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the Cup Product's Applications To Ebisu's Method

Master Thesis

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Mathematics

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Abstract

In [Ebi17], Akihito Ebisu found a remarkable new method to derive certain identities for hypergeometric functions, those identities are known as special values. Ebisu's method involves the use of so-called contiguity relations between hypergeometric functions. In this thesis, we study Ebisu's method and give an alternative approach using cup product relations. This alternative approach gives rise to a more insightful way to do the calculations. With these new insights, we try to answer questions, such as how many special values there exist and what there is beyond the values Ebisu applied his method to in [Ebi17].

Contents

1 Introduction

Hypergeometric functions is a subject that many well-known mathematicians have made a contribution to: Euler, Gauss, Riemann and Kummer, to name a few. Many of the non-elementary functions found in both mathematics and physics can be represented as hypergeometric functions, thus giving an interesting way to look at these functions. The generalized form of a hypergeometric functions is

$$ {}_pF_q(a_1, .., a_p; b_1, ..., b_q; z) := \sum_{m=0}^{\infty} \frac{(a_1)_m...(a_p)_m}{m!(b_1)_m...(b_q)_m} z^m $$

where $(a)_m = a(a+1)...(a+m-1)$. Here are some interesting and useful examples of hypergeometric functions

$$ \begin{aligned} (1-z)^{-a} &= {}_1F_0(a; -; z), \\ \log(1+z) &= z{}_2F_1(1, 1; 2; -z), \\ \tan^{-1}(z) &= z{}_2F_1(1/2, 1; 3/2; -z^2), \\ \sin^{-1}(z) &= z{}_2F_1(1/2, 1/2; 3/2; z^2), \\ \sin(z) &= z{}_0F_1(-; 3/2; -z^2/4), \\ e^z &= {}_0F_0(-; -; z). \end{aligned} $$

Due to how useful they are, many methods have been developed for finding new identities for hypergeometric functions. With the introduction and increased use of computers in the field, the last few decades have seen a rise in algorithms for obtaining new identities. Some examples are the Gosper's algorithm, Zeilberger's algorithm and the Wilf-Zeilberger method (for more information, one might want to take a look at [Koe14] and [EKH04]).
Another recent method for finding identities for hypergeometric functions Ebisu's method from [Ebi17]. Ebisu's method focuses on the Gauss hypergeometric function

$$ F(a, b; c; z) := {}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m(b_m)}{m!(c)_m} z^m, $$

which is also known as *the* hypergeometric function. This is the most well studied hypergeometric function. The identities found with Ebisu's method are what we call special values. A special value is an equality of the following form:
The LHS is a hypergeometric function. Which has the properties that z is a set value in $\mathbb{C}$ and the parameters are multivariate rational functions in $\mathbb{C}$ and/or $\mathbb{Z}_{\geq 0}$.
The RHS is a multivariate function in also $\mathbb{C}$ and/or $\mathbb{Z}_{\geq 0}$. This function is an exponential function combined with Pochhammer symbols (for a definition see 2.3) and, if interpolated, also gamma functions.
Here are some examples with $a, b, c \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$

$$ \begin{aligned} F(a, -n; c; 1) &= \frac{(c-a)_n}{(c)_n} && \text{(Chu-Vandermonde equality, corollary 2.2.3 of [AAR99])}, \\ F(a, b, 1-a+b; -1) &= \frac{\Gamma(1-a+b)\Gamma(1+1/2b)}{\Gamma(1+b)\Gamma(1-a+1/2b)} && \text{(Kummer's theorem, paragraph 2.3 of [Bai35])}, \\ F(a, b; c; 1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} && \text{(Gauss' summation formula, theorem 2.2.2 of [AAR99])}. \end{aligned} $$

While these are already well-known special values, Ebisu's method appears to give both all known and yet unknown special values. Here are some special values which were discovered with Ebisu's method

$$ \begin{aligned} F(a, 1-a; 2a; 1/2 + 1/2i\sqrt{3}) &= \frac{2^{2a-2/3}e^{1/6i(1-a)\pi}\Gamma(2/3)\Gamma(a+1/2)}{3^{3/2a-1/2}\Gamma(5/6)\Gamma(a+1/3)} && \text{((1,3,2-1)(ii) of [Ebi17])}, \\ F(-1-n, -2/3-n; -3-3n; 9/8) &= \frac{3(3/2)_n}{2^{2n+2}(4/3)_n} && \text{((1,2,3-1)(iii) of [Ebi17])}. \end{aligned} $$

Ebisu's method iterates over $(k, l, m) \in \mathbb{Z}^3$ in order try to obtain special values. Getting a special value or not depends on the following step of Ebisu's method, which is also the main focus of this thesis. Assume we have $(k, l, m) \in \mathbb{Z}^3$ given. Let $Q^{(n)}_{k,l,m}, R^{(n)}_{k,l,m} \in C(a, b, c, n, z)$ be rational functions such that we get the following contiguity relation

$$\begin{aligned}
&F(a + kn, b + ln; c + mn)\\
&= Q^{(n)}_{k,l,m} F'(a + k(n-1), b + l(n-1); c + m(n-1); z)\\
&+ R^{(n)}_{k,l,m} F(a + k(n-1), b + l(n-1); c + m(n-1); z).
\end{aligned}$$

Find $(a, b, c, z) \in (\mathbb{Q}[a, b, c])^3 \times \mathbb{Q}$ such that $Q^{(n)}_{k,l,m} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$.
As an example take $(k, l, m) = (0, 1, 1)$, then

$$F(a, b + n; c + n) = \frac{(c + n - 1)(1 - z)}{(b + n - 1)(a - c - n + 1)} F'(a, b; c; z) + \frac{-c - n + 1}{a - c - n + 1} F(a, b; c; z).$$

Here we can pick $(a, b, c, z) = (a, b, c, 1)$ to make $Q^{(n)}_{2,0,1}$ equal to 0 for all n. Important here is the "for all n" part, so we need to make certain that all coefficients of n in the numerator of $Q^{(n)}_{2,0,1}$ become 0 meaning we need to solve 2 equations in the example $1 - z = 0$ and $(c - 1)(1 - z) = 0$. When we take different values for k, l and m the degree of n in the numerator changes, thus the same goes for the number of equations. As we only have 4 parameters/variables to solve all these equations, we start getting to our main question.
How many special values do there exist?

1.1 Structure of the Thesis

After this chapter, the preliminaries, in which we will not only be introducing some necessary functions, but also take our time to give a proper introduction to hypergeometric functions and the many ways to approach them.
Chapter 3 focuses on fully explaining Ebisu's method is. In addition to that, we will make some changes to improve Ebisu's method, as well as giving an example.
Chapter 4 will introduce the cup product from [Beu18]. We will show how the cup product can be made applicable to Ebisu's method and that it also gives stronger relations than contiguity relations. This chapter also contains an alternative proof to one of the theorems which Ebisu uses, but now based on the cup product.
In Chapter 5 we go over observations we made about $Q^{(n)}$ based on different (k, l, m). This chapter contains some claims about the degree in n and z.
Chapters 6 and 7 go on to prove some of the previously made claims concerning the degree in n and z.
Chapter 8 goes back to our initial question about the quantity of special values. To properly address this question, we try to relate this question to the claims we had previously proven. Other than this, we also look at some of the special values not included in [Ebi17].

2 Preliminaries

In this chapter we will mainly focus on introducing hypergeometric functions, but before we get into that some necessary functions.

2.1 Gamma Function

The gamma function $\Gamma(z)$ was discovered by Euler when he extended the domain of the factorial function. For a positive integer n Euler takes

$$\Gamma(n) := (n-1)!.$$

The gamma function is a meromorphic function of the form

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx \quad \text{for } \Re(z) > 0$$

By using integration by parts it shows one of its defining features, that is

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty x^z e^{-x}dx \\
&= \left[-x^z e^{-x}\right]_0^\infty - \int_0^\infty -zx^{z-1}e^{-x}dx \\
&= \lim_{x\to\infty}(-x^z e^{-x}) + 0^z e^0 + z\int_0^\infty x^{z-1}e^{-x}dx \\
&= z\Gamma(z)
\end{aligned}$$

Using $\Gamma(z+1) = z\Gamma(z)$ we can create an analytic continuation for this function, this way we can also start evaluating the function for when $\Re(z) \leq 0$. From this, it also becomes apparent that $\Gamma(z)$ has singularities at all non-positive integers, since $\Gamma(-n) = \frac{\Gamma(1)}{(-n)(-n+1)\ldots 0}$.
A well-known approximation of the Gamma function is Stirling's asymptotic Formula

Lemma 2.1.1 (Stirling's asymptotic Formula, Theorem 1.4.1 of [AAR99])**.**

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-1/2}e^{-z} \qquad \textit{as } \Re z \to \infty$$

2.2 Beta Function

Beta function $B(z)$ also known as the Euler function is defined as

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

with $\Re a, \Re b > 0$. A useful property of this function is its relation to the gamma function, giving us

Lemma 2.2.1 (Theorem 1.1.4 of [AAR99])**.** *For* $a, b \in \mathbb{C}$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

2.3 Pochhammer Symbol

For the case that n is a positive integer, the Pochhammer symbol can be defined as

$$(a)_n := a(a+1)\ldots(a+n-1).$$

This definition can be further generalized to

$$(a)_z := \frac{\Gamma(a+z)}{\Gamma(a)}$$

which is true for $z \in \mathbb{C}$. This means we can also give a value for negative integers

$$(a)_{-n} = \frac{1}{(a-n)_n}.$$

Further

$$(-a)_n = (-1)^n (a-n+1)_n.$$

We can use unsigned Stirling numbers of the first kind to expand the case that n is a positive integer, giving us

$$(a)_n = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} a^i.$$

Another way to turn the Pochhammer symbol into a summation is

Lemma 2.3.1.

$$\sum_{i=0}^{k} \frac{(-1)^i}{(a+i)i!(k-i)!} = \frac{1}{(a)_{k+1}}$$

for $k \in \mathbb{Z}_{\geq 0}$

Proof. Fractional decomposition. □

2.4 Hypergeometric Function

Here we will give an introduction to the hypergeometric function, this is in part inspired by [AAR99]. A hypergeometric function is originally defined as a power series $\sum_{m=0} c_m z^m$ where $\frac{c_{m+1}}{c_m}$ is a rational function in m and $c_0 = 1$. The general form of this is

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) := \sum_{m=0}^{\infty} \frac{(a_1)_m ... (a_p)_m}{(b_1)_m ... (b_q)_m m!} z^m$$

where $a_i, z \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ (this definition includes the $m!$ as the hypergeometric function can be seen as special case of bilateral hypergeometric series, next to that $m! = (1)_m$ is already naturally occurring in most cases).
A question we might ask is for which z this series converges.

Theorem 2.4.1. *The series* ${}_pF_q$ *converges absolutely for all* z *if* $p \leq q$ *and for* $|z| < 1$ *if* $p = q+1$. *For* $p > q+1$ *it diverges when* $z \neq 0$.

Proof. In the case that $p \leq q$, then $\lim_{n\to\infty} c_{m+1}/c_m z = 0$. For $p = q+1$, $\lim_{m\to\infty} |c_{m+1}/c_m z| = |z|$ which means that the series converges absolutely if smaller than 1. If $p > q+1$ and $z \neq 0$, then $\lim_{m\to\infty} |c_{m+1}/c_m z|$ does not exist. □

Of all hypergeometric functions that exist, we are mainly interested in ${}_2F_1$. This is also the most studied form of the hypergeometric function. ${}_2F_1$ is known as Gauss' hypergeometric function and as *the* hypergeometric function, hence from now on when we talk about hypergeometric functions, it will be about this form. This form also has its own shorthands

$$F := F(a, b; c; z) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m.$$

Now onto some of different ways to describe this function. One can easily verify that F satisfies the following differential equation

$$z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0.$$

This differential equation is know as the hypergeometric differential equation. By viewing the hypergeometric function as a solution to this equation, we can make the following extension. If c is a negative integer $-n$ and a or b is in $-i \in \{-n, ..., -1, 0\}$, then

$$F(a, -i; -n; z) := \sum_{m=0}^{i} \frac{(a)_m(-i)_m}{(-n)_m m!} \tag{1}$$

which makes use of that the top term is first to become 0. This extension to the definition does have some caveats as it is not continuous in parameter c. Later more about this when it comes into play.
Next to $F(a, b; c; z)$ the hypergeometric differential equation also has other solutions from which we will later determine some useful relations.

Lemma 2.4.2. *Kummer's 24 solutions to the hypergeometric differential equation.*

$$\begin{aligned}
y_1(a,b,c,z) : &= F(a,b;c;z) && (2)\\
&= (1-z)^{c-a-b}F(c-a,c-b;c;z) && (3)\\
&= (1-z)^{-a}F(a,c-b;c;z/(z-1)) && (4)\\
&= (1-z)^{-b}F(c-a,b;c;z/(z-1)), && (5)\\
y_2(a,b,c,z) : &= F(a,b;a+b+1-c;1-z) && (6)\\
&= z^{1-c}F(a+1-c,b+1-c;a+b+1-c;1-z) && (7)\\
&= z^{-a}F(a,a+1-c;a+b+1-c;1-z^{-1}) && (8)\\
&= z^{-b}F(b+1-c,b;a+b+1-c;1-z^{-1}), && (9)\\
y_3(a,b,c,z) : &= z^{-a}F(a,a+1-c;a+1-b;1/z) && (10)\\
&= (-1)^a(-z)^{b-c}(1-z)^{c-a-b}F(1-b,c-b;a+1-b;1/z) && (11)\\
&= (-1)^a(1-z)^{-a}F(a,c-b;a+1-b;(1-z)^{-1}) && (12)\\
&= (-1)^a(-z)^{1-c}(1-z)^{c-a-1}F(a+1-c,1-b;a+1-b;(1-z)^{-1}), && (13)\\
y_4(a,b,c,z) : &= z^{-b}F(b+1-c,b;b+1-a;1/z) && (14)\\
&= (-1)^b(-z)^{a-c}(1-z)^{c-a-b}F(1-a,c-a;b+1-a;1/z) && (15)\\
&= (-1)^b(1-z)^{-b}F(b,c-a;b+1-a;(1-z)^{-1}) && (16)\\
&= (-1)^b(-z)^{1-c}(1-z)^{c-b-1}F(b+1-c,1-a;b+1-a;(1-z)^{-1}), && (17)\\
y_5(a,b,c,z) : &= z^{1-c}F(a+1-c,b+1-c;2-c;z) && (18)\\
&= z^{1-c}(1-c)^{c-a-b}F(1-a,1-b;2-c;z) && (19)\\
&= z^{1-c}(1-c)^{c-a-1}F(a+1-c,1-b;2-c;z/(z-1)) && (20)\\
&= z^{1-c}(1-c)^{c-b-1}F(b+1-c,1-a;2-c;z/(z-1)). && (21)\\
y_6(a,b,c,z) : &= (1-z)^{c-a-b}F(c-a,c-b;c+1-a-b;1-z) && (22)\\
&= z^{1-c}(1-z)^{c-a-b}F(1-a,1-b;c+1-a-b;1-z) && (23)\\
&= z^{a-c}(1-z)^{c-a-b}F(c-a,1-a;c+1-a-b;1-z^{-1}) && (24)\\
&= z^{b-c}(1-z)^{c-a-b}F(c-b,1-b;c+1-a-b;1-z^{-1}) && (25)
\end{aligned}$$

Proof. Let $f(z)$, $g(z)$ and $h(z)$ be polynomials and consider the following equation

$$f(z)y'' + g(z)y' + h(z)y = 0$$

suppose it has these three singularities α, β, γ which are regular[1] and $a_1, a_2; b_1, b_2; c_1, c_2$ the singularities'

[1]a point α is a regular singularity when next to being a singular point ($f(\alpha) = 0$ and either $g(\alpha)$ or $g(\alpha)$ is not 0) also $\lim_{z\to\alpha}(z-\alpha)g(z)/f(z)$ and $\lim_{z\to\alpha}(z-\alpha)^2 h(z)/f(z)$ exist

respective local exponents. Riemann denoted the set of solutions to such an equation as

$$P\begin{Bmatrix} \alpha & \beta & \gamma & \\ a_1 & b_1 & c_1 & z \\ a_2 & b_2 & c_2 & \end{Bmatrix}.$$

On this several operations can be performed while keeping the solutions the same. First of all, we can use conformal mappings of the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Such a map has the following form

$$t = \frac{\lambda z + \mu}{\delta z + \upsilon}$$

where $\lambda\upsilon - \mu\delta = 1$. Now using a conformal mapping which sends $\{\alpha, \beta, \gamma\}$ to $\{\alpha_1, \beta_1, \gamma_1\}$, we get

$$P\begin{Bmatrix} \alpha & \beta & \gamma & \\ a_1 & b_1 & c_1 & z \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P\begin{Bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \\ a_1 & b_1 & c_1 & t \\ a_2 & b_2 & c_2 & \end{Bmatrix}.$$

The other operation we can apply is multiplication in a singularity such as multiplying by $(z-\alpha)^\mu$. This then increases the local exponents in that singularity. This is because the local exponent of a singularity α is the power to which $(z-\alpha)$ can raised such that multiplied by the base solution at that singularity it gives an actual solution. What this does to Riemann's notation is

$$(z-\alpha)^\mu P\begin{Bmatrix} \alpha & \beta & \infty & \\ a_1 & b_1 & c_1 & z \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P\begin{Bmatrix} \alpha & \beta & \infty & \\ a_1+\mu & b_1 & c_1-\mu & z \\ a_2+\mu & b_2 & c_2-\mu & \end{Bmatrix}$$

(for demonstrative purposes we took $\gamma = \infty$).
Now onto solving the hypergeometric differential equation, which has the following set of solutions

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}. \tag{26}$$

For this set we already know that $F(a,b;c;z)$ is a solution. Hence if we use the operations discussed earlier to create a set with the same solutions and also the same singularities (and local exponent 0 for singularities 0 and 1), then we can derive a new solution by viewing set in (26) to $F(a,b;c;z)$ as a map which gives a solution.

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix} \mapsto F(a,b;c;z) \tag{27}$$

Let us start with the conformal mappings. As we wish to afterwards have the same singularities, all our conformal mappings will be permutations of $\{0, 1, \infty\}$. Hence the only mappings we have are

$$z \mapsto z, 1-z, 1/z, 1/(1-z), 1-z^{-1}, z/(1-z).$$

As an example $z \mapsto 1-z$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix} = P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & 1-z \\ c-a-b & b & 1-c & \end{Bmatrix}$$

giving us the solution $F(a,b;1-(c-a-b);1-z) = F(a,b;a+b+1-c;1-z)$.
Now multiplication, here we multiply by $z^\mu(1-z)^\lambda$ giving us

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ a_1 & b_1 & c_1 & z \\ a_2 & b_2 & c_2 & \end{Bmatrix} = z^\mu(1-z)^\lambda P\begin{Bmatrix} 0 & \infty & 1 & \\ a_1-\mu & b_1+\mu+\lambda & c_1-\lambda & z \\ a_2-\mu & b_2+\mu+\lambda & c_2-\lambda & \end{Bmatrix}$$

As an example of this, multiplying by $(1-z)^{c-a-b}$

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\} = (1-z)^{c-a-b} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & c-b & a+b-c & z \\ 1-c & c-a & 0 & \end{matrix}\right\}$$
$$= (1-z)^{c-a-b} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & c-a & 0 & z \\ 1-c & c-b & a+b-c & \end{matrix}\right\}$$

Note that local exponents do not have any particular order, hence can be switched. This gives us the solution $(1-z)^{c-a-b}F(c-a,c-b;c;z)$.
By doing these described operations we can obtain all of Kummer's solutions. It remains to prove that they occur in groups of four such as (2), (3), (4), (5) in which the solutions are equal. As an example we look at the functions about the singularity $z=0$. As there are only two local exponents for that singularity, there should also only be 2 hypergeometric solutions to the differential equation for that singularity. The first being $F(a,b;c;z)$, which is analytic in $z=0$. The other should be z^λ (with λ not in $\mathbb{Z}$) times an analytic function at $z=0$. These two different solutions are $F(a,b;c;z)$ and $z^{1-c}F(a+1-c;b+1-c;2-c;z)$. $(1-z)^{c-a-b}F(c-a,c-b;c;z)$ is an analytic function at $z=0$, which means $F(a,b,c,z)=k(1-z)^{c-a-b}F(c-a,c-b;c;z)$ with $k\in\mathbb{C}[a,b,c]$. To determine k we can simply look at the first term of their power series, which is in both 1 meaning $F(a,b,c,z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$. By repeating this process we can find all the above equalities. $\square$

At the moment the domain of convergence of $F(a,b;c;z)$ is limited to $|z|<1$. We are going to make an analytic continuation so that it is defined for all $z\in\mathbb{C}\setminus\mathbb{R}_{\geq 1}$.

Theorem 2.4.3 (Euler's Integral Representation). *If $\Re c > \Re b > 0$, then*

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

for $z\in\mathbb{C}\setminus\mathbb{R}_{\geq 1}$. Here we take $\arg(x)=\arg(1-x)=0$.

Proof. First assume that $|z|<1$. Expanding $(1-zx)^{-a}$ using the binomial theorem gives us

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(zx)^n dx$$
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n\int_0^1 x^{n+b-1}(1-x)^{c-b-1}dx$$

Here the switch of summation and integration is possible since $\frac{(a)_n}{n!}z^n x^{n+b-1}(1-x)^{c-b-1}$ converges uniformly for $|z|<1$ and $x\in[0,1]$. When $\Re(c-b),\Re b>0$ the above integral is equal to the beta function in b and $c-b$, meaning we can apply lemma 2.2.1

$$\int_0^1 x^{n+b-1}(1-x)^{c-b-1}dx = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$

resulting in

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n$$
$$= F(a,b;c;z).$$

The integral in the theorem is analytic as long as we do not have any singularities on the line $(0,1)$. This means $1/z\notin[0,1]$ (meaning $z\in\mathbb{C}\setminus\mathbb{R}_{\geq 1}$). Due it being analytic the theorem does not only hold for $|z|<1$, but also for the entire cut plane. $\square$

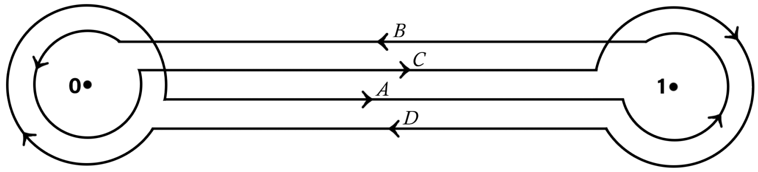

Figure 1: the Pochhammer contour

Interesting about this analytic continuation is that even though it is no longer obvious, it still is symmetric in a and b.
To construct a continuation for when $\Re c > \Re b > 0$ does not hold, we can use the Pochhammer contour.

Theorem 2.4.4. *For $b, c-b \notin \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 1}$ we have*

$$F(a,b;c;z) = \frac{1}{(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_P x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

with P being the Pochhammer contour shown in Figure 1.

Proof. Looking at the Pochhammer contour, there are a few things to note. First of all, since $z \notin \mathbb{R}_{\geq 1}$, the singularity $1/z$ will not be on $[0,1]$, hence our contour will not travel around that singularity. Secondly that our interval does go around singularities 0 and 1.
This last observation means that $x^{b-1} = e^{\log(x)(b-1)}$ and $(1-x)^{c-b-1} = e^{\log(1-x)(c-b-1)}$ are no longer continuous if we use log with any kind of cut (this is due the imaginary part of log giving the angle to the x-axis, so if we go full circle around a singularity, the angle should increase with 2π, but this contradicts log giving a value solely based on which point in $\mathbb{C}$). To resolve this problem, we define the imaginary part of log as arg which we will define as the angle plus rotations (with a counter-clockwise rotation being 2π). This makes for a continuous function as the contour goes both directions around the singularities, hence returning arg to its starting value.
We now still need to make the function single-valued as the value of the integral is depended on what the arg value is on its starting point. For this we pick a starting point on the line A, where we set $\arg(x) = 0$ and $\arg(1-x) = 0$.
We can now evaluate the interval. For this we will first assume $\Re b, \Re(c-b) > 0$. Looking at the integral, it can be split up into 8 curves. 4 lines between r and $1-r$ with $r \in (0, 1/2)$ and 4 circles with radius r. The integral for those circle, will go to 0 as $r \to 0$, to demonstrate this here the counter-clockwise integral around 1, we will call this path γ_r.

$$\begin{aligned}
&\left|\int_{\gamma_r} x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx\right| \\
&= \lim_{r\to 0}\left|\int_{\gamma_r} x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx\right| \\
&= \lim_{r\to 0}\left|\int_0^{2\pi} (1-re^{it})^{b-1}(re^{it})^{c-b-1}(1-z(1-re^{it}))^{-a}\left[\frac{d}{dt}(1-re^{it})\right]dt\right| \\
&= \lim_{r\to 0} r^{c-b}\left|\int_0^{2\pi} (1-re^{it})^{b-1}(e^{it})^{c-b}(1-z(1-re^{it}))^{-a}dt\right| \\
&\text{(for a small enough } r \text{ there should exist } k\text{, such that } |(1-re^{it})^{b-1}(1-z(1-re^{it}))^{-a}| < k) \\
&\leq \lim_{r\to 0} r^{c-b}2\pi k \times \max(e^{-2\pi\Im(c-b)}, 1) \\
&= 0 \quad \text{(follows from that } \Re(c-b) > 0)
\end{aligned}$$

Hence the integral for the path γ_r is 0.
Now to evaluate the line integrals. As r goes to 0, we get integrals from 0 to 1 and the opposite direction. As mentioned earlier, we will start on line A, for this the integral will simply be $\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$.

After going around the singularity 1 once, we will get that $\arg(1-x) = 2\pi$ and $\arg x = 0$ on line B. This means that the integral for line B will be

$$\int_1^0 e^{\log(x)(b-1)} e^{(\log(1-x)+2\pi i)(c-b-1)} e^{-\log(1-zx)a} dx = e^{2\pi i(c-b-1)} \int_1^0 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx$$
$$= -e^{2\pi i(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx$$

By continuing this process we will eventually return back at line A. At this point due to going around 0 and 1 in both directions $\arg(1-x)$ and $\arg(x)$ will again be 0, again confirming that this is still a closed path. The statement that follows from this is that

$$\begin{aligned}\int_P &= (1 - e^{2\pi i(c-b)} + e^{2\pi i(c-b)}e^{2\pi ib} - e^{2\pi i(c-b)}e^{2\pi ib}e^{-2\pi i(c-b)}) \int_0^1 \\ &= (1-e^{2\pi i(c-b)})(1-e^{2\pi ib}) \int_0^1\end{aligned}$$

meaning

$$\begin{aligned}&\frac{1}{(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_P x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx \\ &= F(a,b;c;z).\end{aligned}$$

Since the above integrals are both analytic functions in b and $c-b$, it follows that we now have an analytic continuation which holds for all values of $\Re(c-b)$ and $\Re b$.

□

One of the values we still do not have is that of $z = 1$, for this there exists Gauss' summation formula. For a change we will go onto give a proof which does not use integrals (even though it can easily be done using integrals as when $z = 1$ Euler's integral will turn into the beta function).

Theorem 2.4.5 (Gauss' summation formula)**.** *For* $\Re(c-a-b) > 0$*, we have*

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Proof. The first step of this proof is to show that

$$F(a,b;c;1) = \frac{(c-a)(c-b)}{c(c-a-b)} F(a,b;c+1;1). \tag{28}$$

Define

$$A_n = \frac{(a)_n(b)_n}{n!(c)_n} \quad \text{and} \quad B_n = \frac{(a)_n(b)_n}{n!(c+1)_n}$$

with those being the respective coefficients of the above hypergeometric functions, then

$$(c-a)(c-b)B_n - c(c-a-b)A_n = \frac{(a)_n(b)_n}{n!(c+1)_{n-1}} \left(\frac{(c-a-b)n - ab}{c+n} \right)$$

and

$$c((n+1)A_{n+1} - nA_n) = \frac{(a)_n(b)_n}{n!(c+1)_{n-1}} \left(\frac{(c-a-b)n - ab}{c+n} \right).$$

This together means that

$$c(c-a-b)A_n = (c-a)(c-b)B_n + c(nA_n - (n+1)A_{n+1})$$

and

$$c(c-a-b)\sum_{n=0}^{N} A_n = (c-a)(c-b)\sum_{n=0}^{N} B_n - c(N+1)A_{N+1}.$$

Stirling's asymptotic Formula (Lemma 2.1.1) gives us that

$$(N+1)A_{N+1} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+N)\Gamma(b+N)}{\Gamma(N)\Gamma(c+N)} \sim \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{N^{c-a-b}} \text{ as } N \to \infty.$$

Since $\Re(c-a-b) > 0$, we get that the above goes to 0 when $N \to \infty$, hence proving (28). By iterating this relation n times we get

$$\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}F(a,b;c;1) = \frac{\Gamma(c-a+n)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-a-b+n)}F(a,b;c+n;1).$$

When n goes to ∞ then $\frac{\Gamma(c-a+n)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-a-b+n)} \sim \frac{1}{n^0}$ and Raabe's test shows that $F(a,b;c+n;1)$ converges for every high enough n. To show that $F(a,b;c+n;1)$ converges to 1, we do the following

$$\begin{aligned}|F(a,b;c+n;1)| &\leq 1 + \left|\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{m!(c+n)_m}\right| \\ &= 1 + \left|\frac{ab}{1(c+n)}F(a+1,b+1;c+n+1;1)\right|\end{aligned}$$

As we know that $|F(a+1,b+1;c+n+1;1)|$ converges for all large enough n we can now point out that there must exist an $N \in \mathbb{N}$ and $k \in \mathbb{R}_{\geq 0}$ such that for all $n > N$ we have $|F(a+1,b+1;c+n+1;1)| \leq k$, meaning that

$$\begin{aligned}\lim_{n\to\infty} F(a,b;c+n;1) &= \lim_{n\to\infty} 1 + \frac{ab}{c+n}k \\ &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}F(a,b;c;1) &= \lim_{n\to\infty}\frac{\Gamma(c-a+n)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-a-b+n)}F(a,b;c+n;1) \\ &= 1\end{aligned}$$

proving the theorem. □

2.5 Contiguity Relations

Two hypergeometric functions $F(a_1,b_1;c_1;z)$ and $F(a_2,b_2;c_2,z)$ are contiguous to each other when pairwise all parameters differ by an integer value meaning $a_1-a_2, b_1-b_2, c_1-c_2 \in \mathbb{Z}$. Gauss proved that if you have three contiguous hypergeometric functions there exists a unique linear relation over $\mathbb{Q}(a,b,c,z)$. Such a relation is called a contiguity relation.
In this thesis will be applying contiguity relations a lot and will mainly focus on contiguity relations of the following form

$$F(a+k,b+l;c+m;z) = Q_{k,l,m}F' + R_{k,l,m}(z)F$$

with $(k,l,m) \in \mathbb{Z}^3$ and $Q_{k,l,m}(z), R_{k,l,m}(z) \in \mathbb{Q}(a,b,c,z)$ (in general we write Q and R as short hands). The above is a contiguity relation, since $F'(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z)$.
We can calculate these relations inductively by using the following relations

$$F(a+1,b;c;z) = \frac{z}{a}F' + F,$$

$$F(a-1,b;c;z) = \frac{z(1-z)}{c-a}F' - \frac{a-c+bx}{c-a}F,$$
$$F(a,b+1;c;z) = \frac{z}{b}F' + F,$$
$$F(a,b-1;c;z) = \frac{z(1-z)}{c-b}F' - \frac{b-c+ax}{c-b}F,$$
$$F(a,b;c+1,z) = \frac{c(1-z)}{(c-a)(c-b)}F' - \frac{c(a+b-c)}{(c-a)(c-b)}F \text{ and}$$
$$F(a,b;c-1;z) = \frac{z}{c-1}F' + F.$$

We will not show how to prove these relations as they can simply be checked by comparing power series. Important to note about these relations is that they do not hold when one picks a, b and c such that these contiguity relations contain both contiguous hypergeometric functions using the original hypergeometric function definition and the extension to it by equation (1). As these relations can be used inductively, it follows that the limitations for different definitions of the hypergeometric function hold true for all contiguity relations. There is still a way around this by instead of using equation (1) using

$$F(a,-i;-n;z) := \sum_{m=0}^{\infty} \frac{(a)_m(-i)_m}{m!(-n)_m}$$

with $i,n \in \mathbb{N}$ and $i \leq n$ while also letting the division by zero cancel out the by the zero in the numerator. As this definition is not used anywhere else, we won't do so either.
Here a demonstration of how you can deduce contiguity relations with induction. Assume you want to find a Q and R for $(k,l,m) = (0,1,1)$ so we are looking for

$$F(a,b+1;c+1;z) = QF' + RF.$$

We wish to determine Q, R using the relations we introduced earlier which first gives us

$$F(a,b+1;c+1;z) = \frac{c(1-z)}{(c-a)(c-(b+1))}F'(a,b+1;c;z) + \frac{c(a+b+1-c)}{(c-a)(c-(b+1))}F(a,b+1;c;z).$$

This means we need to have $F(a,b+1;c;z)$ and $F'(a,b+1;c;z)$ in terms of F' and F. Since this is a simple case we already know it for $F(a,b+1;c;z)$

$$F(a,b+1;c;z) = \frac{z}{b}F' + F$$

To obtain that for $F'(a,b+1;c;z)$ we can take the derivative of that of $F(a,b+1;c;z)$ by using the hypergeometric differential equation

$$\begin{aligned}
F'(a,b+1;c;z) &= \frac{d}{dz}\left(\frac{z}{b}F'(a,b;c;z) + F(a,b;c;z)\right) \\
&= \frac{z}{b}F'' + (1+\frac{1}{b})F' \\
&= \frac{z}{b}\left(-\frac{c-(a+b+1)z}{z(1-z)}F' + \frac{ab}{z(1-z)}F\right) + (1+\frac{1}{b})F' \\
&= \frac{az+b-c+1}{b(1-z)}F' + \frac{a}{(1-z)}F.
\end{aligned}$$

Combining this with what we already had, we get

$$F(a,b+1;c+1;z) = \frac{c(1-z)}{b(a-c)}F' + \frac{-c}{a-c}F.$$

3 Ebisu's Method

Akihito Ebisu developed a method to find special values for hypergeometric functions by using contiguity relations [Ebi17], here we shall explain a slightly improved version of it. The gist of this method is making Q equal to zero and then making $F(a+k, b+l; c+m; z)$ equal to 1 in order to find an expression for $F(a, b; c; z)$. Here is how Ebisu's method works in full detail. Assume we already have chosen $(k, l, m) \in \mathbb{Z}^3$, then the first step is to calculate $Q_{k,l,m}$ and $R_{k,l,m}$ such that

$$F(a+k, b+l; c+m; z) = Q_{k,l,m}F' + R_{k,l,m}F.$$

In order to make things wider applicable we look at

$$\begin{aligned}&F(a+kn, b+ln; c+mn; z)\\&= Q^{(n)}_{k,l,m}F'(a+k(n-1), b+l(n-1); c+m(n-1); z)\\&+ R^{(n)}_{k,l,m}F(a+k(n-1), b+l(n-1); c+m(n-1); z)\end{aligned}$$

for which $n \in \mathbb{Z}$. Here we obtain $Q^{(n)}_{k,l,m}$ and $R^{(n)}_{k,l,m}$ by applying the substitution

$$(a, b, c) \mapsto (a+k(n-1), b+l(n-1), c+m(n-1)).$$

to $Q_{k,l,m}$ and $R_{k,l,m}$. Since $Q_{k,l,m}$ and $R_{k,l,m}$ are rational functions in a, b, c, z, $Q^{(n)}_{k,l,m}$ and $R^{(n)}_{k,l,m}$ are rational functions in a, b, c, z and n.

The second step is finding a solution $(a', b', c', z') \in (\mathbb{Q}(a, b, c))^3 \times \mathbb{C}$ such that $Q^{(n)}_{k,l,m} = 0$ for all n. Here we are only interested in certain solutions which we will call admissible quadruples. Admissible quadruples are solutions such that $z' \notin \{0, 1\}$ (all special values that can be obtained with $z' \in \{0, 1\}$ are either trivial or covered by the Gauss' summation formula). An example of admissible quadruple for $(k, l, m) = (1, 2, 2)$ would be $(a, b, 2a, 2)$. This would give us the following space of solutions

$$\{a(1, 0, 2) + b(0, 1, 0) : a, b \in \mathbb{C}\}$$

to pick parameters in. Furthermore notice that

$$[Q_{1,2,2}]^{(n)}_{[c\mapsto 2a, z\mapsto 2]} = 0 \quad \text{for all } n \in \mathbb{Z}_{\geq 1}$$

while this is also a rational function in n. This means that the numerator which is a polynomial has infinite roots in n, hence

$$[Q_{1,2,2}]^{(n+x)}_{[c\mapsto 2a, z\mapsto 2]} = 0 \quad \text{for all } n \in \mathbb{Z}_{\geq 1} \text{ and } x \in \mathbb{C}.$$

So $(a+x, b+2x, 2a+2x, 2)$ is also an admissible quadruple for $(1, 2, 2)$.
When we have an admissible quadruple we get

$$\begin{aligned}&F(a' + (kn+x), b' + l(n+x); c' + m(n+x); z')\\&= 0F'(a' + k(n+x-1), b' + l(n+x-1); c' + m(n+x-1); z')\\&+ R^{(n+x)}_{k,l,m}F(a' + k(n+x-1), b' + l(n+x-1); c' + m(n+x-1); z')\\&= R^{(n+x)}_{k,l,m}F(a' + k(n+x-1), b' + l(n+x-1); c' + m(n+x-1); z')\\&= \prod_{i=1}^{n} R^{(i+x)}_{k,l,m}F(a' + kx, b' + lx; c' + mx; z').\end{aligned}$$

We write this result as

$$F(a' + k(n+x), b' + l(n+x); c' + m(n+x); z') = S^{(n)}_{k,l,m}(x)F(a' + kx, b' + lx; c' + mx; z') \tag{29}$$

with

$$S^{(n)}_{k,l,m}(x) = \prod_{i=1}^{n} R^{(i+x)}_{k,l,m} \quad (\text{When } x = 0, \text{ we write } S^{(n)}_{k,l,m}).$$

Now how to obtain an admissible quadruple. This is done by taking the numerator of rational function $Q^{(n)}_{k,l,m}$ (this numerator set equal to zero we will from now on call the *admissible equation*) and then finding (a', b', c', z') such that all coefficients in n are 0. If the degree of admissible equation is d we would have $d+1$ equations. In practice we can solve this by taking the Gröbner basis of this set of equations, which is possible in most math oriented programming languages (we used both sage math and mathematica and found that mathematica works the best for this). This Gröbner basis then shows all possible solutions.
As the third step, you need to remove one of the hypergeometric functions from equation (29). There are multiple ways of doing so. In this paper we will only touch on the most general way, by setting one of the hypergeometric functions in equation (29) equal to a constant value, preferably 1 (other ways of removing hypergeometric functions include letting n go to infinity, but this does not always work). This is done by reducing the space which is the admissible quadruple such that either $F(a' + kx, b' + lx; c' + mx; z')$ or $F(a'+k(n+x), b'+l(n+x); c'+m(n+x); z')$ becomes a constant. To give an example for this, for admissible quadruple $(a, b, 2a, 2)$ for $(1, 2, 2)$ we could pick $b = 0$ and $x = 0$, meaning $F(a + x, b + 2x; 2a + 2x; 2) = F(a, 0; 2a; 2) = 1$ giving us special value $F(a + n, 2n; 2a + 2n; 2) = S^{(n)}_{k,l,m}$. When doing this, we need to be careful. If we in the previous example had picked $a = 0$ and $x = 0$, we would still get according to equation (1) $F(a+x, b+2x; 2a+2x; 2) = F(0, b; 0; 2) = 1$, but also that $F(a+(n+x), b+2(n+x); 2a+2(n+x); 2) = F(n, b + 2n; 2n; 2)$ is not calculated using equation (1). In subsection 2.5 we noted that contiguity relations only hold true if all contiguous functions in relation either use the original definition for the hypergeometric function or use equation (1). This is no longer the case when $a = 0$ and $x = 0$, thus we get no special value from this.
The x in the above text is an extra we added to Ebisu's method. As such, the original Ebisu's method is equivalent to setting $x = 0$ in the above. The introduction of x can possibly increase the number of special values Ebisu's method gives. Sadly, due to this being added to the thesis in a rather late phase, we did not have the time to research the effects of this and would require further research.
There are still other options to get more out of Ebisu's method (which were actually included with the initial introduction of Ebisu's method) such as interpolation or applying topic of the next subsection.

3.1 Associated Quadruples

The time needed to compute admissible quadruples for (k, l, m) increases significantly as (k, l, m) increases in absolute value. Hence, it is useful to minimize the number of (k, l, m) for which we need to calculate the admissible quadruples using Ebisu's method. That is why in [Ebi17] they use symmetries on (k, l, m) to obtain admissible quadruples. By using symmetry, we can turn an admissible quadruples from a (k, l, m) to an admissible quadruple for a different (k', l', m'), thus we call those quadruples associated.

Theorem 3.1.1. *Fix $(k, l, m) \in \mathbb{Z}^3$ and pick an admissible quadruple (a, b, c, z), then there are 23 admissible quadruples from different (k, l, m) associated to that quadruple. Each of those associated quadruples then has their own equality as shown below. Here, each equality has a different (k, l, m), but they still all use $S^{(n)}_{k,l,m}$.*

(i) $S^{(n)}_{k,l,m} F(a, b; c; z) = F(a + kn, b + ln; c + mn; z)$

(ii) $S^{(n)}_{k,l,m} F(c - a, c - b; c; z) = (1 - z)^{(m-k-l)n} F(c - a + (m - k)n, c - b + (m - l)n; c + mn; z)$

(iii) $S^{(n)}_{k,l,m} F(a, c - b; c; z/(z - 1)) = (1 - z)^{-kn} F(a + kn, c - b + (m - l)n; c + mn; z/(z - 1))$

(iv) $S^{(n)}_{k,l,m} F(c - a, b; c; z/(z - 1)) = (1 - z)^{-ln} F(c - a + (m - k)n, b + ln; c + mn; z/(z - 1))$

(v) $S^{(n)}_{k,l,m} F(a, b; a+b+1-c; 1-z) = \frac{(c)_{mn}(c-a-b)_{(m-k-l)n}}{(c-a)_{(m-k)n}(c-b)_{(m-l)n}} F(a+kn, b+ln; a+b+1-c+(k+l-m)n; 1-z)$

(vi) $S^{(n)}_{k,l,m} F(a + 1 - c; b + 1 - c; a + b + 1 - c; 1 - z) = \frac{(c)_{mn}(c-a-b)_{(m-k-l)n}}{(c-a)_{(m-k)n}(c-b)_{(m-l)n}}$
$\times F(a + 1 - c + (k - m)n, b + 1 - c + (l - m)n; a + b + 1 - c + (k + l - m)n; 1 - z)$

(vii) $S^{(n)}_{k,l,m} F(a, a + 1 - c; a + b + 1 - c; 1 - z^{-1}) = \frac{(c)_{mn}(c-a-b)_{(m-k-l)n}}{(c-a)_{(m-k)n}(c-b)_{(m-l)n}} z^{-kn}$
$\times F(a + kn, a + 1 - c + (k - m)n; a + b + 1 - c + (k + l - m)n; 1 - z^{-1})$

(viii) $S^{(n)}_{k,l,m}F(b+1-c,b;a+b+1-c;1-z^{-1}) = \frac{(c)_{mn}(c-a-b)_{(m-k-l)n}}{(c-a)_{(m-k)n}(c-b)_{(m-l)n}}F(b+1-c+(l-m)n,b+ln;a+b+1-c+(k+l-m)n;1-z^{-1})$

(ix) $S^{(n)}_{k,l,m}F(a,a+1-c;a+1-b;1/z) = \frac{(-1)^{(m-k-l)n}(c)_{mn}(a+1-c)_{(k-m)n}}{(b)_{ln}(a+1-b)_{(k-l)n}}z^{-kn}$
$\times F(a+km,a+1-c+(k-m)n;a+1-b+(k-l)n;1/z)$

(x) $S^{(n)}_{k,l,m}F(1-b,c-b;a+1-b;1/z) = \frac{(-1)^{(m-l)n}(c)_{mn}(a+1-c)_{(k-m)n}}{(b)_{ln}(a+1-b)_{(k-l)n}}(-z)^{(l-m)n}(1-z)^{(m-k-l)n}$
$\times F(1-b-ln,c-b+(m-l)n;a+1-b+(k-l)n;1/z)$

(xi) $S^{(n)}_{k,l,m}F(a,c-b;a+1-b;(1-z)^{-1}) = \frac{(-1)^{(m-l)n}(c)_{mn}(a+1-c)_{(k-m)n}}{(b)_{ln}(a+1-b)_{(k-l)n}}(1-z)^{-kn}$
$\times F(a+kn,c-b+(m-l)n;a+1-b+(k-l)n;(1-z)^{-1})$

(xii) $S^{(n)}_{k,l,m}F(a+1-c,1-b;a+1-b;(1-z)^{-1}) = \frac{(-1)^{(m-l)n}(c)_{mn}(a+1-c)_{(k-m)n}}{(b)_{ln}(a+1-b)_{(k-l)n}}(-z)^{-mn}(1-z)^{(m-k)n}$
$\times F(a+1-c+(k-m)n,1-b-ln;a+1-b+(k-l)n;(1-z)^{-1})$

(xiii) $S^{(n)}_{k,l,m}F(b+1-c,b;b+1-a;1/z) = \frac{(-1)^{m-k-l}(c)_{mn}(b+1-c)_{(l-m)n}}{(a)_{kn}(b+1-a)_{(l-k)n}}z^{-ln}$
$\times F(b+1-c+(l-m)n,b+ln;b+1-a+(l-k)n;1/z)$

(xiv) $S^{(n)}_{k,l,m}F(1-a,c-a;b+1-a;1/z) = \frac{(-1)^{m-k}(c)_{mn}(b+1-c)_{(l-m)n}}{(a)_{kn}(b+1-a)_{(l-k)n}}(-z)^{(k-m)n}(1-z)^{(m-k-l)n}$
$\times F(1-a-kn,c-a+(m-k)n;b+1-a+(l-k)n;1/z)$

(xv) $S^{(n)}_{k,l,m}F(b,c-a;b+1-a;(1-z)^{-1}) = \frac{(-1)^{m-k}(c)_{mn}(b+1-c)_{(l-m)n}}{(a)_{kn}(b+1-a)_{(l-k)n}}(1-z)^{-ln}$
$\times F(b+ln,c-a+(m-k)n;b+1-a+(l-k)n;(1-z)^{-1})$

(xvi) $S^{(n)}_{k,l,m}F(b+1-c,1-a;b+1-a;(1-z)^{-1}) = \frac{(-1)^{m-k}(c)_{mn}(b+1-c)_{(l-m)n}}{(a)_{kn}(b+1-a)_{(l-k)n}}(-z)^{-mn}(1-z)^{(m-l)n}$
$\times F(b+1-c+(l-m)n,1-a-kn;b+1-a+(l-k)n;(1-z)^{-1})$

(xvii) $S^{(n)}_{k,l,m}F(a+1-c,b+1-c;2-c;z) = \frac{(c)_{mn}(a+1-c)_{(k-m)n}(b+1-c)_{(l-m)n}}{(a)_{kn}(b)_{ln}(2-c)_{-mn}}z^{-mn}$
$\times F(a+1-c+(k-m)n,b+1-c+(l-m)n;2-c-mn;z)$

(xviii) $S^{(n)}_{k,l,m}F(1-a,1-b;2-c;z) = \frac{(c)_{mn}(a+1-c)_{(k-m)n}(b+1-c)_{(l-m)n}}{(a)_{kn}(b)_{ln}(2-c)_{-mn}}z^{-mn}(1-z)^{(m-k-l)n}$
$\times F(1-a-kn,1-b-ln;2-c-mn;z)$

(xix) $S^{(n)}_{k,l,m}F(a+1-c,1-b;2-c;z/(z-1)) = \frac{(c)_{mn}(a+1-c)_{(k-m)n}(b+1-c)_{(l-m)n}}{(a)_{kn}(b)_{ln}(2-c)_{-mn}}z^{-mn}(1-z)^{(m-k)n}$
$\times F(a+1-c+(k-m)n,1-b-ln;2-c-mn;z/(z-1))$

(xx) $S^{(n)}_{k,l,m}F(b+1-c,1-a;2-c;z/(z-1)) = \frac{(c)_{mn}(a+1-c)_{(k-m)n}(b+1-c)_{(l-m)n}}{(a)_{kn}(b)_{ln}(2-c)_{-mn}}z^{-mn}(1-z)^{(m-l)n}$
$\times F(b+1-c+(l-m)n,1-a-kn;2-c-mn;z/(z-1))$

(xxi) $S^{(n)}_{k,l,m}F(c-a,c-b;c+1-a-b;1-z) = \frac{(c)_{mn}(a+b-c)_{(k+l-m)n}}{(a)_{kn}(b)_{ln}}(1-z)^{(m-k-l)n}$
$\times F(c-a+(m-k)n,c-b+(m-l)n;c+1-a-b+(m-k-l)n;1-z)$

(xxii) $S^{(n)}_{k,l,m}F(1-a,1-b;c+1-a-b;1-z) = \frac{(c)_{mn}(a+b-c)_{(k+l-m)n}}{(a)_{kn}(b)_{ln}}z^{-mn}(1-z)^{(m-k-l)n}$
$\times F(1-a-kn,1-b-ln;c+1-a-b+(m-k-l)n;1-z)$

(xxiii) $S^{(n)}_{k,l,m}F(c-a,1-b;c+1-a;1-z^{-1}) = \frac{(c)_{mn}(a+b-c)_{(k+l-m)n}}{(a)_{kn}(b)_{ln}}z^{(k-m)n}(1-z)^{(m-k-l)n}$
$\times F(c-a+(m-k)n,1-b-ln;c+1-a+(m-k)n;1-z^{-1})$

(xxiv) $S^{(n)}_{k,l,m}F(c-b,1-b;c+1-a-b;1-z^{-1}) = \frac{(c)_{mn}(a+b-c)_{(k+l-m)n}}{(a)_{kn}(b)_{ln}}z^{(l-m)n}(1-z)^{(m-k-l)n}$
$\times F(c-b+(m-l)n,1-b-ln;c+1-a-b+(m-k-l)n;1-z^{-1})$

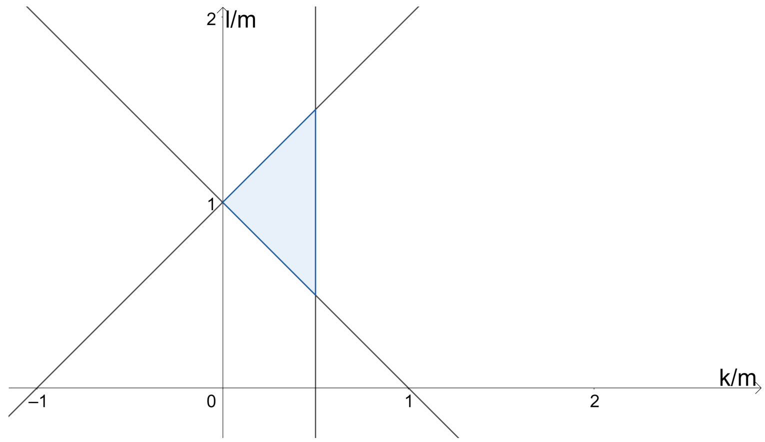

Figure 2: area L

We will wait with giving a proof of this theorem until we have introduced the cup product, as we can then give a more intuitive proof.
We can view this collection of associated quadruples as mappings for the admissible quadruples. For example (ii) shows that from an admissible quadruple (a, b, c, z) for (k, l, m), we can obtain admissible quadruple $(c-a, c-b, c, z)$ for $(m-k, m-l, m)$. This means we only need to compute the admissible quadruples for (k, l, m) to find the special values for 24 mappings of (k, l, m), we can bring this number up to 48 by using the symmetry $F(a, b; c; z) = F(b, a; c; z)$. These mappings together form a closed group G operating over $\mathbb{Z}^3$, this group can be generated by

$$\begin{aligned}
\sigma_1 &: (k, l, m) \to (m-k, l, m) && \text{(iv)}, \\
\sigma_2 &: (k, l, m) \to (k, l, k+l-m) && \text{(v)}, \\
\sigma_3 &: (k, l, m) \to (l, k, m) && \text{(symmetry)}, \\
\sigma_4 &: (k, l, m) \to (m-l, m-k, m) && \text{(ii)}, \\
\sigma_5 &: (k, l, m) \to (-l, -k, -m) && \text{(xviii)}.
\end{aligned}$$

So we have $G = \langle\sigma_1, \sigma_2\rangle \ltimes (\langle\sigma_3\rangle \times \langle\sigma_4\rangle \times \langle\sigma_5\rangle) = S_3 \ltimes (S_2 \times S_2 \times S_2)$, where S_n is the symmetry group of the same degree.

Theorem 3.1.2. *We can fully represent $\mathbb{Z}^3/G$ with*

$$L = \{(k, l, m) \in \mathbb{Z}^3 : 0 \leq k+l-m \leq l-k \leq m\}. \tag{30}$$

Proof. First we will look at $(\langle\sigma_3\rangle \times \langle\sigma_4\rangle \times \langle\sigma_5\rangle)$. From σ_3, we get that both (k, l, m) and (l, k, m) map to each other hence we can choose $0 \leq l-k$. This property is maintained by both σ_4 and σ_5. In the same sense we can respectively pick $0 \leq k+l-m$ and $0 \leq m$ with σ_4 and σ_5.
Furthermore σ_1 switches the value of $l-k$ and $k+l-m$, while maintaining m. Similarly σ_2 changes the values of m and $k+l-m$, while not changing the value of $l-k$. Thus meaning we can pick order $0 \leq k+l-m \leq l-k \leq m$. $\square$

So we only need to look at (k, l, m) contained in L. This set also has some nice properties.

Lemma 3.1.3. *If $(k, l, m) \in L$, then $k, l, m \geq 0$.*

Proof. That $m \geq 0$ is trivial. For the other variables it can be done by working it out using the equation or drawing an image, we chose the latter.
Use that $0 \leq k+l-m \leq l-k \leq m$ is equivalent to $\{1 \leq k/m + l/m\} \cap \{2k/m \leq 1\} \cap \{l/m - km \leq 1\}$. We can now display L as shown in figure 2. $\square$

Hence we do not need to focus on any of the case that k, l or m is negative, which we will later use while examining the contiguity relations.

3.2 Irreducible Special Values

Special values are not all equally interesting to us, particularly some reducible ones can be derived without much hassle or can be described with ${}_1F_0$. For this text the following definition of reducible will suffice (for a better definition please read chapter 4.3 of [Iwa+91]).

Definition 3.2.1. We say an admissible quadruple (a, b, c, z) is irreducible if $a, b, c - a, c - b \notin \mathbb{Z}$. The opposite of this reducible.

As an example of a reducible admissible quadruple, take $b = c + 1$. Define $\theta = x\frac{d}{dx}$. We then have

$$\begin{aligned} F(a, c+1; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (c+1)_n}{n!(c)_n} x^n \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{(c+n)}{c} x^n \\ &= (\theta + c) \sum_{n=0}^{\infty} \frac{(a)_n}{cn!} x^n \\ &= (\theta + c) \frac{1}{c(1-z)^a} \\ &= \frac{1}{(1-z)^a} - \frac{az}{c(1-z)^{a+1}} \end{aligned}$$

We could even rewrite this to $\frac{\theta + c}{c} {}_1F_0(a; -; z)$. Overall, this means that we do not need Ebisu's method for some of these cases.

3.3 Example of Ebisu's Method

For this example we will look at $(k, l, m) = (2, 0, 1)$.
First we calculate Q and R as earlier described, giving us

$$Q_{2,0,1} = \frac{c(az - a - bz + c + z - 1)}{a(a+1)(c-b)}$$

$$R_{2,0,1} = -\frac{c(-a + b - 1)}{(a+1)(c-b)}$$

These need to be turned into $Q_{2,0,1}^{(n)}$ and $R_{2,0,1}^{(n)}$ which is done by simple substitution. We get the admissible equation

$$(2z-1)n^2 + (z + az - a - bz + 2cz - 1)\, n - c - ac + c^2 + cz + acz - bcz$$

(here we look at the numerator of $Q_{2,0,1}^{(n+1)}$ as it simplifies the numerator and substitution step significantly). If we view all the coefficient in n as different equations and we get the Gröbner basis

$$\{2z - 1, 1 + a + b - 2c\}$$

meaning our only solution is $z = \frac{1}{2}$ and $c = \frac{1+a+b}{2}$, giving us the admissible quadruple $(a, b, \frac{1+a+b}{2}, \frac{1}{2})$. Filling this into $R^{(n)}$ we get

$$R_{2,0,1}^{(n)}(a, b, \frac{1+a+b}{2}, \frac{1}{2}) \frac{a + b + 2n - 1}{a + 2n - 1}$$

meaning

$$S^{(n)}(a, b, \frac{1+a+b}{2}, \frac{1}{2}) = \frac{\left(\frac{1+a+b}{2}\right)_n}{\left(\frac{1+a}{2}\right)_n}.$$

We now have

$$F(a + 2n, b; \frac{1+a+b}{2} + n; \frac{1}{2}) = S^{(n)}(a, b, \frac{1+a+b}{2}, \frac{1}{2}) F(a, b; \frac{1+a+b}{2}; \frac{1}{2}).$$

The admissible quadruple $(a, b, \frac{1+a+b}{2}, \frac{1}{2})$ gives us a 2-dimensional space to work with. By reducing this space we get the following results

$$F(a+2n, b; \frac{1+a+b}{2}+n; \frac{1}{2}) = \begin{cases} \frac{\left(\frac{1+b}{2}\right)_n}{\left(\frac{1}{2}\right)_n} & \text{if } a = 0, \\ 1 & \text{if } b = 0, \end{cases}$$

and

$$F(a, b; \frac{1+a+b}{2}; \frac{1}{2}) = \begin{cases} \frac{\left(\frac{1-2n}{2}\right)_n}{\left(\frac{1-2n+b}{2}\right)_n} & \text{if } a = -2n, \\ 1 & \text{if } b = 0. \end{cases}$$

4 Cup Product

There is more than one way to Rome, and so there is more than one way to calculate contiguity relations. The cup product is a bilinear pairing on De Rham cohomologies and by using it we can get a better insight into the structure of contiguity relations. But before we can touch on that, we will need to introduce De Rham cohomology. This section is based on chapter 6 of [Beu18].

4.1 De Rham Cohomology

To not go too deep into abstraction, we will only introduce a case of De Rham cohomology that is useful in this context. This case will focus on the differential form in Euler's integral representation, which is

$$x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx.$$

We will simplify this to the notation dx/y where

$$y := x^\rho(1-x)^\sigma(1-zx)^\tau := x^{1-b}(1-x)^{1+b-c}(1-zx)^a$$

Assume $z \notin \{0,1\}$. Let $K = \mathbb{Q}(\rho,\sigma,\tau)$ to this we add the singularities of the previously mentioned differential form

$$A = K(z)\left[x, \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1-zx}\right].$$

Consider the following differential form $R(x)dx/y$ with $R(x) \in A$, we will denote the $K(z)-$linear space of these forms $R(x)dx/y$ by $\Omega^1(\rho,\sigma,\tau)$. $R(x)dx/y$ is called exact in the case that there exist a $S(x) \in A$ such that $R(x)dx/y = d(S(x)/y)$. We call the space $\Omega^1(\rho,\sigma,\tau)$ modulo all exact forms the twisted De Rham cohomology, this is denoted as

$$H^1_{\text{twist}}(\rho,\sigma,\tau) := \Omega^1(\rho,\sigma,\tau)/\{\text{exact forms}\}.$$

Theorem 4.1.1. *Assume $\rho,\sigma,\tau \notin \mathbb{Z}$. The space $H^1_{twist}(\rho,\sigma,\tau)$ over $K(z)$ is of dimension 2 and spanned by differential forms dx/y and xdx/y.*

Proof. We start with showing that every $R(x)dx/y \in \Omega^1(\rho,\sigma,\tau)$ is modulo exact forms equal to $(p+qx)dx/y$ with $p,q \in K(z)$. Assume that $R(x)$ has a singularity at $x = 0$ of order k, then we can write

$$R(x) = \frac{r}{x^k} + \frac{1}{O(x^{k-1})}$$

with $r \in K(z)$. In order to get rid of this singularity, we need to subtract an exact form which has the same singularity.

$$\begin{aligned}
&d(\frac{(1-x)(1-zx)}{x^{k-1}y}) \\
&= \left(\frac{(2zx-z-1)x-(1-x)(1-zx)(k-1)}{x^k} + \frac{(1-x)(1-zx)}{x^{k-1}}\left(-\frac{\rho}{x}+\frac{\sigma}{1-x}+\frac{z\tau}{1-zx}\right)\right)\frac{dx}{y} \\
&= (1-k-\rho+Ax+Bx^2)\frac{dx}{x^k y}
\end{aligned}$$

with $A, B \in K(z)$. Hence

$$R(x)\frac{dx}{y} - \frac{r}{1-k-\rho}d(\frac{(1-x)(1-zx)}{x^{k-1}y}) = \tilde{R}(x)\frac{dx}{y}$$

where $\tilde{R}(x)$ has a singularity at $x = 0$ of at most degree $k - 1$. By using induction, we get that $R(x)$ can be represented without a singularity in $x = 0$. By also doing this process for $x = 1$ and $x = 1/z$ we can now assume $R(x) \in K(z)[x]$. Suppose $R(x)$ is of degree k in x and $k \geq 2$ meaning we have

$$R(x) = rx^k + O(x^{k-1})$$

with $r \in K(z)$, we can then use exact form

$$d\left(\frac{(1-x)(1-zx)x^{k-1}}{y}\right) = (z(1+k-\rho-\sigma-\tau)x^k + Ax^{k-1} + Bx^{k-2})\frac{dx}{y}$$

with $A, B \in K(z)$, giving us

$$R(x)\frac{dx}{y} - \frac{r}{z(1+k-\rho-\sigma-\tau)}d\left(\frac{(1-x)(1-zx)x^{k-1}}{y}\right) = \tilde{R}(x)\frac{dx}{y}$$

where $\tilde{R}(x)$ is of degree at most $k - 1$. Hence by applying induction we get that $R(x)dx/y \equiv (p + qx)dx/y$ modulo exact forms.
Now to prove that $H^1_{\text{twist}}(\rho, \sigma, \tau)$ is 2-dimensional over $K(z)$. From $R(x)dx/y \equiv (p + qx)dx/y$ we can deduce that 2 is the upper bound. Leaving to prove that $(p + qx)dx/y$ is only exact if $p = q = 0$.
Assume $(p + qx)dx/y$ is exact, meaning that there exists $S(x)$ such that $(p + qx)dx/y = d(S(x)/y)$. If $S(x)$ has a singularity in 0, 1 or $1/z$, then this expansion shows

$$d\left(\frac{S(x)}{y}\right) = \left(S'(x) + S(x)\left(\frac{-\rho}{x} + \frac{\sigma}{1-x} + \frac{z\tau}{1-zx}\right)\right)\frac{dx}{y} \tag{31}$$

that if we fill in an maclaurin series of $S(x)$, we would get that since $\rho, \sigma, \tau \notin \mathbb{Z}$ that $d(S(x)/y)$ should have a singularity in the same place of greater order. So we get that $S(x)$ does not have singularities, meaning $S(x) \in K(z)[x]$. From equation (31) it follows that if it is a polynomial in x, then it must have roots 0, 1 and $1/z$ as else $d(S(x)/y)$ will have singularities in those points. This means $S(x)$ is either of degree ≥ 3 or $S(x) = 0$. Hence $(p + qx)dx/y$ is only exact if $p = q = 0$, meaning $(p + qx)dx/y$ spanned a 2-dimensional $K(z)$-space modulo the exact forms. □

4.2 Cup Product

We can now define the cup product which is a $K(z)$-bilinear pairing between $H^1_{twist}(\rho, \sigma, \tau)$ and $H^1_{twist}(-\rho, -\sigma, -\tau)$. Consider twisted forms

$$\omega_1 = G_1(x)\frac{dx}{x^\rho(1-x)^\sigma(1-zx)^\tau}$$

$$\omega_2 = G_2(x)\frac{dx}{x^{-\rho}(1-x)^{-\sigma}(1-zx)^{-\tau}}$$

with $G_1(x), G_2(x) \in A$. Consider these forms near $x = 0, 1, 1/z$ or ∞, all likely singularities of the above. We define $\langle\omega_1, \omega_2\rangle_p$ as the residue of $(\int \omega_1)\omega_2$ at $x = p$. To further clarify this, an example for the case that $p = 1$ and $G_1(x) = G_2(x) = 1$. We are looking at

$$\langle\omega_1, \omega_2\rangle_1 = \text{Res}_{x=1}\left(\int x^{-\rho}(1-x)^{-\sigma}(1-zx)^{-\tau}dx\right)x^\rho(1-x)^\sigma(1-zx)^\tau dx$$

First, we wish to apply the substitution $x \mapsto 1 - x$, this way we are taking the residue at zero. This will affect both differential forms,

$$= \text{Res}_{x=0}\left(\int(1-x)^{-\rho}(1-(1-x))^{-\sigma}(1-z(1-x))^{-\tau}d(1-x)\right)(1-x)^\rho(1-(1-x))^\sigma(1-z(1-x))^\tau d(1-x)$$

$$= \text{Res}_{x=0}\left(\int(1-x)^{-\rho}x^{-\sigma}(1-z+zx)^{-\tau}(-1)dx\right)(1-x)^\rho x^\sigma(1-z+x)^\tau(-1)dx.$$

In order to calculate the residue take the Maclaurin expansion while keeping x^σ separate.

$$\begin{aligned}
&= \mathrm{Res}_{x=0}\left(\int (1-x)^{-\rho}x^{-\sigma}(1+\frac{z}{1-z}x)^{-\tau}dx\right)(1-x)^{\rho}x^{\sigma}(1+\frac{z}{1-z}x)^{\tau}dx \\
&= \mathrm{Res}_{x=0}\left(\frac{1}{1-\sigma}x^{1-\sigma}+\frac{\rho+\frac{z}{1-z}\tau}{2-\sigma}x^{2-\sigma}+...\right)\left(x^{\sigma}+(-\rho-\frac{z}{1-z})x^{\sigma-1}+...\right)dx \\
&= \mathrm{Res}_{x=0}\left(\frac{1}{1-\sigma}x^{1}+\frac{\rho+\frac{z}{1-z}\tau}{2-\sigma}x^{2}+...\right)\left(x^{0}+(-\rho-\frac{z}{1-z})x^{1}+...\right)dx \\
&= 0
\end{aligned}$$

The cup product itself is defined as

$$\langle\omega_1,\omega_2\rangle = \sum_{p\in\{0,1,1/z,\infty\}}\langle\omega_1,\omega_2\rangle_p.$$

We now know that the cupprodcut is a $K(z)$-bilinear pairing between $\Omega^1(\rho,\sigma,\tau)$ and $\Omega^1(-\rho,-\sigma,-\tau)$. In order for this to also be true for $H^1_{twist}(\rho,\sigma,\tau)$ and $H^1_{twist}(-\rho,-\sigma,-\tau)$ we will need to prove it gives zero when either ω_1 or ω_2 is an exact form. Before we prove this, a simple lemma to help.

Lemma 4.2.1. *$\langle\omega_1,\omega_2\rangle = -\langle\omega_2,\omega_1\rangle$ and $\langle\omega_1,\omega_2\rangle_p = -\langle\omega_2,\omega_1\rangle_p$*

Proof. We know that $\frac{d}{dx}(\int\omega_1)(\int\omega_2) = \omega_1\int\omega_2 + (\int\omega_1)\omega_2$. Since the LHS is the derivative of a local Laurent series expansion, the residue has to be equal to 0. Hence $0 = \langle\omega_1,\omega_2\rangle_p + \langle\omega_2,\omega_1\rangle_p$. □

Theorem 4.2.2. *If either ω_1 or ω_2 is an exact form, then $\langle\omega_1,\omega_2\rangle = 0$*

Proof. Because of Lemma 4.2.1 we can WLOG pick ω_1 to be exact, meaning $\omega_1 = d(S(x)/y)$. We now have

$$\left(\int\omega_1\right)\omega_2 = \frac{S(x)}{y}G_2(x)ydx = S(x)G_2(x)dx.$$

Since $S(x)G_2(x)\in A$ is a rational function, the sum of its residues will be 0. □

We now have proven that the cup product is in fact a $K(z)$-linear pairing between $H^1_{twist}(\rho,\sigma,\tau)$ and $H^1_{twist}(-\rho,-\sigma,-\tau)$.

4.3 the Cup Product and the Simplified Admissible Equation

Here we will next to proving a different way to calculate the contiguity relations, also introduce how we can simplify the admissible equation.

Theorem 4.3.1. *Again let $\rho = 1-b$, $\sigma = b+1-c$, $\tau = a$ and $y = x^\rho(1-x)^\sigma(1-zx)^\tau$. Define*

$$\omega_{k,l,m} = x^l(1-x)^{m-l}(1-zx)^{-k}dx/y \in H^1_{twist}(\rho,\sigma,\tau)$$

and

$$\omega = x^{\rho-1}(1-x)^{\sigma-1}(1-zx)^{\tau-1}dx \in H^1_{twist}(-\rho,-\sigma,-\tau),$$

where $\omega_{k,l,m}$ is the differential form in Euler's integral representation. Then

$$\omega_{k,l,m} = U(z)\omega_{0,1,1} + V(z)\omega_{0,0,0}. \tag{32}$$

where $U(z) = (a-c)z\langle\omega_{k,l,m},\omega\rangle$ and $V(z) = b\langle\omega_{k,l,m},x^{-1}\omega\rangle$.

Proof. Because of Theorem 4.1.1, we know that there exist $U(z),V(z)\in K(z)$ such that

$$\omega_{k,l,m} = U(z)x\frac{dx}{y} + V(z)\frac{dx}{y} = U(z)\omega_{0,1,1} + V(z)\omega_{0,0,0}.$$

Calculations show that

$$\langle \omega_{0,1,1}, \omega \rangle = \frac{1}{(\rho+\sigma+\tau-2)z}, \qquad \langle \omega_{0,0,0}, \omega \rangle = 0,$$

$$\langle \omega_{0,1,1}, x^{-1}\omega \rangle = 0, \qquad \langle \omega_{0,0,0}, x^{-1}\omega \rangle = \frac{1}{1-\rho}.$$

Hence we can now determine $U(z)$ and $V(z)$ as follows

$$\begin{aligned}\langle \omega_{k,l,m}, \omega \rangle &= \langle U(z)\omega_{0,1,1} + V(z)\omega_{0,0,0}, \omega \rangle \\ &= U(z)\langle \omega_{0,1,1}, \omega \rangle + V(z)\langle \omega_{0,0,0}, \omega \rangle \\ &= U(z)\frac{1}{(\rho+\sigma+\tau-2)z}\end{aligned}$$

$$\begin{aligned}\langle \omega_{k,l,m}, x^{-1}\omega \rangle &= U(z)\langle \omega_{0,1,1}, x^{-1}\omega \rangle + V(z)\langle \omega_{0,0,0}, x^{-1}\omega \rangle \\ &= V(z)\frac{1}{1-\rho}.\end{aligned}$$

□

Now giving it the same form as in Ebisu's method for contiguity relations, keep in mind that

$$F' = \frac{ab}{c}F(a+1, b+1; c+1; z),$$

so we want to use $\omega_{1,1,1}$ instead of $\omega_{0,1,1}$.

Corollary 4.3.2.

$$\omega_{k,l,m} = W(z)\omega_{1,1,1} + X(z)\omega_{0,0,0} \tag{33}$$

where $W(z) = a(1-z)z\langle \omega_{k,l,m}, \omega \rangle$ *and* $X(z) = -bz\langle \omega_{k,l,m}, \omega \rangle + b\langle \omega_{k,l,m}, x^{-1}\omega \rangle$

Proof. This proof will work by expressing $\omega_{0,1,1}$ in $\omega_{1,1,1}$ and $\omega_{0,0,0}$ and then substituting it into the previous theorem.
From Theorem 4.3.1 we obtain that

$$\begin{aligned}\omega_{1,1,1} &= (a-c)z\langle \omega_{1,1,1}, \omega \rangle \omega_{0,1,1} + b\langle \omega_{1,1,1}, x^{-1}\omega \rangle \omega_{0,0,0} \\ &= \frac{a-c}{a(1-z)}\omega_{0,1,1} + \frac{b}{a(1-z)}\omega_{0,0,0}.\end{aligned}$$

This can be rewritten to

$$\omega_{0,1,1} = \frac{a(1-z)}{a-c}\omega_{1,1,1} - \frac{b}{a-c}\omega_{0,0,0}$$

substituting this into the previous theorem gives us

$$\begin{aligned}\omega_{k,l,m} &= U(z)(\frac{a(1-z)}{a-c}\omega_{1,1,1} - \frac{b}{a-c}\omega_{0,0,0}) + V(z)\omega_{0,0,0} \\ &= U(z)\frac{a(1-z)}{a-c}\omega_{1,1,1} + (-U(z)\frac{b}{a-c} + V(z))\omega_{0,0,0},\end{aligned}$$

which is the desired result. □

Theorem 4.3.3.

$$F(a+k, b+l; c+m; z) = QF' + RF$$

with $Q = \frac{(c)_m}{a(b)_l(c-b)_{m-l}}W(z)$ *and* $R = \frac{(c)_m}{(b)_l(c-b)_{m-l}}X(z)$.

Proof. This proof will mainly rely on using Euler's integral representation to turn the differential forms into hypergeometric functions. Before we can do that, we need to prove that we can use $\int_0^1$ as a $K(z)$-linear map from $H^1_{\text{twist}}(\rho,\sigma,\tau)$ to $K(z)$. That it is a linear map from $\Omega^1(\rho,\sigma,\tau)$ to $K(z)$ while possibly using the Pochhammer contour is trivial. So to prove that this is also the case for $H^1_{twist}(\rho,\sigma,\tau)$ we need to prove that $\int_0^1$ maps exact forms to 0. Let $S(x)\in A$, then $d(S(x)/y)$ is an exact form. Since $S(x)\in A$, we know that if it has singularities at 0 or 1, that the degree is finite. Hence we can always pick a value M such that if $\Re(\rho),\Re(\sigma)<M$, then $S(x)/y$ does not have any singularities at 0 or 1, meaning that if we take the integral for $\Re(\rho),\Re(\sigma)<M$, then

$$\int_0^1 d(S(x)/y) = [S(x)/y]_0^1 = 0.$$

In order for this to be true for all ρ and σ we can extend the domain of $\int_0^1$ by using the Pochhammer contour, which should also return 0, else this evaluation would not be continuous with our earlier findings of it being 0 for the open set $\{(\rho,\sigma)\in\mathbb{C}^2 : \Re(\rho),\Re(\sigma)<M\}$.
We now have proven that taking the integral of equality (33) maintains the equality, hence we can start turning this into a statement about hypergeometric functions. Euler's integral representation says that

$$\frac{\Gamma(b+l)\Gamma(c-b+m-l)}{\Gamma(c+m)}F(a+k,b+l;c+m;z) = \int_0^1 \omega_{k,l,m}$$

giving us

$$\frac{\Gamma(b+l)\Gamma(c-b+m-l)}{\Gamma(c+m)}F(a+k,b+l;c+m;z) = W(z)\frac{\Gamma(b+1)\Gamma(c-b)}{\Gamma(c+1)}\frac{c}{ab}F' + X(z)\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F.$$

By dividing both side by $\frac{\Gamma(b+l)\Gamma(c-b+m-l)}{\Gamma(c+m)}$ we get the desired result. □

To make Theorem 4.3.3 even more applicable with Ebisu's method, we can substitute n into it, giving us

$$Q^{(n+1)} = \frac{(c+mn)_m(1-z)z}{(b+ln)_l(c-b+(m-l)n)_{m-l}}\langle\omega^{(n+1)}_{k,l,m},\omega^{(n+1)}\rangle. \tag{34}$$

where

$$\omega^{(n+1)} := [\omega]_{(a,b,c)\to(a+kn,b+ln,c+mn)}$$

and

$$\omega^{(n+1)}_{k,l,m} := [\omega_{k,l,m}]_{(a,b,c)\to(a+kn,b+ln,c+mn)} = \omega_{k(n+1),l(n+1),m(n+1)}$$

We have now shown how to calculate contiguity relations using the cup product. We now go on to explain how we simplify the admissible equation using them.

Lemma 4.3.4. *Let (k,l,m) be fixed and (a,b,c,z) an admissible quadruple, then*

$$\langle\omega^{(n)}_{k,l,m},\omega^{(n)}\rangle = 0$$

for all $n\in\mathbb{Z}_{\geq 0}$.

Proof. Let us look at which part of equation (34) an admissible quadruples can come from. Assume they come from

$$\frac{(c+mn)_m(1-z)z}{(b+ln)_l(c-b+(m-l)n)_{m-l}},$$

in that case every coefficient in n in the numerator has to be 0. We cannot achieve this by picking c since that will not make $m^m(1-z)z$ the coefficient of n^m equal to 0 (also picking m equal to 0, does not work since then the Pochhammer symbol becomes 1). So, the options left are to set z equal to 1 or 0, but then it would not be an admissible quadruple. Hence all admissible quadruples result in $\langle\omega^{(n)}_{k,l,m},\omega^{(n)}\rangle = 0$. □

Remark 4.3.5. We can use similar logic as in the previous proof to show that $(b+ln)_l(c-b+(m-l)n)_{m-l}$ does not remove solutions from the numerator of $\langle\omega^{(n)}_{k,l,m},\omega^{(n)}\rangle$.

This opens up some interesting options. One of which is simplifying the admissible equation.
We can tell from Lemma 4.3.4 that

$$\frac{(c+mn)_m(1-z)z}{(b+ln)_l(c-b+(m-l)n)_{m-l}}$$

does not give us any extra admissible quadruples. Hence we can introduce the *simplified* admissible equation.

Definition 4.3.6. The simplified admissible equation is simply said the numerator of

$$\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle = 0$$

As we need to make all coefficients of n equal to 0 to obtain an admissible quadruple.

Here we also made one extra improvement over the original admissible equation using Lemma 4.2.1 which tells that switching around the terms in the cup product is the same as multiplying by -1, thus does not affect the solutions. This has as its benefit that when calculating the integral in the cup product we will not have k, l, m in the integral.
As already stated in the above remark $(b+ln)_l(c-b+(m-l)n)_{m-l}$ cannot divide out solutions from $\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle$. Next to this, the set of Q^{n+1} we tested in chapter 5 does not show any occurrences of $(b+ln)_l(c-b+(m-l)n)_{m-l}$ having common terms with the numerator (unless of course when $m<l$, which makes $(c-b+(m-l)n)_{m-l}$ into a fraction), indicating that dividing by it does not simplify the numerator. Furthermore, in appendix A there is a proof as to why it will never have common terms with the numerator for $0 \leq l \leq m \leq k$. We only proved it for this set as it is the simplest case, for other cases it would require calculating more residues (see Lemma 4.4.1).

4.4 Constructively Calculating the Cup Product

Here we will mainly focus on calculating $\langle \omega, \omega_{k,l,m} \rangle$, of which the process is the same as for $\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle$ and most steps can be applied to the cup product in general. Here we will introduce a new function in order to limit the amount of terms we get when expanding the cup product, next to that we will also show when certain residues will become 0.
Let $r = l$, $s = m - l$, $t = -k$. We pick G_N to be an expansion most useful for calculating the cup product

$$\begin{aligned} G_N(p_1, p_2, C) &:= \text{coefficient of } x^N \text{ in } (1-x)^{p_1}(1+Cx)^{p_2} \\ &= \sum_{i=0}^{N} \binom{p_1}{i} (-1)^i \binom{p_2}{N-i} C^{N-i}. \end{aligned} \tag{35}$$

First the cup product at 0.

$$\begin{aligned} \langle \omega, \omega_{k,l,m} \rangle_0 &= \mathrm{Res}_{x=0} \left(\int x^{\rho-1}(1-x)^{\sigma-1}(1-zx)^{\tau-1} dx \right) x^{r-\rho}(1-x)^{s-\sigma}(1-zx)^{t-\tau} dx \\ &= \mathrm{Res}_{x=0} \left(\int x^{\rho-1} \sum_{N\geq 0} G_N(\sigma-1, \tau-1, -z) x^N dx \right) x^{r-\rho} \sum_{M \geq 0} G_M(s-\sigma, t-\tau, -z) x^M dx \\ &= \mathrm{Res}_{x=0} \sum_{N \geq 0} \frac{1}{\rho+N} G_N(\sigma-1, \tau-1, -z) \sum_{M\geq 0} G_M(s-\sigma, t-\tau, -z) x^{r+N+M} dx \\ &= \sum_{M+N=-r-1} \frac{1}{\rho+N} G_N(\sigma-1, \tau-1, -z) G_M(s-\sigma, t-\tau, -z) \\ &= \sum_{M+N=-l-1} \frac{1}{\rho+N} G_N(\sigma-1, \tau-1, -z) G_M(m-l-\sigma, -k-\tau, -z) \end{aligned}$$

where $M, N \in \mathbb{Z}_{\geq 0}$. The main advantage you can see here is that we will not have one big summation going over 4 variables, which you would normally get if you would write out taking the residue. This is also really for when programming the cup product. Similarly to $x = 0$ we get

$$\begin{aligned}
\langle \omega, \omega_{k,l,m} \rangle_1 &= (1-z)^{-k-1} \sum_{M+N=l-m-1} \frac{1}{\sigma+N} G_N(\rho-1, \tau-1, \frac{z}{1-z}) G_M(l-\rho, -k-\tau, \frac{z}{1-z}) \\
\langle \omega, \omega_{k,l,m} \rangle_{\frac{1}{z}} &= z^{-m}(z-1)^{m-l-1} \sum_{M+N=k-1} \frac{1}{\tau+N} G_N(\rho-1, \sigma-1, \frac{1}{z-1}) G_M(l-\rho, m-l-\sigma, \frac{1}{z-1}) \\
\langle \omega, \omega_{k,l,m} \rangle_\infty &= (-1)^{m-l-k} z^{-k-1} \sum_{M+N=m-k-1} \frac{1}{-\rho-\sigma-\tau+2+N} G_N(\sigma-1, \tau-1, \frac{-1}{z}) \\
&\times G_M(m-l-\sigma, -k-\tau, \frac{-1}{z})
\end{aligned}$$

Lemma 4.4.1. *Under the following conditions terms of $\langle \omega, \omega_{k,l,m} \rangle = \sum_{p \in \{0,1,1/z,\infty\}} \langle \omega, \omega_{k,l,m} \rangle_p$ will be 0.*

- *If $-l \leq 0$, $\langle \omega, \omega_{k,l,m} \rangle_0 = 0$.*
- *If $l - m \leq 0$, $\langle \omega, \omega_{k,l,m} \rangle_1 = 0$.*
- *If $k \leq 0$, $\langle \omega, \omega_{k,l,m} \rangle_{1/z} = 0$.*
- *If $m - k \leq 0$, $\langle \omega, \omega_{k,l,m} \rangle_\infty = 0$.*

Proof. The above summations with G_N for those cup products do not have terms if the above inequalities are met. □

4.5 Proving the Associated Quadruples

We now have all results needed for our proof for the associated quadruples from Theorem 3.1.1.

Proof of Theorem 3.1.1. From Lemma 4.3.4 we know that if we have an admissible quadruple (a, b, c, z), then $W^{(n)} = 0$ (from Corollary 4.3.2). Thus, instead of looking at

$$F(a + kn, b + ln; c + mn; z) = S^{(n)} F$$

we can look at

$$\omega_{k,l,m}^{(n+1)} = \omega_{kn,ln,mn} = Y^{(n)} \omega_{0,0,0} \tag{36}$$

with

$$Y^{(n+1)} = \prod_{i=0}^{n} X^{(i)} = \frac{(b)_{ln}(c-b)_{(m-l)n}}{(c)_{mn}} S^{(n+1)}.$$

This is a stronger statement, which will allow us to connect Kummer's solutions to the hypergeometric differential equation (Lemma 2.4.2) in a more intuitive way.

During the proof for Kummer's solutions one of the things we used were conformal mappings of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with which we permuted the singularities $0, 1, \infty$. Since we were applying the mappings on z, we could not map the fourth singularity of the integrand of Euler's integral representation, $1/z$, to the other singularities. Since we are now working with differential forms definite in x, this is possible. With there being 4 singularities we can create 24 permutations (same number as there exist associated quadruples). These can be all be obtained by making pairs of the following 2 sets of maps:

$$x \mapsto x, 1-x, 1/x, 1/(1-x), 1-x^{-1}, x/(1-x)$$

and

$$x \mapsto x, \frac{x-\frac{1}{z}}{x-1}, \frac{1}{zx}, \frac{x-1}{zx-1}.$$

The last set here decides which singularity gets placed where singularity $\frac{1}{z}$ used to be, while the first are just permutations of the singularities $0, 1, \infty$.
By applying all the aforementioned permutations, we get all the differential forms in the associated quadruples. As an example, the map $x \mapsto 1/x$. Applying this substitution to $\omega_{k,l,m}$ results in

$$\begin{aligned}
[\omega_{k,l,m}]_{[x\mapsto 1/x]} &= \left(x^{b+l-1}(1-x)^{c-b+m-l-1}(1-zx)^{-a-k}dx\right)_{[x\mapsto 1/x]} \\
&= \frac{1}{x}^{b+l-1}(1-\frac{1}{x})^{c-b+m-l-1}(1-\frac{z}{x})^{-a-k}\frac{dx}{-x^2} \\
&= (-1)^{c-b-a+m-l-k}z^{a+k}x^{a-c+k-m}(1-x)^{c-b+m-l-1}(1-\frac{x}{z})^{-a-k}dx \\
&= (-1)^{c-b-a+m-l-k}z^{a+k}[\omega_{k,k-m,k-l}]_{[z\mapsto 1/z, a\mapsto a, b\mapsto a-c+1, c\mapsto a-b+1]}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
S^{(n)}F(a, a+1-c; a+1-b; 1/z) = &\frac{(-1)^{(m-k-l)n}(c)_{mn}(a+1-c)_{(k-m)n}}{(b)_{ln}(a+1-b)_{(k-l)n}}z^{-kn} \\
&\times F(a+km, a+1-c+(k-m)n; a+1-b+(k-l)n; 1/z)
\end{aligned}$$

Meaning we now have the admissible quadruple $(a, a+1-c, a+1-b, 1/z)$ for $(k, k-m, k-l)$. This matches associated quadruple (ix). One can optionally use the equalities between Kummer's solutions to also instantly get (x), (xi) and (xii)
By repeating this process for all the other permutations we obtain the full list.

□

5 Observations and Claims

For this thesis, we wrote a program capable of calculating admissible quadruples and simplified admissible quadruples (introduced in definition 4.3.6). One of the things we used it for was looking for patterns in these admissible equations. From Lemma 3.1.3 we got that we do not need to look at cases in which k, l, m are negative. By also using the symmetry between a and b we got that we only have to look at $l \leq k$ for the regular admissible equations. So we used this and an upper bound of 12 on k, l, m to make our observations. From this, we made the following observations/claims.

Claim 5.0.1. *For $k, l, m \geq 0$ the degree of the admissible equation in n is $\max(k, m) + \max(l, m) - 1$ and for the simplified admissible equation we get degree $\max(k + l - m, k, l, m) - 1$. Thus a lot lower for the simplified equation.*

Claim 5.0.2. *For $k, l, m \geq 0$, if you take the leading coefficient of the (simplified) admissible equation in n, then it is a polynomial over $\mathbb{Z}(z)$*

This interesting since all z then have to be a solution to the leading coefficient which is then independent of a, b, c.

Claim 5.0.3. *For $k, l, m \geq 0$ the degree of the leading coefficient in z of the simplified admissible equation is $m - \min(k, l) - 1$ if $m \geq k + l$ and if $m \leq k + l$ it is $\max(k, l) - 1$ (notice the overlap). For the not simplified admissible equation, you would have to add 1 when $m = 0$ or $m \geq k + l$.*

The following chapters will go onto prove these claims to a certain degree for the simplified admissible equation. The cases for the not simplified equation can be derived by making minor changes to the proves.
First, we will prove the claim gives a correct upper bound on the degree in n. This one is probably the most interesting as it shows how certain values cancel each other out, which then results in the claimed degree.
Secondly, we will start looking into proving that the degree in n is exact. This is overall just a process of carefully removing terms of a lower degree until the result becomes apparent. Hence we will only demonstrate this for $l \leq m \leq k$ as for the other case it will likely follow similarly. Here we do prove the exact degree in z for $l \leq m \leq k$, but that is more like a by-product. In remark 9.4 of [Iwa17], they point out how little the degree in z matters. This is because all so far observed z for admissible quadruples with k, l, m not 0, is either a rational or quadratic number. This is while Claim 5.0.3 show how the degree in z increases rather fast. This means that even though we saw defining polynomials for z of degree 9 (this is the highest degree for which we both calculated the admissible quadruple and the Gröbner basis, for which we used the upper bound 10 on k, l, m), we still got at most quadratic values for z. Furthermore in theorem 2.3 of [Iwa17], there is an algorithm for calculating the equation for z which result from the leading coefficient (not the leading coefficient itself as it differs by a non-constant rational factor) for when $k + l \leq m$.

6 Proving the Upper Bound on the Degree in n

Here we will start proving Claim 5.0.1 for the simplified admissible equation by proving that it gives an upper bound to the degree. This is an important first step, as we will later see, quite a few terms will cancel each other out in the cup product.
Looking at Lemma 4.4.1, it appears that proving the upper bound is easier for certain k, l, m than for others. With the easiest being $l \leq m \leq k$ and $0 = k \leq l \leq m$ for which you only need to calculate 1 residue and the most difficult being $0 < k < m < l$ which requires 3 residues to be calculated (with the regular admissible equation you can turn this into the easiest case by using the symmetry between k and l, sadly we now do not have this property).

6.1 $l \leq m \leq k$

From Lemma 4.4.1 we can tell that for $l \leq m \leq k$ we only need to calculate the residue at $1/z$. So the complete calculation for $l \leq m \leq k$ is

$$\begin{aligned}
&\langle \omega^{(n+1)}, \omega^{(n+1)}_{k,l,m} \rangle && (37)\\
&= \langle \omega^{(n+1)}, \omega^{(n+1)}_{k,l,m} \rangle_{\frac{1}{z}} \\
&= z^{-m}(z-1)^{m-l-1} \sum_{M+N=k-1} \frac{1}{a+kn+N} G_N(-b-ln, b-c+(l-m)n, \frac{1}{z-1}) && (38)\\
&\quad \times G_M(l+b-1+ln, m-l-b-1+c+(m-l)n, \frac{1}{z-1})
\end{aligned}$$

Looking back at definition of G_N in (35), we can tell degree of G_N is at most N. Hence if we were to simply add all fractions in the summation above together we would get that the degree is bounded by $k-1+k-2 = 2k-3$ (higher in case k is small), which is far higher than our claim which states $\max(k+l-m, k, l, m) - 1 = k-1$. Hence it will take some effort to determine the correct bound.

Theorem 6.1.1. *If we have a summation of the form*

$$\sum_{M+N=L-1} \frac{1}{\rho+N} G_N(\sigma-1, \tau-1, f) G_M(s-\sigma, t-\tau, f) \tag{39}$$

where $L \in \mathbb{Z}_{\geq 0}$, $\rho, \sigma, \tau \in \mathbb{Z}[a, b, c][n]$ being linear (or constant) polynomials in n and $f \in \mathbb{Q}(z)$. Then the degree of the numerator of (39) is less or equal to $L-1$.

Proof. As we are looking for an upper bound on the degree in n, we can take that ρ, σ, τ are all of degree 1 in n (the theorem states they are either linear of constant in n, thus here we pick the highest possibility to get the upper bound).
We now start with turning it into a Laurent series in order to get the numerators together. Here we pick $\alpha_N = G_N(\sigma-1, \tau-1, f) G_{L-1-N}(s-\sigma, t-\tau, f)$.

$$\begin{aligned}
&\sum_{M+N=L-1} \frac{1}{\rho+N} G_N(\sigma-1, \tau-1, f) G_M(s-\sigma, t-\tau, f) \\
&= \sum_{M+N=L-1} \frac{\alpha_N}{\rho+N} \\
&= \sum_{M+N=L-1} \frac{\frac{1}{\rho}\alpha_N}{1+\frac{N}{\rho}} \\
&= \sum_{M+N=L-1} \frac{\alpha_N}{\rho} \sum_{p\geq 0} \left(\frac{-N}{\rho}\right)^p
\end{aligned}$$

$$= \sum_{p\geq 0}(-1)^p \sum_{N=0}^{L-1} \frac{\alpha_N}{\rho}\left(\frac{N}{\rho}\right)^p$$
$$= \sum_{p\geq 0}(-1)^p \frac{\sum_{N=0}^{L-1} N^p\alpha_N}{\rho^{p+1}}$$

Since we are only interested in the numerator, we multiply by the denominator as it would be before we took the Laurent series. Which would be $(\rho)_L = \sum_{i=1}^{L} \begin{bmatrix} L \\ i \end{bmatrix}\rho^i$

$$\sum_{i=1}^{L}\begin{bmatrix} L \\ i \end{bmatrix}\rho^i \sum_{p\geq 0}(-1)^p \frac{\sum_{N=0}^{L-1} N^p\alpha_N}{\rho^{p+1}}$$
$$= \sum_{i=1}^{L}\begin{bmatrix} L \\ i \end{bmatrix} \sum_{p\geq 0}(-1)^p \rho^{i-p-1}\sum_{N=0}^{L-1} N^p\alpha_N$$
$$= \sum_{i=1}^{L}\begin{bmatrix} L \\ i \end{bmatrix} \sum_{p=0}^{i-1}(-1)^p \rho^{i-p-1}\sum_{N=0}^{L-1} N^p\alpha_N$$

The last step is possible because this is the numerator, hence all terms with a denominator should cancel out.
Since we are interested in the power of ρ, we change the summations such that the first summation numerates over the power of ρ. For this we pick $j = i - p - 1$ and choose to longer sum over i. Since j is dependent on p, it means that the values p can take in the second summation also change (in a way such that the overall summation does not change). The new upper bound for p for a certain j is the highest value p such that $j = i - p - 1$ with $i \in \{1, 2, ..., L\}$, thus the upper bound on p is $L - 1 - j$. This gives us the following summations.

$$= \sum_{j=0}^{L-1}\rho^j \sum_{p=0}^{L-1-j}\begin{bmatrix} L \\ j+p+1 \end{bmatrix}(-1)^p \sum_{N=0}^{L-1} N^p\alpha_N$$

Let us now look at what effect on the degree in n the second and third summation have. Since $\begin{bmatrix} L \\ j+p+1 \end{bmatrix}$ and -1 are integers, it will largely depend on $\sum_{N=0}^{L} N^p\alpha_N$. Define $\theta = x\frac{d}{dx}$

$$\begin{aligned}
\sum_{N=0}^{L-1} N^p\alpha_N &= \sum_{N+M=L-1} N^p G_N(\sigma-1,\tau-1,f)G_M(s-\sigma,t-\tau,f) \\
&= \text{ coefficient of } x^{L-1} \text{ in } \sum_{N\geq 0} N^p G_N(\sigma-1,\tau-1,f)x^N \sum_{M\geq 0} G_M(s-\sigma,t-\tau,f)x^M \\
&= \text{ coefficient of } x^{L-1} \text{ in } \left(\theta^p \sum_{N\geq 0} G_N(\sigma-1,\tau-1,f)x^N\right) \sum_{M\geq 0} G_M(s-\sigma,t-\tau,f)x^M
\end{aligned}$$

Start with the case that $p = 0$, we then get

$$\begin{aligned}
&\text{coefficient of } x^{L-1} \text{ in } \left(\theta^0 \sum_{N\geq 0} G_N(\sigma-1,\tau-1,f)x^N\right) \sum_{M\geq 0} G_M(s-\sigma,t-\tau,f)x^M \\
=&\text{ coefficient of } x^{L-1} \text{ in } (1-x)^{\sigma-1}(1+fx)^{\tau-1}(1-x)^{s-\sigma}(1+fx)^{t-\tau} \\
=&\text{ coefficient of } x^{L-1} \text{ in } (1-x)^{s-1}(1+fx)^{t-1}
\end{aligned}$$

Which hence has degree 0 in n. For $p=1$ we get

$$
\begin{aligned}
&\text{coefficient of } x^{L-1} \text{ in } \left(\theta^1 \sum_{N\geq 0} G_N(\sigma-1,\tau-1,f)x^N\right) \sum_{M\geq 0} G_M(s-\sigma,t-\tau,f)x^M \\
=\ &\text{coefficient of } x^{L-1} \text{ in } \left(\theta(1-x)^{\sigma-1}(1+fx)^{\tau-1}\right)(1-x)^{s-\sigma}(1+fx)^{t-\tau} \\
=\ &\text{coefficient of } x^{L-1} \text{ in } \left(-x(\sigma-1)(1-x)^{\sigma-2}(1+fx)^{\tau-1} + fx(\tau-1)(1-x)^{\sigma-1}(1+fx)^{\tau-2}\right) \\
&(1-x)^{s-\sigma}(1+fx)^{t-\tau} \\
=\ &\text{coefficient of } x^{L-1} \text{ in } -x(\sigma-1)(1-x)^{s-2}(1+fx)^{t-1} + fx(\tau-1)(1-x)^{s-1}(1+fx)^{t-2}
\end{aligned}
$$

So here the degree is $1=p$ in n. This is starting to look like what we want, since $j+p\leq L-1$. Let us now do this calculation more generally and in order to keep things comprehensible we will drop terms which have a lower degree in n than the upper bound.

$$
\begin{aligned}
&\text{coefficient of } x^{L-1} \text{ in } \left(\theta^p \sum_{N\geq 0} G_N(\sigma-1,\tau-1,f)x^N\right) \sum_{M\geq 0} G_M(s-\sigma,t-\tau,f)x^M \\
=\ &\text{coefficient of } x^{L-1} \text{ in } \left(\theta^p(1-x)^{\sigma-1}(1+fx)^{\tau-1}\right)(1-x)^{s-\sigma}(1+fx)^{t-\tau} \\
=\ &\text{coefficient of } x^{L-1} \text{ in} \\
&\left(\sum_{i=0}^{p} x^p \binom{p}{i}(-1)^i(\sigma-1-i+1)_i(1-x)^{\sigma-1-i} f^{p-i}(\tau-1-p+i-1)_{p-i}(1+fx)^{\tau-1-p+i} + O(n^{p-1})\right) \\
&(1-x)^{s-\sigma}(1+fx)^{t-\tau} \\
=\ &\text{coefficient of } x^{L-1} \text{ in } \sum_{i=0}^{p} x^p \binom{p}{i}(-1)^i\sigma^i(1-x)^{s-1-i} f^{p-i}\tau^{p-i}(1+fx)^{t-1-p+i} + O(n^{p-1})
\end{aligned}
$$

Here we see that the degree is at most $i+p-i=p$ in n.
Now zooming back out we get

$$
\begin{aligned}
&\sum_{j=0}^{L-1}\rho^j \sum_{p=0}^{L-1-j} \begin{bmatrix} L \\ j+p+1 \end{bmatrix} (-1)^p \sum_{N=0}^{L-1} N^p \alpha_N \\
&= \sum_{j=0}^{L-1}\rho^j \sum_{p=0}^{L-1-j} \begin{bmatrix} L \\ j+p+1 \end{bmatrix} (-1)^p O(n^p) \\
&= \sum_{j=0}^{L-1}\rho^j O(n^{L-1-j}) \\
&= O(n^{L-1})
\end{aligned}
$$

Hence the upper bound on the degree in n is $L-1$. □

If we apply Theorem 6.1.1 to equation (38), we get that the upper bound of the degree for $l\leq m\leq k$ is $k-1$, just as Claim 5.0.1 says.

6.2 Other Cases

The cases which are now left are $m\leq k,l$, $k,l\leq m$ and $k\leq m\leq l$. These can now easily be done with Theorem 6.1.1.

6.2.1 $m \leq k, l$

Here we need to work with 2 residues. Note here that in the proof of Theorem 6.1.1 we did not use that in the cup product the numerator and denominator can partially cancel each other out (we even multiplied the fraction with the denominator to get the numerator), hence for the case that $l \leq m \leq k$ this would mean

$$\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_{\frac{1}{z}} = \frac{O(n^{m-k-1})}{(a+kn)_k}.$$

If we go on to apply this to $m \leq k, l$ we get

$$\begin{aligned}\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle &= \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_{1/z} + \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_1 \\ &= \frac{O(n^{k-1})}{(a+kn)_k} + \frac{O(n^{l-m-1})}{(b+1-c+(l-m)n)_{l-m}} \\ &= \frac{O(n^{l-m+k-1})}{(a+kn)_k (b+1-c+(l-m)n)_{l-m}}\end{aligned}$$

Hence the upper bound on the degree is $l - m + k - 1 = k + l - m - 1$.

6.2.2 $k, l \leq m$

Here we get

$$\begin{aligned}\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle &= \left(\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_\infty + \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_{1/z} \right) \\ &= \frac{O(n^{m-k-1})}{(c-a+(m-k)n)_{m-k}} + \frac{O(n^{k-1})}{(a+kn)_k} \\ &= \frac{O\left(n^{m-1}\right)}{(c-a+(m-k)n)_{m-k}(a+kn)_k}.\end{aligned}$$

So we get upper bound on the degree $m - 1 = \max(k + l - m, m, k, l) - 1$.

6.2.3 $k \leq m \leq l$

Here we get

$$\begin{aligned}\langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle &= \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_1 + \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_\infty + \langle \omega^{(n+1)}, \omega_{k,l,m}^{(n+1)} \rangle_{1/z} \\ &= \frac{O(n^{l-m-1})}{(b+1-c+(l-m)n)_{l-m}} + \frac{O(n^{m-k-1})}{(c-a+(m-k)n)_{m-k}} + \frac{O(n^{k-1})}{(a+kn)_k} \\ &= \frac{O(n^{l-1})}{(b+1-c+(l-m)n)_{l-m}(c-a+(m-k)n)_{m-k}(a+kn)_k}\end{aligned}$$

Hence the upper bound on the degree is $l - 1 = \max(k + l - m, k, l, m) - 1$ and thus proving our claim for the upper bound on the degree.

7 Proving the Exact Degree in n for $l \leq m \leq k$

Here we will go onto prove the exact degree in n in the simplified admissible equation for $0 \leq l \leq m \leq k$, but before we can do that we need to prove what the denominator of it.

Claim 7.0.1. *For $k, l, m \geq 0$ the denominator of $\langle \omega^{(n+1)}, \omega^{(n+1)}_{k,l,m} \rangle$ after cancelling common factors is*

$$(a + kn)_k (c - a + (m - k)n)_{m-k} z^m (z - 1)^{k+l-m}$$

while not including the factors which are a fraction for the given k, l, m.

Theorem 7.0.2. *For $0 \leq l \leq m \leq k$ the denominator of $\langle \omega^{(n+1)}, \omega^{(n+1)}_{k,l,m} \rangle$ after cancelling common factors is*

$$(a + kn)_k z^m (z - 1)^{k+l-m}.$$

Proof. First note here that $(a+kn)_k(z-1)^{k+l-m}z^m$ being the denominator of $\langle \omega^{(n+1)}, \omega^{(n+1)}_{k,l,m} \rangle$ is due to the substitution equivalent to $(a)_k(z-1)^{k+l-m}z^m$ being the denominator of $\langle \omega, \omega_{k,l,m} \rangle$.

$$\langle \omega, \omega_{k,l,m} \rangle = z^{-m}(z-1)^{m-l-1} \sum_{M+N=k-1} \frac{1}{a+N} G_N(-b, b-c, \frac{1}{z-1}) G_M(l+b-1, m-l+c-b-1, \frac{1}{z-1})$$

In the above a does not occur in the numerator, meaning $(a)_k$ must be part of the denominator.
Now to prove that $(z-1)^{k+l-m}$ is part of the denominator. In order to determine this, we will fully expand the cup product into 1 summation

$$\langle \omega, \omega_{k,l,m} \rangle = z^{-m}(z-1)^{m-l-1} \sum_{d+e+f+g=k-1} \binom{-b}{d}\binom{b-c}{e}\binom{l+b-1}{f}\binom{m-l+c-b-1}{g} \tag{40}$$
$$\frac{1}{a+d+e}(-1)^{d+f}(z-1)^{-e-g}$$

where $d, e, f, g \in \mathbb{Z}_{\geq 0}$. If we prove that the sum of all terms in summation with $(z-1)^{k-1}$ in denominator is non zero, then that means the denominator cannot be simplified as z is not in the numerator of the terms with $(z-1)^{k-1}$ in the denominator. We are now talking about the terms with $e + g = k - 1$, meaning

$$(z-1)^{-k+1} \sum_{e+g=k-1} \binom{b-c}{e}\binom{m-l+c-b-1}{g} \frac{1}{a+e}$$
$$= (z-1)^{-k+1} \sum_{e=0}^{k-1} \frac{(b-c-e+1)_e (m-l+c-b-k+1+e)_{k-1-e}}{(a+e)e!(k-1-e)!}$$

Since we are only interested in if it is non zero, we only need to prove that the coefficient of $(c-b)^{k-1}$ is non zero. We can now drop all terms which are not needed for this, giving us something to which we can apply Lemma 2.3.1

$$\sum_{e=0}^{k-1} \frac{(b-c)^e (c-b)^{k-1-e}}{(a+e)e!(k-1-e)!} = (c-b)^{k-1} \sum_{e=0}^{k-1} \frac{(-1)^e}{(a+e)e!(k-1-e)!}$$
$$= \frac{(c-b)^{k-1}}{(a)_k}$$

hence non zero.
We have now proven that the denominator of the summation in (40) is $(a)_k(z-1)^{k-1}$. Separating this summation from $(z-1)^{1-k}$ gives us

$$\langle \omega, \omega_{k,l,m} \rangle = \frac{1}{z^m(z-1)^{k+l-m}} \sum_{d+e+f+g=k-1} \binom{-b}{d}\binom{b-c}{e}\binom{l+b-1}{f}\binom{m-l+c-b-1}{g} \tag{41}$$
$$\times \frac{1}{a+d+e}(-1)^{d+f}(z-1)^{k-1-e-g}$$

Thus $(z-1)^{k+l-m}$ is part of the denominator.
We now have all the tools to tell if the suggested denominator contains all necessary factors for the denominator. By using similar expansion steps as in the proof of Theorem 6.1.1, we get that

$$\begin{aligned}\langle \omega, \omega_{k,l,m} \rangle = &\frac{1}{(a)_k z^m (z-1)^{l+1-m}} \sum_{j=0}^{k-1} a^j \sum_{p=0}^{k-1-j} \begin{bmatrix} k \\ j+p+1 \end{bmatrix} (-1)^p \\ &\times \text{coefficient of } x^{k-1} \text{ in } \left(\theta^p (1-x)^b (1+\frac{1}{z-1}x)^{b-c} \right) \left((1-x)^{l+b-1} (1-\frac{1}{z-1}x)^{m-l+c-b-1} \right)\end{aligned}$$

Here the question mainly lies in determining what the denominator is of the second summation. After having applied θ all the b and c in the powers of $(1-x)$ and $(1+\frac{1}{z-1}x)$ will cancel each other out. Hence, when we expand all terms in order to get the coefficient of x^{k-1}, we see that the binomial theorem gives a summation containing binomial coefficients filled with integers since b and c in the powers of $(1-x)$ and $(1+\frac{1}{z-1}x)$ have cancelled each other out as mentioned earlier. Which means that the binomial coefficients will be all integers. Since Stirling numbers are also just integers, we get that the denominator of the second summation is $(z-1)^{k-1}$. Thus after cancellation of common factors the denominator of Q has to be a factor of $(a)_k(z-1)^{k+l-m}z^m$.
Now the final step of the proof, showing that z^{m-1} is part of the denominator. One way of proving this is by showing that for the cup product as depicted in (41) that the summation contains a constant term (non-zero coefficient of z^0). From what we have proven before we know that denominator of that summation is $(a)_k$. In order to prove that summations is not divisible by z we are going to prove that the coefficient of $c^{k-1}z^0$ is not 0. To do this we remove all terms which do not affect the coefficient

$$\begin{aligned}&\text{coefficient of } c^{k-1}z^0 \text{ in } \sum_{d+e+f+g=k-1} \binom{-b}{d}\binom{b-c}{e}\binom{l+b-1}{f}\binom{m-l+c-b-1}{g} \\ &\times \frac{1}{a+d+e}(-1)^{d+f}(z-1)^{k-1-e-g} \\ &= \text{coefficient of } c^{k-1}z^0 \text{ in } \sum_{e+g=k-1} \binom{-b}{0}\binom{-c}{e}\binom{l-1}{0}\binom{m-l+c-1}{g}\frac{1}{a+e} \\ &= \text{coefficient of } c^{k-1}z^0 \text{ in } \sum_{e+g=k-1} \binom{-c}{e}\binom{m-l+c-1}{g}\frac{1}{a+e} \\ &= \text{coefficient of } c^{k-1}z^0 \text{ in } c^{k-1}\sum_{e=0}^{k-1} \frac{(-1)^{-e}}{(a+e)e!(k-1-e)!} \\ &= \frac{1}{(a)_k} \qquad \text{(Lemma 2.3.1)}.\end{aligned}$$

Hence the coefficient in the numerator of $c^{k-1}z^0$ is always 1. This means that z^m is part of the denominator. We have now proven that denominator after cancelling common factors is contained in $(a)_k(z-1)^{k+l-m}z^m$ and also always contains $(a)_k(z-1)^{k+l-m}z^m$. Hence we have proven the theorem. □

Now that we have proven the denominator we can focus on proving the exact degree without having to worry about terms being divided out.

Theorem 7.0.3. *For $0 \leq l \leq m \leq k$ the degree of the simplified admissible equation in n is $k-1$ and the leading coefficient in n is of degree $k-1$ in z.*

Proof. The strategy for this proof is first proving an upper bound on the degree in z for the leading coefficient. Then we prove that the coefficient of $n^{k-1}z^{k-1}$ is not 0. Since we already proved that the upper bound on the degree in n is $k-1$, this means that the degree is always $k-1$.

First determining how the numerator of $\langle \omega, \omega_{k,l,m} \rangle$ looks like.

$$\langle \omega, \omega_{k,l,m} \rangle = \frac{z^{-m}(z-1)^{m-l-1}}{(a)_k} \sum_{j=0}^{k-1} a^j \sum_{p=0}^{k-1-j} \begin{bmatrix} k \\ j+p+1 \end{bmatrix} (-1)^p$$
$$\times \text{ coefficient of } x^{k-1} \text{ in } \left(\theta^p (1-x)^{-b} (1 + \frac{1}{z-1} x)^{b-c} \right) (1-x)^{l+b-1} (1 + \frac{1}{z-1} x)^{m-l+c-b-1}$$

For this we first notice that we can pick $p = k-1-j$, as else we will not get the highest degree in c and b when expanding θ (while proving the upper bound on the degree we showed that then the combined degree in c and b is at most $k-1-j$, which together with the degree of a^j adds up to $k-1$. This guarantees that we are not disposing n to the highest degree by focusing only on terms of higher degree which cancel out).

$$\frac{z^{-m}(z-1)^{m-l-1}}{(a)_k} \sum_{j=0}^{k-1} a^j (-1)^{k-1-j}$$
$$\times \text{ coefficient of } x^{k-1} \text{ in } \left(\theta^{k-1-j} (1-x)^{-b} (1 + \frac{1}{z-1} x)^{b-c} \right) (1-x)^{l+b-1} (1 + \frac{1}{z-1} x)^{m-l+c-b-1}$$
$$= \frac{z^{-m}(z-1)^{m-l-1}}{(a)_k} \sum_{j=0}^{k-1} a^j (-1)^{k-1-j}$$
$$\times \text{ coefficient of } x^{k-1} \text{ in } \left(\frac{1}{z-1} \right)^{m-l-1} \left(\theta^{k-1-j} (1-x)^{-b} (z-1+x)^{b-c} \right) (1-x)^{l+b-1} (z-1+x)^{m-l+c-b-1}$$

We now work out the derivation while removing all terms which do not contribute to the coefficient of n^{m+k-1} (if we would substitute n in). From now on we will use $\equiv$ when we remove terms which do not contribute to the coefficient of n^{k-1}.

$$\theta^{k-1-j} (1-x)^{-b} (z-1+x)^{b-c}$$
$$\equiv x^{k-1-j} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} (-b-i+1)_i (-1)^i (1-x)^{-b-i} (b-c-k+j+i+2)_{k-1-j-i} (z-1+x)^{b-c-k+1+j+i}$$
$$\equiv x^{k-1-j} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} (-b)^i (-1)^i (1-x)^{-b-i} (b-c)^{k-1-j-i} (z-1+x)^{b-c-k+1+j+i}$$

Now putting it back in

$$\text{coefficient of } x^{k-1} \text{ in } \left(\frac{1}{z-1} \right)^{m-l-1} \left(\theta^{k-1-j} (1-x)^{-b} (z-1+x)^{b-c} \right) (1-x)^{l+b-1} (z-1+x)^{m-l+c-b-1}$$
$$\equiv \text{coefficient of } x^{k-1} \text{ in } \left(\frac{1}{z-1} \right)^{m-l-1} x^{k-1-j} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} (-b)^i (-1)^i (1-x)^{-b-i}$$
$$\times (b-c)^{k-1-j-i} (z-1+x)^{b-c-k+1+j+i} (1-x)^{l+b-1} (z-1+x)^{m-l+c-b-1}$$
$$= \text{coefficient of } x^{j} \text{ in } \left(\frac{1}{z-1} \right)^{m-l-1} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} b^i (1-x)^{l-i-1}$$
$$\times (b-c)^{k-1-j-i} (z-1+x)^{m-l-k+j+i}$$

We now switch our focus to proving the upper bound on the degree in z in the coefficient of n^{m+k-1}, hence we temporarily remove certain terms, which do not affect the degree in x or z directly. This makes it possible to simplify it to this

$$\text{coefficient of } x^{j} \text{ in } \left(\frac{1}{z-1} \right)^{m-l-1} (1-x)^{l-i-1} (z-1+x)^{m-l-k+j+i}$$

we now add in the $z^{-m}(z-1)^{m-l-1}$ we had at the start, we also put the terms which are in the denominator of the cup product in the denominator.

$$\text{coefficient of } x^j \text{ in } \frac{(z-1)^{k+l-m}(1-x)^{l-1-i}(z-1+x)^{m-l-k+j+i}}{z^m(z-1)^{k+l-m}}$$

As numerator we then get

$$\text{coefficient of } x^j \text{ in } (z-1)^{k+l-m}(1-x)^{l-i-1}(z-1+x)^{m-l-k+j+i}.$$

In order to prove the upper bound on the degree in z, we simply need to write it out

$$\begin{aligned}
&\text{coefficient of } x^j \text{ in } (z-1)^{k+l-m}(1-x)^{l-i-1}(z-1+x)^{m-l-k+j+i}\\
&= \text{coefficient of } x^j \text{ in } (z-1)^{j+i}(1-x)^{l-i-1}(1+\frac{1}{z-1}x)^{m-l-k+j+i}\\
&= \text{coefficient of } x^j \text{ in } (z-1)^{j+i}\sum_{e=0}^{\infty}\frac{(i-l+1)_e}{e!}x^e\sum_{d=0}^{\infty}(\frac{1}{z-1}x)^d\binom{m-l-k+j+i}{d}\\
&= \text{coefficient of } x^j \text{ in } (z-1)^{j+i}\sum_{e=0}^{j}\frac{(i-l+1)_e}{e!}x^e(\frac{1}{z-1}x)^{j-e}\binom{m-l-k+j+i}{j-e}\\
&= \sum_{e=0}^{j}\frac{(i-l+1)_e}{e!}(z-1)^{e+i}\frac{(m-l-k+i+e+1)_{j-e}}{(j-e)!}.
\end{aligned}$$

Since $i \leq k-1-j$ and $e \leq j$, we get that the upper bound on the degree in z is $k-1$. Hence we can reduce the summation to setting $i = k-1-j$ and $e = j$. We can also further go onto find the then coefficient

$$\text{coefficient of } z^{k-1} \text{ in } \frac{(k-j-l)_j}{j!}(z-1)^{k-1}\frac{(m-l-1)_0}{0!} = \frac{(k-j-l)_j}{j!}$$

We put this back into what we previously had while also removing terms which belong in the denominator

$$\begin{aligned}
&z^{k-1} \times \text{coefficient of } x^j z^{k-1} \text{ in } \sum_{i=0}^{k-1-j}\binom{k-1-j}{i}b^i(1-x)^{l-i-1}\\
&\times (b-c)^{k-1-j-i}(z-1+x)^{m-l-k+j+i}\\
&= z^{k-1} \times \text{coefficient of } x^j z^{k-1} \text{ in } \binom{k-1-j}{k-1-j}b^{k-1-j}(1-x)^{l-k+j}\\
&\times (b-c)^0(z-1+x)^{m-l-1}\\
&= b^{k-1-j}\frac{(k-j-l)_j}{j!}z^{k-1}.
\end{aligned}$$

We put this into the full numerator (here we use $\equiv$ when we remove terms which do not affect the coefficient of $z^{k-1}n^{k-1}$)

$$\begin{aligned}
&\sum_{j=0}^{k-1} a^j(-1)^{k-1-j} \\
&\quad\times \text{coefficient of } x^{k-1} \text{ in } \left(\frac{1}{z-1}\right)^{m-l-1} \left(\theta^{k-1-j}(1-x)^{-b}(z-1+x)^{b-c}\right)(1-x)^{l+b-1}(z-1+x)^{m-l+c-b-1} \\
\equiv &\sum_{j=0}^{k-1} a^j(-1)^{k-1-j} \\
&\quad\times \text{coefficient of } x^{j} \text{ in } \left(\frac{1}{z-1}\right)^{m-l-1} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} b^i(1-x)^{l-i-1}(b-c)^{k-1-j-i}(z-1+x)^{m-l-k+j+i} \\
\equiv &\sum_{j=0}^{k-1} a^j(-1)^{k-1-j} \\
&\quad\times \text{coefficient of } x^{j} \text{ in } \left(\frac{1}{z-1}\right)^{m-l-1} \binom{k-1-j}{k-1-j} b^{k-1-j}(1-x)^{l-(k-1-j)-1}(b-c)^{0}(z-1+x)^{m-l-1} \\
\equiv &\sum_{j=0}^{k-1} a^j(-1)^{k-1-j}b^{k-1-j}\frac{(k-j-l)_j}{j!}z^{k-1}
\end{aligned}$$

Here we have that for certain value of j that $(k-j-l)_j$ is equal to 0. This greatly depends on if $k=l$, if $k=l$ it is never 0, but if e.g. $k-1=l$, then there is only one term in the sum that is not 0. Meaning we get completely different coefficients for $z^{k-1}n^{k-1}$ depending on this.
First the case that $k \neq l$

$$\begin{aligned}
&\sum_{j=0}^{k-1} a^j(-1)^{k-1-j}b^{k-1-j}\frac{(k-j-l)_j}{j!}z^{k-1} \\
= &\sum_{j=0}^{k-l-1} a^j(-1)^{k-1-j}b^{k-1-j}\frac{(k-j-l)_j}{j!}z^{k-1}.
\end{aligned}$$

Now to substitute in n with the map $(a,b,c) \mapsto (a+kn, b+ln, c+mn)$. Since we do not care about a,b,c after the substitution (possible since this is only the numerator), we can simplify it to $(a,b,c) \mapsto (kn, ln, mn)$, giving us

$$\begin{aligned}
&\sum_{j=0}^{k-l-1} (kn)^j(-1)^{k-1-j}(ln)^{k-1-j}\frac{(k-j-l)_j}{j!}z^{k-1} \\
&= z^{k-1}n^{k-1}\sum_{j=0}^{k-l-1} k^j(-l)^{k-1-j}\frac{(k-j-l)_j}{j!} \\
&= z^{k-1}n^{k-1}(-l)^l\sum_{j=0}^{k-l-1} k^j(-l)^{k-l-1-j}\binom{k-l-1}{j} \\
&= z^{k-1}n^{k-1}(-l)^l(k-l)^{k-l-1}.
\end{aligned}$$

This is non zero, as $k-l \neq 0$ and if $-l=0$, then we get[2] $(-l)^l = 1$. Thus we have proven this theorem for $0 \leq l \leq m \leq k$ with $k \neq l$.

[2] As we are here working over integers and not over real numbers (which do have limits), means that when we raise something to the power 0 it becomes equal to 1

Here the case for if $k = l$. Which means

$$\begin{aligned}
&\sum_{j=0}^{k-1} a^j (-1)^{k-1-j} b^{k-1-j} \frac{(k-j-l)_j}{j!} z^{k-1} \\
&= (-1)^{k-1} \sum_{j=0}^{k-l-1} a^j b^{k-1-j} z^{k-1}.
\end{aligned}$$

Again doing the same substitution, we get

$$\begin{aligned}
&(-1)^{k-1} \sum_{j=0}^{k-1} (kn)^j (ln)^{k-1-j} z^{k-1} \\
&= z^{k-1} n^{k-1} (-1)^{k-1} k^k
\end{aligned}$$

This is only zero if $k = 0$, but then the cup product itself is zero, which matches our degree claim. Hence it is non zero for all case with $k > 0$. Meaning we have now proven the theorem. $\square$

8 Results and Further Observations

Here we will go further in on what our results mean and talk about what we uncovered about the quantity of special values.

8.1 Trivial Admissible Quadruples

There is no bound on k, l, m such that no trivial admissible quadruples will occur outside it. With trivial we, in this case, mean that k, l, m have a common divisor d, causing them to have the same admissible quadruples as $(k/d, l/d, m/d)$. Here a proof for why this is. Assume you have (dk, dl, dm) and wish to show that it has the all the admissible quadruples (k, l, m) has. We can prove this by expanding the contiguity relations

$$\begin{aligned}
&F(a+dkn, b+dln; c+dmn; z)\\
&= Q^{(n)}_{dk,dl,dm} F'(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)\\
&+ R^{(n)}_{dk,dl,dm} F(a+dk(n-1), b+dl(n-1); c+dm(n-1), z).
\end{aligned}$$

Now we use the contiguity relations for (k, l, m), which gives us a tree like structure

$$\begin{aligned}
&F(a+dkn, b+dln; c+dmn; z)\\
&= Q^{(dn)}_{k,l,m} F'(a+dkn-1, b+dln-1; c+dmn-1, z) + R^{(dn)}_{k,l,m}\Big(\\
&\quad Q^{(dn-1)}_{k,l,m} F'(a+dkn-2, b+dln-2; c+dmn-2, z) + R^{(dn-1)}_{k,l,m}\Big(\\
&\quad \ldots\\
&\quad \Big(Q^{(d(n-1)+1)}_{k,l,m} F'(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)\\
&+ R^{(d(n-1)+1)}_{k,l,m} F(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)\Big)\Big)\ldots\Big)
\end{aligned}$$

each of the F' that occurs in this can be also expressed using contiguity relations, meaning we can again turn this back into a summation of $F'(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)$ and $F(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)$. Define

$$\begin{aligned}
&F'(a+dkn-i, b+dln-i; c+dmn-i, z)\\
&= X^{(i)} F'(a+dk(n-1), b+dl(n-1); c+dm(n-1), z)\\
&+ Y^{(i)} F(a+dk(n-1), b+dl(n-1); c+dm(n-1), z).
\end{aligned}$$

This gives us that

$$Q^{(n)}_{dk,dl,dm} = \sum_{i=0}^{n-1} Q^{(dn-i)}_{k,l,m} \prod_{j=0}^{i-1} R^{(dn-j)}_{k,l,m} X^{(i)}.$$

since every term in the summation contains $Q^{(dn-i)}_{k,l,m}$, $Q^{(n)}_{dk,dl,dm}$ has to become 0 when an admissible quadruple of (k, l, m) is taken.

The thing that makes trivial admissible quadruples so uninteresting is the fact, that as one might guess, only gives weaker versions of special values. This is because $S^{(n)}_{dk,dl,dm} = S^{(dn)}_{k,l,m}$.

Thus we can say that there are infinitely many special values, but the ones we have now proven are just all weaker versions of already existing ones. Which is an answer with which we are not satisfied with.

As a final note here, there are (dk, dl, dm) that have admissible quadruples which (k, l, m) does not have. We will later in subsection 8.3.2 give an example of this.

8.2 Degree of the Simplified Admissible Equation

From what we have obtained for $l \le m \le k$, we can tell that the simplified admissible equation is of degree $k-1$, which means k equations. Hence intuition would tell that if you can only pick (a, b, c, z) to solve these

equations there would not be any solutions for $k > 4$. In [Ebi17] had already found $(5, 2, 4)$ with degree 5 to have special values, thus showing that the degree for this does not matter.
Furthermore, when looking at (k, l, m) with an even higher degree for the simplified admissible quadruple there also appears to be no clear end to admissible quadruples occurring.

8.3 Special Values not Included by Ebisu

Ebisu only covers (k, l, m) included in the set of Theorem 3.1.2 with $m \leq 6$. We chose to look beyond this, for this we looked at the set of Theorem 3.1.2 with $k, l, m \leq 11$ and also looked at all $k, l, m \leq 10$ (these are lower boundaries than in the chapter 5 as now we are also looking at the gröbner bases, which means more time required for the calculations). In this subsection, we will look at several cases which we found interesting.

8.3.1 $(q, 1, q)$

While looking at the gröbner bases of different (k, l, m) certain patterns do appear to occur. Often when $l = m$ (same for $k = m$) there is a non trivial Gröbner basis, which looks rather similar to that of l and m increased by an integer. One of those is $(q, 1, q)$ with $q \in \mathbb{Z}_{\geq 2}$. Here there always appear to be at least these 2 admissible quadruples (it seems as if only for $q = 2$ and $q = 3$ there are more quadruples).

$$(a, \frac{a-1}{q}, a-1, \frac{q}{q-1}) \tag{42}$$

$$(a, \frac{a+q}{q}, a+2, \frac{q}{q-1}) \tag{43}$$

With this (42) gives the easiest case to explain, with it we have

$$S^{(n)} = \frac{(-1)^n (q-1)^{qn} q^{-qn} (a)_{qn}}{((q-1)/qa + 1/q)_{(q-1)n} \, (a/q + (q-1)/q)_n}$$

If we pick $a = 0$, $S^{(n)}$ itself becomes 0, giving us the special value

$$F(qn, -\frac{1}{q} + n; -1 + qn; \frac{q}{q-1}) = 0 \qquad \text{for } n \in \mathbb{Z}_{\geq 1}.$$

Proving that this special value exists for all $q \in \mathbb{Z}_{\geq 2}$ would require proving that simplified admissible equation would always be 0 for this admissible quadruple. Due to sheer magnitude of terms in the simplified admissible equation has (it contains 3 summation), we did not succeed in proving that Ebisu's method does give this special value for all $q \in \mathbb{Z}_{\geq 2}$ (calculations do show it is true for $q \leq 30$). We can however show that the above special special is true for all $q \in \mathbb{Z}_{\geq 2}$. This is possible as it is a rather simple reducible special value. Here we have

$$\begin{aligned} F(qn, -\frac{1}{q} + n; qn-1; z) &= \sum_{m=0}^{\infty} \frac{(-\frac{1}{q} + n)_m (qn + m - 1)}{m!(qn-1)} z^m \\ &= \frac{1}{qn-1} (\theta + qn - 1)(1-z)^{1/q - n} \\ &= \frac{(z + (1-z)q)}{q} (1-z)^{1/q - n - 1} \end{aligned}$$

which is equal to 0 when we pick $z = q/(q-1)$, thus proving the special value to be true.
What we now have can indicate that there exist infinitely many unique special values which can be obtained with Ebisu's method, but we are particularly interested in proving that Ebisu's method gives predictable results.
Overall there appear to be many similarities in the Gröbner basis of (k, l, m) where either $l = m$ or $k = m$ as for example the Gröbner bases for $(q, 1, q)$, $(q, 2, q)$ and $(q, 3, q)$ look close to identical. Thus to us it would not be surprising if with more study these could all be generalized by studying special values we have already been obtained.

8.3.2 (4,6,8)

Now excluding (k,l,m) with $k=m$ and $l=m$ we look at the Gröbner bases of (k,l,m) in the set of Theorem 3.1.2 with $k,l,m \leq 11$ (excluding those Ebisu covered). Here there are only 3 (k,l,m) where there are non trivial admissible quadruples, and the special value is not a really weak version of a well known identity (later an example of this). These are $(4,6,8)$, $(4,4,8)$ and $(5,5,10)$ which oddly enough all have both trivial and non trivial admissible quadruples.
Here we look deeper into the case of $(4,6,8)$. This is because, next to having interesting admissible quadruples, it also has admissible quadruples which exemplify why all the admissible quadruples we found outside of the 3 above mentioned are not interesting.
Let us first look at the Gröbner basis

$$\{(z-2)((z-16)z+16)(z(z+32)-32),(z-2)(4b-3c+1)(z-2)(2a-c),\\ (c-2a)(-(4b-3c+1)),(2a-c-3)(2a-c)(2a-c+3)\}.$$

This gives us admissible quadruples

$$(a,b,2a,2) \tag{44}$$
$$(a,\frac{3a-5}{2},2a-3,2) \tag{45}$$
$$(a,\frac{3a+4}{2},2a+3,2) \tag{46}$$
$$(a,\frac{6a-1}{4},2a,8+4\sqrt{3}) \tag{47}$$
$$(a,\frac{6a-1}{4},2a,8-4\sqrt{3}) \tag{48}$$
$$(a,\frac{6a-1}{4},2a,-16+12\sqrt{2}) \tag{49}$$
$$(a,\frac{6a-1}{4},2a,-16-12\sqrt{2}) \tag{50}$$

As $(4,6,8)$ has a common divisor, there is an overlap with the Gröbner basis for $(2,3,4)$ which is

$$\{(z-16)z+16,4b-3c+1,2a-c\}.$$

Thus we do not care for (47) and (48). Furthermore, all special values obtained from the first 3 admissible quadruples have $z=2$. This means that all special values deduced from those can be obtained from special values obtained from $(0,2,2)$ and $(1,2,2)$ by applying associated quadruples (as shown [Ebi17]). When applying associated quadruples you will also notice that you will get a weaker version of Kummer's theorem. This is what I meant earlier with being a weaker version of a well known identity. Most non trivial admissible admissible quadruples for (k,l,m) in the set of Theorem 3.1.2 with $m>6$ and $k,l \neq m$ are of this kind. These are recognized by having z equal to either $-1,0,1,2,1/2$ ($z=0$ or $z=1$ means they are a weaker versions of Gauss' summation formula, the others mean that they a weaker versions of Kummer's theorem). For the admissible quadruple $(a,\frac{6a-1}{4},2a,-16+12\sqrt{2})$ we have

$$\begin{aligned} R^{(n)} &= \frac{32(2a-5+8n)(2a-1+8n)}{3\left(4-3\sqrt{2}\right)^6(6a-13+24n)(6a-5+24n)} \\ &= \frac{8(2/4a-5/4+2n)_2}{27\left(4-3\sqrt{2}\right)^6(6/24a-13/24+n)(6/24a-5/24+n)} \end{aligned}$$

and

$$S^{(n)} = \frac{8^n(2/4a+3/4)_{2n}}{27^n\left(4-3\sqrt{2}\right)^{6n}(6/24a+11/24)_n(6/24a+19/24)_n}.$$

Thus we get the following special values:

1. For $a = \frac{1}{6}$

$$F(1/6+4n, 6n; 2/6+8n; -16+12\sqrt{2}) = \frac{8^n (18/24)_{2n}}{27^n \left(4-3\sqrt{2}\right)^{6n} (12/24)_n (20/24)_n}.$$

2. For $a = 0$ we get no special value, as we would then be working with a contiguity relation using both the original hypergeometric function definition and that in equation (1).

3. For $a = -4n$

$$F(-4n, -6n - \frac{1}{4}, -8n, -16+12\sqrt{2}) = \frac{27^n \left(4-3\sqrt{2}\right)^{6n} (-n+11/24)_n (-n+19/24)_n}{8^n (-2n+3/4)_{2n}}.$$

4. For $a = -4n + 1/6$

$$F(-4n+1/6, -6n, -8n+1/3, -16+12\sqrt{2}) = \frac{27^n \left(4-3\sqrt{2}\right)^{6n} (-n+1/2)_n (-n+5/6)_n}{8^n (-2n+5/6)_{2n}}.$$

Notice here that unlike the example we give in the subsection with $(q, 1, q)$, that when we interpolate these special values in n such that it does not need to be an integer, we get an irreducible special value. Special values that appear outside of Ebisu's boundary of $m \leq 6$ with either $k = m$ or $l = m$ seem to not have this property.
Onto $(a, \frac{6a-1}{4}, 2a, -16-12\sqrt{2})$, here we have

$$S^{(n)} = \frac{8^n (2/4a + 3/4)_{2n}}{27^n \left(4+3\sqrt{2}\right)^{6n} (6/24a + 11/24)_n (6/24a + 19/24)_n},$$

which is only slightly different from the $S^{(n)}$ of the previous admissible quadruple. Here we get the following special values:

1. For $a = \frac{1}{6}$

$$F(1/6+4n, 6n; 2/6+8n; -16-12\sqrt{2}) = \frac{8^n (18/24)_{2n}}{27^n \left(4+3\sqrt{2}\right)^{6n} (12/24)_n (20/24)_n}.$$

2. For $a = -4n$

$$F(-4n, -6n - \frac{1}{4}, -8n, -16+12\sqrt{2}) = \frac{27^n \left(4+3\sqrt{2}\right)^{6n} (-n+11/24)_n (-n+19/24)_n}{8^n (-2n+3/4)_{2n}}.$$

3. For $a = -4n + 1/6$

$$F(-4n+1/6, -6n, -8n+1/3, -16+12\sqrt{2}) = \frac{27^n \left(4+3\sqrt{2}\right)^{6n} (-n+1/2)_n (-n+5/6)_n}{8^n (-2n+5/6)_{2n}}.$$

Thus we see that interesting special values do exist outside of the set that Ebisu looked at, but there seem to be less of them, hence leaving it uncertain what is still to be found.

9 Conclusion

We started this thesis with giving a proper introduction to hypergeometric functions and Ebisu's method. In this part we even made some improvements on Ebisu's method, which due to time constraints we could not fully research.

In the rest of the thesis, we did a variety of things concerning Ebisu's method by using the cup product from [Beu18]. We, first of all, showed it can be applied to Ebisu's method by using it to calculate contiguity relations. For this, we also took some time to make the calculation easier to program by introducing the function G_N and making some other small observations.

The main added benefit we gained by applying the cup product is, that unlike other methods which use an inductive process to calculate them, this gives a direct calculation by several summations. Which made certain properties easier to spot and less chaotic.

Using the cup product we managed to get a different way of proving associated quadruple. This allows us a stronger way of looking at them in the form of differential forms.

During the thesis, we made several claims about the degree in n and z. One of these we proved is the upper bound on the degree in n. One of the benefits of having done so, is that it shows quite a few terms will cancel each other out during the calculation of the cup product.

We went further than that for the case $l \leq m \leq k$ as it has the benefit of only needing one residue to be calculated in order to determine the admissible equation. For this case, we proved the exact degree in n and z. Sadly, the degree in n does not appear to have a strong impact on the existence of admissible quadruples. Even when the number of equations that needs to be solved is higher than the number of variables we can choose to solve them.

Next, we made further observations about the special values that exist outside the set of (k, l, m) that Ebisu looked at in [Ebi17]. During this we again focused on our original question about the quantity of special values. What we found was firstly trivial admissible quadruples, which keep giving weaker versions of special values. Hence meaning that we have in a way proven that there are infinitely many special values, but not in a way that is by any means useful. After that we showed a pattern which does indicate the existence of infinite unique special values and we also proved that all those specials values are for a fact true. But again these special values have an predictable pattern to them, hence less interesting. We finished with some still interesting looking special values which we found which do have a real use to be being interpolated.

A Denominator of Q^{n+1} for $l \leq m \leq k$

This proof is mainly included because I have already taken the time to write it down. This will be some sort of continuation of the proof of Theorem 7.0.2.

Theorem A.0.1. *$(b + ln)_l(c - b + (m - l)n)$ is part of the denominator of $Q^{(n+1)}$.*

Proof. Just as in the proof of Theorem 7.0.2 we note that proving this is equivalent to proving that $(b)_l(c - b)_{m-l}$ is part of the denominator of Q.

$$Q = \frac{(c)_m}{(b)_l(c-b)_{m-l}} z^{1-m}(z-1)^{m-l} \sum_{M+N=k-1} \frac{1}{a+N} G_N(-b, b-c, \frac{1}{z-1}) G_M(l+b-1, m-l+c-b-1, \frac{1}{z-1})$$

So we wish to prove that the terms of $(b)_l(c - b)_{m-l}$ do not divide

$$(c)_m z^{1-m}(z-1)^{m-l} \sum_{M+N=k-1} \frac{1}{a+N} G_N(-b, b-c, \frac{1}{z-1}) G_M(l+b-1, m-l+c-b-1, \frac{1}{z-1})$$

Here we can remove the factors in front of the summation as those are not divisible by any factor of $(c - b)_{m-l}(b)_l$, giving us

$$\sum_{M+N=k-1} \frac{1}{a+N} G_N(-b, b-c, \frac{1}{z-1}) G_M(l+b-1, m-l+c-b-1, \frac{1}{z-1})$$

which brings us in a rather similar situation as Theorem 6.1.1. After applying the same expansion steps as in the proof of that theorem we get

$$\begin{aligned} &(a)_k \sum_{M+N=k-1} \frac{1}{a+N} G_N(-b, b-c, \frac{1}{z-1}) G_M(l+b-1, m-l+c-b-1, \frac{1}{z-1}) \\ &= \sum_{j=0}^{k-1} a^j \sum_{p=0}^{k-1-j} \begin{bmatrix} k \\ j+p+1 \end{bmatrix} (-1)^p \qquad (51) \\ &\quad \times \text{coefficient of } x^{k-1} \text{ in } \left(\theta^p (1-x)^{-b} (1 + \frac{1}{z-1} x)^{b-c}\right) \left((1-x)^{l+b-1} (1 + \frac{1}{z-1} x)^{m-l+c-b-1}\right) \end{aligned}$$

In order to show it is not divisible by $(c - b + i)$, we are going to prove that c^{k-1} is the highest power of c and then show that the coefficient of c^{k-1} does not equal the coefficient of bc^{k-2} times -1, which means it is not divisible by $(c - b + i)$.
To do this easily we need to remove as many terms as possible which do not influence the coefficient c times b such that their combined power is the highest possible. In the above summation there appear to be 2 ways for c and b to be involved as factor. The first is when a term such as $(1 - x)^b$ gets taken the derivative of, giving us $-b(1-x)^{b-1}$. The other is when expanding a term such as $(1 - x)^b$ into $\sum_{v=0}^{\infty} \binom{b}{v} x^v$. Important to notice here is that the last case will never occur, this is because after taking the derivative, we are left with something similar to $(1 - x)^{b-3}(1 - x)^{l+b-1} = (1 - x)^{l-4}$, as you can see, b is cancelled out in the power. Hence we are only interested in the cases in which the maximum number of derivatives is taken in order to get the highest sum of powers in c and b. Meaning we are only interested in the case that $p = k - 1$, hence $j = 0$ leaving us with

$$\begin{aligned} &a^0 \begin{bmatrix} k \\ k \end{bmatrix} (-1)^{k-1} \times \text{coefficient of } x^{k-1} \text{ in } \left(\theta^{k-1} (1-x)^{-b} (1 + \frac{1}{z-1} x)^{b-c}\right) \left((1-x)^{l+b-1} (1 + \frac{1}{z-1} x)^{m-l+c-b-1}\right) \\ &= \text{coefficient of } x^{k-1} \text{ in } (-1)^{k-1} \left(\theta^{k-1} (1-x)^{-b} (1 + \frac{1}{z-1} x)^{b-c}\right) \left((1-x)^{l+b-1} (1 + \frac{1}{z-1} x)^{m-l+c-b-1}\right) \\ &= \text{coefficient of } x^{k-1} \text{ in } \frac{(-1)^{k-1}}{(z-1)^{m-l-1}} \left(\theta^{k-1} (1-x)^{-b} (z-1+x)^{b-c}\right) (1-x)^{l-1+b} (z-1+x)^{m-l-b-1+c} \end{aligned}$$

Furthermore, when calculating for example $\theta x(1-x)^b$, we are not interested in all parts we get by applying the product rule. This is because

$$\begin{aligned}\theta x(1-x)^b &= x\frac{d}{dx}x(1-x)^b \\ &= x(1-x)^b - bx^2(1-x)^{b-1}\end{aligned}$$

Here $x(1-x)^b$ will definitely not affect the coefficient of multiples of c and b such that their combined power is the maximum that occurs. So we can again further simplify, giving us

$$\begin{aligned}
&\text{coefficient of } x^{k-1} \text{ in } \frac{(-1)^{k-1}}{(z-1)^{m-l-1}}x^{k-1}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b-i+1)_i(1-x)^{-b-i} \\
&\quad (z-1)^{k-1-i}(b-c-k+1+i-1)_{k-1-i}(z-1+x)^{b-c-k+1+i}(1-x)^{l-1+b}(z-1+x)^{m-l-b-1+c} \\
&= \text{coefficient of } x^0 \text{ in } \frac{(-1)^{k-1}}{(z-1)^{m-l-1}}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b-i+1)_i \\
&\quad (z-1)^{k-1-i}(b-c-k+1+i-1)_{k-1-i}(1-x)^{l-1-i}(z-1+x)^{m-l-k+i} \\
&= \frac{(-1)^{k-1}}{(z-1)^{m-l-1}}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b-i+1)_i \\
&\quad (z-1)^{k-1-i}(b-c-k+1+i-1)_{k-1-i}(z-1)^{m-l-k+i} \\
&= \frac{(-1)^{k-1}}{(z-1)^{m-l-1}}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b-i+1)_i \\
&\quad (b-c-k+1+i-1)_{k-1-i}(z-1)^{m-l-1} \\
&= (-1)^{k-1}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b-i+1)_i \\
&\quad (b-c-k+1+i-1)_{k-1-i}.
\end{aligned}$$

We again remove unnecessary terms, giving us

$$\begin{aligned}&(-1)^{k-1}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(-b)^i(b-c)^{k-1-i} \\ &= (-1)^{k-1}\sum_{i=0}^{k-1}\binom{k-1}{i}b^i(b-c)^{k-1-i}.\end{aligned}$$

This show that the power of b plus that of c is at most $k-1$. Here c^{k-1} has coefficient 1 and $c^{k-2}b$ has coefficient $-(k-1)$. Since $k,l,m \ge 0$ and $l \le m \le k$ we get that when $(k,l,m) \ne (0,0,0)$ that the above term is not divisible by $(c-b+i)$ and it not being divisible by $(b+i)$ follows from the fact that there is no bc^{k-1}. Hence we get that $(b)_l(c-b)_{m-l}$ is part of the denominator of Q. So, $(b+ln)_l(c-b+(m-l)n)_{m-l}$ is part of the denominator of $Q^{(n+1)}$. □

B Code

While working on this thesis we created a multitude of scripts for calculating contiguity relations, admissible equations and simplified admissible equations. Of these the first 3 were in sage math. In order of creation, these 3 applied the inductive method shown in subsection 2.5, used theorem 4.3.3 (while using sage math to calculate the residues and integral, when doing this, sage math would start giving errors with $k, l, m > 8$ for it being too difficult) and used theorem 4.3.3 with G_N form equation 35. The last version we remade in mathematica as mathematica is far faster for these calculations. While mathematica's syntax appears to stray quite a bit from what one would normally expect from a programming language, it does have the benefit of being a strong programming language. That meant that unlike sage math, which is a weak programming language, we did not have to deal with variables being given the wrong class due to it being decided by underlying logic, which then would mean certain functions wouldnt' work.
Here the mathematica code, which was next to being last version also the most compact version. Here is only the code directly used for calculating things such as simplified admissible quadruples, Q, Q^n, R and the Gröbner basis. Hence not the multitude of specialized for loops we used for our observations.

```
(*here cupProduct calculates \langle x^deg \oo,\oo_{k,l,m}*)
cupProduct[k_, l_, m_, deg_] :=
 cupAt0[k, l, m, deg] + cupAt1[k, l, m, deg] +
  cupAtDivZ[k, l, m, deg] + cupAtInf[k, l, m, deg]

cupAt0[k_, l_, m_, deg_] :=
 Sum[1/(1 - b + N + deg)*GN[b - c, a - 1, -z, N]*
   GN[m - l - b - 1 + c, -k - a, -z, -l - 1 - deg - N], {N,
   0, -l - 1 - deg}]
cupAt1[k_, l_, m_, deg_] := (1 - z)^(-k - 1)*
  Sum[1/(b + 1 - c + N)*GN[-b + deg, a - 1, z/(1 - z), N]*
    GN[l - (1 - b), -k - a, z/(1 - z), l - m - 1 - N], {N, 0,
    l - m - 1}]
cupAtDivZ[k_, l_, m_, deg_] :=
 z^(-m - deg)*(z - 1)^(m - l - 1)*
  Sum[1/(a + N)*GN[-b + deg, b - c, 1/(z - 1), N]*
    GN[l - 1 + b, m - l - b - 1 + c, 1/(z - 1), k - 1 - N], {N, 0,
    k - 1}]
cupAtInf[k_, l_, m_, deg_] := (-1)^(m - l - k)*z^(-k - 1)*
  Sum[1/(-deg - a + c + N)*GN[b - c, a - 1, -1/z, N]*
    GN[m - l - b - 1 + c, -k - a, -1/z, m - k - 1 + deg - N], {N, 0,
    m - k - 1 + deg}]

GN[power1_, power2_, coef_, N_] :=
 Sum[Binomial[power1, i]*(-1)^i*Binomial[power2, N - i]*
   coef^(N - i), {i, 0, N}]


simplifiedAdmissibleEquation[k_, l_, m_] :=
  Numerator[
    Together[FunctionExpand[cupProduct[k, l, m, 0]]]] /. {a ->
     a + k*n, b -> b + l*n, c -> c + m*n};

getGroebnerBasis[k_, l_, m_] :=
 GroebnerBasis[
  CoefficientList[simplifiedAdmissibleEquation[k, l, m], n], {a, b, c,
    z}]

getQ[k_, l_, m_] :=
```

```
 Together[Pochhammer[c, m]/Pochhammer[b, l]/
    Pochhammer[c - b, m - l]*(z - 1)*z*cupProduct[k, l, m, 0]]
getQn[k_, l_, m_] :=
 getQ[k, l, m] /. {a -> a + k (n - 1), b -> b + l (n - 1),
   c -> c + m (n - 1)}
getR[k_, l_, m_] :=
 Together[-Pochhammer[c, m]/Pochhammer[b, l]/
    Pochhammer[c - b, m - l]*(b*cupProduct[k, l, m, -1] -
     b z cupProduct[k, l, m, 0])]
```

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