While this result is known for the standard little d-cubes, the question of whether it also holds for the topological skew little cubes operads of Example 2.1.13 remains open.

On the other hand, we know from the work of Lurie (see Section 3.2.2) that there *is* a version of the additivity theorem available when we work with homotopy-coherent operads and the associated tensor product. Since we can identify a homotopy-coherent G-skew little cube operad with the nerve of the corresponding topological G-skew little cube operad, we begin to see how a shift to the right fibrations picture may assist us in generalising the aforementioned product theorem to a broader class of manifolds. Of course, we remark that since we are now planning to work with a different tensor product, we are longer in the situation of directly mirroring the Boardman-Vogt tensor product of right modules as specified by [DHK19]. Nonetheless, since the motivation for that construction (at least in the context of configuration modules) was to determine a relationship between the configuration spaces of a product manifold in terms of the configuration spaces of its factors, we feel reasonably justified in "shifting the goalposts" of the construction.

To make this more clear we need to specify some notation: for a quasi-operad $\mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$, we let $\mathcal{O}^{\mathrm{act}}$ denote the pullback over the simplicial set $N\mathsf{Fin}$ – this is the quasi-categorical version of looking at the category $\mathsf{F}(\mathcal{O})$ instead of the full category of operators. Given a dilation representation of G in $GL_d(\mathbb{R})$, Remark 3.2.2 tells us that there is a weak equivalence $N\left({}^t\mathbb{E}_d^G\right)^{\otimes} \simeq \mathbb{E}_d^{G,\otimes}$, which pulls back to a weak equivalence of (discrete) simplicial spaces $N\mathsf{F}({}^t\mathbb{E}_d^G) \simeq \mathbb{E}_d^{G,\mathrm{act}}$. Thus, if M is a G-framed d-manifold, there is a right fibration

$$\mathcal{G}(\mathcal{C}_M^G) \to N(\mathsf{F}({}^t\mathbb{E}_d^G)) \simeq \mathbb{E}_d^{G,\mathrm{act}}$$

By definition of the quasi-categorical tensor product, we have a bifunctor

$$\theta : \mathbb{E}_d^{G,\otimes} \times \mathbb{E}_{d'}^{G',\otimes} \to \mathbb{E}_d^{G,\otimes} \odot \mathbb{E}_{d'}^{G',\otimes}$$

and this restricts to a map on the active parts of the respective quasi-operads, which we also denote by θ. We now recall Lurie's additivity result for skew little cubes (Theorem 3.2.3), which can be rephrased as saying that there is a a weak equivalence of quasi-operads,

$$\iota : \mathbb{E}_d^{G,\otimes} \odot \mathbb{E}_{d'}^{G',\otimes} \xrightarrow{\simeq} \mathbb{E}_{d+d'}^{G\times G',\otimes}$$

where G and G' are locally compact Hausdorff groups with dilation representations in $GL_d(\mathbb{R})$, $GL_{d'}(\mathbb{R})$ respectively. This map also restricts to a weak equivalence on the active parts of the respective operads, which we again denote by ι. Having put these notations in place, many of the steps we take hereafter will closely mirror those of the previous section. We first state the desired result:

Theorem 4.5.1. *Let G and G' be locally compact Hausdorff groups with representations in $GL_d(\mathbb{R})$, $GL_{d'}(\mathbb{R})$ respectively. Let M be a G-framed d-manifold and let N be a G'-framed d'-manifold. Then, there is an equivalence*

$$\mathbb{L}\Sigma_\theta\left(\mathcal{G}(\mathcal{C}_M^G) \,\square\, \mathcal{G}(\mathcal{C}_N^{G'})\right) \simeq \mathrm{Ho}(\iota^*)\mathcal{G}(\mathcal{C}_{M\times N}^{G\times G'})$$

in the homotopy category of right fibrations over $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}$.

As advertised, the proof is in a very similar spirit to what we described above. First, we note that [DHK19, Lemma 5.5] tells us that the G-framed configuration spaces also satisfy the multi-locality property, so that we have weak equivalences

$$\operatorname*{hocolim}_{U\in\mathrm{Disk}(M)} \mathcal{C}_U^G(k) \simeq \mathcal{C}_M^G(k)$$

for each k, and hence since coproducts commute with homotopy colimits, we have weak equivalences

$$\operatorname*{hocolim}_{U\in\mathrm{Disk}(M)} \mathcal{G}\left(\mathcal{C}_U^G\right)_0 \simeq \mathcal{G}\left(\mathcal{C}_M^G\right)_0$$

Then by the same arguments as in Lemma 4.4.3, we obtain levelwise weak-equivalences of simplicial spaces over $\mathbb{E}_d^{G,\mathrm{act}}$

$$\operatorname*{hocolim}_{U\in\mathrm{Disk}(M)} \mathcal{G}\left(\mathcal{C}_U^G\right) \simeq \mathcal{G}\left(\mathcal{C}_M^G\right)$$

which in turn means that we have weak equivalences in the homotopy category of right fibrations over $\mathbb{E}_d^{G,\mathrm{act}}$.

Using this multi-locality property, we will attempt to devise a local version of Theorem 4.5.1, much as we did in the analogous setting in Proposition 4.4.5. Before we do that, we observe that for each G we have maps of simplicial spaces

$$\tilde{\nu}_{\mathbb{E}_d^G} : N\mathsf{N}(\mathcal{J}) \to \mathbb{E}_d^{G,\mathrm{act}}$$

coming from the unit map for the quasi-operad $\mathbb{E}_d^{G,\otimes}$. This leads to a base-change pair

$$\Sigma_{\tilde{\nu}_{\mathbb{E}_d^G}} : s\mathcal{S}_{/N\mathsf{F}(\mathcal{J})} \rightleftarrows s\mathcal{S}_{/\mathbb{E}_d^{G,\mathrm{act}}} : \tilde{\nu}_{\mathbb{E}_d^G}^*$$

Using the weak equivalence between $\mathbb{E}_d^{G,\mathrm{act}}$ and $N\mathsf{F}({}^t\mathbb{E}_d^G)$ and the same arguments as in Example 4.3.1, we can relate the G-framed configuration fibrations of (disjoint copies of) Euclidean spaces to certain right fibrations over $N\mathsf{N}(\mathcal{J})$:

Lemma 4.5.2. *There is an equivalence*

$$\mathcal{G}(\mathcal{C}^G_{\sqcup_n \mathbb{R}^d}) \simeq \Sigma_{\tilde{\nu}_{\mathbb{E}^G_d}} \mathcal{G}(\mathfrak{S}_n)$$

in the homotopy category of right fibrations over $\mathbb{E}_d^{G,\mathrm{act}}$.

Remark 4.5.3. We also remark that since $N\mathcal{J}^\otimes$ is the unit for the quasi-categorical tensor product $\odot$, we have a map

$$\theta : N\mathcal{J}^\otimes \times N\mathcal{J}^\otimes \to N\mathcal{J}^\otimes$$

and this restricts to a map on $N\mathsf{N}(\mathcal{J})$, which we also denote by θ. In fact, this map $\theta : N\mathsf{N}(\mathcal{J}) \to N\mathsf{N}(\mathcal{J}) \to N\mathsf{N}(\mathcal{J})$ coincides with the map μ (as defined on the nerves of categories) with which we worked in the previous section – this follows from [Lur17, Proposition 2.2.5.13/Remark 2.2.5.14]. By analogy with Remark 4.3.4, we have a commutative diagram of simplicial sets

$$\begin{tikzcd} N\mathsf{N}(\mathcal{J}) \times N\mathsf{N}(\mathcal{J}) \arrow[r, "\tilde{\nu}_{\mathbb{E}^G_d} \times \tilde{\nu}_{\mathbb{E}^{G'}_{d'}}"] \arrow[d, "\theta"'] & \mathbb{E}_d^{G,\mathrm{act}} \times \mathbb{E}_{d'}^{G',\mathrm{act}} \arrow[d, "\theta"] \\ N\mathsf{N}(\mathcal{J}) \arrow[r, "\tilde{\nu}_{\mathbb{E}^G_d \odot \mathbb{E}^{G'}_{d'}}"] & \left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}} \end{tikzcd}$$

Hence, for any sequences $\mathcal{X}, \mathcal{Y}$, we have weak equivalences

$$\Sigma_\theta \left(\Sigma_{\tilde{\nu}_{\mathbb{E}^G_d}} \mathcal{G}(\mathcal{X}) \,\square\, \Sigma_{\tilde{\nu}_{\mathbb{E}^{G'}_{d'}}} \mathcal{G}(\mathcal{Y}) \right) \simeq \Sigma_{\tilde{\nu}_{\mathbb{E}^G_d \odot \mathbb{E}^{G'}_{d'}}} \Sigma_\theta \left(\mathcal{G}(\mathcal{X}) \,\square\, \mathcal{G}(\mathcal{Y}) \right)$$

in the category of right fibrations over $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}$.

With these assertions in place, we are now in a position to state and prove a local version of 4.5.1 – we retain the hypotheses of that statement:

Proposition 4.5.4. *Let* $U \in \mathrm{Disk}(M)$ *and* $V \in \mathrm{Disk}(N)$*. Then, there is an equivalence*

$$\mathbb{L}\Sigma_\theta \left(\mathcal{G}(\mathcal{C}_U^G) \,\square\, \mathcal{G}(\mathcal{C}_V^{G'}) \right) \simeq \mathrm{Ho}(\iota^*) \mathcal{G}\left(\mathcal{C}^{G\times G'}_{U\times V} \right)$$

in the homotopy category of right fibrations over $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}$.

Proof. Our first task will be to produce a map

$$f : \mathcal{G}\left(\mathcal{C}_U^G\right) \,\square\, \mathcal{G}\left(\mathcal{C}_V^{G'}\right) \to \theta^* \iota^* \mathcal{G}\left(\mathcal{C}^{G\times G'}_{U\times V}\right)$$

where we now have

$$\theta^*\iota^*\mathcal{G}(\mathcal{C}^{G\times G'}_{U\times V}) = \left(\mathbb{E}_d^{G,\mathrm{act}} \times \mathbb{E}_{d'}^{G',\mathrm{act}}\right) \times^h_{\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}} \left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}} \times^h_{\mathbb{E}_{d+d'}^{G\times G',\mathrm{act}}} \mathcal{G}\left(\mathcal{C}^{G\times G'}_{U\times V}\right)$$

Let us write $\pi_U : \mathcal{G}\left(\mathcal{C}_U^G\right) \to \mathbb{E}_d^{G,\mathrm{act}}$ and $\pi_V : \mathcal{G}\left(\mathcal{C}_V^{G'}\right) \to \mathbb{E}_{d'}^{G',\mathrm{act}}$ for the respective projection maps. On 0-simplices, the map f sends a 0-simplex (ρ_U, ρ_V) to

$$\left(\pi_U(\rho_U), \pi_V(\rho_V), \mathrm{cons}_{d\odot d'}^{\theta(\pi_U(\rho_U),\pi_V(\rho_V))}, \theta\left(\pi_U(\rho_U), \pi_V(\rho_V)\right), \mathrm{cons}_{d+d'}^{(\iota\circ\theta)(\pi_U(\rho_U),\pi_V(\rho_V))}, \rho_U \times \rho_V\right)$$

where for a fixed n-simplex δ in $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}$, we write $\mathrm{cons}^\delta_{d\odot d'}$ to mean the constant path in $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}_n$ from δ to itself; and for a fixed n-simplex δ in $\mathbb{E}_{d+d'}^{G\times G',\mathrm{act}}$, we write $\mathrm{cons}^\delta_{d+d'}$ to mean the constant path in $(\mathbb{E}_{d+d'}^{G\times G',\mathrm{act}})_n$ from δ to itself.

On 1-simplices, f is given by

$$(\tilde{\varepsilon}_d, \rho_U, \tilde{\varepsilon}_{d'}, \rho_V) \mapsto \left(\tilde{\varepsilon}_d, \tilde{\varepsilon}_{d'}, \mathrm{cons}_{d\odot d'}^{\theta(\tilde{\varepsilon}_d,\tilde{\varepsilon}_{d'})}, \theta\left(\tilde{\varepsilon}_d, \tilde{\varepsilon}_{d'}\right), \mathrm{cons}_{d+d'}^{(\iota\circ\theta)(\tilde{\varepsilon}_d,\tilde{\varepsilon}_{d'})}, (\iota\circ\theta)\left(\tilde{\varepsilon}_d, \tilde{\varepsilon}_{d'}\right), \rho_U \times \rho_V\right)$$

where $\tilde{\varepsilon}_d \in (\mathbb{E}_d^{G,\mathrm{act}})_1$ and $\tilde{\varepsilon}_{d'} \in (\mathbb{E}_{d'}^{G',\mathrm{act}})_1$, and $\rho_U \in \mathcal{G}\left(\mathcal{C}_U^G\right)$, $\rho_V \in \mathcal{G}\left(\mathcal{C}_V^{G'}\right)$ such that $\pi_U(\rho_U) = d_0\tilde{\varepsilon}_d$ and $\pi_V(\rho_V) = d_0\tilde{\varepsilon}_{d'}$. The Segal condition determines f on all higher simplices.

By adjunction, we obtain a map

$$\Sigma_\theta\left(\mathcal{G}\left(\mathcal{C}_U^G\right) \square \mathcal{G}\left(\mathcal{C}_V^{G'}\right)\right) \to \iota^*\mathcal{G}\left(\mathcal{C}_{U\times V}^{G\times G'}\right)$$

It remains to show that this map is a weak equivalence. Let us fix $U = \sqcup_n \mathbb{R}^d$ and $V = \sqcup_m \mathbb{R}^{d'}$. By Lemma 4.5.2, we have

$$\mathcal{G}\left(\mathcal{C}_{\sqcup_n\mathbb{R}^d}^G\right) \simeq \Sigma_{\tilde{\nu}_{\mathbb{E}_d^G}}\mathcal{G}\left(\mathfrak{S}_n\right) \quad \text{and} \quad \mathcal{G}\left(\mathcal{C}_{\sqcup_m\mathbb{R}^{d'}}^{G'}\right) \simeq \Sigma_{\tilde{\nu}_{\mathbb{E}_{d'}^{G'}}}\mathcal{G}\left(\mathfrak{S}_m\right)$$

Hence, to show the required weak equivalence, the two-out-of-three property tells us that it is sufficient to show that the top horizontal map in the commutative diagram of simplicial spaces over $\left(\mathbb{E}_d^G \odot \mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}$

$$\begin{tikzcd}
\Sigma_\theta\left(\Sigma_{\tilde{\nu}_{\mathbb{E}_d^G}}\mathcal{G}\left(\mathfrak{S}_n\right) \square \Sigma_{\tilde{\nu}_{\mathbb{E}_{d'}^{G'}}}\mathcal{G}\left(\mathfrak{S}_m\right)\right) \arrow[r] \arrow[d, "\simeq"'] & \iota^*\Sigma_{\nu_{\mathbb{E}_{d+d'}^{G\times G'}}}\mathcal{G}\left(\mathfrak{S}_{nm}\right) \arrow[d, "\simeq"] \\
\Sigma_\theta\left(\mathcal{G}\left(\mathcal{C}_{\sqcup_n\mathbb{R}^d}^G\right) \square \mathcal{G}\left(\mathcal{C}_{\sqcup_m\mathbb{R}^{d'}}^{G'}\right)\right) \arrow[r] & \iota^*\mathcal{G}\left(\mathcal{C}_{\sqcup_{nm}\mathbb{R}^{d+d'}}^{G\times G'}\right)
\end{tikzcd}$$

is a weak equivalence. Remark 4.5.3 tells us that we have weak equivalences

$$\Sigma_\theta\left(\Sigma_{\tilde{\nu}_{\mathbb{E}_d^G}}\mathcal{G}\left(\mathfrak{S}_n\right) \square \Sigma_{\tilde{\nu}_{\mathbb{E}_{d'}^{G'}}}\mathcal{G}\left(\mathfrak{S}_m\right)\right) \simeq \Sigma_{\tilde{\nu}_{\mathbb{E}_d^G\odot\mathbb{E}_{d'}^{G'}}}\Sigma_\theta\left(\mathcal{G}\left(\mathfrak{S}_n\right) \square \mathcal{G}\left(\mathfrak{S}_m\right)\right)$$

of with respect to the right fibration model structure on $s\mathcal{S}_{/\left(\mathbb{E}_d^G\odot\mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}}$, while Theorem 3.2.3 tells us that this latter simplicial space is weakly equivalent to

$$\iota^*\Sigma_{\nu_{\mathbb{E}_{d+d'}^{G\times G'}}}\Sigma_\theta\mathcal{G}\left(\mathfrak{S}_n \square \mathfrak{S}_m\right)$$

in the right fibration model structure on $s\mathcal{S}_{/\left(\mathbb{E}_d^G\odot\mathbb{E}_{d'}^{G'}\right)^{\mathrm{act}}}$. Thus, to conclude, we only need to show that we have a weak equivalence

$$\Sigma_\theta\left(\mathcal{G}\left(\mathfrak{S}_n\right)\square\mathcal{G}\left(\mathfrak{S}_m\right)\right) \simeq \mathcal{G}\left(\mathfrak{S}_{nm}\right)$$

of right fibrations over $N\mathsf{N}(\mathcal{J})$. This follows by the same argument used at the end of the proof of Proposition 4.4.5, since as we noted, the functor θ on $N\mathsf{N}\left(\mathcal{J}\right)$ coincides with the functor μ. $\square$

With our local version of the result in place, the proof of the global result, Theorem 4.5.1 is formally identical to the proof of 4.4.1, except that we use θ in place of μ and $\mathcal{C}_U^G$ (resp. $\mathcal{C}_V^{G'}$) in place of $\mathcal{C}_U$ (resp. $\mathcal{C}_V$).

Using the connection between the right fibrations picture and the right modules picture, we can say that Theorem 4.5.1 essentially generalises the original product theorem (4.1.14) to an equivalence between $\mathcal{C}_M^G \underline{\star}^{\mathbb{L}} \mathcal{C}_N^{G'}$ and $\mathcal{C}_{M\times N}^{G\times G'}$ for *some* modified notion of the Boardman-Vogt tensor product of modules, $\underline{\star}$, (although the fact that we are utilising the homotopy-coherent tensor product of quasi-operads means that we no longer have quite such a sense of whether we can really claim that we have an equivalence of presheaves on some category – hence our rather vague statement of equivalence). Nonetheless, this is essentially the stated goal of [DHK19, Theorem 5.6], except that that statement was uneasily premised on the conjecture of an additivity theorem for the strict Boardman-Vogt tensor product of the topological skew little-cubes operads.

4.5.1 Outlook and Future Work

In [HK18], a kind of Künneth Theorem was devised for configuration modules of products of manifolds:

Theorem 4.5.5. *Let M and N be framed d, d'-manifolds respectively, and let R be a field of characteristic* 0*. Then there is an isomorphism*

$$H_p\left(H_*\left(\mathrm{Conf}(M);R\right) \star^{\mathbb{L}} H_*\left(\mathrm{Conf}(N);R\right)\right)_q \cong H_{p+q}\left(\mathrm{Conf}(M\times N);R\right)$$

of R-linear $\mathbb{E}_{d+d'}$-modules.

This result has two foundational principles:

1. The product theorem for configuration modules – Theorem 4.1.14.
2. A suitable notion of a linear right module and a linear version of the Boardman-Vogt tensor product of modules.

Since the translation into the right fibrations picture allows us to expand the product theorem to manifolds with a more general kind of framing, we might also hope that there is a suitable version of this result available if we can devise a notion of linear right fibrations over an operad. Indeed, certain of the arguments from [HK18] translate very easily into a right fibrations perspective. However, we have thus far been impeded in our attempts to bring this translation to a successful conclusion by a few issues:

1. A **linear right module over an operad** $\mathcal{O}$ is a functor
$$G : \mathsf{F}(\mathcal{O})_R^{\mathrm{op}} \to \mathrm{grMod}_R$$
where the category $\mathsf{F}(\mathcal{O})_R$ has the same objects as $\mathsf{F}(\mathcal{O})$, but its hom-objects are obtained by computing the homology (with coefficients in R) of the hom-objects of $\mathsf{F}(\mathcal{O})$. Based on our earlier arguments, the obvious counterpart of this should be a simplicial object in grMod_R (or perhaps dgMod_R) with a reference map to $N\mathsf{F}(\mathcal{O})_R \cong H_*(N\mathsf{F}(\mathcal{O}))$. However, a technical issue arises at this juncture: while the category of simplicial spaces is enriched in spaces, thus allowing us the possibility of a left Bousfield localisation, we cannot assert that the category of simplicial (differential) graded R-modules is enriched in (differential) graded R-modules, which prevents the use of an enriched Bousfield localisation following the approach of [Bar07]. As a result, it is not necessarily possible to produce a right-fibrations model structure on the category of simplicial graded R-modules over some fixed (Segal) simplicial graded R-module, even though obvious versions of all these notions present themselves.

2. We note that the category of simplicial graded R-modules *is* enriched over simplicial sets, as per [GJ09, Section III.2], which might advocate the definition of a notion of Segality for simplicial graded modules, e.g. a diagram $\mathcal{X}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{grMod}_R$ such that
$$U(\mathcal{X}_\bullet(n)) \to U(\mathcal{X}_\bullet(1)) \times_{U(\mathcal{X}_\bullet(0))} \dots \times_{U(\mathcal{X}_\bullet(0))} U(\mathcal{X}_\bullet(1))$$
is a weak equivalence of simplicial sets, where U is the forgetful functor $s\mathrm{grMod}_R \to s\mathrm{Set}$. However, this approach leads to many complications about how to handle the module-grading. Furthermore, in the absence of an obvious map $\mathcal{X}_\bullet(1) \to \mathcal{X}_\bullet(0)$, we cannot formulate an analogue of the notion of a right fibration.

3. Related to the issue of simplicial graded R-modules not having an enrichment in graded R-modules is the question of providing a left adjoint to the linearised Grothendieck construction (which can be defined more or less in analogy with the Grothendieck construction we introduced in this chapter), and a right adjoint to the external tensor product of simplicial graded R-modules over some fixed simplicial spaces.

   Via [Ver+92, Section 3.2], there is a notion of an enriched Grothendieck construction for locally Cartesian categories $\mathcal{E}$, which admits a left adjoint:
$$\mathcal{L} : \mathrm{cat}(\mathcal{E})_{/\mathsf{A}} \rightleftarrows \mathcal{E}^{\mathsf{A}^{\mathrm{op}}} : \mathcal{G}$$
where A is enriched in $\mathcal{E}$. Taking the nerve, we obtain a kind of Grothendieck construction which mimics the one we discussed for simplicial spaces:
$$\mathcal{G} : \mathcal{E}^{\mathsf{A}^{\mathrm{op}}} \to s\mathcal{E}_{/N\mathsf{A}}$$
However, it is no longer obvious whether the nerve functor admits a left adjoint so that we can determine a left adjoint to this $\mathcal{E}$-enriched Grothendieck construction.

   The lack of a right adjoint to the external tensor product also means that we are not necessarily in a position to define a left-derived version of the obvious "linearisation" of the Boardman-Vogt tensor product of right fibrations.

These obstacles grieve us somewhat as so many of the proofs and assertions from [HK18] *do* appear to admit an easy translation to the right fibrations picture. It is to be hoped that these impediments are merely technical, and can somehow be circumnavigated.

Chapter 5

The Configuration Category of a Manifold

In this chapter, we discuss one of the major objects of interest in this thesis – the so-called configuration category of a manifold. This object was developed by Boavida and Weiss in [BW18a] and [BW18b], but is closely related to constructions given in [Lur09b], [Lur17], [And10] and [AFT17]. It also bears more than a passing resemblance to the configuration module of a manifold, but it has been constructed in such a way that it is independent of the framing. In particular, an open conjecture at the end of [BW18b] has provided a motivation for our research as it relates to the Dunn additivity theorem for little cube operads.

5.1 Constructions of the Configuration Category

There are several potential constructions of the configuration category, two of which will be discussed here. The various constructions are shown to be equivalent in [BW18a, Section 3.2]. In what follows, we fix M to be a smooth n-dimensional manifold. Let Mfld be the category whose objects are n-dimensional manifolds, and given two such objects M_1, M_2, we set

$$\mathrm{Hom}_{\mathsf{Mfld}}(M_1, M_2) = \mathrm{Emb}(M_1, M_2)$$

We give these hom sets the compact-open topology so that Mfld is enriched in spaces. Let Disk be the full topological subcategory of Mfld whose objects are disjoint unions of copies of $\mathbb{R}^n$.

Construction 5.1.1 (Multipatch Model of Configuration Category)**.** Define a functor $E : \mathsf{Disk}^{\mathrm{op}} \to \mathcal{S}$ by $U \mapsto \mathrm{Emb}(U, M)$ and consider the topological category X_M (which arises via a kind of Grothendieck construction from E): the objects of X_M are pairs $(\sqcup_k \mathbb{R}^n, f)$, where $f \in E(\sqcup_k \mathbb{R}^n)$ and the space of all morphisms is:

$$\coprod_{k,l \geq 0} \mathrm{Emb}(\sqcup_k \mathbb{R}^n, \sqcup_l \mathbb{R}^n) \times E(\sqcup_l \mathbb{R}^n)$$

The target map for this category is given by projection onto the second factor, while the source map is given by composition. There is an obvious forgetful functor $X_M \to \mathsf{Disk}$ which induces a map of simplicial spaces $NX_M \to N\mathsf{Disk}$. On the other hand, taking path components defines a functor $\pi_0 : \mathsf{Disk} \to \mathsf{Fin}$. Composing the induced maps on nerves gives:

$$NX_M \to N\mathsf{Disk} \to N\mathsf{Fin}$$

so that NX_M has the structure of a simplicial space over $N\mathsf{Fin}$. As the nerve of a category, it is immediate that NX_M is a Segal space, but it is not in general true that NX_M is a fibrewise complete Segal space over $N\mathsf{Fin}$ – with this in mind, we simply define the configuration category of M, $\mathrm{con}(M)$, to be the fibrewise completion of NX_M over $N\mathsf{Fin}$.

Remark 5.1.2. In [BW18a], an explicit description is given of the 0- and 1-simplices of this fibrewise completion. First, it is shown that $\mathrm{con}(M)_0$ is weakly equivalent to $\sqcup_{k \geq 0} \mathrm{Emb}(\langle k \rangle, M)$ – i.e. the vertices correspond to ordered configurations of a finite number of points in M. Given such an ordered configuration $f : \langle k \rangle \rightarrowtail M$, the homotopy fibre of the map $d_1 : \mathrm{con}(M)_1 \to \mathrm{con}(M)_0$ over f is the space

$$\Phi_f \simeq \coprod_{l \geq 0} \coprod_{g:\langle l \rangle \to \langle k \rangle} \prod_{i \in \langle k \rangle} \mathrm{Emb}\left(g^{-1}\{i\}, \mathbb{R}^n\right) \tag{5.1}$$

Alternatively, we can characterise this homotopy fibre by working with a tubular neighbourhood U_f of the image of f in M. In this context, an element of the space on the right-hand side is determined by a so-called *reverse exit path* in the tubular neighbourhood U_f (we will explain this term in detail in the next construction of the configuration category).

Construction 5.1.3 (Particle Model)**.** For this construction, we begin by observing that there is a stratification on the collection of maps from $\langle k \rangle$ to M: namely, any such map can be factored as a composition $\langle k \rangle \twoheadrightarrow \langle l \rangle \rightarrowtail M$; we let $\mathrm{maps}(\langle k \rangle, M)_l$ denote the stratum of those maps which factor through an embedding $\langle l \rangle \rightarrowtail M$, and we note that if $l \geq l'$, then there is an inclusion $\mathrm{maps}(\langle k \rangle, M)_{l'} \subseteq \mathrm{maps}(\langle k \rangle, M)_l$. We consider the topological category X'_M whose objects are embeddings of single points in M, i.e. pairs $(\langle k \rangle, f)$, with $f \in \mathrm{Emb}(\langle k \rangle, M)$; a morphism between two such pairs $(\langle k \rangle, f)$, $(\langle l \rangle, g)$ is the data of a pair $u \in \mathrm{Hom}_{\mathsf{Fin}}(\langle k \rangle, \langle l \rangle)$ and a **reverse exit path** $\gamma : [0, a] \to \mathrm{maps}(\langle k \rangle, M)$ from f to $g \circ u$ for some finite a – that is, a path γ which satisfies the property that if $s' \geq s$, then the stratum containing $\gamma(s')$ is contained in the closure of the stratum containing $\gamma(s)$ (i.e., if $\gamma(s) \in \mathrm{maps}(\langle k \rangle, M)_l$ and $\gamma(s') \in \mathrm{maps}(\langle k \rangle, M)_{l'}$, then $l \geq l'$). Composition of two morphisms is given by composition of maps of finite sets and composition of Moore paths.

This category is also equipped with a forgetful functor to the category Fin and hence $NX'_M \to N\mathsf{Fin}$ is a morphism of Segal spaces. A morphism $(u, \gamma) : (\langle k \rangle, f) \to (\langle l \rangle, g)$ in NX'_M is homotopy invertible if and only if the reverse exit path γ lies in a single stratum (which corresponds to the situation where $k = l$ – but this is precisely the condition for homotopy invertibility of the map $u : \langle k \rangle \to \langle l \rangle$ in $N\mathsf{Fin}$). Using properties of homotopically stratified spaces and exit path categories, this observation allows us to deduce that NX'_M is already a fibrewise complete Segal space over $N\mathsf{Fin}$ (see [BW18a, Proposition 3.3] for details of this argument).

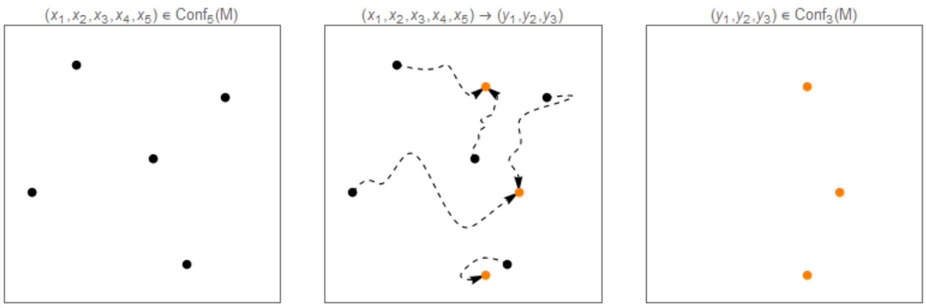

Figure 6: Example of exit path from a configuration of 5 "particles" to a configuration of 3 "particles"

The reason this model is called the particle model is that we can view an object $f : \langle k \rangle \to M$ as describing a configuration of k particles "living on" the manifold M. In this guise, a morphism from $f : \langle k \rangle \to M$ to $g : \langle l \rangle \to M$ is a way of passing from k particles to l particles. The reverse exit path condition on the morphisms ensures that the number of particles can never increase: physically, this corresponds to the requirement that two or more particles are allowed to collide and amalgamate, but once that amalgamation has happened, they cannot then be separated. (Equally, this ensures that "new" particles cannot be created from nothing). In Figure 6, we have provided a schematic example of this reverse exit path principle in action:

- In the left hand diagram, we have an initial configuration of five particles (black).
- In the central diagram, these particles move around the manifold (with their respective paths indicated by the dashed lines), such that there are two collisions between two and three particles respectively, as well as one particle which doesn't collide with any others. The resulting amalgamated particles are depicted in orange.
- In the right diagram, we have a final configuration of three particles (orange).

This avatar of the configuration category is closely related to the exit path category studied in (among other places) [Lur17, Appendix A.9], [AF15] and [AFT17]. Based on our heuristic explanation above, it is not at all surprising to note that this model has attracted some interest in physics, where it has found applications in questions around topological quantum field theories and factorization algebras. For more on these applications, we refer the reader to e.g. [Gin15], [CG16].

Showing that these two models give rise to the same fibrewise complete Segal space over $N\mathsf{Fin}$ relies on constructing a zig-zag

$$NX_M \xleftarrow{j} W \xrightarrow{\sim} W^{\#} \xrightarrow{i} NX'_M$$

of Dwyer-Kan equivalences over $N\mathsf{Fin}$, where we write $W^{\#}$ for the fibrewise completion of W over $N\mathsf{Fin}$. (By a fibrewise version of Theorem 1.4.15, two Segal objects over some fixed complete Segal space are equivalent in the fibrewise complete Segal model structure if and only if they are Dwyer-Kan equivalent in a fibrewise sense.)

The simplicial space W is the nerve of a category Y with the same objects as the category X_M. Given a pair of such objects (i.e. framed embeddings) $f \in \mathrm{Emb}(\sqcup_k \mathbb{R}^n, M)$ and $g \in \mathrm{Emb}(\sqcup_l \mathbb{R}^n, M)$, the space of morphisms between them in Y is empty if the image of f is not contained in the image of g; otherwise, the space of morphisms is the space of morphisms in $\mathrm{con}(\mathrm{im}(g))$ from f_0 to g_0, where we write f_0 to mean the composition of f with the natural inclusion at the origin $\langle k\rangle \hookrightarrow \sqcup_k \mathbb{R}^n$. As the fibrewise completion of W, there is by construction a Dwyer-Kan equivalence $W \xrightarrow{\sim} W^{\#}$. Using the same kind of explicit description of the fibrewise-completion as in Remark 5.1.2, we see that the space of 0-simplices of $W^{\#}$ is $\sqcup_{k\geq 0}\mathrm{Emb}(\langle k\rangle, M)$; while an edge in $W^{\#}$ corresponding to a morphism from $f_0 : \langle k\rangle \rightarrowtail M$ to $g_0 : \langle l\rangle \rightarrowtail M$ (such that f_0 (resp. g_0) arises as the evaluation at the origin of an embedding $f : \sqcup_k \mathbb{R}^n \rightarrowtail M$ (resp. $g : \sqcup_l \mathbb{R}^n \rightarrowtail M$); and the image of f is contained in the image of g) is given by the data of

- a tubular neighbourhood V_g of the image of g_0 in M
- a tubular neighbourhood U_f of the image of f_0 in M contained in V_g
- a reverse exit path from f_0 to g_0 which lies inside V_g

In degree 0, the map $i : W^{\#} \to NX'_M$ is just the identity – in other words, the passage from W to NX'_M in degree 0 is determined by forgetting the framing on each ordered configuration (i.e. by restricting an ordered embedding $\sqcup_k \mathbb{R}^n \rightarrowtail M$ to its image at the origin). In degree 1, the map $i : W^{\#} \to NX'_M$ is given by forgetting the tubular neighbourhoods (this is a weak equivalence of spaces). Since $W^{\#}$ and NX'_M are both fibrewise complete Segal spaces over $N\mathsf{Fin}$, the fact that i is a weak equivalence in degrees 0 and 1 is enough to guarantee that i is a Dwyer-Kan equivalence, and hence the composite $W \to NX'_M$ is also a Dwyer-Kan equivalence.

In degree 0, the map $j : W \to NX_M$ is just the identity. Exhibiting the required fibrewise Dwyer-Kan equivalence is done by showing that the square

$$\begin{tikzcd}
W_1 \arrow[r, "{(d_0,d_1)}"] \arrow[d, "j"'] & W_0 \times W_0 \arrow[d, "j\times j=\mathrm{id}"] \\
(NX_M)_1 \arrow[r, "{(d_0,d_1)}"] & (NX_M)_0 \times (NX_M)_0
\end{tikzcd}$$

is homotopy Cartesian. Given a morphism ζ in X_1 from $f : \sqcup_k \mathbb{R}^n \rightarrowtail M$ to $g : \sqcup_l \mathbb{R}^n \rightarrowtail M$, we wish to show that the homotopy fibre of j over ζ is weakly contractible. By definition of the space W, this homotopy fibre depends on $\mathrm{con}(\mathrm{im}(g))$, which is a product of factors indexed by the connected components of $\mathrm{im}(g)$. Hence, it is only necessary to deduce the desired weak contractibility in the case where g is a configuration of a single point in M. In [BW18a, Lemma 3.4], this is reduced to the simple case where $M = \mathbb{R}^n$, where it is shown directly that

Lemma 5.1.4. *If f is a configuration of k points in $\mathbb{R}^n$, then the space of exit paths starting at f and ending at the configuration $\langle 1\rangle \to \{0\} \hookrightarrow \mathbb{R}^n$ is contractible.*

Thus $j : W \to NX_M$ is also a Dwyer-Kan equivalence, giving the desired zig-zag of simplicial spaces over $N\mathsf{Fin}$. Hence, after fibrewise completion, the multipatch and particle models give rise to equivalent fibrewise complete Segal spaces over $N\mathsf{Fin}$, i.e. the two models of the configuration category coincide.

5.2 Connections between the Configuration Category and Operads

We have already encountered a number of ways to relate categories and operads, but in [BW18a, Section 7.1], another such construction is presented which is connected to the configuration category.

Construction 5.2.1. Given a plain operad $\mathcal{O}$ in spaces, we define a topological category $\mathrm{cat}(\mathcal{O})$ whose space of objects is

$$\coprod_{k\geq 0} \mathcal{O}(k)$$

and whose space of morphisms lying over a morphism $f : \langle k\rangle \to \langle l\rangle$ in Fin is

$$\mathcal{O}(l) \times \prod_{1\leq i\leq l} \mathcal{O}\left(f^{-1}\{i\}\right)$$

The target map is given by projection from the factor $\mathcal{O}(l)$, while the source map is given by composition. There is an obvious reference functor from $\mathrm{cat}(\mathcal{O})$ to Fin.

At first glance, the reader might wonder whether this is not the same as the category of operators of $\mathcal{O}$ (restricted to Fin) which we described in Definition 2.3.3 – we warn that this is not the case! In the category $F(\mathcal{O})$, the space of morphisms over $f : \langle k\rangle \to \langle l\rangle$ is

$$\prod_{1\leq i\leq l} \mathcal{O}\left(f^{-1}\{i\}\right)$$

so the morphisms in this new category $\mathrm{cat}(\mathcal{O})$ also require the specification of an l-ary operation. With a moment's thought, we realise that $\mathrm{cat}(\mathcal{O})$ is actually the slice category $F(\mathcal{O}) \downarrow \langle 1\rangle$, where the space of operations over $f : \langle k\rangle \to \langle l\rangle$ is determined by a commutative triangle of operations in $\mathcal{O}$:

$$\begin{tikzcd} \langle l\rangle \arrow[rr, "(\omega_i)_{1\leq i\leq l}"] \arrow[dr, "\alpha"'] & & \langle k\rangle \arrow[dl, "\alpha\circ(\omega_i)_{1\leq i\leq l}"] \\ & \langle 1\rangle & \end{tikzcd} \tag{5.2}$$

where $\omega_i \in \mathcal{O}(f^{-1}\{i\})$ and $\alpha \in \mathcal{O}(l)$ (so that $\alpha\circ(\omega_1,\ldots,\omega_l) \in \mathcal{O}(k)$). Evidently, $N\mathrm{cat}(\mathcal{O})$ is a Segal space over $N\mathsf{Fin}$. If we impose a further condition on the collection of unary operations of $\mathcal{O}$, then we can say even more:

Lemma 5.2.2. *[BW18a, Lemma 7.4] If $\mathcal{O}$ is an operad in spaces such that $\mathcal{O}(1)$ is weakly contractible, then $N\mathrm{cat}(\mathcal{O}) \to N\mathsf{Fin}$ is a fibrewise complete Segal space.*

The main observation required to deduce this is that if $\mathcal{O}(1)$ is weakly contractible, then a morphism $\mathcal{O}(k) \ni \omega \to \widetilde{\omega} \in \mathcal{O}(l)$ in $\mathrm{cat}(\mathcal{O})$ is homotopy invertible if and only if $l = k$, so that we find

$$(N\mathrm{cat}(\mathcal{O}))_1^{he} = \coprod_{l\geq 0}\coprod_{\sigma\in\mathfrak{S}_l} \mathcal{O}(l) \times \prod_{1\leq i\leq 1} \mathcal{O}(\sigma^{-1}\{i\}) \simeq \coprod_{l\geq 0}\coprod_{\sigma\in\mathfrak{S}_l} \mathcal{O}(l)$$

thus ensuring that there is a weak equivalence between the homotopy fibres of the maps $(N\mathrm{cat}(\mathcal{O}))_1^{he} \to N\mathsf{Fin}_1^{he}$ and $N\mathrm{cat}(\mathcal{O})_0 \to N\mathsf{Fin}_0$.

Remark 5.2.3. Construction 5.2.1 can be generalised to coloured operads. Specifically, if $\mathcal{O}$ is a coloured simplicial operad whose space of unary operations is weakly contractible, then we can fix a colour c of $\mathcal{O}$ and consider the "slice" of all operations over c, i.e. the objects are elements of $\mathcal{O}\left(\{c_i\}_{i\in I}; c\right)$ (for colours $\{c_i\}_{i\in I} \subseteq \mathrm{Col}\left(\mathcal{O}\right)$) and morphisms are determined by analogy with diagram (5.2).

In fact, we can even generalise this construction to quasi-operads: given a reduced quasi-operad $\mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$, selecting a colour can be seen as giving a morphism of pre-operads $\Delta[0] \xrightarrow{c} \mathcal{O}^{\otimes}$ (where $\Delta[0] \to N\mathsf{Fin}_*$ by $* \mapsto \langle 1\rangle$). In this context, the association $\mathcal{O} \mapsto \mathrm{cat}(\mathcal{O})$ then translates to a functor:

$$\left(\begin{tikzcd} \Delta[0] \arrow[rr, "c"] \arrow[dr, "*\mapsto\langle 1\rangle"'] & & \mathcal{O}^{\otimes} \arrow[dl] \\ & N\mathsf{Fin}_* & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} \left(\mathcal{O}_{/c}\right)^{\mathrm{act}} \arrow[d] \\ N\mathsf{Fin} \end{tikzcd}\right)$$

where we write $\mathcal{O}_{/c}$ to mean the pullback $\Delta[0] \times_{N\mathsf{Fin}_*} \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ and $(-)^{\mathrm{act}}$ to denote the pullback over the active morphisms of Fin_* (i.e. the morphisms of Fin).

Returning to our original construction, let us examine the case where $\mathcal{O}$ is the topological little d-cubes operad (which is a plain operad whose space of unary operations is weakly contractible): we observe that there is a degreewise weak equivalence of simplicial spaces over $N\mathsf{Fin}$:

$$NX_{\mathbb{R}^d} \simeq N\mathrm{cat}({}^t\mathbb{E}_d)$$

where X_M is the category we introduced in Construction 5.1.1. Thus, there is a degreewise weak equivalence between $\mathrm{con}(\mathbb{R}^d)$ and $N\mathrm{cat}({}^t\mathbb{E}_d)$ as simplicial spaces over $N\mathsf{Fin}$.

The category of operads in spaces (where we blur the distinction between spaces and simplicial sets somewhat) can be given a simplicial enrichment as follows: first, for any such operad $\mathcal{P}$, we let $\mathcal{P}^{\Delta[k]}$ be the operad for which the set of i-simplices of its l-ary operations is given by

$$\mathcal{P}^{\Delta[k]}(l)_i = \mathrm{Map}(\Delta[k], \mathcal{P}(l))_i = \mathrm{Hom}_{sSet}(\Delta[k] \times \Delta[i], \mathcal{P}(l))$$

As per [BM03, Theorem 3.2], the category of plain simplicial operads with these simplicial mapping spaces can be given a simplicial model structure such that a morphism of plain operads $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence (resp. fibration) if and only if it is a levelwise weak equivalence (resp. fibration) of simplicial sets. This model structure is cofibrantly generated. (It can also be shown that the model structure on all coloured simplicial operads given in [CM11] restricts on plain operads to this model structure.) On the other hand, we know that the complete Segal space model structure is a simplicial model structure, which allows us to define derived simplicial mapping objects in the fibrewise complete Segal space model structure by

$$\mathbb{R}\mathrm{Map}_{N\mathsf{Fin}}(X, Y) = \lim\left(* \to \mathbb{R}\mathrm{Map}(X, N\mathsf{Fin}) \leftarrow \mathbb{R}\mathrm{Map}(Y, N\mathsf{Fin})\right)$$

where we recall the notation $\mathbb{R}\mathrm{Map}$ from Chapter 1 for the derived simplicial mapping spaces in a simplicial model category.

The construction $\mathcal{O} \mapsto N\mathrm{cat}(\mathcal{O})$ of 5.2.1 is functorial and preserves weak equivalences, so there is an induced morphism of simplicial sets:

$$\mathbb{R}\mathrm{Map}(\mathcal{O}, \mathcal{P}) \to \mathbb{R}\mathrm{Map}_{N\mathsf{Fin}}(N\mathrm{cat}(\mathcal{O}), N\mathrm{cat}(\mathcal{P}))$$

For us, a case of special interest of the above map is where one of the operads is a little d-cubes operad, since we have already observed a close link between $N\mathrm{cat}({}^t\mathbb{E}_d)$ and $\mathrm{con}(\mathbb{R}^d)$. In [BW18a], it is shown that under certain propitious circumstances, even more can be said about this map:

Theorem 5.2.4. *[BW18a, Theorem 7.5] Let $\mathcal{O}, \mathcal{P}$ be operads in spaces such that $\mathcal{O}(0), \mathcal{O}(1), \mathcal{P}(0), \mathcal{P}(1)$ are weakly contractible (we say $\mathcal{O}$ and $\mathcal{P}$ are reduced). Then the map*

$$\mathbb{R}\mathrm{Map}(\mathcal{O}, \mathcal{P}) \to \mathbb{R}\mathrm{Map}_{N\mathsf{Fin}}(N\mathrm{cat}(\mathcal{O}), N\mathrm{cat}(\mathcal{P}))$$

is a weak homotopy equivalence.

Since this result and so many of the ideas in its proof are of central interest to us, we will outline some of the steps taken to prove these assertions, though we will by no means give an exhaustive exposition of the finer details (since these run to many pages in [BW18a]).

By [CM13] there is an equivalence of categories between the category of coloured simplicial operads with the model structure of [CM11] and complete Segal dendroidal spaces, so we can rephrase the statement of Theorem 5.2.4 as demanding that the map

$$\mathbb{R}\mathrm{Map}(N_d\mathcal{O}, N_d\mathcal{P}) \to \mathbb{R}\mathrm{Map}_{N\mathsf{Fin}}(N\mathrm{cat}(\mathcal{O}), N\mathrm{cat}(\mathcal{P}))$$

is a weak homotopy equivalence. Furthermore, because we are working with reduced operads, we can restrict our view to the category of rooted closed dendroidal spaces (i.e. space-valued presheaves on the category Ω_{rc} of rooted closed trees, denoted $rcd\mathcal{S}$ or $\mathcal{P}(\Omega_{rc})$) and ask whether the following is an equivalence:

$$\mathbb{R}\mathrm{Map}(N_d^{rc}\mathcal{O}, N_d^{rc}\mathcal{P}) \to \mathbb{R}\mathrm{Map}_{N\mathsf{Fin}}(N\mathrm{cat}(\mathcal{O}), N\mathrm{cat}(\mathcal{P})) \tag{5.3}$$

where $N_d^{rc} = \iota^* N_d$ for $\iota : \Omega_{rc} \hookrightarrow \Omega$ the natural inclusion functor.

At this point in the proof, an explicit functor is described to prove the desired equivalence in (5.3). To be able to describe this functor, we need some preliminary definitions. First, given a simplicial space Z, let $\mathsf{simp}(Z)$ denote its category of simplices: this is the category whose objects are pairs $([n], z)$ with $[n] \in \Delta$ and $z \in Z_n$;

a morphism between two such pairs $([n], z) \to ([m], y)$ is a morphism $\phi : [n] \to [m]$ in Δ such that $\phi^*(y) = z$. For any simplicial space Z, there is a natural equivalence of categories

$$s\mathcal{S}_{/Z} \simeq \mathcal{S}^{\mathsf{simp}(Z)^{\mathrm{op}}}$$

To see why this might be the case, consider a map $f : X^f \to Z$ of simplicial spaces. To such a map, we can associate a contravariant functor $\tilde{f}$ from $\mathsf{simp}(Z)$ to spaces by setting $\tilde{f}([n], z) = f^{-1}\{z\} \subseteq X^f_n$; likewise, given a morphism $([n], \phi^* y) \to ([m], y)$ where ϕ is a map in Δ, the map $\phi^* : X^f_m \to X^f_n$ restricts to a map on fibres $f^{-1}\{y\} \to f^{-1}\{\phi^* y\}$. Conversely, given such a contravariant functor $\tilde{f}$, we can associate a simplicial space X by

$$X_n = \sqcup_{z \in Z_n} \tilde{f}(n)$$

The functoriality of $\tilde{f}$ ensures that X is a simplicial space, and there is an obvious map of simplicial spaces from X to Z.

In particular, we can consider $N\mathsf{Fin}$ as a discrete simplicial space and write $\mathsf{simp}(\mathsf{Fin})$ for its category of simplices. The objects of the category $\mathsf{simp}(\mathsf{Fin})$ are strings of maps of finite sets $(S_0 \to S_1 \to \ldots \to S_n)$; a morphism between two such strings is of the form

$$\Big(S_{\phi(0)} \to S_{\phi(1)} \to \ldots \to S_{\phi(n)}\Big) \longmapsto \Big(S_0 \to S_1 \to \ldots \to S_m\Big)$$

for some $\phi : [n] \to [m]$ in Δ. (We note that $\mathsf{simp}(\mathsf{Fin})$ is a subcategory of the category $\Delta_{\mathbb{F}}$ which we introduced in Definition 3.4.1 when discussing complete Segal operads.) By the foregoing remarks, we know that there is an equivalence of categories $s\mathcal{S}_{/N\mathsf{Fin}} \simeq \mathcal{S}^{\mathsf{simp}(\mathsf{Fin})}$. Using this perspective, Boavida and Weiss produce a functor $j : \mathsf{simp}(\mathsf{Fin}) \to \Omega_{rc}$ which in turn induces adjoint pairs:

$$s\mathcal{S}_{/N\mathsf{Fin}} \simeq \mathcal{S}^{\mathsf{simp}(\mathsf{Fin})^{\mathrm{op}}} \underset{\xrightarrow{j_*}}{\overset{\xrightarrow{j_!}}{\xleftarrow{j^*}}} \mathcal{S}^{\Omega_{rc}^{\mathrm{op}}} = rcd\mathcal{S}$$

The functor j is defined as follows: given a string of finite sets $([n], S) := S_* = \left(S_0 \xrightarrow{s^1} S_1 \xrightarrow{s^2} \ldots \xrightarrow{s^n} S_n\right)$, we can associate a closed tree whose set of edges is $\sqcup_{0 \le i \le n} S_i \sqcup \{\mathfrak{R}\}$ where $\{\mathfrak{R}\}$ is just a one-element set. As mentioned in Lemma 2.5.1 and the subsequent remarks, a closed tree is specified by the data of a poset with an unique maximal element whose partial order relation satisfies a certain criterion. The partial order on the tree $j(S_*)$ is determined as follows:

($*$) the maximal element is $\mathfrak{R}$;

($\circledast$) if $i \le l$ and $y \in S_i, z \in S_l$, then $y \le z$ if and only if $z = s^{i+1,l}(y)$ (where we write $s^{i,l}$ to mean the composite $s^l \circ s^{l-1} \circ \ldots s^i : S_{i-1} \to S_l$).

Example 5.2.5. As an example, consider the string of finite sets $S_* = \{a, b, c, d, e\} \xrightarrow{s^1} \{f, g, h\} \xrightarrow{s^2} \{i, j\}$, where

$$\begin{aligned} s^1(a) &= s^1(b) &= f \qquad & s^1(c) &= g \qquad\qquad & s^1(d) &= s^1(e) = h \\ s^2(f) &= i & & s^2(g) &= s^2(h) &= j \end{aligned}$$

Then $j(S_*)$ is the closed tree presented in Figure 7 (we have written k for the root element in the diagram).

Remark 5.2.6. There is evidently a close relationship between this construction and the one seen in [CHH18], which we discussed in Construction 3.5.4. The key difference between the two is that j effectively appends an element $\langle 1 \rangle$ to each string and then applies the map τ and the closure operation (although the latter is somewhat less important since we work in a context in which the space of nullary operations is weakly contractible). The functor j is also defined on a much simpler category, insofar as $\mathsf{simp}(\mathsf{Fin})$ has fewer morphisms than $\Delta_{\mathbb{F}}$. Nonetheless, in our dealings with the functor j we will draw more than a little inspiration from the approach taken in [CHH18].

With the desired functor j in place, the equivalence of (5.3) is proven by demonstrating that j^* is homotopically fully faithful, i.e. for every reduced plain operad $\mathcal{O}$, it is shown that there is a weak equivalence

$$N^{rc}_d \mathcal{O}(T) \to \mathbb{R} j_* j^* N^{rc}_d \mathcal{O}(T)$$

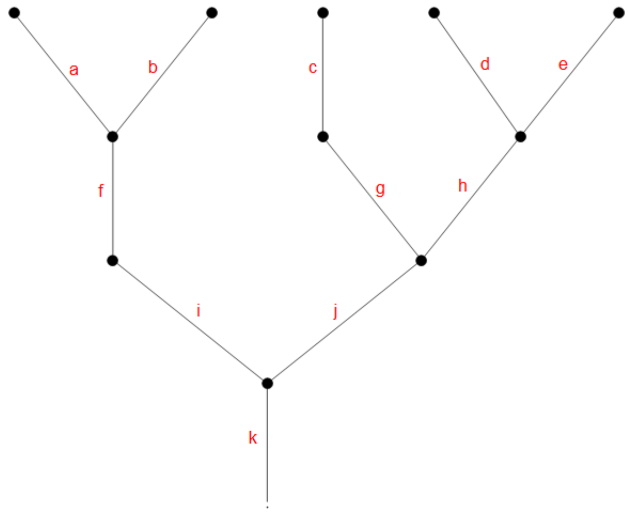

Figure 7: $j(S_*)$ for Example 5.2.5

for each tree T. The argument involves inducting on the vertices of T, where the key proof in the inductive argument stems from the fact that operads satisfy the Segal property (3.3.3), viz.

$$N_d\mathcal{O}(T) \simeq \prod_{v \in \mathrm{vert}(T)} N_d\mathcal{O}(C_{|v|})$$

where the product is over all vertices of T, $|v|$ denotes the number of incoming edges of the vertex v and $C_{|v|}$ is the corolla with $|v|$ vertices. As a consequence of the above, we obtain an adjunction

$$s\mathcal{S}_{/N\mathsf{Fin}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} rcd\mathcal{S}_{\mathrm{CS,Red}} \tag{5.4}$$

where by $rcd\mathcal{S}_{\mathrm{CS,Red}}$, we mean the localisation of the category of rooted closed dendroidal spaces with respect to completeness and Segal equivalences, and with respect to the condition of being reduced (by the work of [CM13] this is Quillen equivalent to the category of ∞-operads with weakly contractible spaces of unary and nullary operations). For purely formal reasons (e.g. by Theorem 1.2.10, or [Lur09a, Proposition 5.2.7.4]), we know that there is a left Bousfield localisation of the category on the left hand side of this adjunction so that the induced map becomes an equivalence. One of the questions we wish to study is:

Question 1. *What localisations do we need to apply to $s\mathcal{S}_{/N\mathsf{Fin}}$ to make the adjunction (5.4) an equivalence?*

We delay an investigation of this question to a later chapter. In the next section, we study another of the facets of the configuration category which will serve to provide us with further motivation to study the question above.

5.3 Configuration Categories of Products

In this section, we discuss the *box tensor product* of simplicial spaces over $N\mathsf{Fin}$. This operation has the pleasing geometric property that the box tensor product of the configuration categories of a pair of manifolds is equivalent, in a suitable sense, to the configuration category of the product of the manifolds. Applying this property to the case where the manifolds we work with are Euclidean spaces, this opens up questions about whether we might utilise the configuration category to obtain an alternative (and arguably more geometric) proof of the Dunn additivity theorem for ∞-operads. Before we can concern ourselves with such questions, we need to introduce a number of technical definitions.

Definition 5.3.1. A surjective map of finite sets $s : \langle k \rangle \twoheadrightarrow \langle l \rangle$ is **selfic** if the injective map $\langle l \rangle \to \langle k \rangle$ defined by $i \mapsto \min\{s^{-1}\{i\}\}$ is increasing.

With this notion in place, we can define a discrete category BoxFin:

- the objects of BoxFin are diagrams $\langle l\rangle \xleftarrow{s} \langle c\rangle \xrightarrow{t} \langle r\rangle$ in which s and t are selfic and such that the induced map $\langle c\rangle \xrightarrow{(s,t)} \langle l\rangle \times \langle r\rangle$ is injective;
- a morphism between two such objects $\langle l\rangle \xleftarrow{s} \langle c\rangle \xrightarrow{t} \langle r\rangle$ and $\langle l'\rangle \xleftarrow{s'} \langle c'\rangle \xrightarrow{t'} \langle r'\rangle$ is a commutative diagram (in Fin):

$$\begin{tikzcd} \langle l\rangle \arrow[d] & \langle c\rangle \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \langle r\rangle \arrow[d] \\ \langle l'\rangle & \langle c'\rangle \arrow[l, "s'"'] \arrow[r, "t'"] & \langle r'\rangle \end{tikzcd}$$

There are three evident forgetful functors $p_0, p_1, p_2 : \mathsf{BoxFin} \to \mathsf{Fin}$, sending a diagram $\langle l\rangle \xleftarrow{u} \langle c\rangle \xrightarrow{v} \langle r\rangle$ to $\langle c\rangle$, $\langle l\rangle$ and $\langle r\rangle$ respectively. We note that $p_0 : N\mathsf{BoxFin} \to N\mathsf{Fin}$ is a fibrewise complete Segal space. (If we had only considered the category whose diagrams consist of surjective, rather than selfic, arrows, we would obtain an equivalent category; however, the nerve of the resulting category would no longer be a fibrewise complete Segal space over $N\mathsf{Fin}$.)

Definition 5.3.2 (Pre-Box Product)**.** Let X and Y be fibrewise complete Segal spaces over Fin. We can define a new simplicial space over $N\mathsf{Fin}$, $X \boxtimes^{\mathrm{pre}} Y$ (verbally, the *pre-box product of X and Y*) as the levelwise pullback of the diagram:

$$\begin{tikzcd} & N\mathsf{BoxFin} \arrow[d, "p_1 \times p_2"] \\ X \times Y \arrow[r] & N\mathsf{Fin} \times N\mathsf{Fin} \end{tikzcd}$$

This simplicial space has the structure of a space over $N\mathsf{Fin}$ via the composite map

$$X \boxtimes^{\mathrm{pre}} Y \to N\mathsf{BoxFin} \xrightarrow{p_0} N\mathsf{Fin}$$

In fact, more can be said: if X and Y are fibrewise complete Segal spaces over $N\mathsf{Fin}$, then so is $X \boxtimes^{\mathrm{pre}} Y$. To see this, we note that the property of being Segal over $N\mathsf{Fin}$ is closed under (homotopy) pullbacks; on the other hand, the property of being fibrewise complete over a space is also closed under products and satisfies the following property: if $Z \to B$ is a fibrewise complete map of Segal spaces and we have a homotopy Cartesian diagram of Segal spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & C \arrow[d] \\ Z \arrow[r] & B \end{tikzcd}$$

then $W \to C$ is also fibrewise complete. Thus, since X and Y are both fibrewise complete over $N\mathsf{Fin}$, we see that $X \boxtimes^{\mathrm{pre}} Y$ is fibrewise complete over $N\mathsf{BoxFin}$ – but $p_0 : N\mathsf{BoxFin} \to N\mathsf{Fin}$ is also fibrewise complete, so the composition $X \boxtimes^{\mathrm{pre}} Y \to N\mathsf{Fin}$ must be fibrewise complete.

Example 5.3.3. The case of greatest interest for us is when X and Y are the configuration categories of some manifolds, M and M'. Using the particle model construction of the configuration categories (5.1.3), we can identify $\mathrm{con}(M)$ as the nerve of a category X'_M, whose objects are embeddings $f : \langle l\rangle \to M$; likewise, we can view $\mathrm{con}(M')$ as the simplicial space whose space of 0-simplices consists of embeddings $g : \langle r\rangle \to M'$. Hence, it follows that a 0-simplex of $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ is the data of a triple

$$\left(\langle l\rangle \xrightarrow{f} M, \langle r\rangle \xrightarrow{g} M', \langle l\rangle \xleftarrow{s} \langle c\rangle \xrightarrow{t} \langle r\rangle\right) \in (NX'_M)_0 \times (NX'_{M'})_0 \times (N\mathsf{BoxFin})_0$$

where the maps f, g are embeddings and s, t are selfic. The condition that the induced map $(s,t) : \langle c\rangle \to \langle l\rangle \times \langle r\rangle$ is injective ensures that there is an induced map on 0-simplices

$$\begin{aligned} \mathrm{con}(M)_0 \boxtimes^{\mathrm{pre}} \mathrm{con}(M')_0 &\longrightarrow \mathrm{con}(M \times M')_0 \\ \left(\langle l\rangle \xrightarrow{f} M, \langle r\rangle \xrightarrow{g} M', \langle l\rangle \xleftarrow{s} \langle c\rangle \xrightarrow{t} \langle r\rangle\right) &\longmapsto \left(\langle c\rangle \xrightarrow{(s,t)} \langle l\rangle \times \langle r\rangle \xrightarrow{f\times g} M \times M'\right) \end{aligned}$$

We recall that in this construction of the configuration category, an edge from $f : \langle l\rangle \rightarrowtail M$ to $f' : \langle l'\rangle \rightarrowtail M$ is determined by a map of finite sets $u : \langle l\rangle \to \langle l'\rangle$ and a reverse exit path $\gamma : [0,a] \to M$ from f to $f' \circ u$ in the stratified space of all maps from $\langle l\rangle$ to M. For concision, let us write such an edge as $(u,\gamma) : (\langle l\rangle, f) \to (\langle l'\rangle, f')$. Then, a 1-simplex in $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ is determined by

- a 1-simplex $(u,\gamma) : (\langle l\rangle, f) \to (\langle l'\rangle, f')$ in $\mathrm{con}(M)$ (i.e. a morphism in X'_M);
- a 1-simplex $(v,\zeta) : (\langle r\rangle, g) \to (\langle r'\rangle, g')$ in $\mathrm{con}(M')$;
- a morphism in BoxFin of the form

$$\begin{tikzcd} \langle l\rangle \arrow[d, "u"] & \langle c\rangle \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "w"] & \langle r\rangle \arrow[d, "v"] \\ \langle l'\rangle & \langle c'\rangle \arrow[l, "s'"] \arrow[r, "t'"'] & \langle r'\rangle \end{tikzcd}$$

For brevity, such a 1-simplex in $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ will be written as a tuple $((u,\gamma),(v,\zeta),w)$.

A priori, there is not an obvious map from the 1-simplices of $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ to $\mathrm{con}(M \times M')$ – the reason being that the Moore paths γ and ζ may be parametrised by intervals of different lengths, e.g. $\gamma : [0,a] \to M$ and $\zeta : [0,b] \to M'$. However, in the special case where the Moore paths *are* parametrised by intervals of the same length, say $[0,a]$, then the commutativity of the diagram above tells us that $(\gamma \times \zeta) \circ (s,t)$ is a path in $\mathrm{maps}(\langle c\rangle, M \times M')$ from $(f \times g) \circ (s,t)$ to

$$((f' \circ u) \times (g' \circ v)) \circ (s,t) = (f' \times g') \circ (u \times v) \circ (s,t) = ((f' \times g') \circ (s',t')) \circ w$$

i.e. $(\gamma \times \zeta) \circ (s,t)$ is a reverse exit path from $(f \times g) \circ (s,t)$ to $(f' \times g') \circ (s',t')$, so $(w, \gamma \times \zeta)$ defines a morphism in $X'_{M \times M'}$, corresponding to a 1-simplex in $\mathrm{con}(M \times M')$.

It might seem like this extra condition of having the same interval parametrising both Moore paths is quite restrictive – however, the restriction can be circumvented to an extent. First, note that $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ can be realised as the nerve of a category which we call $\widetilde{X}_{M,M}$ – the objects (resp. morphisms) of this category are precisely the 0-simplices (resp. 1-simplices) of $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ (the legitimacy of the assertion that $N\widetilde{X}_{M,M'}$ is equivalent to $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ stems from that fact that $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ is a Segal space). Let $Y_{M,M'}$ be the subcategory which has the same objects, but whose collection of morphisms consists only of those tuples $((u,\gamma),(v,\zeta),w)$ in which the reverse exit paths γ and ζ are parametrised by intervals of the same length. The induced inclusion of simplicial spaces $NY_{M,M'} \hookrightarrow N\widetilde{X}_{M,M'}$ is a weak equivalence, so that we obtain a map $(\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M'))_1 \to \mathrm{con}(M \times M')_1$. By the Segal property, it follows that there is a map of fibrewise complete Segal spaces over $N\mathsf{Fin}$

$$\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M') \to \mathrm{con}(M \times M') \tag{5.5}$$

We might wonder if this is an equivalence of fibrewise complete Segal spaces over $N\mathsf{Fin}$. This is almost the case, but we need to attend to a slightly subtle point first.

5.3.1 Conservatisation

A functor $F : \mathcal{C} \to \mathcal{D}$ is usually said to be conservative if it satisfies the property that a morphism $f : c \to c'$ in $\mathcal{C}$ is an isomorphism if and only if $F(f) : F(c) \to F(c')$ is an isomorphism in $\mathcal{D}$. We can devise an analogous definition of conservativity for simplicial spaces (although at first glance it may appear that the concepts are not related):

Definition 5.3.4. A map of simplicial spaces $w : X \to B$ is said to be **conservative** if for any surjective morphism $u : [k] \to [l]$ in Δ, the commutative diagram

$$\begin{tikzcd} X_l \arrow[r, "u^*"] \arrow[d] & X_k \arrow[d] \\ B_l \arrow[r, "u^*"] & B_k \end{tikzcd}$$

is homotopy Cartesian.

The following lemma helps to reconcile these two apparently distinct ideas for a special class of maps of simplicial spaces:

Lemma 5.3.5. *[BW18a, Lemma 8.2] Let $\mathcal{C}$ be a small category and let $w : X \to N\mathcal{C}$ be a fibrewise complete Segal space. Then the following are equivalent:*

(i) X is conservative over $N\mathcal{C}$;

(ii) an element γ is homotopy invertible (i.e. $\gamma \in X_1^{he}$) if and only if $w(\gamma)$ is an isomorphism in $\mathcal{C}$.

Of course, the point of the latter condition is that the homotopy invertible simplices in $N\mathcal{C}$ are precisely the isomorphisms of $\mathcal{C}$, so we see that condition (ii) encodes a kind of "up-to-homotopy" version of the traditional definition of conservativity.

In particular, the aforementioned lemma tells us that the configuration category of a manifold is conservative over $N\mathsf{Fin}$ – to see this, we recall that in the particle model (Construction 5.1.3), the homotopy invertible morphisms of $\mathrm{con}(M)$ are those exit paths which remain in a single stratum, i.e. those morphisms which lie over an isomorphism in the category Fin. However, it is not generally the case that the pre-box product of two conservative spaces should be conservative. To solve this, a functorial conservatisation procedure is devised in [BW18a, Section 8.3], which we now discuss with particular reference to its application to the case of the pre-box product of two configuration categories.

First, for each $r \geq 0$, we define a category $\mathcal{E}(r)$:

- objects are diagrams in Δ of the form $[r] \to [k] \overset{u}{\twoheadleftarrow} [l]$, where the map u is a surjection;
- a morphism between two such objects is a commutative diagram in Δ of the form

$$\begin{tikzcd} {[r]} \arrow[r] \arrow[d, equal] & {[k]} \arrow[d] & {[l]} \arrow[l, two heads, "u"'] \arrow[d] \\ {[r]} \arrow[r] & {[k']} & {[l']} \arrow[l, two heads, "u'"'] \end{tikzcd}$$

Using this category, we can define a **conservatisation functor** Λ as follows: given a map of simplicial spaces $X \to B$, we can define a new simplicial space ΛX over B with

$$(\Lambda X)_r = \operatorname*{hocolim}_{([r]\to[k]\twoheadleftarrow[l])\in\mathcal{E}(r)} X_l \times_{B_l} B_k$$

The map $\Lambda X \to B$ is determined by the composition

$$\operatorname*{hocolim}_{([r]\to[k]\twoheadleftarrow[l])\in\mathcal{E}(r)} X_l \times_{B_l} B_k \to \operatorname*{colim}_{([r]\to[k]\twoheadleftarrow[l])\in\mathcal{E}(r)} B_k \to B_r$$

We can also define a functor $(-)^!$ such that the conservatisation procedure becomes universal in a derived sense, that is, we have a diagram

$$\begin{tikzcd} X \arrow[dr] & X^! \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \Lambda X \arrow[dl] \\ & B & \end{tikzcd}$$

The r-simplices of $X^!$ are defined by taking the homotopy colimit over the subcategory of $\mathcal{E}(r)$ spanned by those diagrams in which the map $[l] \twoheadrightarrow [k]$ is the identity, i.e.

$$(X^!)_r = \operatorname*{hocolim}_{[r]\to[k]\overset{=}{\longleftarrow}[k]} X_k \times_{B_k} B_k = \operatorname*{hocolim}_{[r]\to[k]} X_k$$

It is straightforward to see that the map $X^! \to X$ is a weak equivalence as indicated in the diagram above.

Using the equivalence between the categories $s\mathcal{S}_{/B}$ and $\mathcal{S}^{\mathsf{simp}(B)^{\mathrm{op}}}$ we can rephrase the functor Λ in slightly more convenient terms: given a map of simplicial spaces $w : X \to B$ and an r-simplex $b \in B$, let $X(b) = w^{-1}\{b\}$. To each such simplex b, we can associate a category $\mathcal{E}(b)$:

- objects of $\mathcal{E}(b)$ are diagrams $b \to c \leftarrow d$ in $\mathsf{simp}(B)$ such that the map in Δ underlying $d \to c$ is surjective (so e.g. if $c \in B_k$ and $d \in B_l$, then there exist maps $v : [r] \to [k]$ and $u : [l] \twoheadrightarrow [k]$ such that $v^*c = b$ and $u^*c = d$) (we say that the morphism $d \to c$ is *dominant*);
- morphisms of $\mathcal{E}(b)$ are commutative diagrams in $\mathsf{simp}(B)$ of the form

$$\begin{tikzcd} b \arrow[r] \arrow[d, equal] & c \arrow[d] & d \arrow[l] \arrow[d] \\ b \arrow[r] & c' & d' \arrow[l] \end{tikzcd}$$

  in which both the rightmost arrows are dominant.

We see that this is just a fibrewise version of the category $\mathcal{E}(r)$. In a similar fashion, we define a functor $\Lambda X : \mathsf{simp}(B)^{\mathrm{op}} \to \mathcal{S}$ by

$$(\Lambda X)(b) = \operatorname*{hocolim}_{(b\to c\leftarrow d)\in\mathcal{E}(b)} X(d)$$

Likewise, we define $X^! : \mathsf{simp}(B)^{\mathrm{op}} \to \mathcal{S}$ by

$$X^!(b) = \operatorname*{hocolim}_{(b\to c\xleftarrow{=} d)\in\mathcal{E}(b)} X(d) = \operatorname*{hocolim}_{(b\to c)} X(c)$$

The advantage in pivoting to this perspective is that we can look at a subcategory $\mathcal{E}^\flat(b)$ spanned by those diagrams $(b \to c \leftarrow d)$ in which both arrows are dominant and define a new functor $\Lambda^\flat$ by

$$(\Lambda^\flat X)(b) = \operatorname*{hocolim}_{(b\to c\leftarrow d)\in\mathcal{E}^\flat(b)} X(d)$$

Since the inclusion $\mathcal{E}^\flat(b) \hookrightarrow \mathcal{E}(b)$ has a right adjoint for each r-simplex b, it follows that $(\Lambda^\flat X)(b) \to (\Lambda X)(b)$ is a weak equivalence, so we can instead use $\Lambda^\flat$ as our conservatisation functor. For the case of a non-degenerate simplex b, the only dominant morphism $b \to c$ is the identity map, so that we arrive at the simplified expression of $(\Lambda^\flat X)(b)$ (for non-degenerate b)

$$(\Lambda^\flat X)(b) = \operatorname*{hocolim}_{b\leftarrow d} X(d)$$

where we now take the homotopy colimit over all dominant morphisms $d \to b$.

The value of these constructions is affirmed by the following trio of results from [BW18a, Section 8.3]:

Lemma 5.3.6. *Let $X \to B$ be a map of simplicial spaces. Then*

(i) $\Lambda X \to B$ is conservative;

(ii) $X^! \to X$ is a degreewise weak equivalence;

(iii) if $X \to B$ is already conservative, then the map $X^! \to \Lambda X$ is a degreewise weak equivalence.

Example 5.3.7. Having put these constructions in place, we now wish to analyse them as they apply to the pre-box product of the configuration categories of two manifolds, $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$. Let us consider the

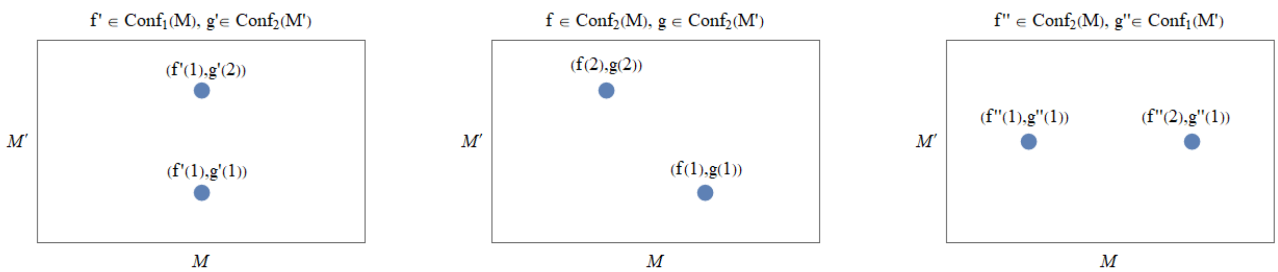

Figure 8: Three vertices of $\mathrm{con}(M \times M')$ lying over $\langle 2\rangle$

schematic of the product of the manifolds M and M' presented in Figure 8 – these three diagrams all represent 0-simplices in $\mathrm{con}(M \times M')$ lying over the object $\langle 2\rangle$ in $N\mathsf{Fin}$, and all three arise as the image of some points in the pre-box product of $\mathrm{con}(M)$ with $\mathrm{con}(M')$. Namely:

- the diagram on the left is the image of a point

$$\left(f' : \langle 1\rangle \to M, g' : \langle 2\rangle \to M', \beta' = \left(\langle 1\rangle \twoheadleftarrow \langle 2\rangle \xrightarrow{=} \langle 2\rangle\right)\right) \in \mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')_0$$

- the diagram in the middle is the image of

$$\left(f : \langle 2\rangle \to M, g : \langle 2\rangle \to M', \beta = \left(\langle 2\rangle \xleftarrow{=} \langle 2\rangle \xrightarrow{=} \langle 2\rangle\right)\right) \in \mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')_0$$

- the diagram on the right is the image of

$$\left(f'' : \langle 2\rangle \to M, g'' : \langle 2\rangle \to M', \beta'' = \left(\langle 2\rangle \xleftarrow{=} \langle 2\rangle \twoheadrightarrow \langle 1\rangle\right)\right) \in \mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')_0$$

In $\mathrm{con}(M \times M')$, it is clear that there are homotopy invertible morphisms between these three configurations; however, the morphism in $\mathrm{con}(M)$ from f to f' is evidently not invertible since we move from one stratum to another; likewise, the morphism g to g'' is not invertible in $\mathrm{con}(M'')$. As a result, the corresponding morphisms in $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')$ also fail to be homotopy invertible. This observation is at the core of why the map of Equation (5.5) fails to be a weak equivalence of fibrewise complete Segal spaces.

All is not lost however: the value of the conservatisation procedure described above is that it effectively forces the homotopy invertibility of these morphisms in the pre-box product. This becomes intuitively clear when we consider the categories $\mathcal{E}((f,g,\beta)), \mathcal{E}((f',g',\beta'))$ and $\mathcal{E}((f'',g'',\beta''))$ – we see that taking homotopy colimits of the functor $\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M') : \mathsf{simp}(\mathsf{Fin})^{op} \to \mathcal{S}$ over each of these three categories yields objects which are weakly homotopy equivalent to each other.

(Obviously our remarks in the previous paragraph don't constitute a rigorous mathematical argument of why conservatisation might assist us in converting Equation (5.5) into an equivalence, but they should at least serve as a roadmap for our intuition.)

In [BW18b, Lemma 2.9], it is shown that the map $\Lambda\left(\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(N)\right) \to \mathrm{con}(M \times N)$ is a weak equivalence in simplicial degree 0. Critical to this deduction is a technical result about exit path categories of (homotopically) stratified spaces:

Lemma 5.3.8. *Let Q be a locally contractible homotopically stratified space, with exit path category $\mathcal{EP}_Q$ and full path category $\mathcal{P}_Q$. Then, the map*

$$\underset{[r]\in\Delta}{\mathrm{hocolim}}\,(\mathcal{EP}_Q)_r \to \underset{[r]\in\Delta}{\mathrm{hocolim}}\,(\mathcal{P}_Q)_r$$

induced by the inclusion is a weak equivalence

This lemma is of interest in our setting because the space of embeddings of k points into the product manifold $M \times M'$ admits a stratification as follows: an embedding $\langle k\rangle \rightarrowtail M \times M'$ determines maps $\langle k\rangle \to M$, $\langle k\rangle \to M'$ which may not be embeddings – but we can factor every map $\langle k\rangle \to M$ as $\langle k\rangle \to \langle i\rangle \rightarrowtail M$, where $\langle k\rangle \to \langle i\rangle$ is selfic (and likewise for the map $\langle k\rangle \to M'$) – thus a stratum of the space of embeddings $\langle k\rangle \rightarrowtail M \times M'$ corresponds to a pair of selfic maps $(\langle k\rangle \to \langle i\rangle, \langle k\rangle \to \langle j\rangle)$. Applying the above technical result to the case where Q is the space of all embeddings $\langle k\rangle \rightarrowtail M \times M'$, we find that the weak equivalence described in the lemma corresponds on 0-simplices to the map

$$\Lambda^\flat\left(\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')\right)_{/\langle k\rangle} \to \Lambda^\flat \mathrm{con}(M \times M')_{/\langle k\rangle}$$

induced by the map 5.5.

It should be noted that, in general, the conservatisation procedure does not preserve the property of being a fibrewise complete Segal space. However, this can be remedied formally: it is possible to produce an endofunctor K of simplicial spaces which sends a simplicial space X over B to a fibrewise complete Segal space KX over B such that X and KX are weakly equivalent in the complete Segal space model structure (this can be achieved via the Segal completion functor that Rezk produces in [Rez01, Section 14]). With this functor available to us, we can simply successively apply the functors K and Λ: given a simplicial space X over B, we have a zig-zag of morphisms of simplicial spaces over B, natural in X:

$$\begin{tikzcd}
X \arrow[d] & & \\
KX & & \\
(KX)^! \arrow[u, "\simeq"'] \arrow[r] & \Lambda KX \arrow[d] & \\
& K\Lambda KX & \\
& (K\Lambda KX)^! \arrow[u, "\simeq"'] \arrow[r] & \Lambda K\Lambda KX \arrow[d] \\
& & K\Lambda K\Lambda KX \\
& & (\Lambda K\Lambda KX)^! \arrow[u, "\simeq"'] \arrow[r] & \ldots
\end{tikzcd}$$

The resulting homotopy colimit of this procedure, which we denote by $(\Lambda K)^\infty X$ will be a conservative fibrewise complete Segal space over B. Hence $(\Lambda K)^\infty$ defines a functorial method of simultaneously performing conservatisation and fibrewise-Segal-completion. With this functor in place, we can make the following definition:

Definition 5.3.9. Let X and Y be fibrewise complete Segal spaces over $N\mathsf{Fin}$. We define the conservative fibrewise complete Segal space $X \boxtimes Y$ (verbally, the *box product of X and Y*) over $N\mathsf{Fin}$ to be $(\Lambda K)^\infty\left(X \boxtimes^{\mathrm{pre}} Y\right)$.

5.3.2 Configuration Categories and Additivity of Little Cubes Operads

Having assembled the battalion of definitions from the previous subsection, we can now return to the relationship between the pre-box product of configuration categories of two manifolds and the configuration category of the product of the manifolds which we met in Equation (5.5) – by construction, the map we produced there induces a map

$$\mathrm{con}(M) \boxtimes \mathrm{con}(M') \to \mathrm{con}(M \times M') \tag{5.6}$$

of fibrewise complete Segal spaces over $N\mathsf{Fin}$. One of the major results of [BW18b] is that

Theorem 5.3.10. *[BW18b, Theorem 2.8] The map (5.6) is a weak equivalence of fibrewise complete Segal spaces over $N\mathsf{Fin}$.*

We have already highlighted some of the details of how this proof works in simplicial degree 0. In fact, it can be shown that $\Lambda\left(\mathrm{con}(M) \boxtimes^{\mathrm{pre}} \mathrm{con}(M')\right) \to \mathrm{con}(M \times M')$ is a levelwise weak equivalence of simplicial spaces over $N\mathsf{Fin}$, but we will not discuss here the methods by which this is deduced, since they rely on further technical arguments which will have limited external utility/interest for us. For the details of these arguments, we refer the reader to [BW18b, Section 3] and [BW18a, Section 8.4].

Of more interest to us is the possibility of combining this result with Theorem 5.2.4: that result told us that there is a homotopically fully faithful functor $j^* : (rcd\mathcal{S})_{\mathrm{CS,Red}} \to s\mathcal{S}_{/N\mathsf{Fin}}$, which gives us a means to consider the little cubes operad (which is a reduced plain operad) as a simplicial space $j^* N_d^{rc}\, {}^t\mathbb{E}_d \simeq \mathrm{con}(\mathbb{R}^d)$ over $N\mathsf{Fin}$. Identifying $N_d^{rc}\, {}^t\mathbb{E}_d$ with the corresponding homotopy-coherent operad $\mathbb{E}_d^\otimes$ (via the equivalence between complete dendroidal Segal spaces and quasi-operads), Theorem 5.3.10 tells us that we have a weak equivalence of conservative Segal spaces over $N\mathsf{Fin}$:

$$j^*\mathbb{E}_d^\otimes \boxtimes j^*\mathbb{E}_{d'}^\otimes \simeq \mathrm{con}(\mathbb{R}^d) \boxtimes \mathrm{con}(\mathbb{R}^d) \simeq \mathrm{con}\left(\mathbb{R}^{d+d'}\right) \simeq j^*\mathbb{E}_{d+d'}^\otimes$$

This relation bears more than a passing similarity to the (∞-categorical) additivity theorem (3.2.1) for little cubes, leading us to speculate:

Question 2. *Given reduced Segal dendroidal spaces X, Y, is there an equivalence of simplicial spaces over $N\mathsf{Fin}$ (with respect to some localisation...)*

$$j^*(X \odot Y) \simeq j^*X \boxtimes j^*Y$$

where $\odot$ is the ∞-categorical tensor product of ∞-operads?

If the answer is yes, then Theorems 5.2.4 and 5.3.10 effectively give a new proof of the additivity theorem, one which relies on rather more geometric arguments than the weak approximation and assembly arguments used in the proof of Lurie. Our elliptical reference to *some* localisation hints that this question is intimately linked with Question 1. In the hopes of resolving the second question, we are thus compelled to study the first in greater detail. This is the subject of our next chapter.

Chapter 6

A Quillen Equivalence for Reduced Operads

In the previous chapter, we saw that there is an adjunction

$$j_! : s\mathcal{S}_{/N\mathsf{Fin}} \rightleftarrows rcd\mathcal{S}_{\mathrm{CS,Red}} = \mathcal{P}_{\mathrm{CS,Red}}(\Omega_{rc}) : j^* \tag{6.1}$$

where we localise the category on the right with respect to the complete Segal property, and the condition of being reduced. Via the Quillen equivalence between complete dendroidal Segal spaces and ∞-operads, the category on the right corresponds to the category of reduced ∞-operads. Using the fact that the functor j^* is homotopically fully faithful, we were able to assert that there is some localisation on the category $s\mathcal{S}_{/N\mathsf{Fin}}$ such that this adjunction descends to a Quillen equivalence. Our goal in this chapter is to study an appropriate collection of localising conditions and deduce the stated Quillen equivalence, thus resolving Question 1, and possibly giving us some hope of providing a positive answer to Question 2.

6.1 Localising the Category $s\mathcal{S}_{/N\mathsf{Fin}}$

We commence by producing a collection of (sets of) morphisms in $s\mathcal{S}_{/N\mathsf{Fin}}$ such that by taking left Bousfield localisations with respect to each set, we obtain a model category whose properties mirror those of the category $\mathcal{P}(\Omega_{rc})_{\mathrm{CS,Red}}$. In other words, we have to deduce what kind of objects in $s\mathcal{S}_{/N\mathsf{Fin}}$ will correspond under the functor $j_!$ to reduced operads (AKA reduced complete Segal dendroidal spaces). Furthermore, having seen the significance of the notion of conservativity in the previous chapter, we might query whether it isn't necessary to also localise with respect to some kind of conservative conditions – as it turns out, the question of being conservative (and of being complete and fibrewise complete over $N\mathsf{Fin}$) will actually be subsumed by the reduced and Segal localisations which we introduce here.

Remark 6.1.1. Before we proceed any further, we wish to draw attention to the localisation with respect to completeness on the right hand side of 6.1. Let $u : \Delta \hookrightarrow \Omega$ be the inclusion of the simplicial indexing category in the category of trees, and $\overline{\zeta} : \Omega \to \Omega_{rc}$ the functor which sends a tree T to the closed tree $\overline{T}_+$, which is defined by appending a root to the closure of T; and let $\overline{u}$ be the composition $\overline{\zeta} \circ u : \Delta \to \Omega_{rc}$. We see immediately that $\overline{u}[0] = \overline{C}_1$. We recall that a Segal object $\mathcal{X} \in \mathcal{P}(\Omega_{rc})$ is complete if $\overline{u}^*\mathcal{X}$ is a complete Segal space, i.e. if we write E to mean the nerve of the groupoid

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

then there is a weak equivalence

$$\mathrm{Map}(E, \overline{u}^*\mathcal{X}) \xrightarrow{\sim} \mathrm{Map}(\Delta[0], \overline{u}^*\mathcal{X}) \simeq \mathcal{X}(\overline{u}[0]) = \mathcal{X}(\overline{C}_1) \tag{6.2}$$

However, if $\mathcal{X}$ is a reduced Segal presheaf, then the Segal property tells us that

$$\overline{u}^*\mathcal{X}([n]) \simeq \mathcal{X}\left(\overline{u}[1]\right) \times_{\mathcal{X}(\overline{u}[0])} \ldots \times_{\mathcal{X}(\overline{u}[0])} \mathcal{X}\left(\overline{u}[1]\right)$$

while the fact that $\mathcal{X}$ is reduced tells us that we have weak equivalences $\mathcal{X}(\overline{u}[1]) \simeq * \simeq \mathcal{X}(\overline{u}[0])$ – in particular, we must also have $(\overline{u}^*\mathcal{X})_1^{he} \simeq *$, so the condition that $\mathcal{X}$ is complete becomes vacuous in this setting. With this point in mind, we will hereafter denote the category of reduced complete Segal dendroidal spaces, $\mathcal{P}_{\mathrm{CS,Red}}(\Omega_{rc})$, by $\mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc})$ (to denote reduced Segal dendroidal spaces), since there is no difference between these categories. Hence, we see that there is no requirement for us to impose any kind of localisation on $s\mathcal{S}_{/N\mathsf{Fin}}$ with respect to a completeness condition.

We apply the following localisations on the category $s\mathcal{S}_{/N\mathsf{Fin}}$:

(1) The **reduced condition** for simplicial spaces over $N\mathsf{Fin}$: weak equivalence over the maps

$$\emptyset \to \{\langle 0\rangle\} \quad \text{and} \quad \emptyset \to \{\langle 1\rangle\}$$

where $\langle 0\rangle = \emptyset$. Under the map j, this corresponds to contractibility of the root and the 0-corolla (i.e. the closure of the root), and in turn relates to the fact that we are working with operads whose spaces of nullary operations are weakly contractible.

(2) **Contractibility of internal unary edges**: weak equivalences over the natural inclusions

$$(S) \hookrightarrow (S \to \langle 1\rangle)$$

Since the functor j acts by appending a root (i.e. $\langle 1\rangle$) to a string of finite sets, this essentially ensures that if the forest associated to some string of finite sets S_* already has a root (i.e. it is already a tree) then that tree is weakly equivalent to $j_!(S_*) = j(S_*)$ – this in turn relates to the fact that we only want to look at operads whose spaces of unary operations are weakly contractible.

(3) The **pullback Segal condition** for simplicial spaces over $N\mathsf{Fin}$: if $\mathcal{X}$ is a simplicial space over $N\mathsf{Fin}$, and S_* is a string of finite sets S_*, let us write $\mathcal{X}(S_*)$ to mean the fibre of the map $\mathcal{X} \to N\mathsf{Fin}$ over the simplex S_*. We say $\mathcal{X}$ satisfies the pullback Segal condition if it is local with respect to the natural maps in $s\mathcal{S}_{/N\mathsf{Fin}}$ of the form

$$(S_0 \to S_1) \cup_{S_1} (S_1 \to S_2) \cup_{S_2} \ldots \cup_{S_{n-1}} (S_{n-1} \to S_n) \longrightarrow (S_0 \to S_1 \to \ldots \to S_n)$$

for any string of morphisms of finite sets, S_*, of length ≥ 2; in other words, there are weak equivalences

$$\mathcal{X}(S_0 \to S_1 \to \ldots \to S_n) \xrightarrow{\sim} \mathcal{X}(S_0 \to S_1) \times_{\mathcal{X}(S_1)} \mathcal{X}(S_1 \to S_2) \times_{\mathcal{X}(S_2)} \ldots \times_{\mathcal{X}(S_{n-1})} \mathcal{X}(S_{n-1} \to S_n)$$

(4) The **product Segal condition** for simplicial spaces over $N\mathsf{Fin}$: first note that, given a length one string $s : S_0 \to S_1$, we can identify

$$\mathcal{X}(S_0 \xrightarrow{s} S_1) \simeq \prod_{x \in S_1} \mathcal{X}\left(\begin{tikzcd} s^{-1}\{x\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \langle 1\rangle \arrow[d, "1\mapsto x"] \\ S_0 \arrow[r, "s"] & S_1 \end{tikzcd}\right) \tag{6.3}$$

For each $x \in S_1$, let us write $\mathscr{I}_x^s$ to denote the simplicial space

$$\begin{tikzcd} s^{-1}\{x\} \arrow[r] & \langle 1\rangle \arrow[d, "1\mapsto x"] \\ & S_1 \end{tikzcd}$$

(by which we mean the discrete simplicial space $\Lambda[2,1]$ lying over these finite sets in Fin) and let $\mathscr{I}^s$ denote the pushout $\mathscr{I}_{x_1}^s \cup_{S_1} \ldots \cup_{S_1} \mathscr{I}_{x_m}^s$ where $S_1 = \{x_1, \ldots, x_m\}$. For each $x \in S_1$, write $\mathscr{E}_x^s$ to mean the simplicial space

$$\begin{tikzcd} s^{-1}\{x\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \langle 1\rangle \arrow[d, "1\mapsto x"] \\ S_0 \arrow[r, "s"] & S_1 \end{tikzcd}$$

(by which we mean the discrete simplicial space $\Delta[1] \times \Delta[1]$ lying over these finite sets in Fin) and let $\mathscr{E}^s$ denote the pushout $\mathscr{E}^s_{x_1} \cup_s \ldots \cup_s \mathscr{E}^s_{x_m}$. By (6.3), there are weak equivalences $\mathcal{X}\left(S_0 \xrightarrow{s} S_1\right) \simeq \mathcal{X}(\mathscr{E}^s)$. The natural inclusions $\mathscr{I}^s_{x_k} \hookrightarrow \mathscr{E}^s_{x_k}$ induce maps $\mathscr{I}^s \to \mathscr{E}^s$. We say a simplicial space $\mathcal{X}$ over $N\mathsf{Fin}$ satisfies the product Segal condition if it is local with respect to all such maps, i.e. we have weak equivalences

$$\mathcal{X}(\mathscr{E}^s) \xrightarrow{\sim} \mathcal{X}(\mathscr{I}^s)$$

for all 1-simplices $s : S_0 \to S_1$ in $N\mathsf{Fin}$. Note that we can write

$$\mathcal{X}(\mathscr{I}^s) = \mathcal{X}(S_1) \times \prod_{x \in S_1} \mathcal{X}(s^{-1}\{x\})$$

so locality with respect to this class of morphisms tells us that for any 1-simplex $s : S_0 \to S_1$ in $N\mathsf{Fin}$, there are weak equivalences

$$\mathcal{X}\left(S_0 \xrightarrow{s} S_1\right) \simeq \mathcal{X}(S_1) \times \prod_{x \in S_1} \mathcal{X}(s^{-1}\{x\})$$

In summary, the objects in $s\mathcal{S}_{/N\mathsf{Fin}}$ which are local with respect to the first two classes of maps correspond to reduced dendroidal spaces, while the objects in $s\mathcal{S}_{/N\mathsf{Fin}}$ which are local with respect to the third and fourth classes of maps correspond under $j_!$ to those dendroidal spaces which satisfy a Segal condition. In combination with Remark 6.1.1, it follows that if an object is local with respect to all four of the classes above, then its image under $j_!$ is a reduced ∞-operad. Verbally, we will refer to such an object as a **reduced Segal simplicial space over** $N\mathsf{Fin}$. We next wish to show that these localisations also contain all the information required to meet the condition of being fibrewise complete over $N\mathsf{Fin}$ and conservative.

Lemma 6.1.2. *If a simplicial space $\mathcal{X}$ over $N\mathsf{Fin}$ is local with respect to the above conditions then it is fibrewise complete over $N\mathsf{Fin}$ and conservative.*

Proof. First, we prove that $\mathcal{X} \to N\mathsf{Fin}$ is fibrewise complete, i.e. the following square is homotopy Cartesian

$$\begin{tikzcd} \mathcal{X}_1^{\mathrm{he}} \arrow[r, "d_0"] \arrow[d] & \mathcal{X}_0 \arrow[d] \\ (N\mathsf{Fin})_1^{\mathrm{he}} \arrow[r, "d_0"] & (N\mathsf{Fin})_0 \end{tikzcd}$$

We know that the elements of $(N\mathsf{Fin})_1^{\mathrm{he}}$ are bijections of finite sets, so fix such a bijection $\sigma : S \to S$ in Fin and consider the fibre over σ. If $\mathcal{X}$ is local in the sense above (in particular, meeting the "reduced" and "product Segal" locality conditions), we must have

$$\mathcal{X}(S \xrightarrow{\sigma} S) \simeq \mathcal{X}(S) \times \prod_{i \in S} \mathcal{X}(\sigma^{-1}\{i\}) \simeq \mathcal{X}(S)$$

which is exactly the fibre of $\mathcal{X}_0 \to (N\mathsf{Fin})_0$ over $d_0(S \xrightarrow{\sigma} S) = S$. Thus, we conclude that $\mathcal{X}$ is fibrewise complete.

For conservativity, we want to show that for every surjective map $u : [k] \twoheadrightarrow [l]$ in Δ, we have a homotopy Cartesian diagram

$$\begin{tikzcd} \mathcal{X}_l \arrow[r, "u^*"] \arrow[d] & \mathcal{X}_k \arrow[d] \\ (N\mathsf{Fin})_l \arrow[r, "u^*"] & (N\mathsf{Fin})_k \end{tikzcd}$$

If u is an identity map, there is nothing to prove, so let us assume that u is a non-trivial surjection. Fix a string

$$S_* = \left(S_0 \xrightarrow{s^1} S_1 \xrightarrow{s^2} \ldots \xrightarrow{s^l} S_l\right)$$

in $(N\mathsf{Fin})_l$, and write $S'_* = u^* S_*$, so S'_* is of the form

$$\left(S_{u(0)} \xrightarrow{s^{u(0),u(1)}} S_{u(1)} \xrightarrow{s^{u(1),u(2)}} \ldots \xrightarrow{s^{u(k-1),u(k)}} S_{u(k)}\right)$$

where $s^{i,j}$ is the composite map $S_i \to S_{i+1} \to \ldots \to S_j$. We wish to show that we have weak equivalences $\mathcal{X}(S'_*) \simeq \mathcal{X}(S_*)$, but using the Segal pullback property of $\mathcal{X}$, this amounts to showing that we have weak equivalences

$$\begin{aligned}\mathcal{X}\left(S_{u(0)} \xrightarrow{s^{u(0),u(1)}} S_{u(1)}\right) \times_{\mathcal{X}\left(S_{u(1)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(k-1)}\right)} \mathcal{X}\left(S_{u(k-1)} \xrightarrow{s^{u(k-1),u(k)}} S_{u(k)}\right)\\ \xrightarrow{\sim} \mathcal{X}\left(S_0 \xrightarrow{s^1} S_1\right) \times_{\mathcal{X}(S_1)} \ldots \times_{\mathcal{X}(S_{l-1})} \mathcal{X}\left(S_{l-1} \xrightarrow{s^l} S_l\right)\end{aligned} \tag{6.4}$$

Since u is a non-trivial surjection, we know that at least $k-l$ of the arrows in the string S'_* must be identity arrows (and if there are $t > k-l$ identity arrows in S'_*, then there are $t-(k-l)$ identity arrows in the string S_*). If we can show that we have the desired weak equivalence for $k-l=1$, then we can reason inductively that the stated equivalence must hold for all $k-l$.

We can use the product Segal property of $\mathcal{X}$ to see that if $s^{u(i-1),u(i)}$ is the unique identity arrow which does not come from an identity arrow in the string S_* (in other words $u : [k] \to [k-1]$ is the elementary degeneracy $d^i : [k] \to [k-1]$ with $d^i(i) = d^i(i+1) = i$ and $d^i(j) = j$ for all $j \leq i$), then the corresponding factor $\mathcal{X}\left(s^{u(i-1),u(i)}\right) := \mathcal{X}\left(S_{u(i-1)} \xrightarrow{s^{u(i-1),u(i)}} S_{u(i)}\right)$ in the map (6.4) satisfies

$$\begin{aligned}\mathcal{X}\left(s^{u(i-1),u(i)}\right) &\simeq \mathcal{X}\left(S_{u(i)}\right) \times \prod_{x \in S_{u(i)}} \mathcal{X}\left(\left(s^{u(i-1),u(i)}\right)^{-1}\{x\}\right)\\ &\simeq \mathcal{X}\left(S_{u(i)}\right) \times \prod_{x \in S_{u(i)}} \mathcal{X}\left(\{x\}\right)\\ &\simeq \mathcal{X}\left(S_{u(i)}\right)\end{aligned} \tag{6.5}$$

since by assumption $s^{u(i-1),u(i)}$ is an identity map. Thus we have

$$\begin{aligned}\mathcal{X}(S'_*) &\simeq \mathcal{X}\left(s^{u(0),u(1)}\right) \times_{\mathcal{X}\left(S_{u(1)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(i-1)}\right)} \mathcal{X}\left(s^{u(i-1),u(i)}\right) \times_{\mathcal{X}\left(S_{u(i)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(k-1)}\right)} \mathcal{X}\left(s^{u(k-1),u(k)}\right)\\ &\simeq \mathcal{X}\left(s^{u(0),u(1)}\right) \times_{\mathcal{X}\left(S_{u(1)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(i-1)}\right)} \mathcal{X}\left(S_{u(i)}\right) \times_{\mathcal{X}\left(S_{u(i)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(k-1)}\right)} \mathcal{X}\left(s^{u(k-1),u(k)}\right)\\ &\simeq \mathcal{X}\left(s^{u(0),u(1)}\right) \times_{\mathcal{X}\left(S_{u(1)}\right)} \cdots \times_{\mathcal{X}\left(S_{u(i-1)}\right)} \mathcal{X}\left(s^{u(i),u(i+1)}\right) \times_{\mathcal{X}(S_{u(i+1)})} \cdots \times_{\mathcal{X}\left(S_{u(k-1)}\right)} \mathcal{X}\left(s^{u(k-1),u(k)}\right)\\ &\simeq \mathcal{X}(s^1) \times_{\mathcal{X}(S_1)} \ldots \times_{\mathcal{X}(S_{i-1})} \mathcal{X}(s^i) \times_{\mathcal{X}(S_i)} \ldots \times_{\mathcal{X}(S_{k-2})} \mathcal{X}(s^{k-1})\\ &\simeq \mathcal{X}(S_*)\end{aligned}$$

where we use (6.5) in the second line; in the third line, we use the weak equivalence $\mathcal{X}(S_{u(i-1)}) \simeq \mathcal{X}(S_{u(i-1)}) \times_{\mathcal{X}(S_{u(i)})} \mathcal{X}(S_{u(i)})$; and at the fourth line, we use the definition of $u = d^i$ (which tells us that e.g. $s^{u(0),u(1)} = s^{0,1} = s^1$ and $s^{u(k-1),u(k)} = s^{k-2,k-1} = s^{k-1}$). Since all monotone surjections $u : [k] \twoheadrightarrow [l]$ arise as compositions of such degeneracy maps d^i, the pullback Segal condition tells us that $\mathcal{X}$ satisfies (6.4) for all such u. □

We recall that we can identify the category $s\mathcal{S}_{/N\mathsf{Fin}}$ with the category of (space-valued) presheaves on $\mathsf{simp}(\mathsf{Fin})$, which we will hereafter denote by $\mathcal{P}\left(\mathsf{simp}(\mathsf{Fin})\right)$. We claim that the list of localising maps which we described above is a complete list of the localisations required on $s\mathcal{S}_{/N\mathsf{Fin}}$, i.e. writing $\mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ to mean the left Bousfield localisation of $\mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$ with respect to the above-listed classes of morphisms, we claim that there is a Quillen equivalence

$$j_! : \mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \rightleftarrows \mathcal{P}_{\text{Seg,Red}}(\Omega_{rc}) : j^* \tag{6.6}$$

The first step towards proving this assertion is assuring ourselves that the functor j^* is right Quillen with respect to these localisations. Before we do this, let us fix some notation: given an object $([n], S)$ (in $\mathsf{simp}(\mathsf{Fin})$ or $\Delta^1_{\mathbb{F}}$), corresponding to a string

$$\left(S_0 \xrightarrow{s^1} S_1 \xrightarrow{s^2} \ldots \xrightarrow{s^n} S_n\right)$$

(in the case of an object in $\Delta^1_{\mathbb{F}}$, the set S_n equals $\langle 1\rangle$) and an element $x_l \in S_l$ (with $0 < l \leq n$), we write S_{i,x_l} to denote the fibre of the composite $(S_i \to \ldots \to S_l)$ over x_l. Using this notation, we can identify the Segal core of an object $([n], S)$ in $\Delta^1_{\mathbb{F}}$ with the colimit over all diagrams of the form

$$\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\}\right) \cup \left(S_{1,x_2} \xrightarrow{s^2} \{x_2\}\right) \cup \ldots \cup \left(S_{n-2,x_{n-1}} \xrightarrow{s^1} \{x_{n-1}\}\right)$$

where the colimit is taken over all x_{n-1} in S_{n-1} and all possible combinations $x_1, \ldots, x_{n-1}$ such that $s^i(x_i) = x_{i+1}$ for each $1 \leq i \leq n-2$.

In addition, we will adopt the following somewhat sloppy convention for the forthcoming lemma: given a Segal object $\mathcal{Z} \in \mathcal{P}(\Omega_{rc})$ and a set S, we will write $\mathcal{Z}(S)$ to mean the image of $\mathcal{Z}$ on the corolla whose leaves are indexed by the set S, and taking advantage of the Segal property, we will write $\mathcal{Z}(S_0 \xrightarrow{s^1} S_1)$ to mean the product $\mathcal{Z}(S_1) \times \prod_{x_1 \in S_1} \mathcal{Z}(S_{0,x_1})$, etc. (This is merely to save us from having to draw diagrams of trees when working through our calculations.)

Lemma 6.1.3. *Let $\mathcal{Z}$ be a fibrant object in $\mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc})$. Then $j^*\mathcal{Z}$ is a fibrant object in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$.*

Proof. Let us begin by checking that $j^*\mathcal{Z}$ is local with respect to the reduced condition:

$$(j^*\mathcal{Z})(\emptyset) = \mathcal{Z}(\emptyset) = * \simeq \mathcal{Z}(\overline{u}[0]) = \mathcal{Z}\left(j(\langle 0\rangle)\right) = (j^*\mathcal{Z})\left(\langle 0\rangle\right)$$

since $\mathcal{Z}$ is reduced, so $\mathcal{Z}(\overline{\eta}) = \mathcal{Z}(\overline{u}[0]) \simeq *$. Likewise, we see that

$$(j^*\mathcal{Z})(\emptyset) = \mathcal{Z}(\emptyset) = * \simeq \mathcal{Z}(\overline{u}[1]) = \mathcal{Z}\left(j(\langle 1\rangle)\right) = (j^*\mathcal{Z})\left(\langle 1\rangle\right)$$

since $\mathcal{Z}$ is reduced and satisfies a Segal condition. Thus $j^*\mathcal{Z}$ is a reduced presheaf on $\mathsf{simp}(\mathsf{Fin})$.

The contractibility of internal unary edges condition follows similarly: if $\mathcal{Z}$ is a fibrant object in $\mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc})$ and T is a tree with an unary root vertex, then the space $\mathcal{Z}(T)$ is weakly equivalent to the space $\mathcal{Z}(T')$, where T' is obtained from T by contracting the unique edge above the root vertex. Since the tree $j(S)$ is obtained from the tree $j(S \to \langle 1\rangle)$ by just such a contraction, we see that $j^*\mathcal{Z}$ is local with respect to this condition as well.

Next, we show that $j^*\mathcal{Z}$ satisfies the pullback Segal property. Fix an object in $\mathsf{simp}(\mathsf{Fin})$ of length 2, $(S_0 \xrightarrow{s^1} S_1 \xrightarrow{s^2} S_2)$. When we apply j to this string, we obtain a tree whose collection of incoming edges at the root vertex is indexed by the elements of the set S_2. For each element $x_2 \in S_2$, we obtain a subtree with root x_2 of the form

$$\left(S_{0,x_2} \xrightarrow{s^1} S_{1,x_2} \xrightarrow{s^2} \{x_2\}\right)$$

and for each $x_1 \in S_{1,x_2}$, there is a corresponding corolla $(S_{0,x_1} \to \{x_1\})$. Putting this information together with the fact that $\mathcal{Z}$ is Segal, we have

$$\begin{aligned}
\mathcal{Z}\left(j\left(S_0 \xrightarrow{s^1} S_1 \xrightarrow{s^2} S_2\right)\right) &\simeq \mathcal{Z}\left(S_2\right) \times \prod_{x_2 \in S_2} \mathcal{Z}\left(S_{0,x_2} \xrightarrow{s^1} S_{1,x_2} \xrightarrow{s^2} \{x_2\}\right) \\
&\simeq \mathcal{Z}\left(S_2\right) \times \prod_{x_2 \in S_2} \mathcal{Z}\left(S_{1,x_2}\right) \times \prod_{x_1 \in S_{1,x_2}} \mathcal{Z}\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\}\right) \\
&\simeq \mathcal{Z}\left(S_2 \to \langle 1\rangle\right) \times \prod_{x_2 \in S_2} \mathcal{Z}\left(S_{1,x_2} \to \langle 1\rangle\right) \times \prod_{x_1 \in S_{1,x_2}} \mathcal{Z}\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\} \to \langle 1\rangle\right) \\
&= (j^*\mathcal{Z})\left(S_2\right) \times \prod_{x_2 \in S_2} (j^*\mathcal{Z})\left(S_{1,x_2}\right) \times \prod_{x_1 \in S_{1,x_2}} (j^*\mathcal{Z})\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\}\right) \\
&= (j^*\mathcal{Z})\left(S_0 \xrightarrow{s^1} S_1\right) \times_{(j^*\mathcal{Z})(S_1)} (j^*\mathcal{Z})\left(S_1 \xrightarrow{s^2} S_2\right)
\end{aligned}$$

where, in the third line, we used the fact that $\mathcal{Z}$ is Segal and the fact that $\mathcal{Z}(\overline{\eta})$ is weakly contractible, which allows us to append a root to the trees inside each factor without affecting the weak homotopy type of the space – this trick allows us to write all the factors as objects in the image of j^*, as in the fourth line.

By an identical argument, we can show that for any length n object in $\mathsf{simp}(\mathsf{Fin})$, $(S_0 \to S_1 \to \ldots \to S_n)$, we have

$$(j^*\mathcal{Z})\left(S_0 \to S_1 \to \ldots \to S_n\right) \simeq (j^*\mathcal{Z})\left(S_0 \to S_1 \to \ldots S_{n-1}\right) \times_{(j^*\mathcal{Z})(S_{n-1})} (j^*\mathcal{Z})\left(S_{n-1} \to S_n\right)$$

so by induction on n, we see that $j^*\mathcal{Z}$ satisfies the pullback Segal condition.

Finally, we claim that $j^*\mathcal{Z}$ satisfies the product Segal condition for any 1-simplex, $s^1 : S_0 \to S_1$, in $N\mathsf{Fin}$. To see this, we note that $j(S_0 \xrightarrow{s^1} S_1)$ is the tree whose leaves nearest the root vertex are labeled by elements of S_1. Using the fact that $\mathcal{Z}$ is Segal and reduced, we calculate that

$$\begin{aligned}
(j^*\mathcal{Z})\left(S_0 \xrightarrow{s^1} S_1\right) &\simeq \prod_{x_1 \in S_1} \mathcal{Z}\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\}\right) \\
&\simeq \prod_{x_1 \in S_1} \mathcal{Z}\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\} \to \langle 1 \rangle\right) \\
&= \prod_{x_1 \in S_1} (j^*\mathcal{Z})\left(S_{0,x_1} \xrightarrow{s^1} \{x_1\}\right)
\end{aligned}$$

as required. □

6.2 Deducing a Quillen equivalence

At this point, we set out our strategy for deducing that 6.6 is a Quillen equivalence. The key to our approach is the following diagram:

$$\begin{tikzcd}
\mathsf{simp}(\mathsf{Fin}) \arrow[r, "k"] \arrow[d, "\kappa"'] & \Delta^1_{\mathbb{F}} \arrow[r, "\tau"] \arrow[d, "\overline{\tau}"'] & \Omega \arrow[dl, "\mathrm{cl}"] \arrow[ddl, bend left, "\overline{\zeta}"] \\
\mathsf{simp}(\mathsf{Fin}) \arrow[dr, "j"'] & \Omega_c \arrow[d, "\zeta"] & \\
& \Omega_{rc} &
\end{tikzcd}$$

where τ is the functor defined in 3.5.4 and $\overline{\tau}$ is by definition the composition of τ with the closure operation, cl; k and κ both act by "appending a root" to a string of maps of finite sets, i.e.

$$k\,(S_0 \to S_1 \to \ldots \to S_n) = (S_0 \to S_1 \to \ldots \to S_n \to \langle 1 \rangle)$$

and likewise for κ (but the target categories of these functors differ – hence the distinction in nomenclature); and ζ is the functor which appends a root to a closed tree. We have already shown that j^* preserves reduced Segal objects, while it is clear that cl^* sends reduced Segal objects in $\mathcal{P}(\Omega_c)$ to reduced Segal objects in $\mathcal{P}(\Omega)$; in what follows, we will also demonstrate that k^* preserves reduced Segal objects, leading us to the following diagram

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) & \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}}) \arrow[l, "k^*"'] & \mathcal{P}_{\mathrm{Seg,Red}}(\Omega) \arrow[l, "\tau^*"'] \\
\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \arrow[u, "\kappa^*"] & \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_c) \arrow[u, "\overline{\tau}^*"] \arrow[ur, "\mathrm{cl}^*"'] & \\
& \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc}) \arrow[ul, "j^*"] \arrow[u, "\zeta^*"] \arrow[uur, bend right, "\overline{\zeta}^*"'] &
\end{tikzcd} \tag{6.7}$$

In [CHH18], it is shown that the functor $\tau^* : \mathcal{P}_{\mathrm{Seg}}(\Omega) \to \mathcal{P}_{\mathrm{Seg}}(\Delta^1_{\mathbb{F}})$ is a Quillen equivalence; since the localisations $\mathcal{P}_{\mathrm{Seg}}(\Omega) \to \mathcal{P}_{\mathrm{Seg,Red}}(\Omega)$ and $\mathcal{P}_{\mathrm{Seg}}(\Delta^1_{\mathbb{F}}) \to \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}})$ are accessible, it must be the case that $\tau^* : \mathcal{P}_{\mathrm{Seg,Red}}(\Omega) \to \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}})$ is also a Quillen equivalence. Essentially by construction (and the fact that we have localised with respect to the reduced condition), the functor cl^* is a Quillen equivalence. We claim that as a result of localising with respect to the reduced condition, the functors ζ^* and κ^* are also Quillen equivalences, so $\overline{\tau}^*$ is a Quillen equivalence by the two-out-of-three property. Hence j^* is a Quillen equivalence if and only if k^* is, by the two-out-of-three property.

With a view towards exploiting this diagram, our initial efforts are therefore directed towards showing that κ^* and ζ^* are Quillen equivalences. Having accomplished this, we will then outline our plan for demonstrating that k^* is a Quillen equivalence. (Proving that k^* is a Quillen equivalence is a rather more tractable problem, since we remove the need to translate back and forth from the setting of trees.)

Lemma 6.2.1. *(i) The functor $\kappa^* : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$ sends reduced Segal objects to reduced Segal objects.*

(ii) The functor $\zeta^* : \mathcal{P}(\Omega_{rc}) \to \mathcal{P}(\Omega_c)$ sends reduced Segal objects to reduced Segal objects.

Proof. We will show (i) – the proof of (ii) is almost identical. Let $\mathcal{X}$ be a reduced Segal presheaf on $\mathsf{simp}(\mathsf{Fin})$. Then

$$\kappa^*\mathcal{X}\left(\langle 0\rangle\right) = \mathcal{X}\left(\langle 1\rangle\right) \simeq *$$

since $\langle 0\rangle = \emptyset$ and $\mathcal{X}$ satisfies the reduced localisation condition. Also, since $\mathcal{X}$ is local with respect to inclusions of the form $(S_*) \hookrightarrow (S_* \to \langle 1\rangle)$, we see that

$$\kappa^*\mathcal{X}\left(\langle 1\rangle\right) = \mathcal{X}\left(\langle 1\rangle \to \langle 1\rangle\right) \simeq \mathcal{X}\left(\langle 1\rangle\right) \simeq *$$

so $\kappa^*\mathcal{X}$ is reduced. Likewise, $\kappa^*\mathcal{X}$ is also local with respect to inclusions of the form $(S_*) \hookrightarrow (S_* \to \langle 1\rangle)$ since

$$\kappa^*\mathcal{X}\left(S_* \to \langle 1\rangle\right) = \mathcal{X}\left(S_* \to \langle 1\rangle \to \langle 1\rangle\right) \simeq \mathcal{X}\left(S_* \to \langle 1\rangle\right) = \kappa^*\mathcal{X}(S_*)$$

To see that $\kappa^*\mathcal{X}$ satisfies the product Segal condition, let us consider a 1-simplex $s : S_0 \to S_1$ in $N\mathsf{Fin}$ – because $\mathcal{X}$ is Segal and reduced, we have

$$\begin{aligned}
\kappa^*\mathcal{X}\left(S_0 \xrightarrow{s^1} S_1\right) &= \mathcal{X}\left(S_0 \xrightarrow{s^1} S_1 \to \langle 1\rangle\right) \\
&\simeq \mathcal{X}\left(S_0 \xrightarrow{s^1} S_1\right) \\
&\simeq \prod_{x_1 \in S_1} \mathcal{X}\left(S_{0,x_1} \to \{x_1\}\right) \\
&\simeq \prod_{x_1 \in S_1} \mathcal{X}\left(S_{0,x_1} \to \{x_1\} \to \langle 1\rangle\right) \\
&= \prod_{x_1 \in S_1} \kappa^*\mathcal{X}\left(S_{0,x_1} \to \{x_1\}\right)
\end{aligned}$$

as required. Similarly, to see that $\kappa^*\mathcal{X}$ is local with respect to the pullback Segal condition, we can again use tha fact that $\mathcal{X}$ is reduced and Segal to obtain

$$\begin{aligned}
\kappa^*\mathcal{X} & \left(S_0 \xrightarrow{s^1} \dots \xrightarrow{s^n} S_n\right) \\
&= \mathcal{X}\left(S_0 \xrightarrow{s^1} \dots \xrightarrow{s^n} S_n \to \langle 1\rangle\right) \\
&\simeq \mathcal{X}\left(S_0 \xrightarrow{s^1} \dots \xrightarrow{s^n} S_n\right) \\
&\simeq \prod_{x_n \in S_n} \prod_{i=1}^{n-2} \prod\nolimits_{x_{n-i} \in S_{n-i,x_{n-i+1}}} \mathcal{X}\left(S_{0,x_1} \to \{x_1\}\right) \times \dots \times \mathcal{X}\left(S_{n-1,x_n} \to \{x_n\}\right) \times \mathcal{X}\left(S_n\right) \\
&\simeq \prod_{x_n \in S_n} \prod_{i=1}^{n-2} \prod\nolimits_{x_{n-i} \in S_{n-i,x_{n-i+1}}} \mathcal{X}\left(S_{0,x_1} \to \{x_1\} \to \langle 1\rangle\right) \times \dots \times \mathcal{X}\left(S_{n-1,x_n} \to \{x_n\} \to \langle 1\rangle\right) \times \mathcal{X}\left(S_n \to \langle 1\rangle\right) \\
&= \prod_{x_n \in S_n} \prod_{i=1}^{n-2} \prod\nolimits_{x_{n-i} \in S_{n-i,x_{n-i+1}}} \kappa^*\mathcal{X}\left(S_{0,x_1} \to \{x_1\}\right) \times \dots \times \kappa^*\mathcal{X}\left(S_{n-1,x_n} \to \{x_n\}\right) \times \kappa^*\mathcal{X}\left(S_n\right) \\
&= \kappa^*\mathcal{X}\left(S_0 \to S_1\right) \times_{\kappa^*\mathcal{X}(S_1)} \dots \times_{\kappa^*\mathcal{X}(S_{n-1})} \kappa^*\mathcal{X}\left(S_{n-1} \to S_n\right)
\end{aligned}$$

□

Lemma 6.2.2. *The functor $\kappa_* : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$ sends reduced Segal objects to reduced Segal objects.*

Proof. Let $\mathcal{X}$ be a reduced Segal presheaf on $\mathsf{simp}(\mathsf{Fin})$. By definition,

$$\kappa_*\mathcal{X}\left([n], S\right) = \lim_{([m],S') \in (\kappa \downarrow ([n],S))^{\mathrm{op}}} \mathcal{X}\left([m], S'\right)$$

An object in $(\kappa \downarrow ([n], S))$ is a pair $(([m], S'), (u, \mathrm{id}) : \kappa([m], S') \to [n], S)$ where $([m], S')$ is an object in $\mathsf{simp}(\mathsf{Fin})$ and $u : [m+1] \to [n]$ in Δ, and by id we mean the identity natural transformation, $S'_i \xrightarrow{=} S_{u(i)}$ for $0 \le i \le m$ and $\langle 1 \rangle \xrightarrow{=} S_{u(m+1)}$. Note that we have a composite map

$$([m], S') \xrightarrow{(\delta^{m+1}, \mathrm{id})} \kappa([m], S') \xrightarrow{(u, \mathrm{id})} ([n], S)$$

where $\delta^i : [m] \to [m+1]$ is the unique injection whose image does not contain i. Moreover, there is a commutative diagram

$$\begin{tikzcd}
([m], S') \arrow[rr, "{(u \circ \delta^{m+1},\ \mathrm{id})}"] \arrow[d, "{(\delta^{m+1},\ \mathrm{id})}"'] & & ([n], S) \arrow[d, "{(\delta^{n+1},\ \mathrm{id})}"] \\
\kappa([m], S') \arrow[rr, "{(\delta^{n+1} \circ u \circ \delta^{m+1},\ \mathrm{id})}"] \arrow[dr, "{(u,\ \mathrm{id})}"'] & & \kappa([n], S) \arrow[dl, "{(d^n,\ \mathrm{id})}"] \\
& ([n], S) &
\end{tikzcd}$$

where $d^n : [n+1] \to [n]$ is the unique order-preserving surjection which hits n twice. This diagram tells us that the object $(([n], S), (d^n, \mathrm{id}) : \kappa([n], S) \to ([n], S))$ is terminal in the category $(\kappa \downarrow ([n], S))$, so it is initial in the opposite category. Hence, we compute

$$\kappa_* \mathcal{X}([n], S) \simeq \mathcal{X}([n], S)$$

As $\mathcal{X}$ is reduced and Segal, it follows that $\kappa_* \mathcal{X}$ must also be. □

Hence, we see that κ_* restricts to a right adjoint on reduced Segal objects. Combined with the previous lemma, we claim

Lemma 6.2.3. *The right adjoint $\kappa_* : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$ restricts to an inverse to κ^* on reduced Segal objects.*

Proof. We wish to show that for any reduced Segal presheaf $\mathcal{X}$ on $\mathsf{simp}(\mathsf{Fin})$ we have natural equivalences, $\mathcal{X} \xrightarrow{\sim} \kappa_* \kappa^* \mathcal{X}$ and $\kappa^* \kappa_* \mathcal{X} \xrightarrow{\sim} \mathcal{X}$. Since κ^* preserves reduced Segal objects, $\kappa^* \mathcal{X}$ must also be Segal and reduced, so for any string $(S_0 \to \ldots \to S_n)$ in $\mathsf{simp}(\mathsf{Fin})$, we have

$$\kappa_* \kappa^* \mathcal{X}(S_0 \to \ldots \to S_n) \simeq \kappa^* \mathcal{X}(S_0 \to \ldots \to S_n) = \mathcal{X}(S_0 \to \ldots \to S_n \to \langle 1 \rangle) \simeq \mathcal{X}(S_0 \to \ldots \to S_n)$$

where the first weak equivalence follows from the computations of Lemma 6.2.2. Thus $\mathcal{X} \xrightarrow{\sim} \kappa_* \kappa^* \mathcal{X}$. On the other hand,

$$\kappa^* \kappa_* \mathcal{X}(S_0 \to \ldots \to S_n) = \kappa_* \mathcal{X}(S_0 \to \ldots \to S_n \to \langle 1 \rangle) \simeq \mathcal{X}(S_0 \to \ldots \to S_n \to \langle 1 \rangle) \simeq \mathcal{X}(S_0 \to \ldots \to S_n)$$

so $\kappa^* \kappa_* \mathcal{X} \xrightarrow{\sim} \mathcal{X}$. □

Lemma 6.2.4. *The functor $\zeta^* : \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc}) \to \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_c)$ is a Quillen equivalence.*

Proof. The functor ζ is left adjoint to the natural inclusion $\nu : \Omega_{rc} \hookrightarrow \Omega_c$, so $\zeta_! = \nu^* : \mathcal{P}(\Omega_c) \to \mathcal{P}(\Omega_{rc})$. We know by [BW18a, Lemma 7.12] that ν^* preserves weak equivalences with respect to the reduced Segal localisations on the respective presheaf categories, so we get a Quillen adjunction

$$\nu^* : \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_c) \rightleftarrows \mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc}) : \zeta^*$$

We claim that this is actually a Quillen equivalent pair. To see this, let $\mathcal{Z}$ be a reduced Segal presheaf on Ω_c and let $T \in \Omega_c$. Then, we have

$$\zeta^* \nu^* \mathcal{Z}(T) = \mathcal{Z}(\nu \circ \zeta(T)) = \mathcal{Z}(\zeta(T)) \simeq \mathcal{Z}(T)$$

where at the last step, we use the fact that $\mathcal{Z}$ is reduced, so appending a stump to the root of T has no effect on the weak homotopy type of $\mathcal{Z}(T)$. On the other hand, if $\mathcal{Z}'$ is a reduced Segal presheaf on Ω_{rc} and $T \in \Omega_{rc}$, then

$$\nu^* \zeta^* \mathcal{Z}'(T) = \mathcal{Z}'(\zeta(T)) \simeq \mathcal{Z}'(T)$$

since $\mathcal{Z}'$ is reduced. Thus, ν^* is an inverse to ζ^* on reduced Segal presheaves, so ζ^* is a Quillen equivalence as claimed. □

With these assertions in place, we can now take advantage of diagram (6.7) to prove that (6.6) is indeed a Quillen equivalence. Our approach to proving that k^* is a Quillen equivalence will shadow the one taken in [CHH18] (which we discussed in Section 3.5.2). To give an outline:

- ☑ We begin with a discussion of how the functor k^* acts on representable objects of $\mathcal{P}(\Delta^1_{\mathbb{F}})$. We show that k^* does indeed send reduced Segal objects to reduced Segal objects. We also show that k^* preserves local equivalences, so that we obtain a Quillen adjunction
$$k^* : \mathcal{P}_{\text{Seg,Red}}(\Delta^1_{\mathbb{F}}) \rightleftarrows \mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) : k_*$$
- ☑ We describe an analogue in $\mathsf{simp}(\mathsf{Fin})$ of the *elementary objects* which we find in $\Delta^1_{\mathbb{F}}$, which in turn correspond to corollas. We show that there is a Quillen equivalence between the reduced Segal presheaves on these categories of elementary objects.
- ☑ We prove that the restriction to elementary objects for both $\mathsf{simp}(\mathsf{Fin})$ and $\Delta^1_{\mathbb{F}}$ induces a monadic adjunction. We then avail of Lemma 3.5.7 to conclude that k^* is a Quillen equivalence.

6.3 The functor k^*

In preparation for our attempt to prove that k^* is a Quillen equivalence, we begin by attempting to understand how k^* acts on representable presheaves. Let us fix an object $([n], S)$ in $\Delta^1_{\mathbb{F}}$. Then a simplex of the simplicial space $k^*([n], S)$ is the data of

- an object $([m], T)$ in $\mathsf{simp}(\mathsf{Fin})$;
- a morphism in Δ, $u : [m+1] \to [n]$;
- a choice of element $r \in S_{u(m+1)}$ (which we refer to as a root);

such that, if we define an object $\left([u(m+1)], S^{(r)}\right)$ in $\Delta^1_{\mathbb{F}}$ by requiring that each square in the following commutative diagram is Cartesian,

$$\begin{tikzcd}
S_0^{(r)} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & S_1^{(r)} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \dots \arrow[r] & S_{u(m+1)-1}^{(r)} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & S_{u(m+1)}^{(r)} = \{r\} \arrow[d, tail] \\
S_0 \arrow[r] & S_1 \arrow[r] & \dots \arrow[r] & S_{u(m+1)-1} \arrow[r] & S_{u(m+1)}
\end{tikzcd}$$

and we let $\left([u(m+1)-1], \overline{S^{(r)}}\right)$ be the associated object in $\mathsf{simp}(\mathsf{Fin})$ obtained by removing the term $\{r\}$ from the sequence $S_*^{(r)}$, then there is a morphism $([m], T) \to \left([u(m+1)-1], \overline{S^{(r)}}\right)$ in $\mathsf{simp}(\mathsf{Fin})$. To provide some

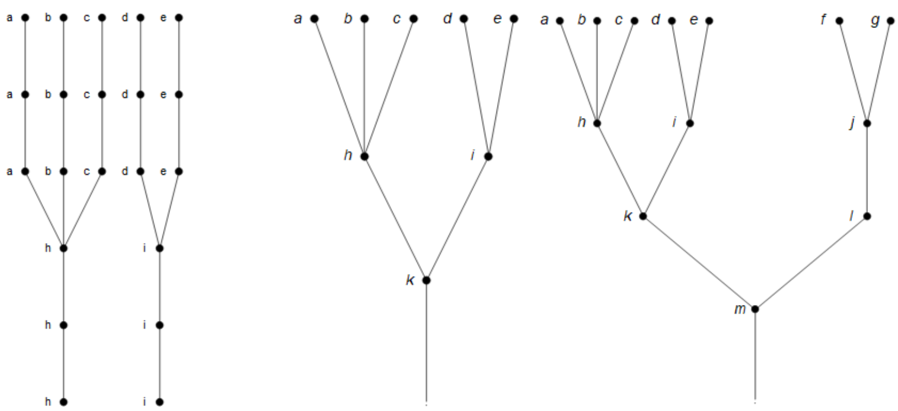

Figure 9: Schematic representation of a simplex of $k^*([3], S)$ (where $([3], S)$ is depicted by the tree on the right)

intuition on this, we turn to some pictures: we have represented an element $([3], S)$ by the tree on the right of Figure (9). The object on the left corresponds to $([6], T)$. We define a morphism $u : [7] \to [3]$ by

$$i \mapsto \begin{cases} 0 & 0 \le i \le 2 \\ 1 & 3 \le i \le 6 \\ 2 & i = 7 \end{cases}$$

The root, r, in this example is k, and the object $([2], S^{(k)})$ is the central image in our schematic, and corresponds to a subtree (indeed, this is precisely the point of this Cartesian squares definition of $S^{(k)}$). On the other hand, it is also clear that there is a morphism $([6], T) \to \left([1], \overline{S^{(k)}}\right)$ in $\mathsf{simp}(\mathsf{Fin})$.

Finally, we note that in the special case where $([n], S)$ is an elementary object in $\Delta^1_{\mathbb{F}}$, i.e. a string of the form $(S_0 \to \langle 1 \rangle)$ for some finite set S_0, then essentially, the map u either satisfies $u(m+1) = 0$ or $u(m+1) = 1$. In the former case, $([m], T)$ is just a string of the form $\left(\langle 1 \rangle \xrightarrow{=} \ldots \xrightarrow{=} \langle 1 \rangle\right)$, and in the latter case $([m], T)$ takes the form $\left(S_0 \xrightarrow{=} \ldots \xrightarrow{=} S_0\right)$.

Remark 6.3.1. Before we go on to describe the homotopical properties of the functor k^*, we take a moment to explicitly point out that by a *reduced* Segal presheaf on $\Delta^1_{\mathbb{F}}$, we mean one which is local with respect to the generating local Segal equivalences for complete Segal operads (see (3.3) and (3.4)) and the maps

- $\emptyset \to ([0], \langle 0 \rangle)$ and $\emptyset \to ([0], \langle 1 \rangle)$, ($\langle 0 \rangle = \emptyset$, so this *does* define an object in $\Delta^1_{\mathbb{F}}$);
- $(\delta^{n+1}, \mathrm{id}) : ([n], S) \to ([n+1], S \to \langle 1 \rangle)$ for any object $([n], S)$ of $\Delta^1_{\mathbb{F}}$.

Proposition 6.3.2. *The functor k^* preserves reduced Segal objects.*

Proof. Let $\mathcal{F}$ be a reduced Segal presheaf on $\Delta^1_{\mathbb{F}}$, and consider an object $([n], S)$ in $\mathsf{simp}(\mathsf{Fin})$. Then

$$\begin{aligned}
(k^*\mathcal{F})([n], S) &= \mathcal{F}\left(S_0 \xrightarrow{s^1} \ldots \xrightarrow{s^n} S_n \to \langle 1 \rangle\right) \\
&\simeq \mathcal{F}\left(S_n \to \langle 1 \rangle\right) \times \textstyle\prod_{1 \le j \le n-1} \prod_{x_j \in S_j : s^j(x_j) = x_{j+1}} \mathcal{F}\left(S_{j, x_{j+1}} \to \langle 1 \rangle\right) \\
&\simeq \mathcal{F}\left(S_n \to \langle 1 \rangle\right) \times \textstyle\prod_{1 \le j \le n-1} \prod_{x_j \in S_j : s^j(x_j) = x_{j+1}} \mathcal{F}\left(S_{j, x_{j+1}} \to \langle 1 \rangle \to \langle 1 \rangle\right) \\
&= \left(k^*\mathcal{F}\right)\left(S_n\right) \times \textstyle\prod_{1 \le j \le n-1} \prod_{x_j \in S_j : s^j(x_j) = x_{j+1}} (k^*\mathcal{F})\left(S_{j, x_{j+1}} \to \langle 1 \rangle\right)
\end{aligned}$$

where we use the fact that $\mathcal{F}$ satisfies the Segal condition in the second line; and in the third line, we use the fact that $\mathcal{F}$ is reduced, so that it is local with respect to inclusions of the form $(S) \hookrightarrow (S \to \langle 1 \rangle)$ for any finite set S. Thus $k^*\mathcal{F}$ is a Segal presheaf.

We also see that $k^*\mathcal{F}([0], \langle 0 \rangle) = \mathcal{F}([0], \langle 1 \rangle) \simeq *$ and $k^*\mathcal{F}([0], \langle 1 \rangle) = \mathcal{F}\left([1], \langle 1 \rangle \to \langle 1 \rangle\right) \simeq *$, since $\mathcal{F}$ is Segal and reduced. □

We can also study how k^* behaves with respect to the generating local equivalences in the category $\mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}})$.

Proposition 6.3.3. *Local equivalences in $\mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}})$ are sent to local equivalences in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ under k^*.*

Proof. We show that k^* preserves equivalences with respect to the separate reduced and Segal localisations. First, $k^*\emptyset = \emptyset$, while $k^*\left([0], \langle 0 \rangle\right) = ([0], \langle 1 \rangle)$, so that

$$k^*\Big(\emptyset \to ([0], \langle 0 \rangle)\Big) = \Big(\emptyset \to ([0], \langle 1 \rangle)\Big)$$

and by definition, this is a local equivalence in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$. Likewise, $k^*\left([0], \langle 1 \rangle\right) = ([1], \langle 1 \rangle \to \langle 1 \rangle)$, which is locally equivalent to $([0], \langle 1 \rangle)$ in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$, so that $k^*\left(\emptyset \to ([0], \langle 1 \rangle)\right)$ is also a local equivalence.

To show that k^* preserves equivalences with respect to the Segal condition, it suffices to show that we have equivalences $k^*\left(([n], S)_{\mathrm{Seg}}\right) \simeq k^*([n], S)$ for each object $([n], S)$ in $\Delta^1_{\mathbb{F}}$. We note that by the properties of the colimit, a simplex of $([n], S)_{\mathrm{Seg}}$ corresponds to giving a collection of simplices of $([1], S_{i, x_{i+1}} \to \langle 1 \rangle)$, for a collection of elements $x_i \in S_i$, $1 \leq i \leq n$, where $x_n = 1$ and $s^i(x_i) = x_{i+1}$ for $1 \leq i \leq n-1$. By our discussion at the start of the section, a simplex of $k^*(([n], S)_{\mathrm{Seg}})$ is the data of a tuple of objects $\{([m_{x_{i+1}}], T_{x_{i+1}})\}_{0 \leq i \leq n-1}$ such that for each i, the functor $T_{x_{i+1}} : [m_{x_{i+1}}] \to [1]$ satisfies either $T_{x_{i+1}}(j) = \langle 1 \rangle$ or $T_{x_{i+1}}(j) = S_{i,x_{i+1}}$ for all j.

Since we have localised $s\mathcal{S}_{/N\mathsf{Fin}}$ with respect to the reduced condition, it can be seen that $k^*\left(([n], S)_{\mathrm{Seg}}\right)$ is locally equivalent as a simplicial space over $N\mathsf{Fin}$ to the simplicial space $\mathcal{X}$, where a simplex of $\mathcal{X}$ is given by a tuple of the form $\{([0], Q_{x_{i+1}})\}_{0 \leq i \leq n-1}$, where the set $Q_{x_{i+1}}$ is either equal to $\langle 1 \rangle$ or $S_{i,x_{i+1}}$.

In turn, since we have localised $s\mathcal{S}_{/N\mathsf{Fin}}$ with respect to the Segal condition, $\mathcal{X}$ is equivalent in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$ to the simplicial space $\mathcal{Y}$, where a simplex of $\mathcal{Y}$ is given by an object $([m], T)$ of $\mathsf{simp}(\mathsf{Fin})$ such that we have an inert morphism $(u, \eta) : k([m], T) \to ([n], S)$ in $\Delta^1_{\mathbb{F}}$, i.e., we have a commutative diagram of Cartesian squares

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, tail, "\eta_0"] \arrow[dr, phantom, "\lrcorner", very near start] & T_1 \arrow[r] \arrow[d, tail, "\eta_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \dots \arrow[r] & T_m \arrow[r] \arrow[d, tail, "\eta_m"] \arrow[dr, phantom, "\lrcorner", very near start] & \langle 1 \rangle \arrow[d] \\ S_{u(0)} \arrow[r] & S_{u(0)+1} \arrow[r] & \dots \arrow[r] & S_{u(0)+m} \arrow[r] & S_{u(0)+m+1} \end{tikzcd}$$

Now, recalling our description of a simplex of $k^*([n], S)$ for a general object $([n], S)$ in $\Delta^1_{\mathbb{F}}$, we see that $\mathcal{Y}$ is locally equivalent as a simplicial space over $N\mathsf{Fin}$ to $k^*([n], S)$ by the reduced condition. Thus we have constructed a string of local equivalences from $k^*\left(([n], S)_{\mathrm{Seg}}\right)$ to $k^*([n], S)$, by which we conclude that k^* preserves local equivalences. $\square$

As a corollary to the above, it follows that the right adjoint $k_* : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}(\Delta^1_{\mathbb{F}})$ preserves reduced Segal objects, i.e. it descends to a right adjoint on the localisation. With this result in place, we can tick off the first item on our checklist.

6.4 Elementary Objects

Our next step is to identify the elementary objects of the category $\mathsf{simp}(\mathsf{Fin})$. The obvious choice would be to select the objects of the form $([0], S)$: under the map k, these are evidently sent to the elementary objects of $\Delta^1_{\mathbb{F}}$. However, this subcategory doesn't carry enough information: in $\Delta^1_{\mathbb{F}}$, we may have an automorphism of $([1], \langle p \rangle \to \langle 1 \rangle)$, determined by $\sigma : \langle p \rangle \to \langle p \rangle$, where $\sigma \in \mathfrak{S}_p$; for the corresponding object in $\mathsf{simp}(\mathsf{Fin})$, the only automorphism possible is the identity map (by definition of the maps in $\mathsf{simp}(\mathsf{Fin})$). To remedy this, we must take an alternative approach and consider instead the subcategory $\Sigma \subseteq \mathsf{Fin}$, which has the same objects, but whose collection of morphisms consists of the bijective maps of sets. By naturality, there is an inclusion $N\Sigma \hookrightarrow N\mathsf{Fin}$ and hence a subcategory inclusion $\mathsf{simp}(\Sigma) \hookrightarrow \mathsf{simp}(\mathsf{Fin})$. We note that an object of $\mathsf{simp}(\Sigma)$ is of the form

$$\left(S_0 \xrightarrow[\simeq]{\sigma^1} \dots \xrightarrow[\simeq]{\sigma^n} S_0 \right)$$

while a morphism between two such objects will be of the form (with $v : [m] \to [n]$ in Δ)

$$\left(S_0 \xrightarrow[\simeq]{\sigma^{v(0),v(1)}} \dots \xrightarrow[\simeq]{\sigma^{v(m-1),v(m)}} S_0 \right) \to \left(S_0 \xrightarrow[\simeq]{\sigma^1} \dots \xrightarrow[\simeq]{\sigma^n} S_0 \right)$$

where as before we write $\sigma^{v(i-1),v(i)}$ to mean the composite $\sigma^{v(i)} \circ \dots \circ \sigma^{v(i-1)} \circ \sigma^{v(i-1)+1}$ from the $v(i-1)^{\text{th}}$ copy of S_0 to the $v(i)^{\text{th}}$ copy of S_0.

The point of using this category is that we can now capture the same notion of permutations acting on the elementary objects of $\Delta^1_{\mathbb{F}}$. More precisely, we can define a functor $\alpha : \mathsf{simp}(\Sigma) \to \Delta^{\mathrm{el}}_{\mathbb{F}}$, which acts on an object $([n], S_0)$ by sending it to $([1], S_0 \to \langle 1 \rangle)$. To a morphism $(v, \mathrm{id}) : ([m], S_0) \to ([n], S_0)$ in $\mathsf{simp}(\Sigma)$, the functor α associates the permutation

$$\left(\sigma^n \circ \dots \circ \sigma^0\right) \circ \left(\sigma^{v(m-1),v(m)} \dots \circ \sigma^{v(0),v(1)}\right)^{-1} \in \mathfrak{S}_{S_0}$$

Referring back to our assertion that the category $\mathsf{simp}(\Sigma)$ now carries the same notion of "permutations acting on the elementary objects", we see for example, that if $\sigma \in \mathfrak{S}_p$ and $\delta^0 : [0] \to [1]$ is the unique order-preserving injection which does not have 0 in its image, then

$$\alpha(\delta^0, \mathrm{id}) : \alpha\left([0], \langle p \rangle\right) \to \alpha\left([1], \langle p \rangle \xrightarrow{\sigma} \langle p \rangle\right)$$

is precisely the permutation σ. It should be noted however that α is not an equivalence of categories, since it is not faithful.

Let $\Delta_{\mathbb{F}}^{\Sigma,1}$ denote the full subcategory of $\Delta_{\mathbb{F}}^{1}$ spanned by objects of the form $\left(S_0 \xrightarrow{\cong} S_0 \xrightarrow{\cong} \dots \xrightarrow{\cong} S_0 \to \langle 1\rangle\right)$. We note that $\Delta_{\mathbb{F}}^{\Sigma,1}$ is precisely the (fully faithful) image of the category $\mathsf{simp}(\Sigma)$ under the functor k, leading to an equivalence of presheaf categories $k^* : \mathcal{P}(\Delta_{\mathbb{F}}^{\Sigma,1}) \xrightarrow{\cong} \mathcal{P}(\mathsf{simp}(\Sigma))$. In the previous section we saw that k^* preserves local equivalences and local objects with respect to the reduced Segal localisation, so that k^* restricts to a Quillen equivalence $\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\Sigma,1}) \xrightarrow{\simeq} \mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\Sigma))$.

We note that there is a natural inclusion $\varsigma : \Delta_{\mathbb{F}}^{\text{el}} \hookrightarrow \Delta_{\mathbb{F}}^{\Sigma,1}$. Since ς is a fully faithful inclusion and the Segal condition on $\mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}})$ is vacuous, the next result is immediate:

Lemma 6.4.1. *The functor $\varsigma^* : \mathcal{P}(\Delta_{\mathbb{F}}^{\Sigma,1}) \to \mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}})$ preserves reduced Segal objects.*

Lemma 6.4.2. *The functor $\varsigma_* : \mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}}) \to \mathcal{P}(\Delta_{\mathbb{F}}^{\Sigma,1})$ sends reduced Segal objects to reduced Segal objects.*

Proof. We first show that ς_* preserves Segal objects. By definition, a presheaf $\mathcal{G}$ on $\Delta_{\mathbb{F}}^{1}$ is Segal if $\mathcal{G}\mid_{\Delta_{\mathbb{F},\text{int}}^{1\ \text{op}}}$ is a right Kan extension of its restriction to the category $\Delta_{\mathbb{F}}^{\text{el op}}$. Thus, essentially by definition, for any presheaf $\mathcal{F}$ on $\Delta_{\mathbb{F}}^{\text{el}}$, $\varsigma_*\mathcal{F}$ is a Segal object in $\mathcal{P}(\Delta_{\mathbb{F}}^{\Sigma,1})$.

We now show that ς_* preserves reduced objects. First, note that the category $(\Delta_{\mathbb{F}}^{\text{el op}})_{([0],\langle 0\rangle)/}$ has a single object, $\left(([0],\langle 0\rangle), ([0],\langle 0\rangle) \xrightarrow{=} ([0],\langle 0\rangle)\right)$, so for any reduced presheaf $\mathcal{F}$ on $\Delta_{\mathbb{F}}^{\Sigma,1}$, we calculate

$$\varsigma_*\mathcal{F}\left([0],\langle 0\rangle\right) = \lim_{(\Delta_{\mathbb{F}}^{\text{el op}})_{([0],\langle 0\rangle)/}} \mathcal{F} \simeq \mathcal{F}([0],\langle 0\rangle) \simeq *$$

On the other hand, the category $(\Delta_{\mathbb{F}}^{\text{el op}})_{([0],\langle 1\rangle)/}$ has an initial object, $\left(([0],\langle 1\rangle), ([0],\langle 1\rangle) \xrightarrow{\text{id}} ([0],\langle 1\rangle)\right)$ from which we deduce

$$\varsigma_*\mathcal{F}\left([0],\langle 1\rangle\right) \simeq \mathcal{F}\left([0],\langle 1\rangle\right) \simeq *$$

Thus ς_* preserves reduced Segal objects. □

Hence, we have an adjoint pair $\varsigma^* : \mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\Sigma,1}) \rightleftarrows \mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\text{el}}) : \varsigma_*$.

Lemma 6.4.3. *The functor ς_* restricts to an inverse of ς^* on reduced Segal objects.*

Proof. We wish to show that we have natural equivalences $\text{id}_{\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\text{el}})} \xrightarrow{\sim} \varsigma_*\varsigma^*$ and $\varsigma^*\varsigma_* \xrightarrow{\sim} \text{id}_{\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\Sigma,1})}$. Since ς is the inclusion of a full subcategory, the functor ς_* is fully faithful, so we automatically obtain an equivalence $\varsigma^*\varsigma_* \xrightarrow{\sim} \text{id}_{\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\Sigma,1})}$.

On the other hand, given a reduced Segal presheaf $\mathcal{F}$ on $\Delta_{\mathbb{F}}^{\Sigma,1}$, by definition of the Segal property, we must have a weak equivalence $\mathcal{F} \simeq \varsigma_*\varsigma^*\mathcal{F}$. □

Hence, we have a zig-zag of Quillen equivalences relating the presheaf categories of the respective elementary objects:

$$\mathcal{P}_{\text{Seg,Red}}\left(\Delta_{\mathbb{F}}^{\text{el}}\right) \xleftarrow[\simeq]{\varsigma^*} \mathcal{P}_{\text{Seg,Red}}\left(\Delta_{\mathbb{F}}^{\Sigma,1}\right) \xrightarrow[\simeq]{k^*|_{\mathsf{simp}(\Sigma)}} \mathcal{P}_{\text{Seg,Red}}\left(\mathsf{simp}(\Sigma)\right)$$

This completes the second item on our checklist.

6.5 Monadicity and a Quillen Equivalence

We can now move on to the third step on our checklist. If we write $e_{\mathbb{F}} : \Delta_{\mathbb{F}}^{\Sigma,1} \hookrightarrow \Delta_{\mathbb{F}}^{1}$ and $e_{\mathsf{simp}} : \mathsf{simp}(\Sigma) \hookrightarrow \mathsf{simp}(\mathsf{Fin})$ for the respective subcategory inclusions, then combining the foregoing results yields the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{1}) \arrow[r, "k^*"] \arrow[d, "e_{\mathbb{F}}^*"'] & \mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \arrow[d, "e_{\mathsf{simp}}^*"] \\
\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\Sigma,1}) \arrow[r, "k^*|_{\mathsf{simp}(\Sigma)}", "\simeq"'] \arrow[d, "\varsigma^*"', "\simeq"] & \mathcal{P}_{\text{Seg,Red}}(\mathsf{simp}(\Sigma)) \\
\mathcal{P}_{\text{Seg,Red}}(\Delta_{\mathbb{F}}^{\text{el}}) &
\end{tikzcd} \tag{6.8}$$

We know from [CHH18, Lemma 5.3] that the map $\varsigma^* \circ e_{\mathbb{F}}^*$ admits a left adjoint, and that the adjunction is monadic. Since the functor ς^* is a Quillen equivalence, and since $e_{\mathbb{F}}^*$ admits a left adjoint (which we denote by $F_{\mathbb{F}}$), the adjunction $F_{\mathbb{F}} \dashv e_{\mathbb{F}}$ must also be monadic. By an argument which is formally rather similar to the one used in [CHH18], we also find that:

Proposition 6.5.1. *The functor $e_{\mathsf{simp}}^* : \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma))$ admits a left adjoint F_{simp}, and the adjunction $F_{\mathsf{simp}} \dashv e_{\mathsf{simp}}^*$ is monadic.*

Proof. The existence of a left adjoint is straightforward: if we denote by L_{simp} the localisation $\mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$, then $L_{\mathsf{simp}} \circ e_{\mathsf{simp},!}$ is left adjoint to e_{simp}^*. Regarding the question of monadicity, we prevail on a characterisation of this condition from [Lur17, Theorem 4.7.3.5], which says that e_{simp}^* exhibits $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ as monadic over $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma))$ if and only if the following conditions are met:

(a) The functor e_{simp}^* is conservative.

(b) Every e_{simp}^*-split object $\mathcal{X}_\bullet$ in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ admits a colimit in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ which is preserved by e_{simp}^*.

To prove conservativity, suppose $\mathcal{X}$ and $\mathcal{Y}$ are reduced Segal presheaves on $\mathsf{simp}(\Sigma)$ which satisfy the condition that $e_{\mathsf{simp}}^*\mathcal{X}$ and $e_{\mathsf{simp}}^*\mathcal{Y}$ are equivalent as reduced Segal presheaves on $\mathsf{simp}(\Sigma)$. Let $\omega : \mathsf{Fin} \hookrightarrow \mathsf{simp}(\mathsf{Fin})$ be the inclusion functor given by $S \mapsto ([0], S)$. By definition, the fact that $\mathcal{X}$ is Segal and reduced means that for every string $(S_0 \to S_1 \to \ldots \to S_n)$, $\mathcal{X}$ satisfies:

$$\begin{aligned}\mathcal{X}(S_0 \to S_1 \to \ldots \to S_n) &\simeq \mathcal{X}(S_0 \to S_1) \times_{\mathcal{X}(S_1)} \ldots \times_{\mathcal{X}(S_{n-1})} \mathcal{X}(S_{n-1} \to S_n) \\ &\simeq \left(\mathcal{X}(S_1) \times \prod_{x_1 \in S_1} \mathcal{X}(S_{0,x_1})\right) \times_{\mathcal{X}(S_1)} \ldots \times_{\mathcal{X}(S_{n-1})} \left(\mathcal{X}(S_n) \times \prod_{x_n \in S_n} \mathcal{X}(S_{n-1,x_n})\right)\end{aligned} \tag{6.9}$$

In particular, we can characterise the reduced Segal presheaves on $\mathsf{simp}(\mathsf{Fin})$ as those $\mathcal{X}$ such that $\omega^*\mathcal{X}$ is reduced and such that we have a weak equivalence $\omega_*\omega^*\mathcal{X} \simeq \mathcal{X}$. Since we can factor ω^* through the map e_{simp}^*, we see that if $e_{\mathsf{simp}}^*\mathcal{X}$ and $e_{\mathsf{simp}}^*\mathcal{Y}$ are equivalent as reduced Segal presheaves on $\mathsf{simp}(\Sigma)$, then necessarily $\omega^*\mathcal{X}$ and $\omega^*\mathcal{Y}$ must be equivalent as a reduced presheaves on Fin, and hence, by the above characterisation, we have equivalences

$$\mathcal{X} \simeq \omega_*\omega^*\mathcal{X} \simeq \omega_*\omega^*\mathcal{Y} \simeq \mathcal{Y}$$

Thus e_{simp}^* is conservative.

Let us now study condition (b). We consider an e_{simp}^*-split object $\mathcal{X}_\bullet$ in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ – that is, a simplicial object $\mathcal{X}_\bullet : N(\Delta)^{\mathrm{op}} \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ such that $e_{\mathsf{simp}}^*\mathcal{X}_\bullet : N(\Delta)^{\mathrm{op}} \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma))$ extends to $\mathcal{X}'_\bullet : N(\Delta_{-\infty})^{\mathrm{op}} \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma))$. As $\mathcal{X}_\bullet$ can be viewed as a diagram in $\mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$, we see that its colimit in $\mathsf{simp}(\mathsf{Fin})$ exists, and we denote this colimit by $\mathcal{Y}$.

Since the localisation functor $L_{\mathsf{simp}} : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ is a left adjoint, it preserves colimits; in particular, we see that $L_{\mathsf{simp}}\mathcal{Y}$ must be a colimit of the diagram $\mathcal{X}_\bullet$ in $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$.

We also know that e_{simp}^* is a left adjoint, so $e_{\mathsf{simp}}^*\mathcal{Y}$ must be a colimit of the diagram $e_{\mathsf{simp}}^*\mathcal{X}_\bullet$ in $\mathcal{P}(\mathsf{simp}(\Sigma))$ – but we can also view $\mathcal{X}'_\bullet$ as a diagram in $\mathcal{P}(\mathsf{simp}(\Sigma))$ whose colimit is $\mathcal{X}'_{-\infty}$. Thus, it must be the case that these two colimits are weakly equivalent, i.e. $\mathcal{X}'_{-\infty} \simeq e_{\mathsf{simp}}^*\mathcal{Y}$. However, by our assumption of splitness, we know that $\mathcal{X}'_{-\infty}$ is a local object (i.e. a reduced Segal presheaf), so it must also be the case that $\mathcal{Y}$ is local. Thus, we have a weak equivalence $L_{\mathsf{simp}}\mathcal{Y} \simeq \mathcal{Y}$, and the diagram $\mathcal{X}_\bullet$ has a colimit in the category $\mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$. Finally, the image of this colimit under the map e_{simp}^* is (weakly equivalent to) $\mathcal{X}'_{-\infty}$, so e_{simp}^* preserves colimits, as required.

Hence, e_{simp}^* meets both of the conditions stated above, so the adjunction is monadic. □

Having shown that the two vertical maps in the commutative square in diagram (6.8) are monadic, we ultimately want to leverage the fact that the lower horizontal map is a Quillen equivalence to prove the same about the upper horizontal. To do this, we will apply Lemma 3.5.7 to the following diagram:

$$\begin{tikzcd} \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}}) \arrow[rr, "k^*"] \arrow[dr, "e_{\mathbb{F}}^*"'] & & \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \arrow[dl, "(k^*|_{\mathsf{simp}(\Sigma)})^{-1} \circ e_{\mathsf{simp}}^*"] \\ & \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^{\Sigma,1}_{\mathbb{F}}) & \end{tikzcd}$$

We have already shown that conditions (a) and (b) of Lemma 3.5.7 are satisfied, so to take advantage of the result of the lemma, we only need to prove that condition (c) holds. Before taking this step, we must deduce one auxiliary claim, based on the following diagram

$$\begin{tikzcd} \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \arrow[r, "k_*"] \arrow[d, "e^*_{\mathsf{simp}}"] & \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}}) \arrow[r, "k^*"] \arrow[d, "e^*_{\mathbb{F}}"] & \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin})) \arrow[d, "e^*_{\mathsf{simp}}"] \\ \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma)) \arrow[r, "(k^*|_{\mathsf{simp}(\Sigma)})^{-1}", "\simeq"'] & \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^{\Sigma,1}_{\mathbb{F}}) \arrow[r, "k^*|_{\mathsf{simp}(\Sigma)}", "\simeq"'] & \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\Sigma)) \end{tikzcd}$$

Lemma 6.5.2. *There is a natural equivalence $k^* |_{\mathsf{simp}(\Sigma)} \circ e^*_{\mathbb{F}} \circ k_* \simeq e^*_{\mathsf{simp}} k^* k_* \xrightarrow{\sim} e^*_{\mathsf{simp}}$*

Proof. Let $\mathcal{X}$ be a reduced Segal presheaf on $\mathsf{simp}(\mathsf{Fin})$. For an object $([n], S_0) = \left(S_0 \xrightarrow[\simeq]{\sigma^1} \dots \xrightarrow[\simeq]{\sigma^n} S_0 \right)$ in $\mathsf{simp}(\Sigma)$, we have

$$\left(e^*_{\mathsf{simp}} \circ k^* \circ k_* \mathcal{X}\right)([n], S_0) = (k_* \mathcal{X})(k([n], S_0)) = \lim_{\left(\mathsf{simp}(\mathsf{Fin})_{/k([n],S_0)}\right)^{\mathrm{op}}} \mathcal{X}$$

Objects of the category $\mathsf{simp}(\mathsf{Fin})_{/k([n],S_0)}$ are pairs $\left(([m], T), k([m], T) \xrightarrow{(u,\eta)} k([n], S_0)\right)$, where $([m], T)$ is an object in $\mathsf{simp}(\mathsf{Fin})$, and (u, η) is a morphism in $\Delta^1_{\mathbb{F}}$. The objects of this category come in two distinct forms:

(i) The morphism $u : [m+1] \to [n+1]$ sends $m+1$ to $n+1$;

(ii) The morphism $u : [m+1] \to]n+1]$ sends $m+1$ to $i < n+1$

In case (i), the Cartesian property of the morphisms in $\Delta^1_{\mathbb{F}}$ tells us that we have a commutative diagram of the form

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, tail, "\eta_0"'] \arrow[dr, phantom, "\lrcorner"] & T_1 \arrow[r] \arrow[d, tail, "\eta_1"'] \arrow[dr, phantom, "\lrcorner"] & \dots \arrow[r] & T_m \arrow[r] \arrow[d, tail, "\eta_m"'] \arrow[dr, phantom, "\lrcorner"] & \langle 1 \rangle \arrow[d, "="'] \\ S_0 \arrow[r, "\cong"] & S_0 \arrow[r, "\cong"] & \dots \arrow[r, "\cong"] & S_0 \arrow[r] & \langle 1 \rangle \end{tikzcd}$$

The Cartesian property enforces the condition that $T_j = S_0$ for all j. In fact, this means that the map (u, η) comes from a morphism $([m], T) \to ([n], S_0)$ in $\mathsf{simp}(\mathsf{Fin})$. In case (ii), the Cartesian property means that we have a diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, tail, "\eta_0"'] \arrow[dr, phantom, "\lrcorner"] & T_1 \arrow[r] \arrow[d, tail, "\eta_1"'] \arrow[dr, phantom, "\lrcorner"] & \dots \arrow[r] & T_m \arrow[r] \arrow[d, tail, "\eta_m"'] \arrow[dr, phantom, "\lrcorner"] & \langle 1 \rangle \arrow[d, hook] \\ S_0 \arrow[r, "\cong"] & S_0 \arrow[r, "\cong"] & \dots \arrow[r, "\cong"] & S_0 \arrow[r, "\cong"] & S_0 \end{tikzcd}$$

which means that $T_i = \langle 1 \rangle$ for all $0 \le i \le m$. In fact, these two distinct cases correspond to two separate connected components of the diagram. We let $\mathsf{simp}(\mathsf{Fin})^{\mathrm{root}}_{/k([n],S_0)}$ denote the connected component consisting of objects which satisfy condition (i), and we write $\mathsf{simp}(\mathsf{Fin})^{\mathrm{nroot}}_{/k([n],S_0)}$ for the connected component corresponding to objects which satisfy condition (ii). Thus, we have

$$\lim_{\left(\mathsf{simp}(\mathsf{Fin})_{/k([n],S_0)}\right)^{\mathrm{op}}} \mathcal{X} \simeq \left(\lim_{\left(\mathsf{simp}(\mathsf{Fin})^{\mathrm{root}}_{/k([n],S_0)}\right)^{\mathrm{op}}} \mathcal{X} \right) \times \left(\lim_{\left(\mathsf{simp}(\mathsf{Fin})^{\mathrm{nroot}}_{/k([n],S_0)}\right)^{\mathrm{op}}} \mathcal{X} \right)$$

We remark however that since $\mathcal{X}$ is reduced, it satisfies $\mathcal{X}(\langle 1 \rangle \to \dots \to \langle 1 \rangle) \simeq *$. In particular, this means that the image of any object of $\mathsf{simp}(\mathsf{Fin})^{\mathrm{nroot}}_{/k([n],S_0)}$ under $\mathcal{X}$ is weakly contractible, so the limit of $\mathcal{X}$ over $\mathsf{simp}(\mathsf{Fin})^{\mathrm{nroot}}_{/k([n],S_0)}$ is also weakly contractible.

On the other hand, any object $(([m], T), k([m], T) \to k([n], S_0))$ of $\mathsf{simp}(\mathsf{Fin})^{\mathrm{root}}_{/k([n],S_0)}$ comes from a morphism $([m], T) \to ([n], S_0)$ in $\mathsf{simp}(\mathsf{Fin})$. Therefore, the indexing category $\mathsf{simp}(\mathsf{Fin})^{\mathrm{root}}_{/k([n],S_0)}$ has a terminal object, namely $\left(([n], S_0), k([n], S_0) \xrightarrow{k(\mathrm{id})} k([n], S_0)\right)$ – this object is thus initial in the opposite category, so we have

$$\lim_{\left(\mathsf{simp}(\mathsf{Fin})^{\mathrm{root}}_{/k([0],S_0)}\right)^{\mathrm{op}}} \mathcal{X} \simeq \mathcal{X}([n], S_0)$$

In summary, we compute

$$k^* k_* \mathcal{X}([n], S_0) \simeq \mathcal{X}([n], S_0) \times * = \mathcal{X}([n], S_0)$$

as required. □

Proposition 6.5.3. *The functor $k^* : \mathcal{P}_{\mathrm{Seg,Red}}(\Delta^1_{\mathbb{F}}) \to \mathcal{P}_{\mathrm{Seg,Red}}(\mathsf{simp}(\mathsf{Fin}))$ is a Quillen equivalence.*

Proof. As advertised, the proof of this utilises Lemma 3.5.7. We need to demonstrate that we have a natural equivalence $F_{\mathsf{simp}} \circ k^* |_{\mathsf{simp}(\Sigma)} \xrightarrow{\sim} k^* \circ F_{\mathbb{F}}$. However, we know that k^* admits a right adjoint, so it suffices to show that the corresponding natural equivalence holds for the right adjoints, i.e. that $e^*_{\mathbb{F}} \circ k_* \xrightarrow{\sim} (k^* |_{\mathsf{simp}(\Sigma)})^{-1} \circ e^*_{\mathsf{simp}}$ – but this is precisely the result of Lemma 6.5.2. Thus by Lemma 3.5.7, we conclude that k^* is indeed a Quillen equivalence. □

Chapter 7

A Tensor Product for Reduced Operads

In our first dalliance with the box product, one of the most intriguing properties that we encountered is the fact that it satisfies a kind of product relation for configuration categories, viz. given manifolds M and M' (of dimensions d and d' respectively), there is a local weak equivalence of simplicial spaces over $N\mathsf{Fin}$:

$$\mathrm{con}(M) \boxtimes \mathrm{con}(M') \simeq \mathrm{con}(M \times M')$$

Using the levelwise weak equivalences of simplicial spaces $\mathrm{con}(\mathbb{R}^d) \simeq j^*\mathbb{E}_d^\otimes$ (where we identify the ∞-operad $\mathbb{E}_d^\otimes$ with its closed rooted dendroidal nerve) for all $d \geq 0$, we obtain a kind of Dunn additivity relation –

$$j^*\mathbb{E}_d^\otimes \boxtimes j^*\mathbb{E}_{d'}^\otimes \simeq j^*\mathbb{E}_{d+d'}^\otimes \simeq j^*\left(\mathbb{E}_d^\otimes \odot \mathbb{E}_{d'}^\otimes\right)$$

of reduced Segal simplicial spaces over $N\mathsf{Fin}$, leading us to ask whether there is a deeper relationship at play between the box product of reduced Segal simplicial spaces and the tensor product of reduced ∞-operads. Having established a Quillen equivalence between these categories, we are now in a stronger position to examine the merits of this conjecture.

We warn the reader that the arguments in favour of this assertion which we will now present should be treated with a degree of caution, predicated as they are on the rather sketchy proofs of Lemma 7.1.7 and Lemma 7.1.8. We hope at a later juncture to provide more thorough proofs of these claims in another location. As such, this chapter may be viewed as an *outline* for an alternative proof of the additivity theorem.

We remark that the box product cannot define a tensor product on the category $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$ since by construction

$$\mathcal{X} \boxtimes (\langle 1 \rangle) \simeq \mathcal{X} \boxtimes^{\mathrm{pre}} (\langle 1 \rangle) \simeq \mathcal{X} \quad \text{while} \quad \mathcal{X} \boxtimes \emptyset \simeq \emptyset$$

but by localising with respect to the reduced condition, we obtain a weak equivalence $\langle 1 \rangle \simeq \emptyset$ in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$. In this regard, the most we can hope for is that the box product defines a bifunctor on $s\mathcal{S}_{/N\mathsf{Fin}}$ which preserves Segal equivalences and respects the reduced localisation (i.e. if $\mathcal{X}$ and $\mathcal{Y}$ are reduced, then so is $\mathcal{X} \boxtimes \mathcal{Y}$) – as we will see, this latter property is more or less immediate when we refer back to the definition of the (pre-)box product. Despite this obstacle, we might still hope to draw some connections between the box product and the tensor product.

The key to our investigation will be a generalisation of Construction 5.2.1 which we signposted in Remark 5.2.3. Before we can utilise this construction however, we are obliged to ensure that the box product descends to a bifunctor on the category of reduced Segal simplicial spaces over $N\mathsf{Fin}$ – thus far, we have only constructed the box product for simplicial spaces over $N\mathsf{Fin}$, without demonstrating that it is well-behaved with respect to Segal equivalences in the localised model structure.

More precisely, our approach to this initial problem will be to show that for every reduced Segal object $\mathcal{X} \in s\mathcal{S}_{/N\mathsf{Fin}}$ the functor $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ (where L_{Red} denotes the reduced localisation) preserves Segal objects and Segal equivalences, so that it descends to a functor

$$L_{\mathrm{Red}}(\mathcal{X} \boxtimes^{\mathrm{pre}} -)\,|_{\mathrm{Seg}} \colon \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg}} \to \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}}$$

If we can further demonstrate that taking the pre-box product with a fixed object $\mathcal{X}$ preserves reduced objects (i.e. if $\mathcal{Y}$ is a reduced object in $\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg}}$, then $\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}$ is also reduced), then this functor restricts to a functor

$$L_{\mathrm{Red}}(\mathcal{X} \boxtimes^{\mathrm{pre}} -)\mid_{\mathrm{Seg,Red}} : \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}} \to \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}}$$

Working in the setting of reduced Segal simplicial spaces over $N\mathsf{Fin}$, Lemma 6.1.2 guarantees that we have also localised with respect to conservatisation, thus leading to local equivalences

$$\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y} = L_{\mathrm{Red}}(\mathcal{X} \boxtimes^{\mathrm{pre}} -)\mid_{\mathrm{Seg,Red}} (\mathcal{Y}) \simeq \mathcal{X} \boxtimes \mathcal{Y}$$

for any reduced Segal object $\mathcal{Y}$ in $s\mathcal{S}_{/N\mathsf{Fin}}$.

7.1 Box Product for Reduced Segal Simplicial Spaces over $N\mathsf{Fin}$

As proposed above, to ascertain that the box product descends to a bifunctor on the reduced and Segal localisations of $N\mathsf{Fin}$, we need to convince ourselves that the pre-box satisfies a number of properties:

1. $- \boxtimes^{\mathrm{pre}} -$ sends reduced Segal objects to reduced Segal objects;

2. $L_{\mathrm{Red}}\left(- \boxtimes^{\mathrm{pre}} -\right)$ preserves Segal equivalences in each variable, i.e. given a fixed fibrant object $\mathcal{X}$ in $\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}}$ and a Segal equivalence $\mathcal{Y} \xrightarrow{\sim} \mathcal{Y}'$ of simplicial spaces over $N\mathsf{Fin}$, we wish to prove that the induced map $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}\right) \to L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}'\right)$ is a reduced Segal equivalence (as remarked we cannot expect $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ to preserve reduce equivalences).

3. $L_{\mathrm{Red}}\left(- \boxtimes^{\mathrm{pre}} -\right)$ preserves colimits in each variable, e.g. given a fixed fibrant object $\mathcal{X}$ and a diagram $\mathcal{Y}_\bullet : I \to \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg}}$, the map $\underset{i\in I}{\mathrm{colim}} L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}_i\right) \to L_{\mathrm{Seg}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \left(\underset{i\in I}{\mathrm{colim}} \mathcal{Y}_i\right)\right)$ is a reduced Segal equivalence.

For item 1, we can in part defer to [BW18b, Proposition 1.4], which tells us that if $\mathcal{X}$ and $\mathcal{Y}$ are fibrewise complete Segal spaces over $N\mathsf{Fin}$, then $\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}$ is also a fibrewise complete Segal space over $N\mathsf{Fin}$. In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are reduced and Segal, then they are fibrewise complete over $N\mathsf{Fin}$, as per Lemma 6.1.2 – so to conclude that condition 1 is met, it is only necessary to demonstrate the following:

Proposition 7.1.1. *If $\mathcal{X}$ and $\mathcal{Y}$ are reduced Segal objects in $s\mathcal{S}_{/N\mathsf{Fin}}$, then $\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}$ is also reduced.*

Proof. By definition, $\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}\left(\langle n\rangle\right)$ comprises all triples (β, x, y) where β is an object of BoxFin such that $p_0\beta = \langle n\rangle$, $x \in \mathcal{X}(p_1\beta)$ and $y \in \mathcal{Y}(p_2\beta)$. In particular, if $n = 0$, we know by the definition of the objects in BoxFin that there is precisely one such β available to us, namely $\langle 0\rangle \xleftarrow{=} \langle 0\rangle \xrightarrow{=} \langle 0\rangle$ – so

$$\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}(\langle 0\rangle) = \mathcal{X}(\langle 0\rangle) \times \mathcal{Y}(\langle 0\rangle) \simeq *$$

since $\mathcal{X}$ and $\mathcal{Y}$ are reduced. Likewise, there is only one object β in BoxFin which satisfies $p_0\beta = \langle 1\rangle$: $\langle 1\rangle \xleftarrow{=} \langle 1\rangle \xrightarrow{=} \langle 1\rangle$. Thus

$$\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}(\langle 1\rangle) = \mathcal{X}(\langle 1\rangle) \times \mathcal{Y}(\langle 1\rangle) \simeq *$$

since $\mathcal{X}$ and $\mathcal{Y}$ are reduced. Similarly, given any finite set S, we know that a 1-simplex β in $N\mathsf{BoxFin}$ which satisfies $p_0\beta = (S \to \langle 1\rangle)$ must take the form of a commutative diagram in Fin

$$\begin{tikzcd}
\langle l_0\rangle \arrow[d] & S \arrow[l, two heads, "s"'] \arrow[r, two heads, "t"] \arrow[d] & \langle r_0\rangle \arrow[d] \\
\langle 1\rangle & \langle 1\rangle \arrow[l, "="'] \arrow[r, "="] & \langle 1\rangle
\end{tikzcd}$$

Thus

$$\begin{aligned}
\mathcal{X} \boxtimes^{\mathrm{pre}} \mathcal{Y}\left(S \to \langle 1\rangle\right) &= \coprod_{p_0\beta = (S\to\langle 1\rangle)} \mathcal{X}\left(\langle l_0\rangle \to \langle 1\rangle\right) \times \mathcal{Y}\left(\langle r_0\rangle \to \langle 1\rangle\right) \\
&\simeq \coprod_{p_0\beta = (S\to\langle 1\rangle)} \mathcal{X}\left(\langle l_0\rangle\right) \times \mathcal{Y}\left(\langle r_0\rangle\right) \\
&\simeq \coprod_{\langle l_0\rangle \twoheadleftarrow S \twoheadrightarrow \langle r_0\rangle \text{ in } \mathsf{BoxFin}} \mathcal{X}\left(\langle l_0\rangle\right) \times \mathcal{Y}\left(\langle r_0\rangle\right)
\end{aligned}$$

where in the second line we use the fact that $\mathcal{X}$ and $\mathcal{Y}$ are reduced (so they are local with respect to localisation condition (2) on $s\mathcal{S}_{/N\mathsf{Fin}}$), and in the third line we use the fact that $\langle 1\rangle$ is a terminal object in Fin, which means that every object in BoxFin is equipped with a map to $\langle 1\rangle \xleftarrow{=} \langle 1\rangle \xrightarrow{=} \langle 1\rangle$. □

It will suffice to prove item 2 for the generating localising morphisms, i.e. we have to establish:

(a) for any string of morphisms of finite sets of length $n \geq 2$, $(S_0 \to \ldots \to S_n)$, we have a reduced Segal equivalence

$$L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} (S_0 \to \ldots \to S_n)\right) \simeq L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \left((S_0 \simeq S_1) \cup_{S_1} \ldots \cup_{S_{n-1}} (S_{n-1} \to S_n)\right)\right)$$

(b) for any morphism of finite sets $s : S_0 \to S_1$, we have a reduced Segal equivalence

$$L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \mathscr{I}^s\right) \simeq L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} \mathscr{E}^s\right)$$

where the presheaves $\mathscr{I}^s$ and $\mathscr{E}^s$ were defined for the product Segal localisation (localisation (4) on $s\mathcal{S}_{/N\mathsf{Fin}}$).

We will return to this question momentarily, but first let us deduce 3 under the assumption that $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ *does* preserve Segal equivalences. Since presheaves are stable under pullbacks, the functor $\mathcal{X} \boxtimes^{\mathrm{pre}} - = p_{0!}(p_1, p_2)^*(\mathcal{X}, -) : \mathcal{P}(\mathsf{simp}(\mathsf{Fin})) \to \mathcal{P}(\mathsf{simp}(\mathsf{Fin}))$ preserves colimits (and correspondingly when we view $\mathcal{X} \boxtimes^{\mathrm{pre}} -$ as a functor on $s\mathcal{S}_{/N\mathsf{Fin}}$). Thus, if $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ preserves Segal equivalences, then by the universal property of the Bousfield localisation, the dashed arrow in the middle horizontal of the diagram

$$\begin{tikzcd}
s\mathcal{S}_{/N\mathsf{Fin}} \arrow[rr, "L_{\mathrm{Red}}(\mathcal{X}\boxtimes^{\mathrm{pre}}-)"] \arrow[d, "L_{\mathrm{Seg}}"'] & & \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Red}} \arrow[d, "L_{\mathrm{Seg}}"] \\
\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg}} \arrow[rr, dashed, "L_{\mathrm{Red}}(\mathcal{X}\boxtimes^{\mathrm{pre}}-)|_{\mathrm{Seg}}"] & & \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}} \\
\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}} \arrow[u, hook] \arrow[rr, dashed, "L_{\mathrm{Red}}(\mathcal{X}\boxtimes^{\mathrm{pre}}-)|_{\mathrm{Seg,Red}}", "\simeq(\mathcal{X}\boxtimes-)|_{\mathrm{Seg,Red}}"'] & & \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}} \arrow[u, hook]
\end{tikzcd}$$

exists and preserves colimits. It follows that the restriction to reduced Segal objects indicated in the lower row also exists and preserves colimits.

Hence, the key to deducing that the box product descends to a bifunctor which preserves Segal objects and respects the reduced localisation is in showing that taking the pre-box product with a fixed presheaf preserves the generating localising Segal equivalences. To establish that $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ sends a pullback Segal equivalence to a reduced Segal equivalence, we will show that $\mathcal{X} \boxtimes^{\mathrm{pre}} -$ sends pullback Segal equivalences to Segal equivalences: obviously then, localising with respect to the reduced condition on both sides of a Segal equivalence will give rise to a reduced Segal equivalence. The proof that $\mathcal{X} \boxtimes^{\mathrm{pre}} -$ sends pullback Segal equivalences to Segal equivalences will proceed in a number of steps. We know that the map

$$(S_0 \to S_1) \cup_{S_1} (S_1 \to S_2) \longrightarrow (S_0 \to S_1 \to S_2)$$

comes from the map $\Lambda[2,1] \to \Delta[2]$, which is a Joyal weak equivalence of simplicial sets (and hence corresponds to a Segal weak equivalence of discrete simplicial spaces). By [Lur09a, Corollary 2.1.2.7], it follows that if $\mathcal{X}$ is a reduced Segal space over $N\mathsf{Fin}$, then we also have Joyal weak equivalences over $N\mathsf{Fin} \times N\mathsf{Fin}$ in each simplicial degree

$$\mathcal{X}_\bullet \times \left((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)\right) \xrightarrow{\sim} \mathcal{X}_\bullet \times (S_0 \to S_1 \to S_2)$$

thus giving Segal equivalences over $N\mathsf{Fin} \times N\mathsf{Fin}$

$$\mathcal{X} \times \left((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)\right) \xrightarrow{\sim} \mathcal{X} \times (S_0 \to S_1 \to S_2)$$

We want to leverage the fact that the pre-box product is computed by taking levelwise pullbacks with $N\mathsf{BoxFin}$ over $N\mathsf{Fin} \times N\mathsf{Fin}$, to figure out whether we also have Segal equivalences

$$\mathcal{X} \boxtimes^{\mathrm{pre}} \left((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)\right) \xrightarrow{\sim} \mathcal{X} \boxtimes^{\mathrm{pre}} (S_0 \to S_1 \to S_2)$$

of simplicial spaces over $N\mathsf{Fin}$.

There are a few steps towards this assertion, the first of which is a technical result pertaining to the map $(p_1, p_2) : N\mathsf{BoxFin} \to N\mathsf{Fin} \times N\mathsf{Fin}$:

Proposition 7.1.2. *The map $\rho := (p_1, p_2) : N\mathsf{BoxFin} \to N\mathsf{Fin} \times N\mathsf{Fin}$ is a Cartesian fibration.*

Proof. It is immediate that ρ is an inner fibration. Let us fix an object $\langle l_1\rangle \xleftarrow{s_1} \langle c_1\rangle \xrightarrow{t_1} \langle r_1\rangle$ in BoxFin, and consider a morphism in $\mathsf{Fin}\times\mathsf{Fin}$

$$(\psi_L,\psi_R) : (\langle l_0\rangle,\langle r_0\rangle) \longrightarrow \rho\left(\langle l_1\rangle \twoheadleftarrow \langle c_1\rangle \twoheadrightarrow \langle r_1\rangle\right) = (\langle l_1\rangle,\langle r_1\rangle)$$

Consider the BoxFin-object $\langle l_0\rangle \twoheadleftarrow (\psi_L\times\psi_R)^{-1}\circ(s_1,t_1)\left(\langle c_1\rangle\right) \twoheadrightarrow \langle r_0\rangle$ (where the left and right arrows are just given by the restrictions of the projection maps from the product $\langle l_0\rangle\times\langle r_0\rangle$), which is equipped with an obvious map

$$f := \left(\psi_L,(s_1,t_1)^{-1}\circ(\psi_L\times\psi_R),\psi_R\right) : \left(\langle l_0\rangle \twoheadleftarrow (\psi_L\times\psi_R)^{-1}\circ(s_1,t_1)\left(\langle c_1\rangle\right) \twoheadrightarrow \langle r_0\rangle\right) \to \left(\langle l_1\rangle \xleftarrow{s_1} \langle c_1\rangle \xrightarrow{t_1} \langle r_1\rangle\right)$$

Given any other morphism in BoxFin,

$$g = (\varphi_L,\varphi_C,\varphi_R) : \left(\langle l_2\rangle \xleftarrow{s_2} \langle c_2\rangle \xrightarrow{t_2} \langle r_2\rangle\right) \to \left(\langle l_1\rangle \xleftarrow{s_1} \langle c_1\rangle \xrightarrow{t_1} \langle r_1\rangle\right)$$

such that there is a factorisation in $\mathsf{Fin}\times\mathsf{Fin}$

$$\begin{tikzcd}
(\langle l_2\rangle,\langle r_2\rangle) \arrow[rr, "\rho(g)"] \arrow[dr, dashed, "h'=(\chi_L,\chi_R)"'] & & (\langle l_1\rangle,\langle r_1\rangle) \\
& (\langle l_0\rangle,\langle r_0\rangle) \arrow[ur, "\rho(f)"'] &
\end{tikzcd}$$

it is clear that the indicated dashed arrow in the diagram

$$\begin{tikzcd}
\left(\langle l_2\rangle \xleftarrow{s_2} \langle c_2\rangle \xrightarrow{t_2} \langle r_2\rangle\right) \arrow[rr, "g"] \arrow[dr, dashed, "h"'] & & \left(\langle l_1\rangle \xleftarrow{s_1} \langle c_1\rangle \xrightarrow{t_1} \langle r_1\rangle\right) \\
& \left(\langle l_0\rangle \twoheadleftarrow (\psi_L\times\psi_R)^{-1}\circ(s_1,t_1)\left(\langle c_1\rangle\right) \twoheadrightarrow \langle r_0\rangle\right) \arrow[ur, "f"'] &
\end{tikzcd}$$

exists and is unique, and $\rho(h) = h'$: namely, we take h to be $\left(\chi_L,(\chi_L\times\chi_R)\circ(s_2,t_2),\chi_R\right)$. Thus, the map f is ρ-Cartesian and $\rho(f) = (\psi_L,\psi_R)$, so ρ is a Cartesian fibration. $\square$

Before we state and prove that taking the pre-box product with a fixed fibrant object preserves pullback Segal equivalences, we record a fact which we will utilise in our proof.

Lemma 7.1.3. *[Lur09a, Corollary 3.3.1.4] Let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d, "p'"] \\
S \arrow[r] & S'
\end{tikzcd}$$

be a pullback of simplicial sets, where p' is a Cartesian fibration. Then the diagram is homotopy Cartesian with respect to the Joyal model structure.

Proposition 7.1.4. *Let $\mathcal{X}$ be a fibrant object in the reduced Segal model structure on $s\mathcal{S}_{/N\mathsf{Fin}}$. Then for any $n\geq 2$, we have Segal equivalences*

$$\mathcal{X}\boxtimes^{\mathrm{pre}}\left((S_0\to S_1)\cup_{S_1}\dots\cup_{S_{n-1}}(S_{n-1}\to S_n)\right) \xrightarrow{\sim} \mathcal{X}\boxtimes^{\mathrm{pre}}(S_0\to\dots\to S_n)$$

Proof. It will suffice to prove the claim for $n=2$, as we can argue inductively for higher n. Let us apply the abovementioned lemma to the commutative diagram

$$\begin{tikzcd}
& \mathcal{X}\boxtimes^{\mathrm{pre}}\left((S_0\to S_1)\cup_{S_1}(S_1\to S_2)\right) \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & N\mathsf{BoxFin} \arrow[dl, "="] \arrow[dd, "\rho"] \\
\mathcal{X}\boxtimes^{\mathrm{pre}}(S_0\to S_1\to S_2) \arrow[rr, crossing over] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & N\mathsf{BoxFin} & \\
& \mathcal{X}\times\left((S_0\to S_1)\cup_{S_1}(S_1\to S_2)\right) \arrow[rr] \arrow[dl, "\sim"] & & N\mathsf{Fin}\times N\mathsf{Fin} \arrow[dl, "="] \\
\mathcal{X}\times(S_0\to S_1\to S_2) \arrow[rr] & & N\mathsf{Fin}\times N\mathsf{Fin} \arrow[from=uu, crossing over, "\rho"] &
\end{tikzcd}$$

Proposition 7.1.2 tells us that ρ is a Cartesian fibration, so since the map

$$\mathcal{X} \times ((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)) \to \mathcal{X} \times (S_0 \to S_1 \to S_2)$$

is in each degree a Joyal weak equivalence of simplicial sets over $N\mathsf{Fin} \times N\mathsf{Fin}$ by our remarks preceding Proposition 7.1.2, the lemma, in conjunction with the above commutative diagram, tells us that

$$\mathcal{X}_k \boxtimes^{\text{pre}} ((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)) \to \mathcal{X}_k \boxtimes^{\text{pre}} (S_0 \to S_1 \to S_2)$$

is for each k a Joyal weak equivalence of simplicial sets over $N\mathsf{BoxFin}$. Thus

$$\mathcal{X} \boxtimes^{\text{pre}} ((S_0 \to S_1) \cup_{S_1} (S_1 \to S_2)) \to \mathcal{X} \boxtimes^{\text{pre}} (S_0 \to S_1 \to S_2)$$

is a Segal weak equivalence of simplicial spaces over $N\mathsf{BoxFin}$. By composition with the map $p_0 : N\mathsf{BoxFin} \to N\mathsf{Fin}$, it is thus a Segal weak equivalence of simplicial spaces over $N\mathsf{Fin}$. □

Before proceeding to study whether $L_{\text{Red}}(\mathcal{X} \boxtimes^{\text{pre}} -)$ preserves local equivalences with respect to the product Segal localisation, let us recap the notation that we used to describe that localisation. Given a morphism of finite sets $s : S_0 \to S_1$, with $S_1 = \{x_1, \ldots, x_n\}$, we were able to define a simplicial space over $N\mathsf{Fin}$ (by gluing along s), $\mathscr{E}^s = \mathscr{E}^s_{x_1} \cup_s \ldots \cup_s \mathscr{E}^s_{x_n}$, where for each $x_i \in S_1$, $\mathscr{E}^s_{x_i}$ is the simplicial space over $N\mathsf{Fin}$

$$\begin{tikzcd} s^{-1}\{x_i\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \langle 1 \rangle \arrow[d, hook, "1 \mapsto x_i"] \\ S_0 \arrow[r, "s"] & S_1 \end{tikzcd}$$

On the other hand, by gluing along S_1, it was also possible to define a simplicial space $\mathscr{I}^s = \mathscr{I}^s_{x_1} \cup_{S_1} \ldots \cup_{S_1} \mathscr{I}^s_{x_n}$ where $\mathscr{I}^s_{x_i}$ is the simplicial space over $N\mathsf{Fin}$

$$\begin{tikzcd} s^{-1}\{x_i\} \arrow[r] & \langle 1 \rangle \arrow[d, hook, "1 \mapsto x_i"] \\ & S_1 \end{tikzcd}$$

Having recalled the relevant notation in use, we will state our target result.

Proposition 7.1.5. *Let $\mathcal{X}$ be a fixed fibrant object in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\text{Seg,Red}}$ and let $s : S_0 \to S_1$ be a morphism of finite sets. Then there is a reduced Segal equivalence of simplicial spaces over $N\mathsf{Fin}$:*

$$L_{\text{Red}}(\mathcal{X} \boxtimes^{\text{pre}} \mathscr{I}^s) \xrightarrow{\sim} L_{\text{Red}}(\mathcal{X} \boxtimes^{\text{pre}} \mathscr{E}^s)$$

As in the proof of Proposition 7.1.4, this result will follow from a number of smaller assertions. First, we note that it will suffice to demonstrate this for representable presheaves, so we only need to show that given any length-m string of morphisms of finite sets, $T_* = (T_0 \to \ldots \to T_m)$, the map

$$L_{\text{Red}}(T_* \boxtimes^{\text{pre}} \mathscr{I}^s) \to L_{\text{Red}}(T_* \boxtimes^{\text{pre}} \mathscr{E}^s)$$

is a reduced Segal equivalence. We can further ease our workload by taking advantage of Proposition 7.1.4, which tells us that $L_{\text{Red}}(- \boxtimes^{\text{pre}} -)$ is well-behaved with respect to pullback Segal equivalence – with this in mind, it is only necessary to demonstrate the above claim in the case where T_* is a string of length 0 or 1. Let us first consider the case $m = 0$, i.e. we want to show that for any finite set T_0, we have a reduced Segal equivalence

$$L_{\text{Red}}((T_0) \boxtimes^{\text{pre}} \mathscr{I}^s) \xrightarrow{\sim} L_{\text{Red}}((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s)$$

We note that an n-simplex of $(T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$ is essentially the data of an n-simplex ξ of $N\mathsf{BoxFin}$ such that $p_1(\xi_k) = T_0$ for each $0 \le k \le n$ and there is a map in $\mathsf{simp}(\mathsf{Fin})$, either of the form

- $p_2(\xi) \to (S_{0,i} \to \langle 1 \rangle)$ for some $i \in S_1$ (where we write $S_{0,i} = s^{-1}\{i\}$); or
- $p_2(\xi) \to (S_0 \to S_1)$

In other words, we can view an n-simplex of $(T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$ as a string of finite sets $(V_0 \to \ldots \to V_n)$ where $V_k \subseteq u^* (S_0 \times T_0 \to S_1 \times T_0)_k$, for some map $u : [n] \to [1]$ in Δ.

On the other hand, an n-simplex of $(T_0) \boxtimes^{\text{pre}} \mathscr{I}^s$ will be an n-simplex ξ' of $N\mathsf{BoxFin}$ such that $p_1(\xi'_k) = T_0$ for each $0 \leq k \leq n$ and there is a map in $\mathsf{simp}(\mathsf{Fin})$ either of the form

- $p_2(\xi') \to (S_{0,i} \to \{i\})$ for some $i \in S_1$; or
- $p_2(\xi') \to (S_1)$

In the former case an n-simplex of $(T_0) \boxtimes^{\text{pre}} \mathscr{I}^s$ will be a string of finite sets $(V_0 \to \ldots \to V_n)$, where $V_k \subseteq u^* (S_{0,i} \times T_0 \to \{i\} \times T_0)_k$ for some $i \in S_1$ and $u : [n] \to [1]$ in Δ; while in the latter case, an n-simplex will be a string of finite sets $(V_0 \to \ldots \to V_n)$, where $V_k \subseteq S_1 \times T_0$ for each $0 \leq k \leq n$.

Having examined the objects $(T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$ and $(T_0) \boxtimes^{\text{pre}} \mathscr{I}^s$ in some detail, we are hopefully now in a better position to prove that taking the pre-box product with a fixed finite set is well-behaved with respect to the product Segal equivalence. Key to determining this will be the Quillen equivalent pair $(j_! \dashv j^*)$ which we studied in the previous chapter.

Proposition 7.1.6. *Let T_0 be a fixed finite set and let $S_* = \left(S_0 \xrightarrow{s} S_1\right)$ be a 1-simplex in $N\mathsf{Fin}$ (viewed as a representable object in $s\mathcal{S}_{/N\mathsf{Fin}}$). Then the map $L_{\text{Red}} \left((T_0) \boxtimes^{\text{pre}} \mathscr{I}^s\right) \to L_{\text{Red}} \left((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s\right)$ is a reduced Segal equivalence.*

This result will be a consequence of:

Lemma 7.1.7. *Let T_0 and S_* be as in the statement of Proposition 7.1.6. Then we have reduced Segal equivalences of rooted closed dendroidal spaces*

(i) $j_! \left((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s\right) \simeq \overline{C}_{T_0} \otimes j_!(S_*)$

(ii) $j_! \left((T_0) \boxtimes^{\text{pre}} \mathscr{I}^s\right) \simeq \overline{C}_{T_0} \otimes j_!(S_*)_{\text{Seg}}$

where $j_! : s\mathcal{S}_{/N\mathsf{Fin}} \to rcd\mathcal{S}$ and $\otimes$ denotes the tensor product of (rooted closed) dendroidal spaces, and for a representable object $\mathcal{F} \in rcd\mathcal{S}$, $\mathcal{F}_{\text{Seg}}$ denotes its Segal core (alias spine).

Proof of Proposition 7.1.6. We note that the tensor product on closed dendroidal spaces is compatible with Segal equivalences, i.e. if $\mathcal{F}$ and $\mathcal{F}'$ are Segal equivalent as objects in $rcd\mathcal{S}$ and $\mathcal{G}$ is a fixed fibrant object (with respect to the Segal localisation on $rcd\mathcal{S}$), then the objects $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{F}' \otimes \mathcal{G}$ are also Segal equivalent. In particular, we note that since we have weak equivalences $j_!(S)_{\text{Seg}} \simeq j_!(S)$, we obtain the following commutative diagram in $(rcd\mathcal{S})_{\text{Seg,Red}}$:

$$\begin{tikzcd}
j_! \left((T_0) \boxtimes^{\text{pre}} \mathscr{I}^s\right) \arrow[r, "\simeq"] \arrow[d] & \overline{C}_{T_0} \otimes j_!(S_*)_{\text{Seg}} \arrow[d, "\simeq"] \\
j_! \left((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s\right) \arrow[r, "\simeq"] & \overline{C}_{T_0} \otimes j_!(S_*)
\end{tikzcd}$$

By the two-out-of-three property, it follows that the left vertical arrow is a weak equivalence. However, we know that $j_!$ descends to a Quillen equivalence $j_! : \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\text{Seg,Red}} \xrightarrow{\simeq} (rcd\mathcal{S})_{\text{Seg,Red}}$, so it must also be the case that $(T_0) \boxtimes^{\text{pre}} \mathscr{I}^s \simeq (T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$ in $\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\text{Seg,Red}}$ as required. □

Proof of Lemma 7.1.7. For *(i)*, let Y be a fixed object in Ω_{rc}. By definition of the left Kan extension, we have

$$j_! \left((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s\right) = \operatorname*{colim}_{V_* \in \mathsf{simp}(\mathsf{Fin})^{\text{op}} \downarrow ((T_0) \boxtimes^{\text{pre}} \mathscr{E}^s)} j(V_*)(Y)$$

By our prior explanation of the simplicial space $(T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$, we know that there is an arrow $V_* \to (T_0) \boxtimes^{\text{pre}} \mathscr{E}^s$ in $s\mathcal{S}_{/N\mathsf{Fin}}$ precisely if V_* is of the form $(V_0 \to \ldots \to V_n)$ for some $n \geq 0$, such that either

- $V_k \subseteq u^* (S_0 \times T_0 \to S_1 \times T_1)_k$; or
- $V_k \subseteq u^* (S_{0,l} \times T_0 \to T_0)_k$ for some $l \in S_1$

for each $0 \leq k \leq n$, where $u : [n] \to [1]$ in Δ. In turn, this string V_* corresponds to $p_0\xi$ for some n-simplex $\xi \in N\mathsf{BoxFin}$ of one of the following forms

$$
\begin{tikzcd}
T_0 \arrow[d, equal] & V_0 \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] \arrow[d] & S_{u(0)} \arrow[d] \\
T_0 \arrow[d, equal] & V_1 \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] \arrow[d] & S_{u(1)} \arrow[d] \\
\vdots \arrow[d, equal] & \vdots \arrow[d] & \vdots \arrow[d] \\
T_0 & V_n \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] & S_{u(n)}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
T_0 \arrow[d, equal] & V_0 \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] \arrow[d] & (S_{0,l} \to \langle 1 \rangle)_{u(0)} \arrow[d] \\
T_0 \arrow[d, equal] & V_1 \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] \arrow[d] & (S_{0,l} \to \langle 1 \rangle)_{u(1)} \arrow[d] \\
\vdots \arrow[d, equal] & \vdots \arrow[d] & \vdots \arrow[d] \\
T_0 & V_n \arrow[l, two heads, "\pi_{T_0}"'] \arrow[r, two heads, "\pi_{u^*S_*}"] & (S_{0,l} \to \langle 1 \rangle)_{u(n)}
\end{tikzcd}
$$

We can identify each V_k as corresponding to a poset of either $T_0 \times S_1$ or $T_0 \times S_{0,l}$ for some $l \in S_1$, in turn corresponding to the data of a pair of operations, one in the closed corolla $\overline{C}_{T_0}$, and the other in the closed tree $j_!(S_*) = j(S_*)$ (we are somewhat blurring the distinction between a tree and the operad associated to it by referring to an "operation" in a tree); and these operations must be compatible in the sense that we can either project the "T_0-operation" of $V_k \subseteq (S_{u(k)} \times T_0)$ first and then project the "$S_{u(k)}$-operation", or we can reverse the order in which we perform these projections, and it won't matter: loosely speaking we record this condition by means of a diagram of the form

$$
\begin{tikzcd}
\left(T_0 \times S_{u(k)} \supseteq V_k\right) \arrow[r, "\pi_{u^*S_*}"] \arrow[d, "\pi_{T_0}"'] & \left(S_{u(k)} \supseteq \pi_{u^*S_*} V_k\right) \arrow[d] \\
(T_0 = \pi_{T_0} V_k) \arrow[r] & \langle 1 \rangle
\end{tikzcd}
\tag{7.1}
$$

where the equality $T_0 = \pi_{T_0} V_k$ stems from the fact that π_{T_0} is a surjection for all k.

By the reduced localisation, this dendroidal space is weakly equivalent to $j_!(S_*) \otimes \overline{C}_{T_0}$ – to see this, note that the reduced localisation induces a weak equivalence between C_{T_0} and $\overline{C}_{T_0}$ (i.e. we can substitute in nullary operations without changing the weak-homotopy type), and thus we can replace $j_!(T_0 \boxtimes^{\mathrm{pre}} \mathscr{E}^s)$ with the rooted closed dendroidal space defined by taking the left Kan extension over all strings V_* such that each V_k in V_* satisfies $T_0 \supseteq \pi_{T_0} V_k$. Working with such strings, the commutativity condition recorded in diagram (7.1) corresponds to the Boardman-Vogt interchange condition for an operation in the tensor product of the closed dendroidal spaces $j_!(S_*)$ and $\overline{C}_{T_0} = j_!(T_0)$. In other words, we have the desired weak equivalence

$$j_! \left((T_0) \boxtimes^{\mathrm{pre}} \mathscr{E}^s\right) \simeq \overline{C}_{T_0} \otimes j_!(S_*)$$

For *(ii)*, we can proceed in a similar manner: by definition of the left Kan extension, we have

$$j_! \left((T_0) \boxtimes^{\mathrm{pre}} \mathscr{I}^s\right) = \operatorname*{colim}_{V_* \in \mathsf{simp}(\mathsf{Fin})^{\mathrm{op}} \downarrow ((T_0) \boxtimes^{\mathrm{pre}} \mathscr{I}^s)} j(V_*)(Y)$$

There is an arrow $V_* \to (T_0) \boxtimes^{\mathrm{pre}} \mathscr{I}^s$ in $s\mathcal{S}_{/N\mathsf{Fin}}$ precisely if V_* is of the form $(V_0 \to \ldots \to V_n)$ for some $n \geq 0$ such that either

- $V_k \subseteq u^* \left(S_{0,l} \times T_0 \to \{l\} \times T_1\right)_k$ for some $l \in S_1$ and each $0 \leq k \leq n$, with $u : [n] \to [1]$ in Δ; or
- $V_k \subseteq S_1 \times T_0$ for each $0 \leq k \leq n$.

Making the same arguments as above, and utilising the fact that we have localised the category of rooted closed dendroidal spaces with respect to the reduced condition (specifically taking advantage of the contractibility of internal unary vertices), we can identify each V_k as corresponding to a pair of operations, one in $\overline{C}_{T_0}$, the other in $j\left(S_{0,l} \to \{l\}\right)$ or $j\left(S_1 \to \langle 1 \rangle\right) \simeq j\left(S_1\right)$ (depending on the set V_k), and these pairs of operations must be compatible in a sense analogous to the characterisation of diagram (7.1). By definition, the collection of all operations coming from the corollas $j(S_{0,l} \to \{l\})$ or $j(S_1 \to \langle 1 \rangle)$ corresponds to the spine/Segal core of the closed tree $j_!(S_*)$ – hence, we see that there is an equivalence of reduced Segal dendroidal spaces between $j_! \left(T_0 \boxtimes^{\mathrm{pre}} \mathscr{I}^s\right)$ and $\overline{C}_{T_0} \otimes j_!(S_*)_{\mathrm{Seg}}$, as expected. $\square$

By analogous arguments, we can also deduce the following result:

Lemma 7.1.8. *Let $T_* = \left(T_0 \xrightarrow{t} T_1\right)$ and $S_* = \left(S_0 \xrightarrow{s} S_1\right)$ be 1-simplices in $N\mathsf{Fin}$. Then there are reduced Segal equivalences in $rcd\mathcal{S}$:*

$$j_!\left(T_* \boxtimes^{\mathrm{pre}} \mathscr{E}^s\right) \simeq j_!\left(T_*\right) \otimes j_!\left(S_*\right) \quad \textit{and} \quad j_!\left(T_* \boxtimes^{\mathrm{pre}} \mathscr{I}^s\right) \simeq j_!\left(T_*\right) \otimes j_!\left(S_*\right)_{\mathrm{Seg}}$$

Using this result, we can imitate the argument used to deduce Proposition 7.1.6 to see also:

Proposition 7.1.9. *Let $T_* = \left(T_0 \xrightarrow{t} T_1\right)$ and $S_* = \left(S_0 \xrightarrow{s} S_1\right)$ be 1-simplices in $N\mathsf{Fin}$. Then the map $L_{\mathrm{Red}}\left(T_* \boxtimes^{\mathrm{pre}} \mathscr{I}^s\right) \to L_{\mathrm{Red}}\left(T_* \boxtimes^{\mathrm{pre}} \mathscr{E}^s\right)$ is a reduced Segal equivalence.*

Hence it follows that for any fixed fibrant object $\mathcal{X} \in \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)$, we have reduced Segal equivalences

$$\mathcal{X} \boxtimes^{\mathrm{pre}} \mathscr{I}^s \simeq \mathcal{X} \boxtimes^{\mathrm{pre}} \mathscr{E}^s$$

i.e. we have proven Proposition 7.1.5. This result, together with Proposition 7.1.4 ensures that $L_{\mathrm{Red}}\left(\mathcal{X} \boxtimes^{\mathrm{pre}} -\right)$ descends to a functor $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg}} \to (s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$, and hence $- \boxtimes -$ descends to a bifunctor on $s\mathcal{S}_{/N\mathsf{Fin}}$ which preserves Segal equivalences and respects reduced objects, as required.

7.2 Comparison with Tensor Product of ∞-Operads

At this juncture we remind ourselves of the substance of Remark 5.2.3: given an ∞-operad, $\mathcal{O}^{\otimes}$ and a map of ∞-preoperads $c : \Delta[0] \to \mathcal{O}^{\otimes}$, which we view as "picking out the colour c", we can associate the simplicial space $(\mathcal{O}_{/c})^{\mathrm{act}}$, which is equipped with a natural map to $N\mathsf{Fin}$. In the case where $\mathcal{O}^{\otimes}$ arises as the homotopy-coherent nerve of a plain operad $\mathcal{O}$ in spaces, this construction is equivalent to taking the nerve of the the topological category $\mathrm{cat}(\mathcal{O})$ which we defined in Construction 5.2.1.

In what follows, we will write $\mathrm{ptOp}_{\infty}^{\mathrm{Red}}$ to mean the category of pointed reduced ∞-operads, whose objects are given by morphisms of ∞-preoperads

$$\begin{tikzcd} \Delta[0] \arrow[rr, "c"] \arrow[dr] & & \mathcal{O}^{\otimes} \arrow[dl] \\ & N\mathsf{Fin}_* & \end{tikzcd}$$

such that $\mathcal{O}^{\otimes}$ is a reduced ∞-operad (we will frequently identify such an object with the ∞-operad $\mathcal{O}^{\otimes}$ for brevity). The morphisms in this category are given by commutative diagrams

$$\begin{tikzcd} & \Delta[0] \arrow[dl, "c"'] \arrow[dr, "c'"] \arrow[dd] & \\ \mathcal{O}^{\otimes} \arrow[rr] \arrow[dr] & & \mathcal{O}'^{\otimes} \arrow[dl] \\ & N\mathsf{Fin}_* & \end{tikzcd}$$

We let $\varphi : \mathrm{ptOp}_{\infty}^{\mathrm{Red}} \to s\mathcal{S}_{/N\mathsf{Fin}}$ denote the aforementioned functor,

$$\left(\begin{tikzcd} \Delta[0] \arrow[rr, "c"] \arrow[dr] & & \mathcal{O}^{\otimes} \arrow[dl] \\ & N\mathsf{Fin}_* & \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} \left(\mathcal{O}_{/c}\right)^{\mathrm{act}} \arrow[d] \\ N\mathsf{Fin} \end{tikzcd}\right)$$

Remark 7.2.1. As alluded to already, the functor φ is a generalisation of the construction which associates to a plain simplicial operad $\mathcal{O}$ the topological category $\mathrm{cat}\,(\mathcal{O})$. By [BW18a, Lemma 7.9/Lemma 7.12], there is a weak equivalence

$$\varphi(N\mathcal{O}^{\otimes}) = N\mathrm{cat}\,(\mathcal{O}) \simeq j^* N_d^{rc}\mathcal{O}$$

in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$ for a plain simplicial operad $\mathcal{O}$ (we recall that taking the homotopy-coherent nerve of the category of operators of a simplicial operad $\mathcal{O}$ yields a quasi-operad). Since we know that j^* and N_d^{rc} are Quillen equivalences with respect to the relevant reduced Segal localisations, it follows that φ preserves colimits, at least in the restricted setting where we work with plain simplicial operads.

Example 7.2.2. To familiarise ourselves with this functor, let us consider its action on an especially simple kind of operad: a closed n-corolla, $\overline{C}_n$. If we write $\mathfrak{R}$ for the root of the corolla and $\ell_1, \dots, \ell_n$ for the leaves, then there are essentially two distinct ways in which $\overline{C}_n$ can be pointed:

- The map $\Delta[0] \to \overline{C}_n$ selects one of the leaves, say ℓ_i.
- The map $\Delta[0] \to \overline{C}_n$ selects the root.

In the former case, the simplicial space $\varphi\left(\overline{C}_n\right)$ is seen to be trivial: the only possible operations with target ℓ_i are elements of the space $\overline{C}_n(-;\ell_i)$, which is a singleton, and the morphisms between the collection of all such operations is necessarily also contractible. In other words, we find that $\varphi\left(\overline{C}_n\right) = \Delta[0]$.

The latter case is more interesting. Since we are working with closed operads, there are several possible spaces of operations with target $\mathfrak{R}$, simply by inserting nullary operations in place of the leaves, e.g., $\overline{C}_n(\ell_1, \ell_2, \ell_3, \dots, \ell_n; \mathfrak{R})$, $\overline{C}_n(-, \ell_2, \ell_3, \dots, \ell_n; \mathfrak{R})$, ... are all subspaces of $\varphi\left(\overline{C}_n\right)_0$. Based on this observation, we can identify the space of 0-simplices of $\varphi\left(\overline{C}_n\right)$ with the poset of all (ordered) subsets of $\langle n\rangle$, which we denote by $\mathcal{P}\mathrm{o}\left(\langle n\rangle\right)$.

Given finite ordered subsets $I, J \subseteq \langle n\rangle$ such that there is a bijection of sets $\sigma : J \xrightarrow{\cong} I'$, where I' is an ordered subset of I, then there is an associated edge in $\varphi\left(\overline{C}_n\right)$ from $\overline{C}_n(\{\ell_i\}_{i\in I}; \mathfrak{R})$ to $\overline{C}_n(\{\ell_j\}_{j\in J}; \mathfrak{R})$ induced by composition with the tuple of nullary operations $(\omega_{i'})_{i'\in I\setminus I'} \in \overline{C}_n\left(-;\ell_{i'}\right)$. In fact, we can define a category, $\mathscr{F}_n$ whose objects (resp. morphisms) coincide with the vertices (resp. edges) of $\varphi\left(\overline{C}_n\right)$, i.e. an object of $\mathscr{F}_n$ is an ordered subset $I \subseteq \langle n\rangle$, and a morphism $I \to J$ in $\mathscr{F}_n$ is the data of a bijection of sets from J to an ordered subset of I. Since operads satisfy a Segal property, we find that $\varphi\left(\overline{C}_n\right)$ and $N\mathscr{F}_n$ are levelwise weak equivalent simplicial spaces.

Example 7.2.3. Extending the previous example, we can consider this construction for the Boardman-Vogt tensor product of two corollas. Since the ∞-categorical version of an n-corolla operad arises from the homotopy-coherent nerve of the discrete n-corolla operad, it follows that the ∞-categorical tensor product of two ∞-operadic corollas can be identified with the nerve of the Boardman-Vogt tensor product of the respective discrete corollas.

Let us fix two non-negative integers n, m. The colours of the operad $\overline{C}_n \odot \overline{C}_m$ are of the form $p \otimes q$, where p, q are colours of $\overline{C}_n$, $\overline{C}_m$ respectively. If both p and q are leaves of the respective corollas, then the simplicial space $\varphi\left(\overline{C}_n \odot \overline{C}_m\right)$ is equivalent to $\Delta[0]$. If p is the root of $\overline{C}_n$ and q is a leaf of $\overline{C}_m$, then $\varphi\left(\overline{C}_n \odot \overline{C}_m\right)$ is equivalent to $\varphi\left(\overline{C}_n\right)$ (and vice versa when the roles of p and q have been swapped). Hence, the only case we need to investigate is when both p and q are the roots of their respective corollas.

The operations with target $\mathfrak{R} \otimes \mathfrak{R}$ are generated by operations $v \otimes \mathfrak{R}$, $\mathfrak{R} \otimes w$, where $v \in \left(\overline{C}_n\right)_{/\mathfrak{R}}$ and $q \in \left(\overline{C}_m\right)_{/\mathfrak{R}}$. In Example 7.2.2, we drew a correspondence between the elements of $\left(\overline{C}_n\right)_{/\mathfrak{R}}$ and the ordered subsets of $\langle n\rangle$; in a similar way, we can associate a vertex of $\left(\overline{C}_n \odot \overline{C}_m\right)_{/\mathfrak{R}\otimes\mathfrak{R}}$ with the category of products of ordered subsets of $\langle n\rangle$ and $\langle m\rangle$, which we denote by $\mathcal{P}\mathrm{o}\left(\langle n\rangle \times \langle m\rangle\right)$. We note that there are natural maps from $\mathcal{P}\mathrm{o}\left(\langle n\rangle \times \langle m\rangle\right)$ to $\mathcal{P}\mathrm{o}\left(\langle n\rangle\right)$ and $\mathcal{P}\mathrm{o}\left(\langle m\rangle\right)$, which we denote by π_n, π_m respectively: if $v\otimes w \in \left(\overline{C}_n\right)_{/\mathfrak{R}} \times \left(\overline{C}_m\right)_{/\mathfrak{R}}$, then these maps are given by

$$\mathcal{P}\mathrm{o}\left(\langle n\rangle \times \langle m\rangle\right) \ni v \otimes w \mapsto v \in \mathcal{P}\mathrm{o}\left(\langle n\rangle\right) \qquad \text{and} \qquad \mathcal{P}\mathrm{o}\left(\langle n\rangle \times \langle m\rangle\right) \ni v \otimes w \mapsto w \in \mathcal{P}\mathrm{o}\left(\langle m\rangle\right)$$

We record the data of a 0-simplex of the simplicial space $\varphi\left(\overline{C}_n \odot \overline{C}_m\right)$ by a diagram

$$\begin{tikzcd}
\Big(\langle n\rangle \times \langle m\rangle \supseteq L\Big) \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \Big(\langle m\rangle \supseteq \pi_m(L)\Big) \arrow[d] \\
\Big(\langle n\rangle \supseteq \pi_n(L)\Big) \arrow[r] & \mathfrak{R} = \langle 1\rangle
\end{tikzcd} \tag{7.2}$$

where the commutativity of this diagram captures the compatibility of the generating operations in the Boardman-Vogt tensor product (viz. the Boarmdan-Vogt interchange relation). An edge between two such vertices, L and

L', is guaranteed by the existence of maps of sets (each of which factors as a bijection to an ordered subset, followed by an inclusion)

- $f : L' \hookrightarrow L$ (corresponding to an edge $L \to L'$ in $\varphi\left(\overline{C}_{nm}\right)$)
- $f_n : \pi_n(L') \hookrightarrow \pi_n(L)$ (corresponding to an edge $\pi_n(L) \to \pi_n(L')$ in $\varphi\left(\overline{C}_n\right)$)
- $f_m : \pi_m(L') \hookrightarrow \pi_m(L)$ (corresponding to an edge $\pi_m(L) \to \pi_m(L')$ in $\varphi\left(\overline{C}_m\right)$)

and these maps must be compatible with each other in the sense that $\pi_n \circ f = f_n \circ \pi_n$ and $\pi_m \circ f = f_m \circ \pi_m$. We can define a category $\mathscr{G}_{n,m}$ whose objects are diagrams of the form (7.2), and whose morphisms are described by cubes

$$\begin{tikzcd}
\left(\langle n\rangle \times \langle m\rangle \supseteq L'\right) \arrow[rr, "\pi_m"] \arrow[dr, "f"] \arrow[dd, "\pi_n"'] & & \left(\langle m\rangle \supseteq \pi_m(L')\right) \arrow[dr, "f_m"] \arrow[dd] & \\
& \left(\langle n\rangle \times \langle m\rangle \supseteq L\right) \arrow[rr, "\pi_m"] \arrow[dd, "\pi_n"] & & \left(\langle m\rangle \supseteq \pi_m(L)\right) \arrow[dd] \\
\left(\langle n\rangle \supseteq \pi_n(L')\right) \arrow[rr] \arrow[dr, "f_n"] & & \langle 1\rangle \arrow[dr] & \\
& \left(\langle n\rangle \supseteq \pi_n(L)\right) \arrow[rr] & & \langle 1\rangle
\end{tikzcd} \tag{7.3}$$

such that $L \to L'$ in $\mathscr{F}_{nm}$, $\pi_m(L) \to \pi_m(L')$ in $\mathscr{F}_m$ and $\pi_n(L) \to \pi_n(L')$ in $\mathscr{F}_n$. By the previous descriptions, it is clear that $N\mathscr{G}_{n,m}$ is levelwise weak equivalent to $\varphi\left(\overline{C}_n \odot \overline{C}_m\right)$.

Our first goal will be to show that, given objects $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ of $\mathrm{ptOp}_{\infty}^{\mathrm{Red}}$, there is a morphism of simplicial spaces over $N\mathsf{Fin}$,

$$\Gamma : \varphi\left(\mathcal{O}^{\otimes} \odot \mathcal{P}^{\otimes}\right) \to \varphi\left(\mathcal{O}^{\otimes}\right) \boxtimes \varphi\left(\mathcal{P}^{\otimes}\right)$$

and that this morphism is natural in $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$. We will then try to show that this is actually a weak equivalence with respect to the reduced Segal model structure on $s\mathcal{S}_{/N\mathsf{Fin}}$. In both steps, we will have recourse to the examples we have just studied, since the category of reduced operads is generated under colimits by the closed corollas (AKA the elementary objects in the category $\mathcal{P}_{\mathrm{Seg,Red}}(\Omega_{rc})$). In fact, using this assumption in conjunction with Remark 7.2.1, it will suffice to deduce the existence of the map Γ in the case where $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are (the homotopy-coherent nerves of) closed corollas, $\overline{C}_n$, $\overline{C}_m$, i.e. we just need to provide a morphism of simplicial spaces over $N\mathsf{Fin}$:

$$\Gamma : N\mathscr{G}_{n,m} \to N\mathscr{F}_n \boxtimes N\mathscr{F}_m$$

To gain some intuition for how such a map might be devised, it is incumbent on us to study the simplicial space $N\mathscr{F}_n \boxtimes N\mathscr{F}_m$ in greater depth. Via the conservative localisation (which is subsumed under the reduced Segal localisation on $s\mathcal{S}_{/N\mathsf{Fin}}$), we know that $N\mathscr{F}_n \boxtimes N\mathscr{F}_m$ is weakly equivalent to $N\mathscr{F}_n \boxtimes^{\mathrm{pre}} N\mathscr{F}_m$, so it suffices to study the latter simplicial space.

A vertex of $N\mathscr{F}_n \boxtimes^{\mathrm{pre}} N\mathscr{F}_m$ is an object of BoxFin of the form $\beta = (\langle l\rangle \twoheadleftarrow \langle c\rangle \twoheadrightarrow \langle r\rangle)$, where $p_1\beta = \langle l\rangle \in \mathcal{P}\mathrm{o}\left(\langle n\rangle\right)$ and $p_2\beta = \langle r\rangle \in \mathcal{P}\mathrm{o}\left(\langle m\rangle\right)$. A 1-simplex of $N\mathscr{F}_n \boxtimes^{\mathrm{pre}} N\mathscr{F}_m$ is a morphism $\beta \to \beta'$ in BoxFin where $p_1(\beta)$, $p_1(\beta') \in \mathcal{P}\mathrm{o}\left(\langle n\rangle\right)$ and $p_2(\beta)$, $p_2(\beta') \in \mathcal{P}\mathrm{o}\left(\langle m\rangle\right)$.

As part of the defining properties of objects in the category BoxFin, there is an injection $\langle c\rangle \rightarrowtail \langle l\rangle \times \langle r\rangle$, so we can identify $\langle c\rangle$ with an object of $\mathcal{P}\mathrm{o}\left(\langle l\rangle \times \langle r\rangle\right)$. Using this perspective, we can construct a category $\mathscr{H}_{n,m}$ as follows: an object of $\mathscr{H}_{n,m}$ is recorded by a diagram

$$\begin{tikzcd}
\left(\langle n\rangle \times \langle m\rangle \supseteq \langle l\rangle \times \langle r\rangle \hookleftarrow \langle c\rangle\right) \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \left(\langle m\rangle \supseteq \langle r\rangle \supseteq \pi_m\langle c\rangle\right) \arrow[d] \\
\left(\langle n\rangle \supseteq \langle l\rangle \supseteq \pi_n\langle c\rangle\right) \arrow[r] & \langle 1\rangle
\end{tikzcd}$$

where we write $\pi_n\langle c\rangle$ to mean the projection onto $\mathcal{P}o(\langle n\rangle)$ of the image of $\langle c\rangle$ in $\langle n\rangle \times \langle m\rangle$, and where the inclusion $\langle c\rangle \rightarrowtail \langle l\rangle \times \langle r\rangle$ comes from an object $(\langle l\rangle \twoheadleftarrow \langle c\rangle \twoheadrightarrow \langle r\rangle)$ in BoxFin such that $\langle l\rangle \subseteq \langle n\rangle$ and $\langle r\rangle \subseteq \langle m\rangle$. A morphism between two such diagrams corresponding to a morphism of BoxFin-objects

$$\begin{tikzcd}
\langle l\rangle \arrow[d, "t"] & \langle c\rangle \arrow[l, two heads] \arrow[r, two heads] \arrow[d, "u"] & \langle r\rangle \arrow[d, "v"] \\
\langle l'\rangle & \langle c'\rangle \arrow[l, two heads] \arrow[r, two heads] & \langle r'\rangle
\end{tikzcd}$$

is determined by a cube similar to the one in (7.3), i.e. of the form

$$\begin{tikzcd}
\big(\langle n\rangle \times \langle m\rangle \supseteq \langle l\rangle \times \langle r\rangle \leftarrowtail \langle c\rangle\big) \arrow[rr, "\pi_m"] \arrow[dr, "u"] \arrow[dd, "\pi_n"'] & & \big(\langle m\rangle \supseteq \langle r\rangle \supseteq \pi_m\langle c\rangle\big) \arrow[dr, "t\,|_{\pi_m\langle c\rangle}"] \arrow[dd] & \\
& \big(\langle n\rangle \times \langle m\rangle \supseteq \langle l'\rangle \times \langle r'\rangle \leftarrowtail \langle c'\rangle\big) \arrow[rr, "\pi_m"] \arrow[dd, "\pi_n"] & & \big(\langle m\rangle \supseteq \langle r'\rangle \supseteq \pi_m\langle c'\rangle\big) \arrow[dd] \\
\big(\langle n\rangle \supseteq \langle l\rangle \supseteq \pi_n\langle c\rangle\big) \arrow[rr] \arrow[dr, "v\,|_{\pi_n\langle c\rangle}"'] & & \langle 1\rangle \arrow[dr] & \\
& \big(\langle n\rangle \supseteq \langle l'\rangle \supseteq \pi_n\langle c'\rangle\big) \arrow[rr] & & \langle 1\rangle
\end{tikzcd}$$

Construction 7.2.4. Using the above descriptions, we can construct a map of simplicial spaces over $N\mathsf{Fin}$, $\widetilde{\Gamma} : \varphi(\overline{C}_n \odot \overline{C}_m) \to \varphi(\overline{C}_n) \boxtimes^{\mathrm{pre}} \varphi(\overline{C}_m)$ by taking the nerve of the obvious functor $\mathscr{G}_{n,m} \hookrightarrow \mathscr{H}_{n,m}$, defined on objects by

$$\left(\begin{tikzcd}
\big(\langle n\rangle \times \langle m\rangle \supseteq L\big) \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \big(\langle m\rangle \supseteq \pi_m(L)\big) \arrow[d] \\
\big(\langle n\rangle \supseteq \pi_n(L)\big) \arrow[r] & \langle 1\rangle
\end{tikzcd}\right) \mapsto \left(\begin{tikzcd}
\big(\langle n\rangle \times \langle m\rangle \supseteq \langle n\rangle \times \langle m\rangle \leftarrowtail \langle c\rangle\big) \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \big(\langle m\rangle \supseteq \langle m\rangle \supseteq \pi_m\langle c\rangle\big) \arrow[d] \\
\big(\langle n\rangle \supseteq \langle n\rangle \supseteq \pi_n\langle c\rangle\big) \arrow[r] & \langle 1\rangle
\end{tikzcd}\right)$$

where L is the image of $\langle c\rangle$ in $\langle n\rangle \times \langle m\rangle$.

Remark 7.2.5. The functor $\mathscr{G}_{n,m} \to \mathscr{H}_{n,m}$ is not an equivalence of categories – heuristically speaking, an object of $\mathscr{H}_{n,m}$ "carries (a lot) more information" than an object of $\mathscr{G}_{n,m}$, since it contains all the data of a BoxFin-object, rather than just a subset of $\langle n\rangle \times \langle m\rangle$. Hence, it is unreasonable to expect that $\widetilde{\Gamma}$ is an equivalence of simplicial spaces over $N\mathsf{Fin}$. All is not lost, however: the solution to our problem lies in the passage from $\varphi\left(\overline{C}_n\right) \boxtimes^{\mathrm{pre}} \varphi\left(\overline{C}_m\right)$ to $\varphi\left(\overline{C}_n\right) \boxtimes \varphi\left(\overline{C}_m\right)$, i.e. in the conservatisation procedure.

We recall from Lemma 5.3.5 that a fibrewise complete Segal space $w : X \to N\mathcal{C}$ (where $\mathcal{C}$ is a small category) is conservative if and only if it satisfies the condition that:

($\circledast$) a 1-simplex f of X is homotopy invertible if and only if $w(f)$ is an isomorphism in $\mathcal{C}$.

The reference map $w : \varphi\left(\overline{C}_n\right) \boxtimes^{\mathrm{pre}} \varphi\left(\overline{C}_m\right) \simeq N\mathscr{H}_{n,m} \to N\mathsf{Fin}$ is determined as follows: the image under w of an object in $\mathscr{H}_{n,m}$,

$$\begin{tikzcd}
\big(\langle n\rangle \times \langle m\rangle \supseteq \langle l\rangle \times \langle r\rangle \leftarrowtail \langle c\rangle\big) \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \big(\langle m\rangle \supseteq \langle r\rangle \supseteq \pi_m\langle c\rangle\big) \arrow[d] \\
\big(\langle n\rangle \supseteq \langle l\rangle \supseteq \pi_n\langle c\rangle\big) \arrow[r] & \langle 1\rangle
\end{tikzcd}$$

is $\langle c\rangle$ in Fin. Because $\varphi\left(\overline{C}_n\right) \boxtimes^{\mathrm{pre}} \varphi\left(\overline{C}_m\right)$ arises as the nerve of the category $\mathscr{H}_{n,m}$, an edge is homotopy-invertible if and only if it defines an isomorphism in the category $\mathscr{H}_{n,m}$. In particular, if $\langle l\rangle \subsetneq \langle n\rangle$ and $\langle r\rangle \subsetneq \langle m\rangle$, then the following 1-simplex is not homotopy-invertible in $\varphi\left(\overline{C}_n\right) \boxtimes^{\mathrm{pre}} \varphi\left(\overline{C}_m\right)$:

$$\begin{tikzcd}
\Big(\langle n\rangle \times \langle m\rangle \supseteq \langle l\rangle \times \langle r\rangle \leftarrowtail \langle c\rangle\Big) \arrow[rr, "\pi_m"] \arrow[dr, "\mathrm{id}_{\langle c\rangle}"] \arrow[dd, "\pi_n"'] & & \Big(\langle m\rangle \supseteq \langle r\rangle \supseteq \pi_m\langle c\rangle\Big) \arrow[dr, "\mathrm{id}_{\pi_m\langle c\rangle}"] \arrow[dd] & \\
& \Big(\langle n\rangle \times \langle m\rangle \supseteq \langle n\rangle \times \langle m\rangle \leftarrowtail \langle c\rangle\Big) \arrow[rr, "\pi_m"] \arrow[dd, "\pi_n"'] & & \Big(\langle m\rangle \supseteq \langle m\rangle \supseteq \pi_m\langle c\rangle\Big) \arrow[dd] \\
\Big(\langle n\rangle \supseteq \langle l\rangle \supseteq \pi_n\langle c\rangle\Big) \arrow[rr] \arrow[dr, "\mathrm{id}_{\pi_n\langle c\rangle}"] & & \langle 1\rangle \arrow[dr] & \\
& \Big(\langle n\rangle \supseteq \langle n\rangle \supseteq \pi_n\langle c\rangle\Big) \arrow[rr] & & \langle 1\rangle
\end{tikzcd} \tag{7.4}$$

On the other hand, we see that the image under w of this edge in $N\mathsf{Fin}$ is $\mathrm{id}_{\langle c\rangle} : \langle c\rangle \to \langle c\rangle$, which *is* an isomorphism in Fin, so $w : N\mathscr{H}_{n,m} \to N\mathsf{Fin}$ cannot be conservative. However, by applying the functorial fibrewise-completion/conservatisation functor $(\Lambda K)^{\infty}$ to $w : N\mathscr{H}_{n,m} \to N\mathsf{Fin}$, we force the image of the edge (7.4) to become homotopy-invertible. Based on this observation, we make the following definition...

Definition 7.2.6. Let $\Gamma : \varphi(\overline{C}_n \odot \overline{C}_m) \to \varphi(\overline{C}_n) \boxtimes \varphi(\overline{C}_m)$ be the map of simplicial spaces over $N\mathsf{Fin}$ given by the composition

$$\varphi(\overline{C}_n \odot \overline{C}_m) \xrightarrow{\widetilde{\Gamma}} \varphi(\overline{C}_n) \boxtimes^{\mathrm{pre}} \varphi(\overline{C}_m) \xrightarrow{(\Lambda K)^{\infty}} \varphi(\overline{C}_n) \boxtimes \varphi(\overline{C}_m)$$

It is almost an immediate consequence of the remarks above that Γ satisfies the following property:

Lemma 7.2.7. *The map Γ determines a weak equivalence of reduced Segal simplicial spaces over $N\mathsf{Fin}$.*

Proof. The observations at the end of Remark 7.2.5 are crucial – by localising with respect to conservativity, all the vertices in $N\mathscr{H}_{n,m}$ contained in the fibre over $\langle c\rangle$ become weak homotopy equivalent to one another – and hence to the image under $\widetilde{\Gamma}$ of the unique vertex of $N\mathscr{G}_{n,m}$ over $\langle c\rangle$. Hence, as conservative Segal spaces over $N\mathsf{Fin}$, $N\mathscr{G}_{n,m}$ and $(\Lambda K)^{\infty} N\mathscr{H}_{n,m}$ are weakly equivalent.

We saw in Lemma 6.1.2 that localisation with respect to conservativity is subsumed under the reduced Segal localisation on $s\mathcal{S}_{/N\mathsf{Fin}}$. Hence, $N\mathscr{G}_{n,m}$ and $(\Lambda K)^{\infty} N\mathscr{H}_{n,m}$ are weakly equivalent in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$, i.e. $\varphi\left(\overline{C}_n \odot \overline{C}_m\right) \simeq \varphi\left(\overline{C}_n\right) \boxtimes \varphi\left(\overline{C}_m\right)$ in $(s\mathcal{S}_{/N\mathsf{Fin}})_{\mathrm{Seg,Red}}$, as required. ☐

As we remarked, the category of reduced operads is generated under colimits by the closed corollas. Since we have ascertained that the box product descends to a map on reduced Segal objects which preserves colimits in both variables, and since we know that φ preserves colimits (at least when we restrict our attention to plain reduced simplicial operads), this lemma provides us with the following corollary:

Corollary 7.2.8. *For plain reduced simplicial operads $\mathcal{O}$ and $\mathcal{P}$, there is a weak equivalence*

$$\varphi\left(N\mathcal{O}^{\otimes} \odot N\mathcal{P}^{\otimes}\right) \xrightarrow{\sim} \varphi\left(N\mathcal{O}^{\otimes}\right) \boxtimes \varphi\left(N\mathcal{P}^{\otimes}\right)$$

in $\left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg,Red}}$.

As a special case of the corollary, we have the (by now, far-too-familiar) result: (writing $\mathbb{E}_d^{\otimes}$ to mean both the homotopy coherent nerve of the category of operators of the topological little d-cubes operad ${}^t\mathbb{E}_d$, and also the associated complete Segal dendroidal space)

Theorem 7.2.9 (Dunn Additivity)**.** *There is a weak equivalence of reduced ∞-operads*

$$\mathbb{E}_d^{\otimes} \odot \mathbb{E}_{d'}^{\otimes} \simeq \mathbb{E}_{d+d'}^{\otimes}$$

Proof.

$$\begin{aligned} j^* N_d^{rc}\left(\mathbb{E}_d^{\otimes} \odot \mathbb{E}_{d'}^{\otimes}\right) &\simeq \varphi\left(\mathbb{E}_d^{\otimes} \odot \mathbb{E}_{d'}^{\otimes}\right) \\ &\simeq \varphi\left(\mathbb{E}_d^{\otimes}\right) \boxtimes \varphi\left(\mathbb{E}_{d'}^{\otimes}\right) \\ &\simeq \mathrm{con}\left(\mathbb{R}^d\right) \boxtimes \mathrm{con}(\mathbb{R}^{d'}) \\ &\simeq \mathrm{con}\left(\mathbb{R}^{d+d'}\right) \\ &\simeq \varphi\left(\mathbb{E}_{d+d'}^{\otimes}\right) \\ &\simeq j^* \mathbb{E}_{d+d'}^{\otimes} \end{aligned}$$

where we use [BW18a, Theorem 7.5] at the third and fifth lines, and [BW18b, Theorem 1.1] in the fourth line. Since $j^* : \mathcal{P}_{\mathrm{Seg},\mathrm{Red}}(\Omega_{rc}) \to \left(s\mathcal{S}_{/N\mathsf{Fin}}\right)_{\mathrm{Seg},\mathrm{Red}}$ is a Quillen equivalence and $N_d^{rc} : \mathrm{Op}_\infty^{\mathrm{Red}} \to \mathcal{P}_{\mathrm{Seg},\mathrm{Red}}(\Omega_{rc})$ is also a Quillen equivalence, it must be the case that

$$\mathbb{E}_d^{\otimes} \odot \mathbb{E}_{d'}^{\otimes} \simeq \mathbb{E}_{d+d'}^{\otimes}$$

□

Appendix A

Quasi-Categories and coCartesian Morphisms

In this appendix, we give a short review of quasi-categories, paying special attention to the notion of coCartesian morphisms, which play a significant role in the definition of a quasi-operad.

A.1 Quasi-Categories

We recall from our discussion of Segal spaces that the Segal condition gives a way of characterising those simplicial sets that arise as the nerve of a small category. We can repackage the Segal condition as saying that a simplicial set satisfies the right lifting property with respect to a certain collection of maps. Let us first fix some notation: if $\Delta[n]$ denotes the standard n-simplex, then we write $\Delta[\{k_0, \ldots, k_i\}]$ to mean the restriction of $\Delta[n]$ to the subcomplex with vertices labelled $\{k_0, \ldots, k_i\}$. With this notation, we have the following rephrasing of (1.1):

Theorem A.1.1. *A simplicial set K is equivalent to the nerve of some small category if and only for each $0 < i < n$, there exists a unique dotted lift in the following diagram:*

$$\begin{tikzcd} \Lambda[n,i] \arrow[r] \arrow[d, hook] & K \\ \Delta[n] \arrow[ur, dotted, "!"'] & \end{tikzcd}$$

In particular, the statement of uniqueness is key in allowing us to determine a well-defined composition law: as we encountered when discussing the notion of "composition" in Segal spaces, making sense of a composition rule from the data of a weak homotopy equivalence leads to a composition that is only well-defined up to homotopy, but if the lift is unique then the composition automatically becomes well-defined, leading to the construction of a category whose objects are the vertices of K, whose morphisms are the edges of K, and whose composition law is determined by the above lifting property. In general however, as with Segal spaces, this requirement for the existence of a unique lift is excessively restrictive. This is where the definition of a quasi-category (or ∞-category) enters:

Definition A.1.2. A **quasi-category** is a simplicial set K such that for any $0 < i < n$, any map $f_0 : \Lambda[n, i] \to K$ admits an extension to a map $\Delta[n] \to K$.

It is immediate that the nerve of a small category is automatically a quasi-category. More generally, the "categorical" structure of a quasi-category $\mathcal{C}$ is determined in a similar manner to the kind of "up-to-homotopy" categorical structure we described for Segal spaces: the vertices and edges of $\mathcal{C}$ play the role of objects and morphisms respectively, and the existence of the extensions determines a composition law which is well-defined up to a suitable notion of homotopy. Along with this notion of homotopy, there is a corresponding notion of invertibility up to homotopy. In particular, we say a quasi-category is an **∞-groupoid** if every edge is invertible. From any quasi-category $\mathcal{C}$, we can extract an ∞-groupoid by considering the maximal subsimplicial set $\mathcal{C}^{\simeq}$ consisting only of the invertible edges. A familiar first example of an ∞-groupod is a Kan complex (which satisfies the extension property for all $0 \leq i \leq n$).

We recall from our discussion of simplicially-enriched model categories that for any two simplicial sets X, Y, there is a mapping space $Y^X = \mathrm{Map}(X, Y)$ whose set of n-simplices is $\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n] \times X, Y)$. An important result of Joyal tells us that for any quasi-category $\mathcal{C}$ and any simplicial set X, the simplicial set $\mathcal{C}^X$ is also a quasi-category. In particular, this allows us to define the notion of categorical equivalence between simplicial sets: a map $Y \to X$ is a **categorical equivalence** if and only if for every quasi-category $\mathcal{C}$, there is an equivalence of Kan complexes

$$(\mathcal{C}^X)^{\simeq} \xrightarrow{\sim} (\mathcal{C}^Y)^{\simeq}$$

This notion of categorical equivalence defines a class of weak equivalences for the so-called *categorical model structure* (also known as a the *Joyal model structure*) on the category of simplicial sets: the cofibrations in this structure are the monomorphisms of simplicial sets and the fibrant objects are the quasi-categories.

Remark A.1.3. Given the evident overlap with our discussion of complete Segal spaces, it is not altogether surprising to learn that the associated model categories are Quillen equivalent: in fact, in the paper [JT06], it is shown that there are two Quillen equivalences between the model category for quasi-categories and the model category for complete Segal spaces. The easiest to describe is induced by the functor which sends a bisimplicial set $X_{*,*}$ to its zeroth row, $X_{*,0}$ (in particular, this has the pleasing side-effect of demonstrating that a complete Segal space is determined by its first row).

There is also a close link between quasi-categories and simplicial/topological categories: the construction underlying this link is the **homotopy coherent nerve**. We construct this functor in a few steps.

Given an object $[n]$ in the ordinal category and $0 \leq i \leq j \leq n$, let $P(i,j)$ denote the poset consisting of the subsets of $\{i, i+1, \dots, j\} \subseteq \{0, 1 \dots, n\}$. Then we can define a simplicial category $\mathfrak{C}[\Delta[n]]$, whose objects are the objects of $[n]$, and given a pair $i, j \in [n]$, the simplicial set of all maps between them is given by

$$\mathrm{Map}_{\mathfrak{C}[\Delta[n]]}(i,j) = \begin{cases} \emptyset & j < i \\ N(P(i,j)) & i \leq j \end{cases}$$

Given $i \leq j \leq k$, the composition map

$$\mathrm{Map}_{\mathfrak{C}[\Delta[n]]}(j,k) \times \mathrm{Map}_{\mathfrak{C}[\Delta[n]]}(i,j) \to \mathrm{Map}_{\mathfrak{C}[\Delta[n]]}(i,k)$$

is determined by the map of posets

$$N(P(j,k))_0 \times N(P(i,j))_0 \ni (I_{j,k}, I_{i,j}) \mapsto I_{i,j} \cup I_{j,k} \in N(P(i,k))_0$$

This category $\mathfrak{C}[\Delta[n]]$ can be viewed as a kind of "thickening‘" of the category $[n]$ by removing the condition that composition of arrows between elements ought to be strictly associative (for a more precise explanation of this point, see [Lur09a, Remark 1.1.5.2]). This construction defines a functor $\mathfrak{C} : \Delta \to s\mathrm{Cat}$, and using the Yoneda embedding, this extends to a functor $\mathfrak{C} : s\mathrm{Set} \to s\mathrm{Cat}$. This new functor admits a right adjoint – the **homotopy-coherent nerve** $N : s\mathrm{Cat} \to s\mathrm{Set}$, which is completely determined by the adjunction law:

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta[n], N\mathcal{C}) \rightleftarrows \mathrm{Hom}_{s\mathrm{Cat}}(\mathfrak{C}[\Delta[n]], \mathcal{C})$$

Using the Quillen equivalence between spaces and simplicial sets induced by the adjunction between the geometric realisation and the singular simplices functor, we observe that there is also a nerve functor from topological categories to simplicial sets – we also refer to this as the homotopy-coherent nerve (a common abuse of notation). Finally, we note that the homotopy-coherent nerve of an ordinary unenriched category (viewed as a discrete topological category) coincides with the usual nerve of that category (for this reason we won't make a notational distinction between these various notions of nerve).

The following theorem tells us that the homotopy-coherent nerve behaves particularly well with respect to an important collection of objects among the simplicial sets:

Theorem A.1.4. *[Lur09a, Theorem 1.1.5.10] Let $\mathcal{C}$ be an object in the simplicial category of Kan complexes. Then $N\mathcal{C}$ is a quasi-category.*

Remark A.1.5. It is unsurprising to learn that this fact is part of the much stronger assertion that the homotopy-coherent nerve is a right Quillen functor between suitable model structures on the categories of simplicial categories (whose fibrant objects are those categories whose mapping spaces are Kan complexes) and simplicial sets with the categorical model structure (so that the fibrant objects are quasi-categories). In fact, [Lur09a, Theorem 1.1.5.13] tells us that the counit map of the adjunction $\mathfrak{C} \dashv N$ is a weak equivalence, which justifies the assertion that simplicial categories and quasi-categories are equivalent models for ∞-categories.

Emanating from the foregoing proposition, we can define the quasi-category of spaces $\mathcal{S}$ to be the homotopy-coherent nerve of the simplicial category of $\mathcal{K}$an

- whose objects are Kan complexes; and
- for any pair of Kan complexes X, Y, the simplicial mapping space is given by

$$\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X, Y)_n = \mathrm{Hom}(\Delta[n] \times X, Y)$$

(Via the singular simplices/geometric realisation adjunction this is equivalent to the topological category of CW-complexes.)

Many familiar notions from category theory can be redefined in such a way that they automatically extend neatly to the world of quasi-categories. Our main application in studying quasi-categories will be to ∞-operads, so for our purposes, we will restrict our attention to just a handful of these notions.

A.2 The Grothendieck Construction and coCartesian Fibrations

A functor of ordinary categories $G : \mathcal{C} \to \mathcal{D}$ is said to be **left fibered in sets** if for every $x \in \mathcal{C}$ and every morphism $g : Gx \to \tilde{y}$ in $\mathcal{D}$, there exists $y \in \mathcal{C}$ and a morphism $f : x \to y$ which lifts g to $\mathcal{C}$. Dually, we say G is **right fibered in sets** if we can find a lift to $\mathcal{C}$ for every morphism $g : \tilde{x} \to Gy$. We write $\mathrm{Fib}(\mathcal{D})$ to denote the subcategory of $\mathrm{Cat}_{/\mathcal{D}}$ consisting of those functors $G : \mathcal{C} \to \mathcal{D}$ which are left fibered in sets.

Closely related to this notion of functors fibered in sets is the Grothendieck construction for ordinary categories: given a functor $F : \mathcal{D} \to \mathrm{Set}$, we can associate its category of elements $\int F$ whose objects are pairs (d, x) where d is an object in $\mathcal{D}$ and $x \in Fd$; a morphism in $\int F$ from (d, x) to (d', x') is the data of a morphism $f : d \to d'$ such that $F(f)(x) = x'$. The natural forgetful functor $\int F \to \mathcal{D}$ is left fibered in sets; and in fact this construction determines an equivalence of categories

$$\mathrm{Fun}(\mathcal{D}, \mathrm{Sets}) \to \mathrm{Fib}(\mathcal{D})$$

The quasi-categorical notion of being left (resp, right) fibered in sets is that of the left (resp. right) fibration of simplicial sets: we say a map of simplicial sets $p : X \to Y$ is a **left fibration** if for all $0 \leq i < n$, the indicated lift exists in the given commutative square:

$$\begin{tikzcd} \Lambda[n,i] \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Dually, p is a **right fibration** if the dotted lift exists for all $0 < i \leq n$. We say p is an **inner fibration** if the indicated lift exists for $0 < i < n$.

Analogously, one can examine pseudofunctors $F : \mathcal{D} \to \mathrm{Cat}$ (we consider pseudofunctors since Cat naturally organises into a 2-category) – that is, to each object $d \in \mathcal{D}$, F associates a category Fd; a morphism $f : d \to d'$ in $\mathcal{D}$ induces a functor $Ff : Fd \to Fd'$; and for each composable pair of morphisms $f : d \to d'$, $g : d' \to d''$, there is a natural isomorphism $\tau : Fg \circ Ff \to F(g \circ f)$. By the same prescription as above, we can define the Grothendieck construction $\int F$ in an essentially identical manner; again, this is equipped with an obvious forgetful functor $\pi : \int F \to \mathcal{D}$. However, we now need to adapt our notion of left fibered in sets accordingly...

Given a functor $\pi : \mathcal{C} \to \mathcal{D}$, and a morphism $f : x \to y$ in $\mathcal{C}$, we say f is **π-coCartesian** if for every morphism $g : x \to z$ in $\mathcal{C}$ and every factorization

$$\begin{tikzcd} \pi x \arrow[r, "\pi(g)"] \arrow[d, "\pi(f)"'] & \pi z \\ \pi y \arrow[ur, "\tilde{h}"'] & \end{tikzcd} \tag{A.1}$$

there exists an unique factorization

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] & z \\ y \arrow[ur, dashed, "h"'] & \end{tikzcd} \tag{A.2}$$

such that $\pi(h) = \tilde{h}$. We say π is a **coCartesian fibration** if every morphism $\tilde{f} : \pi x \to \tilde{y}$ in $\mathcal{D}$ has a lift to a π-coCartesian morphism $f : x \to y$ in $\mathcal{C}$. Dually, we say the functor π is Cartesian precisely when π^{op} is coCartesian. With this in mind, we can define a 2-category $\mathrm{Fib}^{coc}(\mathcal{D})$ whose

- objects are coCartesian fibrations $F : \mathcal{C} \to \mathcal{D}$;
- morphisms are the commutative triangles

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathcal{C}' \arrow[dl, "\pi'"] \\ & \mathcal{D} & \end{tikzcd}$$

  such that φ sends π-coCartesian morphisms to π'-coCartesian morphisms;
- natural transformations are those natural transformations which are compatible with the projection to $\mathcal{D}$.

With this definition in place, it turns out that the Grothendieck construction determines an equivalence of 2-categories:

$$\mathrm{PseudoFun}(\mathcal{D}, \mathrm{Cat}) \to \mathrm{Fib}^{coc}(\mathcal{D})$$

We have already seen that the data of a pseudofunctor $F : \mathcal{D} \to \mathrm{Cat}$ is somewhat unwieldy as a result of trying to satisfy the coherence requirements of working with 2-categories; working with quasi-categories, producing such a coherent functor could be crippling. With this in mind, we instead develop our approach to obtaining a quasi-categorical Grothendieck construction from the perspective of these coCartesian fibrations.

First, note that we can the reframe discussion of π-coCartesian edges: the commutative triangle (A.1), can be viewed as a morphism $\sigma : \Delta[2] \to N\mathcal{D}$, while the maps $f : x \to y$ and $g : x \to z$ define a map $f \vee g : \Lambda[2,0] \to N\mathcal{C}$. Altogether, the existence of the lifting factorization in the diagram (A.2) corresponds to the existence of the dotted lift in the commutative diagram

$$\begin{tikzcd} \Lambda[2,0] \arrow[r, "f \vee g"] \arrow[d, hook] & N\mathcal{C} \arrow[d, "N\pi"] \\ \Delta[2] \arrow[r, "\sigma"] \arrow[ur, dotted] & N\mathcal{D} \end{tikzcd}$$

Using this as inspiration, let $\pi : X \to Y$ be a map of simplicial sets, and let $f : x \to y$ be an edge in X. We say f is p-**coCartesian** if for every diagram

$$\begin{tikzcd} \Lambda[n,0] \arrow[r, "\sigma"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the dotted lift exists, where $\sigma|_{\Delta[\{0,1\}]} = f$. Dually, we say the edge f is p-**Cartesian** if the lift exists when we replace $\Lambda[n,0]$ in the above diagram by $\Lambda[n,n]$. We say that the map $p : X \to Y$ is a **coCartesian fibration** if it satisfies the following:

(i) p is an inner fibration of simplicial sets;

(ii) for every vertex x in X and every edge $\tilde{f} : p(x) \to \tilde{y}$ in Y, there exists a p-coCartesian edge $f : x \to y$ such that $pf = \tilde{f}$.

We observe that a coCartesian fibration of simplicial sets $p : X \to Y$ is a left fibration if and only if edge of X is p-coCartesian.

Remark A.2.1. Lurie determines an equivalence between the quasi-category of coCartesian fibrations over a quasi-category $\mathcal{D}$ and the collection of all quasi-categorical functors from $\mathcal{D}$ to the quasi-category of quasi-categories (denoted Cat_∞) (following [Har, Theorem 2.6.15], we dub this equivalence the *Lurie-Grothendieck correspondence*). The equivalence is implemented by the so-called *straightening* and *unstraightening constructions*.

Intuitively, these constructions are a quasi-categorical analogue of the Grothendieck construction which we have dealt with on a 2-categorical level. While we will make further reference to the straightening and unstraightening constructions in our discourse on the quasi-categorical Dunn additivity theorem, we will not enter too deeply into the details of these constructions or the stated equivalence as these both go a little beyond the breadth of exposition or level of difficulty which we are aiming for here: instead, we encourage the reader to let their 2-categorical intuition carry them through these choppy waters. Alternatively, we exhort the curious reader to consult the relevant sections of [Lur09a] (e.g. Section 2.2.1, Section 3.2), or to seek a somewhat briefer exposition in the notes [Har]. We will point out that the Lurie-Grothendieck correspondence restricts to an equivalence of quasi-categories between the quasi-category of left fibrations over a simplicial set X and the quasi-category of functors from X into the quasi-category of spaces $\mathcal{S}$ – we can think of these as the quasi-categorical versions of cosheaves on X; dually there is an equivalence between the quasi-category of right fibrations over X and the quasi-category of presheaves on X (for details of this, see [Lur09a, Proposition 5.1.1]), which is more than a little reminiscent of our discussions in Chapter 4.

Appendix B

Dunn Additivity for Quasi-Operads

In this appendix, we sketch an outline of the proof given in [Lur17] to deduce the additivity theorem for quasi-categorical little cubes. We begin with a description of the wreath product of quasi-operads, and then collect some facts about weak approximations for operads. In the third section, we study the proof of the additivity theorem itself. We make no claims to originality in this outline: given the fairly technical nature of the proof, we have tried to stay close to Lurie's original formulation of almost all the results and proofs. In essence, this appendix merely serves to make our exposition of the notion of additivity for quasi-operads somewhat more self-contained.

B.1 Wreath Products

We recall that given a quasi-category $\mathcal{C}$ we were able to associate an operad, $\mathcal{C}^{\sqcup}$, as per Example 3.1.3. In the special case where $\mathcal{C}$ arises as the homotopy coherent nerve of some ordinary category $\mathcal{J}$, we can identify $\mathcal{C}^{\sqcup}$ with the homotopy coherent nerve of a category $\mathcal{J}^{\sqcup}$ defined as follows:

- objects of $\mathcal{J}^{\sqcup}$ are finite sequences $(j_1, \dots, j_m)$ where each j_i is an object of $\mathcal{J}$;
- a morphism between objects $(j_1, \dots, j_n)$ and $(j'_1, \dots, j'_m)$ is the data of a map $\alpha : \langle n \rangle \to \langle m \rangle$ in Fin_* and for each $i \in \alpha^{-1}\{k\}$, a morphism $j_i \to j'_k$ in $\mathcal{J}$.

If we consider $\mathcal{J} = \mathsf{Fin}_*$, then we note that there is an obvious functor $\Phi : \mathsf{Fin}_*^{\sqcup} \to \mathsf{Fin}_*$, given on objects by $(\langle k_1 \rangle, \dots, \langle k_n \rangle) \mapsto \langle k_1 + \ldots + k_n \rangle$, and this in turn induces a map of quasi-categories $\Phi : (N\mathsf{Fin}_*)^{\sqcup} = N\left(\mathsf{Fin}_*^{\sqcup}\right) \to N\mathsf{Fin}_*$

Definition B.1.1 (Wreath Product of Quasi-Operads)**.** Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be quasi-operads. As a simplicial set, we define the **wreath product** $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ by

$$\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} = \mathcal{O}^{\otimes} \times_{N\mathsf{Fin}^*} \mathcal{P}^{\sqcup}$$

where we write $\mathcal{P}^{\sqcup}$ to mean $(\mathcal{P}^{\otimes})^{\sqcup}$. The reference map from this simplicial set to $N\mathsf{Fin}_*$ is given by the composite

$$\mathcal{O}^{\otimes} \times_{N\mathsf{Fin}^*} \mathcal{P}^{\sqcup} \to \mathcal{P}^{\sqcup} \to (N\mathsf{Fin}_*)^{\sqcup} \xrightarrow{\Phi} N\mathsf{Fin}_*$$

An edge f in $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ is the data of an edge $g : (d_1, \dots, d_n) \to (d'_1, \dots, d'_m)$ in $\mathcal{P}^{\sqcup}$ lying over a morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in $N\mathsf{Fin}_*$, together with an edge h in $\mathcal{O}^{\otimes}$ lying over α. We say f is an **inert morphism** in $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ if h is inert in $\mathcal{O}^{\otimes}$ and each morphism $d_i \to d'_k$ (where $\alpha(i) = k$) is inert in $\mathcal{P}$.

Recall that by construction $\mathcal{C}^{\sqcup}$ satisfies

$$\mathrm{Hom}_{N\mathsf{Fin}_*}(K, \mathcal{C}^{\sqcup}) = \mathrm{Hom}_{s\mathrm{Set}}(K \times_{N\mathsf{Fin}_*} N\Gamma^*, \mathcal{C})$$

for each simplicial set K with a map to $N\mathsf{Fin}_*$. In particular, since there is an obvious forgetful functor from $N\Gamma^*$ to $N\mathsf{Fin}_*$, there is an induced map

$$\mathrm{Hom}_{N\mathsf{Fin}_*}(K, \mathcal{C} \times N\mathsf{Fin}_*) = \mathrm{Hom}_{s\mathrm{Set}}(K \times_{N\mathsf{Fin}_*} N\mathsf{Fin}_*, \mathcal{C}) \to \mathrm{Hom}_{N\mathsf{Fin}_*}(K, \mathcal{C}^{\sqcup})$$

for each simplicial set K with a reference map to $N\mathsf{Fin}_*$, and hence there is a map $\mathcal{C} \times N\mathsf{Fin}_* \to \mathcal{C}^{\sqcup}$. In particular, this means that if $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are quasi-operads, then we have a composite map

$$\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to N\mathsf{Fin}_* \times \mathcal{P}^{\otimes} \to \mathcal{P}^{\sqcup}$$

as well as a projection $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$. Composing both these maps with the respective reference maps to $N\mathsf{Fin}_*$, we find that they have the same image, so by the universal property of the pullback, there is a map of simplicial sets over $N\mathsf{Fin}_*$, $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$. The value of this bifunctor is that we have the following result from Lurie:

Theorem B.1.2. *[Lur17, Theorem 2.4.4.3] Let M denote the inert morphisms of the quasi-operad $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$. Then there is a weak equivalence of quasi-preoperads*

$$\mathcal{O}^{\otimes,\natural} \odot \mathcal{P}^{\otimes,\natural} \simeq \left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}, M\right)$$

Thus we can view the wreath product as a kind of explicit construction of the tensor product of ∞-operads.

B.2 Weak Approximation of Quasi-Operads

Definition B.2.1. Let $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ be a quasi-operad and let $\mathcal{C}$ be a quasi-category. A categorical fibration (i.e. a fibration in the Joyal model structure on sSet) $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ is said to be a **weak approximation** of $\mathcal{C}$ to $\mathcal{O}^{\otimes}$ if it satisfies the following conditions:

1. Let $p' := p \circ f$, let c be an object of $\mathcal{C}$ and let $\langle n \rangle = p'(c)$. For each $1 \le i \le n$, there exists a locally f-coCartesian morphism $\alpha_i : c \to c_i$ lifting the inert morphism $\rho^i : \langle n \rangle \to \langle 1 \rangle$ such that $f(\alpha_i)$ is inert in $\mathcal{O}^{\otimes}$.

2. Let c be an object of $\mathcal{C}$ and let $\alpha : X \to f(c)$ be a morphism in $\mathcal{O}^{\otimes}$; let $\mathcal{E}$ be the full subcategory of $\mathcal{C}_{/c} \times_{\mathcal{O}^{\otimes}_{/f(c)}} \mathcal{O}^{\otimes}_{/X}$ spanned by those objects corresponding to pairs $\left(c' \xrightarrow{\beta} c, X \xrightarrow{\gamma} f(c)\right)$ such that γ is inert in $\mathcal{O}^{\otimes}$. Then $\mathcal{E}$ is weakly contractible.

If $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ is an arbitrary morphism of quasi-categories, we say that f is a weak approximation if we can factor it as

$$\mathcal{C} \xrightarrow{f'} \mathcal{C}' \xrightarrow{f''} \mathcal{O}^{\otimes}$$

where f' is a categorical equivalence, and f'' is a a categorical fibration which is a weak equivalence in the sense above.

In certain circumstances, we can provide a more tractable version of this notion. Before we can explain this in detail, we need to define an auxiliary quasi-category, Tup_n – this is the subcategory of $N\mathsf{Fin}_{*/\langle n \rangle}$ with the same objects, whose morphisms are given by those commutative triangles

$$\begin{tikzcd} \langle m \rangle \arrow[rr, "\alpha"] \arrow[dr] & & \langle m \rangle \arrow[dl] \\ & \langle n \rangle & \end{tikzcd}$$

such that α is a bijection of finite sets with points.

Proposition B.2.2. *[Lur17, Proposition 2.3.3.14] Let $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ be a quasi-operad such that $\mathcal{O}_{\langle 1 \rangle}$ is a Kan complex, and suppose $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ is a categorical fibration which satisfies*

- ($\circledast$) *for all objects c in $\mathcal{C}$ and all inert morphisms in $N\mathsf{Fin}_*$, $\beta : p \circ f(c) \to \langle n \rangle$, there exists a $(p \circ f)$-coCartesian morphism $\overline{\beta} : c \to c'$ which lifts β such that $f(\overline{\beta})$ is inert in $\mathcal{O}^{\otimes}$.*

Then f is a weak approximation of $\mathcal{C}$ to $\mathcal{O}^{\otimes}$ if and only if the following holds:

- ($\circledcirc$) *for each object c in $\mathcal{C}$, there is a weak homotopy equivalence (where $(p \circ f)(c) = \langle n \rangle$)*

$$\mathcal{C}_{/c} \times_{N\mathsf{Fin}_{*/\langle n \rangle}} \mathrm{Tup}_n \xrightarrow{\sim} \mathcal{O}^{\otimes}_{/f(c)} \times_{N\mathsf{Fin}_{*/\langle n \rangle}} \mathrm{Tup}_n$$

Definition B.2.3. Let $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ and $q : \mathcal{O}'^{\otimes} \to N\mathsf{Fin}_*$ be quasi-operads, and let $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ be a weak approximation. Writing $p' = p \circ f$, we say that a functor $\mathcal{C} \to \mathcal{O}'^{\otimes}$ is a **$\mathcal{C}$-algebra object** of $\mathcal{O}'^{\otimes}$ if

(i) The square of simplicial sets

$$\begin{tikzcd} \mathcal{C} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{O}'^{\otimes} \arrow[d, "q"] \\ \mathcal{O}^{\otimes} \arrow[r, "p"] & N\mathsf{Fin}_* \end{tikzcd}$$

commutes.

(ii) For any object c in $\mathcal{C}$ with $p'(c) = \langle n \rangle$, if we choose objects c_i in $\mathcal{C}$ for all $1 \leq i \leq n$ and morphisms $\alpha_i : c \to c_i$ lying over the inert maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$, then $g(\alpha_i)$ is inert in $\mathcal{O}'^{\otimes}$.

We write $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ for the full subcategory of $\mathrm{Fun}_{N\mathsf{Fin}_*}(\mathcal{C}, \mathcal{O}'^{\otimes})$ spanned by $\mathcal{C}$-algebra objects of $\mathcal{O}'^{\otimes}$.

Proposition B.2.4. *[Lur17, Theorem 2.3.3.23] Let $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ and $q : \mathcal{O}'^{\otimes} \to N\mathsf{Fin}_*$ be quasi-operads, and let $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ be a weak approximation. Suppose f induces an equivalence of quasi-categories,*

$$\mathcal{C}_{\langle 1 \rangle} = \mathcal{C} \times_{N\mathsf{Fin}_*} \{\langle 1 \rangle\} \xrightarrow{\sim} \mathcal{O}_{\langle 1 \rangle}$$

Then the map $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ induced by f is an equivalence of quasi-categories.

B.3 Dunn Additivity

After stating the Dunn additivity theorem for quasi-categorical little cubes (Theorem 3.2.1), we indicated the functors that are used to elicit the stated equivalence: namely, there is a pair of functors

$$\mathbb{E}_d^{\otimes} \times \mathbb{E}_1^{\otimes} \xrightarrow{\theta'} \mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes} = NW \xrightarrow{\theta} \mathbb{E}_{d+1}$$

where θ' and θ are determined on objects of the underlying topological categories by

$$(\langle n \rangle, \langle m \rangle) \mapsto \left(\underbrace{\langle m \rangle, \ldots, \langle m \rangle}_{\text{n times}} \right) \quad \text{and} \quad (\langle s_1 \rangle, \ldots, \langle s_n \rangle) \mapsto \langle s_1 + \ldots + s_n \rangle$$

respectively. As we pointed out at that time, the concept of weak approximation is critical to the proof of the Dunn additivity theorem. The foundation on which the proof is built is the following

Proposition B.3.1. *For fixed $d \geq 0$, the map $\theta : \mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes} \to \mathbb{E}_{d+1}^{\otimes}$ is a weak approximation.*

Assuming this holds, we have as a corollary

Corollary B.3.2. *The map θ induces a weak equivalence of ∞-preoperads, $\left(\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}, M\right) \xrightarrow{\sim} \mathbb{E}_{d+1}^{\otimes,\natural}$, (where M is the collection of inert morphisms of the wreath product of $\mathbb{E}_d^{\otimes}$ and $\mathbb{E}_1^{\otimes}$).*

Proof. Let θ_0 denote the composition $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes} \xrightarrow{\theta} \mathbb{E}_{d+1}^{\otimes} \to N\mathsf{Fin}_*$. Note that for any object X of $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$ and every inert morphism $\alpha : \theta_0(X) \to \langle n \rangle$ in $N\mathsf{Fin}_*$, there exists a θ_0-coCartesian morphism $\overline{\alpha} : X \to Y$ lifting α.

Note also that the category $\mathcal{E} := \mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes} \times_{N\mathsf{Fin}_*} \{\langle 1 \rangle\}$ has a final object, so since $\left(\mathbb{E}_{k+1}^{\otimes}\right)_{\langle 1 \rangle}$ is a contractible Kan complex, it follows that θ induces a weak homotopy equivalence $\mathcal{E} \xrightarrow{\sim} \left(\mathbb{E}_{k+1}^{\otimes}\right)_{\langle 1 \rangle}$.

Let M_0 be the collection of all θ_0-coCartesian morphisms in $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$ lying over an inert morphism $\langle m \rangle \to \langle 1 \rangle$ in $N\mathsf{Fin}_*$, and let $q : \mathcal{O}'^{\otimes} \to N\mathsf{Fin}_*$ be a quasi-operad. Combining Proposition B.3.1 with Proposition B.2.4, we find that the map θ induces a weak equivalence from $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ to the full subcategory $\mathcal{A}$ of $\mathrm{Fun}_{N\mathsf{Fin}_*}\left(\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}, \mathcal{O}'^{\otimes}\right)$ spanned by those morphisms which send morphisms in M_0 to inert morphisms in $\mathcal{O}'^{\otimes}$ (in a sense, we have made M_0 the inert morphisms of $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$ – viewed as a quasi-category without operadic structure – and we are now defining $\mathcal{A}$ to be the quasi-category of algebra objects in $\mathcal{O}'^{\otimes}$ of the quasi-category $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$, as in Definition B.2.3). The goal now is to show that a functor over $N\mathsf{Fin}_*$, $f : \mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes} \to \mathcal{O}'^{\otimes}$ is an object of $\mathcal{A}$ if and only if it carries inert morphisms of $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$ to inert morphisms of $\mathcal{O}'^{\otimes}$.

First, suppose that f sends inert morphisms of $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$ to inert morphisms of $\mathcal{O}'^{\otimes}$ – to show that f is an element of $\mathcal{A}$, we note that M_0 consists of inert morphisms in $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$, so it is enough to show that f sends an equivalence $E \to E'$ in $\mathcal{E}$ to an equivalence in $\mathcal{O}'^{\otimes}$. By definition, the morphism $E \to E'$ in $\mathcal{E}$ corresponds to a commutative triangle

$$\begin{tikzcd} E \arrow[rr, "\beta"] \arrow[dr, "\phi"'] & & E' \arrow[dl, "\phi'"] \\ & (\langle 1 \rangle) & \end{tikzcd}$$

in $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$. To show that $f(\beta)$ is an equivalence, it is sufficient by the two-out-of-three property to show that $f(\phi)$ and $f(\phi')$ are equivalences in $\mathcal{O}'^{\otimes}$ – but this is immediate from the fact that ϕ and ϕ' are inert, and from the properties of the map f.

Conversely, suppose f is an element of $\mathcal{A}$, and let $\alpha : X \to X'$ be an inert morphism in $\mathbb{E}_d^{\otimes} \wr \mathbb{E}_1^{\otimes}$. By the properties of inert morphisms, to show that f sends inert morphisms to inert morphisms, we only need to consider the case where $X' \in \mathcal{E}$, in which case we can factor α as $\alpha' \circ \alpha''$, where $\alpha'' \in M_0$ and α' is a morphism in $\mathcal{E}$. Since $f \in \mathcal{A}$, we see that $f(\alpha')$ must be an equivalence, while the inertness of α'' implies that $f(\alpha'')$ is inert, from which it follows that $f(\alpha)$ is inert. □

Using this corollary, we can then supply the proof of the quasi-categorical additivity theorem for little cubes:

Proof of Theorem 3.2.1. We work by induction on d. If $d = 0$, then $\mathbb{E}_d^{\otimes}$ is pointed, so the equivalence is clear. If $d = 1$, then for each d' we can factor the map of quasi-preoperads $\mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{d'}^{\otimes,\natural} \to \mathbb{E}_{1+d'}^{\otimes,\natural}$ as a composite

$$\mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{d'}^{\otimes,\natural} \to \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_{d'}^{\otimes}, M\right) \to \mathbb{E}_{1+d'}^{\otimes,\natural}$$

By Theorem B.1.2, the first map is an equivalence of quasi-preoperads, while Corollary B.3.2 tells us that the latter map is also an equivalence. Thus we have the desired result for $d = 1$. For $d > 1$, we have a commutative square of quasi-preoperads:

$$\begin{tikzcd} \mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{d-1}^{\otimes,\natural} \odot \mathbb{E}_{d'}^{\otimes,\natural} \arrow[r] \arrow[d] & \mathbb{E}_d^{\otimes,\natural} \odot \mathbb{E}_{d'}^{\otimes,\natural} \arrow[d] \\ \mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{d-1+d'}^{\otimes,\natural} \arrow[r] & \mathbb{E}_{d+d'}^{\otimes,\natural} \end{tikzcd}$$

By our inductive hypothesis, the left and right vertical arrows are weak equivalences, as is the top horizontal, so by two-out-of-three, the lower horizontal arrow must also be a weak equivalence. □

Thus the proof of the additivity theorem flows from our ability to prove Proposition B.3.1. Since the map $\theta : \mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes} \to \mathbb{E}_{1+d}^{\otimes}$ satisfies both the conditions of Proposition B.2.2 (which supplied us with an alternative characterisation of weak approximation) it suffices to show that for each object $X = (\langle n_1 \rangle, \dots, \langle n_k \rangle)$ of $\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}$ such that $\sum_i n_i = n$, the induced map

$$\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)_{/X} \times_{N\mathsf{Fin}_{*/\langle n \rangle}} \mathrm{Tup}_n \to \left(\mathbb{E}_{1+d}^{\otimes}\right)_{/\langle n \rangle} \times_{N\mathsf{Fin}_{*/\langle n \rangle}} \mathrm{Tup}_n$$

is a weak homotopy equivalence. To prove this we want to work with a slightly simpler category than the one on the left of the above map. Let us define a subcategory $\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/X}$ as follows:

- an object is a morphism $\alpha : (\langle m_1 \rangle, \dots, \langle m_j \rangle) \to X$ such that the underlying maps in Fin_*, $\langle j \rangle \to \langle k \rangle$ and $\langle m_1 + \ldots + m_j \rangle \to \langle n \rangle$ are active morphisms.
- a morphism between objects $\alpha : (\langle m_1 \rangle, \dots, \langle m_j \rangle) \to X$ and $\alpha' : \left(\langle m_1' \rangle, \dots, \langle m_{j'}' \rangle\right) \to X$ is a commutative triangle

$$\begin{tikzcd} (\langle m_1 \rangle, \dots, \langle m_j \rangle) \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & \left(\langle m_1' \rangle, \dots, \langle m_{j'}' \rangle\right) \arrow[dl, "\alpha"] \\ & X & \end{tikzcd}$$

  such that the underlying maps in Fin_* induced by β satisfy
  - $\langle j \rangle \to \langle j' \rangle$ is active in Fin_*;
  - $\langle m_1 + \ldots + m_j \rangle \to \langle m_1' + \ldots + m_{j'}' \rangle$ is a bijection.

Let us also define $\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^1_{/X} \subseteq \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/X}$ to be the full subcategory spanned by those morphisms $\alpha : (\langle m_1\rangle, \ldots, \langle m_j\rangle) \to X$ such that $m_i > 0$ for all i. Then we have a chain of natural subcategory inclusions

$$\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^1_{/X} \hookrightarrow \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/X} \hookrightarrow \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)_{/X} \times_{N\mathsf{Fin}_{*}\ /\langle n\rangle} \mathrm{Tup}_n$$

It can be shown that the first inclusion admits a right adjoint, from which it follows that the two are weak homotopy equivalent; likewise, the inclusion $\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^1_{/X} \hookrightarrow \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)_{/X} \times_{N\mathsf{Fin}_{*}\ /\langle n\rangle} \mathrm{Tup}_n$ also admits a right adjoint, so this map is also a weak homotopy equivalence. By the two-out-of-three property, it follows that the inclusion $\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/X} \hookrightarrow \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)_{/X} \times_{N\mathsf{Fin}_{*}\ /\langle n\rangle} \mathrm{Tup}_n$ must also be a weak homotopy equivalence. Thus, we can prove Proposition B.3.1 by demonstrating that for each $X = (\langle n_1\rangle, \ldots, \langle n_k\rangle)$ such that $n_1 + \ldots + n_k = n$, the induced map

$$\phi_X : \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/X} \to \left(\mathbb{E}_{1+d}^{\otimes}\right)_{/\langle n\rangle} \times_{N\mathsf{Fin}_{*}\ /\langle n\rangle} \mathrm{Tup}_n$$

is a weak homotopy equivalence. Moreover, ϕ_X is equivalent to a product of maps $\left(\phi_{\langle n_i\rangle}\right)_{1\le i\le k}$, so it is only necessary to show that we have the desired weak homotopy equivalence in the case $k = 1$, i.e. we have reduced the proof of the aforementioned proposition to proving the following:

Proposition B.3.3. *The map*

$$\phi_n : \left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/\langle n\rangle} \to \left(\mathbb{E}_{1+d}^{\otimes}\right)_{/\langle n\rangle} \times_{N\mathsf{Fin}_{*}\ /\langle n\rangle} \mathrm{Tup}_n$$

is a weak homotopy equivalence for each $n \geq 0$.

In what follows, we fix n as above. We recall that we defined a topological category W such that $NW = \mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}$: objects of the category W are sequences of objects in Fin_*, $(\langle m_1\rangle, \ldots, \langle m_j\rangle)$; a morphism between two such objects $(\langle m_1\rangle, \ldots, \langle m_j\rangle) \to (\langle m'_1\rangle, \ldots, \langle m'_k\rangle)$ is given by the following data

- a morphism in Fin_*, $\alpha : \langle j\rangle \to \langle k\rangle$
- for each $1 \le i \le k$, an operation $\varepsilon_i \in^t \mathbb{E}_1\left(\alpha^{-1}\{i\}\right)$
- a morphism in Fin_*, $\beta : \langle m\rangle \to \langle m'\rangle$, where $m = m_1 + \ldots m_j$ and $m' = m'_1 + \ldots + m'_k$
- for each $1 \le l \le m'$, an operation $\zeta_l \in^t \mathbb{E}_d\left(\beta^{-1}\{l\}\right)$.

Let us define a topological subcategory $W_0 \subseteq W$ with the same objects, but whose morphisms are given by tuples $(\alpha, (\varepsilon_i)_{1\le i\le k}, \beta, (\zeta)_{1\le l\le m'})$ as above such that the morphism α is active and the morphism β is a bijection. We define a topological functor $T : W_0^{\mathrm{op}} \to \mathcal{S}$ as follows: given an object $X = (\langle m_1\rangle, \ldots, \langle m_j\rangle)$ in W_0, let $T(X) \subseteq \mathrm{Hom}_W(X, (\langle n\rangle))$ be the subspace comprising those maps $X = (\langle m_1\rangle, \ldots, \langle m_j\rangle) \to (\langle n\rangle)$ which induce active maps $\alpha : \langle j\rangle \to \langle 1\rangle$ and $\beta : \langle m_1 + \ldots + m_j\rangle \to \langle n\rangle$ in Fin_*.

Given a topological category, $\mathcal{C}$, we write $\mathrm{Sing}(\mathcal{C})$ to denote the associated simplicial category, whose mapping spaces are given by applying the singular simplices functor to the mapping spaces of $\mathcal{C}$. By the universal property of the adjunction $\mathfrak{C} \dashv N$, there is a canonical map $\nu : \mathfrak{C}[NW_0] \to \mathrm{Sing}(W_0)$, and by applying the unstraightening construction, we obtain a canonical isomorphism of simplicial sets $\left(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_d^{\otimes}\right)^0_{/(\langle n\rangle)} \simeq \mathrm{Un}_\nu \mathrm{Sing}(T)$.

In parallel, we will define a topological subcategory $\mathcal{E}$ of ${}^t\mathbb{E}_{1+d}^{\otimes}$, with a functor $U : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$, so that we can determine a canonical isomorphism of simplicial sets $\mathbb{E}_{1+d\ /\langle n\rangle}^{\otimes} \times_{N\mathsf{Fin}_{*\langle n\rangle}} \mathrm{Tup}_n \simeq \mathrm{Un}_{\nu'} \mathrm{Sing}(U)$, where $\nu' : \mathfrak{C}[N\mathcal{E}] \to \mathrm{Sing}(\mathcal{E})$ is the canonical counit map. The idea for the proof of Proposition B.3.3 will be to relate $\mathrm{Un}_\nu \mathrm{Sing}(T)$ and $\mathrm{Un}_{\nu'} \mathrm{Sing}(U)$. Explicitly, we let $\mathcal{E}$ be the topological subcategory of ${}^t\mathbb{E}_{1+d}^{\otimes}$ whose morphisms are those inducing equivalences in Fin_*, and we define the functor U by $X \mapsto \mathrm{Hom}_{{}^t\mathbb{E}_{1+d}^{\otimes}}(X, \langle n\rangle)$.

The functor $W \xrightarrow{\xi} {}^t\mathbb{E}_{1+d}^{\otimes}$ (where $N\xi = \theta$) restricts to a functor $W_0 \to \mathcal{E}$, in such a way that the functor ϕ_n can be identified with a map $\mathrm{Un}_\nu \mathrm{Sing}(T) \to \mathrm{Un}_{\nu'} \mathrm{Sing}(U)$, coming from a natural transformation $T \to U \circ \xi\ |_{W_0}$. By simplifying T and U it will be possible to prove the stated claim in Proposition B.3.3. The key to this simplification is the following construction...

Construction B.3.4. We construct a poset P as follows:

- An element of P is a pair $(I, \sim)$, where I is a finite collection of subintervals of $(-1, 1)$, and $\sim$ is an equivalence relation on the path components of I, which satisfies a kind of convexity property: writing $[x]$ for the path component of an element x lying inside I, if $x < y < z$ and $[x] \sim [z]$, then $[x] \sim [y] \sim [z]$.

- For two such elements $(I,\sim),(I',\sim')$ in P, we write $(I,\sim)\leq(I',\sim')$ if $I\subseteq I'$ and $[x]\sim'[y]\implies[x]\sim[y]$.

Given an object $w=(\langle m_1\rangle,\ldots,\langle m_b\rangle)$ in W_0, we can identify an element of the space

$$T(w)=\mathrm{Hom}_{\mathsf{F}({}^t\mathbb{E}_1)}\left(\langle b\rangle,\langle 1\rangle\right)\times\prod_{1\leq j\leq b}\mathrm{Hom}_{\mathsf{F}({}^t\mathbb{E}_d)}\left(\langle m_j\rangle,\langle n\rangle\right)$$

with the data of a tuple $(\delta,\varepsilon_1,\ldots,\varepsilon_b)$, where $\delta:\sqcup_b(-1,1)\to(-1,1)$ is a rectilinear embedding, and for each $1\leq j\leq b$, $\varepsilon_j:\sqcup_{m_j}(-1,1)^d\to(-1,1)^d$ is a rectilinear embedding. For each $(I,\sim)$ in P a functor $T_{(I,\sim)}:W_0^{\mathrm{op}}\to\mathcal{S}$ is defined by setting $T_{(I,\sim)}(w)$ to be the subspace of $T(w)$ comprising those elements of $T(w)$ which satisfy:

(i) $\delta(-1,1)\subseteq I$ (which thus induces a map λ from the set $\{1,\ldots,b\}$ to the collection of path components of I).

(ii) Given $1\leq i,j\leq b$ such that $\lambda(i)\sim\lambda(j)$, it is either the case that $i=j$ or the embeddings ε_i and ε_j have disjoint images.

Given an object $\langle m\rangle$ of $\mathcal{E}$, an element of $U(\langle m\rangle)$ is an element of $\mathrm{Hom}_{\mathsf{F}({}^t\mathbb{E}_{1+d})}\left(\langle m\rangle,\langle n\rangle\right)$, i.e. a rectilinear embedding $\gamma:\sqcup_m(-1,1)^{1+d}\to\sqcup_n(-1,1)^{1+d}$. Writing $\sqcup_n(-1,1)^{1+d}\cong\langle n\rangle_o\times(-1,1)^d\times(-1,1)$ (where we write $\langle n\rangle_o$ to emphasise the fact that we are working with the unpointed finite set $\{1,\ldots,n\}$), we obtain two projection maps

$$\langle n\rangle_o\times(-1,1)^d\xleftarrow{p_0}\langle n\rangle_o\times(-1,1)^{1+d}\xrightarrow{p_1}(-1,1)$$

With these notations in place, we can define for each element $(I,\sim)\in P$ a functor $U_{(I,\sim)}$, whose image on $\langle m\rangle$ consists of all the elements of $U(\langle m\rangle)$ which satisfy:

(i) $p_1\circ\gamma(-1,1)^{1+d}\subseteq I$ (thus inducing a map λ' from the set $\langle m\rangle_o$ to the collection of path components of I).

(ii) For $1\leq i,j\leq m$ such that $\lambda'(i)\sim\lambda'(j)$, it is either the case that $i=j$ or that $p_0\circ\gamma\left(\{i\}\times(-1,1)^{1+d}\right)$ and $p_0\circ\gamma\left(\{j\}\times(-1,1)^{1+d}\right)$ are disjoint in $\langle n\rangle_o\times(-1,1)^d$.

It is evident that the functors $T_{(I,\sim)}$ constructed above are built in such a way that they are analogous. Indeed, the point of these constructions is that we can whittle our foregoing observations about Proposition B.3.3 down to showing that

Proposition B.3.5. *[Lur17, Proposition 5.1.2.18] For every element $(I,\sim)$ in P, the functor ξ induces a weak homotopy equivalence* $\mathrm{Un}_\nu\mathrm{Sing}\left(T_{(I,\sim)}\right)\simeq\mathrm{Un}_{\nu'}\mathrm{Sing}\left(U_{(I,\sim)}\right)$

The reason this is possible is because we can essentially build the functors T and U from the sub-functors $T_{I,\sim}$ and $U_{I,\sim}$ via a kind of local-to-global principle. More precisely, we have

Proposition B.3.6. *[Lur17, Proposition 5.1.2.16/5.1.2.17]*

(i) The functor $\mathrm{Sing}(T)$ *is a homotopy colimit of the diagram of functors* $\{T_{(I,\sim)}:\mathrm{Sing}(W_0)^{\mathrm{op}}\to s\mathrm{Set}\}_{(I,\sim)\in P}$.

(ii) The functor $\mathrm{Sing}(U)$ *is a homotopy colimit of the diagram of functors* $\{U_{(I,\sim)}:\mathrm{Sing}(\mathcal{E})^{\mathrm{op}}\to s\mathrm{Set}\}_{(I,\sim)\in P}$.

In case *(i)*, the proof stems from defining a weakly equivalent functor T' whose image on w (as defined above) is given by

$$\mathrm{Conf}_b\left((-1,1)\right)\times\prod_{1\leq j\leq b}\mathrm{Conf}_{m_j}\left(\sqcup_n(-1,1)^d\right)$$

and using the fact that the functor T' will satisfy the corresponding local-to-global principle (which in turn is deduced using a generalised version of the Seifert-van Kampen Theorem). Likewise, for case *(ii)*, we again define a functor U' which is weakly equivalent to U and whose image on $\langle m\rangle$ is given by $\mathrm{Conf}_m\left(\sqcup_n(-1,1)^{1+d}\right)$, and again asserting that the corresponding local-to-global principle holds for the functor U'.

Proof of B.3.5. We construct a functor $T^0_{(I,\sim)}:W_0^{\mathrm{op}}\to\mathcal{S}$ which associates to an object $w=(\langle m_1\rangle,\ldots,\langle m_b\rangle)$ the space of rectilinear embeddings from $\sqcup_b(-1,1)$ into I, $\mathrm{Rect}\left(\sqcup_b(-1,1),I\right)$. The natural transformation of topological functors $T_{(I,\sim)}\to T^0_{(I,\sim)}$ induces a fibration of simplicial functors $\mathrm{Sing}\left(T_{(I,\sim)}\right)\to\mathrm{Sing}\left(T^0_{(I,\sim)}\right)$, which in turn leads to a right fibration $\mathrm{Un}_\nu\mathrm{Sing}\left(T_{(I,\sim)}\right)\to\mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)$.

An object of $\mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)$ is a pair $(w = (\langle m_1\rangle, \ldots, \langle m_b\rangle), f)$, where $f \in \mathrm{Rect}(\sqcup_b(-1,1), I)$. Let $\mathcal{X}$ be the subcategory of $\mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)$ consisting of those pairs (w, f) such that the map f induces a bijection $\langle b\rangle_o \to \pi_0 I$. We note that $\mathcal{X}$ is weakly equivalent to the Kan complex, $\mathrm{Tup}_{\pi_0 I}$. Since the inclusion $\mathcal{X} \hookrightarrow \mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)$ admits a left adjoint, it is a weak homotopy equivalence, and hence by base-change, we also have a weak equivalence

$$\mathcal{X} \times_{\mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)} \mathrm{Un}_\nu\mathrm{Sing}\left(T_{(I,\sim)}\right) \simeq \mathrm{Un}_\nu\mathrm{Sing}\left(T_{(I,\sim)}\right)$$

We can also define $U^0_{(I,\sim)} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ by $\langle m\rangle \mapsto \mathrm{Hom}_{\mathsf{Fin}}(\langle m\rangle, \pi_0 I)$ (viewed as a discrete space). Note that $\mathrm{Un}_{\nu'}\mathrm{Sing}\left(U^0_{(I,\sim)}\right)$ is also weak homotopy equivalent to $\mathrm{Tup}_{\pi_0 I}$. In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathrm{Un}_\nu\mathrm{Sing}\left(T^0_{(I,\sim)}\right)} \mathrm{Un}_\nu\mathrm{Sing}\left(T_{(I,\sim)}\right) \arrow[r, "F"] \arrow[d] & \mathrm{Un}_{\nu'}\mathrm{Sing}\left(U_{(I,\sim)}\right) \arrow[d] \\
\mathcal{X} \arrow[r, "\simeq"] & \mathrm{Un}_{\nu'}\mathrm{Sing}\left(U^0_{(I,\sim)}\right)
\end{tikzcd}$$

Since both vertical arrows are right fibrations, to show that F is a weak homotopy equivalence (and hence that we have the weak homotopy equivalence of the statement), it is only necessary to show that F induces a weak equivalence on the fibres. Explicitly, given

- an object $w = (\langle m_1\rangle, \ldots, \langle m_b\rangle)$ with image $F(w) = \langle m\rangle$, where $m = m_1 + \ldots + m_b$; and
- $\eta \in \mathrm{Rect}(\sqcup_b(-1,1), I) = T^0_{(I,\sim)}(w)$ such that η induces a bijection between $\langle b\rangle_o$ and $\pi_0 I$, with image $F(\eta) = \eta' \in \mathrm{Hom}_{\mathsf{Fin}}(\langle m\rangle, \pi_0 I)$

we wish to show that the map induced by F

$$\Phi : \mathrm{Sing}\left(T_{(I,\sim)}(w) \times_{T^0_{(I,\sim)}(w)} \{\eta\}\right) \to \mathrm{Sing}\left(U_{(I,\sim)}(w) \times_{U^0_{(I,\sim)}(w)} \{\eta'\}\right)$$

is a weak homotopy equivalence. However, Φ is the product of an identity map with an inclusion into $\prod_{1\le j\le m} \mathrm{Rect}((-1,1), I_j)$, where I_j is the image of j under the map $\eta' : \langle m\rangle_o \to \pi_0 I$. Since each of the spaces $\mathrm{Rect}((-1,1), I_j)$ is contractible, the claim follows. $\square$

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