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# Cohomology of compact Lie groups

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# Introduction

In this thesis we will take a look at the de Rham cohomology of Lie groups. For general smooth manifolds this cohomology is often hard to determine, however for compact Lie groups we will show that this can be made a lot simpler. The group structure of a Lie group plays an important role in this, among other things it allows us to define so called *'left-invariant forms'* on the manifold. We will see that the complex of these forms is closely related to the exterior algebra of the Lie algebra. Using this, the cohomology can essentially be calculated completely by using the algebraic structure of the Lie algebra.

The thesis is split into four parts. Before the first part, we give a short overview of various properties of Lie groups and Lie algebras, that we will need for later. In Part I we will show the relationship that we described above. The first step will be to show that the cohomology induced by the complex of left-invariant forms of a Lie group $G$, is naturally isomorphic to a suitable cohomology defined in terms of the space $\Lambda^k\mathfrak{g}^*$. Here $\mathfrak{g}$ denotes the Lie algebra of $G$. The next step is then to show that this cohomology of left-invariant forms is actually the same as the (de Rham) cohomology of all forms. For this we will have to make the assumption that $G$ is compact. That will then conclude the first part. The main source for Part I is the article written by Chevalley and Eilenberg, [2]. This article was published in 1948, and as far as we know it is the first one describe this method in full detail.

The second part of the thesis will be a short intermezzo, where we introduce the concepts of tensor products and exterior algebras of vector spaces. We will need this for the other two parts, and especially for the final part.

In Part III then, we will look at a first application of the theory developed in the first part. We will start by discussing *bi-invariant forms*, forms that are both left- and right-invariant. The induced cohomology of these forms will also turn out to be isomorphic to the de Rham cohomology. Actually, we will see something even stronger, namely that every equivalence class of the de Rham cohomology contains exactly one bi-invariant form. We will use this fact to prove the famous *Hodge decomposition theorem* in the case of compact, connected Lie groups. To do this, we will have to introduce a Riemannian structure on $G$, which can then be used to define the *Hodge Laplacian* $\Delta : \Omega^k(G) \to \Omega^k(G)$. The theorem then states that every cohomology class contains exactly one *harmonic* form, i.e. a form $\omega$ such that $\Delta\omega = 0$. We will prove this by showing that the harmonic forms correspond exactly to the bi-invariant ones. This is a fact that was actually proved by H. Hodge himself, in his book [8]. However, since the lack of modern notation makes the theorem in this book almost completely unrecognizable to its modern day formulation, we do not use it as a source here. The main source that we will be using for this part is the book written by S. Helgason, [7].

In the final part we will take a closer look at the space $\Lambda^k\mathfrak{g}^*$, and the 'cohomology' that we will define on it. The space is sometimes referred to as the *Koszul complex*, and it is therefore fitting that we use an article written by J. Koszul himself, [10], as the primary source for this part. The main thing we will try to show is a theorem originally proven by Heinz Hopf in 1941 (see [9]), that states that the cohomology of every compact Lie group is isomorphic to the cohomology of the (cartesian) product of a certain number of odd-dimensional spheres. Hopf proved the theorem by using the structure of the group itself, we will prove it by using the structure of the Lie algebra. This shows a nice application of the work we have done in Part I, and is arguably a bit more elegant.

In the thesis, we will here and there refer to some theory about *representations* (of Lie groups). We do not use any of this to prove crucial results, but nevertheless we have included a brief introduction to representations in the Appendix, for the interested reader.

## 0.1 Preliminaries

For this thesis, we will assume that the reader has a fair knowledge of (smooth) manifolds, as well as some background in functional analysis and topology. In this section we will list some general results, in order to re-familiarize the reader with them and to introduce the notation that we will be using throughout the thesis. If necessary, everything discussed here can be found in [12], which is an excellent introduction into the theory of smooth manifolds. More advanced discussions can also be found in the book by Bott&Tu, [1].

With $M$ we will always mean a smooth manifold, and unless otherwise stated, its dimension will be denoted by $n$. For every point $p \in M$, there is a tangent space $T_pM$, consisting of tangent vectors $X \in T_pM$. The bundle over $M$ of these tangent spaces is denoted by $TM$, a *vector field* $\vec{X}$ is then a smooth section of this bundle. The space of all vector fields on $M$ is denoted by $\mathfrak{X}(M)$. At every point $p$, $\vec{X}_p$ is an element of $T_pM$. If $f : M \to N$ is some smooth map between two manifolds, then it induces a *tangent map* $T_pf : T_pM \to T_{f(p)}N$ for every $p$. In literature, one can also see the notation $(df)_p$ for this map.

The space of all *differential k-forms* on $M$ will be denoted by $\Omega^k(M)$. An element of this space will generally be denoted by $\omega$. For every $p$, $\omega_p$ will then be an element of $\Lambda^k(T_pM)^*$, the space of all alternating, $k$-linear functions $(T_pM)^k \to \mathbb{R}$. On the space of differential forms we have the *wedge product* $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^k(M)^{k+l}(M)$, this makes the space $\Omega^\bullet(M) = \oplus_k \Omega^k(M)$ into a *graded algebra*. It satisfies *graded commutativity*,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega, \tag{1}$$

if $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$.

**Definition 0.1.1.** A (real) algebra $\mathcal{A}$ is a real vector space, endowed with a bilinear multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

On $\Omega^k(M)$, we have the *de Rahm derivative* or *exterior derivative* $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Since $d \circ d = 0$, the spaces $\Omega^k(M)$ together with $d = d_k$ form a *complex*, for which we can define the *k-th de Rahm cohomology* as the quotient

$$H^k_{dR}(M) := Ker(d_{k+1})/Im(d_k)$$

The wedge product also induces a graded algebra structure on the cohomology, this is possible because the $d$-operator satisfies the properties of an *anti-derivation*, meaning $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, where $\omega$ and $\eta$ are as above.

Let $X$ and $Y$ be two topological spaces. We call two functions $f, g : X \to Y$ *homotopic* if there exists a continuous map $F : \mathbb{R} \times X \to Y$ such that $F(0, \cdot\,) = f$, $F(1, \cdot\,) = g$. We will be needing the following theorem, that the reader should already be familiar with.

**Theorem 0.1.2.** *Let $M$, $N$ smooth manifolds, and $f, g : M \to N$ two smooth maps that are homotopic. Then there exist a linear map $H$ such that we can write*

$$f^*\omega - g^*\omega = dH(\omega) + H(d\omega),$$

*for all $\omega \in \Omega^k(G)$. In particular, it follows that the maps $f^*, g^* : H^k_{dR}(N) \to H^k_{dR}(M)$, induced by the pullbacks of $f$ and $g$, are equal.*

Finally, if $V$ is a vector space (vector spaces in this thesis will always be assumed to be over $\mathbb{R}$), then by $V^*$ we denote its dual space. If $\gamma : V \to W$ is a linear map between vector spaces, then its dual map $W^* \to V^*$ will be denoted by $\gamma^*$.

## 0.2 Basics of Lie groups and Lie algebras

In this section we will give a short introduction to some of the basic properties of Lie groups. Some of these might already be familiar to the reader, while others will be completely new. Nevertheless, we won't provide detailed proofs for most of theorems, mainly because the proofs are almost always quite elementary. We refer to [13] (or basically any other introductory text on Lie groups) for a more thorough treatment.

First we recall the definition of a Lie group.

**Definition 0.2.1.** We call a smooth manifold $G$ a *Lie group* if it is equipped with a group structure, in such a way that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth.

If $G_1$ and $G_2$ are Lie groups, the product $G_1 \times G_2$ with component-wise multiplication is again a Lie group. Using the group structure of $G$, for every element $g \in G$ we can define the *left-multiplication* $l_g : G \to G$ as the map given by $l_g(x) = gx$, for $x \in G$. This will be a smooth map, and since its inverse is given by $l_{g^{-1}}$, it is actually a diffeomorphism $G$ with itself. We can also define the *right-multiplication* $r_g$ in a similar way.

Since it is a diffeomorphism, the push-forward $(l_g)_* : \mathfrak{X}(G) \to \mathfrak{X}(G)$ of $l_g$ is well-defined. It is given on a vector field $\vec{X}$ by

$$\big((l_g)_* X\big)_h = T_{g^{-1}h}(l_g)(X_{g^{-1}h}),$$

for $g, h \in G$. This allows the next definition.

**Definition 0.2.2.** We call a vector field $\vec{X} \in \mathfrak{X}(G)$ *left-invariant* if $(l_g)_*\vec{X} = \vec{X}$ for all $g \in G$. The space of all left-invariant vector fields is denoted by $\mathfrak{X}_L(G)$

The following theorem is an important first step towards showing the relationship between the complex of left-invariant forms and the exterior algebra $\Lambda^k\mathfrak{g}^*$, that we mentioned in the introduction.

**Theorem 0.2.3.** *Let $G$ be a Lie group. Then the map $ev : \mathfrak{X}_L(G) \to T_eG$, $v \mapsto v(e)$ is a linear isomorphism.*

*Proof.* For a left-invariant vector field $\tilde{Y}$ it holds that $\tilde{Y}_g = T_e(l_g)\tilde{Y}_e$ (for any $g \in G$). It follows that $\tilde{Y}$ is completely determined by its value in $e$, from which we conclude that the map $ev$ must be injective. We will now prove surjectivity. Let $X \in T_eG$. Define the vector field $\tilde{X}$ by

$$\tilde{X}_g := (l_g)_*X = T_e l_g X. \tag{2}$$

One can show that this indeed defines a smooth vector field. Since by the chain rule we have that $T_e(l_{gh}) = T_e(l_g) \circ T_e(l_h)$ for $g, h \in G$, it quickly follows that $\tilde{X}$ is left-invariant. Since it clearly holds that $\tilde{X}_e = X$, we see that $ev$ is surjective, and since it is also clearly linear, it is therefore a linear isomorphism between $\mathfrak{X}_L(G)$ and $T_eG$. □

From now on we will always write $\tilde{X}$ for the left-variant vector field induced by an element $X \in T_eG$, as in equation (2). We will now introduce the important *exponential map*.

**Definition 0.2.4.** The *exponential map* $\exp : T_eG \to G$ is defined by

$$\exp(X) = \alpha_X(1),$$

where $\alpha_X$ is defined to be the maximal integral curve of $\tilde{X}$, the left-invariant vector field induced by $X$.

One can show that the maximal integral curve of a left-invariant vector field on a Lie group is always defined on all of $\mathbb{R}$, which makes the definition above possible. The exponential map has the following properties.

**Lemma 0.2.5.** *For all $s, t \in \mathbb{R}$, $X \in T_eG$:*

*(i)* $\exp(sX) = \alpha_X(s)$.

*(ii)* $\exp(s + t)X = \exp(sX)\exp(tX)$.

*Also,* $\exp$ *is a smooth map, and a local diffeomorphism around 0. Its tangent map at the origin,* $T_0 \exp$*, is the identity on* $T_eG$.

*Proof.* See Lemma 3.6 in [13]. □

The next Lemma will be important later on.

**Lemma 0.2.6.** *Let $X \in \mathfrak{g}$, $\tilde{X}$ the induced left-invariant vector field. Denote by $\phi_t$ the flow of $\tilde{X}$. Then*

$$\phi_t(g) = (r_{\exp tX})(g).$$

*Proof.* The crucial remark is that if $\alpha$ is an integral curve of $\tilde{X}$, then $l_g \circ \alpha$ is also an integral curve, this follows by using the chain rule. The curve $t \mapsto l_g(\exp tX)$, is therefore an integral curve of $X$, starting at $g$. By uniqueness of integral curves, it is thus equal to $\phi_t(g)$. Since $l_g(\exp tX) = r_{\exp tX}(g)$, the proof is then complete. □

We will now start discussing the *Lie algebra* of a Lie group $G$. For $x \in G$, define the *conjugation map* $C_x : G \to G$ as the map $g \mapsto xgx^{-1} = l_x \circ r_{x^{-1}}(g)$. Since it is the composition of diffeomorphisms, this will also be a diffeomorphism. Since we have $C_x(e) = e$, it therefore follows that $T_eC_x \in GL(T_eG)$.

**Definition 0.2.7.** For $x \in G$, define the map $\mathrm{Ad}(x) \in GL(T_eG)$ by $\mathrm{Ad}(x) := T_eC_x$

We call the map $\mathrm{Ad} : G \to GL(T_eG)$ the *adjoint representation of G.* Note that $\mathrm{Ad}(e)$ is the identity map on $T_eG$. Since the tangent space of $GL(T_eG)$ at the identity, $T_IGL(T_eG)$, is equal to $End(T_eG)$ the following definition makes sense.

**Definition 0.2.8.** We define the (linear) map $\mathrm{ad} : T_eG \to End(T_eG)$ by $\mathrm{ad} := T_e\mathrm{Ad}$.

Using the chain rule, we see that for $X \in T_eG$ we have

$$\mathrm{ad}(X) = \tfrac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp tX). \tag{3}$$

**Definition 0.2.9.** Let $X, Y \in T_eG$. We define the *Lie bracket* $[X, Y] \in T_eG$ by

$$[X, Y] := \text{ad}(X)Y.$$

It can be shown that the map $(X, Y) \mapsto [X, Y]$ is bilinear and anti-symmetric. It also satisfies the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

This makes $T_eG$, together with the Lie bracket $[.\,,.]$, into a *Lie algebra*, which from now on we will denote by $\mathfrak{g}$.

**Definition 0.2.10.** A *Lie algebra* $\mathfrak{a}$ is vector space together with a bilinear map $[.\,,.] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ that for all $X, Y, Z \in \mathfrak{a}$ satisfies:

1. $[X, Y] = -[Y, X]$.
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

***Remark.*** In literature one will sometimes see different definitions of the Lie bracket. For example, one can define the *Lie bracket of vector fields* by $[\vec{X}, \vec{Y}]_p(f) = \vec{X}_p(\vec{Y}f) - \vec{Y}_p(\vec{X}f)$ for all $f \in C^\infty(G)$, and use the isomorphism between $T_eG$ and $\mathfrak{X}_L(G)$ to induce a Lie bracket on $T_eG$. One can show that this definition is equivalent to the one we have given here.

**Definition 0.2.11.** A *Lie group homomorphism* is a smooth map between two Lie groups, that is also a group homomorphism

**Definition 0.2.12.** Let $\mathfrak{a}, \mathfrak{b}$ be two Lie algebras. A linear map $\rho : \mathfrak{a} \to \mathfrak{b}$ is called a *Lie algebra homomorphism* if it satisfies

$$\rho([X, Y]_{\mathfrak{a}}) = [\rho(X), \rho(Y)]_{\mathfrak{b}}.$$

**Lemma 0.2.13.** *Let $\varphi : G \to H$ be a Lie group homomorphism. Then the tangent map $T_e\varphi$ is a Lie algebra homomorphism between the associated Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$*

*Proof.* Lemma 4.10 in [13]. □

**Definition 0.2.14.** By $G_e$ we denote the subgroup of $G$ generated by elements of the form $\exp(X)$ for $X \in \mathfrak{g}$. It is called the *component of the identity* of $G$.

The following theorem will be important for applications.

**Theorem 0.2.15.** *If $G$ is connected, then $G_e = G$.*

*Proof.* See Lemma 5.8 in [13]. The proof involves showing that $G_e$ is both an open and closed subset of $G$, from which the statement follows. The converse implication is actually also shown, but we do not need it here. □

# Part I

# Left-invariant forms and cohomology

In the previous chapter we established a linear isomorphism between the space of left-invariant vector fields of a Lie group $G$, and its Lie algebra $\mathfrak{g}$. In this section we will follow a similar procedure to relate the space of left-invariant $k$-forms on G to the space of $k$-linear alternating functions on $\mathfrak{g}$. In other words, we will establish an isomorphism $\Omega^k_L(G) \cong \Lambda^k\mathfrak{g}^*$, where $\Omega^k_L(G)$ denotes the space of left-invariant $k$-forms. The main goal of this chapter will then be to prove that the complex of left-invariant forms induces the same cohomology as the de Rham complex of all forms, provided that $G$ is compact. We will also define a suitable 'exterior differential' on $\Lambda\mathfrak{g}^*$, that will give us an isomorphism between the De Rham cohomology $H^k_{dR}(G)$ and the 'cohomology' of $\mathfrak{g}$. This very useful result essentially means that we can compute the De Rham cohomology of a compact Lie group by using purely the algebraic structure of its Lie algebra. This fact will be exploited in later chapters.

The following sections are primarily based on the article by Chevalley and Eilenberg [2], that was written in 1948. However, they use some results from homology theory and since we want to avoid this, some of the proofs have been significantly modified. Moreover, we try to avoid speaking about representations, and therefore the formulation of most theorems differs form the original formulation.

## 1.1 Left-invariant forms and the Lie algebra

We start with a definition.

**Definition 1.1.1.** We call a $k$-form $\omega \in \Omega^k(G)$ *left-invariant* if $(l_g)^*\omega = \omega$, for any $g \in G$ (recall that $l_g : G \to G$ denotes the left-multiplication on $G$, i.e. $l_g(x) = gx$). The set of all left-invariant $k$-forms on $G$ is denoted by $\Omega^k_L(G)$.

**Proposition 1.1.2.** *The map* $ev : \Omega^k_L(G) \longrightarrow \Lambda^k\mathfrak{g}^*$, $ev(\omega) = \omega_e$ *is an isomorphism of vector spaces.*

*Proof.* Since for any left invariant form $\omega \in \Omega^k_L(G)$, and vectors $v_1, ..., v_k \in T_gG$, it holds that

$$\omega_g(v_1, ..., v_k) = (l_{g^{-1}})^*(\omega_g)(v_1, ..v_k) = \omega_e(T_g(l_{g^{-1}})^k(v_1, ..., v_k)),$$

it follows that $\omega$ is entirely defined by its value at $e$. So $ev$ is injective.

For surjectivity, let $f \in \Lambda^k\mathfrak{g}^*$. Now define $(\omega)_g(v_1, ..., v_k) := f(T_g(l_{g^{-1}})^k(v_1, ...v_k))$, for $v_1, ..., v_k \in T_gG$. This defines a $k$-form on $G$. Since

$$(l_h)^*(\omega)_g(v_1, ..., v_k) = (\omega)_{hg}(T_g(l_h)(...)) = f(T_{hg}(l_{g^{-1}h^{-1}}) \circ T_g(l_h)(...)) = f(T_g(l_{g^{-1}})(...)),$$

where the last equality follows from the chain rule, we conclude that $\omega$ is left invariant. Also clearly $(\omega)_e = f$.

To see that $\omega$ is a smooth form, note that the map $v \mapsto T_g(l_{g^{-1}})v$ is smooth as a map $G \to TG$, since it is obtained by differentiating the smooth map $(x, y) \mapsto l_{x^{-1}}(y)$ with respect to $y$ at $y = g$, in the direction of $v \in T_gG$. It follows that the map $x \mapsto (\omega)_x$ is smooth, and this implies that $\omega$ is a smooth $k$-form. Therefore $ev$ is a bijection between $\Omega^k_L(G)$ and $\Lambda^k\mathfrak{g}^*$, and since it is clearly linear, it is a linear isomorphism. □

We now turn our attention to the complex of left-invariant forms. We want to show that these form a *sub-complex* of the complex of all forms. To do this, we need to check that the exterior derivative $d : \Omega^k(G) \to \Omega^{k+1}(G)$ preserves left-invariance, that is, it should map $\Omega^k_L(G)$ to $\Omega^{k+1}_L(G)$. This is a simple consequence of the fact that $d$ commutes with pullbacks, in particular therefore $(l_g)^*(d\omega) = d(l_g)^*\omega$. It follows that by restricting $d$ to left-invariant forms, we obtain an exterior derivative $\Omega^k_L(G) \to \Omega^{k+1}_L(G)$, which we will also denote by $d$. With this, we can define the *k-th cohomology group of left-invariant forms*, $H^k_L(G)$, as the usual cohomology of this complex.

As stated before, we want to relate this cohomology of left-invariant forms to a suitable cohomology defined in terms of the Lie algebra, which we will call the *Lie algebra cohomology.* For this we will need the following proposition:

**Proposition 1.1.3.** *For* $\omega \in \Omega^k_L(G)$ *and smooth left-invariant vector fields* $X_0, X_1, ...X_k$ *on* $G$*, we have the following formula for the exterior derivative (the hats indicate that the variable should be left out):*

$$d\omega(X_0, ..., X_k) = \sum_{i<j} (-1)^{i+j}\, \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_k). \tag{1.1}$$

*Proof.* Recall that for any $k$-form $\omega \in \Omega^k(G)$ , we have (see [3]):

$$d\omega(X_0, ..., X_k) = \sum_{i<j} (-1)^{i+j}\, \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_k) + \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k)).$$

We will prove that the terms of the second sum on the right hand side are equal to zero for a left-invariant form $\omega \in \Omega_L^k(G)$. Note that we have $\omega_g(X_1, ..., X_k) = (l_{g^{-1}})^* \omega_g(X_1, ..., X_k) = \omega_e(T_g(l_{g^{-1}})^k(X_1, ...X_k)) = \omega_e(X_1, ...X_k)$, since the vector fields are left-invariant. This shows that $\omega(X_1, ..., X_k)$ is a constant function, hence $X_i(\omega(X_1, ..., X_k)) = 0$. This hold for all $i$, so it follows that the second sum of the right hand side is indeed equal to zero.

□

From this proposition and the fact that there exists a left-invariant global frame on $G$, it immediately follows that for tangent vectors $x_0, x_1, ..., x_k \in T_eG$, we have

$$(d\omega)_e(x_0, ..., x_k) = \sum_{i<j} (-1)^{i+j} \, \omega([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_k).$$

Motivated by this, we define an 'exterior derivative' on $\Lambda^k \mathfrak{g}^*$. We will denote it by $\delta$, and for $f \in \Lambda^k \mathfrak{g}^*$ and $x_0, ..., x_k \in \mathfrak{g}$ we simply define it as

$$\delta f(x_0, ..., x_k) = \sum_{i<j} (-1)^{i+j} \, f([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_k). \tag{1.2}$$

Note that this indeed defines a linear map $\Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^*$. We will now check that it satisfies the following two properties, that are comparable to properties satisfied by the exterior derivative.

**Proposition 1.1.4.** *For $\delta$ as defined above, we have for $f_1 \in \Lambda^k \mathfrak{g}^*$, $f_2 \in \Lambda^m \mathfrak{g}^*$ the following:*

*(i)* $\delta(f_1 \wedge f_2) = (\delta f_1) \wedge f_2 + (-1)^k f_1 \wedge (\delta f_2)$.

*(ii)* $\delta(\delta f_1) = 0$.

*Proof.* (i) and (ii) can be proved by induction on $k$, by basically purely algebraic considerations. This is done in [2, p. 105] □

With item (ii) of the previous proposition, we can define the *k-th cohomology group of* $\mathfrak{g}$ essentially analogous to how we defined it for $G$. So let $Z^k(\mathfrak{g}) := \{f \in \Lambda^k \mathfrak{g}^* \mid \delta f = 0\}$ and $B^k(\mathfrak{g}) := \{f \in \Lambda^k \mathfrak{g}^* \mid f = \delta h, \text{ for some } h \in \Lambda^{k-1} \mathfrak{g}^*\}$. Then define $H^k(\mathfrak{g}) = Z^k(\mathfrak{g})/B^k(\mathfrak{g})$.

We can now state the following theorem, which is an easy consequence of Proposition 1.1.2 and the remark after 1.1.3.

**Theorem 1.1.5.** *For a Lie group $G$ with Lie algebra $\mathfrak{g}$, the map $ev : \Omega_L^k(G) \to \Lambda^k \mathfrak{g}^*$ induces a linear isomorphism*

$$H_L^k(G) \cong H^k(\mathfrak{g}).$$

Since the wedge-product of two left-invariant forms is again left-invariant, the space $\Omega_L^\bullet(G) := \oplus_k \Omega_L^k(G)$ has the structure of a graded algebra. It follows that the space $H_L^\bullet(G) := \oplus_k H_L^k(G)$ is also a graded algebra. Similarly, since $\Lambda \mathfrak{g}^* := \oplus_k \Lambda^k \mathfrak{g}^*$ is a graded algebra, the same thing holds for $H^*(\mathfrak{g}) := H^k(\mathfrak{g})$. Here we have used assertion $(i)$ of Proposition 1.1.4. It is easy to see that the map $ev : \Omega_L^\bullet(G) \to \Lambda \mathfrak{g}^*$ is a homomorphism of algebras (it holds that $ev(\omega \wedge \eta) = ev(\omega) \wedge ev(\eta)$). The theorem above then implies that $ev$ induces the *isomorphism of graded algebras*

$$H_L^\bullet(G) \simeq H^*(\mathfrak{g}).$$

## 1.2 Relating the deRham cohomology to the cohomology of left-invariant forms

While Theorem 1.1.5 looks nice, on its own it is not very useful to us. We are interested in the entire cohomology group of $G$, not just the cohomology of the left-invariant forms. However, as it will turn out, for $G$ compact these two are actually the same thing:

**Theorem 1.2.1.** *Let G be a compact connected Lie group. Then the inclusion maps $\Omega^k_L(G) \hookrightarrow \Omega^k(G)$ induce linear isomorphisms:*

$$H^k_{dR}(G) \cong H^k_L(G).$$

Since the inclusion map is clearly an algebra homomorphism between $\Omega^\bullet_L(G)$ and $\Omega^\bullet(G)$, Theorem 1.2.1 also implies that the spaces $H^\bullet_{dR}(G)$ and $H^\bullet_L(G)$ are isomorphic as graded algebras. The rest of this section will be dedicated to proving this theorem. We will always assume that G is compact (and not necessarily mention this every time). In this section we will consider the more general case where $G$ is acting on some manifold $M$, again by an action $l : G \times M \to M, l(g, x) = l_g(x)$, smooth as a map between manifolds. We can then define $\Omega^k_L(M)$ in the same way we defined it for just $G$. The aforementioned theorem then still holds, we have:

**Theorem 1.2.2.** *Let M be a manifold, G a compact connected Lie group acting on M from the left. Then the inclusion maps $\Omega^k_L(M) \hookrightarrow \Omega^k(M)$ induce linear isomorphisms:*

$$H^k_{dR}(M) \cong H^k_L(M).$$

Denote by $i : \Omega^k_L(M) \to \Omega^k(M)$ the natural inclusion. We then have the following commutative diagram.

$$\begin{array}{ccccc} \xrightarrow{\cdots} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{\cdots} \\ & \uparrow i & & \uparrow i & \\ \xrightarrow{\cdots} & \Omega^k_L(M) & \xrightarrow{d} & \Omega^{k+1}_L(M) & \xrightarrow{\cdots} \end{array} \tag{1.3}$$

We now want introduce a map into the opposite direction, meaning a map $\Omega^k(M) \to \Omega^k_L(M)$. To do this we need to find a way to turn any $k$-form into a left-invariant one. Intuitively, we want to 'average' the form over $G$. This can be done by integrating over a suitable (left-invariant) density on $G$, called the *Haar Measure*. We first recall some properties of densities (taken from [13]).

**Definition 1.2.3.** Let $V$ be a (real) vector space. A *density* on $V$ is a map $\lambda : V^n \to \mathbb{R}$ such that, for any linear transformation $T \in End(V)$:

$$T^*\lambda := \lambda \circ T^k = |\det T|\lambda.$$

The space of all densities on $V$ is denoted by $\mathcal{D}V$. One can show that this is a 1-dimensional vector space.

If $M$ is a smooth manifold, then every tangent space $T_xM$ has the structure of a vector space. We can then define the *bundle of densities on M*, denoted by $\mathcal{D}TM$ as the bundle with fibers $(\mathcal{D}TM) \simeq \mathcal{D}T_xM$. A *(smooth) density on M* can then be defined as a smooth section of this bundle. In other words, a density on $M$ is an element $\lambda$ such that for every $x \in M$, $\lambda_x$ is a density on $T_xM$, in such a way that (viewed in local coordinates) $\lambda_x$ depends smoothly on $x$.

There are some obvious similarities between densities and differential forms on $M$. The following example shows the relation between the two.

**Example 1.2.4.** Let $\omega \in \Omega^n(M)$ be a differential form on $M$ (with $dim(M) = n$). Then $|\omega|$, defined by

$$|\omega|_p(v_1, ..., v_k) = |w_p(v_1, ...v_k)| \in \mathbb{R},$$

is a smooth density on $M$.

The main difference between densities and differential forms is that (top)-forms transform under pull-backs as the determinant of the linear transformation, while densities transform as the absolute value of the determinant (the pullback of a density is defined in the same way as for forms). This is also a reason why densities are useful, since it means we do not have to worry about orientation when we define integration over densities. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$. Let $\lambda_n$ be the density defined by $\lambda_n(e_1, ..., e_n) = 1$. One can easily deduce that every density on an open set $U \subseteq \mathbb{R}^n$ can be written as $f\lambda_n$, for an $f \in C^\infty(U)$. If $f \in C_c(U)$ (i.e $f$ compactly supported on $U$, then define than the integral over $U$ as

$$\int_U f\lambda_n = \int_U f(x)dx,$$

where $dx$ is the standard Lebesgue measure on $\mathbb{R}^n$. Integration of compactly supported densities on manifolds can now be defined completely analogous to how it is defined for differential forms, by pulling back under charts and using a partition of unity. Note however that we can use the substitution of variables theorem without having to worry about orientation, because of the way densities transform.

We can also talk about left invariance of densities, in the same way as for differential forms. Call a density $\lambda$ on $G$ *left invariant* if $(l_g)^*\lambda = \lambda$, for all $g \in G$. One can then show that the map $\lambda \mapsto \lambda_e$ is a linear isomorphism between the space of left-invariant densities and the space $\mathcal{D}\mathfrak{g}$ (compare Theorem 0.2.3). This allows the next definition.

**Definition 1.2.5.** The unique density $\lambda$ on $G$, with $(l_g)^*(\lambda) = \lambda$ for all $g \in G$ and $\int_G \lambda = 1$, is called the *(normalized) Haar measure on G*, and will be denoted by $dg$.

Note that here we make the assumption that $G$ is compact, in order to do the normalization. We will be needing the following lemma about this Haar measure.

**Lemma 1.2.6.** *For a left-invariant density $\lambda$ on $G$, and $f \in C_c(G)$, we have for all $g$ in $G$:*

$$\int_G \big((l_g)^*f\big)\lambda = \int_G f\lambda. \tag{1.4}$$

*Here $(l_g)^*f := f \circ l_g$*

*Proof.* Since $\lambda$ is left-invariant, we can write $(l_g)^*(f)\lambda = (l_g)^*(f)(l_g)^*(\lambda) = (l_g)^*(f\lambda)$. Observing that $l_g$ is an diffeomorphism, and applying the substitution of variables theorem then gives the desired result. $\square$

We are now ready to give the following definition, which will define the desired map $\Omega^k(M) \to \Omega^k_L(M)$.

**Definition 1.2.7.** For any $\omega \in \Omega^k(M)$, we define the map $m : \Omega^k(M) \to \Omega^k_L(M)$ by:

$$m(\omega) = \int_G (l_g)^*\omega \, dg. \tag{1.5}$$

The integral is defined point-wise, i.e.

$$m(\omega)_p(v_1, ..., v_k) = \int_G \big((l_g)^*\omega\big)_p(v_1, ..., v_k) \, dg.$$

Note that since in every point $((l_g)^*\omega)_p(v_1, ..., v_k)$ is smooth as a function of $g$ (with image in $\mathbb{R}$), the integral is well-defined point-wise. To see that $m(\omega)$ is a smooth form, note that since $G$ is compact we can use a partition of one to decompose the integral as a finite sum of integrals over charts. Therefore, by locally differentiating under the integral sign, it follows that the integral commutes with taking (partial) derivatives, from which it follows the $m(\omega)$ is a smooth form. It also follows from the same reasoning that $m(d\omega) = dm(\omega)$, since locally $d$ works by taking partial derivatives, so commutes with the integral (note that moreover $(l_g)^*$ also commutes with $d$). The fact that $m$ indeed maps to $\Omega^k_L(M)$ is asserted in the next lemma.

**Lemma 1.2.8.** *For $m$ as defined above we have the following properties:*

*(i) For any $\omega \in \Omega^k(M)$, we have $(l_g)^*(m(\omega)) = m(\omega)$, i.e. $m(\omega) \in \Omega^k_L(M)$.*

*(ii) If $\omega \in \Omega^k_L(M)$, then $m(\omega) = \omega$.*

*(iii) $m(d\omega) = dm(\omega)$.*

*Proof.* Fix $h \in G$, then

$$(l_h)^* \int_G (l_g)^*\omega \, dg = \int_G (l_h)^*(l_g)^*\omega \, dg = \int_G (l_{gh})^*\omega \, dg = \int_G (l_g)^*\omega \, dg.$$

Here the last equality follows by Lemma 1.2.6. The first follows by simply writing out the various definitions:

$$\Big((l_h)^* \int_G (l_g)^*\omega \, dg\Big)_p(v_1, ..., v_k) = \Big(\int_G (l_g)^*\omega \, dg\Big)_{hp}\big(T_e(l_h)(v_1, ..., v_k)\big) =$$
$$\int_G ((l_g)^*\omega)_{hp}(T_e(l_h)(v_1, ..., v_k)) \, dg = \int_G \omega_{ghp}(T_e(l_g) \circ T_e(l_h)(v_1, ..., v_k)) \, dg$$
$$= \int_G ((l_h)^*(l_g)^*\omega)_p(v_1, ..., v_k) \, dg.$$

With this we have proven (i). Assertion (ii) immediately follows from $(l_g)^*\omega = \omega$, and the fact that $\int_G dg = 1$. Assertion (iii) has been shown above. $\square$

We now turn our attention to the cohomology of $M$. Since we now have the following commutative diagram,

$$\begin{array}{ccccccc} \xrightarrow{\cdots} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{\cdots} \\ & m\downarrow\uparrow i & & m\downarrow\uparrow i & \\ \xrightarrow{\cdots} & \Omega^k_L(M) & \xrightarrow{d} & \Omega^{k+1}_L(M) & \xrightarrow{\cdots} \end{array} \tag{1.6}$$

it follows that $i$ and $m$ induce maps in cohomology, $i_* : H^k_L(M) \to H^k(M)$, $m_* : H^k(M) \to H^k_L(M)$. We want to prove that these are isomorphisms. Note that by Lemma 1.2.8 $(ii)$, we already have $m \circ i = id$, so immediately $m_* \circ i_* = (id)_* = id$. However, in general it does not hold that $i \circ m = id$. To prove that in cohomology this identity does hold, we need to show that we can write $(m - id)\omega = dH(\omega) + H(d\omega)$, where $H$ is a linear map $\Omega^k(M) \to \Omega^{k-1}(M)$ (or rather a collection of maps, defined for every $k$). A collection of such linear maps that satisfies this equality for all $\omega \in \Omega^k(M)$ is called a *homotopy operator* (of $m$ and $id$). If we can prove that there exists a homotopy operator, it then follows that in cohomology the two maps are equal, so that $m_*$ is indeed a linear isomorphism between the two spaces.

By Theorem 0.1.2, we know there exists such an operator for the pullbacks of any two maps that are homotopic. Now if we assume that $G$ is connected, it is easy to see that $l_g \simeq l_e = id$, for every $g \in G$, by choosing a path $\gamma$ from $g$ to $e$ (recall that we write $f_1 \simeq f_2$ to indicate that two functions $f_1$ and $f_2$ are homotopic). We therefore have:

$$m(\omega) - \omega = \int_G (l_g)^*\omega \, dg \; - \; (id)^*\omega = \int_G (l_g - id)^*\omega \, dg = \int_G dk_g(\omega) + k_g(d\omega) \, dg \tag{1.7}$$

Where $k_g$ is a homotopy operator between $(l_g)^*$ and $id$, for every $g \in G$. However, it is not clear if we can turn this expression into to the desired form, mainly because we don't know how the maps $k_g$ depend on $g$. In the proof of the theorem we will therefore use a construction that explicitly shows the dependence on $g$. The main inspiration for this comes from [5, Ch. IV, §1], but it should be noted that the actual proofs that we will give differ from the ones given there.

In the part that follows we will briefly 'forget' that we are working with a Lie group, and instead consider a general manifold $X$. This will make it easier to see what is going on.

We will now introduce some notation. Let $M$ and $X$ be two smooth manifolds, let $dx$ be a density on $X$. Assume we have a smooth family of maps $l_x : M \to M$ indexed by $x \in X$, i.e. the map $l : X \times M \to M$, with $l(x, m) = l_x(m)$, is smooth as a map between manifolds. Denote for any $x \in X$ by $i_x : M \to X \times M$ the map defined by $i(m) = (x, m))$. By $\pi : X \times M \to M$ we denote the projection to $M$, so $\pi(x, m) = m$. Finally, $const_{x_0} : X \to X$ is the map that has value $x_0$ everywhere.

***Remark***. Note that, with the notation above, we have $l \circ i_x = l_x$. Also, it holds that $\pi \circ i_x = id_M$. The map $(l_{x_0} \circ \pi) : X \times M \to M$ is given by $(l_{x_0} \circ \pi)(x, m) = l_{x_0}(m)$ for $x_0 \in M$.

**Lemma 1.2.9.** *Let $U \subseteq X$ be an open set such that $x_0 \in U$, and $U$ is contractible to $\{x_0\}$. Then the restrictions of $l$ and $l_{x_0} \circ \pi$ to $U \times M$ are homotopic as maps $U \times M \to M$.*

*Proof.* There exists a homotopy $F : \mathbb{R} \times U \to U$ such that $F(0, \cdot\,) = id_U,\ F(1, \cdot\,) = const_{x_0}$, since $U$ is contractible. Now define

$$\tilde{F} : \mathbb{R} \times U \times M \to M, \ \ \tilde{F}(t, x, m) = l\big(F(t, x), m\big).$$

Then $\tilde{F}$ is an homotopy between $l$ and $l_{x_0} \circ \pi$ (easy to check), so we have $l \simeq l_{x_0} \circ \pi$. □

**Lemma 1.2.10.** *Suppose that $dx$ is compactly supported in $U$, and let $x_0 \in U$. Then there exists a homotopy operator $H_U^{x_0} : \Omega^k(M) \to \Omega^{k-1}(M)$ such that for all $\omega \in \Omega^k(M)$;*

$$\int_X (l_x)^*\omega \, dx - \int_X (l_{x_0})^*\omega \, dx \ = \ dH_U^{x_0}(\omega) + H_U^{x_0}(d\omega).$$

*Proof.* Be the remark above, we can write $(l_x)^*\omega - (l_{x_0})^*\omega = (l \circ i_x)^*\omega - (l_{x_0} \circ \pi \circ i_x)^*\omega = (i_x)^* \circ (l^* - (l_{x_0} \circ \pi)^*)\omega$.
By Lemma 1.2.9, we can therefore write $(l_x)^*\omega - (l_{x_0})^*\omega = (i_x)^*(dk(\omega) + k(d\omega))$ for a homotopy operator $k = k_U^{x_0} : \Omega^k(M) \to \Omega^{k-1}(X \times M)$. Since everything involved is linear, and $d$ commutes with pullbacks, we now have

$$\int_X (l_x)^*\omega \ \ dx - \int_X (l_{x_0})^*\omega \ \ dx = \int_X d\,(i_x)^*(k(\omega)) \ \ dx - \int_X (i_x)^*(k(d\omega)) \ \ dx. \tag{1.8}$$

Considering the integrals pointwise (i.e in a point $p \in M$, and on tangent vectors $v_1, ..., v_k \in T_pM$), we see that the integrands on the right hand side depend smoothly on $x$ (as functions $X \to \mathbb{R}$). We can therefore, by locally differentiating under the integral sign, write

$$\int_X d\,(i_x)^*(k(\omega))\, dx = d \int_X (i_x)^*(k(\omega))\, dx.$$

Define now $H_U^{x_0} : \Omega^k(M) \to \Omega^{k-1}(M)$ as

$$H_U^{x_0}(\omega) := \int_X (i_x)^*(k(\omega))\, dx,$$

then by the above we can write $\int_X (l_x)^*\omega \, dx - \int_X (l_{x_0})^*\omega \, dx = dH_U^{x_0}(\omega) + H_U^{x_0}(d\omega)$.

□

**Lemma 1.2.11.** *Suppose that $X$ is a compact connected manifold, $x_0 \in X$. Let $dx$ be a density on $X$. Then there exists a homotopy operator $H : \Omega^k(M) \to \Omega^{k-1}(M)$ such that for all $\omega \in \Omega^k(M)$*

$$\int_X (l_x)^*\omega \, dx - \int_X (l_{x_0})^*\omega \, dx \ = \ dH(\omega) + H(d\omega).$$

*Proof.* Let $\{U_i\}$ be an open, finite cover of $X$, where the $U_i$'s are chosen in such a way that they are all contractible to points $x_i \in U_i$ (using charts, this is clearly always possible). Now choose a smooth partition of unity $\{\psi_i\}$ subordinate to this cover. By the previous lemma, for every $i$ we can write

$$\int_X (l_x)^* \omega \; \psi_i(x)\, dx - \int_X (l_{x_i})^* \omega \; \psi_i(x)\, dx \;=\; dH^{x_i}_{U_i}(\omega) + H^{x_i}_{U_i}(d\omega). \tag{1.9}$$

We now remark that by choosing a smooth path in $X$, we have $l_{x_i} \simeq l_{x_0}$ for every $i$, and so we can write $\int_X \big((l_{x_i})^*(\omega) - (l_{x_0})\big)^* \omega \; \psi_i(x)\, dx = dh^{x_i}(\omega) + h^{x_i}(d\omega)$, again by using local differentiation under the integral sign. Note that $h^{x_i}$ will be a function $\Omega^k(M) \to \Omega^{k-1}(M)$. Then:

$$\begin{aligned}
\int_X (l_x)^* \omega \; \psi_i(x)\, dx - &\int_X (l_{x_0})^* \omega \; \psi_i(x)\, dx \\
&= \int_X (l_x)^* \omega \; \psi_i(x)\, dx - \int_X (l_{x_i})^* \omega \; \psi_i(x)\, dx + \int_X \big((l_{x_i})^* - (l_{x_0})^*\big)\, \omega \; \psi_i(x)\, dx \\
&= dH^{x_i}_{U_i}(\omega) + H^{x_i}_{U_i}(d\omega) + dh^{x_i}(\omega) + h^{x_i}(d\omega) \\
&= d(H^{x_i}_{U_i} + h^{x_i})(\omega) + (H^{x_i}_{U_i} + h^{x_i})(d\omega).
\end{aligned}$$

Now finally, since $\int_X \; dx = \sum_i \int_X \psi_i(x)\, dx$ we can define $H = \sum_i \big(H^{x_i}_{U_i} + h^{x_i}\big)$, and we obtain the desired result. □

With Lemma 1.2.11 proven, Theorem 1.2.1 follows immediately by setting $X = G$, and observing that $l_e = id_M$. Together with Theorem 1.1.5 we now also have the following corollary, which is the main result of this chapter:

**Corollary 1.2.12.** *Let $G$ be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Then we have the following linear isomorphism of cohomology groups;*

$$H^k_{dR}(G) \cong H^k(\mathfrak{g}).$$

*The isomorphism is induced by the map $\omega \mapsto m(\omega)_e$, $\Omega^k(G) \to \Lambda^k \mathfrak{g}^*$.*

# Part II

# Intermezzo: Tensor products and exterior algebras

In the previous part we established most of the basic theory that we will use in this thesis. Part III and IV will focus on applications of this theory.

Before we start with this however, we first need to introduce some algebraic structures that are important tools for what we will discuss later. More specifically, we will introduce some basic notions of tensor products and exterior algebras of vector spaces. A large part of this will be necessary for Part IV of this thesis, where we will heavily rely on these structures for our discussion.

A lot of what we will discuss here is rather standard theory, and it can therefore be found in almost any introductory text on (differential) algebra. Our treatment is taken from multiple sources. Most of the discussion about tensor products is based on the treatment in the book by Lee, [12, Ch. 12]. The section about exterior algebras is based mainly on Serge Lang's book, [11, Ch. XIX]. The final section, where we construct tensor products of exterior algebras is also based on this, although some of the more specific parts are taken from the first few sections of Koszuls article, [10, Ch. I]. This article will also be the main source in Part IV.

Because of the elementary (and very algebraic) nature of this part, a couple of propositions are stated without proof. We ask the reader to go through the sources we just mentioned, if he or she is interested in these proofs, or a more thorough treatment.

## 2.1 Tensor products

We start with an introduction to (formal) tensor products of vector spaces. As always, these vector spaces are always assumed to be finite dimensional, and have scalar field $\mathbb{R}$. As said before, this treatment is based on [12, Chapter 12].

**Definition 2.1.1.** Let $S$ be any set. A *formal linear combination of elements of* $S$ is a function $f : S \to \mathbb{R}$ with $f(s) = 0$ for all but finitely many $s \in S$. By $\mathcal{F}(S)$ we denote the set of formal linear combinations of $S$. With pointwise addition and scalar multiplication $\mathcal{F}(S)$ becomes a vector space, the *free vector space on* $S$.

A formal linear combination of $S$ can intuitively be seen as a formal sum $\sum a_i x_i$ with $x_i \in S$, $a_i \in \mathbb{R}$. From this it is clear that if we identify an element $x \in S$ with the function $\delta_x \in \mathcal{F}(S)$, defined by $\delta_x(x) = 1, \delta_x(y) = 0$ for all $y \neq x$, then $S$ forms a (linear) basis of $\mathcal{F}(S)$.

Let now $V_1, ..., V_k$ be vector spaces, and form the free vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$. Denote by $\mathcal{R}$ the subspace of $\mathcal{F}(V_1 \times \cdots \times V_k)$ generated by elements of the form

$$(v_1, ..., v_i + v_i', ...v_k) - (v_1, ..., v_i, ...v_k) - (v_1, ..., v_i', ...v_k) \text{ and}$$
$$(v_1, ..., av_i, ...v_k) - a(v_1, ..., v_i, ..., v_k),$$

where $v_i \in V_i$, and $a \in \mathbb{R}$.

**Definition 2.1.2.** We define the *tensor product of* $V_1, ..., V_k$, denoted by $V_1 \otimes \cdots \otimes V_k$, as the quotient space $\mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R}$. The equivalence class of a $k$-tuple $(v_1, ..., v_k)$ is denoted by $v_1 \otimes ... \otimes v_k$

From how we defined the tensor product, it follows that we have:

$$v_1 \otimes \cdots \otimes (av_i + v_i') \otimes \cdots \otimes v_k = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) + v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_k$$

Also, it holds that every element of tensor product can be expressed as a linear combination of elements of the form $v_1 \otimes \cdots \otimes v_k$. A basis is formed by elements of the form $e_1^{i_1} \otimes \cdots \otimes e_k^{i_k}$, where $e_j^{i_l}$ is a element in the basis of $V_j$. For tensor product spaces there is the following "universal property", that uniquely characterizes the tensor product up to linear isomorphism:

**Proposition 2.1.3** (Universal propery of the Tensor Product)**.** *Let* $A : V_1 \times \cdots \times V_k \to W$ *be a multilinear map into some vector space* $W$*. Then there exists a unique linear map* $\tilde{A}$ *such that the diagram below commutes (*$\pi$ *denotes the natural projection map).*

$$\begin{array}{ccc} V_1 \times \cdots \times V_k & \xrightarrow{A} & W \\ \downarrow \pi & \nearrow \tilde{A} & \\ V_1 \otimes \cdots \otimes V_k & & \end{array} \tag{2.1}$$

This property can be used to prove, among other things, that taking tensor products is associative. A second important proposition shows the relationship between the tensor product of the dual spaces of the $V_i$'s, and the space of multilinear functions $V_1 \times \cdots \times V_k$, denoted by $L(V_1, ..., V_k, \mathbb{R})$.

**Proposition 2.1.4.** *There is a canonical isomorphism of vector spaces* $V_1^* \otimes \cdots \otimes V_k^* \simeq L(V_1, ..., V_k, \mathbb{R})$*, given by* $(\omega^1 \otimes ... \otimes \omega^k)(v_1, ..., v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$*.*

We will now focus our attention on tensor product spaces of the form $V \otimes \cdots \otimes V$ ($k$-times) with $V$ a vector space. A shorthand notation for this is $V^{\otimes k}$ or $T^k(V)$.

For differentiable functions on such tensor spaces, we have the following useful lemma ('the product rule for tensor products).

**Lemma 2.1.5** (Product rule for tensor)**.** *Let* $V$ *be a vector space,* $f_1, f_2 : \mathbb{R} \to V$ *two differentiable functions. Then:*

$$\tfrac{d}{dt}(f_1(t) \otimes f_2(t)) = (\tfrac{d}{dt} f_1) \otimes f_2 + f_1 \otimes (\tfrac{d}{dt} f_2).$$

*Proof.* The proof is actually very similar to the proof of the 'normal' product rule in $\mathbb{R}$. We have

$$\lim_{h\to 0}\frac{f_1(t+h)\otimes f_2(t+h)-f_1(t)\otimes f_2(t)}{h}=\lim_{h\to 0}\frac{f_1(t+h)\otimes f_2(t+h)-f_1(h)\otimes f_2(t+h)}{h}$$
$$+\lim_{h\to 0}\frac{f_1(h)\otimes f_2(t+h)-f_1(h)\otimes f_2(t+h)}{h}.$$

Then using the bi-linearity of the tensor product, the statement follows. □

With the notation introduced above, we can define the *tensor space of* $V$, $T(V)$ as the direct sum $T(V) := \oplus_k T^k(V)$. With tensor multiplication $\otimes : T^k(V) \times T^l(V) \to T^{k+l}(V)$ this becomes a *graded algebra*, we call it the *tensor algebra of* $V$

## 2.2 Exterior algebra

We will go on to define the *exterior algebra of* $V$. Let $I$ be the two-sided ideal in $T(V)$ generated by elements of the form $v \otimes v$ for $v \in V$. This is the linear subspace generated by elements of the form $v_1 \otimes \cdots \otimes v_k$ with $v_i = v_j$ for some $i \neq j$.

**Definition 2.2.1.** The *exterior algebra of* $V$, denoted by $\Lambda(V)$, is defined as the quotient space $T(V)/I$. The equivalence class of an element $x \otimes y$ is denoted by $x \wedge y$.

Since it holds that

$$0 = (x+y)\wedge(x+y) = x\wedge x + x\wedge y + y\wedge x + y\wedge y = x\wedge y + y\wedge x$$

($x \wedge x = 0$ for all $x \in V$), we see that for all $x, y \in V$, $x \wedge y = -(y \wedge x)$. In other words, the wedge product $\wedge$ is anti-commutative.

The *k-th exterior power of* $V$, denoted by $\Lambda^k(V)$ is the subspace of $\Lambda(V)$ generated by elements of the form $v_1 \wedge ... \wedge v_k$, it is the space of elements of *degree* $k$. We have $\Lambda^1(V) = V$,and define $\Lambda^0(V) = \mathbb{R}$. If dim($V$)= $n$, we have that $\Lambda^p(V) = 0$ for all $p > n$. $\Lambda(V)$ then decomposes as the direct sum $\Lambda(V) = \oplus_{k=1}^n \Lambda^k(V)$. Endowed with the wedge product $\wedge$, $\Lambda(V)$ also has the structure of a graded algebra. For $x \in \Lambda^k(V), y \in \Lambda^l(V)$ we have the following identity (*graded commutativity*):

$$x \wedge y = (-1)^{kl} y \wedge x. \tag{2.2}$$

For exterior powers, we also have a universal property:

**Proposition 2.2.2.** *Let $A : V^k \to W$ be a alternating linear map into a vector space $W$. Then there is a unique linear map $A'$ that makes the following diagram commute:*

$$\begin{array}{ccc} V^k & \xrightarrow{A} & W \\ \downarrow \pi & \nearrow_{A'} & \\ \Lambda^k(V) & & \end{array} \tag{2.3}$$

Applying this proposition for $W = \mathbb{R}$ shows that there is a natural isomorphism between the space of $k$-linear alternating functions $V^k \to \mathbb{R}$ and the dual $(\Lambda^k(V))^*$ of $\Lambda^k(V)$. As we will see below, this dual space is isomorphic to $\Lambda^k(V^*)$, and thus we have an isomorphism between the space of $k$-linear alternating functions $V^k \to \mathbb{R}$, and $\Lambda^k(V^*)$. This explains why the notation $\Lambda^k V^*$ is used for both spaces. We will also keep using this notation for both spaces, from context it should always be clear which one we are talking about.

**Proposition 2.2.3.** *Let* $\Phi : \Lambda^k(V^*) \to (\Lambda^k V)^*$ *be the map determined by*

$$\Phi(f^1 \wedge ... \wedge f^k)(x_1 \wedge ... \wedge x_k) = \det(f^i(x_j)).$$

*Then* $\Phi$ *is well-defined and a linear isomorphism* $\Lambda^k(V^*) \simeq (\Lambda^k V)^*$.

*Proof.* To introduce this map one can use the universal property of the exterior algebra, by noting that the above is indeed alternating with respect to the $f^i$'s. By then choosing a basis of $V$ (and $V^*$), one can show that it is a linear isomorphism. □

By the previous proposition, from now on we can identify $\Lambda^k V^* \simeq (\Lambda^k V)^*$. Let $\gamma : V \to W$ be a linear map between vector spaces $V$ and $W$. Then, using the universal property, we can uniquely extend $\gamma$ to an algebra homomorphism $\tilde{\gamma} : \Lambda V \to \Lambda W$. It satisfies $\tilde{\gamma}(v_1 \wedge ... \wedge v_k) = \gamma(v_1) \wedge ... \wedge \gamma(v_k)$, and will actually be a *graded* algebra homomorphism (meaning it maps $\Lambda^k V$ to $\Lambda^k W$ for a given $k$).

We can then also determine the dual map $\tilde{\gamma}^* : (\Lambda W)^* \to (\Lambda V)^*$. Using the identification above, this will be exactly the map that we would obtain by extending $\gamma^* : W^* \to V^*$ to $\widetilde{(\gamma^*)} : \Lambda W^* \to \Lambda V^*$. When talking about the dual map, we will always mean this one, the one *after* the identification. It follows that this dual will *also* be a graded algebra homomorphism.

## 2.3 Tensor products of algebras

We will now discuss the construction of the tensor product of two algebras. This construction can be done for any algebra, but we will focus our attention to exterior algebras, since those are the ones we will be using.

**Definition 2.3.1.** Let $V, W$ be a vectorspaces, $\Lambda V, \Lambda W$ their exterior algebras. We define the *tensor product of the exterior algebras* as the space $\Lambda V \otimes \Lambda W$, endowed with the multiplication

$$(u \otimes v) \cdot (u' \otimes v') = (-1)^{kl}(u \wedge u') \otimes (v \wedge v'),$$

where $k$ and $l$ are the degrees of $u'$ and $v$.

By defining the product in this way, the resulting space will still have the structure of an algebra, and moreover it will still satisfy graded commutativity (this is the reason for the $(-1)^{kl}$ factor). The next proposition gives another important reason why this definition is natural.

**Proposition 2.3.2.** *There exist a natural isomorphism (of graded algebras) between* $\Lambda(V \oplus W)$ *and* $\Lambda V \otimes \Lambda W$.

*Proof.* Let $\{e_i\}$ and $\{f_i\}$ be bases of $V$ and $W$ respectively. A basis of $\Lambda(V \oplus W)$ is given by elements of the form $e_{i_1} \wedge ... \wedge e_{i_l} \wedge f_{i_{l+1}} \wedge ... \wedge f_{i_k}$. Define now a map $\Lambda(V \times V) \to \Lambda V \otimes \Lambda V$ on basis elements by

$$e_{i_1} \wedge ... \wedge e_{i_l} \wedge f_{i_{l+1}} \wedge ... \wedge f_{i_k} \mapsto (e_{i_1} \wedge ... \wedge e_{i_l} \wedge 1) \otimes (f_{i_{l+1}} \wedge ... \wedge f_{i_k} \wedge 1)$$

(recall that 1 is the unit element of $\Lambda^0 V$ and $\Lambda^0 W$, which in this thesis will always be $1 \in \mathbb{R}$). One can check that the map given above this indeed defines an isomorphism of graded algebras between the two spaces. □

We will now discuss some important facts having to do with this identification. Assume we have a vector space $V = V_1 \oplus V_2$. Let $i_1 : V_1 \to V$ be the canonical inclusion of $V_1$ into $V$. Under the identification of the previous proposition, the extension $\tilde{\iota_1} : \Lambda(V_1) \to \Lambda(V_1) \otimes \Lambda(V_2) \simeq \Lambda(V)$ will be given by $\tilde{\iota_1}(u) = u \otimes 1$. In a similar way, we can define the map $i_2$, and then we have $\tilde{\iota_2}(u) = 1 \otimes u$.

It can now also be worked out that if $a \in \Lambda(V_1^*), b \in \Lambda(V_2^*)$, the dual map $\tilde{\iota_1}^* : \Lambda(V_1^*) \otimes \Lambda(V_2^*) \to \Lambda(V_1^*)$ will be given by

$$\tilde{\iota_1}^*(a \otimes b) = b_0 a, \tag{2.4}$$

where $b_0$ a is the (scalar) component of $b$ in $\Lambda^0(V_2)$. To see this, note that for $u \in V_1$, $(a \otimes b)(u \otimes 1) = a(u)b(1) = a(u)b_0 \in \mathbb{R}$, where we have used $(\Lambda(V_1 \oplus V_2))^* \simeq \Lambda(V_1^*) \otimes \Lambda(V_2^*)$ and the identification from Proposition 2.2.3.

Let us now look at the 'direct' dual of $i_1$, $i_1^* : V_1^* \oplus V_2^* \to V_1^*$. Let $a \oplus b \in V_1^* \oplus V_2^*$. Then it holds for every $v \in V_1$ that

$$i_1^*(a \oplus b)(v) = (a \oplus b)(i_1(v)) = a(v).$$

We have used here that $(V_1 \oplus V_2)^* \simeq V_1^* \oplus V_2^*$. Therefore, we see that the map $i_1^*$ is the natural projection map of $V_1^* \oplus V_2^*$ to the first coordinate.

Finally, we will discuss the 'diagonal' map $\varphi : V \to V \times V$ given by $\varphi(v) = (v, v)$. The dual of its extension to $\Lambda(V)$ will be a map $\Lambda(V^*) \otimes \Lambda(V^*) \to \Lambda(V^*)$, for which the following important proposition holds.

**Proposition 2.3.3.** *Let $\varphi : V \to V \times V$ the map $v \mapsto (v, v)$, and let $\tilde{\varphi}^* : \Lambda(V^*) \otimes \Lambda(V^*) \to \Lambda(V^*)$ be the induced dual map on the exterior algebra. Then:*

$$\tilde{\varphi}^*(a \otimes b) = a \wedge b$$

*for $a \otimes b \in \Lambda(V^*) \otimes \Lambda(V^*)$.*

*Proof.* By the discussion at the end of the previous section, it follows that $\tilde{\varphi}^*$ is an algebra homomorphism. We therefore have for an element $a \otimes b \in \Lambda(V^*) \otimes \Lambda(V^*)$

$$\tilde{\varphi}^*(a \otimes b) = \tilde{\varphi}^*\big((a \otimes 1) \wedge (1 \otimes b)\big) = \tilde{\varphi}^*(a \otimes 1) \wedge \tilde{\varphi}^*(1 \otimes b)$$

We now claim that $\tilde{\varphi}^*(a \otimes 1) = a$. Let $\pi_1 : V \times V \to V$ be the projection to the first component, i.e. $\pi_1(u, v) = u$. Clearly, $\pi_1 \circ \varphi$ is the identity. Therefore $\tilde{\varphi}^* \circ \tilde{\pi_1}^*$ must also be the identity. The map $\tilde{\pi_1}^* : \Lambda(V^*) \to \Lambda(V^*) \otimes \Lambda(V^*)$ can be shown to be the map $a \mapsto a \otimes 1$, this follows from the fact that the dual of $\pi_1$ is the natural inclusion $V^* \to V^* \times V^*$ to the first component (compare the discussion about $i_1^*$ above). The claim then follows. We can of course prove a similar claim for the second component, which then proves the proposition. □

# Part III

# Hodge decomposition theorem for compact Lie groups

As said before, in this part we will look at an application of what we have shown in Part I. The theorem we want to prove is a version of the *Hodge decomposition theorem*, in the case of compact connected Lie groups. The exact formulation of this theorem is given in Theorem 3.4.8. Before we start proving this theorem, we will first take a look at *bi-invariant* forms, forms that are both left- and right-invariant. These are an important part of the proof. We will also briefly discuss the concept of *Riemannian manifolds*, since Hodges theorem is about manifolds that have such a Riemannian structure. Afterwards, we will introduce the *Hodge star operator*, which is a map $\Omega^k(G) \to \Omega^{k-1}(G)$. With this operator, we can then define the *Hodge Laplacian*, the map $\Delta : \Omega^k(G) \to \Omega^k(G)$. Our version of the Hodge decomposition theorem is about exactly this operator. The main source for this part is the book by Helgason, [7, Ch. II, § 7]. Our treatment differs from Helgason's mainly in that we introduce the $*$-operator in a different (but equivalent) way, and that we provide more details in the final part of the proof.

## 3.1 Bi-invariant forms

The primary source of this section is again the article by Chevalley and Eilenberg, [2, § 11-12]

**Definition 3.1.1.** A *bi-invariant k-form* $\omega \in \Omega^k(G)$ is a form that is both left- and right-invariant, i.e. $(l_g)^*\omega = \omega = (r_g)^*\omega$, for all $g \in G$. We denote by $\Omega^k_I(G)$ the space of all bi-invariant $k$-forms.

Since the pullback of both left- and right-multiplication commutes with $d$, we can define the *cohomology of bi-invariant forms*, denoted $H_I(G)$, in the same way we defined the cohomology of left-invariant forms.

***Remark.*** Note that in particular a bi-invariant form is a left-invariant form that is invariant under conjugation, this follows from the fact that $C_h = l_h \circ r_{h^{-1}}$.

**Lemma 3.1.2.** *Let $\omega$ be a bi-invariant form. Then $\omega$ is closed, i.e. $d\omega = 0$*

*Proof.* Assume $\omega$ is a bi-invariant $k$-form. Then $\omega$ is certainly left-invariant, and is therefore completely determined by its value at $e$. By the remark above, it is also invariant under conjugation, i.e. $(C_h)^*\omega = \omega$ for all $h \in G$. Since conjugation leaves the identity fixed, it follows that

$$\omega_e(x_1, .., x_n) = \omega_e(T_e(C_h)x_1, ..., T_e(C_h)x_k) \quad \text{for } x_1, ..., x_k \in T_eG.$$

Using linearity, we can assume without loss of generality that we can write $\omega_e = \omega_1 \wedge ... \wedge \omega_k \in \Lambda^k \mathfrak{g}^*$. The above equation can the be expressed as:

$$\omega_1 \wedge ... \wedge \omega_k = (\omega_1 \circ \mathrm{Ad}(h)) \wedge ... \wedge (\omega_k \circ \mathrm{Ad}(h)) \tag{3.1}$$

(recall that $\mathrm{Ad}(h) := T_eC_h$). Set now $h := \exp(tx)$, for some $x \in \mathfrak{g}$. Differentiating equation 3.1 with respect to $t$ in $t = 0$ on both sides then gives, using Lemma 2.1.5 and the chain rule:

$$0 = \sum_{i=1}^{k} (\omega_1 \circ id) \wedge ...(\omega_i \circ \tfrac{d}{dt}\big|_{t=0} \ \mathrm{Ad}(\exp tx))).... \wedge (\omega_k \circ id)$$
$$= \sum_{i=1}^{k} \omega_1 \wedge ...(\omega_i \circ \mathrm{ad}(x)).... \wedge \omega_k.$$

Recall that, by definition, $\mathrm{ad}(x)y = [x, y]$. Therefore, by the equation above

$$\sum_{i=1}^{k} \omega_e(x_1, ..., [x, x_i], ..., x_k) = 0 \quad \text{for all } x \in \mathfrak{g}. \tag{3.2}$$

Now also recall that we have for a left-invariant form, by Prop. 1.1.3

$$d\omega_e(x_0, ..., x_k) = \sum_{i<j} (-1)^{i+j} \ \omega_e([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_k).$$

Using that the Lie bracket is anti-symmetric, we can then also write:

$$2d\omega_e(x_0, ..., x_k) = \sum_{i<j} (-1)^{i+j} \ \omega_e([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_k) +$$
$$\sum_{j<i} (-1)^{i+j+1} \ \omega_e([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ...x_k).$$

Be rearranging terms, using that $\omega$ is skew-symmetric, i.e. interchanging variables changes the sign, we find:

$$2d\omega_e(x_0, ..., x_k) = \sum_{i \neq j} \pm \ \omega_e(x_0, ..., [x_i, x_j], ..., \hat{x_j}, ..., x_k).$$

Applying equation 3.2, we see that indeed $d\omega_e = 0$, and by left-invariance of $\omega$ therefore $d\omega = 0$. □

***Remark***. Using some *representation theory*, one can give a more elegant proof. It goes like this. First we note that, by (3.1), the bi-invariant forms correspond exactly to the elements of $\Lambda^k\mathfrak{g}^*$ that are invariant under the *adjoint representation* Ad of $G$ in $\Lambda^k\mathfrak{g}^*$. But these correspond to the elements of $\Lambda^k\mathfrak{g}^*$ that are invariant under the *induced representation* of Ad (which is ad) of $\mathfrak{g}$ in $\Lambda^k\mathfrak{g}^*$. This then immediately gives (3.2). The rest of the proof is then the same. See the Appendix for a short introduction to representations.

**Theorem 3.1.3.** *Let $G$ a compact Lie group. Then the inclusion map $i : \Omega_I^k(G) \to \Omega^k(G)$ induces a linear isomorphism*

$$H^k_{dR}(G) \simeq H^k_I(G).$$

*Proof.* Define the left action of $G \times G$ on $G$ by $l(g,h)(x) = gxh^{-1}$, for $(g,h) \in G \times G, x \in G$. Note that the bi-invariant forms on $G$ are exactly the forms that are (left)-invariant under this action. Now apply Theorem 1.2.2. □

**Theorem 3.1.4.** *Every equivalence class of $H^k_{dR}(G)$ contains exactly one bi-invariant form.*

*Proof.* This follows by looking at the space $H^k_I(G)$. Since $d\omega = 0$ for every bi-invariant form $\omega$, it follows that $H^k_I(G) := Ker(d_k)/Im(d_{k-1}) = \Omega^k_I(G)/\{0\} = \Omega^k_I(G)$ (recall that we restrict $d_k$ in this definition to the space $\Omega^k_I(G)$). Thus Theorem 3.1.3 implies the linear isomorphism (and, in particular, bijective relation)

$$H^k_{dR}(G) \simeq \Omega^k_I(G),$$

from which the statement follows. □

## 3.2 Riemannian manifolds

The theorem that we want to prove in Part III involves so called *Riemannian manifolds.* We will give a short introduction to Riemannian manifolds in this section.

The main idea of a Riemannian manifold is that we introduce a 'metric' on our manifold (actually more of an inner product), so that we are able to talk about distances, and hence consider the manifold as a metric space.

**Definition 3.2.1.** Let M be a smooth manifold. A *Riemannian metric on M* is a smooth covariant, symmetric 2-tensor field $\beta$ on M that is positive definite at each point. We call the pair $(M, \beta)$ a *Riemannian manifold.* ([12, Ch. 13])

Note that from the definition it follows that $\beta$ defines, at every point $p$ in $M$, an inner product $\beta_p$ on $T_pM$. Often we will use the term 'metric' instead of 'Riemannian metric'. If a Riemannian manifold is also a Lie group, then we can talk about left- and right-invariance of the metric. We say that $\beta$ is *left-invariant* if $(l_g)^*\beta = \beta$, in other words if $\beta_h(u,v) = \beta_{gh}(T_h l_g\, u, T_h l_g\, v)$ for all $g, h \in G$, and $u, v \in T_hG$. In similar ways, we define what it means for a metric to be *right-invariant* or *bi-invariant.*

**Proposition 3.2.2.** *Let G be a compact Lie group. Then G admits a bi-invariant metric $\beta_g$.*

*Proof.* Let $\langle .\,,.\rangle$ be any inner product on $\mathfrak{g}$. Construct a right-invariant metric by defining $\beta_g(u,v) := \langle T_g(r_{g^{-1}})u, T_g(r_{g^{-1}})v\rangle$ for $u, v \in T_gG$.

To now construct a metric that is also left-invariant, we use the same trick we used to make left-invariant forms, the 'averaging' process. Define $\beta'_p(u,v) := \int_G (l_g^*\beta)_p(u,v)\; dg = \int_G \beta_{gp}(T_p l_g u, T_p l_g v)\; dg$ for $u, v \in T_pG$. Left-invariance of $\beta'$ then follows by left-invariance of the Haar measure, in the same way as we have seen for forms. $\square$

## 3.3 The Hodge star operator

In this section, we will introduce the so called *Hodge star operator.* The main source for this is [4, § 2.7], and the book by Helgason, [7, Ch. II, § 7]. Let $V$ be a vector space, endowed with an inner product $\langle .\,,.\rangle$. This inner product induces an inner product on the space $\Lambda^k V$, which can be given by

$$\langle \lambda_1 \wedge \cdots \wedge \lambda_k, \mu_1 \wedge \cdots \wedge \mu_k\rangle = \det\langle \lambda_i, \mu_j\rangle,$$

for $\lambda_i, \mu_j \in \Lambda^1 V$. Let $e_1, ..., e_n$ be a orthonormal basis of $V$. Then for $\lambda \in \Lambda^k V, \mu \in \Lambda^{n-k}V$, we can write $\lambda \wedge \mu = f(\lambda,\mu)\, e_1 \wedge \cdots \wedge e_n$, for some specific linear function $f$ with values in $\mathbb{R}$. For a fixed $\lambda$ there is a unique $*\lambda \in \Lambda^{n-k}V$ such that $f(\lambda,\mu) = \langle \mu, *\lambda\rangle$ (for infinite dimensional vector spaces, this is the *Riesz representation theorem*).

**Definition 3.3.1.** We define the $*$-operator, $* : \Lambda^k V \to \Lambda^{n-k}V$ as the map that sends a $\lambda$ to (the uniquely determined) $*\lambda$.

**Proposition 3.3.2.** *The $*$ operator is linear, and determined on orthonormal basis elements by*

$$*\,(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{i_{k+1}} \wedge \cdots \wedge e_{i_n} \tag{3.3}$$

*where $(i_1, ..., i_n)$ is a permutation of $(1, ..., n)$ and the sign is determined by whether this permutation is odd or even. Also, for $\eta \in \Lambda^k(V)$, $**\eta = (-1)^{k(n-k)}\eta$.*

*Proof.* Showing that $*$ is linear is easy, everything else then follows quickly by choosing a basis and working out the requirements. For more details, see [4, § 2.7]. □

From the proposition, it follows that for $\lambda, \mu \in \Lambda^k V$ we have

$$\mu \wedge *\lambda = (-1)^{k(n-k)} \langle \mu, \lambda \rangle e_1 \wedge \cdots \wedge e_n. \tag{3.4}$$

Since any inner product on $V$ induces an inner product on the dual $V^*$, the Hodge star operator is also defined on the space $\Lambda^k V^*$. For a Riemannian manifold $M$, with metric $\beta_g$, this means that the $*$-operator is defined on $\Lambda^k T_p M^*$ for every $p \in M$. If moreover $M$ is orientable, then we can choose a positively oriented volume form $\Theta = \theta_1 \wedge ... \wedge \theta_n$ (where the $\theta_i \in \Omega^1(M)$ form an orthonormal frame), and define the $*$-operator on $\Omega^k(M)$ by requiring for $\omega \in \Omega^{n-k}(M)$, $\eta \in \Omega^k(M)$,

$$\omega \wedge \eta = \langle \omega, *\eta \rangle\ \Theta, \quad \text{i.e. } (\omega \wedge \eta)_p = \langle \omega_p, *\eta_p \rangle_p\ \Theta_p, \text{ for all } p \in M. \tag{3.5}$$

Here we mean with $\langle .\ ,. \rangle_p$ the inner product on the dual space $\Lambda^k T_p M^*$, induced by the metric at the point $p$.

Let us now assume that we have a compact, connected Lie group $G$, with a given *bi-invariant* metric, which can always be constructed as in Proposition 3.2.2. We can then choose a global. orthonormal left-invariant frame $\tilde{X}_1, ..., \tilde{X}_n$, with dual frame $\theta_1, ..., \theta_n$ (chosen so that $\theta_i(X_j) = \delta_{ij}$), and set $\Theta = \theta_1 \wedge ... \wedge \theta_n$. Then $\Theta$ is clearly also left-invariant, and we can choose the orientation on $G$ such that $\Theta$ is positively oriented. We claim moreover that $\Theta$ is also right-invariant. Since $\Theta$ is a left-invariant top-form, we have $(r_g)^*\Theta = (r_g)^*(l_{g^{-1}})^*\Theta = \mathrm{Ad}(g^{-1})^*\Theta = \det(\mathrm{Ad}(g^{-1}))\Theta$. The claim then follows by using the lemma below, which is an important property of compact connected Lie groups.

**Lemma 3.3.3.** *Let $G$ be a compact connected Lie group. Then $\det(Ad(g)) = 1$, for all $g \in G$.*

*Proof.* Consider the map $g \mapsto |\det(\mathrm{Ad}(g))|$. It can be easily checked that this is a continuous group homomorphism into the multiplication group of positive real numbers. Its image therefore has to be a compact subgroup of $(\mathbb{R}_{>0}, \cdot)$. It is easy to see that this can only be the subgroup $\{1\}$. We now know that $\det(\mathrm{Ad}(g))$ is either 1 or $-1$. However, since the map $g \mapsto \det(\mathrm{Ad}(g))$ is continuous, with connected domain, its image must also be connected. Since $\det(\mathrm{Ad}(e)) = 1$, it therefore follows that $\det(\mathrm{Ad}(g)) = 1$ for all $g \in G$. □

***Remark.*** In general, Lie groups that satisfy $|\det(Ad(g))| = 1$ for all $g \in G$ are called *unimodular.*

**Lemma 3.3.4.** *The $*$-operator as defined above for compact connected Lie groups commutes with left- and right pull-backs, i.e. for $\omega \in \Omega^k(G)$ we have:*

$$*(l_g^*\omega) = l_g^*(*\omega) \text{ and } *(r_g^*\omega) = r_g^*(*\omega) \quad \text{for all } g \in G$$

*Proof.* In the definition of the $*$-operator we started with a bi-invariant metric on $G$. This metric induces an inner product on $\Omega^k(G)$ that will be invariant under left- and right-pullbacks. This can be worked out using some theory from linear algebra, where we use that the adjoint of an isometry is also an isometry. We will show the statement of this lemma for left-multiplication, the case of right-multiplication can be done in essentially the same way. Let $\omega, \eta \in \Omega^k(G)$, using (3.4) we have

$$(-1)^{k(n-k)}\, \omega \wedge (*\, l_g^*\eta) = \langle \omega, l_g^*\eta\rangle\, \Theta = \langle l_{g^{-1}}^*\omega,\ l_{g^{-1}}^* l_g^*\eta\rangle\, \Theta = \langle l_{g^{-1}}^*\omega, \eta\rangle\, \Theta.$$

Also, by left-invariance of $\Theta$,

$$\begin{aligned}(-1)^{k(n-k)}\, \omega \wedge (l_g^* * \eta) &= (-1)^{k(n-k)}\, (l_g)^*(l_{g^{-1}}^*\omega \wedge *\eta) = (l_g)^*\big(\langle l_{g^{-1}}^*\omega, \eta\rangle\, \Theta\big) \\ &= \langle l_{g^{-1}}^*\omega, \eta\rangle\, (l_g)^*\Theta = \langle l_{g^{-1}}^*\omega, \eta\rangle\, \Theta.\end{aligned}$$

So we see that for all $\omega \in \Omega^k(G)$, we have $\omega \wedge (*\, l_g^*\eta) = \omega \wedge (l_g^* * \eta)$. We can therefore conclude that $*(l_g^*\eta) = l_g^*(*\eta)$.

□

With the $*$-operator, we can define (another) inner product on $\Omega^\bullet(G)$. For $\omega \in \Omega^k(G)$, $\eta \in \Omega^l(G)$ it is given by:

$$\langle \omega, \eta\rangle = \begin{cases} 0 & \text{if } k \neq l, \\ \int_G \omega \wedge *\eta & \text{if } k = l \end{cases} \tag{3.6}$$

By extending it bilinearly, we obtain an inner product on the whole of $\Omega^\bullet(G)$. It is strictly positive definite, to see this write $\omega = \sum_{i_1,\ldots i_p} a_{i_1\ldots i_p}\theta_{i_1} \wedge \ldots \wedge \theta_{i_p}$, which we can do because the $\theta_i$ form a basis of the dual tangent space at every point. This follows from the way we defined them. Now

$$\omega \wedge *\omega = \sum_{i_1,\ldots i_p} a_{i_1\ldots i_p}^2 \Theta$$

and the statement follows since $\Theta$ is positively oriented.

***Remark.*** If $\Theta$ is normalized, i.e. $\int_G \Theta = 1$, then this inner product is actually the same (up to perhaps a sign), as the one we saw before. This follows by integrating equation (3.5) over $G$ on both sides.

## 3.4 The Hodge Laplacian and Hodge's theorem for compact connected Lie groups

Before we define the Hodge Laplacian, we will first introduce the *$d^*$-operator*, $d^* : \Omega^k(G) \to \Omega^{k-1}(G)$. It is defined for a form $\omega \in \Omega^k(G)$ by

$$d^*\omega = (-1)^{n(k+1)+1}\ * d * (\omega). \tag{3.7}$$

We can then formulate the following important proposition:

**Proposition 3.4.1.** *Let $\eta, \omega \in \Omega^\bullet(G)$. Then*

$$\langle d\omega, \eta \rangle = \langle \omega, d^*\eta \rangle.$$

*In other words, $d^*$ is the adjoint of $d$. Here $\langle . , . \rangle$ is the inner product we defined above.*

*Proof.* Since the inner product is bilinear, we only need to check the statement for $\omega \in \Omega^{k-1}(G)$, $\eta \in \Omega^k(G)$. We then have:

$$\begin{aligned} d\omega \wedge *\eta - \omega \wedge *(d^*\eta) &= d\omega \wedge *\eta - \omega \wedge (-1)^{n(k+1))} * * d * \eta \\ &= d\omega \wedge *\eta - (-1)^{k-1} d * \eta = d(\omega \wedge *\eta). \end{aligned}$$

Now integrate over $G$ on both sides, and with Lemma 3.4.2 below the statement then follows (note that $\omega \wedge *\eta$ is a $(n-1)$-form). We have used here that $* * \omega = (-1)^{k(n-k)}\omega$, and that $(-1)^{k^2} = (-1)^k$, in order to work out the sign. □

**Lemma 3.4.2.** *Let $\omega$ be $(n-1)$ -form. Then $\int_G d\omega = 0$.*

*Proof.* This is Stokes theorem for manifolds without boundary. □

**Definition 3.4.3.** We define the *Hodge Laplacian* $\Delta : \Omega^k(G) \to \Omega^k(G)$ as $\Delta := dd^* + d^*d$. We say that a form $\omega$ is *harmonic* if $\Delta\omega = 0$.

**Proposition 3.4.4.** *A form $\omega$ is harmonic if and only if $d\omega = d^*\omega = 0$*

*Proof.* We have

$$\langle \Delta\omega, \omega \rangle = \langle dd^*\omega, \omega \rangle + \langle d^*d\omega, \omega \rangle = \langle d^*\omega, d^*\omega \rangle + \langle d\omega, d\omega \rangle.$$

Since $\langle . , . \rangle$ is positive definite, the result follows. □

We will now begin the main proof of this section, the statement that the harmonic forms are exactly the bi-invariant ones. Before we start, we will need to prove two lemmas. These lemmas cannot be found anywhere in Helgasons book, but since both the assertions and proofs are non-trivial, we feel it is appropriate to include them. With $\mathcal{L}_V$ we denote the Lie derivative along a vector field $V$.

**Lemma 3.4.5.** *Let $V \in \mathfrak{X}(M)$ a vector field on a manifold $M$, with flow $\phi_t$. Let $\omega$ be a $k$-form. Suppose $\mathcal{L}_V(\omega) = 0$, then for all $x_0 \in M$ there exist a neighborhood $U$ of $x_0$ and a $\delta > 0$ such that*

$$(\phi_t)^*\omega|_U = \omega|_U \qquad \forall t \in (-\delta, \delta)$$

*Proof.* Choose $U$ and $\delta$ such that $(-\delta, \delta) \times U \subset \mathrm{dom}(\phi)$, then choose $x \in U$. We have $(\phi_t)^*\omega(x) = \omega_{\phi_t(x)} \circ (D\phi_t(x))^k \in \Lambda^k T_x^* M$. Now:

$$\begin{aligned} \tfrac{d}{dt}(\phi_t)^*\omega(x) = \tfrac{d}{dt}s\big|_{s=0}(\phi_{t+s})^*(x) = \tfrac{d}{ds}\big|_{s=0}(\phi_t)^*(\phi_s)^*(x) = \tfrac{d}{ds}\big|_{s=0}(\phi_s)^*_{\phi_t(x)}(D\phi_t(x))^k \\ = \mathcal{L}_V(\omega)_{\phi_t(x)} \circ (D\phi_t(x))^k = 0. \end{aligned}$$

It follows that $t \mapsto (\phi_t)^*\omega(x)$, a smooth function with image in $\Lambda^k T_x^* M$ (a fixed vector space), has derivative equal to zero, hence $(\phi_t)^*\omega(x) = (\phi_0)^*\omega(x) = \omega(x)$. □

**Lemma 3.4.6.** *With notation as in the previous lemma, suppose $\phi_t \circ \Delta = \Delta \circ \phi_t$ for all $t$. Then $\mathcal{L}_V \circ \Delta = \Delta \circ \mathcal{L}_V$.*

*Proof.* We will prove the statement on open sets $U \subseteq M$ where we can choose a local frame $\lambda_1, ..., \lambda_n \in \Omega^k(U)$. By covering $M$ with subsets like this, the statement the follows for the entire manifold. In such a local frame, we can write $\omega = \sum_i \omega^i \lambda_i$. Then also $\Delta\omega = \sum_{i,j}(P^i_j \omega^i)\lambda_j$, where $P^i_j$ is some scalar differential operator. This follows from the fact that we can locally write $d$ and $*$ in such a way.

We claim that in this basis we have

$$\frac{d}{dt}\Big|_{t=0} ((\phi_t)^*\omega)^i = (\mathcal{L}_V\omega)^i,$$

where $(\cdot)^i$ denotes picking the $i$-th component with respect to the basis. To see this, note that $(\cdot)^i : \Lambda^k(T_xM^*) \to \mathbb{R}$ is a linear functional. Also, $(\omega)^i_x = (\omega_x)^i$. Therefore

$$\frac{d}{dt}\Big|_{t=0} (\phi_t^*\omega)^i_x = \frac{d}{dt}\Big|_{t=0} (((\phi_t)^*\omega)_x)^i = \left(\frac{d}{dt}\Big|_{t=0} ((\phi_t)^*\omega)_x\right)^i = (\mathcal{L}_V\omega_x)^i = (\mathcal{L}_V\omega)^i_x.$$

We now have the following:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \Delta(\phi_t^*\omega) = \frac{d}{dt}\Big|_{t=0} \sum_{i,j} P^i_j(\phi_t^*\omega))\lambda_j &= \sum_{i,j} \frac{d}{dt}\Big|_{t=0} P^i_j(\phi_t^*\omega)^i\lambda_j \\ &= \sum_{i,j} P^i_j(\mathcal{L}_V\omega^i)\lambda_j = \Delta \circ \mathcal{L}_V(\omega).\end{aligned}$$

We have used here that $\frac{d}{dt}\Big|_{t=0} \circ P^i_j = P^i_j \circ \frac{d}{dt}\Big|_{t=0}$, which is simply the statement that we can interchange partial derivatives. Now using the hypothesis of the lemma, we conclude that $\mathcal{L}_V \circ \Delta(\omega) = \frac{d}{dt}\Big|_{t=0} \phi_t^*(\Delta\omega) = \frac{d}{dt}\Big|_{t=0} \Delta(\phi_t^*\omega) = \Delta \circ \mathcal{L}_V(\omega)$. □

**Theorem 3.4.7.** *Let $G$ be a compact connected Lie group. Then a form $\omega$ is harmonic if and only if it is bi-invariant.*

*Proof.* Let $\omega$ a bi-invariant form. By Lemma 3.1.2, $d\omega = 0$. Since $*$ commutes with lthe pullback of left- and right-multiplication, we see that $*\omega$ is also bi-invariant. Thus $d^*\omega = *d(*\omega)$ is also zero. So, by Prop. 3.4.4, $\omega$ is harmonic.

Let now $\omega$ be a harmonic form, then in particular $d\omega = d^*\omega = 0$. Let $X \in \mathfrak{g}$, and write $\tilde{X}$ for the induced left-invariant vector field on $G$. By Cartan's magic formula,
$\mathcal{L}_{\tilde{X}}\omega = i(\tilde{X})d\omega + di(\tilde{X})\omega = di(\tilde{X})\omega$, since $d\omega$=0. Now we can write the following.

$$\langle \mathcal{L}_{\tilde{X}}\omega, \mathcal{L}_{\tilde{X}}\omega\rangle = \langle \mathcal{L}_{\tilde{X}}\omega, di(\tilde{X})\omega\rangle = \langle d^*\mathcal{L}_{\tilde{X}}, i(\tilde{X})\omega)\rangle. \tag{3.8}$$

Using Lemma 3.4.6, we have that $\mathcal{L}_{\tilde{X}}\omega$ is also a harmonic form. Thus $d^*\mathcal{L}_{\tilde{X}} = 0$, and by the above therefore $\langle \mathcal{L}_{\tilde{X}}\omega, \mathcal{L}_{\tilde{X}}\omega\rangle = 0$. Thus $\mathcal{L}_{\tilde{X}}\omega = 0$. Now recall that for the flow $\phi_t$ of a left-invariant vector field , we have $\phi_t(x) = r_{\exp tX}$. Since both $d$ and $*$ commute with pull-backs of right-multiplication, it follows that $\phi_t$ commutes with $\Delta$. With Lemma 3.4.5, we then see that for all $x \in G$, $(r^*_{\exp tX}\omega)_x = \omega_x$, for $t$ small enough. Since $G$ is connected, it is completely generated by products of such elements (i.e. of the form $\exp tX$), and so we can conclude that $\omega$ is right-invariant. Repeating this entire argument with a *right-invariant* vector field, it follows in the same way that $\omega$ is also left-invariant. So $\omega$ is bi-invariant. □

**Corollary 3.4.8.** *Let $G$ be a compact connected Lie group. Then every equivalence class of the cohomology of $G$ contains exactly one harmonic form.*

*Proof.* Follows from the previous theorem in combination with Theorem 3.1.4. □

With the above corollary, we have proven what we wanted to prove in this part. As we have said before, the corollary can be reformulated for a general Riemannian manifold $M$, and it then becomes (a version of) the famous *Hodge Decomposition Theorem*. Since this theorem provides an explicit way to find representatives of the cohomology, it is an important tool when studying this. Moreover, the $\Delta$-operator can be shown to be a so called *elliptic operator*. Among other things, this simplifies solving differential equations involving the operator, which is another reason for studying it. The proof of the general theorem requires a lot more theory and proofs than what we have used here. This is therefore a good example of how powerful the structure of a Lie group can be.

# Part IV

# Theorem of Hopf

In this final part we will look at a second application of the theory that we have built up. Specifically, we will use the established isomorphisms to prove a theorem about the cohomology of a compact Lie group $G$, by showing the statement for its Lie algebra cohomology, as defined in part I. The theorem we want to prove says that the cohomology of any compact Lie group is linearly isomorphic to the cohomology of the Cartesian product of certain odd-dimensional spheres. The exact formulation is given in Corollary 4.3.6. The theorem was originally stated, and proved, by the German mathematician Heinz Hopf in his article [9], written in 1941. The proof uses mainly the group multiplication. In this article, Hopf actually proves the theorem for what he calls '$\Gamma$-manifolds', which are manifolds that admit left- and right-multiplication, but with a weaker condition on the inverse map than for Lie groups. We will not go to this more general case, and instead keep focusing on compact Lie groups. This will allow us to make full use of the properties of the Lie algebra, which enables us to prove the theorem in a different way. Our entire treatment is based on the article by Koszul, [10, Ch. 1-10], who as far as we know was the first person to prove the theorem using the Lie algebra.

His proof, despite being arguably more elegant than the one given by Hopf, is still rather long and requires a lot of build up. The aim of this part will therefore not be to give a fully detailed proof, but instead we will only show the main part of the proof, and only briefly touch upon the topics that lie too far outside of the focus of this thesis. Essentially, this means we restrict our attention to Section 10 of Koszuls article, and only introduce the necessary parts from the sections before that. We refer to the article for more details. As perhaps a bit of a warning, the article is written in French, however with a basic knowledge of this language it is very well readable.

## 4.1 Preliminaries

In this first section, we will introduce some of the concepts that will be needed later on. We start by discussing properties of a general Lie algebra $\mathfrak{a}$, later on we will turn our focus to properties that hold especially for a Lie algebra $\mathfrak{g}$ of a compact Lie group.

Recall that, by Proposition 2.2.3, for an exterior algebra $\Lambda V$ of a vector space $V$ we can identify the dual of $\Lambda V$ with $\Lambda(V^*)$. For $f_i \in V^*$, $x_i \in V$ this duality is given by

$$\langle f_1 \wedge ... \wedge f_k,\ x_1 \wedge ... \wedge x_k \rangle \ := \ \Phi(f_1 \wedge ... \wedge f_k)(x_1 \wedge ... \wedge x_k) = \det(f_j(x_i))$$

For $f \in \Lambda V^*$ and $x \in \Lambda V$ we say that $x$ is *orthogonal to* $f$ if $\langle f, x\rangle = 0$. If $A \in End(\Lambda V)$, then we call a map $A^* \in End(V^*)$ *the dual of* $A$ if $\langle f_1 \wedge ... \wedge f_k,\ A(x_1 \wedge ... \wedge x_k)\rangle = \langle A^*(f_1 \wedge ... \wedge f_k),\ x_1 \wedge ... \wedge x_k\rangle$.

**Definition 4.1.1.** Let $\mathfrak{a}$ a Lie algebra. On $\Lambda\mathfrak{a}$, define the map $\partial : \Lambda^k\mathfrak{a} \to \Lambda^{k-1}\mathfrak{a}$ by bilinearly extending

$$\partial(x_1 \wedge \cdots \wedge x_k) = \sum_{i<j} (-1)^{i+j+1}\,[x_i, x_j] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_k,$$

for $x_i \in \mathfrak{a}$.

**Proposition 4.1.2.** *The map* $\delta : \Lambda\mathfrak{a}^* \to \Lambda\mathfrak{a}^*$ *defined previously is the dual of* $(-\partial)$

*Proof.* Identify the space of $k$-linear alternating functions on $\mathfrak{a}$ with the dual of $\Lambda\mathfrak{a}$. The map $\delta : \Lambda^k\mathfrak{a}^* \to \Lambda^{k+1}\mathfrak{a}$ is then given by:

$$\delta\omega(x_0 \wedge \cdots \wedge x_k) = \sum_{i<j} (-1)^{i+j}\,\omega([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_k).$$

Where $\omega \in \Lambda^k\mathfrak{a}^*$, $x_0, ..., x_k \in \mathfrak{a}$. It is then obvious that we have $\delta(\omega) = \omega \circ (-\partial)$, and the statement follows. ☐

From the above proposition it follows that $\partial \circ \partial = 0$, since a similar assertion holds for $\delta$. Having defined $\partial$ and $\delta$, the spaces $\Lambda\mathfrak{a}$ and $\Lambda\mathfrak{a}^*$ now have the structure of a complex. Elements of $\Lambda\mathfrak{a}$ are often referred to as *chaines*, the elements of $\Lambda\mathfrak{a}^*$ are called *cochains*. We call chains that are zero's of $\partial$ *cycles*, and similarly *co-cycles* are the zero's of $\delta$.

In the same way as we have seen before we can now define the *cohomology of* $\mathfrak{a}$, $H^*(\mathfrak{a})$, as the quotient of the cocycles by $Im(\delta)$. The wedge product induces a graded algebra structure on $H^*(\mathfrak{a})$. For the sake of clarity we will denote the multiplication with the $\smile$ symbol. In the same fashion we can define the *homology of* $\mathfrak{a}$, by taking the quotient of the cycles in $\Lambda\mathfrak{a}$ by the image of $\partial$. This homology will be denoted by $H(\mathfrak{a})$. It is important here to note that since $\partial$ is not generally an anti-derivation, this space will in general not have an induced algebraic structure. Both the homology and cohomology will be graded spaces, we can write $H(\mathfrak{a}) = \oplus_k H_k(\mathfrak{g})$ and $H^*(\mathfrak{a}) = \oplus_k H^k(\mathfrak{a})$. Note that if $\mathfrak{g}$ is the Lie algebra of a compact Lie group, then $H^*(\mathfrak{g})$ is the cohomology of $\mathfrak{g}$ that we have already seen in Part I.

***Remark***. Recall that an *anti-derivation* on a graded algebra $\mathcal{A} = \oplus_k \mathcal{A}^k$, with multiplication $\wedge$, is a linear map $\theta : \mathcal{A} \to \mathcal{A}$ that satisfies

$$\theta(x \wedge y) = \theta(x) \wedge y + (-1)^k x \wedge \theta(y),$$

for $x \in \mathcal{A}, y \in \mathcal{A}^k$.

The duality of $\Lambda\mathfrak{a}$ and $\Lambda\mathfrak{a}^*$ carries over in the duality of $H^*(\mathfrak{a})$ and $H(\mathfrak{a})$, as one would expect. To see this, note that

$$\begin{aligned}\langle f + \delta g, x + \partial y\rangle &= \langle f, x\rangle + \langle \delta g, x\rangle + \langle f, \partial y\rangle + \langle \delta g, \partial y\rangle \\ &= \langle f, x\rangle + \langle g, (-\partial)x\rangle + \langle -\delta f, y\rangle + \langle g, \partial^2 y\rangle \\ &= \langle f, x\rangle,\end{aligned}$$

for $f, g \in \Lambda\mathfrak{a}^*$, $x, y \in \Lambda\mathfrak{a}$, where we are assuming that $\delta f = 0 = \partial x$. This shows that the duality is independent of the choice of representative of an equivalence class, from which the statement follows. This then also implies that $H^*(\mathfrak{a})$ and $H(\mathfrak{a})$ have the same dimension.

We will now state an important proposition about the space $H(\mathfrak{a})$, that holds for certain $\mathfrak{a}$, in particular if $\mathfrak{a}$ is the induced Lie algebra of a compact Lie group. We will state the proposition without proof, mainly because the proof involves a lot of statements about general (Lie) algebras, which lies somewhat out of the scope of this thesis. The entire argument can be read in Koszul's article ([10, § 6-9]).

**Proposition 4.1.3.** *Let $\mathfrak{g}$ be the Lie-algebra of a compact Lie group $G$. Then $H(\mathfrak{g})$ can be given the structure of a graded algebra, where the multiplication is induced by the wedge product on $\Lambda\mathfrak{g}$.*

Proposition 4.1.3 is more generally true for a type of Lie algebra that we call *reductive*. We will not go into the details of what this means, but know that the Lie algebra of a compact Lie group is a special case of this type of Lie algebra. The proof Koszul gives uses among other things that a compact Lie algebra is unimodular (see the remark in section 3.3), which is equivalent to saying that the trace of $ad(x)$ is zero, for all $x$ in $\mathfrak{g}$. Another thing it uses is that on invariant cycles, the $\partial$ map behaves like an anti-derivation. By then showing that every homology class contains at least one invariant cycle, the algebra structure can be defined accordingly.

***Remark***. Because of the previous proposition, we can define the *homology algebra* of $\mathfrak{g}$, denoted by $H_*(\mathfrak{g})$. Its multiplication will be denoted by the $\smile$ symbol. From the definition of $\partial$, it is clear that the map commutes with any graded algebra homomorphism $\gamma : \Lambda\mathfrak{g} \to \Lambda\mathfrak{h}$, Therefore, any such algebra homomorphism induces a homomorphism of the algebras $H_*(\mathfrak{g})$ and $H_*(\mathfrak{h})$. This homomorphism will be denoted by $\gamma$ as well.

We will also be need the following well-known theorem, called the *Künneth theorem.*

**Theorem 4.1.4** (Künneth)**.** *Let $M, N$ be smooth manifolds, $H^\bullet_{dR}(M)$ and $H^\bullet_{dR}(N)$ their respective cohomology algebras. Then there exist an isomorphism of graded algebras*

$$H^\bullet_{dR}(M \times N) \simeq H^\bullet_{dR}(M) \otimes H^\bullet_{dR}(N).$$

The isomorphism is induced by the isomorphisms between the exterior algebras of the tangent spaces, that we have seen in Lemma 2.3.2. From the theorem it also follows that if $G$ and $H$ are two Lie groups, then the isomorphism $\Lambda(\mathfrak{g}^* \times \mathfrak{h}^*) \simeq \Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{h}^*$ induces an isomorphism $H^*(\mathfrak{g} \times \mathfrak{h}) \simeq H^*(\mathfrak{g}) \otimes H^*(\mathfrak{h})$. This can also be proved directly as follows. We use that by what we have discussed in Section 3.1, the cohomology of $\mathfrak{g}$ is isomorphic to the exterior algebra of elements that are invariant under the adjoint representation. In other words, we have the isomorphism of algebras $H^*(\mathfrak{g}) \simeq (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$. Similarly, it holds that $H^*(\mathfrak{g} \times \mathfrak{h}) \simeq (\Lambda(\mathfrak{g}^* \times \mathfrak{h}^*))^{\mathfrak{g}\times\mathfrak{h}}$. It can then be shown using properties of the representation that $\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{h}^*\right)^{\mathfrak{g}\times\mathfrak{h}} = (\Lambda\mathfrak{g}^*)^{\mathfrak{g}} \otimes (\Lambda\mathfrak{h}^*)^{\mathfrak{h}} \simeq H^*(\mathfrak{g}) \otimes H^*(\mathfrak{h})$, which proves the statement. We again refer to the Appendix for a short introduction to representations.

## 4.2 Primitive elements

We are now ready to begin with the final preparations for the theorem we want to prove.

**Definition 4.2.1.** We define the space $D^*(\mathfrak{g})$ as the linear subspace of $H^*(\mathfrak{g})$ consisting of all linear combinations of elements of the form $a \smile b$, where $a, b \in H^*(\mathfrak{g})$ both have degree $> 0$.

**Definition 4.2.2.** The space $P(\mathfrak{g})$ is defined as the subspace of $H_*(\mathfrak{g})$ that is orthogonal to $D^*(\mathfrak{g})$, *and* to to the subspace $H^0(\mathfrak{g})$. Elements of $P(\mathfrak{g})$ are called *primitive.*

***Remark***. Note that both spaces are graded subspaces, i.e. we can write $D^*(\mathfrak{g}) = \oplus_k D^k(\mathfrak{g})$, where $D^k(\mathfrak{g}) = D^*(\mathfrak{g}) \cap H^k(\mathfrak{g})$, and similarly we have $P(\mathfrak{g}) = \oplus_k P_k(\mathfrak{g})$. Note that $D^*(\mathfrak{g})$ is a two-sided ideal in $H^*(\mathfrak{g})$. From the definition, it follows that $P_0(\mathfrak{g}) = \{0\}$.

In the next two lemmas, we will prove two crucial properties of the primitive elements of $H_*(\mathfrak{g})$. Let $\varphi$ be the 'diagonal' map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, i.e. $\varphi(x) = (x, x)$. As usual, by $\tilde{\varphi}$ we denote the extension of this map to $\Lambda\mathfrak{g}$, as well as the induced map $\tilde{\varphi} : H_*(\mathfrak{g}) \to H_*(\mathfrak{g} \times \mathfrak{g})$.

**Lemma 4.2.3.** *Let $\tilde{\varphi} : H_*(\mathfrak{g}) \to H_*(\mathfrak{g} \times \mathfrak{g})$ be the map defined above. Then for $u \in P(\mathfrak{g})$,*

$$\tilde{\varphi}(u) = u \otimes 1 + 1 \otimes u.$$

*Proof.* Recall that we have the identification $H^*(\mathfrak{g} \times \mathfrak{g}) \simeq H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g})$. Since for every element in $a \otimes b \in H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g})$ we can write $a \otimes b = (a \otimes 1) \smile (1 \otimes b)$, we see that we have $H^p(\mathfrak{g}) \otimes H^q(\mathfrak{g}) \subset D^*(\mathfrak{g} \times \mathfrak{g})$ for every $p, q > 0$. It can then be easily verified that we must have:

$$D^*(\mathfrak{g} \times \mathfrak{g}) = \sum_{p,q>0} H^p(\mathfrak{g}) \otimes H^q(\mathfrak{g}) + D^*(\mathfrak{g}) \otimes 1 + 1 \otimes D^*(\mathfrak{g}).$$

We can now ask ourselves what an element of the space $P(\mathfrak{g} \times \mathfrak{g})$ should look like. Since it has to be orthogonal to every element in $D^*(\mathfrak{g} \times \mathfrak{g})$, it can only be of the form $u \otimes 1$ or $1 \otimes u$ for some $u \in P(\mathfrak{g})$, where the component 1 is needed so that it is orthogonal to all subspaces $H^p(\mathfrak{g}) \otimes H^q(\mathfrak{g})$ $(p, q > 0)$. Every linear combination of elements of this form is clearly also orthogonal to $D^*(\mathfrak{g} \times \mathfrak{g})$. It follows that

$$P(\mathfrak{g} \times \mathfrak{g}) = P(\mathfrak{g}) \otimes 1 + 1 \otimes P(\mathfrak{g}).$$

Now note that by what we have discussed before (see Part II), the dual map $\tilde{\varphi}^* : H^*(\mathfrak{g} \times \mathfrak{g}) \to H^*(\mathfrak{g})$ is also a homomorphism (of algebras). By definition of the space $D^*(\mathfrak{g})$, it follows that $\tilde{\varphi}^*(D^*(\mathfrak{g} \times \mathfrak{g})) \subseteq D^*(\mathfrak{g})$. Therefore $\tilde{\varphi}(P(\mathfrak{g})) \subseteq P(\mathfrak{g} \times \mathfrak{g})$, and thus for $u \in P(\mathfrak{g})$;

$$\tilde{\varphi}(u) \in P(\mathfrak{g}) \otimes 1 + 1 \otimes P(\mathfrak{g}).$$

Let $\pi_1 : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the natural projection to the first coordinate. One can show that for $a \otimes b \in H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g})$ it holds that $\tilde{\pi_1}(a \otimes b) = b_0\, a$, where $b_0$ is the component of $b$ with degree zero (compare to the discussion in Section 2.3). Since $P_0\mathfrak{g} = \{0\}$, it therefore holds that $\tilde{\pi_1}(1 \otimes P(\mathfrak{g})) = \{0\}$. Now observe that $\varphi \circ \pi_1$ is the identity, we therefore must have that

$$\tilde{\pi_1} \circ \tilde{\varphi}(u) = u.$$

By the above, this implies that the component of $\tilde{\varphi}(u)$ in $P(\mathfrak{g}) \otimes 1$ has to be equal to $u \otimes 1$. We can reason in a similar way for the other component, which shows that indeed $\tilde{\varphi}(u) = u \otimes 1 + 1 \otimes u$. □

**Lemma 4.2.4.** *Let $u$ be a (non-zero) element of $P(\mathfrak{g})$. Then the degree of $u$ is odd.*

*Proof.* With the notation $(u)^p$ we will mean the $p$-fold product $u \smile ... \smile u$. The map $\tilde{\varphi}$ is an algebra homomorphism from $H_*(\mathfrak{g})$ to $H_*(\mathfrak{g} \times \mathfrak{g})$. Therefore, if we choose $p > 0$ such that $(u)^p = 0$ (such a $p$ exists. since we can always choose $p$ larger that the dimension of $G$), we have

$$(\tilde{\varphi}(u))^p = \tilde{\varphi}(u^p) = (u \otimes 1 + 1 \otimes u)^p = 0.$$

Now let us assume that $u$ has *even* degree. Looking at Definition 2.3.1, it then holds that $(u \otimes 1) \smile (I \otimes u) = (I \otimes u) \smile (u \otimes I)$. Therefore, it can be worked out that

$$0 = (u \otimes 1 + 1 \otimes u)^p = \sum_{n=0}^{p} \binom{p}{n} (u^{p-n} \otimes u^n)$$

(this is the well-known binomial formula). Now assume that we have chosen $p$ to be minimal, that is, $(u)^p = 0$ but $(u)^{p-1} \neq 0$. Since all terms in the equation above are linearly independent, in particular we then have that the term $p(u^{p-1} \otimes u)$ must be equal to 0. But this means that either $u = 0$ or $u^{p-1} = 0$, which is a contradiction in both cases. Therefore the degree of $u$ must be odd. □

The importance of the degrees of the primitive elements being odd will become clear in the following lemma. As we will see, it will allow us to define an algebra homomorphism from $\Lambda(P(\mathfrak{g}))$ to $H_*(\mathfrak{g})$.

**Lemma 4.2.5.** *Let $(\mathcal{A}, \cdot\,)$ be a graded algebra, satisfying graded commutativity. Assume $\gamma : V \to \mathcal{A}$ is a linear map from a vector space $V$, which has only elements of odd degree in its image (apart from 0). Then there exists an algebra homomorphism $\bar{\gamma} : \Lambda V \to \mathcal{A}$ that is equal to $\gamma$ when restricted to $V = \Lambda^1(V)$*

*Proof.* Let us first look at the tensor algebra of $V$, $T(V)$. The algebra homomorphism $\gamma' : T(V) \to \mathcal{A}$ that sends an element $v_1 \otimes ... \otimes v_k$ to $\gamma(v_1) \cdot ... \cdot \gamma(v_k)$ can be defined using the universal property of $T^k(V)$. Since every element $\gamma(v)$ is either zero or has odd degree, from graded commutativity it follows that

$$\gamma'(v \otimes v) = \gamma(v) \cdot \gamma(v) = -\gamma(v) \cdot \gamma(v).$$

It follows that we must have $\gamma'(v \otimes v) = 0$. Therefore $\gamma'$ descents to an algebra homomorphism $\bar{\gamma} : T(V)/I = \Lambda(V) \to \mathcal{A}$, where $I$ is as before the two-sided ideal generated by elements of the form $v \otimes v$. Clearly, on $V$ it will still just be $\gamma$, which proves the statement. □

## 4.3 Proof of the theorem

We can now formulate the next lemma. By $i : P(\mathfrak{g}) \to H_*(\mathfrak{g})$, we denote the natural inclusion.

**Lemma 4.3.1.** *The inclusion map $i$ extends to an injective homomorphism of algebras $\bar{\iota} : \Lambda P(\mathfrak{g}) \to H_*(\mathfrak{g})$.*

*Proof.* Using Lemma 4.2.4, we can define the algebra homomorphism $\bar{\iota}$ as in Lemma 4.2.5. To show that it is injective, choose a basis of homogeneous elements $\{u_i\}_{1 \leq i \leq l}$ of the linear space $P(\mathfrak{g})$, arranged such that $\deg(u_i) \leq \deg(u_{i+1})$ for all $i$. To show that $\bar{\iota}$ is injective it is enough to show that it is non-zero on all algebraic combinations of these elements, since those form a basis of $\Lambda P(\mathfrak{g})$. For this it is enough to show that $\bar{\iota}(u_1 \wedge ... \wedge u_l) \neq 0$. Assume now to the contrary that $\bar{\iota}(u_1 \wedge ... \wedge u_l) = u_1 \smile ... \smile u_l = 0$. Since $u_l \neq 0$, there must then exist an index $s$ such that

$$u_s \smile u_{s+1} \smile ... \smile u_l = 0, \quad \text{but } u_{s+1} \smile ... \smile u_l \neq 0.$$

Denote by $p$ the degree of $u_s$. Now look at the element $\tilde{\varphi}(u_s \smile ... \smile u_l) \in H_*(\mathfrak{g}) \otimes H_*(\mathfrak{g})$, where $\tilde{\varphi}$ is the map we defined previously. Since $\tilde{\varphi}$ is a homomorphism, this element has to be equal to zero. Its component in the space $H_p(\mathfrak{g}) \otimes H_*\mathfrak{g}$ must therefore also be equal to zero. Using the equation from Lemma 4.2.3, one can work out that this component is given by

$$\sum_{deg(u_i)=p} (-1)^{s+i}\, u_i \otimes (u_s \smile ... \smile \hat{u}_i \smile ... \smile u_l).$$

Since the $u_i$'s are all linearly independent, from this it follows that $u_s \otimes (u_{s+1} \smile ... \smile u_l) = 0$. Since $u_s \neq 0$, this implies that $u_{s+1} \smile ... \smile u_l = 0$, which is a contradiction. □

Having showed that $\bar{\iota}$ is an injective homomorphism, we would now like to show that it is an isomorphism of algebras. For this, we need to show that it is surjective.

**Lemma 4.3.2.** *The homomorphism* $\bar{\iota} : \Lambda P(\mathfrak{g}) \to H_*(\mathfrak{g})$ *is surjective.*

*Proof.* We will prove the statement by showing that the dimension of $\Lambda P(\mathfrak{g})$ is greater or equal to the dimension of $H_*(G)$. Since $\bar{\iota}$ is in particular an injective linear map, between finite dimensional vector spaces, this then implies that $\bar{\iota}$ must also be surjective.

Choose a basis $\{a_i\}$ (which is *minimal*), of homogeneous elements of the (cohomology) *algebra* $H^*(\mathfrak{g})$, This means that for every element $a \in H^*(\mathfrak{g})$, there exist $\lambda_{i_1,\dots,i_k} \in \mathbb{R}$ such that we can write

$$a = \sum_{i_1,\dots i_k} \lambda_{i_1,\dots,i_k} a_{i_1} \smile \dots \smile a_{i_k}.$$

We claim that for every $p > 0$, the amount of elements of degree $p$ in $\{a_i\}$ is equal to the dimension of the quotient $H^p(\mathfrak{g})/D^p(\mathfrak{g})$, which in turn is equal to the dimension of $P_p(\mathfrak{g})$ by the duality. For this we need to show that $H^*(\mathfrak{g})$ is equal to the direct sum of $D^*(\mathfrak{g})$ and the *linear* span of $\{a_i\}$, which we will denote by $\langle a_i \rangle$. By the definition of the spaces, it is clear that we have $H^*(\mathfrak{g}) = \langle a_i \rangle + D^*(\mathfrak{g})$. So we only need to show that $\langle a_i \rangle \cap D^*(\mathfrak{g}) = \{0\}$.

Assume to the contrary that there exists a non-zero element $a$ in $D^*(\mathfrak{g})$ that is also in $\langle a_i \rangle$. Since it is in $D^*(\mathfrak{g})$, we can write it as $\sum \lambda_{i_1,\dots,i_k} a_{i_1} \smile \dots \smile a_{i_k}$, with every term being the cup product of *at least* two elements of $\{a_i\}$ with degree $> 0$. However, since by assumption it is an element of $\langle a_i \rangle$ too, it can also be written as $\sum_{i \in I} \lambda_i a_i$ for some index set $I$, chosen so that $\lambda_i \neq 0$ for all $i$. We thus have:

$$\sum_{i_1,\dots i_k} \lambda_{i_1,\dots,i_k} \, a_{i_1} \smile \dots \smile a_{i_k} = \sum_{i \in I} \lambda_i a_i.$$

By comparing degrees, we can assume that all the terms have the same degree. It then follows that the $a_i$'s on the right hand side cannot also appear somewhere on the left hand site, since that would create an unequal degree. Now fix an $i_0 \in I$, then it holds that

$$\lambda_{i_0} a_{i_0} = \sum_{i_1,\dots i_k} \lambda_{i_1,\dots,i_k} \, a_{i_1} \smile \dots \smile a_{i_k} - \sum_{i \in I \setminus \{i_0\}} \lambda_i a_i.$$

Since $a_{i_o}$ appears nowhere on the right hand side, this is in contradiction with the minimality of $\{a_i\}$, so it follows that we must have $H^*(\mathfrak{g}) = \langle a_i \rangle \oplus D^*(\mathfrak{g})$.

Using that $P(\mathfrak{g})$ is orthogonal to $D^*(\mathfrak{g})$, it follows that $(P_p(\mathfrak{g}))^* = H^p(\mathfrak{g})/D^p(\mathfrak{g})$. Because of this, and Lemma 4.2.4, we also know that, apart from one element of degree zero, the basis $\{a_i\}$ consists entirely of elements of odd degree. These elements (so without the degree zero element) generate a linear subspace $V$ of $H^*(\mathfrak{g})$ that, by the claim above, has the same dimension as $P(\mathfrak{g})$. Therefore there exists a linear isomorphism from $P(\mathfrak{g})$ to $V$, which by Lemma 4.2.5 can be extended to an algebra homomorphism $\Lambda P(\mathfrak{g}) \to H^*(\mathfrak{g})$. Its image will be the *subalgebra* of $H^*(\mathfrak{g})$ generated by elements of $V$, as well as the unit element. But this is the whole of $H^*(\mathfrak{g})$, since $\{a_i\}$ is contained in this set. We have therefore shown that there exists a surjective linear map from $\Lambda P(\mathfrak{g})$ to $H^*(\mathfrak{g})$, which means that the dimension of $\Lambda P(\mathfrak{g})$ is greater or equal to the dimension of $H^*(\mathfrak{g})$. Since the dimensions of $H^*(\mathfrak{g})$ and $H_*(\mathfrak{g})$ are equal, the proof is complete.

□

Now that we have shown that $\bar{\iota} : \Lambda P(\mathfrak{g}) \to H_*(\mathfrak{g})$ is an algebra isomorphism, we turn to the dual map, $\bar{\iota}^* : H^*(\mathfrak{g}) \to \Lambda(P(\mathfrak{g}))^*$. By duality, this will certainly be a linear isomorphism. We would like to show that is also an isomorphism of algebras.

**Lemma 4.3.3.** *The map $\bar{\iota}^* : H^*(\mathfrak{g}) \to \Lambda(P(\mathfrak{g}))^*$ is an isomorphism of algebras.*

*Proof.* Denote by $\xi : P(\mathfrak{g}) \to P(\mathfrak{g}) \times P(\mathfrak{g})$ the diagonal map $u \mapsto (u, u)$. The extension to $\Lambda P(\mathfrak{g})$, $\tilde{\xi} : \Lambda P(\mathfrak{g}) \to \Lambda P(\mathfrak{g}) \otimes \Lambda P(\mathfrak{g})$ is given by on elements $u \in P(\mathfrak{g})$ by $\tilde{\xi}(u) = u \otimes 1 + 1 \otimes u$. Denote by $\bar{\iota} \otimes \bar{\iota}$ the map $\Lambda P(\mathfrak{g}) \otimes \Lambda P(\mathfrak{g}) \to H_*(\mathfrak{g}) \otimes H_*(\mathfrak{g})$, $u \otimes u' \mapsto \bar{\iota}(u) \otimes \bar{\iota}(u')$. Then we have

$$(\bar{\iota} \otimes \bar{\iota}) \circ \tilde{\xi}(u) = \bar{\iota}(u) \otimes 1 + 1 \otimes \bar{\iota}(u) = \tilde{\varphi} \circ \bar{\iota}(u),$$

where the last equality follows by Lemma 4.2.3. So we see $(\bar{\iota} \otimes \bar{\iota}) \circ \tilde{\xi} = \tilde{\varphi} \circ \bar{\iota}$, and therefore also

$$\tilde{\xi}^* \circ (\bar{\iota} \otimes \bar{\iota})^* = \bar{\iota}^* \circ \tilde{\varphi}^*.$$

By Lemma 2.3.3, we have

$$\tilde{\xi}^*(u \otimes v) = \tilde{\xi}^*(u) \wedge \tilde{\xi}^*(v) \quad \text{for } u \otimes v \in \Lambda P(\mathfrak{g})^* \otimes \Lambda P(\mathfrak{g})^*.$$

Similarly, using this lemma one can deduce that

$$\tilde{\varphi}^*(a \otimes b) = a \smile b \quad \text{for } a \otimes b \in H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g}).$$

To keep track of the domains; we have $\tilde{\varphi}^* : H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g}) \to H^*(\mathfrak{g})$ and $\tilde{\xi}^* : \Lambda P(\mathfrak{g})^* \otimes \Lambda P(\mathfrak{g})^* \to \Lambda P(\mathfrak{g})^*$. Putting this all together, we see for $a \smile b \in H^*(\mathfrak{g})$;

$$\bar{\iota}^*(a \smile b) = \bar{\iota}^* \circ \tilde{\varphi}^*(a \otimes b) = \tilde{\xi}^* \circ (\bar{\iota} \otimes \bar{\iota})^*(a \otimes b) = \bar{\iota}^*(a) \wedge \bar{\iota}^*(b).$$

This shows that $\bar{\iota}^* : H^*(\mathfrak{g}) \to \Lambda P(\mathfrak{g})$ is indeed an algebra isomorphism. □

With the previous lemma, we are now finally ready to state and prove the main theorem of this section. As mentioned earlier, we will relate the cohomology of a compact Lie group to the cohomology of a product of $n$-spheres. First recall the following, well-known theorem (see for example [12, Th. 17.21]).

**Theorem 4.3.4.** *The cohomology $H^k_{dR}(S^n)$ of an $n$-dimensional sphere is given by*

$$H^k_{dR}(S^n) = \begin{cases} 0 & \text{if } k \neq 0, n, \\ \mathbb{R} & \text{if } k = 0, n. \end{cases} \tag{4.1}$$

**Theorem 4.3.5.** *Let $\mathfrak{g}$ be the Lie algebra of a compact connected Lie group. Then there exist odd integers $n_1, ..., n_l$ such that we have the isomorphism of graded algebras*

$$H^*(\mathfrak{g}) \ \simeq H^\bullet_{dR}(S^{n_1} \times ... \times S^{n_l})$$

*Proof.* By the previous Lemma, we know $H^*(\mathfrak{g}) \simeq \Lambda(P(\mathfrak{g})^*)$. Since $P(\mathfrak{g})^*$ is a finite-dimensional vector space, we can choose a finite linear basis $\{a_i\}_{1\leq i\leq l}$ of $P(\mathfrak{g})^*$. As we have seen before, the $a_i$'s will have odd degree. Since we can now write $P(\mathfrak{g})^* \simeq \oplus_i \langle a_i \rangle$, (where, as before $\langle a_i \rangle$ denotes the linear span of $\{a_i\}$) it holds that

$$\Lambda P(\mathfrak{g}) \simeq \Lambda\langle a_1 \rangle \otimes ... \otimes \Lambda\langle a_l \rangle.$$

Since for every $i$, $\langle a_i \rangle$ is a one dimensional vector space, its exterior algebra is simply given by the direct sum $\langle 1 \rangle \oplus \langle a_i \rangle$. Let $n_i$ be the degree of $a_i$ (as an element of $H^*(\mathfrak{g})$). Then by the above theorem we have a straightforward algebra isomorphism between the algebras $H^\bullet(S^{n_i})$ and $\Lambda\langle a_i \rangle$. Doing this for every $i$ then gives us the following isomorphism of algebras, in an obvious way:

$$\Lambda\langle a_1 \rangle \otimes ... \otimes \Lambda\langle a_l \rangle \simeq H^\bullet_{dR}(S^{n_1}) \otimes ... \otimes H^\bullet_{dR}(S^{n_l}).$$

By the Künneth theorem, it then follows that $\Lambda P(\mathfrak{g})^* \simeq H^\bullet_{dR}(S^{n_1} \times ... \times S^{n_l})$.
Putting everything together, we obtain the isomorphism of graded algebras $H^*(\mathfrak{g}) \simeq H^\bullet_{dR}(S^{n_1} \times ... \times S^{n_l})$. □

**Corollary 4.3.6** (Hopf)**.** *Let $G$ be a compact connected Lie group. Then there exist odd integers $n_1, ..., n_l$ such that for all $k$ we have the isomorphism of graded algebras*

$$H^\bullet_{dR}(G) \simeq H^\bullet_{dR}(S^{n_1} \times ... \times S^{n_l}).$$

*Proof.* Follows by the previous theorem, in combination with what we have done in Part I. Specifically, we use the isomorphisms of graded algebras $H^*(\mathfrak{g}) \simeq H^\bullet_L(G) \simeq H^\bullet_{dR}(G)$. □

# Appendix: Representations

In this section we will give a brief introduction to representations of Lie groups. It should be pointed out that the theory of representations is far broader than what we will be discussing. We will limit ourselves to only the context that is useful for this thesis, for example by restricting to finite dimensional vector spaces. Most of what we will discuss is based on [13, Ch. 20], and [6, Ch. 5]. Missing proofs can also be found there.

Let us define what a representation is.

**Definition 4.3.7.** Let $V$ be a finite-dimensional vector space. A *(finite dimensional) smooth representation* $(\pi, V)$ *of* $G$ *in* $V$ is a left action $\pi : G \times V \to V$ that is smooth, and such that $\pi(x) : V \to V$ is a linear endomorphism of $V$ for every $x \in G$.

***Remark***. It is actually only necessary to demand that $\pi$ is continous, smoothness then follows automatically. However, we will not prove this. Alternatively, one can also define a representation as a Lie group homomorphism $\pi : G \to GL(V)$, which is equivalent to definition given here.

An important example of a Lie group representation is the adjoint representation map, $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Recall that $\mathrm{Ad}(x)$ is defined as $T_e C_x$, where $C_x$ is the conjugation by $x$. See section 0.2 for more information.

In much the same way that we define a representation of a Lie group, we can also define a representation of a Lie algebra.

**Definition 4.3.8.** A *represention of* $\mathfrak{g}$ *in* $V$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to End(V)$, where $End(V)$ is equipped with the commutator bracket.

The next proposition shows a common way to construct Lie algebra homomorphisms.

**Proposition 4.3.9.** *Let* $\pi : G \to GL(V)$ *a Lie group representation of* $G$ *in* $V$*. Then* $\pi_* := T_e\pi : \mathfrak{g} \to End(\mathfrak{g})$ *is a representation of the Lie algebra* $\mathfrak{g}$*.*

This shows in particular that $ad : \mathfrak{g} \to End(V)$ is a Lie algebra representation, the '*adjoint representation' of* $\mathfrak{g}$. The next proposition shows an important relationship between the exponential map and the representations of both $G$ and $\mathfrak{g}$.

**Proposition 4.3.10.** *We have* $\pi_*(X) = \frac{d}{dt}|_{t=0}\pi(exp(tX))$ *for* $X \in \mathfrak{g}$*. Also*

$$\pi(expX) = e^{\pi_*(X)}.$$

**Definition 4.3.11.** Let $\pi_1, \pi_2$ be representations of $G$ in $V_1$ and $V_2$ respectively. The *tensor product representation* $\pi_1 \otimes \pi_2 : G \to GL(V_1 \otimes V_2)$ is defined by $(\pi_1 \otimes \pi_2)(x) = \pi_1(x) \otimes \pi_2(x)$ for $x \in G$.

**Proposition 4.3.12.** *With notation as above, the associated Lie algebra representation satisfies*

$$(\pi_1 \otimes \pi_2)_*(X) = \pi_{1*}(X) \otimes 1 \; + \; 1 \otimes \pi_{2*}(X) \quad \textit{for } X \in \mathfrak{g}.$$

*Proof.* Using the product rule (for tensor products, Lemma 2.1.5), we have for $u \in V_1, v \in V_2$:

$$\begin{aligned}(\pi_1 \otimes \pi_2)_*(X)(u \otimes v) &= \frac{d}{dt}\Big|_{t=0} \; \pi_1(\exp\; tX)u \otimes \pi_2(\exp\; tX)v \\ &= \pi_{1*}(X)u \otimes v + u \otimes \pi_{2*}(X)v.\end{aligned}$$

So the result follows. □

This proposition can also be easily generalized to a $k$-fold tensor product.

**Definition 4.3.13.** Let $(\pi, V)$ be a representation of $G$ in $V$. An element $v \in V$ is called *$G$-invariant* if $\pi(x)v = v$ for all $x \in G$. Denote by $V^G$ the space of all $G$-invariant elements, i.e

$$V^G := \{v \in V \mid \pi(x)v = v \text{ for all } x \in G\}.$$

Let $\pi_*$ be the induced representation of $\mathfrak{g}$ in $V$. We call $v \in V$ *$\mathfrak{g}$-invariant* if $\pi_*(X)v = 0$ for all $X \in \mathfrak{g}$. Define the space $V^{\mathfrak{g}}$ by:

$$V^{\mathfrak{g}} := \{v \in V \mid \pi_*(X)v = 0 \text{ for all } X \in \mathfrak{g}\}.$$

**Proposition 4.3.14.** *Assume $G$ is connected. Then an element $v \in V$ is $G$-invariant if and only if $v$ is $\mathfrak{g}$-invariant. In other words, $V^G = V^{\mathfrak{g}}$.*

*Proof.* Can be derived using Proposition 4.3.10. See Lemma 29.1 in [13]. □

## Dual representation

Since a finite dimensional vector space is isomorphic to its dual, we can also define a representation of $G$ in this dual space. This is done in the following way.

**Definition 4.3.15.** Let $\pi$ be a representation of $G$ in $V$. Then we define the *dual representation* $\pi^\vee : G \to GL(V^*)$ of $\pi$ by

$$\pi^\vee(x)v^* = v^* \circ \pi(x^{-1})$$

for $x \in G$, $v^* \in V^*$.

One can easily check that this indeed defines a representation. We can also calculate the *induced dual representation* $\pi_*^\vee : \mathfrak{g} \to GL(V^*)$, $\pi_*^\vee := T_e\pi^\vee$. An important example of such a representation is the adjoint representation of $\mathfrak{g}$ in $\Lambda^k\mathfrak{g}^*$, given by the map $\mathrm{ad}(\cdot)^* : G \to GL(\Lambda^k\mathfrak{g}^*)$.

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