Utrecht University

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Smoothed analysis of order type realizability

Bachelor Thesis

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Abstract

An abstract order type is a mapping $\chi : \genfrac{[}{]}{0pt}{}{n}{3} \to \{+, -, 0\}$, satisfying some set of conditions, where $\genfrac{[}{]}{0pt}{}{n}{3}$ is the collection of all triples of a set of n elements. Given an ordered point set $P = (p_1, \ldots, p_n)$, we define the order type χ_P of P as a mapping $\binom{P}{3} \to \{+, -, 0\}$. The order type χ_P indicates for each ordered triple whether it is oriented clockwise(-), oriented counter-clockwise(+) or collinear(0). We say a point set P *realizes* an order type χ, if $\chi_P(p_i, p_j, p_k) = \chi(i, j, k)$, for all $i < j < k$. A set of points in the plane is among the most basic and natural geometric objects of study in Discrete and Computational Geometry. They are relevant in theory and practice alike. Interestingly, order types are capturing most structure of point sets. In particular, many geometric properties of a point set depend only on the order type. Thus the question of which order types are actually realizable as point sets is important, to understand the combinatorial nature of point sets in the plane. The first result regarding this question was formulated already 1700 years ago as geometry was enriched by its first projective property. More answers followed 70 years ago, when this question was linked to a problem in real algebraic geometry, resulting in a theoretical upper bound under certain conditions. A more recent development suggested that this upper bound in the worst-case analysis of order type realizability was too pessimistic

Another recent development is smoothed analysis, which gained popularity to explain the practical performance of algorithms, even if they perform badly in the worst case. Smoothed analysis is an interpolation between average-case analysis and worst-case analysis. The idea is to study the expected performance on small perturbations of the worst input. The performance is measured in terms of the magnitude δ of the perturbation and the input size.

In this thesis, the most important quantity in order type realizability is considered to be the norm, a measure of the amount of grid points one needs in order to be able to realize a given order type on a grid. We find an upper bound on the norm in the average case of $\mathcal{O}(n^3 \cdot 2^n)$, whereas the norm is tightly bounded in the worst case by $\Theta(2\hat{}2^n)$. Using smoothed analysis, we can interpolate between these results by our parameter $\delta > 0$, since the result we find is an upper bound on the norm of $\mathcal{O}(n^3 \cdot 2^n/\delta)$.

A great many thanks to:

Selma Tasci, (for all the emotional support one could wish for when working together with)

Till Miltzow, who smiled as he threw me into the deep,
and
Ivor van der Hoog, who eventually taught me how to swim.

Contents

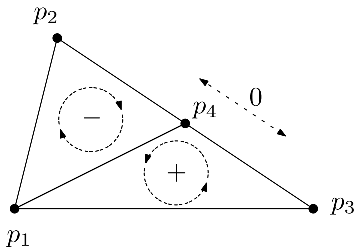

Figure 1: A configuration that describes the possible orientations of triples of points: the orientation of $\{p_1, p_2, p_4\}$ is '-', that of $\{p_1, p_3, p_4\}$ is '+', and the collinear triple $\{p_2, p_3, p_4\}$ has orientation '0'.

1 Introduction

In this thesis we study the problem of order type realizability and the computational complexity thereof. This is a problem in the field of computational geometry; a branch of computer science devoted to the study of algorithms which can be stated in terms of geometry. More specifically, this problem is of the category combinatorial computational geometry which deals with geometric objects as discrete identities. Computational geometry in this sense was first mentioned in a book [27] in 1985 by Preparata and Shamos where they mention that the first geometrical problems date back to antiquity.

The basis of book was the PhD thesis that Shamos [32] published in 1978. In his thesis, he formulated the real RAM model which is widely used in computational geometry. The real RAM model is a model of a computer, where each memory cell is capable of storing a real number up to an infinite precision. The real RAM model is a well-suited model for theoretical analysis in computational geometry, as geometric objects (even those that can be described using integer values) often have real-valued substructures (e.g. a unit square has a diameter of $\sqrt{2}$ which cannot be expressed as an integer value). Fredman and Willard [14] introduced a more realistic memory model which they called the word RAM model. In the word RAM model, a memory cell can (just as an actual computer) store only a finite amount of information, specifically a "word" of b bits for some fixed integer b. We study the conditions under which these models will give the same results regarding order type realizability. In its most general form, an order type is a mapping from an ordered set to $\{+, -, 0\}$ (skip ahead to Section 1.2 if you prefer to read the more general definition first). If the ordered set is a set of points P, order type can be defined as a geometric property of P:

Consider an ordered set $P = (p_1, p_2, \ldots, p_n)$ of n real-valued points in $[0, 1]^2$. The *order type* χ_P of the ordered point set P is defined as a mapping from every triple of points in P to the orientation of that triple, or formally, $\chi_P : \binom{P}{3} \to \{+, -, 0\}$ [15]. The value that corresponds to a given orientation of a triple of points is explained using Figure 1. For example, if we follow the points $\{p_1, p_2, p_4\}$ in the order they occur from a central position, we turn clockwise, which is a negative orientation, so $\chi_P(p_1, p_2, p_4)$ is equal to '-'. When we do the same for the points $\{p_1, p_3, p_4\}$, we move counter-clockwise, which is a positive orientation, so $\chi_P(p_1, p_3, p_4)$ is equal to '+'. The value '0' is assigned to collinear triples, such as $\{p_2, p_3, p_4\}$.

In this thesis, we are interested in abstract order types which can be realized by a point set in the word RAM model. Before we can elaborate on the problem studied in this thesis, we need a better description of the theoretical concepts underlying both finite-precision computations and order type realizability. Hence we take a step back to explore these involved concepts.

1.1 Representing point sets as discrete identities.

In this thesis we consider probably the most basic object one could encounter in geometry: a point set. Consider an ordered set $P = (p_1, p_2, \ldots, p_n)$ of n real-valued points in $[0,1]^2$, we can store P by storing for every point its two coordinates. Most algorithms designed for point sets and the analysis of their complexity assumes that the exact value of the coordinates can be stored. In the real RAM model this is immediately true. In the word RAM model, however, it could be that we cannot express the exact value of each coordinate, since we can only store a finite number of bits.

When we can only store a finite number of bits, we must round the real values of the coordinates to a number that can be expressed with whichever finite precision we are allowed to use. This rounding may change topological information of the geometric objects involved. A classical problem that is encountered when performing finite-precision calculations is that a point may incorrectly be determined to lie on the left side of an oriented line. This phenomenon is especially troublesome when studying order types of a real-valued point set P. Consider the directed line between two points $p, q \in P$ and a point $r \in P$. If the point r is incorrectly determined to lie left of this line then we have incorrectly determined the order type of $\Delta(p, q, r)$. These kind of rounding errors may lead to incorrect or nonsensical solutions when the algorithms that is used assumes that the computer has infinite precision.

The assumption of infinite precision is what allows us to think of parameters to be samples from a continuous domain, rather than discrete values: Greene and Yao [19] transform geometric concepts and algorithms from the continuous domain to the discrete domain. As an example they consider the following classical problem in a discrete setting: given a set of n line segments L, report the at most n^2 points of intersections between lines in L. It is easy to verify that even if the lines in L can be described with discrete coordinates, points of intersection between these lines might not. Greene and Yao model the coordinates of these points of intersections as values from a continuous domain and round them down to discrete values. In addition, they store an additional structure on the continuous description of the values which they use to detect inconsistencies of their rounded points (such as when a set of collinear real values stops to be collinear). Milenkovic [23] refers to this method as the "hidden variable method". He also proposes another method of correct and verifiable geometric reasoning using finite precision arithmetic, which he calls "data normalization". Here a geometric structure is transformed into a discrete configuration that is such that all finite precision calculations yield the same answers as infinite precision calculations would find.

This normalization method resembles the approach we will use to solve the problem that we have encountered. We aspire to let the topological information of the configuration that our geometric structure (the point set P) was transformed into, be equal to the topological information of P. In our situation, the transformation is a *digitization* of the point set P, and we will find a sufficient condition for the digitization of P to have the same topological information.

Digitization and grids. Given some value k, we can partition $[0,1]^2$ into k^2 equally sized squares. We will denote the width of the squares by $\omega = 1/k$ and the corners of the squares form a grid that is a subset of $\omega\mathbb{Z}^2$. We can express the coordinates of any point on this grid using $\log(k)$ bits per axis, but in general, real-valued points do not lie on grid points. We define the $k - \textit{digitization}$ as a one-to-one mapping from P to P' which maps real-valued points in $[0,1]^2$ to the nearest grid point in $\omega\mathbb{Z}^2$ (see Figure 2 for an illustration). Such a procedure is often referred to as *grid snapping*.

When we snap a point set P onto a grid we obtain a point set $P' = \{p'_1, p'_2, \ldots, p'_n\}$ where p'_i is the grid point that is closest to $p_i \in P$, and in Figure 2 we see that it may happen that $p'_i = p'_j$ while $p_i \neq p_j$. In the same figure we see that the order type of P, depending on the orientation of triples of points, can be different from P', depending on the grid width ω and the positions of the points in P. It is clear that in this case, the topological information of P' does not resemble that of P. We note that the condition that *all* topological information is conserved is too strong:

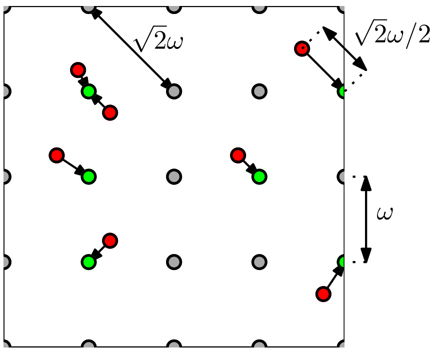

Figure 2: Grid snapping in the case $k = 1/\omega = 4$. The points in P are indicated in red, and they corresponding grid points in P' are indicated in green. Note that points in P' may be collinear or even on the same location while corresponding points in P are not. The largest distance between a pair of corresponding points $\mathrm{dist}(p_i, p'_i)$ is at most $\sqrt{2}\omega/2$. On the right side we see that digitizing a point set may change its order type, depending on the locations of the points in P and the grid width ω.

no matter how precise (less wide) we make the grid, there will always be an infinite number of real values in between grid points. If the point set P contains several of such points between two grid points, their snapping P' will not preserve the topological information of P. However, as the width of the grid decreases, inconsistencies can only occur on a smaller and smaller scale.

We conclude that the conservation of topological information depends on the locations of the points in P, as well as the grid width ω. Also, since it is impossible to conserve all topological information for any choice of P, we find a condition to conserve the order type rather than all topological information.

Order types. Since topological information is lost nearly always in the process of digitization, one could argue that the results of calculations based on a digitized point set are not reliable. However, when the order type is conserved, this is not true. Aichholzer, Aurenhammer and Krasser [1] enumerate all possible order types for point sets with cardinality up to 10, and show that we can use the order type to characterize important combinatorial properties of a finite point configuration. Aichholzer and Krasser [3] later find methods to characterize order types for point sets of higher cardinality. This database shows us that point sets with the same order type share topological information, and other works show us how the order type can be used to solve various geometrical problems [2, 4, 26]. This leads us to believe that the conservation of order type, instead of all topological information, will suffice as a condition that calculations with P' as input configuration will yield the same answers as input configuration P for geometrical problems that we are interested in.

1.2 Order type realizability.

An *abstract order type* is a mapping χ that maps every triple from a set of n elements to an element of $\{+, -, 0\}$, satisfying some set of conditions. We denote the collection of all triples that can be formed from n elements by $\binom{[n]}{3}$, so formally, we can write $\chi : \binom{[n]}{3} \to \{+, -, 0\}$.

Recall the definition of order type of a point set. We say a point set P *realizes* an order type χ if $\chi_P(p_i, p_j, p_k) = \chi(i, j, k)$, for all $i < j < k$. We say that we have successfully represented a point set P as a discrete identity P' when P and P' realize the same order type.

Since three orientations are possible for a triple of points, one trivially state that there exist at most $3^{\binom{n}{3}}$ different order types for a set of n points. If P is a real-valued point set where no three points are collinear, Goodman and Pollack [16] show that the number of order types that P could

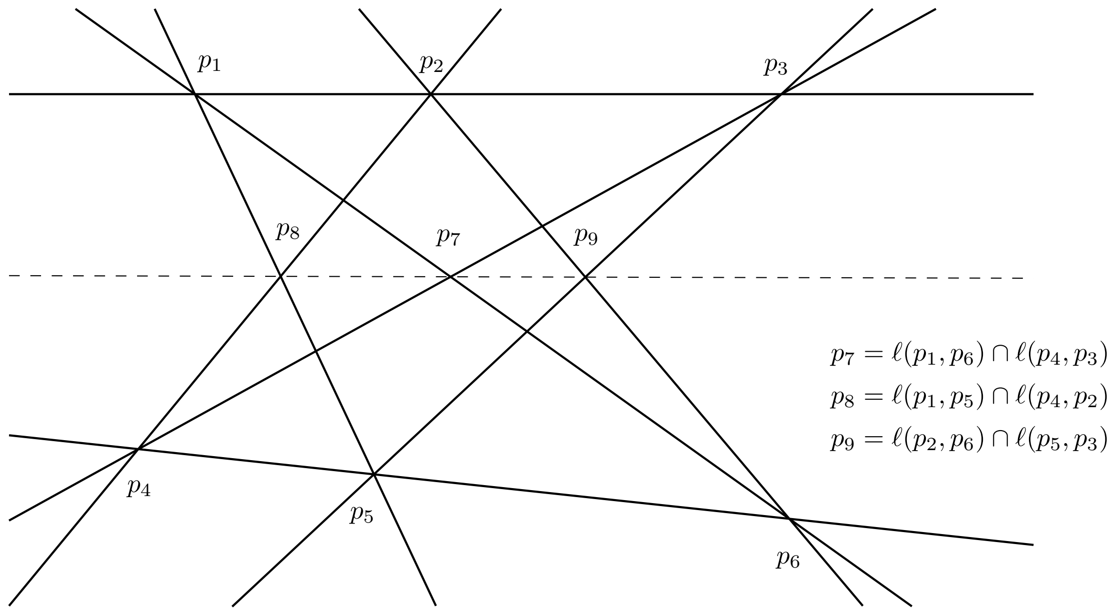

Figure 3: Illustration of the configuration used in Pappos' hexagon theorem: when $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$ are two sets of collinear points, then the points of intersection $\{p_7, p_8, p_9\}$ are collinear as well.

realize is upper bounded by $2^{4n\log(n)+\mathcal{O}(n)}$. Throughout this thesis (following the example of [17]) we will assume that the point set P contains no collinear points. This assumption is often referred to as the points lying in general position [8]. It is unknown how many out of all possible abstract order types for a set of n elements can be realized by a configuration of n points where points *can* be collinear.

Pappos' hexagon theorem. First we consider a result that explicitly shows that not every abstract order type χ can be realized by a point set. Pappos of Alexandria lived in the fourth century, and he proved the following result in the field of projective geometry, while that field developed only many centuries later [28].

Theorem I (Pappos's hexagon theorem)**.** *Let p_1, p_2, p_3 be three collinear points and let p_4, p_5 and p_6 also be three collinear points. The lines $\ell(p_1, p_6)$, $\ell(p_1, p_5)$, $\ell(p_2, p_6)$ intersect the lines $\ell(p_4, p_3)$, $\ell(p_4, p_2)$, $\ell(p_5, p_3)$, respectively, and the three points of intersection are collinear.*

Nine points are described (refer to Figure 3), which together form a point set Q which realizes a certain order type. We denote by p_7, p_8 and p_9 the three points of intersection, the theorem states that when the first eight points are placed in the plane as described, we know that $\chi_Q(p_7, p_8, p_9) = 0$. An abstract order type, however, *can* map some set of 9 elements to the same values for all triples, except for $\{7, 8, 9\}$. Hence we must conclude that not every abstract order type can be realized by a point set. One might say that it is incorrectly assumed that the points of intersection exist, but in projective geometry, these always exist and have the desired properties. That means this result holds in the projective plane, and if something does not exist in $\mathbb{RP}^2$, then certainly it does not in $\mathbb{R}^2$.

Norm of a point set and bit complexity. For the abstract order types that *can* be realized by a point set, there is some "degree of realizability". For example, Grünbaum has shown that configurations exist that cannot be expressed in rational coordinates, hence require infinite precision [21]. In the remainder of this section, we will formalize the scale on which we measure the degree of realizability. We also mention Mnëv's universality theorem here, in order to show that order type realizability is not just of interest when considering representations of point sets with finite precision, but also when considering problems in real algebraic geometry.

Goodman, Pollack and Sturmfels [18] have provided us with the definition of a *norm*, which we will use to formalize the notion 'degree of realizability'. The norm ν of a point set P is a measure of the minimal number of grid points needed to describe a point set that realizes the same order type as P. Or formally,

$$\nu(P) = \min \max\{|x'_1|, |x'_2|, \ldots, |x'_n|, |y'_1|, |y'_2|, \ldots, |y'_n|\}$$

where the minimum is taken over all point sets $P' \subset \mathbb{Z}^2$ that realize the same order type as P. Remember that Grünbaum showed that there are configurations that cannot be expressed in rational coordinates, in that case this definition does not apply, and for those cases, we define the norm to be zero. The notion "degree of realizability" of an order type χ is formalized by the norm of any point set that realizes χ.

Given the norm of some point set P, we know that it can be realized on an integer grid with width $\nu(P)$. This means that coordinates can have $2\nu(P)$ different values, so the amount of bits required to express the largest coordinate is $\log(2 \cdot \nu(P))$. Goodman, Pollack and Sturmfels [17] show that they can construct for any value of n a point set whose norm is doubly exponential in n, which means that there is a point set with bit complexity at least of the order $\mathcal{O}(2^n)$.

Constructing a set which requires a large grid size to be realized was enough to show a lower bound, but to give an upper bound, we must represent every possible point set up to order type. This has been done by Aichholzer and Krasser [3] for point sets with cardinality up to 11, the amount of different order types for higher cardinalities is too large to expect a lot of progress using this approach. However, Goodman, Sturmfels and Pollack [17] showed that the norm of a point set of n points in general space (thus excluding collinear points, which excludes those sets that require infinite precision) is upper bounded by $2\hat{}2^{cn}$ for some constant c. Obviously, there is an entirely different way to solve this problem; they used a mathematical connection between order types and semi-algebraic sets which allowed them to use a result defined in terms of those sets [20, lemma 10] to find the maximal grid size needed to represent any point set of n points in general space.

Mnëv's universality theorem. The connection between semi-algebraic sets and order types is a lot stronger than the application described above. One of the most astounding results regarding order types, is by Mnëv [24]. Let S be a semi-algebraic set, i.e., the set of real numbers satisfying a given set of polynomial equations and inequalities. Then there is an abstract order type O_S, such that the realization space is "stably-equivalent" to S. Roughly speaking, stably-equivalent means that topological properties are the same. In particular, S is empty, if and only if O_S can be realized. This shows that the anecdotal results that we saw before, i.e., Pappos hexagon theorem and the doubly exponential lower bound found by Goodman, Pollack and Sturmfels are not just isolated phenomena, but actually stem from a deeper mathematical connection to real algebraic geometry.

Even though it is beyond the scope of this thesis to discuss Mnëv's universality theorem in more detail, we note that the problem we study not only concerns representing point sets with finite precision. It may even translate to a result in the field of algebraic geometry, just like a result in algebraic geometry offered a solution to proving the doubly exponential upper bound on the norm of points in general space.

1.3 Relation to recent work.

We have seen that some abstract order types cannot be realized by a planar point set, some require infinite precision, and some can only be realized by a point set with a norm that has a doubly exponential dependence on the cardinality n. The most recent of these results was published in 1989, but there are recent developments in this area as well. For example, Fabila-Monroy and Huemer define an average case and show that, in the average case, a point set can be snapped to grid with size polynomial in n without changing its order type, as n tends to infinity. Devillers, Duchon, Glisse and Goaoc found a similar result as a byproduct of their research, in which they study the amount of bits an algorithm needs to read, in order to determine the order type of a point set in the average case.

The average case. If we generate an abstract order type by just randomly picking $\binom{n}{3}$ orientations, the probability is rather large that it cannot be realized. Also, it makes sense to assume that an order type that can only be realized when using infinitely precise coordinates, does not occur often in practice, as Devillers *et al.* showed. When interested in representing point sets on a grid, a more natural way to define an average case is to pick points randomly from a probability space.

Fabila-Monroy and Huemer [13] chose some value $M \in \mathbb{N}$, and considered a set S of n points distributed independently and uniformly at random in $[0, M]^2$ to be an average case. They denote by S' the set of n integer grid points that lie closest to the points in S, and show that in the average case, the order types of S and S' are equal with a probability that tends to 1 as n tends to infinity. In this thesis, we agree on the definition of average case, but if we want to compare this result to the upper bound on the norm found by Goodman *et al.*, we need a different result.

We can use this result to find an upper bound on the norm by observing that S and S' have the same order type, and thus the same norm. The norm of S' is easily upper bounded by the grid size, so we can state that in the average case, the norm of S is upper bounded by M, with probability that tends to 1 as n tends to infinity. Still, in order to compare this result to that of Goodman *et al.*, we need prove a different result.

In Section 3, the theorem of Fabila-Monroy and Huemer is adjusted, and a proof of this adjusted theorem is given, which allows us to state the desired result. This result holds not only asymptotically, but for all n, so we can use it to compute an expected value of the norm. Instead of S, we use P, a set of n points, distributed independently and uniformly at random in $[0, 1]^2$. We upper bound the norm in the same way, and find that the expected value of the norm of P is upper bounded by $16n^3 \cdot 2^{cn}$, where c is the same as the constant that Goodman *et al.* used in their upper bound on the norm.

Determining order types. Devillers, Duchon, Glisse and Goaoc used the same setting as the average case and, as a byproduct of their research, found a result that is similar to that of Fabila-Monroy and Huemer. On could say that Devillers *et al.* researched what we called the 'degree of realizability' of order types, but formalized it in a different way: they determine the amount of coordinate bits an algorithm must read in order to determine the order type of an ordered point set, in the average case. They give a near-tight upper bound of $4n \log(n) + \mathcal{O}(n)$ on this amount of coordinate bits [9].

In the case that the order type of a point set P cannot be determined by reading b bits, it is likely that P', the 2^b-digitization of P, has a different order type than P. If we choose b large enough that the order types of P and P' are equal, then the norm of P and P' is equal, and smaller than the amount of bits required to represent P'. Using this argument, we expect the norm of P to be upper bounded by $\mathcal{O}(n \cdot n^{4n})$. This is, asymptotically, larger than the value of $16n^3 \cdot 2^{cn}$ that we found, which makes sense, since this is a rather crude translation a result with different applications.

These recent works show that order type realizability is an active field of research, and these new results created a new problem. When we compare the result of Goodman *et al.*, the upper bound on the norm of any point set in general space, which is $2\hat{\ }2^{cn}$ for some constant c, to our result of $16n^3 \cdot 2^{cn}$ in the average case, where c is the same constant, we see that for large n, the difference becomes rather large. Goodman *et al.* showed that their upper bound holds for all point sets in general space and that it is a tight bound, so we could call this an lower bound in the worst case. We found an upper bound on the norm in the average case. Using this nomenclature, the problem that arises is explaining the difference between worst-case and average-case analysis.

1.4 Smoothed analysis.

Large differences between average-case and worst-case analysis are not rare. Quite recently, Spielman and Teng developed a hybrid of both models of analysis, which enables us to explain such differences. They stated that worst-case analysis can improperly suggest that an algorithm will perform poorly by examining its performance under the most contrived circumstances. Average-case analysis was introduced to provide a less pessimistic measure of the performance of algorithms, and many practical algorithms perform well on the random inputs considered in average-case analysis. However, average-case analysis may be unconvincing as the inputs encountered in many application domains may bear little resemblance to the random inputs that dominate the analysis [34].

Spielman and Teng proposed a new analysis called *smoothed analysis*, where the performance of an algorithm is studied under slight perturbations of arbitrary inputs. They explain their analysis by applying it on the Simplex algorithm, which was known for particularly good performance in practice that was impossible to verify theoretically [22]. Using this new approach, they show that the Simplex algorithm has "smoothed complexity" polynomial in the input size and the standard deviation of Gaussian perturbations of those inputs, which was the desired theoretical verification of its good performance.

This theoretical verification in terms of smoothed analysis is based on the assumption that worst-case examples are rare, and spread out over the possible inputs. If this is the case, then the smoothed analysis will suggest significantly better performance than the worst-case analysis. This is because smoothed analysis is a combination of the average- and worst-case analysis, and offers a parametrization of the scale between the two, and this is what enables us to verify that worst-case examples are rare, under the condition that they are spread out.

Smoothed complexity. The smoothed expected complexity of an instance can be defined as follows [10]: Let us fix some δ, which describes the maximum magnitude of perturbation. We denote by $(\Omega_\delta,\mu_\delta)$ a corresponding probability space where each $x \in \Omega_\delta$ defines for each instance I a new 'perturbed' instance I_x. We denote by $\mathcal{C}(I_x)$ the complexity of instance I_x. The smoothed expected complexity of instance I equals

$$\mathcal{C}_\delta(I) = \mathop{\mathbb{E}}_{x\in\Omega_\delta}[\mathcal{C}(I_x)] = \int_{\Omega_\delta} \mathcal{C}(I_x)\mu_\delta(x)\,\mathrm{d}x$$

If we denote by Γ_n the set of instances of size n, then the smoothed complexity equals:

$$\mathcal{C}_{\mathrm{smooth}}(n,\delta) = \max_{I\in\Gamma_n} \mathop{\mathbb{E}}_{x\in\Omega_\delta}[\mathcal{C}(I_x)]$$

This formalizes what was mentioned before: not only do the majority of instances behave nicely, but actually in every neighborhood (bound by the maximal perturbation δ) the majority of instances behave nicely. The smoothed complexity is measured in terms of n and δ. If the expected complexity is small in terms of $1/\delta$ then this means worst-case examples are spread out, in which case we have a theoretical verification of the hypothesis that worst-case examples are well-spread.

Smoothed complexity of order type realizability. Smoothed analysis is not only a handy tool for explaining differences between average-case and worst-case analysis, but also corresponds to a more realistic analysis in our case, where the input is a point set. This is because real-life applications include measuring errors, apparatuses are not necessarily calibrated and modern measuring tools already process information to display more relevant data, but rounding occurs along the way. Hence it is natural to consider an input set of points P not as a set of exact values,

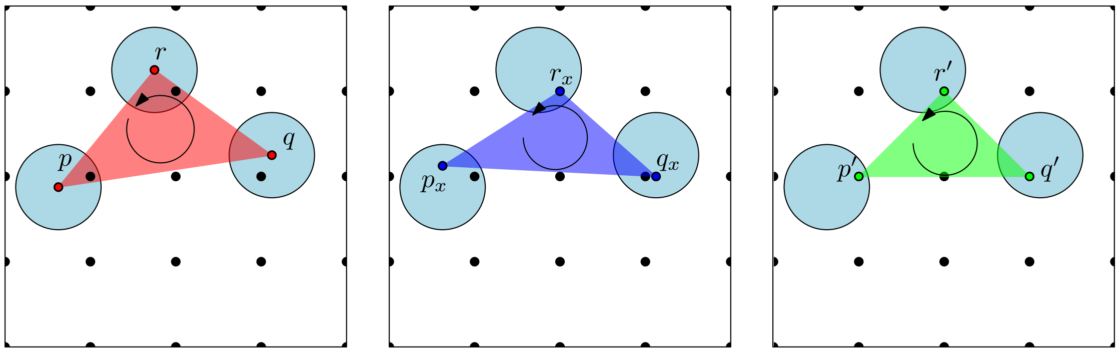

Figure 4: A possible instance $P = \{p, q, r\}$ with positive orientation, being randomly perturbed to $P_x = \{p_x, q_x, r_x\}$, which are then snapped to the grid points $\{p', q', r'\} = P'$, both positively oriented. Note that for some perturbations, snapped points may lie outside the blue disks. We are interested in the likelihood of the order types of P_x and P' being equal.

but instead as a set of points with an inherent measuring error δ. In this case, a more realistic approach is to study the problem under slight random perturbations of P.

If we apply this analysis on the problem of order type realizability, the complexity that we are interested in is the norm ν. We determine the norm of a point set P_x as described before in Section 1.3: if we determine the value of k such that P', the k-digitization of P_x, has the same order type as P_x, then we can compute the norm in terms of k.

In our case, we are interested in planar point sets, so perturbations are bound by a disk with radius δ, hence $\Omega_\delta := \{x_i \mid x_i \in \text{disk}(\delta), i \in \{1, 2, \dots, n\}\}$. Then, an adversary specifies a point set $P = \{p_1, p_2, \dots, p_n\}$, which is then perturbed slightly by some perturbation vector $x \in \Omega_\delta$. This results in $P_x = \{p_1 + x_1, p_2 + x_2, \dots, p_n + x_n\}$, so we can define the smoothed norm of instance P as the expected norm of P_x by integrating over Ω_δ, or formally

$$\nu_\delta(P) = \mathop{\mathbb{E}}_{x \in \Omega_\delta} [\nu(P_x)] = \int_{\Omega_\delta} \nu(P_x)\mu_\delta(x)\, \mathrm{d}x$$

If we denote the set of all possible point sets of cardinality n by Γ_n, the smoothed norm equals:

$$\nu_{\text{smooth}}(n, \delta) = \max_{P \in \Gamma_n} \mathop{\mathbb{E}}_{x \in \Omega_\delta} \nu(P_x)$$

Figure 4 shows an example of an instance $P = \{p, q, r\}$ of three points, chosen by an adversary. The blue disks around these points have radius δ, so the perturbed points $P_x = \{p_x, q_x, r_x\}$ must lie somewhere inside these disks. Then, these points are snapped to grid points to form the set $P' = \{p', q', r'\}$. For every perturbation, we determine the likelihood that P' has the same order type as P_x, and integrate over the possible perturbations, to find the smoothed expected norm of P. To find the smoothed complexity of order type realizability, we find the worst-case instance an adversary can specify and compute the smoothed expected norm of that instance.

Other applications. Clearly, smoothed analysis is a lot more complicated than a worst-case analysis, but the results are usually a lot better. This analysis can be used to show that various algorithms actually run in polynomial time, which forms the theoretical explanation of good performance in practice that was missing so far.

For example, the smoothed analysis of the Nemhauser-Ullmann Algorithm [25] for the knapsack problem shows that it runs in polynomial time [6]. A more general result that holds for a large class of combinatorial problems containing all binary optimization problems, is that problems of this class can be solved in smoothed polynomial time if and only if it can be solved in pseudopolynomial time [7]. Other famous examples are the smoothed analysis of k-means algorithm [5], the 2-OPT

TSP local search algorithm [11], and the local search algorithm for MaxCut [12]. Not surprisingly, teaching material on this subject has become available [29, 31, 30].

Problem statement. In this thesis we are interested in abstract order types which can be realized by a point set that has finite bit complexity (a finite, non-zero norm). For a given abstract order type that is realized by a real-valued point set P, it is non-trivial to determine its norm. However, if we can find a k-digitization of P which has the same order type then the norm of P is upper bound by $2k$ (Observation 1).

To illustrate this idea for finding an upper bound on the norm of a point set P. We apply this technique in the average case in Section 3 where we show that for a random point set the expected norm is upper bound by $\mathcal{O}(n2^n)$. We have already seen in the worst-case there exists an example of a point set P that has a norm lower bound by $\mathcal{O}(2^{2^n})$.

In this thesis, we aim to explain this large difference between average-case and worst-case analysis by applying smoothed analysis on order type realizability. We assume that for any arbitrary real-valued point set P, each point will be perturbed within a uniform distribution of radius δ (the probability space Ω_δ) around the point. The main problem studied in this thesis is finding an upper bound on the norm of an arbitrary real-valued point set P in this implementation of the smoothed analysis framework.

1.5 Results.

First we will consider the average case, i.e., P is a set of n points, chosen independently and uniformly at random from $[0,1]^2$. We find the following upper bound on the expected value of the norm of P, comparable to the result of Devillers *et al.* [9] that an algorithm can determine the order type of a point set with at most $4n\log(n) + \mathcal{O}(n)$ coordinate bits, in the average case.

Theorem 1. *Let P be a set of n points chosen independently and uniformly at random from $[0,1]^2$. The expected value of the norm of P is smaller than $10n^3 \cdot 2^{cn}$ for some constant c.*

When we do not consider the average case, but the case that an adversary can choose the worst-case positions of the points in P, and consider this set under slight perturbations, we can compute an upper bound on the norm of P_x that holds with probability γ.

Theorem 2. *Given a point set $P \subset [0,1]^2$, $|P| = n$ and some magnitude of perturbation δ, define $P_x = \{p_1 + x_1, p_2 + x_2, \ldots, p_n + x_n\}$ where $(x_1, x_2, \ldots, x_n) \in \Omega_\delta$ is a perturbation as defined in the introduction. Then the norm of P_x is smaller than $\frac{4n^3}{(1-\gamma)\delta}$ with probability larger than γ.*

We can use the lemmas that we needed to prove this theorem in a slightly different way in order to calculate the expected value of the norm of P_x, which is the smoothed expected norm of P.

Theorem 3. *Given a point set $P \subset [0,1]^2$, $|P| = n$ and some magnitude of perturbation δ, define $P_x = \{p_1 + x_1, p_2 + x_2, \ldots, p_n + x_n\}$, where $(x_1, x_2, \ldots, x_n) \in \Omega_\delta$ as defined in the introduction. Then the expected value of the norm of P_x is smaller than $4n^3/\delta \cdot 2^{cn}$.*

The smoothed norm of order type realizability is equal to the smoothed expected norm of P that maximizes this value, and since the smoothed expected norm of every point set P is upper bounded by the above theorem, we have that the smoothed norm is upper bounded by $9n^3/\delta^2$.

2 Preliminaries

Before we prove these theorems, we must understand what proves them and why. In Section 3 we prove a slightly alternative version of the theorem that R. Fabila-Monroy and C. Huemer proved in [13], which is why we follow the same steps. They show that we can choose some value M large enough, such that if S is a set of n points chosen independently and uniformly at random from $[0, M]^2$, and we move the points in S to S', the set of n integer grid points that are closest to the points in S, then S and S' realize the same order type. Then, trivially, the norm of both S and S' is smaller than M. They show this result holds when n tends to infinity, and we will show a similar result that holds for all n with a given probability. By integration, we find the expected value of the norm in the average case.

In the case that we allow an adversary to choose a point set P, which is then perturbed to P_x by a perturbation vector $x \in \Omega_\delta$, we will upper bound the norm of P_x. We do that in the same way as in the average case, but the condition that P', which is the digitization of P_x in this case, has the same order type as P_x, becomes a bit more complicated.

Therefore, in contrast to the average case, we will not attempt to place the points of P_x on a grid directly, but we find the probability that the points in P_x can move a distance d without changing the order type of P_x. To show that the order type of P_x does not change under this condition, we show that the orientations of all triangles in P_x do not change when the vertices of those triangles are free to move a distance d. There are two ways in which we will show that the orientation of a triangle will not change, both depending on the minimal height of the triangle.

After we show that points in P_x can move a distance d in any direction, we relate this to the norm by the observation that digitizing a point set P_x is, in a way, moving every point in P_x a limited distance (to a grid point). We show two examples of perturbations of a given point set in Figure 5.

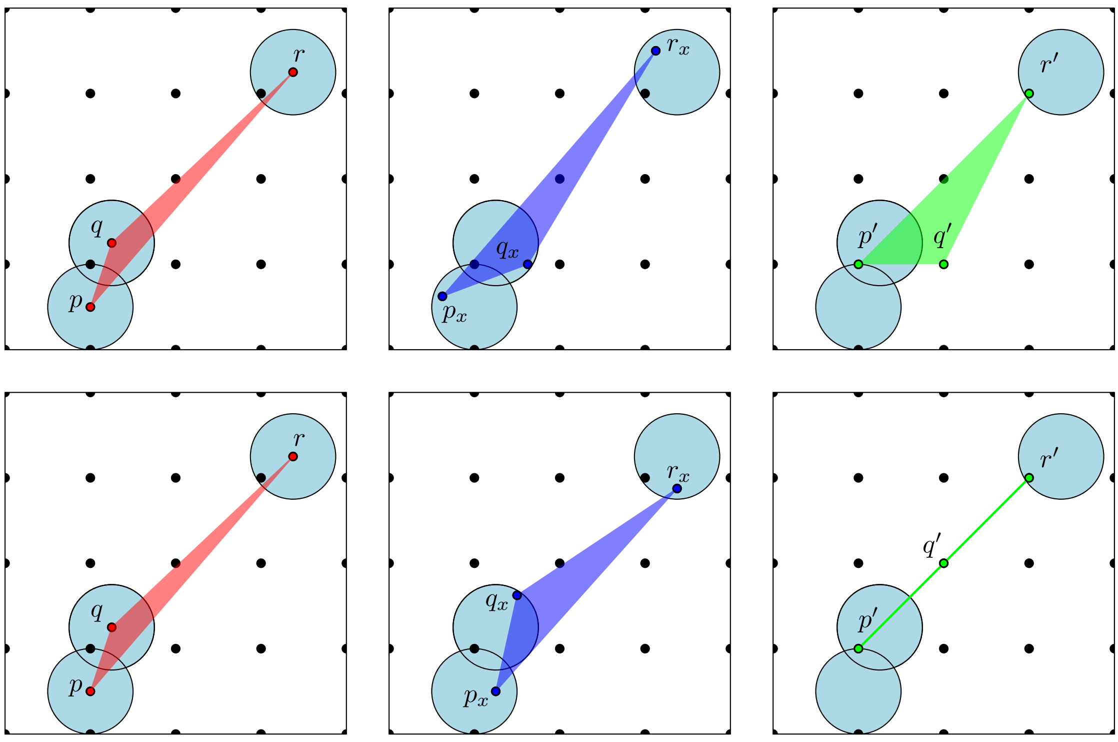

Figure 5: Given a point set $P = \{p, q, r\}$, a perturbed point set P_x may have a different order type than P, but we are interested in the value of k, such that P_x and its k-digitization P' have the same order type. The top row shows a perturbation for which the value of k is large enough, the bottom row shows another perturbation of the same point set P with the same magnitude of perturbation for which the value of k is not large enough.

We denote by $P' \subset \omega\mathbb{Z}^2$ the set of n grid points that are closest to the points in P_x. The largest value a coordinate in P_x, and thus in P', can have, is $1 + \delta$. If we multiply every coordinate in P'

by $1/\omega$, all points are integer grid points and the largest coordinate is $(1+\delta)/\omega < 2/\omega$, since we are only interested in values of $\delta < 1$. From this we can trivially deduce that the norm of P' is at most $2/\omega$.

Observation 1. *Let P' be the $1/\omega$-digitization of P_x, if P' and P_x have the same order type, the norm $\nu(P_x) = \nu(P')$ is at most $2/\omega$.*

If we move the points of P_x to a grid with spacing ω, the maximal distance any point needs to move is $\sqrt{2}\omega/2 < \omega$ (refer to Figure 2). So after we have showed that any point in P_x can move a distance d without changing the order type, we can move every point to a grid with spacing $\omega = d$ without changing order type. This leads to the following observation:

Observation 2. *Let d be the distance every point in P_x can move without changing the order type of P_x, then the norm of P_x is at most $2/d$.*

With this observation we can let go of the idea of snapping points to grid because the norm is upper bounded in terms of d, a property of the perturbed point set P_x. We note, however, that the underlying idea of why this is an upper bound on the norm is that we can snap P_x to a grid and upper bound the norm of the snapped point set.

3 Expected value of the norm in the average case

This section is based on [13], where Theorem II is proven. Given a number M, We define the integer grid $\Gamma_M := [0, M]^2 \cap \mathbb{Z}^2$.

Theorem II. *Let $\epsilon > 0$, n a natural number and let $M := \lfloor n^{3+\epsilon} \rfloor$. Let S be a set of n points chosen independently and uniformly at random from the square $[0, M]^2$. Let S' be the subset of n points of Γ_M that are closest to S. Then, the probability that S' and S have the same order type tends to 1, as n tends to infinity.*

Here we prove a slightly alternative version of this theorem, in which the upper bound holds not only asymptotically, but also in general with probability γ. A small error has been corrected, besides that, the proof is comparable to the original.

3.1 Upper bound on the norm.

Following the example of Fabila-Monroy and Huemer, we first state the theorem and after that the lemmas that lead up to the proof of this theorem. The theorem and lemmas below are numbered alphanumerically, to indicate that they are based on the work of others.

Theorem A. *Let P be a set of points chosen independently and uniformly at random from $[0, 1]^2$. With probability γ, the norm of P is upper bounded by $\frac{10n^3}{1-\gamma}$.*

Before we continue to prove this theorem, we divide the problem in a few smaller steps: first we will consider under what conditions the orientation of a triangle changes. Then we can apply the union bound to the probability that will be the result of those considerations, to find the probability that the orientations of any pair of corresponding triangles in P and P' are different. If no pair of corresponding triangles has a different orientation, the order types of P and P' are equal. Using observation 1, we upper bound the norm of P which proves the theorem.

We will first consider three points $\{p, q, r\} \subset [0, 1]^2$, and their digitizations $\{p', q', r'\} \subset \omega\mathbb{Z}^2$. Since p' is the grid point closest to p, the maximum distance between p' and p is $\sqrt{2}\omega/2$ (refer to Figure 2). Now we consider two snapped points p' and q' and the disks with radius $\sqrt{2}\omega/2 < \omega$ around them as in figure 6. Since all we know about the position of the original points p and q is that they are at most a distance ω from p' and q', the green disks are the possible locations of

p and q. If r lies inside the purple wedge, the orientation of the triangle $\Delta(p,q,r)$ is likely to be different from triangle $\Delta(p',q',r')$. Now we also take the digitizing of r into account, and we note that if $\mathrm{dist}(\ell(p',q'),r)$ is less than ω, the orientations of $\Delta(p,q,r)$ and $\Delta(p',q',r')$ may be different.

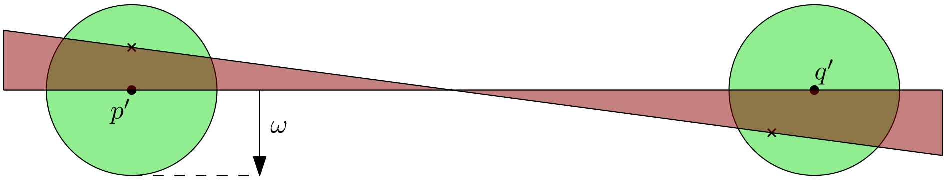

Figure 6: Digitizing points p and q changes the orientation of triangle $\Delta(p,q,r)$ whenever r lies inside the purple area.

We denote by $A_{pq}(omega)$ the area that is such that if r lies inside it, the orientations of $\Delta(p,q,r)$ and $\Delta(p',q',r')$ may be different. To find the probability that r lies inside this area, we upper bound the value of $A_{pq}(omega)$ first.

To find an upper bound on $A_{pq}(omega)$, we find the locations of p and q inside the disks that maximize this area. Clearly, it is on the edge of the disk, but the location on the edge depends on the distance between p' and q'. To simplify our calculations, we replace the circles by squares with the same centers and width 2ω, the double wedge that can be created by the points p,p',q' and q is always smaller than the purple wedge in figure 7 below. We must also take the orange area into account, these are the possible locations of r where $\mathrm{dist}(\ell(p',q'),r)$ is less than ω. The desired upper bound on $A_{pq}(omega)$ is found by the intersection of these two colored areas.

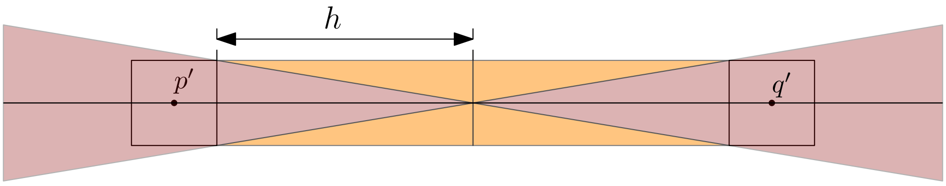

Figure 7: If there is a third point r in one of the colored areas, the orientation of $\Delta(p,q,r)$ can be different from the orientation of $\Delta(p',q',r')$ because of the digitizing of p and q (purple area) or r (orange area)

Lemma B. *If p and q are at a distance $dist(p,q) \geq 8\omega$, then the probability that r, distributed uniformly at random inside the unit square, is inside area $A_{pq}(\omega)$, is less than $10\omega/dist(p,q)$.*

Proof. From Figure 6 we can deduce that the distance between p' and q' can be bounded in terms of $\mathrm{dist}(p,q)$: $\mathrm{dist}(p,q) - 2\omega \leq \mathrm{dist}(p',q') \leq \mathrm{dist}(p,q) + 2\omega$. This gives us the following bounds on $h = \mathrm{dist}(p',q')/2 - \omega$:

$$\mathrm{dist}(p,q)/2 - 2\omega \leq h \leq \mathrm{dist}(p,q)/2$$

The orange area, denoted by $A_O(\omega)$, is now easily bounded: $A_O(\omega) = 2\omega \cdot h \leq \omega \cdot \mathrm{dist}(p,q)$. For the purple area we observe that we can bound the area of both wedges by one wedge with the same apex angle and height $\sqrt{2}$, since that is the maximal length of any line in $[0,1]^2$. We can express that wedge as two triangles on top of each other, where one triangle can be seen as an orange triangle of which the height has been multiplied by the factor $\sqrt{2}/h$. So we can upper bound the purple area $A_P(\omega)$ by taking twice the area of an orange triangle, multiplied by the factor $2/h^2$. This leads to

$$A_P(\omega) \leq 2 \cdot h\omega/2 \cdot 2/h^2 = 2\omega/h \leq 2\omega/(\mathrm{dist}(p,q)/2 - 2\omega)$$

In the original proof, refer to [13], the upper bound instead of the lower bound on h was used erroneously to upper bound the $1/h$ term.

The total area $A_{pq}(\omega)$ is equal to the sum of the colored areas, so we have that:

$$A_{pq}(\omega) = A_O(\omega) + A_P(\omega) \leq \omega \text{ dist}(p,q) + \frac{2\omega}{\text{dist}(p,q)/2 - 2\omega}$$

Now we will apply the condition $\text{dist}(p,q) \geq 8\omega$, which implies $\text{dist}(p,q)/2 - 2\omega \geq \text{dist}(p,q)/4$

$$\leq \left(\text{dist}(p,q)^2 + 8\right) \frac{\omega}{\text{dist}(p,q)}$$

Observe that $\text{dist}(p,q) \leq \sqrt{2}$

$$\leq \frac{10\omega}{\text{dist}(p,q)}$$

Which proves our lemma. □

This probability will no longer depend on $\text{dist}(p,q)$ if we integrate over the probability space of this distance, as in the original proof. The points p and q are distributed independently and uniformly at random in $[0,1]^2$, so we can use the following density function for the distance between two points p and q [33]:

$$f_D(x) = \begin{cases} 2\pi x - 8x^2 + 2x^3 & 0 \leq x \leq 1 \\ 4x\left(\arcsin(1/x) - \arccos(1/x) + 2\sqrt{x^2-1} - x^2/2 - 1\right) & 1 \leq x \leq \sqrt{2} \end{cases}$$

Lemma C. *If p, q and r are distributed independently and uniformly at random inside the unit square, the probability that r is inside area $A_{pq}(\omega)$, is less than 30ω.*

The proof of this lemma requires us to solve an integral, which could use the help of a computer program that could do that for us, so we have moved the proof of this lemma along with the code to Appendix A, so the reader can verify this result.

The integral we need to solve is the following

$$\Pr(r \in A(\omega)) = \int_0^{\sqrt{2}} Pr(r \in A_{pq}(\omega) | \text{dist}(p,q) = x) f_D(x) \text{d}x$$

We have found the probability $Pr(r \in A_{pq}(\omega)|\text{dist}(p,q) > 8\omega) < 10\omega/\text{dist}(p,q)$, and since every probability is upper bounded by 1, we have that $Pr(r \in A_{pq}(\omega)|\text{dist}(p,q) \leq 8\omega) \leq 1$, so we have the equations we need to solve this integral.

We have found the probability that r lies inside the area A_{pq}, which is equal the probability that the orientations of $\Delta(p,q,r)$ and $\Delta(p',q',r')$ may be different. We can extend this result to the probability that any of the corresponding triangles in P and P' may have a different orientation by the union bound, or Boole's inequality, which states the following about events E_i:

$$Pr(\cup_{i=1}^m E_i) \leq \sum_{i=1}^m Pr(E_i)$$

Lemma D. *Let P be a set of n points, chosen independently and uniformly at random from $[0,1]^2$, and let $P' \subset \omega\mathbb{Z}^2$ be the set of n grid points that are closest to the points in P. With probability larger than $1 - 5n^3\omega$, P and P' have the same order type.*

Proof. We choose some indexing of all triangles that the points in P can form. Note that there are $\binom{n}{3} < n^3/6$ distinct triples in a set of n points, so the amount of triangles in P is also less than $n^3/6$.

We define E_i to be the event that triangle i may change orientation, and Lemma C gives us an upper bound on $Pr(E_i)$ for any i. Combining the statements above, we have that

$$\sum_{i=1}^{\binom{n}{3}} Pr(E_i) \leq \binom{n}{3} 30\omega < 5n^3\omega$$

The probability that none of the corresponding triangles in P and P' have a different orientation is the complement of the probability that any pair of corresponding triangles does have two different orientations. We can conclude that the order type P and P' have is the same when all corresponding triangles in P and P' have the same orientation, which happens with probability larger than $1 - 5n^3\omega$. □

We have found a lower bound on the grid width ω that is such that we are sure that P', the $1/\omega$-digitization of P, has the same order type as P. We use Observation 1 from Section 2: when P and $P' \subset \omega\mathbb{Z}^2$ have the same order type, the norm of P is smaller than $2/\omega$. The norm $\nu(P)$ is thus upper bounded by $N = 2/\omega$ with probability larger than $1 - 10n^3/N$, and equating the latter to γ gives us that $N = 10n^3/(1-\gamma)$. So with probability larger than γ, the norm of P is upper bounded by $10n^3/(1-\gamma)$, which proves Theorem A.

We can also use Lemma D to prove Theorem II, we do so by choosing the grid width ω in such a way that the probability that P and P' have the same order type tends to 1, as n tends to infinity. We choose $\omega = 1/\lfloor n^{3+\epsilon} \rfloor$ where ϵ is some value larger than zero. If we now let $S = P/\omega$ and $S' = P'/\omega$, we have proved Theorem II.

3.2 Expected value of the norm.

The results found above still are not ideal; an upper bound that holds with a certain probability is not yet comparable to an upper bound that always holds. So in order to compare the average case to the theoretic upper bound on the norm of $2\hat{}2^{cn}$ (where c is some constant as follows from [17]), we need to compute the expected value of the norm in the average case.

From the previous section, we know that the norm $\nu(P)$ is upper bounded by N with probability larger than $1 - 10n^3/N$, so trivially, $Pr(\nu(P) \geq N) < 10n^3/N$. We can use this probability to compute the expected value of the norm. First we show how to calculate an expected value given such a probability, after which we can simply substitute this probability.

Assume we are given a probability that a non-negative random variable d is larger than some threshold value t and that the domain of d is upper bounded by m for some value $m > 0$. The probability density function of random variable d is denoted by f_d. Then, by definition,

$$\int_0^m Pr(d \geq t)\ \mathrm{d}t = \int_0^m \int_t^m f_d(z)\ \mathrm{d}z\ \mathrm{d}t$$

Since we know that $f_d(z) \geq 0$, the requirements of Tonelli's theorem [35] are met, which allows us to change the order of integration.

$$\int_0^m \int_t^m f_d(z)\ \mathrm{d}z\ \mathrm{d}t = \int_0^m \int_0^z f_d(z)\ \mathrm{d}t\ \mathrm{d}z$$

Since $f_d(z)$ does not depend on t, we can move it out of the inner integral, which then becomes rather trivial.

$$\int_0^m \int_0^z f_d(z)\ \mathrm{d}t\ \mathrm{d}z = \int_0^m f_d(z) \int_0^z 1\ \mathrm{d}t\ \mathrm{d}z = \int_0^m z f_d(z)\ \mathrm{d}z$$

This is the definition of the expected value of a non-negative random variable d with a maximal value of m. We can put all this together to find the following equation:

$$\mathbb{E}[d] = \int_0^m Pr(d \geq t)\ \mathrm{d}t$$

The expected value of the norm $\nu(P)$ can be computed by substituting $Pr(\nu(P) \geq N) < 10n^3/N$, and m by $2\hat{}2^{cn}$, the upper bound found by Goodman, Strumfels and Pollack. Since this upper bound on $Pr(\nu(P) \geq N)$ is larger than one for values of N smaller than $10n^3$, we use 1 as an upper bound of that probability in the domain $[0, 10n^3]$:

$$\begin{aligned}\mathbb{E}[\nu(P)] &= \int_0^{2\hat{}2^{cn}} Pr(\nu(P) \geq N)\, \mathrm{d}N \\ &< \int_0^{10n^3} 1\, \mathrm{d}N + \int_{10n^3}^{2\hat{}2^{cn}} 10n^3/N\, \mathrm{d}N \\ &= 10n^3 \left(1 - 0 + \log(2\hat{}2^{cn}) - \log(10n^3)\right)\end{aligned}$$

Since $n > 1$, we have that $\log(10n^3) > 1$, so

$$< 10n^3 \cdot 2^{cn}$$

Remember that Goodman *et al.* show that c is some constant, which proves the following theorem:

Theorem 1. *Let P be a set of n points chosen independently and uniformly at random from $[0,1]^2$. The expected value of the norm of P is smaller than $10n^3 \cdot 2^{cn}$ for some constant c.*

4 Robustness of a triangle

Throughout this thesis we are given a point set $P = \{p_1, p_2, \ldots, p_n\}$. This point set is then slightly perturbed to a point set $P_x = \{p_1 + x_1, p_2 + x_2, \ldots, p_n + x_n\}$, where x is a random vector from the probability space Ω_δ. Lastly, we digitize P_x and obtain a point set P' and we are interested in whether or not the order type of P_x and P' are the same, which, using Observation 1 can be used to upper bound the norm of P_x.

Suppose you are given three points p, q and r, which are perturbed to the points p_x, q_x and r_x and finally snapped to the points p', q' and r'. We are interested in whether or not the orientation of $\Delta(p_x, q_x, r_x)$ is any different from the orientation of $\Delta(p', q', r')$.

We say a triangle $T_x := \Delta(p_x, q_x, r_x)$ is *d-robust*, if for every set of vectors $\vec{p}, \vec{q}, \vec{r} \in \text{disk}(d)$ holds that $T' := \Delta(p_x + \vec{p}, q_x + \vec{q}, r_x + \vec{r})$ has the same orientation as T_x. For any point p in the plane, its $\frac{1}{\omega}$-digitization could be regarded as an arbitrary translation p of a distance at most $\frac{\sqrt{2}}{2}\omega < \omega$ and thus if T_x is *ω-robust*, it has the same orientation as T'.

Observation 3. *Let T_x be an ω-robust triangle and let T' be the $\frac{1}{\omega}$-digitization of T_x, then T' has the same orientation as T_x.*

We are interested in the probability that we obtain a triangle $\Delta(p_x, q_x, r_x)$ for which its $\frac{1}{\omega}$-digitization has the same orientation. It follows from this observation that we can focus on the probability that $\Delta(p_x, q_x, r_x)$ is 2ω-robust instead.

Now we need a relation between robustness and measurable identities. We indicate all of the possible vectors $\vec{p}, \vec{q}, \vec{r} \in \text{disk}(\delta)$ by disks with radius δ around the vertices (refer to Figure 8), and consider which choice of vectors results in a different orientation of $T' = \Delta(p_x + \vec{p}, q_x + \vec{q}, r_x + \vec{r})$. In the figure, we see a triangle of which the smallest height is $2d$, and it is clear that there is only one choice of vectors $\vec{p}, \vec{q}, \vec{r}$ such that the orientation of T' is different from T_x: all three vectors are perpendicular to the base corresponding to the smallest height. From here we can easily deduce that if T_x is d-robust, there can be no choice of vectors of length d such that T_x has a different orientation than T', so the height for each base of T_x must be more than $2d$. Also, if the height for every base of T_x is larger than $2d$, then clearly, there is no choice of vectors of length d such that the orientation of T_x is different from T', so T_x is robust.

Observation 4. *A triangle T_x is d-robust if and only if for each base of the triangle T_x, its height is more than $2d$.*

In Section 4.1 and 4.2, we consider the placements of r_x that are such that $\Delta(p_x, q_x, r_x)$ is not d-robust. Another method of finding the probability that $\Delta(p_x, q_x, r_x)$ is d-robust, is by simply

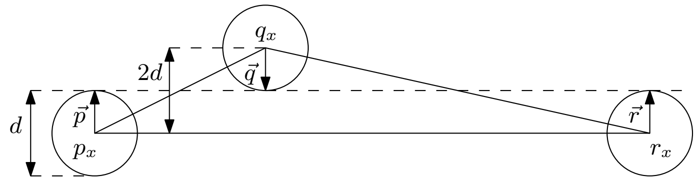

Figure 8: Illustration of the robustness of a triangle: if the smallest height is equal to $2d$, then there is one choice of $\vec{p}, \vec{q}, \vec{r} \in \text{disk}(\delta)$ such that the orientation of T_x is different from T', so this triangle is *not* d-robust.

considering distances from lines (representing the height of a triangle). This method is a lot simpler and even leads to a better result, but it may be more difficult to understand why this method works on a first reading. This method is discussed in Section 4.3.

4.1 Finding the worst-case scenario for obtaining robust triangles.

We denote by r_h the height of the triangle T_x with respect to the base $\overline{p_x q_x}$, p_h and q_h are defined similarly. We are interested in an upper bound on the probability that the triangle T_x is not d-robust and thus (after applying observation 4) $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$ for some positive value d.

To get an understanding of how to upper bound this probability, we first assume two things:

- $\text{dist}(p_x, q_x)$ is guaranteed to be large.
- $d(p_x, r_x), d(q_x, r_x) < d(p_x, q_x)$.

Because of the second assumption, the height r_h is always smaller than q_h and p_h. This is because clearly the area A_t of a triangle is constant and we know that if we denote by h the height and b the base length, $A_t = h \cdot b/2$, so the smallest height corresponds to the largest base.

This scenario can always be achieved in the smoothed analysis model by placing the original points p and q far away from one another and by placing the original point r somewhere in between (refer to Figure 9). Specifically, if we place p, q and r such that $2\delta < d(p, r), d(q, r) < 2d(p, q)$ then it is easy to see that both assumptions are guaranteed. Under these specific conditions, $Pr(r_h < 2d)$ is minimal if the original points p, q, r are collinear.

Lemma 4. *Let p, q and r be three collinear points with $2\delta < d(p, r), d(q, r) < 2d(p, q)$ and let d be a positive real number then*

$$Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{8d}{3\delta}.$$

Proof. Without loss of generality, we assume that the point r lies between p and q and that they all lie on a horizontal line, as depicted in Figure 9.

Consider the perturbed points p_x and q_x. We denote by $A_x(d) = \{v \in \mathbb{R}^2 \mid d(\ell(p_x, q_x), v) \leq 2d\}$ the union of all points in the plane that have distance at most d from the line $\ell(p_x, q_x)$. If $r_x \notin A_x(d)$, we know that the height r_h is larger than $2d$. Because of the assumptions we made earlier, r_h is always smaller than q_h and p_h, so when $r_x \notin A_x(d)$, we know that $\min\{p_h, q_h, r_h\} > 2d$.

The area of the intersection between the perturbation disk of the point r and $A_x(d)$ is less than $4d \cdot 2\delta$ whereas the total area of the area of the disk that r_x can be perturbed to is $\pi\delta^2$. It follows that the probability that r_x is placed within $A_x(d)$ (and therefore the probability that $\min\{p_h, q_h, r_h\}$ is less than $2d$) is upper bound by $\frac{8d\delta}{\pi\delta^2} < \frac{8d}{3\delta}$ which concludes the proof. □

Lemma 4 operates under the assumption that $d(p_x, r_x), d(q_x, r_x) < d(p_x, q_x)$. In Figure 10 we illustrate what happens if we remove this assumption. In the figure, we placed the initial points p and r arbitrarily close and kept the original point q at a large distance.

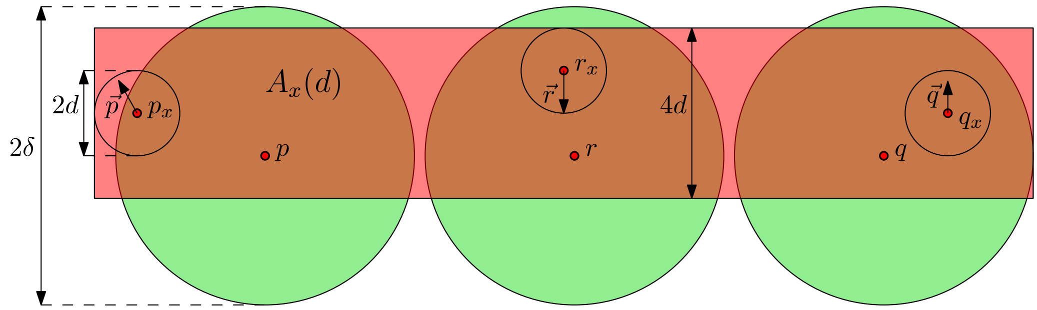

Figure 9: In this figure we see three original points p, q and r and the disks indicating the locations they can be perturbed to. After perturbing p and q we obtain two points p_x and q_x. We show a slab of height $4d$ which contains the line p_xq_x in its middle, denoted by $A_x(d)$. Observe that if r_x lies within this slab, that we can move p_x, q_x and r_x by at most distance d, to obtain a triangle which has a different orientation than $\Delta(p_x, q_x, r_x)$.

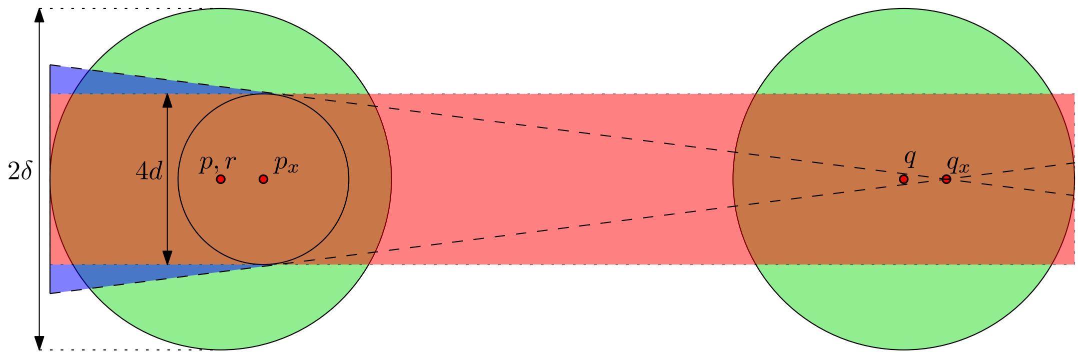

Figure 10: In this figure we see three original points p, q and r and the disks indicating the locations they can be perturbed to. After perturbing p and q we obtain two points p_x and q_x. As in Figure 9, we indicate the area that is such that if r_x lies within this area, we can move p_x, q_x and r_x by at most distance d, to obtain a triangle which has a different orientation than $\Delta(p_x, q_x, r_x)$.

Given the point placements p_x and q_x, we are interested in both the height r_h and p_h.

Just as in Lemma 4 we can define an area $A_x(d)$ as the union of placements of r_x such that $r_h < 2d$ and this area is again shown in red in Figure 10. However, we can also define an area $B_x(d)$, as the union of all placements of r_x such that $p_h < 2d$, this region is bounded by the dashed lines. These regions are not identical, the area of $B_x(d)$ that does not intersect with $A_x(d)$ is indicated in blue. Hence, we have found a larger area for poor placements of r_x, and thus a larger probability that there exists a base of T_x, such that the height of T_x with respect to that base is less than $2d$.

In Figure 10 we can see that the area of bad placements of r_x increases as we move p_x and q_x closer together. The probability that p_x and q_x are closer together increases as we move the original points p and q closer together. Hence, we consider the worst-case scenario where the original points p, q and r lie arbitrarily close. Note that this scenario is equivalent to having p_x, q_x and r_x be three samples from the uniform distribution over Ω_δ which is comparable to the scenario from [13] and [9].

4.2 The worst-case scenario for obtaining robust triangles.

Having defined the worst-case placement of p, q and r an adversary can choose, we can upper bound the probability $Pr(\min\{p_h, q_h, r_h\} < 2d)$ in the worst case. Since it holds in the worst case, it holds for any placement of p, q and r, which is result we need in order to be able to extend this

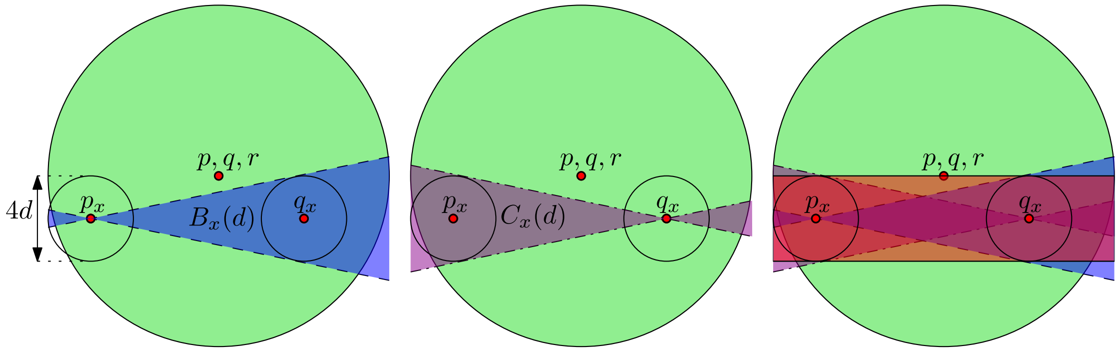

Figure 11: Here we show the areas $B_x(d)$ and $C_x(d)$, and the union $A_x(d) \cup B_x(d) \cup C_x(d)$. The green disk is the area that r_x can be perturbed to, if r_x lies inside the differently colored areas, $\Delta(p_x, q_x, r_x)$ is not d-robust.

result to all triangles in P_x.

Theorem 5. *Let p, q and r be three points chosen by an adversary which are perturbed by some perturbation vector $x \in \Omega_\delta$ and let $d \geq 0$ a real number. Then $Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{44d}{\delta}$.*

We prove this theorem in two steps. First we show, that if the distance between p_x and q_x is some value $t \geq 8d$, then $Pr(\min\{p_h, q_h, r_h\} \leq 2d \mid \text{dist}(p_x, q_x) = t) < \frac{12d}{t}$. We then consider all values t for the distance $\text{dist}(p_x, q_x)$ to upper bound $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$.

Lemma 6. *Let p_x, q_x and r_x be three points chosen independently and uniformly at random from disk(δ). If the distance between p_x and q_x is at least $4d$, then*

$$Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{11d}{dist(p_x, q_x)}$$

Proof. Given p_x and q_x we can define three areas: the area $A_x(d)$ is the union of all possible placements of r_x such that $r_h \leq 2d$. Since p_x and q_x are given, this is the slab as was illustrated in Figure 9. The second area, $B_x(d)$, is the union of all possible placements of r_x such that $p_h < 2d$, this area is a wedge that originates from q_x and is bound by two tangents to a circle around p_x of radius $2d$ as illustrated in Figure 10. If r_x lies inside this wedge, the line $\ell(r_x, q_x)$ intersects the circle of diameter $2d$ around p_x, so $\text{dist}(\ell(r_x, q_x), p_x) = p_h \leq 2d$. The area $C_x(d)$ is the union of all possible placements of r_x such that $q_h < 2d$ and it is defined symmetrically. An illustration of these areas is provided in Figure 11.

If r_x is not placed in $A_x(d) \cup B_x(d) \cup C_x(d)$, then per definition $\min\{p_h, q_h, r_h\} > 2d$. We upper bound the area of $(A_x(d) \cup B_x(d) \cup C_x(d)) \cap \text{disk}(\delta)$ to conclude this proof. We simplify our calculations by replacing the disks around p_x and q_x by squares with side lengths equal to the diameters of the disks: this does not change $A_x(d)$, but it does increase $B_x(d)$ and $C_x(d)$ which is not a problem since we are establishing an upper bound. Now we use Figure 12 to see that $A_x(d)$ is entirely contained in the wedges $B_x(d)$ and $C_x(d)$, except for the red area. Clearly, the red area is smaller than the blue area, note that they are congruent triangles and compare their largest sides. The blue area is counted doubly in the sum $B_x(d) + C_x(d)$, so we can state

Observation 5. $A_x(d) \cup B_x(d) \cup C_x(d) < B_x(d) + C_x(d)$

If we assume the left side of the double wedge that forms $B_x(d)$ and the right side of the double wedge that forms $C_x(d)$ to be zero (refer to Figure 11), then $A_x(d)$ is still contained in the sum $B_x(d) + C_x(d)$ (refer to Figure 12). Also, the left side of $B_x(d)$ we assume to be zero is entirely contained in the left side of $C_x(d)$ and the right side of $C_x(d)$ we assume to be zero is entirely contained in the right side of $B_x(d)$. This means that Observation 5 is still valid under these assumptions.

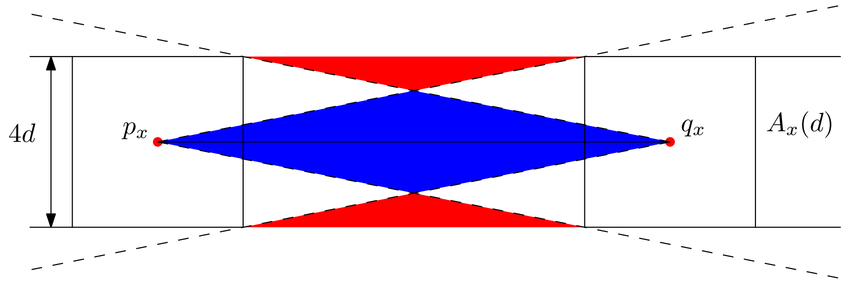

Figure 12: Here we see that the area $A_x(d)$ is for a large part contained in the wedges originating from p_x and q_x.

Using these assumptions, the areas $B_x(d)$ and $C_x(d)$ are easily upper bounded: the height of the wedge originating from p_x is $4d$ after a horizontal distance of $\text{dist}(p_x, q_x) - 2d$, so the area of that part of the wedge is $4d \cdot (\text{dist}(p_x, q_x) - 2d)/2$. The maximal horizontal distance is 2δ if we intersect the wedge with disk(δ), so the maximal area of the wedge is

$$2d(\text{dist}(p_x, q_x) - 2d) \cdot \left(\frac{2\delta}{\text{dist}(p_x, q_x) - 2d}\right)^2 = \frac{8d\delta^2}{\text{dist}(p_x, q_x) - 2d}$$

We assumed that $\text{dist}(p_x, q_x) \geq 4d$, so we see that this area is smaller than $16d\delta^2/\text{dist}(p_x, q_x)$. Using Observation 5 and that the area $C_x(d)$ can be upper bounded in the same way as $B_x(d)$, we state that $(A_x(d) \cup B_x(d) \cup C_x(d)) \cap \text{disk}(\delta) < 2B_x(d) \cap \text{disk}(\delta) \leq 32d\delta^2/\text{dist}(p_x, q_x)$

The area of the disk that r_x can be perturbed to is $\pi\delta^2$, and the perturbation is chosen from a uniform distribution, so the probability that $r_x \in (A_x(d) \cup B_x(d) \cup C_x(d)) \cap \text{disk}(\delta)$ is smaller than

$$\frac{32d\delta^2}{\text{dist}(p_x, q_x)} \cdot \frac{1}{\pi\delta^2} < \frac{11d}{\text{dist}(p_x, q_x)}$$

which concludes the proof. □

Now we can proceed to the second step of our proof. Lemma 6 first assumes that $\text{dist}(p_x, q_x) \geq 4d$ and then provides an upper bound for $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$. Here we show a how to upper bound the probability $Pr(\min\{p_h, q_h, r_h\} < 2d)$ in general, so we need to lose our restriction on $\text{dist}(p_x, q_x)$.

Lemma 7. *Let p_x, q_x and r_x be three points, chosen independently and uniformly at random from disk$(0, \delta)$ and $d \geq 0$ a real number. Then $Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{44d}{\delta}$.*

Proof. At this point, we note a few things: the first thing we note is that $\text{dist}(p_x, q_x)$ always lies between 0 and 2δ. The second thing we note is that if $\text{dist}(p_x, q_x) < 11d$, then Lemma 6 gives an upper bound for $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$ of one, which means that the lemma does not supply any additional information. Lastly we note that if $\text{dist}(p_x, q_x) < 4d$, we have no bound on $Pr(\min\{p_h, q_h, r_h\} < 2d)$ other than the fact that probabilities are always upper bound by 1.

Now we express the probability $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$ without dependence on $\text{dist}(p_x, q_x)$ by integrating over the probability space of $\text{dist}(p_x, q_x)$. This means that we will solve the following integral: $\int_0^\infty Pr(\min\{p_h, q_h, r_h\} \leq 2d \mid \text{dist}(p_x, q_x) = t) f_\delta(t)\ \text{d}t$, where $f_\delta(t)$ is the probability density function of $\text{dist}(p_x, q_x)$. Because of these three prior observations, we can upper bound this probability by stating:

$$Pr(\min\{p_h, q_h, r_h\} < 2d) \leq \int_0^{4d} 1 \cdot f_\delta(t)\ \text{d}t\ + \int_{4d}^{11d} 1 f_\delta(t)\ \text{d}t\ + \int_{11d}^{2\delta} 11d/t \cdot f_\delta(t)\ \text{d}t$$

It is hard to derive the exact value for the probability density function $f_\delta(t)$. But since we are interested in an upper bound on the probability $Pr(\min\{p_h, q_h, r_h\} \leq 2d)$, we can use an upper bound on the probability density function. Since p_x and q_x are both a sample from disk(δ), $Pr(\text{dist}(p_x, q_x) = t \mid t > 2\delta) = 0$. Consider any placement of p_x, if we want that $\text{dist}(p_x, q_x) = t$, then we can only place q_x on the intersection between the domain disk(δ) and the circumference of a disk with center p_x and radius t. The area of this intersection is at most $2\pi t$ and hence $Pr(\text{dist}(p_x, q_x) = t) \leq \frac{2\pi t}{\pi\delta^2} = \frac{2t}{\delta^2}$. The probability density function $f_\delta(t)$ on the domain $t \in [0, 2\delta]$ that specifies the probability that $\text{dist}(p_x, q_x) = t$ is therefore upper bound by $\frac{2t}{\delta^2}$ and thus we state:

$$f_\delta(t) < \begin{cases} 2t/\delta^2 & t \leq 2\delta \\ 0 & t > 2\delta \end{cases}$$

By applying this upper bound for the probability density function:

$$\begin{aligned} Pr(\min\{p_h, q_h, r_h\} < 2d) &\leq \int_0^{4d} 2t/\delta^2 \text{ dt } + \int_{4d}^{11d} 2t/\delta^2 \text{ dt } + \int_{11d}^{2\delta} 22d/\delta^2 \text{ dt} \\ &= 16d^2/\delta^2 - 0 + 121d^2/\delta^2 - 16d^2/\delta^2 + 44d/\delta - 242d^2/\delta^2 \\ &< 44d/\delta \end{aligned}$$

This is the probability we were looking for, hence our lemma is proven. □

Now we use the earlier observation that placing p, q and r arbitrarily close is the worst-case configuration an adversary could choose. Because of this, we know that the probability $Pr(\min\{p_h, q_h, r_h\} < 2d)$ is upper bounded by $44d/\delta$, independent of the configuration the adversary chooses. This proves theorem 5:

Theorem 5. *Let p, q and r be three points chosen by an adversary which are perturbed by some perturbation vector $x \in \Omega_\delta$ and let $d \geq 0$ a real number. Then $Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{44d}{\delta}$.*

4.3 An alternative worst-case situation.

In this section, we present an alternative proof to Theorem 5, which will even lead to a better result. To prove this theorem, we need to show that the minimum height of $\Delta(p_x, q_x, r_x)$ is more than some distance $2d$ by showing that for every base, the height is at least $2d$. In the previous sections, we focused on the placement of r_x. However, we can exploit the way we implement smoothed analysis to prove this theorem in a lot more straightforward way: simply considering the distance of any perturbed point to any line.

The situation is as before: an adversary may choose three points $p, q, r \subset [0, 1]^2$. After these points are chosen, they are perturbed by some vector $x \in \Omega_\delta$ as defined in the introduction. In this case we have $p_x := p + x_p$ where x_p is chosen randomly and uniformly from disk(δ), q_x and r_x are defined in the same way.

We will first consider r_h, the distance from the line $\ell(p_x, q_x)$ to the perturbed point r_x. Since the perturbation is a random variable, our result will be the probability $Pr(r_h \leq 2d)$. When we consider the heights p_h and q_h (defined similarly to r_h), the situation is exactly the same. Then we can apply the union bound to find $Pr(\min\{p_h, q_h, r_h\} < 2d)$, which will prove theorem 5.

Lemma 8. *Let $p, q, r \subset [0, 1]^2$ be perturbed to p_x, q_x, r_x as described above. Then the height r_h is at most $2d$ with probability smaller than $8d/3\delta$.*

Proof. The proof of this lemma is comparable to that of Lemma 4, so we also refer to Figure 9, where we see a slab of width $4d$ centered on the the line $\ell(p_x, q_x)$. We know that if r_x lies outside of this slab, then $r_h > 2d$. We see that under any condition, independent of the placement of p, q and r, the area of intersection between this slab and the disk that r_x can be perturbed to is smaller than $8d\delta$, no matter how the points p, q and r were placed. Since the perturbation is chosen uniformly from this disk with area $\pi\delta^2 > 3\delta^2$, the probability that r_x lies inside this area of intersection is thus smaller than $8d/3\delta$. $\square$

We note that the same result holds for p_h and q_h by permuting p_x, q_x and r_x in the proof above, so we can apply the union bound. In our case, the union bound states the following:

$$Pr(r_h \leq 2d) \cup Pr(p_h \leq 2d) \cup Pr(q_h \leq 2d) \quad \leq \quad Pr(r_h \leq 2d) + Pr(p_h \leq 2d) + Pr(q_h \leq 2d)$$

These probabilities are correlated, but the union bound is a result that is true for any set of probabilities. Since all three probabilities are upper bounded by $8d/3\delta$, we have that $Pr(\min\{p_h, q_h, r_h\} < 2d) = Pr(r_h \leq 2d) \cup Pr(p_h \leq 2d) \cup Pr(q_h \leq 2d) \leq 8d/\delta$. This way, we have simplified our proof and improved our result:

Theorem 5a. *Let p, q and r be three points chosen by an adversary which are perturbed by some perturbation vector $x \in \Omega_\delta$ and let $d \geq 0$ a real number. Then $Pr(\min\{p_h, q_h, r_h\} \leq 2d) < \frac{8d}{\delta}$.*

4.4 Upper bound on the norm of a perturbed point set.

Both Theorem 5 and 5a are proven in the previous sections and they have similar results. In this section, we will show how we can use such a result to upper bound the norm of a perturbed point set.

Given a perturbed point set P_x with $|P_x| = n$, we can think of it as a collection of $\binom{n}{3} < n^3/6$ triangles. Theorem 5a applies to all of these triangles independently, and we can apply the union bound on this collection of triangles to prove the following lemma.

Lemma 9 (Union bound)**.** *Let $P \subset [0,1]^2$ be perturbed to P_x as described in the introduction. Every triangle in P_x is d-robust with probability larger than $1 - 2n^3d/\delta$.*

Proof. We will apply the union bound, or Boole's inequality, which states the following about events E_i:

$$Pr(\cup_{i=1}^m E_i) \leq \sum_{i=1}^m Pr(E_i)$$

We choose some labeling of triangles of which the vertices are points in P_x, and define E_i to be the event that triangle i is not d-robust. For any triangle with vertices p_x, q_x and r_x we know that with probability smaller than $8d/\delta$, the minimum height of $\Delta(p_x, q_x, r_x)$ is $2d$ or less. Using Observation 4, we know that $\Delta(p_x, q_x, r_x)$ is not d-robust in that situation, which means $Pr(E_i) < 8d/\delta$.

We have bounded $Pr(E_i)$ independently of i, so the choice of indexing of the collection of triangles formed by the points in P_x is not relevant, only the total amount of triangles: $\binom{n}{3} < n^3/6$.

In this case, $\cup_{i=1}^m E_i$ denotes the event that any of the triangles in P_x is not d-robust. Substituting $m = \binom{n}{3}$ and the value we found for $Pr(E_i)$, we have that $Pr(\cup_{i=1}^m E_i) < \binom{n}{3} \cdot 8d/\delta < 2n^3d/\delta$. The complement of this event is that all triangles in P_x are d-robust, which happens with probability $1 - 2n^3d/\delta$. $\square$

Observation 3 states that an ω-robust triangle T_x has the same orientation as T', its $1/\omega$-digitization. If all triangles in P_x are ω-robust, then P', the $1/\omega$-digitization of P_x, has the same order type as P_x. Observation 1 from Section 2 states that in this case, the norm $\nu(P_x)$ is upper bounded by $2/\omega$. Substituting $d = \omega$ in Lemma 9, this results in

$$Pr(\nu(P_x) < 2/\omega) > 1 - \frac{2n^3\omega}{\delta} \tag{1}$$

Equating the right-hand side to γ, we find $\omega = \frac{(1-\gamma)\delta}{2n^3}$, or $2/\omega = \frac{4n^3}{(1-\gamma)\delta}$, which proves the following theorem:

Theorem 2. *Given a point set $P \subset [0,1]^2$, $|P| = n$ and some magnitude of perturbation δ, define $P_x = \{p_1 + x_1, p_2 + x_2, \ldots, p_n + x_n\}$ where $(x_1, x_2, \ldots, x_n) \in \Omega_\delta$ is a perturbation as defined in the introduction. Then the norm of P_x is smaller than $\frac{4n^3}{(1-\gamma)\delta}$ with probability larger than γ.*

4.5 The smoothed norm.

The probability defined in Lemma 9 can also be used to calculate the expected value of the norm of P_x, $\mathbb{E}[\nu(P_x)]$. This is the smoothed norm of point set P. Since this results holds for any case, it is the smoothed norm of any point set with cardinality n.

From the previous section, Equation 1, we know that the norm $\nu(P_x)$ is upper bounded by N with probability larger than $1 - 4n^3/(\delta N)$, using the substitution $\omega = 2/N$. We can thus state $Pr(\nu(P) \geq N) < 4n^3/(\delta N)$. We can use this probability to compute the expected value of the norm. First we show how to calculate an expected value given such a probability, after which we can simply substitute this probability. This discussion is exactly equal to that in Section 3.2.

Assume we are given a probability that a non-negative random variable d is larger than some threshold value t and that the domain of d is upper bounded by m for some value $m > 0$. The probability density function of random variable d is denoted by f_d. Then, by definition,

$$\int_0^m Pr(d \geq t)\ \mathrm{d}t = \int_0^m \int_t^m f_d(z)\ \mathrm{d}z\ \mathrm{d}t$$

Since we know that $f_d(z) \geq 0$, the requirements of Tonelli's theorem [35] are met, which allows us to change the order of integration.

$$\int_0^m \int_t^m f_d(z)\ \mathrm{d}z\ \mathrm{d}t = \int_0^m \int_0^z f_d(z)\ \mathrm{d}t\ \mathrm{d}z$$

Since $f_d(z)$ does not depend on t, we can move it out of the inner integral, which then becomes rather trivial.

$$\int_0^m \int_0^z f_d(z)\ \mathrm{d}t\ \mathrm{d}z = \int_0^m f_d(z) \int_0^z 1\ \mathrm{d}t\ \mathrm{d}z = \int_0^m z f_d(z)\ \mathrm{d}z$$

This is the definition of the expected value of a non-negative random variable d with a maximal value of m. We can put all this together to find the following equation:

$$\mathbb{E}[d] = \int_0^m Pr(d \geq t)\ \mathrm{d}t$$

The expected value of the norm $\nu(P)$ can be computed by substituting $Pr(\nu(P) \geq N) < 4n^3/(\delta N)$, and m by $2\hat{\ }2^{cn}$, the upper bound found by Goodman, Sturmfels and Pollack. Since this upper bound on $Pr(\nu(P) \geq N)$ is larger than one for values of N smaller than $4n^3/\delta$, we use 1 as an upper bound of that probability in the domain $[0, 4n^3/\delta]$:

$$\begin{aligned}
\nu_\delta(P) = \mathop{\mathbb{E}}_{x \in \Omega_\delta}[\nu(P_x)] &= \int_0^{2\hat{\ }2^{cn}} Pr(\nu(P_x) \geq N)\ \mathrm{d}N \\
&< \int_0^{4n^3/\delta} 1\ \mathrm{d}N + \int_{4n^3/\delta}^{2\hat{\ }2^{cn}} 4n^3/(\delta N)\ \mathrm{d}N \\
&= 4n^3/\delta\left(1 - 0 + \log(2\hat{\ }2^{cn}) - \log(4n^3/\delta)\right)
\end{aligned}$$

Since $n > 1$ and $\delta < 1$, we have that $\log(4n^3/\delta) > 1$, so

$$< 4n^3/\delta \cdot 2^{cn}$$

Remember that Goodman *et al.* show that c is some constant, which proves the following theorem:

Theorem 3. *Given a point set $P \subset [0,1]^2$, $|P| = n$ and some magnitude of perturbation δ, define $P_x = \{p_1 + x_1, p_2 + x_2, \ldots, p_n + x_n\}$, where $(x_1, x_2, \ldots, x_n) \in \Omega_\delta$ as defined in the introduction. Then the expected value of the norm of P_x is smaller than $4n^3/\delta \cdot 2^{cn}$.*

5 Prospects

In this thesis we managed to find a theoretical explanation of the large difference between the average-case and worst-case analysis of order type realizability. What we cannot claim, however, is that our explanation is complete: we did not show the bound on the smoothed norm to be tight, nor did we show the bound on the norm in the average case to be tight.

These results can easily be generalized to higher dimensions under the condition that the definition of order type changes accordingly. At this point, it is unclear whether this generalization finds any interesting applications.

6 Retrospects

This section is a short overview of the originality of the presented work. Existing results are indexed by Roman numbers, improved results are indexed by letters and original work is indexed by Arabic numbers.

Theorems I and II are paraphrased from sources [28] and [13], respectively. Theorem A is an improvement of Theorem II and its proof is thus based on the proof of Theorem II.

I consider Theorems 1 and 3 to be important new results, but Theorem 2 is new as well. The proof of Theorem 5 is partially based on Theorem II, but the proof of Theorem 5a is new work. I consider Lemma 8 to be the most elegant result, since its proof makes good use of the specific conditions of this problem.

A Proof of Lemma C

Here we repeat Lemma C and show how the integral results in the value that it was claimed to be.

Lemma C. *If p, q and r are distributed independently and uniformly at random inside the unit square, the probability that r is inside area $A_{pq}(\omega)$, is less than 30ω.*

Proof. We have found the probability $Pr(r \in A_{pq}|\text{dist}(p,q) > 8\omega) < 10\omega/\text{dist}(p,q)$, and since every probability is upper bounded by 1, we have that $Pr(r \in A_{pq}|\text{dist}(p,q) \leq 8\omega) \leq 1$. We can use these formulas to calculate the probability $Pr(r \in A)$:

$$\Pr(r \in A_{pq}) = \int_0^{\sqrt{2}} Pr(r \in A|\text{dist}(p,q) = x) f_D(x)\text{d}x$$

We will use the following probability density function, where x is used as a shorthand notation for $\text{dist}(p,q)$.

$$f_D(x) = \begin{cases} 2\pi x - 8x^2 + 2x^3 & 0 \leq x \leq 1 \\ 4x\left(\arcsin(1/x) - \arccos(1/x) + 2\sqrt{x^2-1} - x^2/2 - 1\right) & 1 \leq x \leq \sqrt{2} \end{cases} \tag{A.1}$$

Now we can let some program like *Mathematica* do the work, here ω has been replaced by w, so this code can be copied into a *Mathematica* cell directly.

```
In[1]:= Expand[Integrate[(2 Pi*x - 8 x^2 + 2 x^3), {x, 0, 12 w}] +
      Integrate[
       15w/x*(2 Pi*x - 8 x^2 + 2 x^3), {x, 12 w, 1}] +
      Integrate[
       15w/x*(4 x (ArcSin[1/x] - ArcCos[1/x] +
            2 Sqrt[x^2 - 1] - x^2/2 - 1)), {x, 1, Sqrt[2]}] ] // N
```

This results in the following probability:

$$\int_0^{\sqrt{2}} \Pr(r \in A_{pq}|\text{dist}(p,q) = x) f_D(x)\text{d}x = -1365.33\omega^4 + 1194.67\omega^3 - 301.593\omega^2 + 29.7321\omega$$

$$< -1300\omega^4 - 300\omega^2(1 - 4\omega) + 30\omega$$

Since it is safe to assume that the grid width will be smaller than 1/4,

$$< 30\omega \tag{A.2}$$

This concludes the proof of our lemma. □

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